A Review on

A Refined Sinusoidal Theory for Laminated Composite and Sandwich Plates

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Contents

Abstract

A refined sinusoidal theory for composite laminates using a new zig-zag function [1] is proposed which automatically satisfies the traction-free boundary conditions on the top and bottom surfaces of the laminate. Next, equilibrium equations are obtained using the principle of virtual displacement and the Navier's solution is invoked to solve the boundary value problem. Analytic results are compared to the exact solution from three-dimensional elasticity [2], classical and first-order shear deformation theories for laminated composites. The present theory is capable of calculating displacement and stress fields with excellent accuracy.

2 Introduction

Composites are widely used in industry, specifically aerospace applications where composites offer the appealing strength-to-weight ratio. Compared to good in-planes modulus, composites are susceptible to transverse loading due to their weak transverse modulus. For example, for a typical wind turbine blade, the ratio of the transverse shear modulus to the in-plane shear modulus is about 0.1; therefore, the determination of transverse shear stresses is of paramount importance when analyzing composite laminates.

The isotropic one-layer plate theories are not able to characterize the piecewise (zig-zag) behavior of displacements and stresses in composite laminates [2]. First-order shear deformation theories lack accuracy due to high dependency on the selected shear correction factor. Furthermore, the shear strains are assumed constant along the thickness of the plate. Despite the success of higher-order shear deformation theories in yielding accurate results, layer-wise theories have been introduced in which displacements and stresses are calculated for every single layer. The major drawback of layer-wise theories is the dependency of computation on degrees of freedom of the laminate which rises by increasing the number of layups [3].

In order to utilize the advantages of layer-wise theories and to reduce the computational cost, a combination of higher-order shear deformation along with layer-wise theories have been introduced to describe the piecewise transverse displacement and stress of composite laminates. It is shown that the use of a zig-zag function is more effective than increasing the order of in-plane and transverse displacement components in higher-order theories [4].

The present report replicates the result of a refined sinusoidal plate theory using a new zig-zag function by Xiaohui *et al.* [1] which is inspired upon Touratier [5] sinusoidal shear deformation and Murakami's zig-zag function [6].

3 The Proposed Zig-Zag Function

One of the difficulties in characterizing displacements in composite laminates is the rapid change of in-plane displacements, usually due to stacking sequence. An efficient approach to overcome the abovementioned problem, as reported in the literature [4], is to combine the sinusoidal shear deformation theory with a zig-zag function, *e.g.*, Murakami's zig-zag function, to create the piecewise behavior in the in-plane displacement. A zig-zag function, $Z^k(z)$, is usually defined as

$$Z^k(z) = (-1)^k f(\zeta_k) \tag{1}$$

where

$$\zeta_k = a_k z - b_k \quad , a_k = \frac{2}{z_{k+1} - z_k} \quad , b_k = \frac{z_{k+1} + z_k}{z_{k+1} - z_k}$$

z_k and z_{k+1} denote the lower and upper coordinates of k-th layer demonstrated in Figure. 1.

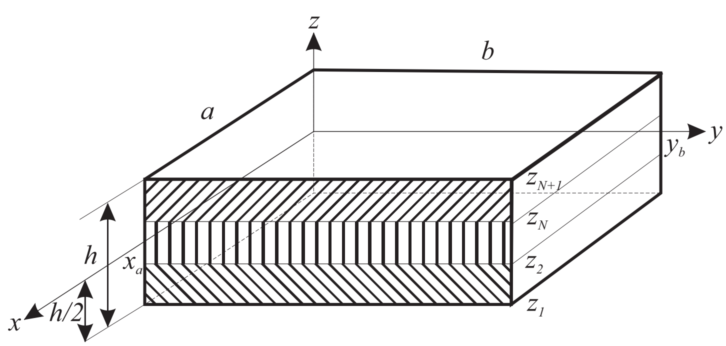

Figure 1: A Schematic of a composite laminate and the coordinate system.

The Murakami's zig-zag function, $M^k(z)$, is obtained by setting $f(\zeta_k) = \zeta_k$ and is given as

$$M^k(z) = (-1)^k \zeta_k \tag{2}$$

where ζ_k is defined according to Eq. 1. Eq. 2 indicates a linear distribution function through the plate thickness (Figure. 2a). The main drawback of the Murakami's function is the violation of the traction-free boundary condition on the top and bottom surfaces of the laminate. Figure. 2b shows the derivative of

$M^k(z)$ w.r.t z where the zero transverse shear condition is clearly violated on the top and bottom surfaces. Hence, a new zig-zag function, $H^k(z)$, is proposed which satisfies a priori the zero boundary conditions on the top and bottom surfaces.

In the $H^k(z)$ function, $f(\zeta_k)$ is replaced with a $\sinh(\zeta_k)$ function accompanied by two other terms, as given in Eq. 3 (see **??** for a comment on $H^k(z)$ terms)

$$H^k(z) = N^k(z) + N_1^k(z) + N_N^k(z) \tag{3}$$

where

$$\begin{aligned}
N^k(z) &= (-1)^k \sinh(\zeta_k) \\
N_1^k(z) &= -(z^2/2/z_1 + (z^3 - 1.5 z_1 z^2)/6/z_{N+1}^2)(-a_1 \cosh(-1)) \\
N_N^k(z) &= -((z^3 - 1.5 z_1 z^2)/6/z_{N+1}^2)((-1)^N a_N \cosh(1))
\end{aligned}$$

a_1, a_N and ζ_k can be calculated from Eq. 1. z_1 to z_{N+1} are the coordinates of each layer in the z-direction as demonstrated in Figure. 1. $H^k(z)$ is plotted for a three-layer composite laminate in Figure. 2a. While $M^k(z)$ varies linearly through each layer, $H^k(z)$ utilizes a nonlinear distribution to provide a better insight into the in-plane displacements at each layer. The derivative of $H^k(z)$ w.r.t z is plotted in Figure. 2b. As it can be seen, $H^k(z)$ is able to fulfill the zero boundary conditions on the top and bottom surfaces.

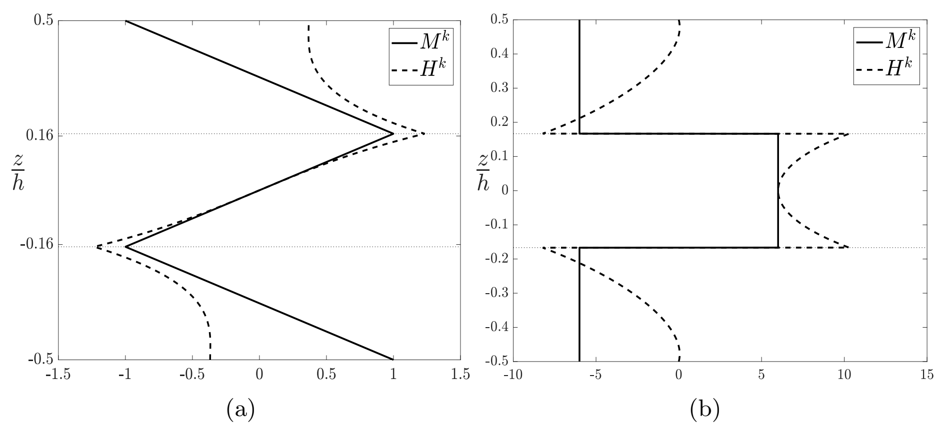

Figure 2: Comparison of (a) $M^k(z)$ and $H^k(z)$ functions and (b) derivatives of $M^k(z)$ and $H^k(z)$ functions.

4 The Theory

4.1 Assumptions

In developing the proposed theory, it is assumed that there is no elongation in the transverse direction and strains are infinitesimal.

