Optimization

References:

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- J. Nocedal, S. J. Wright, Numerical Optimization, Springer, 1999.
- D. G. Luenberger, Y. Ye, Linear and Nonlinear Programming, Springer, Third Edition 2008.

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Grading:

- Homework+Quiz (20%)
- Seminar (10%)
- Midterm+ Final(30%+40%)

Introduction

- people and nature optimize
- optimization is an important tool in decision science
- first identify some *objective:* a quantitative measure of performance
- objective depends on certain characteristics of the system, called *variables or unknowns*.

Our goal is to find values of the variables that optimize the objective.

- Often the variables are restricted, or *constrained*

Introduction

Modeling an optimization problem

- ✓Modeling: identifying objective, variables, and constraints for a given problem
- ✓First step in the optimization process
- ✓too simplistic model
- ✓too complex model

Using an optimization algorithm

- ✓There are a collection of optimization algorithms
- ✓each of which is tailored to a particular type of optimization problem
- ✓responsibility of choosing the appropriate algorithm falls on user

Introduction

- ✓Checking the solution of the problem
- ✓mathematical expressions known as *optimality conditions*
- ✓*Sensitivity Analysis*

Mathematical Definition

optimization is the minimization or maximization of a function subject to constraints on its variables.

Introduction

- x *is the vector of variables, also called unknowns or parameters*
- f *is the objective function, a (scalar) function of x that we want to maximize*

or minimize
- *ci are constraint functions, which are scalar functions of x that define*

certain equations and inequalities that the unknown vector *x must satisfy*.

Introduction

- *x is the vector of variables, also called unknowns or parameters*
- *f is the objective function, a (scalar) function of x that we want to maximize or* minimize
- *c_i are constraint functions, which are scalar functions of x that define certain equations* and inequalities that the unknown vector *x must satisfy*.

Example :

$$\min \,(x_1 - 2)^2 + (x_2 - 1)^2 \qquad \text{subject to} \quad \begin{array}{l} x_1^2 - x_2 \le 0, \\ x_1 + x_2 \le 2. \end{array}$$

$$f(x) = (x_1 - 2)^2 + (x_2 - 1)^2, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad c(x) = \begin{bmatrix} c_1(x) \\ c_2(x) \end{bmatrix} = \begin{bmatrix} -x_1^2 + x_2 \\ -x_1 - x_2 + 2 \end{bmatrix}.$$

