

Spring 2011

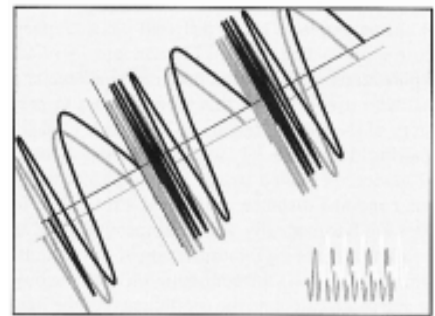
信號與系統 Signals and Systems

Chapter SS-3 Fourier Series Representation of Periodic Signals

Feng-Li Lian

NTU-EE

Feb11 – Jun11



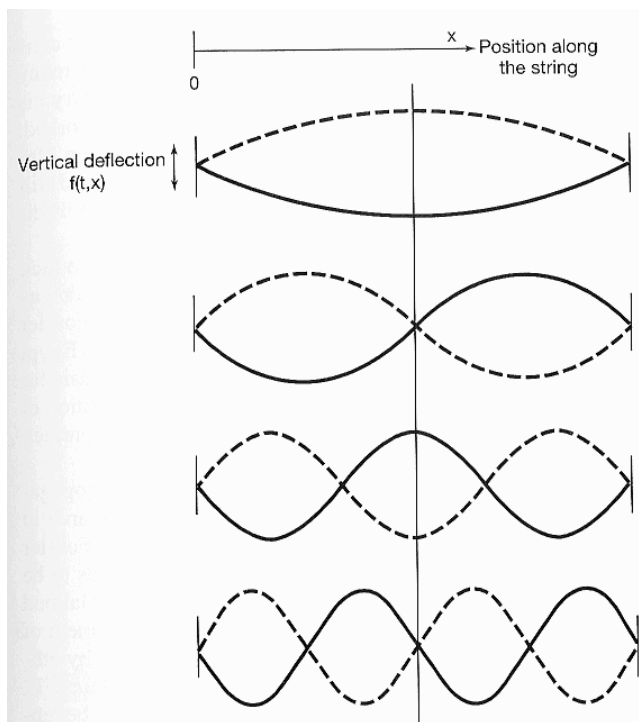
Figures and images used in these lecture notes are adopted from
“Signals & Systems” by Alan V. Oppenheim and Alan S. Willsky, 1997

Outline

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NTUEE-SS3-FS-2

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

- **L. Euler's** study on the motion of a **vibrating string** in 1748

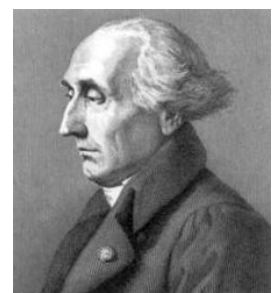


Leonhard Euler
1707-1783
Born in Switzerland
Photo from wikipedia

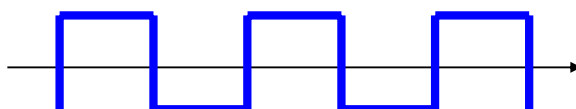
- **L. Euler** showed (in 1748)
 - The configuration of a **vibrating string** at some point in time is a **linear combination** of these **normal modes**
- **D. Bernoulli** argued (in 1753)
 - All **physical motions** of a **string** could be represented by **linear combinations** of **normal modes**
 - But, he did not pursue this mathematically
- **J.L. Lagrange** strongly criticized (in 1759)
 - The use of **trigonometric series** in examination of vibrating strings
 - **Impossible** to represent signals with **corners** using **trigonometric series**



Daniel Bernoulli
1700-1782
Born in Dutch
Photo from wikipedia



Joseph-Louis Lagrange
1736-1813
Born in Italy
Photo from wikipedia



- In 1807, **Jean Baptiste Joseph Fourier**
 - Submitted a paper of using **trigonometric series** to represent “any” periodic signal
 - It is examined by **S.F. Lacroix, G. Monge, P.S. de Laplace, and J.L. Lagrange,**
 - But **Lagrange rejected** it!
- In 1822, **Fourier** published a book “**Theorie analytique de la chaleur**”
 - “**The Analytical Theory of Heat**”



Jean Baptiste Joseph Fourier
1768-1830
Born in France
Photo from wikipedia

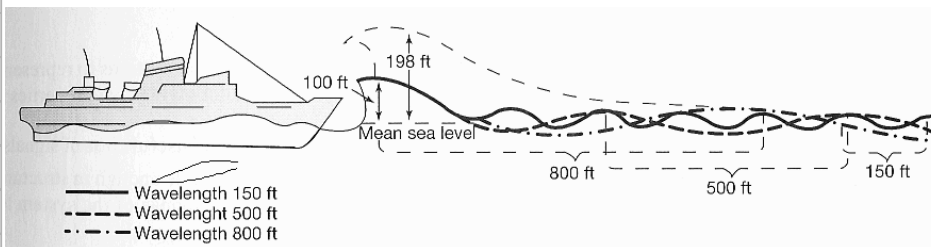


Figure 1.2: A medallion by David d'Angers, the only known portrait of Lacroix, made two years prior to his death. [Académie des Sciences de l'Institut de France]



Sylvestre François de Lacroix
1765-1843
Born in France
Photo from

Gaspard Monge, Comte de Péluse
1746-1818
Born in France
Photo from wikipedia

Pierre-Simon, Marquis de Laplace
1749-1827
Born in France
Photo from wikipedia

A short biography of Sylvestre-François Lacroix
In Science Networks. Historical Studies, V35,
Lacroix and the Calculus, Birkhäuser Basel
2008, ISBN 978-3-7643-8638-2

- **Fourier's main contributions:**

- Studied vibration, heat diffusion, etc.
- Found series of harmonically related sinusoids to be useful in representing the temperature distribution through a body
- Claimed that “any” periodic signal could be represented by such a series (i.e., **Fourier series** discussed in Chap 3)
- Obtained a representation for aperiodic signals (i.e., **Fourier integral or transform** discussed in Chap 4 & 5)
- (Fourier did not actually contribute to the mathematical theory of Fourier series)



- **Impact from Fourier's work:**

- Theory of integration, point-set topology, eigenfunction expansions, etc.
- Motion of planets, periodic behavior of the earth's climate, wave in the ocean, radio & television stations
- Harmonic time series in the 18th & 19th centuries
 - > Gauss etc. on discrete-time signals and systems
- Faster Fourier transform (FFT) in the mid-1960s
 - > Cooley (IBM) & Tukey (Princeton) reinvented in 1965
 - > Can be found in Gauss's notebooks (in 1805)