4.2 Displacement Fields

Displacement fields are borrowed from a five-unknown shear deformation theory and combined with the zig-zag function displacement field. The general form of the displacement fields can be written as

$$\begin{aligned} u^k &= u_0 + u_b + u_s + u_z^k \\ v^k &= v_0 + v_b + v_s + v_z^k \\ w &= w_0 \end{aligned} \tag{4}$$

where u^k, v^k are the in-plane displacements and w is the transverse displacement. u_0, v_0 and w_0 are the corresponding mid-plane displacements. u_b and v_b are the bendings, u_s and v_s are the shear displacements. Finally, u_z^k and v_z^k are the zig-zag displacement components. The expression for each of the abovementioned displacement fields is provided below.

$$u_b = -z\frac{\partial w_0}{\partial x} \quad , v_b = -z\frac{\partial w_0}{\partial y} \quad , u_s = \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)u_1 \quad , v_s = \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)v_1$$

$$u_z^k = H^k(z)u_z \quad , v_z^k = H^k(z)v_z$$

As it can be seen from equations, the zig-zag displacement components are the only layer-dependent variables.

4.3 Strain-Displacement Relation

Assuming infinitesimal strains, the strain-displacement relations can be written as

$$\begin{aligned}
\epsilon_x^k &= \frac{\partial u^k}{\partial x} = \frac{\partial u_0}{\partial x} + \phi_1 \frac{\partial u_1}{\partial x} + \phi_2 \frac{\partial u_z}{\partial x} + \phi_3 \frac{\partial^2 w_0}{\partial x^2} \\
\epsilon_y^k &= \frac{\partial v^k}{\partial y} = \frac{\partial v_0}{\partial y} + \phi_1 \frac{\partial v_1}{\partial y} + \phi_2 \frac{\partial v_z}{\partial y} + \phi_3 \frac{\partial^2 w_0}{\partial y^2} \\
\gamma_{xy}^k &= \frac{\partial u^k}{\partial y} \frac{\partial v^k}{\partial x} = \frac{\partial u_0}{\partial y} + \phi_1 \frac{\partial u_1}{\partial y} + \phi_2 \frac{\partial u_z}{\partial y} + \frac{\partial v_0}{\partial x} + \phi_1 \frac{\partial v_1}{\partial x} + \phi_2 \frac{\partial v_z}{\partial x} + 2\phi_3 \frac{\partial^2 w_0}{\partial x \partial y} \\
\gamma_{xz}^k &= \frac{\partial u_s}{\partial z} \frac{\partial u_z^k}{\partial z} = \cos\left(\frac{\pi z}{h}\right) u_1 + \frac{\partial H^k(z)}{\partial z} u_z = \frac{\partial \phi_1}{\partial z} u_1 + \frac{\partial \phi_2}{\partial z} u_z \\
\gamma_{yz}^k &= \frac{\partial v_s}{\partial z} \frac{\partial v_z^k}{\partial z} = \cos\left(\frac{\pi z}{h}\right) v_1 + \frac{\partial H^k(z)}{\partial z} v_z = \frac{\partial \phi_1}{\partial z} v_1 + \frac{\partial \phi_2}{\partial z} v_z
\end{aligned}$$

where

$$\phi_1 = \frac{h}{\pi} \sin\left(\frac{\pi z}{h}\right) \quad , \phi_2 = H^k(z) \quad , \phi_3 = -z$$

One may easily identify the zero shear strains on the top and bottom surfaces of the laminate from the definition of transverse shear strains, γ_{xz}^k and γ_{yz}^k.

4.4 The Principle of Virtual Displacement

Figure. 3 illustrates the applied loading and boundary condition along with the coordinate system and geometry of the composite laminate. $p^b(x, y)$ and $p^t(x, y)$ are the in-plane distributed loads. Superscript b and superscript t denote the bottom and top surfaces. $q^b(x, y)$ and $q^t(x, y)$ are the transverse loads. T_{x_0}, T_{x_a}, T_{y_0} and T_{y_b} are the axial loads and T_{xz_0}, T_{xz_a}, T_{yz_0} and T_{yz_b} are the transverse shear tractions at the cross-section $x = 0$, $x = a$, $y = 0$ and $y = b$, respectively.

Applying the principle of virtual displacement for the plate shown in Figure. 3 under static condition gives

$$\delta U - \delta W = 0 \tag{5}$$

where δU is the strain energy and δW is the work done by external forces. The integral domain is defined on $\Omega \times (\frac{-h}{2}, \frac{h}{2})$ and Γ is the boundary embracing the circumference of the plate. Ω and h are the plate area and thickness, respectively.

4.4.1 Strain Energy

The virtual strain energy is given as

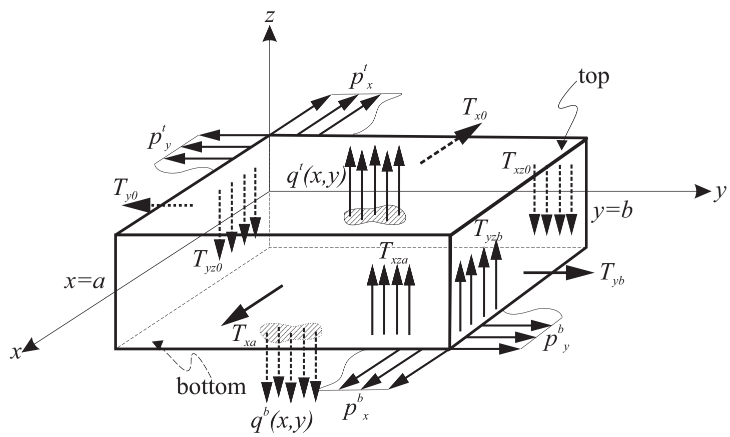

Figure 3: Plate applied loading, boundary conditions, geometry and coordinate system (plies are not shown).

$$
\begin{aligned}
\delta U &= \int_\Omega \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x^k \delta\epsilon_x^k + \sigma_y^k \delta\epsilon_y^k + \tau_{yz}^k \delta\gamma_{yz}^k + \tau_{zx}^k \delta\gamma_{zx}^k + \tau_{xy}^k \delta\gamma_{xy}^k) dzdxdy \\
&= \int_\Omega \int_{-\frac{h}{2}}^{\frac{h}{2}} \Bigg(\sigma_x^k \frac{\partial \delta u_0}{\partial x} + \sigma_x^k \phi_1 \frac{\partial \delta u_1}{\partial x} + \sigma_x^k \phi_2 \frac{\partial \delta u_z}{\partial x} + \sigma_x^k \phi_3 \frac{\partial^2 \delta w_0}{\partial x^2} \\
&\qquad + \sigma_y^k \frac{\partial \delta v_0}{\partial y} + \sigma_y^k \phi_1 \frac{\partial \delta v_1}{\partial y} + \sigma_y^k \phi_2 \frac{\partial \delta v_z}{\partial y} + \sigma_y^k \phi_3 \frac{\partial^2 \delta w_0}{\partial y^2} \\
&\qquad + \tau_{yz}^k \frac{\partial \phi_1}{\partial z} \delta v_1 + \tau_{yz}^k \frac{\partial \phi_2}{\partial z} \delta v_z + \tau_{xz}^k \frac{\partial \phi_1}{\partial z} \delta u_1 + \tau_{xz}^k \frac{\partial \phi_2}{\partial z} \delta u_z \\
&\qquad + \tau_{xy}^k \frac{\partial \delta u_0}{\partial y} + \tau_{xy}^k \phi_1 \frac{\partial \delta u_1}{\partial y} + \tau_{xy}^k \phi_2 \frac{\partial \delta u_z}{\partial y} + \tau_{xy}^k \frac{\partial \delta v_0}{\partial x} \\
&\qquad + \tau_{xy}^k \phi_1 \frac{\partial \delta v_1}{\partial x} + \tau_{xy}^k \phi_2 \frac{\partial \delta v_z}{\partial x} + 2\tau_{xy}^k \phi_3 \frac{\partial^2 \delta w_0}{\partial x \partial y} \Bigg) dzdxdy
\end{aligned} \tag{6}
$$

By replacing stresses in the form of resultant forces and moments according to the set of equations in Eq. 7, Eq. 6 can be rewritten in the form of Eq. 8.