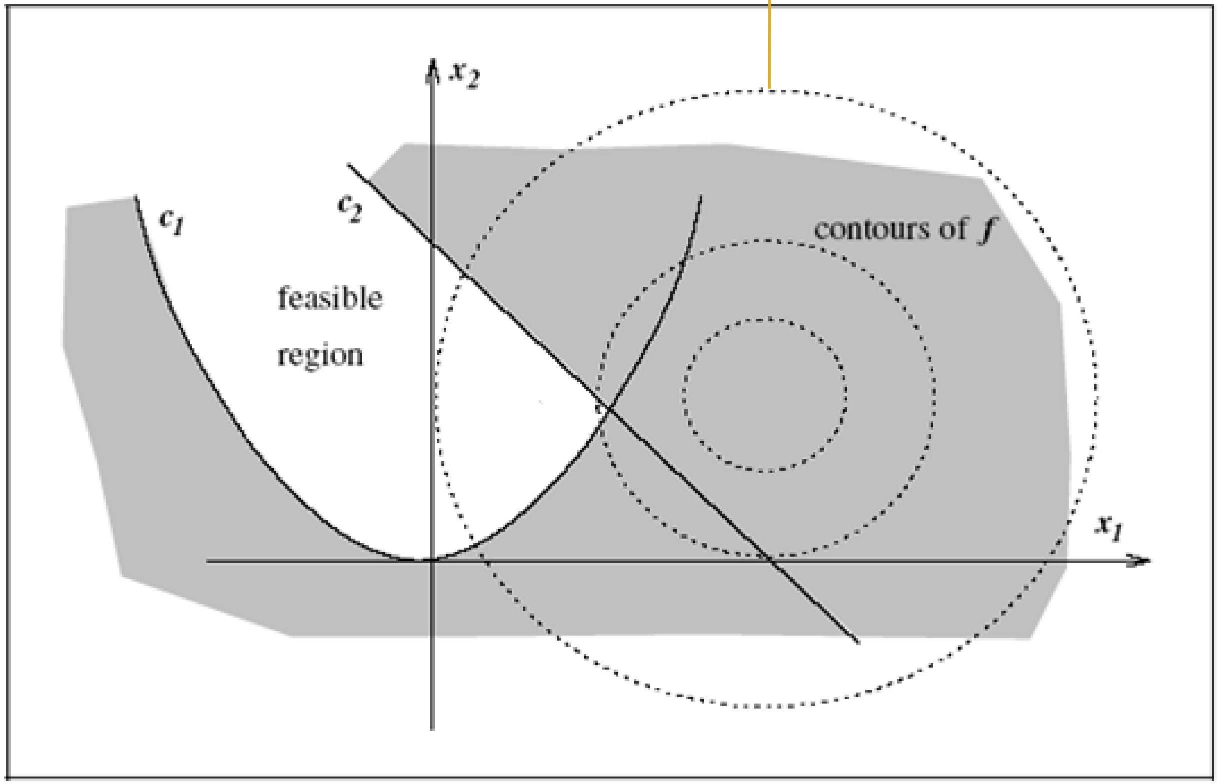
$$c_i(x) \ge 0$$

Introduction

contours of the objective function,

$$f(x) = (x_1 - 2)^2 + (x_2 - 1)^2,$$

$$\text{subject to} \quad \begin{aligned} x_1^2 - x_2 &\le 0, \\ x_1 + x_2 &\le 2. \end{aligned}$$



Contour : the set of points for which *f (x) has a constant value*

feasible region : the set of points satisfying all the constraints

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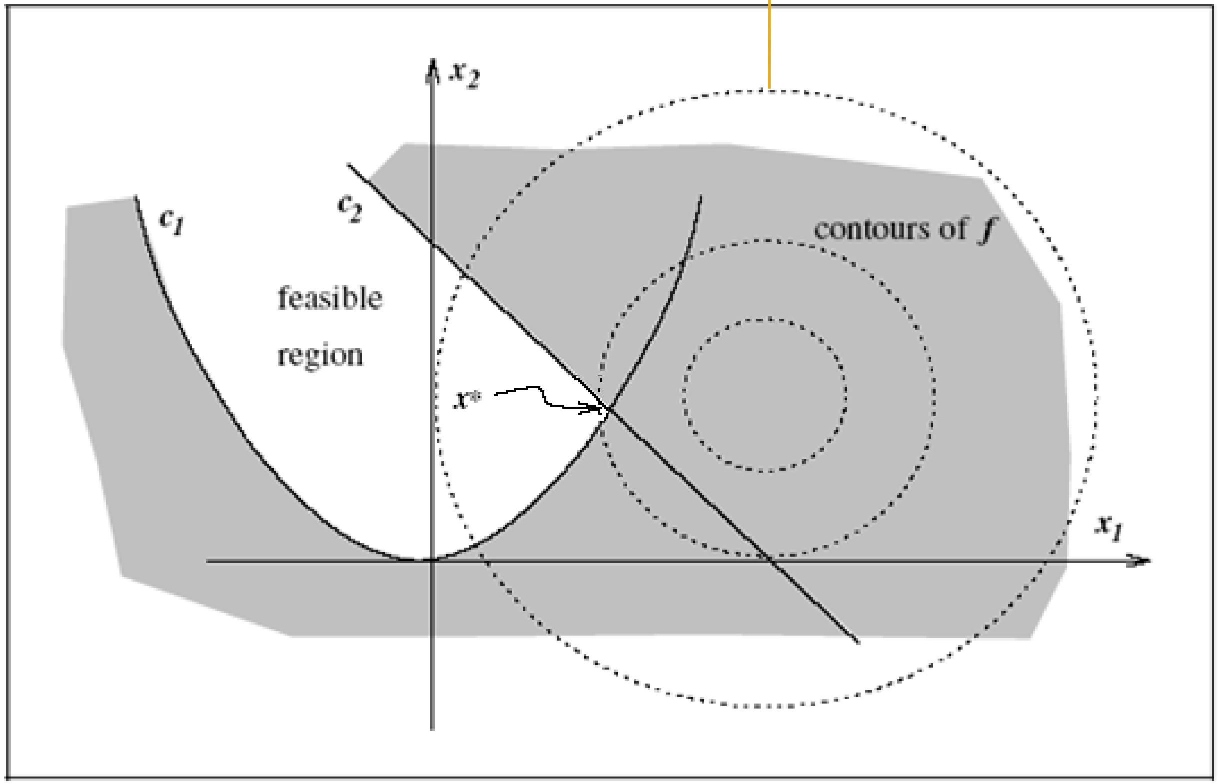