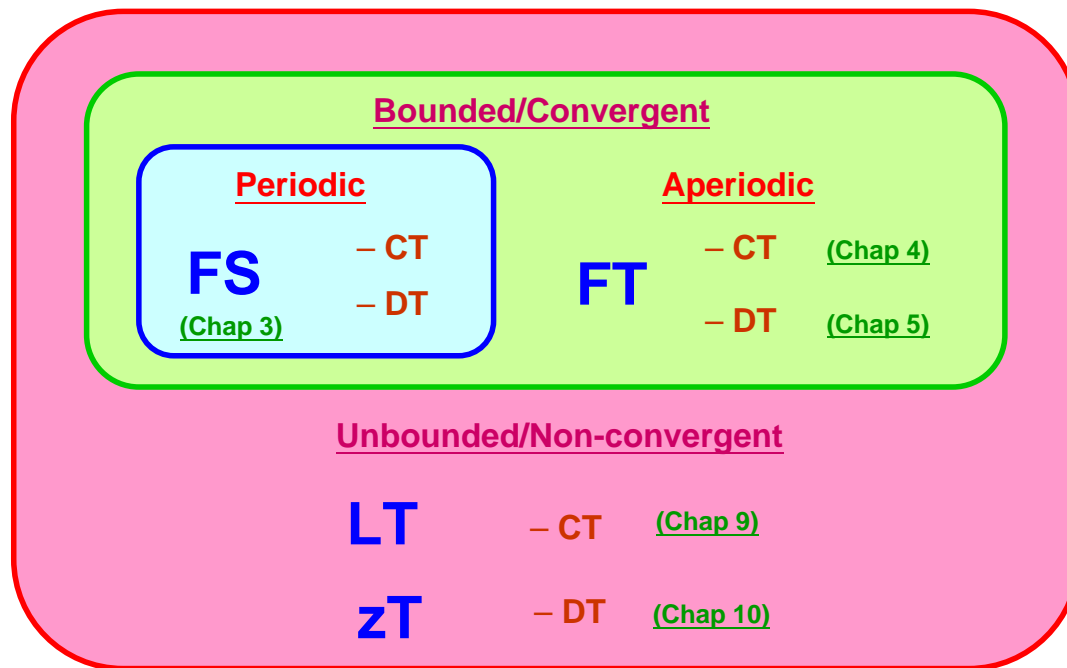


Carl Friedrich Gauss (Gauß)
1777-1855
Born in Germany
Photo from wikipedia

James W. Cooley & John W. Tukey (1965):
"An algorithm for the machine calculation of complex Fourier series",
Math. Comput. 19, 297–301.

Signals & Systems (Chap 1)

LTI & Convolution (Chap 2)



Time-Frequency (Chap 6)

Communication (Chap 8)

CT-DT (Chap 7)

Control (Chap 11)

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
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Basic Idea:

- To represent signals as linear combinations of basic signals

$$\phi_i(t)$$

Key Properties:

- The set of basic signals can be used to construct a broad and useful class of signals
- The response of an LTI system to each signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signals constructed as linear combination of basic signals

$$x(t)$$

$$x(t) \rightarrow \text{LTI} \xrightarrow{h(t)} y(t)$$

One of Choices:

- The set of complex exponential signals

$$\begin{cases} \text{signals of form } e^{st} \text{ in CT} \\ \text{signals of form } z^n \text{ in DT} \end{cases}$$

The Response of an LTI System:

$$\begin{array}{ccccc} \text{input} & \rightarrow & \text{LTI} & \rightarrow & \text{output} \\ x(t) & & h(t) & & y(t) \end{array}$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\begin{cases} \text{CT: } e^{st} \rightarrow H(s)e^{st} \\ \text{DT: } z^n \rightarrow H(z)z^n \end{cases}$$

eigenfunction
eigenvalue

Let $x(t) = e^{st}$ Let $x[n] = z^n$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) d\tau$$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k] x[n - k]$$

$$= \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$= \sum_{k=-\infty}^{+\infty} h[k] z^{n-k}$$

$$= e^{st} \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

$$= z^n \sum_{k=-\infty}^{+\infty} h[k] z^{-k}$$

$$\Rightarrow y(t) = H(s)x(t) = H(s)e^{st}$$

$$\Rightarrow y[n] = H(z)x[n] = H(z)z^n$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k] z^{-k}$$

■ Eigenfunctions and Superposition Properties:

$$e^{s_k t} \xrightarrow{\text{LTI}} H(s_k) e^{s_k t}$$

$$a_1 e^{s_1 t} \longrightarrow a_1 H(s_1) e^{s_1 t}$$

$$a_2 e^{s_2 t} \longrightarrow a_2 H(s_2) e^{s_2 t}$$

$$a_3 e^{s_3 t} \longrightarrow a_3 H(s_3) e^{s_3 t}$$

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

$$\Rightarrow x(t) = \sum_k a_k e^{s_k t} \longrightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

$$\Rightarrow x[n] = \sum_k a_k z_k^n \longrightarrow y[n] = \sum_k a_k H(z_k) z_k^n$$

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- **Fourier Series Representation of Continuous-Time Periodic Signals**
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
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Fourier Series Representation of CT Periodic Signals

- Harmonically related complex exponentials

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk\left(\frac{2\pi}{T}\right)t}, \quad k = 0, \pm 1, \pm 2, \dots$$

$$\omega_0 = \frac{2\pi}{T}$$

- The Fourier Series Representation:

$$x(t) = \dots a_{-2} \phi_{-2}(t) + a_{-1} \phi_{-1}(t) + a_0 \phi_0(t) + a_1 \phi_1(t) + a_2 \phi_2(t) + \dots$$

$$= \sum_{k=-\infty}^{+\infty} a_k \phi_k(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

$k = +1, -1$: the **first harmonic** components
or, the **fundamental** components

$k = +2, -2$: the **second harmonic** components

\dots etc.

■ Example 3.2:

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk(2\pi)t}$$

$$a_0 = 1$$

$$a_1 = a_{-1} = \frac{1}{4}$$

$$a_2 = a_{-2} = \frac{1}{2}$$

$$a_3 = a_{-3} = \frac{1}{3}$$

$$\Rightarrow x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t})$$

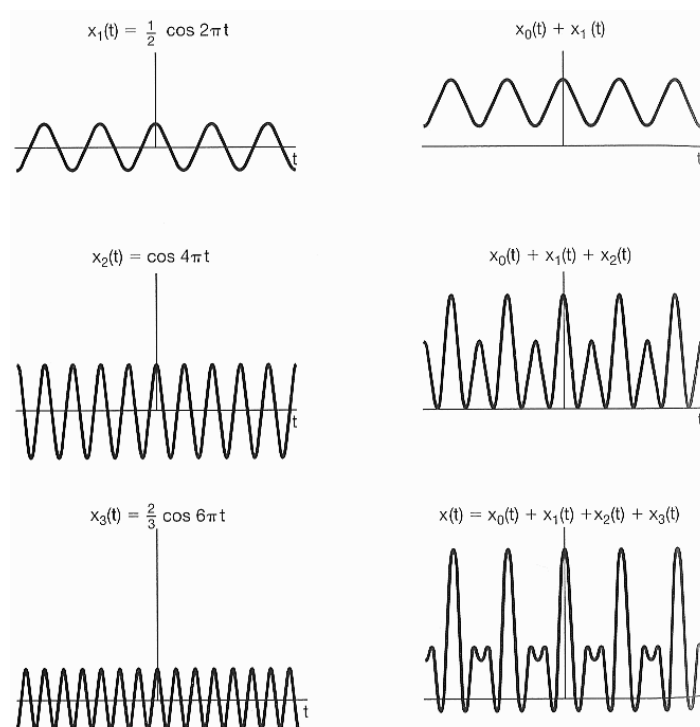
$$\Rightarrow x(t) = 1 + \frac{1}{2} \cos 2\pi t + \cos 4\pi t + \frac{2}{3} \cos 6\pi t$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

$$x(t) = 1 + \frac{1}{2} \cos 2\pi t + \cos 4\pi t + \frac{2}{3} \cos 6\pi t$$



▪ Procedure of Determining the Coefficients: $w_0 = \frac{2\pi}{T}$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk w_0 t}$$

$$x(t) e^{-jn w_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk w_0 t} e^{-jn w_0 t}$$

$$\begin{aligned} \int_0^T x(t) e^{-jn w_0 t} dt &= \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk w_0 t} e^{-jn w_0 t} dt \\ &= \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n) w_0 t} dt \right] \end{aligned}$$

$$\int_0^T e^{j(k-n) w_0 t} dt = \int_0^T \cos((k-n) w_0 t) dt + j \int_0^T \sin((k-n) w_0 t) dt$$