$$
\begin{Bmatrix} N_x \\ M_{x1} \\ M_{x2} \\ M_{x3} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_x^k \\ \sigma_x^k \phi_1 \\ \sigma_x^k \phi_2 \\ \sigma_x^k \phi_3 \end{Bmatrix} dz, \begin{Bmatrix} N_y \\ M_{y1} \\ M_{y2} \\ M_{y3} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_y^k \\ \sigma_y^k \phi_1 \\ \sigma_y^k \phi_2 \\ \sigma_y^k \phi_3 \end{Bmatrix} dz, \begin{Bmatrix} V_{x1} \\ V_{x2} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \frac{\partial \phi_1}{\partial z} \tau_{xz}^k \\ \frac{\partial \phi_2}{\partial z} \tau_{xz}^k \end{Bmatrix} dz \tag{7}
$$

$$\begin{Bmatrix} V_{y1} \\ V_{y2} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \frac{\partial \phi_1}{\partial z}\tau_{yz}^k \\ \frac{\partial \phi_2}{\partial z}\tau_{yz}^k \end{Bmatrix} dz, \begin{Bmatrix} N_{xy} \\ M_{xy1} \\ M_{xy2} \\ M_{xy3} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \tau_{xy}^k \\ \tau_{xy}^k\phi_1 \\ \tau_{xy}^k\phi_2 \\ \tau_{xy}^k\phi_3 \end{Bmatrix} dz$$

$$\begin{aligned} \delta U = \int_\Omega \int_{-\frac{h}{2}}^{\frac{h}{2}} \Bigg(& N_x \frac{\partial \delta u_0}{\partial x} + M_{x1}\phi_1 \frac{\partial \delta u_1}{\partial x} + M_{x2}\frac{\partial \delta u_z}{\partial x} + M_{x3}\frac{\partial^2 \delta w_0}{\partial x^2} \\ & + N_y \frac{\partial \delta v_0}{\partial y} + M_{y1}\frac{\partial \delta v_1}{\partial y} + M_{y2}\frac{\partial \delta v_z}{\partial y} + M_{y3}\frac{\partial^2 \delta w_0}{\partial y^2} \\ & + V_{y1}\delta v_1 + V_{y2}\delta v_z + V_{x1}\delta u_1 + V_{x2}\delta u_z \\ & + N_{xy}\frac{\partial \delta u_0}{\partial y} + M_{xy1}\frac{\partial \delta u_1}{\partial y} + M_{xy2}\frac{\partial \delta u_z}{\partial y} + N_{xy}\frac{\partial \delta v_0}{\partial x} + M_{xy1}\frac{\partial \delta v_1}{\partial x} + M_{xy2}\frac{\partial \delta v_z}{\partial x} \\ & + 2M_{xy3}\frac{\partial^2 \delta w_0}{\partial x \partial y}\Bigg) dz dx dy \end{aligned} \tag{8}$$

The complete derivation of δU is given in Appendix. A.1.

4.4.2 External Work

The virtual work by external forces can be written as

$$\delta W = \delta W_{px} + \delta W_{py} + \delta W_q + \delta W_{T_{x0}} + \delta W_{T_{y0}} + \delta W_{T_{xa}} + \delta W_{T_{yb}} \tag{9}$$

where,

$$\begin{aligned} \delta W_{px} &= -\int_\Omega \left[p_x^b \delta u_0^1(x,y,z_1) + p_x^t \delta u_0^N(x,y,z_{N+1})\right] dxdy \\ \delta W_{py} &= -\int_\Omega \left[p_y^b \delta v_0^1(x,y,z_1) + p_y^t \delta v_0^N(x,y,z_{N+1})\right] dxdy \\ \delta W_q &= -\int_\Omega \left[q^b \delta w(x,y,z_1) + p_y^t \delta w(x,y,z_{N+1})\right] dxdy \end{aligned}$$

$$
\begin{aligned}
\delta W_{Tx0} &= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{x_0}\delta u^k(x_0,y,z) + T_{xz0}\delta w(x_0,y,z)\right] dz \\
\delta W_{Ty0} &= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{y_0}\delta v^k(x,y_0,z) + T_{yz0}\delta w(x,y_0,z)\right] dz \\
\delta W_{Txa} &= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{x_a}\delta u^k(x_a,y,z) + T_{xza}\delta w(x_a,y,z)\right] dz \\
\delta W_{Tyb} &= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{y_b}\delta v^k(x,y_b,z) + T_{yzb}\delta w(x,y_b,z)\right] dz
\end{aligned}
$$

For the full derivation of δW see Appendix. A.2.

4.5 Equilibrium Equations

The Euler-Lagrange equations can be obtained by substituting δU and δW from Section. 4.4.1 and Section. 4.4.2 into the principle of virtual displacement (Eq. 5) and integrating over Ω and plate thickness and subsequently setting the coefficients of δu_0, δu_1, δu_z, δv_0, δv_1,δv_z and δw_0 to zero, using the fundamental lemma of variational calculus.

$$
\begin{aligned}
\int_\Omega \Big\{ &(-N_{x,x} - N_{xy,y} - p_{x0})\delta u_0 + (-M_{x1,x} - M_{xy1,y} - p_{x1} + V_{x1})\delta u_1 + (-M_{x2,x} - M_{xy2,y} \\
&- p_{x2} + V_{x2})\delta u_z + (-N_{y,y} - N_{xy,x} - p_{y0})\delta v_0 + (-M_{y1,y} - M_{xy1,x} - p_{y1} + V_{y1})\delta v_1 + (-M_{y2,y} \\
&- M_{xy2,x} - p_{y2} + V_{y2})\delta v_z + (M_{x3,xx} + M_{y3,yy} + 2M_{xy3,xy} + p_{x3,x} + p_{y3,y} - q)\delta w_0 \Big\} dxdy
\end{aligned}
$$

In above equation, comma is used for differentiation, *e.g.*, $N_{x,x} = \dfrac{\partial N_x}{\partial x}$. The Euler-Lagrange equations are

$$
\begin{aligned}
&\delta u_0 : N_{x,x} + N_{xy,y} + p_{x0} = 0 \\
&\delta u_1 : M_{x1,x} + M_{xy1,y} + p_{x1} - V_{x1} = 0 \\
&\delta u_z : M_{x2,x} + M_{xy2,y} + p_{x2} - V_{x2} = 0 \\
&\delta v_0 : N_{y,y} + N_{xy,x} + p_{y0} = 0 \\
&\delta v_1 : M_{y1,y} + M_{xy1,x} + p_{y1} - V_{y1} = 0 \\
&\delta v_z : M_{y2,y} + M_{xy2,x} + p_{y2} - V_{y2} = 0 \\
&\delta w_0 : M_{x3,xx} + M_{y3,yy} + 2M_{xy3,xy} + p_{x3,x} + p_{y3,y} - q = 0
\end{aligned}
\tag{10}
$$

The definition for resultant forces and moments is given in Eq. 7.

4.6 Boundary Conditions

The primary and secondary boundary conditions can be derived by integrating terms on the plate edges, Γ, and setting them to zero. Table. 1 summarizes both the primary and secondary boundary conditions. The boundary condition requires either one of the quantities mentioned in each row of Table. 1. Explanation of terms with bar is given in Eq. 18 to Eq. 20.

4.7 Consecutive Equations

Generally, the x-axis of the laminate does not coincide with the fiber direction (layup stacking sequence differs from 0 and/or 90 degrees); therefore, some nonzero coupling terms appear in the stiffness matrix and stress-strain relation becomes [7]

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^k = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & Q_{24} & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & Q_{34} & 0 & 0 \\ Q_{14} & Q_{24} & Q_{34} & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & Q_{56} \\ 0 & 0 & 0 & 0 & Q_{56} & Q_{66} \end{bmatrix}^k \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^k \tag{11}$$

$\boldsymbol{Q}^k$ is the transformed elastic matrix w.r.t the laminate coordinate of k-th layup. The components of $\boldsymbol{Q}$ are given in Appendix. B [8].

5 The Navier Solution

5.1 Solution for A Simply Supported Square Plate

In this section, the analytic solution of the proposed plate theory for a simply supported square plate, Figure. 4, under a sinusoidal transverse load, will be explained. For a simply supported plate the following boundary conditions need to be satisfied.

At $x = 0$ and $x = a$

$$v_0 = 0, v_1 = 0, v_z = 0, w_0 = 0$$

At $y = 0$ and $y = b$

$$u_0 = 0, u_1 = 0, u_z = 0, w_0 = 0$$

Hence, the following boundary conditions are defined which meet the abovementioned initial boundary conditions [8].