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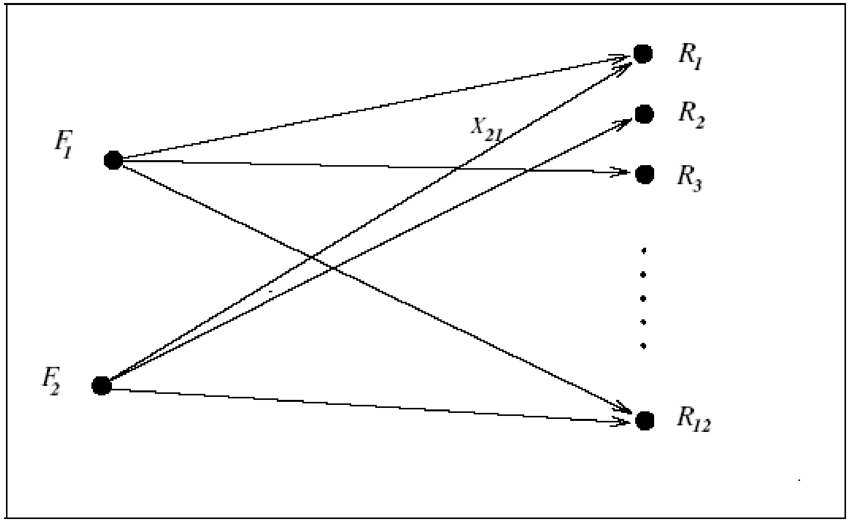
Introduction

Example : a transportation problem



✓A chemical company has 2 factories *F1 and F2*

✓outlets *R1, R2, . . . , R12.*

✓*Fi can produce ai tons of a certain chemical product* each week

✓outlet *Rj has a known weekly demand of bj tons of the product.*

✓the cost of shipping one ton of the product from factory *Fi to retail outlet Rj is cij*

how much of the product to ship from each factory to each outlet so as to satisfy all the requirements and minimize cost.

✓ *xi j is the number of tons of the product* shipped from factory *Fi to retail outlet Rj*

Introduction

$$
\begin{aligned}
& \min \sum_{ij} c_{ij} x_{ij} \\
& \text{subject to} \sum_{j=1}^{12} x_{ij} \le a_i, \quad i = 1, 2, \\
& \sum_{i=1}^{2} x_{ij} \ge b_j, \quad j = 1, \ldots, 12, \\
& x_{ij} \ge 0, \quad i = 1, 2, \quad j = 1, \ldots, 12.
\end{aligned}
$$

Introduction

a *linear programming problem*

$$\min \sum_{ij} c_{ij} x_{ij}$$

$$\text{subject to} \sum_{j=1}^{12} x_{ij} \leq a_i, \qquad i = 1, 2,$$

$$\sum_{i=1}^{2} x_{ij} \geq b_j, \qquad j = 1, \ldots, 12,$$

$$x_{ij} \geq 0, \qquad i = 1, 2, \quad j = 1, \ldots, 12.$$

Introduction

CONTINUOUS VERSUS DISCRETE OPTIMIZATION

- ✓a variable *xi : the number of power plants of type i*
- ✓whether or not a particular factory should be located in a particular city.

$$x_i \in \mathbf{Z} \qquad\qquad x_i \in \{0, 1\}$$

- ✓*integer programming*
- ✓*discrete optimization :* the unknown *x is drawn from a countable (but often* very large) set
- ✓*continuous optimization problems*
- ✓*mixed integer programming*

Introduction

- ❑Constrained optimization
- ❑Unconstrained optimization

✓Local optimization : seek only a local solution, objective function is smaller than at all other feasible nearby points.

✓Global optimization : the point with lowest function value among *all* feasible points

local solutions are also global solutions in some situations

Introduction

- ✓the model cannot be fully specified
- ✓depends on quantities unknown at the time of formulation
- ✓Future interest rates, future demands for a product
- ✓know a number of possible scenarios for the uncertain demand, along with estimates of the probabilities of each scenario.
- ✓*Stochastic optimization* algorithms use these quantifications of the uncertainty to produce solutions that optimize the *expected performance of the model.*
- ✓*Deterministic optimization*

Introduction

Optimization algorithms

- Optimization algorithms are iterative.
- begin with an initial guess of the variable *x*
- generate a sequence of improved estimates until they terminate
- strategy used to move from one iterate to the next distinguishes one algorithm from another.
- Most strategies make use of the values of *f* , *ci , and possibly the first and second derivatives of these functions*.

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Robustness Efficiency Accuracy