▪ Procedure of Determining the Coefficients:

$$\int_0^T e^{j(k-n) w_0 t} dt = \int_0^T \cos((k-n) w_0 t) dt + j \int_0^T \sin((k-n) w_0 t) dt$$

$$= \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

$$\Rightarrow \int_0^T x(t) e^{-jn w_0 t} dt = a_n T \quad \Rightarrow a_n = \frac{1}{T} \int_0^T x(t) e^{-jn w_0 t} dt$$

$$\Rightarrow a_k = \frac{1}{T} \int_0^T x(t) e^{-jk w_0 t} dt$$

• Furthermore,

$$\int_T e^{j(k-n) w_0 t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases} \quad \Rightarrow a_k = \frac{1}{T} \int_T x(t) e^{-jk w_0 t} dt$$

■ In Summary:

- The **synthesis** equation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

- The **analysis** equation:

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt$$

- $x(t) \xleftrightarrow{\text{FS}} a_k$: CT Fourier series pair
- $\{a_k\}$: the **Fourier series coefficients**
or the **spectral coefficients** of $x(t)$
- $a_0 = \frac{1}{T} \int_T x(t) dt$, the **dc** or **constant** component of $x(t)$

■ Fourier Series of Real Periodic Signals:

- If $x(t)$ is **real**, then $x^*(t) = x(t)$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$(a+b)^* = (a^* + b^*)$$

$$(a \times b)^* = (a^* \times b^*)$$

$$\Rightarrow x(t) = x(t)^* = \left(\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \right)^*$$

$$= \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}$$

$$= \sum_{m=-\infty}^{+\infty} a_{-m}^* e^{jm\omega_0 t}$$

$$m = -k$$

$$= \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t}$$

$$k = m$$

$$\Rightarrow a_{-k}^* = a_k \quad \text{or,} \quad a_k^* = a_{-k}$$

Alternative Forms of the Fourier Series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$\Rightarrow x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} \right]$$

$$= a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t} \right]$$

$$\begin{aligned} a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t} &= (R+jI)(C+jS) + (R-jI)(C-jS) \\ &= (RC-IS) + j(RS+IC) + (RC-IS) - j(RS+IC) \\ &= 2(RC-IS) \\ &= a_0 + \sum_{k=1}^{\infty} 2 \operatorname{Re} \left\{ a_k e^{jk\omega_0 t} \right\} \end{aligned}$$

Alternative Forms of the Fourier Series:

- If $a_k = A_k e^{j\theta_k}$

$$\begin{aligned} \Rightarrow x(t) &= a_0 + \sum_{k=1}^{\infty} 2 \operatorname{Re} \left\{ A_k e^{j\theta_k} e^{jk\omega_0 t} \right\} \\ &= a_0 + \sum_{k=1}^{\infty} 2 \operatorname{Re} \left\{ A_k e^{j(k\omega_0 t + \theta_k)} \right\} \\ &= a_0 + 2 \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k) \end{aligned}$$

- If $a_k = B_k + j C_k$

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$(a+jb)(c+jd) = (ac-bd) + j(ad+bc)$$

$$\begin{aligned} \Rightarrow x(t) &= a_0 + \sum_{k=1}^{\infty} 2 \operatorname{Re} \left\{ (B_k + j C_k) e^{jk\omega_0 t} \right\} \\ &= a_0 + 2 \sum_{k=1}^{\infty} \left[B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t) \right] \end{aligned}$$

■ Example 3.4:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta); \quad \cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta}); \quad \sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos \left(2\omega_0 t + \frac{\pi}{4} \right)$$

$$\Rightarrow x(t) = 1 + \frac{1}{2j} [e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

$$+ \frac{1}{2} [e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}]$$

$$\Rightarrow x(t) = 1 + \left(1 + \frac{1}{2j}\right) e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right) e^{-j\omega_0 t}$$

$$+ \left(\frac{1}{2} e^{j(\pi/4)}\right) e^{j2\omega_0 t} + \left(\frac{1}{2} e^{-j(\pi/4)}\right) e^{-j2\omega_0 t}$$

■ Example 3.4:

$$\Rightarrow \begin{cases} a_0 = 1, \\ a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j, \\ a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j, \\ a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j), \\ a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1 - j), \\ a_k = 0, \quad |k| > 2. \end{cases}$$

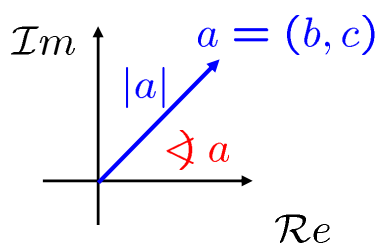
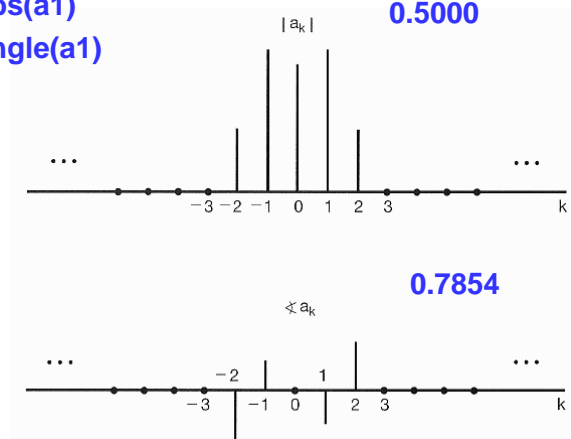
>> a1 = 1-0.5j

>> abs(a1)

>> angle(a1)

1.1180

0.5000

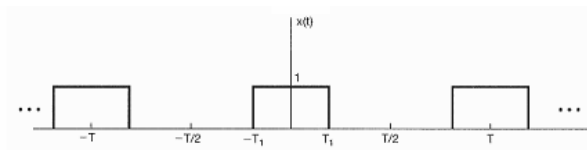


$$a = |a| e^{j\angle a}$$

$$a = |a| [\cos(\angle a) + j \sin(\angle a)]$$

$$a = b + jc = \sqrt{b^2 + c^2} \left[\frac{b}{\sqrt{b^2 + c^2}} + j \frac{c}{\sqrt{b^2 + c^2}} \right]$$

- Example 3.5: $a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt$



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

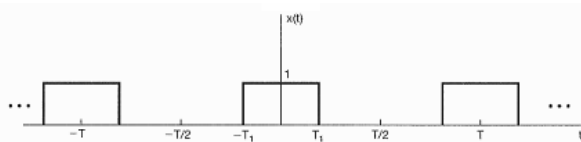
$$k = 0 \quad a_0 = \frac{1}{T} \int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

$$k \neq 0 \quad a_k = \frac{1}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt = \frac{1}{T} \frac{1}{(-jk\omega_0)} e^{-jk\omega_0 t} \Big|_{-T_1}^{T_1}$$

$$= \frac{1}{jk\omega_0 T} [e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}] / \quad \omega_0 = \frac{2\pi}{T}$$

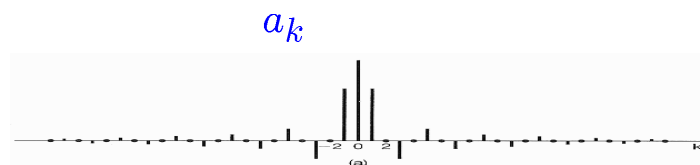
$$= \frac{2 \sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{\sin(k(2\pi/T)T_1)}{k\pi}, \quad k \neq 0$$