Table 1: Primary (left) and Secondary (right) boundary conditions on plate boundaries ($x = x_0,\ y = y_0,\ x = x_a,\ y = y_b$). See Eq. 18 to Eq. 20 for explanation of terms with bar.

$$\begin{cases} u_0(x_0, y) = \overline{u}_{00} \\ u_0(x_a, y) = \overline{u}_{0a} \end{cases} \quad \text{or} \quad \begin{cases} N_x(x_0, y) = \overline{N}_{x0} \\ N_x(x_a, y) = \overline{N}_{xa} \end{cases}$$

$$\begin{cases} u_1(x_0, y) = \overline{u}_{10} \\ u_1(x_a, y) = \overline{u}_{1a} \end{cases} \quad \text{or} \quad \begin{cases} M_{x1}(x_0, y) = \overline{M}_{x10} \\ M_{x1}(x_a, y) = \overline{M}_{x1a} \end{cases}$$

$$\begin{cases} u_z(x_0, y) = \overline{u}_{z0} \\ u_z(x_a, y) = \overline{u}_{za} \end{cases} \quad \text{or} \quad \begin{cases} M_{x2}(x_0, y) = \overline{M}_{x20} \\ M_{x2}(x_a, y) = \overline{M}_{x2a} \end{cases}$$

$$\begin{cases} w_{0,x}(x_0, y) = \overline{w}_{0,x0} \\ w_{0,x}(x_a, y) = \overline{w}_{0,xa} \end{cases} \quad \text{or} \quad \begin{cases} M_{x3}(x_0, y) = \overline{M}_{x30} \\ M_{x3}(x_a, y) = \overline{M}_{x3a} \end{cases}$$

$$\begin{cases} w_0(x_0, y) = \overline{w}_{0x0} \\ w_0(x_a, y) = \overline{w}_{0xa} \end{cases} \quad \text{or} \quad \begin{cases} \overline{V}_{xz0} = -M_{x3,x}(x_0, y) - M_{xy3,y}(x_0, y) - p_{x_3}(x_0, y) \\ \overline{V}_{xza} = -M_{x3,x}(x_a, y) - M_{xy3,y}(x_a, y) - p_{x_3}(x_a, y) \end{cases}$$

$$\begin{cases} w_0(x, y_0) = \overline{w}_{0y0} \\ w_0(x, y_b) = \overline{w}_{0yb} \end{cases} \quad \text{or} \quad \begin{cases} \overline{V}_{yz0} = -M_{y3,y}(x, y_0) - M_{xy3,x}(x, y_0) - p_{y_3}(x, y_0) \\ \overline{V}_{yzb} = -M_{y3,y}(x, y_b) - M_{xy3,x}(x, y_b) - p_{y_3}(x, y_b) \end{cases}$$

$$\begin{cases} v_0(x, y_0) = \overline{v}_{00} \\ v_0(x, y_b) = \overline{v}_{0a} \end{cases} \quad \text{or} \quad \begin{cases} N_y(x, y_0) = \overline{N}_{y0} \\ N_y(x, y_b) = \overline{N}_{yb} \end{cases}$$

$$\begin{cases} v_1(x, y_0) = \overline{v}_{10} \\ v_1(x, y_b) = \overline{v}_{1b} \end{cases} \quad \text{or} \quad \begin{cases} M_{y1}(x, y_0) = \overline{M}_{y10} \\ M_{y1}(x, y_b) = \overline{M}_{y1b} \end{cases}$$

$$\begin{cases} v_z(x, y_0) = \overline{v}_{z0} \\ v_z(x, y_b) = \overline{v}_{zb} \end{cases} \quad \text{or} \quad \begin{cases} M_{y2}(x, y_0) = \overline{M}_{y20} \\ M_{y2}(x, y_b) = \overline{M}_{y2b} \end{cases}$$

$$\begin{cases} w_{0,y}(x, y_0) = \overline{w}_{0,y0} \\ w_{0,y}(x, y_b) = \overline{w}_{0,yb} \end{cases} \quad \text{or} \quad \begin{cases} M_{y3}(x, y_0) = \overline{M}_{y30} \\ M_{y3}(x, y_b) = \overline{M}_{y3b} \end{cases}$$

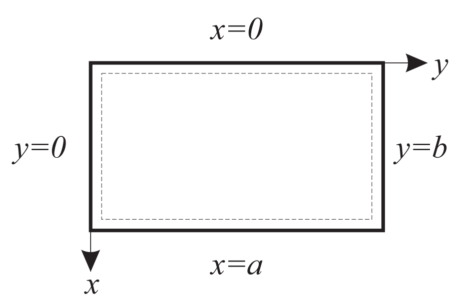

Figure 4: The simply supported plate geometry and coordinate system

$$
\begin{aligned}
u_0 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{0mn}\cos(\alpha x)\sin(\beta y) \\
u_1 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{1mn}\cos(\alpha x)\sin(\beta y) \\
u_z &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{zmn}\cos(\alpha x)\sin(\beta y) \\
v_0 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{0mn}\sin(\alpha x)\cos(\beta y) \\
v_1 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{1mn}\sin(\alpha x)\cos(\beta y) \\
v_z &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{zmn}\sin(\alpha x)\cos(\beta y) \\
w_0 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{0mn}\sin(\alpha x)\sin(\beta y)
\end{aligned} \tag{12}
$$

where $\alpha = m\pi/a$ and $\beta = n\pi/b$. m and n are positive integers which determine the number of terms for each double-Fourier series in Eq. 12. a and b are the plate length and width in x and y-directions.

Based on the terms in Eq. 12, the transverse sinusoidal load is defined according to Eq. 13.

$$
q(x, y) = q_0 \sin(\alpha x)\sin(\beta b) \tag{13}
$$

Substituting Eq. 12 in Eq. 10, one is able to form the following system of equations

$$
[\boldsymbol{K}]\{\boldsymbol{d}\} = \{\boldsymbol{p}\} \tag{14}
$$

in which $\boldsymbol{K}$ is a (7×7) matrix and both $\boldsymbol{d}$ and $\boldsymbol{p}$ are column vectors of (7×1) where

$$
\begin{aligned}
\boldsymbol{d} &= \{u_0, u_1, u_z, v_0, v_1, v_z, w_0\}^T \\
\boldsymbol{p} &= \{0, 0, 0, 0, 0, 0, q(x, y)\}^T
\end{aligned}
$$

5.2 The Solution Procedure

Codes are developed in both MATLAB and Maple[1], which communicate through `.txt` files, to solve the given problem and the relevant flowcharts are provided

[1] available on `github.com/amirbaharvand66/RSPTZ.git`

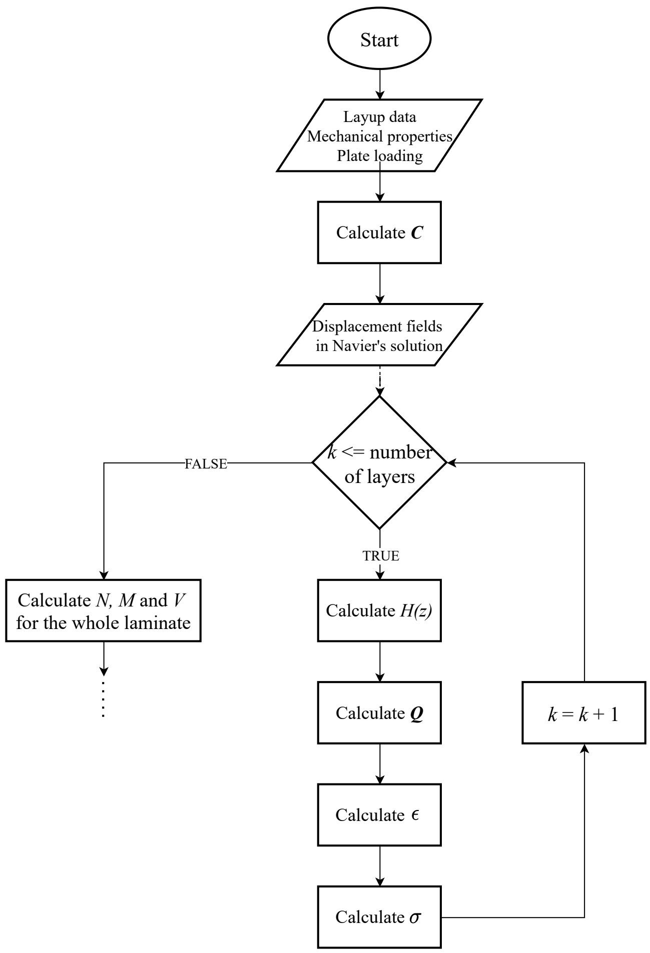

Figure 5: The Navier's solution flowchart (MATLAB part). k indicates the layer number.

