ПЗ 35

a) $A[x] = x' - 4tx$, $D_A = \{x \in C^1[0;1] : x(0) = 0\}$.

$$\begin{cases} x' - 4tx = y \\ x(0) = 0 \end{cases} \qquad x' - 4tx = 0$$

$$\int \frac{dx}{x} = 4\int t\,dt$$

$$x = Ce^{2t^2} = C(t)\cdot e^{2t^2} \Rightarrow y = C'(t)e^{2t^2}$$

$$C'(t) = \frac{y}{e^{2t^2}} \qquad x' = C'(t)e^{2t^2} + 4tC(t)\cdot e^{2t^2}$$

$$C(t) = \int \frac{y\,dt}{e^{2t^2}} \Rightarrow x(t) = e^{2t^2}\cdot\left(\int_0^t \frac{y(s)}{e^{2s^2}}ds + C\right), \; C \in R$$

$$x(0) = 0 \Leftrightarrow 0 = e^0 \cdot C \Rightarrow C = 0$$

$$A^{-1}[y] = e^{2t^2}\cdot\int_0^t \frac{y(s)}{e^{2s^2}}ds$$

$$A[A^{-1}[y]] = A\left[e^{2t^2}\cdot\int_0^t \frac{y(s)}{e^{2s^2}}ds\right] = \frac{d}{dt}\left(e^{2t^2}\int_0^t \frac{y(s)}{e^{2s^2}}ds\right) - 4te^{2t^2}\cdot$$

$$\cdot\int_0^t \frac{y(s)}{e^{2s^2}}ds = \cancel{4te^{2t^2} + e^{2t^2}\int_0^t \frac{y(s)}{e^{2s^2}}ds}$$

$$= 4te^{2t^2}\int_0^t \frac{y(s)}{e^{2s^2}}ds + e^{2t^2}\cdot\frac{y(t)}{e^{2t^2}} - 4te^{2t^2}\cdot\int_0^t \frac{y(s)}{e^{2s^2}}ds = y(t) \checkmark$$

$$A^{-1}[A[x]] = A^{-1}[x' - 4tx] = e^{2t^2}\int_0^t \frac{x'(s) - 4sx(s)}{e^{2s^2}}ds = e^{2t^2}\cdot$$

$$\cdot\left(\int_0^t \frac{x'(s)ds}{e^{2s^2}} - 4\int_0^t \frac{sx(s)ds}{e^{2s^2}}\right) = e^{2t^2}\cdot\left(\frac{x(s)}{e^{2s^2}}\Big|_0^t + \int_0^t \frac{-4x(s)\cdot s\,ds}{e^{2s^2}} -\right.$$

$$\left. - 4\int_0^t \frac{sx(s)ds}{e^{2s^2}}\right) = e^{2t^2}\cdot\frac{x(s)}{e^{2s^2}}\Big|_0^t = x(t) - e^{2t^2}x(0) = x(t) \checkmark$$

б) $A[x] = x'' - 3x' + 2x$, $D_A = \{x \in C^2[0;1] : x(0) = x'(0) = 0\}$.

$$A[x] = y, \; x \in D_A \Leftrightarrow \begin{cases} x'' - 3x' + 2x = y \\ x'(0) = x(0) = 0 \end{cases}$$

П3.33

в) $x(t) = \int_0^t K(t,s)y(s)ds$ $\qquad K(s,s)=0;\ K'(s,s)=1$

$K''-3K'+2K=0 \qquad K(s)=0 \quad K'(s)=1$

$\lambda^2-3\lambda+2=0 \qquad \lambda_1=1 \quad \lambda_2=2$

$K_{og}=C_1e^t+C_2e^{2t}$ $\qquad \begin{cases} C_1'e^s+C_2'e^{2s}=0 \\ C_1'e^s+2C_2'e^{2s}=1 \end{cases}$

$\begin{cases} C_1'e^s=-C_2'e^{2s} \\ C_2'e^{2s}=1 \end{cases} \Rightarrow C_2'=e^{-2s},\ C_2=\frac{-1}{2e^{2s}},\ C_1'=-e^{-s},\ C_1=e^{-s}$

$K(t,s)=e^{-s}e^t-\frac{e^{2t}}{2e^{2s}} \Rightarrow x(t)=\int_0^t\left(e^{-s}e^t-\frac{e^{2t}}{2e^{2s}}\right)y(s)ds$

$A^{-1}[y]=\int_0^t\left(e^{-s}e^t-\frac{e^{2t}}{2e^{2s}}\right)y(s)ds$

c) $A[x]=x''$, $D_A=\{x\in C^2[0;1]: x'(0)=x(1)=0\}$.