- Example 3.5: $T = 4T_1$



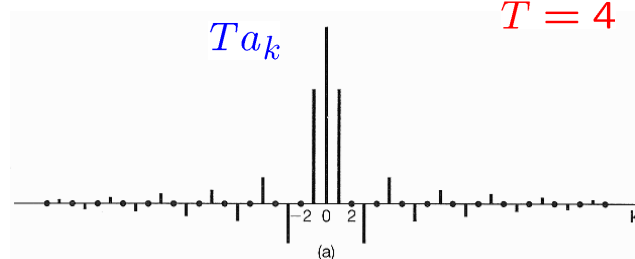
$$a_k = \frac{\sin(k2\pi \frac{T_1}{T})}{k\pi}$$

$$= \frac{\sin(k\frac{\pi}{2})}{k\pi}$$



$$T a_k = T \frac{\sin(k2\pi \frac{T_1}{T})}{k\pi}$$

$$= T \frac{\sin(k\frac{\pi}{2})}{k\pi}$$

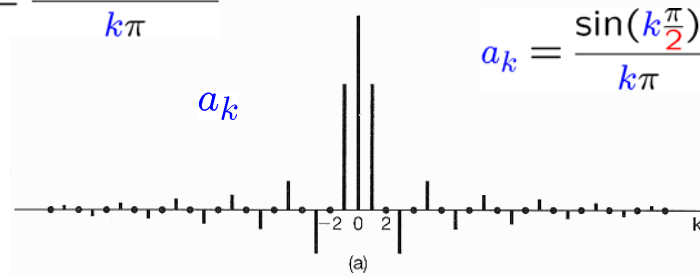
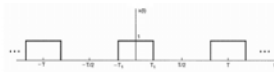


$$T_1 = 1$$

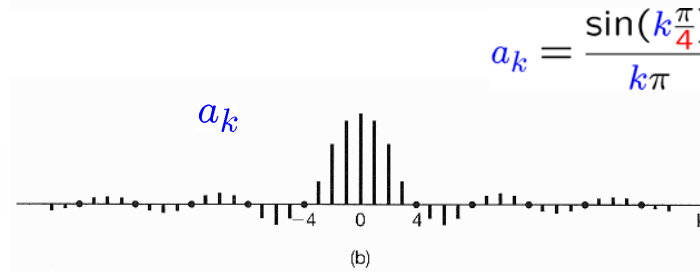
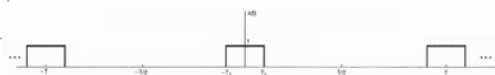
$$T = 4$$

■ Example 3.5: $a_k = \frac{\sin(k2\pi\frac{T_1}{T})}{k\pi}$

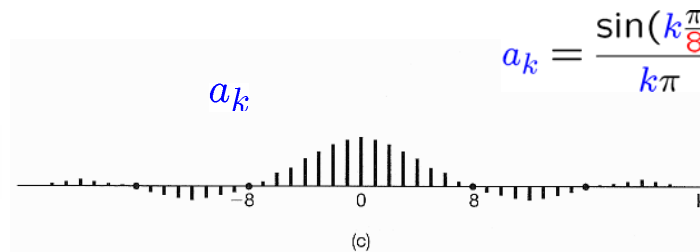
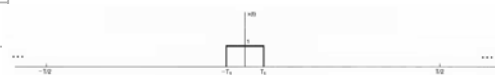
$$T = 4T_1$$



$$T = 8T_1$$

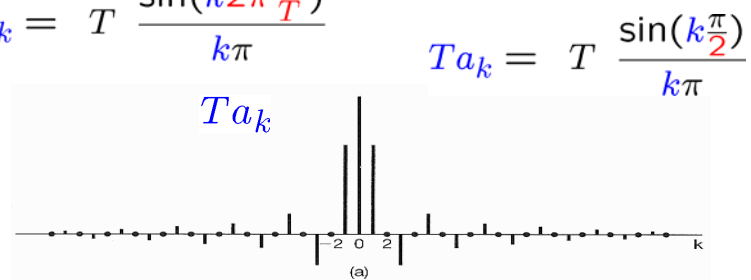


$$T = 16T_1$$

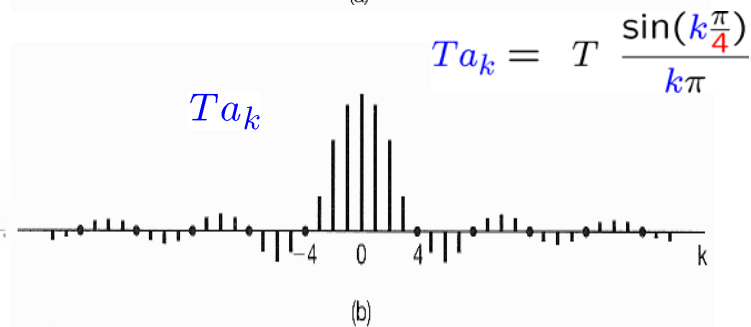
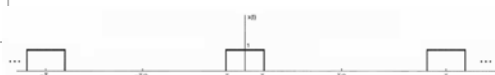


■ Example 3.5: $T a_k = T \frac{\sin(k2\pi\frac{T_1}{T})}{k\pi}$

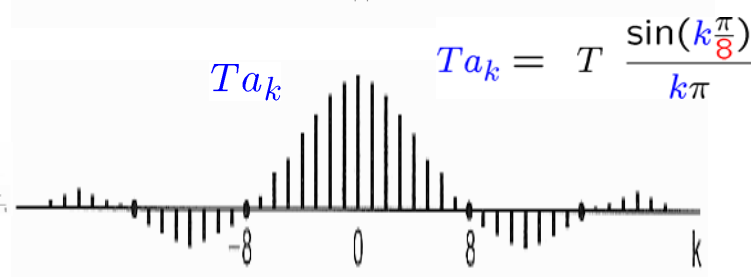
$$T = 4T_1$$



$$T = 8T_1$$

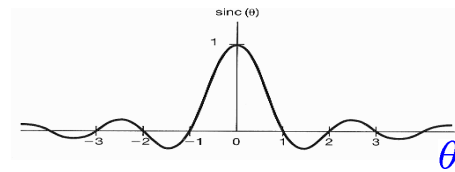
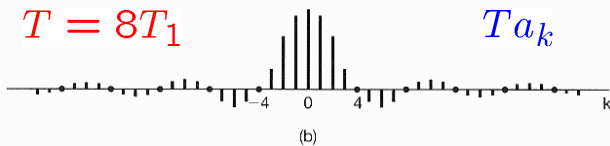


$$T = 16T_1$$



Example 3.5:

$$T = 8T_1$$



$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

$$T a_k = T \frac{\sin(k\frac{\pi}{4})}{k\pi}$$

$$= \frac{1}{4} T \frac{\sin(\pi\frac{k}{4})}{\pi\frac{k}{4}}$$

$$= \frac{1}{4} T \text{sinc}(\frac{k}{4})$$

$$\omega_0 = \frac{2\pi}{T}$$

$$T a_k = \frac{2 \sin(\omega T_1)}{\omega}$$

$$\omega = k\omega_0$$

$$\omega T_1 = k \left(\frac{2\pi}{T} \right) \cdot T_1$$

$$= \frac{2k\pi}{A}$$

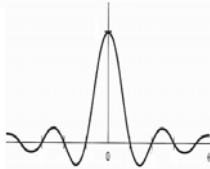
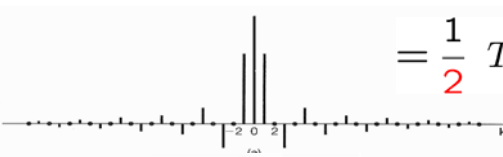
Fourier Series Representation of CT Periodic Signals