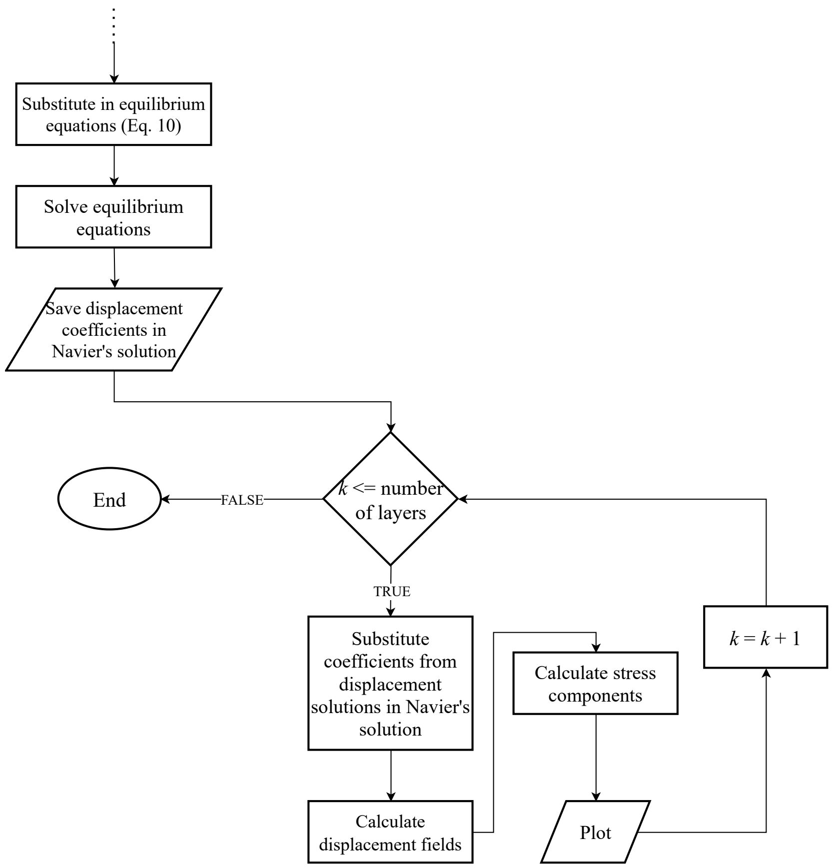

Figure 6: The Navier's solution flowchart (Maple part). k indicates the layer number.

in Figure. 5 and Figure. 6. The solution is performed for $m = 1$ and $n = 1$ in the double-Fourier series available in the Navier's solution. The precision of floating points in MATLAB and Maple are set to 16 and 10 digits, respectively.

The stacking layup of the composite laminate is [0/90/0] and mechanical properties are listed in Table. 2. The material is assumed to be transversely-isotropic ($E_{33} = E_{22}$, $\nu_{31} = \nu_{12}$ and $\mu_{31} = \mu_{12}$). Other constants can be found using Eq. 24.

Table 2: Mechanical properties of the laminate.

Property	**Symbol**	**Unit**	**Value**
Longitudinal Young's modulus	E_{11}	GPa	172.5
Transverse Young's modulus	E_{22}	GPa	6.9
In-plane Poisson's ratio	ν_{12}	-	0.25
Out-of-plane Poisson's ratio	ν_{23}	-	0.25
In-plane shear modulus	μ_{12}	GPa	3.45
Out-of-plane shear modulus	μ_{23}	GPa	1.38

5.3 Result and Discussion

In addition to the present theory, Refined sinusoidal shear deformation plate theory including the zig-zag function (RSPTZ), two other codes including the Classical Laminate Plate Theory (CLPT) and the First-order Shear Deformation plate Theory (FSDT) are developed to provide a better comparison of the evaluated displacements and stresses by RSPTZ. One may, however; states that due to the thick geometrical aspect ratios ($a/h = 4$ and $b/a = 1$) of the plate under study, CLPT is inappropriate compared with the other two theories. CLPT is given for the sake of comparison and its effortless coding procedure. The results for the displacements and stresses are shown in Figure. 7 and are normalized using the equations in Eq. 15. In the current report, both τ_{xz} and τ_{yz} are not calculated from the three-dimensional elasticity solution.

As it can be seen, the RSPTZ by combining both sinusoidal shear deformation plate theory and zig-zag function is capable of showing the in-plane displacement (Figure. 7b) and its change of sign from one layer to another. For the thick plate, both CLPT and FSDT underestimate $\overline{u}$, $\overline{\sigma}_x$, $\overline{\sigma}_y$ which is more pronounced on the free surfaces and interlaminar sections. CLPT is not capable of calculating τ_{xz} and τ_{yz}, because of the assumptions in Kirchhoff's plate theory. As mentioned earlier, the determination of transverse stresses is absolutely-essential during the design level of composite structures. While the FSDT shows a constant distribution of τ_{xz} and τ_{yz} through each layer, that is due to the shear correction factor (here 5/6), RSPTZ can provide a nonlinear distribution of the aforementioned parameters throughout each individual layer. Furthermore, the zero transverse traction is also fulfilled using the RSPTZ on the top and bottom part of the laminate (Figure. 7e and Figure. 7f).

$$\overline{u} = \frac{E_{22}h^2}{a^3} u\left(0, \frac{b}{2}, z\right) \quad , \quad \overline{w} = \frac{100E_{22}h^3}{q_0 a^4} w\left(\frac{a}{2}, \frac{b}{2}, z\right) \quad , \quad \overline{\sigma}_x = \frac{h^2}{q_0 a^2}\sigma_x\left(\frac{a}{2}, \frac{b}{2}, z\right)$$
$$\overline{\sigma}_y = \frac{h^2}{q_0 a^2}\sigma_y\left(\frac{a}{2}, \frac{b}{2}, z\right) \quad , \quad \overline{\tau}_{xy} = \frac{h^2}{q_0 a^2}\tau_{xy}(0, 0, z) \quad , \quad \overline{\tau}_{xz} = \frac{h}{q_0 a}\tau_{xz}\left(0, \frac{b}{2}, z\right)$$
$$\overline{\tau}_{yz} = \frac{h}{q_0 a}\tau_{yz}\left(\frac{a}{2}, 0, z\right) \tag{15}$$

To provide a better insight into the present theory, RSPTZ is benchmarked against the exact solution from the three-dimensional elasticity solution by Pagano [9]. The result is summarized in Table. 3. The last two rows include the integrated values of τ_{xz} and τ_{yz} along the thickness.

Table 3: Comparison of normalized displacement and stress for a three-layer composite ([0, 90, 0]) under double sinusoidal load ($a/h = 4$ and $b/a = 1$). Numbers in parentheses are the absolute error [%] from the exact three-dimensional elasticity solution.

Parameter	**Exact**	**RSPTZ**	**FSDT**	**CLPT**
$\overline{u}(-h/2)$	-	0.0100	0.0050	0.0070
$\overline{w}(0)$	2.006	2.0120(0.3)	1.7620(12.2)	0.4260(78.8)
$\overline{\sigma}_x(h/2)$	0.801	0.8090(1.0)	0.4320(46.0)	0.5360(33.1)
$\overline{\sigma}_y(h/6)$	0.534	0.5450(2.1)	0.4680(12.4)	0.1790(66.6)
$\overline{\tau}_{xy}(h/2)$	0.0505	0.0520(2.2)	0.0360(28.3)	0.0210(58.4)
$\overline{\tau}_{xz}(0)$	0.256	0.2550(0.4)	0.1430(44.0)	0(100)
$\overline{\tau}_{yz}(0)$	0.217	0.1910(12.0)	0.1590(27.0)	0(100)
$\int \overline{\tau}_{xz}$	-	0.2190(14.6)	0.0720(72.0)	0(100)
$\int \overline{\tau}_{yz}$	-	0.0900(58.5)	0.0320(85.4)	0(100)

As expected, the results from CLPT are not satisfying and the FSDT gives a high-value error. Conversely, the RSPTZ is able to keep the error under 2% except for $\overline{\tau}_{yz}$ where the error increases to 12%. It also appears that the RSPTZ underestimates τ_{yz} if three-dimensional elasticity equations are not utilized.