$A[x]=y,\ x\in D_A \Leftrightarrow \begin{cases} x''=y \\ x'(0)=x(1)=0 \end{cases}$

$x''=0 \quad x=C_1t+C_2 \qquad x(1)=C_1+C_2=0 \quad x'(0)=0=C_1 \Rightarrow C_2=0$

$x(t)=0$ при $y=0 \Rightarrow$ оператор обратим

$A^{-1}[y]=\int_0^1 K(t,s)y(s)ds = \omega(t)\cdot\begin{cases} u(t)v(s), & 0\le s\le t \\ u(s)v(t), & t<s<1 \end{cases}$

$x''=y \Rightarrow \begin{cases} C_1't+C_2'=0 \\ C_1'=y,\ C_2'=yt \end{cases} \Rightarrow \begin{cases} C_1=\int_0^t y(t)dt \\ C_2=\int_0^t ty(t)dt \end{cases}$

$x(t)=t\int_0^t y(s)ds+\int_0^t sy(s)ds+C_1t+C_2,\ x(1)=\int_0^1 y(s)ds+\int_0^1 sy(s)ds+C_1+C_2=0$

$x'(t)=C_1+ty(t)+\int_0^t y(s)ds+ty(t),\ x'(0)=C_1=0$

$\begin{cases} C_1=0 \\ C_2=-\int_0^1 y(s)ds-\int_0^1 sy(s)ds \end{cases}$ $\qquad A^{-1}[y]=t\int_0^t y(s)ds+\int_0^t sy(s)ds-$

$-\int_0^1 y(s)ds-\int_0^1 sy(s)ds =$

$$= \int_0^t (t+s)y(s)\,ds - \int_0^t y(s)\,ds - \int_t^1 y(s)\,ds - \int_0^t s y(s)\,ds - \int_t^1 s y(s)\,ds =$$

$$= \int_0^t (t+s-1-s)y(s)\,ds - \int_t^1 (1+s)y(s)\,ds = \int_0^t (t-1)y(s)\,ds - \int_t^1 (1+s)y(s)\,ds =$$

$$= \int_0^1 K(t,s)y(s)\,ds, \quad K(t,s) = \begin{cases} t-1, & 0 \le s \le t \\ s+1, & t < s \le 1 \end{cases}.$$

ПЗ 36

а) $A[x]=t^2x(t)$, $[a;b]=[0;1]$.

$A[x]=\lambda x \Leftrightarrow t^2x(t)=\lambda x(t) \Leftrightarrow (t^2-\lambda)x(t)=0$ (1)

$(t^2-\lambda)=0$ при $\lambda=t^2$, но (1) должно выполняться для всех $t\in[0;1] \Rightarrow$ $x=0$ — единственное реш. уравнения (1) независимо от $\lambda \Rightarrow$ оператор A не имеет собст. чисел и собст. функций.

б) $A[x]=tx(0)$, $[a;b]=[0;1]$.

$A[x]=\lambda x \Leftrightarrow tx(0)=\lambda x(t)$ (2)

1. $\lambda=0$: $tx(0)=0 \Leftrightarrow x(0)=0 \Rightarrow \lambda=0$ явл. собст. числом оператора A, $\{x\in C[0;1]: x(0)=0\}$ — множ-во собст. функций оператора A.

2. $\lambda\neq 0$: $x(t)=\dfrac{tx(0)}{\lambda} \Rightarrow$ решение: $x(t)=ct, c\in\mathbb{R}$

$c=\dfrac{x(0)}{\lambda} \Rightarrow c=\dfrac{0}{\lambda}=0$, $x=0$ явл. единст. решением уравнения (2) $\Rightarrow \lambda=0$ — единст. собст. число оператора A, $\{x\in C[0;1]: x(0)=0\}$ — множ-во собст. функций.

с) $A[x]=\int\limits_{-1}^{1}(t+s)x(s)ds$, $[a;b]=[-1;1]$

$A[x]=\lambda x \Leftrightarrow \int\limits_{-1}^{1}(t+s)x(s)ds=\lambda x(t)$

1. $\lambda=0$: $\int\limits_{-1}^{1}(t+s)x(s)ds=0$ — решение

$\lambda \neq 0$: $(t+s)$ – вырожд. ядро $\begin{vmatrix} \lambda & -2 \\ \frac{2}{3} & -\lambda \end{vmatrix} = 0$

$$\lambda x = t\int_{-1}^{1} x(s)\,ds + \int_{-1}^{1} s x(s)\,ds = C_1 t + C_2$$

$$x = C_1 t + C_2 \Rightarrow \lambda(C_1 t + C_2) = t\int_{-1}^{1}(C_1 s + C_2)\,ds +$$

$$+\int_{-1}^{1}(C_1 s^2 + C_2 s)\,ds = t\cdot\left(\frac{C_1 s^2}{2} + C_2 s\right)\Big|_{-1}^{1} + \left(\frac{C_1 s^3}{3} + \frac{C_2 s^2}{2}\right)\Big|_{-1}^{1} =$$

$$= 2C_2 t + \frac{2}{3}C_1 \quad \begin{cases} C_1\lambda = 2C_2 \\ C_2\lambda = \frac{2}{3}C_1 \end{cases} \Rightarrow \begin{cases} C_2 = \frac{C_1\lambda}{2} \\ \frac{C_1\lambda^2}{2} = \frac{2}{3}C_1 \end{cases} \Rightarrow$$

$$\Rightarrow \lambda^2 = \frac{4}{3}, \quad \lambda = \pm\frac{2}{\sqrt{3}}$$ (собст. числа)

$\lambda = \frac{2}{\sqrt{3}}$: $C_1 = \frac{C_2}{\sqrt{3}} \Rightarrow \left(C_1 = 1; C_1 = \frac{1}{\sqrt{3}}\right)$: $x = t + \frac{1}{\sqrt{3}}$

собст. функция

$\lambda = -\frac{2}{\sqrt{3}}$: $C_2 = -\frac{C_1}{\sqrt{3}} \Rightarrow \left(C_1 = 1; C_2 = -\frac{1}{\sqrt{3}}\right)$: $x = 1 - \frac{t}{\sqrt{3}}$

собст. функция