Example 3.5:

$$T = 4T_1$$

$$T a_k = T \frac{\sin(k\frac{\pi}{2})}{k\pi}$$

$$= \frac{1}{2} T \text{sinc}(\frac{k}{2})$$

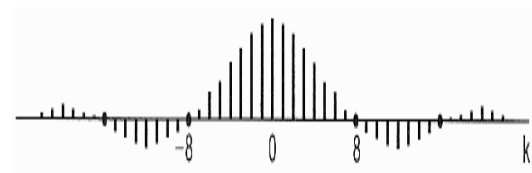


$$T a_k = T \frac{\sin(k2\pi\frac{T_1}{T})}{k\pi}$$

$$T a_k = T \frac{\sin(k\frac{\pi}{8})}{k\pi}$$

$$= \frac{1}{8} T \text{sinc}(\frac{k}{8})$$

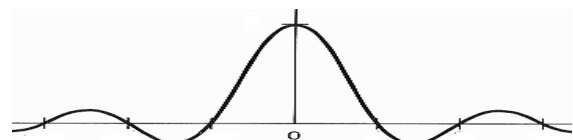
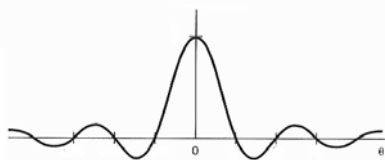
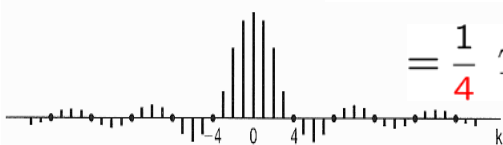
$$T = 16T_1$$



$$T = 8T_1$$

$$T a_k = T \frac{\sin(k\frac{\pi}{4})}{k\pi}$$

$$= \frac{1}{4} T \text{sinc}(\frac{k}{4})$$

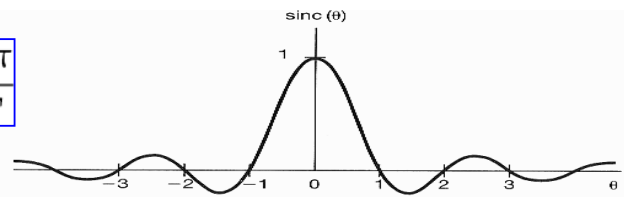


Example 3.5:

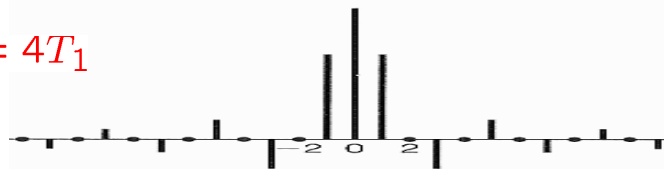
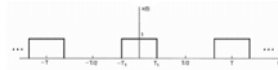
$$T a_k = T \frac{2 \sin(k \omega_0 T_1)}{k \omega_0 T}$$

$$= T_1 \frac{2 \sin(k \omega_0 T_1)}{k \omega_0 T_1}$$

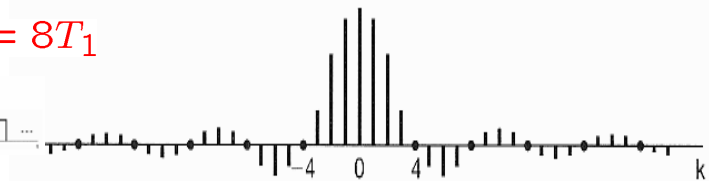
$$\omega_0 = \frac{2\pi}{T}$$



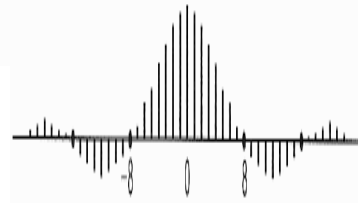
$$T = 4T_1$$



$$T = 8T_1$$



$$T = 16T_1$$



Outline

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- **Convergence of the Fourier Series**
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

- Fourier maintained that
“any” periodic signal could be represented
by a Fourier series
- The truth is that
Fourier series can be used to represent
an extremely large class of periodic signals
- The question is that
when a periodic signal $x(t)$ does in fact have a
Fourier series representation?

$x(t)$

$$x_{FS}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt$$

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk(2\pi/T)t}$$

- One class of periodic signals:
 - Which have finite energy over a single period:

$$\int_T |x(t)|^2 dt < \infty \quad \Rightarrow \quad a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt < \infty$$

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

$$e_N(t) = x(t) - x_N(t)$$

$$e(t) = x(t) - \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$E_N(t) = \int_T |e_N(t)|^2 dt$$

$$E(t) = \int_T |e(t)|^2 dt = 0$$

$$\rightarrow 0 \quad \text{as } N \rightarrow \infty \quad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \quad \forall t ???$$

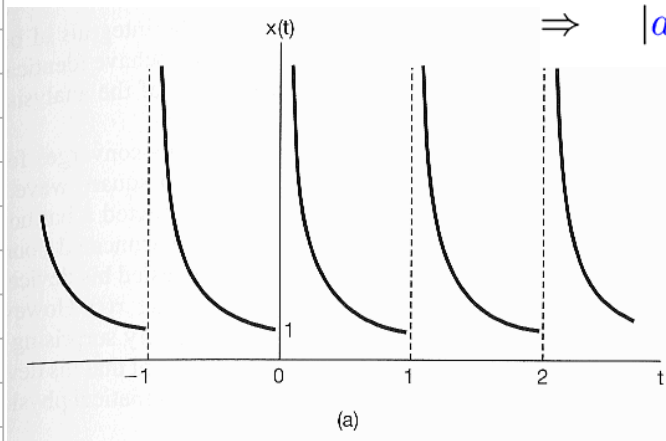
■ The other class of periodic signals:

- Which satisfy **Dirichlet conditions**:
- **Condition 1**:
 - Over any period, $x(t)$ must be **absolutely integrable**, i.e.,

$$\int_T |x(t)| dt < \infty$$



Johann Peter Gustav Lejeune Dirichlet
1805-1859
Born in Germany
Photo from wikipedia



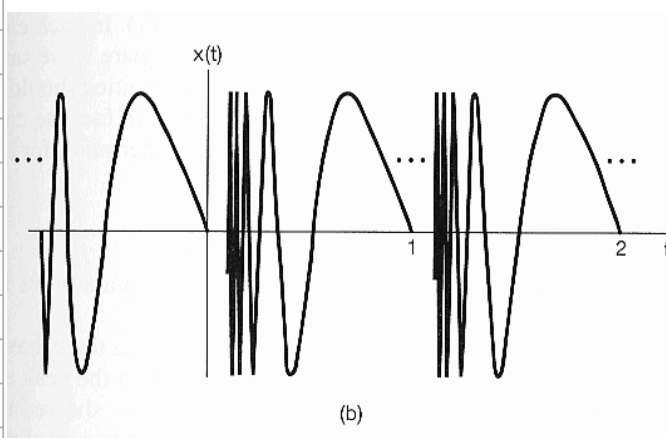
$$\Rightarrow |a_k| \leq \frac{1}{T} \int_T |x(t) e^{-jk\omega_0 t}| dt$$

$$= \frac{1}{T} \int_T |x(t)| dt < \infty$$

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1$$

■ The other class of periodic signals:

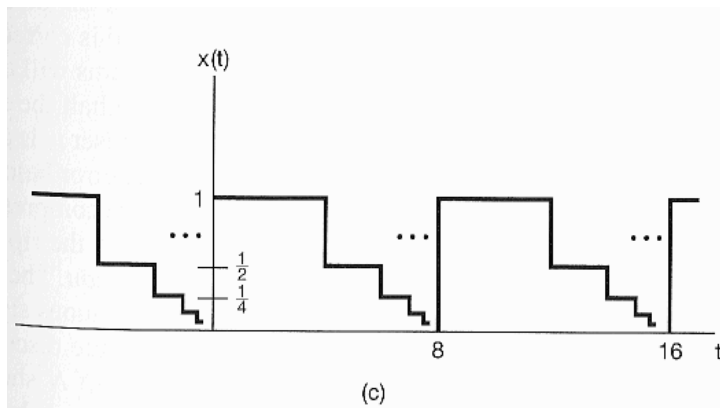
- Which satisfy **Dirichlet conditions**:
- **Condition 2**:
 - In any finite interval, $x(t)$ is of **bounded variation**; i.e.,
 - There are **no more** than a **finite number** of **maxima** and **minima** during any single period of the signal



$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1$$

$$\int_0^1 |x(t)| dt < 1$$

- The other class of periodic signals:
 - Which satisfy Dirichlet conditions:
 - Condition 3:
 - In any finite interval, $x(t)$ has only finite number of discontinuities.
 - Furthermore, each of these discontinuities is finite

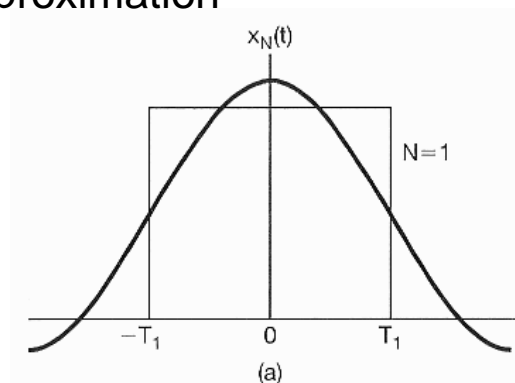


- How the Fourier series converges for a periodic signal with discontinuities
 - In 1898, Albert Michelson (an American physicist) used his harmonic analyzer to compute the truncated Fourier series approximation for the square wave

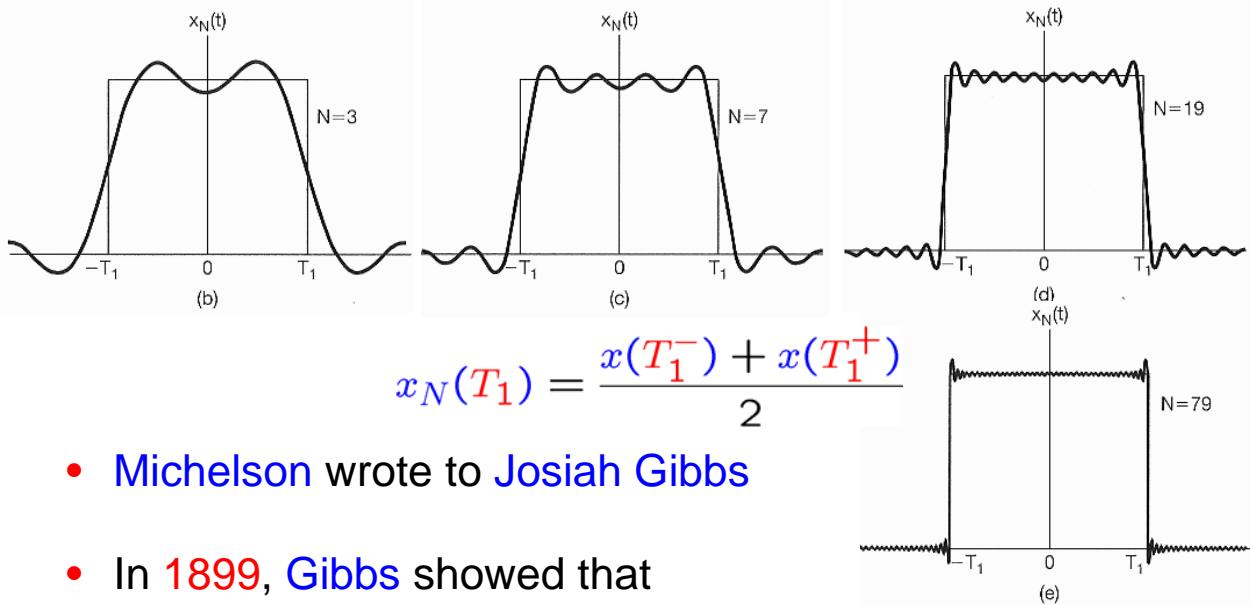


Albert Abraham Michelson
1852-1931
Polish-born German-American
Photo from wikipedia

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

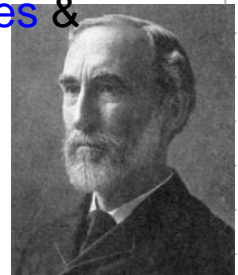


$$x_1(t) = a_{-1} e^{-j \cdot 1 \cdot \omega_0 t} + a_0 + a_1 e^{j \cdot 1 \cdot \omega_0 t}$$



- **Michelson** wrote to **Josiah Gibbs**
- In **1899**, **Gibbs** showed that
 - the partial sum **near discontinuity** exhibits **ripples** &
 - the **peak amplitude** remains **constant** with increasing N
- **The Gibbs phenomenon**

Josiah Willard Gibbs
1839-1903
Born in USA
Photo from wikipedia

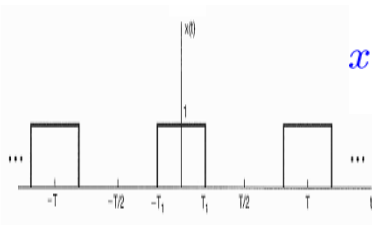


Outline

- A Historical Perspective
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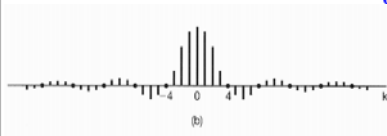
CT Fourier Series Representation:

- The **synthesis** equation:



$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

- The **analysis** equation:



$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt$$

- $x(t) \xleftrightarrow{\mathcal{FS}} a_k$: **Fouries series pair**

Outline

Section	Property
3.5.1	Linearity
3.5.2	Time Shifting
	Frequency Shifting
3.5.6	Conjugation
3.5.3	Time Reversal
3.5.4	Time Scaling
	Periodic Convolution
3.5.5	Multiplication
	Differentiation
	Integration
3.5.6	Conjugate Symmetry for Real Signals
3.5.6	Symmetry for Real and Even Signals
3.5.6	Symmetry for Real and Odd Signals
	Even-Odd Decomposition for Real Signals
3.5.7	Parseval's Relation for Periodic Signals

■ Linearity:

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

- $x(t), y(t)$: periodic signals with period T

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \quad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$y(t) \xleftrightarrow{\mathcal{FS}} b_k \quad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jm\omega_0 t}$$

$$\Rightarrow z(t) = Ax(t) + By(t) \xleftrightarrow{\mathcal{FS}} c_k = Aa_k + Bb_k$$

$$z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$$

Add

■ Time Shifting:

- $x(t)$: periodic signal with period T

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$\Rightarrow x(t - t_0) \xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 t_0} a_k = e^{-jk\left(\frac{2\pi}{T}\right)t_0} a_k$$

$$\text{b/c } b_k = \frac{1}{T} \int_T x(t - t_0) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_T x(\tau) e^{-jk\omega_0(\tau + t_0)} d\tau$$

$$= e^{-jk\omega_0 t_0} \frac{1}{T} \int_T x(\tau) e^{-jk\omega_0 \tau} d\tau$$

Time Reversal:

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk \left(\frac{2\pi}{T}\right) t}$$

$$\Rightarrow x(-t) \xleftrightarrow{\mathcal{FS}} a_{-k}$$

$$\begin{aligned} x(-t) &= \sum_{k=-\infty}^{+\infty} a_k e^{-j k \left(\frac{2\pi}{T}\right) t} \\ &= \sum_{m=-\infty}^{+\infty} a_{-m} e^{j m \left(\frac{2\pi}{T}\right) t} \end{aligned}$$

- If $x(t)$ is **even**, i.e., $x(-t) = x(t)$

$$\Rightarrow a_k \text{ is even, i.e., } a_{-k} = a_k$$

- If $x(t)$ is **odd**, i.e., $x(-t) = -x(t)$

$$\Rightarrow a_k \text{ is odd, i.e., } a_{-k} = -a_k$$

Time Scaling:

- $x(t)$: **periodic** signals with period T
and fundamental frequency w_0

- $x(\alpha t)$: **periodic** signals with period $\frac{T}{\alpha}$
and fundamental frequency αw_0

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk w_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk \left(\frac{2\pi}{T}\right) t}$$

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{j k w_0 (\alpha t)} = \sum_{k=-\infty}^{+\infty} a_k e^{j k \alpha \left(\frac{2\pi}{T}\right) t}$$

$$= \sum_{k=-\infty}^{+\infty} a_k e^{j k (\alpha w_0) t} = \sum_{k=-\infty}^{+\infty} a_k e^{j k \left(\frac{2\pi}{\frac{T}{\alpha}}\right) t}$$

■ Multiplication:

$$z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$$

- $x(t), y(t)$: periodic signals with period T

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$x(t) = \sum_{l=-\infty}^{+\infty} a_l e^{jl\omega_0 t}$$

$$y(t) \xleftrightarrow{\mathcal{FS}} b_k$$

$$y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jm\omega_0 t}$$

$\Rightarrow x(t)y(t)$: also periodic with T

$$z(t) = x(t)y(t) \xleftrightarrow{\mathcal{FS}} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

Add

■ Differentiation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

- $x(t)$: periodic signals with period T

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{FS}} jk\omega_0 a_k$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Integration:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

- $x(t)$: periodic signals with period T

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{FS}} \frac{1}{jk\omega_0} a_k$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Conjugation & Conjugate Symmetry:

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \quad x(t)^* \xleftrightarrow{\mathcal{FS}} a_{-k}^*$$

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{+\infty} a_k e^{j \quad k \quad \omega_0 t} \\ &= \sum_{k=-\infty}^{+\infty} a_{-k} e^{j \quad k \quad \omega_0 t} \end{aligned}$$

■ Conjugation & Conjugate Symmetry:

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \quad x(t)^* \xleftrightarrow{\mathcal{FS}} a_{-k}^*$$

- $x(t) = x(t)^* \Rightarrow a_{-k} = a_k^*$

$x(t)$ is **real** $\Rightarrow \{a_k\}$ are **conjugate symmetric**

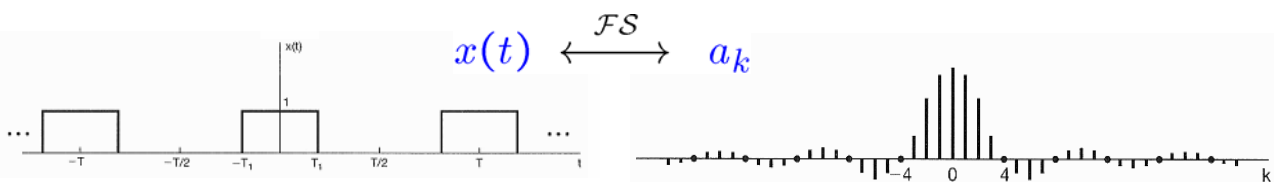
- $x(t) = x(t)^* \ \& \ x(-t) = x(t) \Rightarrow a_{-k} = a_k^* \ \& \ a_{-k} = a_k$
 $\Rightarrow a_k = a_k^*$

$x(t)$ is **real** & **even** $\Rightarrow \{a_k\}$ are **real** & **even**

- $x(t)$ is **real** & **odd** $\Rightarrow \{a_k\}$ are **purely imaginary** & **odd**
 $\Rightarrow a_k^* = -a_k$

■ Parseval's relation for CT periodic signals:

- As shown in Problem 3.46:



$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

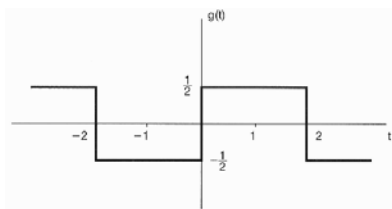
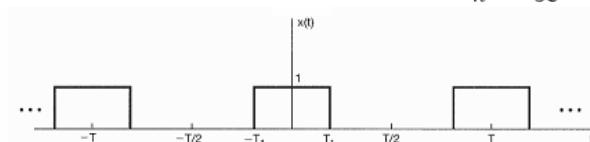
- **Parseval's relation** states that
the **total average power** in a periodic signal
equals
the **sum of the average powers**
in **all** of its **harmonic components**

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$ } Periodic with period T and $y(t)$ } fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a_{-k}^*
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk \frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^t x(\tau) d\tau$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a_{-k}^* \\ \Re\{a_k\} = \Re\{a_{-k}\} \\ \Im\{a_k\} = -\Im\{a_{-k}\} \\ a_k = a_{-k} \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\begin{cases} \Re\{a_k\} \\ j\Im\{a_k\} \end{cases}$
Parseval's Relation for Periodic Signals			
$\frac{1}{T} \int_T x(t) ^2 dt = \sum_{k=-\infty}^{+\infty} a_k ^2$			

Properties of CT Fourier Series

- **Example 3.6:** $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$



$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$x(t) = \begin{cases} 1, & |t| < T/2 \\ 0, & T/2 < |t| < T \end{cases}$$

$$a_0 = \frac{2T_1}{T}$$

$$a_k = \frac{\sin(k(2\pi/T)T_1)}{k\pi}, \quad k \neq 0$$

$$g(t) = x(t - 1) - 1/2$$

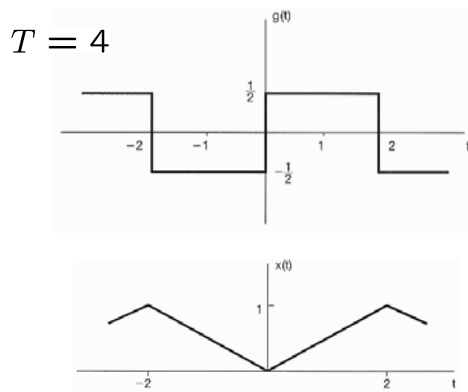
$$\text{with } T = 4, T_1 = 1$$

$$x(t - 1) \xleftrightarrow{\mathcal{FS}} b_k = a_k e^{-jk\pi/2}$$

$$g(t) = x(t - 1) - 1/2 \xleftrightarrow{\mathcal{FS}} \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - 1/2, & \text{for } k = 0 \end{cases}$$

$$g(t) \xleftrightarrow{\mathcal{FS}} \begin{cases} \frac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}$$