6 Conclusion

Determination of the in-plane displacement and shear stress in laminate composites is vital during the design procedure. Therefore, new zig-zag method is presented which is capable of characterizing the zig-zag behavior of the displacement fields through the laminate thickness that has not been fulfilled by previously proposed sinusoidal theories. A priori traction free on the bottom and top surfaces of the laminate is another advantage of the present theory compared to the other zig-zag theories, *i.e.*, Murakami's zig-zag theory. Displacements and stress components can be accurately calculated; nevertheless, it

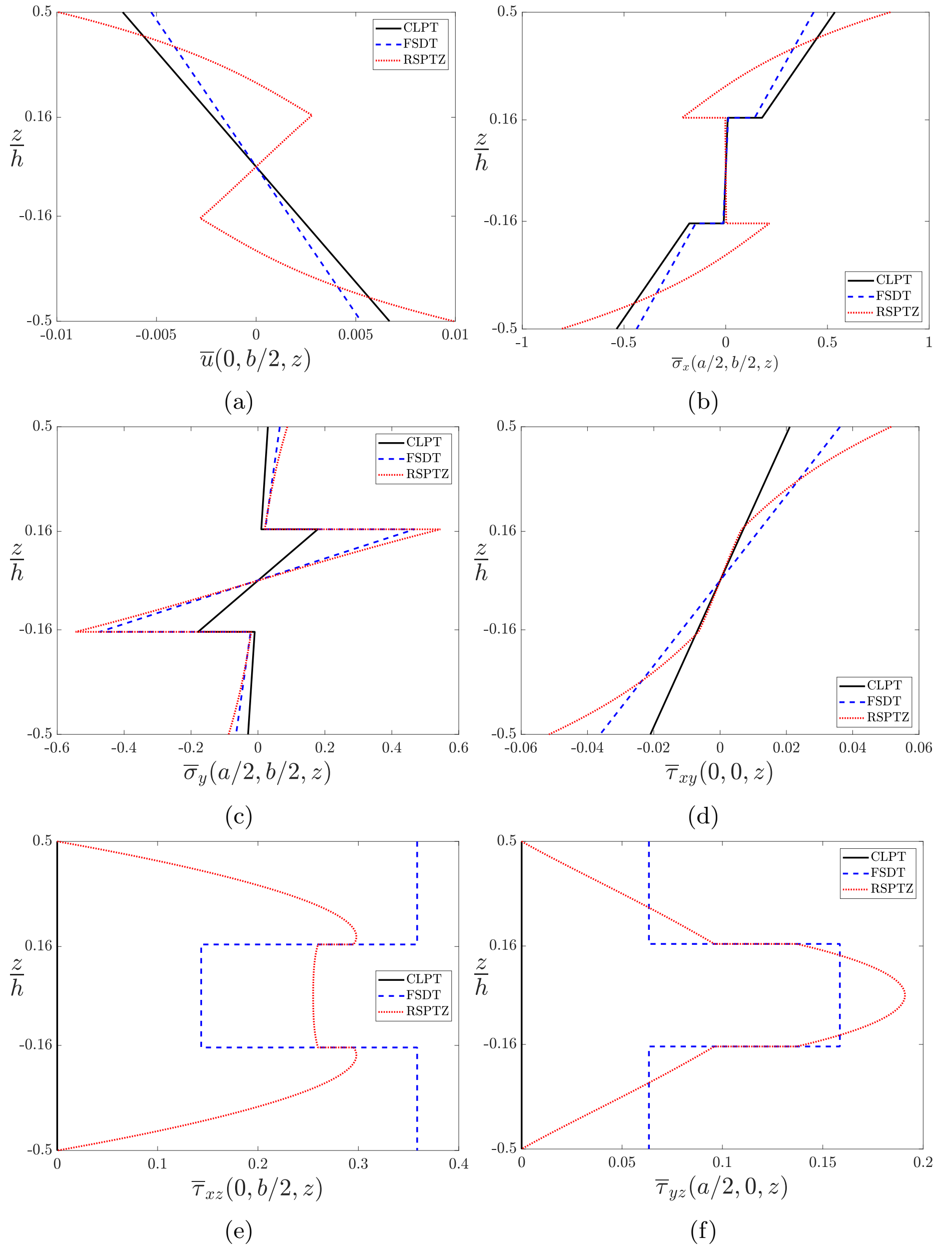

Figure 7: Comparison of normalized displacement and stress plots for a three-layer composite ([0, 90, 0]) under double sinusoidal load ($a/h = 4$ and $b/a = 1$). τ_{xz} and τ_{yz} are not calculated from the three-dimensional elasticity solution.

looks that the accuracy of the in-plane transverse stress falls without using the three-dimensional elasticity equations.

Appendices

A Derivation of δU and δW

A.1 Derivation of δU

$$
\begin{aligned}
&N_x \delta u_{0,x} = (N_x \delta u_0)_{,x} - N_{x,x} \delta u_0 \\
&M_{x1} \delta u_{1,x} = (M_{x1} \delta u_1)_{,x} - M_{x1,x} \delta u_1 \\
&M_{x2} \delta u_{z,x} = (M_{x2} \delta u_z)_{,x} - M_{x2,x} \delta u_z \\
&M_{x3} \delta w_{0,xx} = (M_{x3} \delta w_{0,x})_{,x} - M_{x3,x} \delta w_{0,x} = (M_{x3} \delta w_{0,x})_{,x} - (M_{x3,x} \delta w_0)_{,x} + M_{x3,xx} \delta w_0
\end{aligned}
$$

$$
\begin{aligned}
&N_y \delta v_{0,y} = (N_y \delta v_0)_{,y} - N_{y,y} \delta v_0 \\
&M_{y1} \delta v_{1,y} = (M_{y1} \delta v_1)_{,y} - M_{y1,y} \delta v_1 \\
&M_{y2} \delta v_{z,y} = (M_{y2} \delta v_z)_{,y} - M_{y2,y} \delta v_z \\
&M_{y3} \delta w_{0,yy} = (M_{y3} \delta w_{0,y})_{,y} - M_{y3,y} \delta w_{0,y} = (M_{y3} \delta w_{0,y})_{,y} - (M_{y3,y} \delta w_0)_{,y} + M_{y3,yy} \delta w_0
\end{aligned}
$$

$$
\begin{aligned}
&N_{xy} \delta u_{0,y} = (N_x y \delta u_0)_{,y} - N_{xy,y} \delta u_0 \\
&M_{xy1} \delta u_{1,y} = (M_{xy1} \delta u_1)_{,y} - M_{xy1,y} \delta u_1 \\
&M_{xy2} \delta u_{z,y} = (M_{xy2} \delta u_z)_{,y} - M_{xy2,y} \delta u_z \\
&M_{xy} \delta u_{0,x} = (N_x y \delta v_0)_{,x} - N_{xy,x} \delta v_0 \\
&M_{xy1} \delta v_{1,x} = (M_{xy1} \delta v_1)_{,x} - M_{xy1,x} \delta v_1 \\
&M_{xy2} \delta v_{z,x} = (M_{xy2} \delta v_z)_{,x} - M_{xy2,x} \delta v_z \\
&2M_{xy3} \delta w_{0,xy} = 2[(M_{xy3} \delta w_{0,x})_{,y} - M_{xy3,y} \delta w_{0,x}] = 2[(M_{xy3} \delta w_{0,x})_{,y} - (M_{xy3,y} \delta w_0)_{,x} + M_{xy3,xy} \delta w_0]
\end{aligned}
$$

$$
\begin{aligned}
\int_\Omega \bigg(&- N_{x,x} \delta u_0 - M_{x1,x} \delta u_1 - M_{x2,x} \delta u_z + M_{x3,xx} \delta w_0 \\
&- N_{y,y} \delta v_0 - M_{y1,y} \delta v_1 - M_{y2,y} \delta v_z + M_{y3,yy} \delta w_0 \\
&+ V_{y1} \delta v_1 + V_{y2} \delta v_z + V_{x1} \delta u_1 + V_{x2} \delta u_z \\
&- N_{xy,y} \delta u_0 - M_{xy1,y} \delta u_1 - M_{xy2,y} \delta u_z \\
&- N_{xy,x} \delta v_0 - M_{xy1,x} \delta v_1 - M_{xy2,x} \delta v_z \\
&+ 2M_{xy3,xy} \delta w_0 \bigg) dxdy
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&\int_\Gamma \Bigg(N_x n_x \delta u_0 + M_{x1} n_x \delta u_1 + M_{x2} n_x \delta u_z + M_{x3} n_x \delta w_{0,x} - M_{x3,x} n_x \delta w_0 \\
&\quad + N_y n_y \delta v_0 + M_{y1} n_y \delta v_1 + M_{y2} n_y \delta v_z + M_{y3} n_y \delta w_{0,y} + M_{y3} n_y \delta w_{0,y} - M_{y3,y} n_y \\
&\quad + N_x y n_y \delta u_0 + M_{xy1} n_y \delta u_1 + M_{xy2} n_y \delta u_z + N_x y n_x \delta v_0 + M_{xy1} n_x \delta v_1 + M_{xy2} n_x \delta v_z \\
&\quad + 2M_{xy3} n_y \delta w_{0,x} - 2M_{xy3,y} n_x \delta w_0 \delta w_0 \Bigg) ds = \\
&\int_\Gamma \Bigg(N_x n_x \delta u_0 + M_{x1} n_x \delta u_1 + M_{x2} n_x \delta u_z + M_{x3} n_x \delta w_{0,x} - M_{x3,x} n_x \delta w_0 \\
&\quad + N_y n_y \delta v_0 + M_{y1} n_y \delta v_1 + M_{y2} n_y \delta v_z + M_{y3} n_y \delta w_{0,y} + M_{y3} n_y \delta w_{0,y} - M_{y3,y} n_y \delta w_0 \\
&\quad + N_x y n_y \delta u_0 + M_{xy1} n_y \delta u_1 + M_{xy2} n_y \delta u_z + N_x y n_x \delta v_0 + M_{xy1} n_x \delta v_1 + M_{xy2} n_x \delta v_z \\
&\quad + 2M_{xy3} n_y \delta w_{0,x} - M_{xy3,y} n_x \delta w_0 - M_{xy3,x} n_y \delta w_0 \Bigg) ds
\end{aligned} \tag{17}
$$

where $M_{xy3} n_y \delta w_{0,x}$ vanishes in boundary conditions as the normal vector is $n = (0, 1)$ on planes $\delta w_{0,x}(x_0, y)$ and $\delta w_{0,x}(x_a, y)$.

A.2 Derivation of δW

$$
\begin{aligned}
&\delta u_z^1 = H^1(z) u_z = N^1(z) + N_1^1(z) + N_N^1(z) \\
&N^1(z) = (-1)^1 \sinh(\zeta_1) \\
&N_1^1(z) = -(z^2/2/z_1 + (z^3 - 1.5 z_1 z^2)/6/z_2^2)(-a_1 \cosh(-1)) \\
&N_N^1(z) = -((z^3 - 1.5 z_1 z^2)/6/z_2^2)((-1)^1 a_1 \cosh(1)) \\
&\zeta_1 = a_1 z - b_1, a_1 = \frac{2}{z_2 - z_1}, b_1 = \frac{z_2 + z_1}{z_2 - z_1}
\end{aligned}
$$

$$
\begin{aligned}
&\delta u_z^N = H^N(z) u_z = N^N(z) + N_1^N(z) + N_N^N(z) \\
&N^N(z) = (-1)^N \sinh(\zeta_N) \\
&N_1^N(z) = -(z^2/2/z_1 + (z^3 - 1.5 z_1 z^2)/6/z_{N+1}^2)(-a_1 \cosh(-1)) \\
&N_N^N(z) = -((z^3 - 1.5 z_1 z^2)/6/z_{N+1}^2)((-1)^N a_N \cosh(1)) \\
&\zeta_N = a_N z - b_N, a_N = \frac{2}{z_{N+1} - z_N}, b_N = \frac{z_{N+1} + z_N}{z_{N+1} - z_N}
\end{aligned}
$$

$$
\begin{aligned}
\delta W_{px} = &-\int_\Omega \Bigg(p_x^b \delta u_0 + \phi_3^1 p_x^b \frac{\partial \delta w_0}{\partial x} + \phi_1^1 p_x^b \delta u_1 + \phi_2^1 p_x^b \delta u_z \\
&\qquad + p_x^t \delta u_0 + \phi_3^N p_x^t \frac{\partial \delta w_0}{\partial x} + \phi_1^N p_x^t \delta u_1 + \phi_2^N p_x^t \delta u_z \Bigg) \\
= &-\int_\Omega \Bigg(\{p_x^b + p_x^t\} \delta u_0 + \{\phi_3^1 p_x^b + \phi_3^N p_x^t\} \frac{\partial \delta w_0}{\partial x} \delta u_z \\
&\qquad + \{\phi_1^1 p_x^b + \phi_1^N p_x^t\} \delta u_1 + \{\phi_2^1 p_x^b + \phi_2^N p_x^t\} \Bigg) dxdy =
\end{aligned}
$$

where

$$
\begin{aligned}
p_{x_0} &= p_x^b + p_x^t \\
p_{x_1} &= \phi_2^1 p_x^b + \phi_2^N p_x^t \\
p_{x_2} &= \phi_1^1 p_x^b + \phi_1^N p_x^t \\
p_{x_3} &= \phi_3^1 p_x^b + \phi_3^N p_x^t
\end{aligned}
$$

$$
\begin{aligned}
&\delta v_z^1 = H^1(z) v_z = N^1(z) + N_1^1(z) + N_N^1(z) \\
&N^1(z) = (-1)^1 \sinh(\zeta_1) \\
&N_1^1(z) = -(z^2/2/z_1 + (z^3 - 1.5 z_1 z^2)/6/z_2^2)(-a_1 \cosh(-1)) \\
&N_N^1(z) = -((z^3 - 1.5 z_1 z^2)/6/z_2^2)((-1)^1 a_1 \cosh(1)) \\
&\zeta_1 = a_1 z - b_1, a_1 = \frac{2}{z_2 - z_1}, b_1 = \frac{z_2 + z_1}{z_2 - z_1}
\end{aligned}
$$

$$
\begin{aligned}
&\delta v_z^N = H^N(z) v_z = N^N(z) + N_1^N(z) + N_N^N(z) \\
&N^N(z) = (-1)^N \sinh(\zeta_N) \\
&N_1^N(z) = -(z^2/2/z_1 + (z^3 - 1.5 z_1 z^2)/6/z_{N+1}^2)(-a_1 \cosh(-1)) \\
&N_N^N(z) = -((z^3 - 1.5 z_1 z^2)/6/z_{N+1}^2)((-1)^N a_N \cosh(1)) \\
&\zeta_N = a_N z - b_N, a_N = \frac{2}{z_{N+1} - z_N}, b_N = \frac{z_{N+1} + z_N}{z_{N+1} - z_N}
\end{aligned}
$$

$$
\begin{aligned}
\delta W_{py} = &-\int_\Omega \Bigg(p_y^b \delta v_0 + \phi_3^1 p_y^b \frac{\partial \delta w_0}{\partial y} + \phi_1^1 p_y^b \delta v_1 + \phi_2^1 p_y^b \delta v_z \\
&\quad + p_y^t \delta v_0 + \phi_3^N p_y^t \frac{\partial \delta w_0}{\partial y} + \phi_1^N p_y^t \delta v_1 + \phi_2^N p_y^t \delta v_z \Bigg) \\
= &-\int_\Omega \Bigg(\{p_y^b + p_y^t\} \delta v_0 + \{\phi_3^1 p_y^b + \phi_3^N p_y^t\} \frac{\partial \delta w_0}{\partial y} \delta v_z \\
&\quad + \{\phi_1^1 p_y^b + \phi_1^N p_y^t\} \delta v_1 + \{\phi_2^1 p_y^b + \phi_2^N p_y^t\} \Bigg) dydy =
\end{aligned}
$$

where

$$
\begin{aligned}
p_{y_0} &= p_y^b + p_y^t \\
p_{y_1} &= \phi_2^1 p_y^b + \phi_2^N p_y^t \\
p_{y_2} &= \phi_1^1 p_y^b + \phi_1^N p_y^t \\
p_{y_3} &= \phi_3^1 p_y^b + \phi_3^N p_y^t
\end{aligned}
$$

$$
\delta W_q = -\int_\Omega (q^b \delta w_0 + q^t \delta w_0) dxdy = -\int_\Omega (q^b + q^t) \delta w_0 dxdy = -\int_\Omega (q \delta w_0) dxdy
$$

$$
\begin{aligned}
\delta W_{Tx0} &= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{x_0} \delta u^k(x_0, y, z) + T_{xz0} \delta w(x_0, y, z) \right] dz \\
&= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{x_0} \delta u_0 + \phi_3 T_{x_0} \frac{\partial \delta w_0}{\partial x} + \phi_1 T_{x_0} + \phi_2 T_{x_0} + T_{xz0} \delta w_0 \right] dz
\end{aligned}
$$

$$
\begin{aligned}
\delta W_{Ty0} &= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{y_0} \delta v^k(y_0, y, z) + T_{yz0} \delta w(y_0, y, z) \right] dz \\
&= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{y_0} \delta v_0 + \phi_3 T_{y_0} \frac{\partial \delta w_0}{\partial y} + \phi_1 T_{y_0} + \phi_2 T_{y_0} + T_{yz0} \delta w_0 \right] dz
\end{aligned}
$$

$$
\begin{aligned}
\delta W_{Txa} &= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{x_a} \delta u^k(x_a, y, z) + T_{xza} \delta w(x_a, y, z) \right] dz \\
&= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{x_a} \delta u_a + \phi_3 T_{x_a} \frac{\partial \delta w_0}{\partial x} + \phi_1 T_{x_a} + \phi_2 T_{x_a} + T_{xza} \delta w_0 \right] dz
\end{aligned}
$$

$$
\begin{aligned}
\delta W_{Tyb} &= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{y_b}\delta v^k(y_b, y, z) + T_{yzb}\delta w(y_b, y, z)\right] dz \\
&= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[T_{y_b}\delta v_b + \phi_3 T_{y_b}\frac{\partial \delta w_0}{\partial y} + \phi_1 T_{y_b} + \phi_2 T_{y_b} + T_{yzb}\delta w_0\right] dz
\end{aligned}
$$

where

$$
\begin{Bmatrix} \overline{N}_{x0} \\ \overline{M}_{x10} \\ \overline{M}_{x20} \\ \overline{M}_{x30} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} T_{x0} \\ \phi_1 T_{x0} \\ \phi_2 T_{x0} \\ \phi_3 T_{x0} \end{Bmatrix} dz, \begin{Bmatrix} \overline{N}_{y0} \\ \overline{M}_{y10} \\ \overline{M}_{y20} \\ \overline{M}_{y30} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} T_{y0} \\ \phi_1 T_{y0} \\ \phi_2 T_{y0} \\ \phi_3 T_{y0} \end{Bmatrix} dz \tag{18}
$$

$$
\begin{Bmatrix} \overline{N}_{xa} \\ \overline{M}_{x1a} \\ \overline{M}_{x2a} \\ \overline{M}_{x3a} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} T_{x0} \\ \phi_1 T_{xa} \\ \phi_2 T_{xa} \\ \phi_3 T_{xa} \end{Bmatrix} dz, \begin{Bmatrix} \overline{N}_{yb} \\ \overline{M}_{y1b} \\ \overline{M}_{y2b} \\ \overline{M}_{y3b} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} T_{y0} \\ \phi_1 T_{yb} \\ \phi_2 T_{yb} \\ \phi_3 T_{yb} \end{Bmatrix} dz \tag{19}
$$

$$
\begin{Bmatrix} \overline{V}_{xz0} \\ \overline{V}_{yz0} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} T_{xz0} \\ T_{yz0} \end{Bmatrix} dz, \begin{Bmatrix} \overline{V}_{xza} \\ \overline{V}_{yzb} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} T_{xza} \\ T_{yzb} \end{Bmatrix} dz \tag{20}
$$

B Components of $\boldsymbol{Q}$

B.1 Components of $\boldsymbol{C}$

$\boldsymbol{C}$ is the stiffness matrix and is defined according to each lay-up coordinate system (1, 2, 3).

$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{Bmatrix} \tag{21}
$$

where the components of $\boldsymbol{C}$ are

$$
C_{11} = \frac{E_{11}(1-\nu_{23}\nu_{32})}{\Delta} \quad , \quad C_{12} = \frac{E_{11}(\nu_{21}+\nu_{31}\nu_{23})}{\Delta} \quad , \quad C_{13} = \frac{E_{11}(\nu_{31}+\nu_{21}\nu_{32})}{\Delta}
$$

$$
C_{22} = \frac{E_{22}(1-\nu_{13}\nu_{31})}{\Delta} \quad , \quad C_{23} = \frac{E_{22}(\nu_{32}+\nu_{12}\nu_{31})}{\Delta} \quad , \quad C_{33} = \frac{E_{33}(1-\nu_{12}\nu_{21})}{\Delta}
$$

$$
C_{44} = \mu_{12} \quad , \quad C_{55} = \mu_{23} \quad , \quad C_{66} = \mu_{13} \tag{22}
$$

$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{21}\nu_{32}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} \tag{23}$$

E_{11} is the longitudinal Young's modulus, E_{22} and E_{33} are transverse Young's moduli, μ_{12} and μ_{13} are in-plane shear moduli, μ_{23} is the out-of-plane shear modulus, ν_{12} and ν_{13} are in-plane Poisson's ratios and ν_{23} is the out-of-plane Poisson's ratio. Other constants are calculated from the following set of equations.

$$\frac{\nu_{12}}{E_{11}} = \frac{\nu_{21}}{E_{22}} \quad , \quad \frac{\nu_{13}}{E_{11}} = \frac{\nu_{31}}{E_{33}} \quad , \quad \frac{\nu_{23}}{E_{22}} = \frac{\nu_{32}}{E_{33}} \tag{24}$$

B.2 Components of $\boldsymbol{Q}$

$$\begin{aligned}
Q_{11} &= C_{11}c^4 + 2(C_{12} + 2C_{44})s^2c^2 + C_{22}s^4 \\
Q_{12} &= C_{12}(c^4 + s^4) + (C_{11} + C_{22} - 4C_{44})s^2c^2 \\
Q_{13} &= C_{13}c^2 + C_{23}s^2 \\
Q_{14} &= (C_{11} - C_{12} - 2C_{44})sc^3 + (C_{12} - C_{22} + 2C_{44})cs^3 \\
Q_{22} &= C_{11}s^4 + C_{22}c^4 + (2C_{12} + 4C_{44})s^2c^2 \\
Q_{23} &= C_{13}s^2 + C_{23}c^2 \\
Q_{24} &= (C_{11} - C_{12} - 2C_{44})s^3c + (C_{12} - C_{22} + 2C_{44})c^3s \\
Q_{33} &= C_{33} \\
Q_{34} &= (C_{31} - C_{32})sc \\
Q_{44} &= (C_{11} - 2C_{12} + C_{22} - 2C_{44})c^2s^2 + C_{44}(c^4 + s^4) \\
Q_{55} &= C_{55}c^2 + C_{66}s^2 \\
Q_{56} &= (C_{66} - C_{55})cs \\
Q_{66} &= C_{55}s^2 + C_{66}c^2
\end{aligned} \tag{25}$$

where $c = \cos(\theta)$, $s = \sin(\theta)$ and $Q_{ij} = Q_{ji}$ for $i, j = 1..6$

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