Example 3.7:



$$g(t) \xleftrightarrow{\mathcal{FS}} d_k$$

$$x(t) \xleftrightarrow{\mathcal{FS}} e_k$$

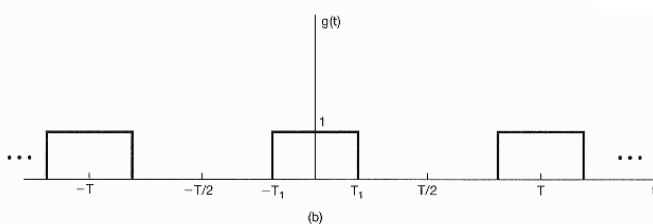
$$\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{FS}} jkw_0 e_k$$

$$g(t) = \frac{d}{dt}x(t) \iff d_k = jk(\pi/2)e_k$$

$$e_k = \begin{cases} \frac{2}{jk\pi} d_k = \frac{2\sin(\pi k/2)}{j(k\pi)^2} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ \frac{1}{2}, & \text{for } k = 0 \end{cases}$$

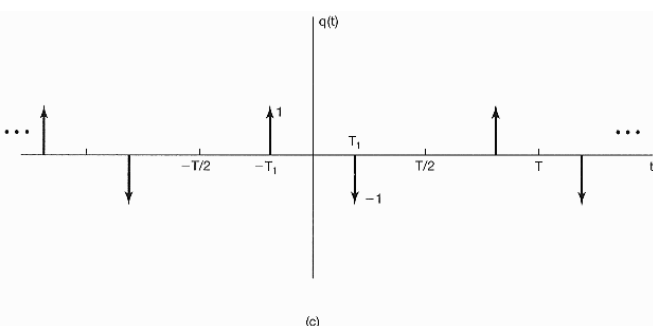
Example 3.8:

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$



$$g(t) \xleftrightarrow{\mathcal{FS}} c_k$$

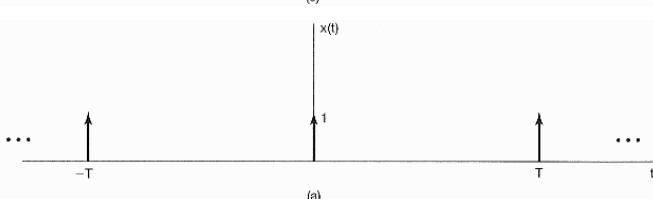
$$q(t) = \frac{d}{dt}g(t) \iff b_k = jkw_0 c_k$$



$$q(t) \xleftrightarrow{\mathcal{FS}} b_k$$

$$q(t) = x(t + T_1) - x(t - T_1)$$

$$\iff b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k$$



$$x(t) \xleftrightarrow{\mathcal{FS}} a_k = \frac{1}{T}$$

■ Example 3.8:

$$b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k$$

$$= \frac{1}{T} [e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}]$$

$$= \frac{2j \sin(k\omega_0 T_1)}{T}$$

$$b_k = jk\omega_0 c_k$$

$$k \neq 0 \quad c_k = \frac{b_k}{jk\omega_0} = \frac{2j \sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

$$k = 0 \quad c_0 = \frac{2T_1}{T}$$

Outline

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

▪ Harmonically related complex exponentials

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk\left(\frac{2\pi}{N}\right)n}, \quad k = 0, \pm 1, \pm 2, \dots$$

$$\phi_{k+N}[n] = e^{j(k+N)\left(\frac{2\pi}{N}\right)n} = e^{jk\left(\frac{2\pi}{N}\right)n} e^{jN\left(\frac{2\pi}{N}\right)n}$$

$$\Rightarrow \phi_k[n] = \phi_{k+N}[n] = \dots = \phi_{k+rN}[n]$$

▪ The Fourier Series Representation:

$$x[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

▪ Procedure of Determining the Coefficients:

$$x[0] = \sum_{k=\langle N \rangle} a_k$$

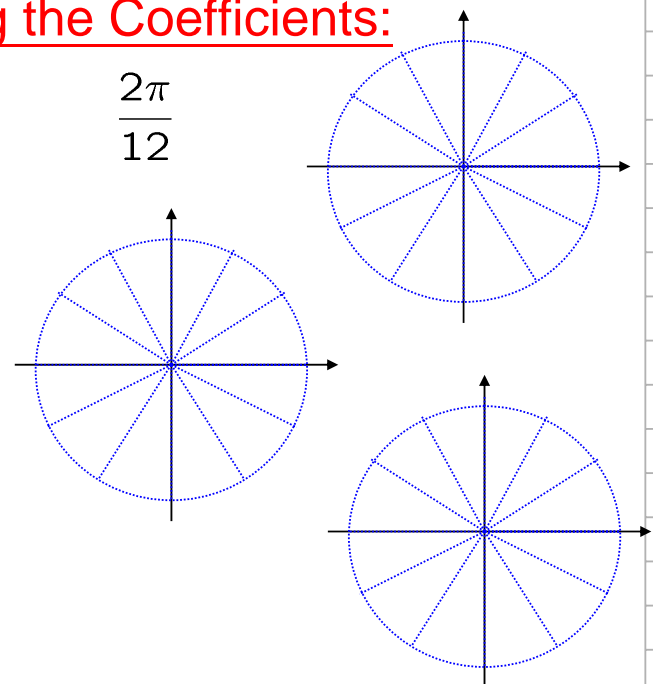
$$x[1] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)}$$

$$x[2] = \sum_{k=\langle N \rangle} a_k e^{jk2\left(\frac{2\pi}{N}\right)}$$

⋮

$$x[N-1] = \sum_{k=\langle N \rangle} a_k e^{jk(N-1)\left(\frac{2\pi}{N}\right)}$$

$$x[N] = \sum_{k=\langle N \rangle} a_k e^{jk(N)\left(\frac{2\pi}{N}\right)}$$



$$\text{and } \sum_{n=\langle N \rangle} e^{jm\left(\frac{2\pi}{N}\right)n} = \begin{cases} N, & m = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$$

■ Procedure of Determining the Coefficients:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \quad \sum_{n=\langle N \rangle} e^{-jr\left(\frac{2\pi}{N}\right)n}$$

$$\sum_{n=\langle N \rangle} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{n=\langle N \rangle} \sum_{k=\langle N \rangle} a_k e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

$$\sum_{n=\langle N \rangle} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{k=\langle N \rangle} a_k \sum_{n=\langle N \rangle} e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

$$= a_r N$$

$$\Rightarrow a_r = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n}$$

Properties of DT Fourier Series

■ In Summary:

- The **synthesis** equation:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

- The **analysis** equation:

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = a_{k+N}$$

- $x[n] \xleftrightarrow{\mathcal{FS}} a_k$: DT Fourier series pair
- $\{a_k\}$: the **Fourier series coefficients**
or the **spectral coefficients** of $x[n]$

■ Example 3.11:

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}n\right) + 3\cos\left(\frac{2\pi}{N}n\right) + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$$

$$\Rightarrow x[n] = 1 + \frac{1}{2j} \left[e^{j\left(\frac{2\pi}{N}n\right)} - e^{-j\left(\frac{2\pi}{N}n\right)} \right] + \frac{3}{2} \left[e^{j\left(\frac{2\pi}{N}n\right)} + e^{-j\left(\frac{2\pi}{N}n\right)} \right] + \frac{1}{2} \left[e^{j\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)} + e^{-j\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)} \right]$$

$$\Rightarrow x[n] = 1 + \left(\frac{3}{2} + \frac{1}{2j}\right) e^{j\left(\frac{2\pi}{N}n\right)} + \left(\frac{3}{2} - \frac{1}{2j}\right) e^{-j\left(\frac{2\pi}{N}n\right)} + \frac{1}{2} e^{j\left(\frac{\pi}{2}\right)} e^{j2\left(\frac{2\pi}{N}n\right)} + \frac{1}{2} e^{-j\left(\frac{\pi}{2}\right)} e^{-j2\left(\frac{2\pi}{N}n\right)}$$

Fourier Series Representation of DT Periodic Signals

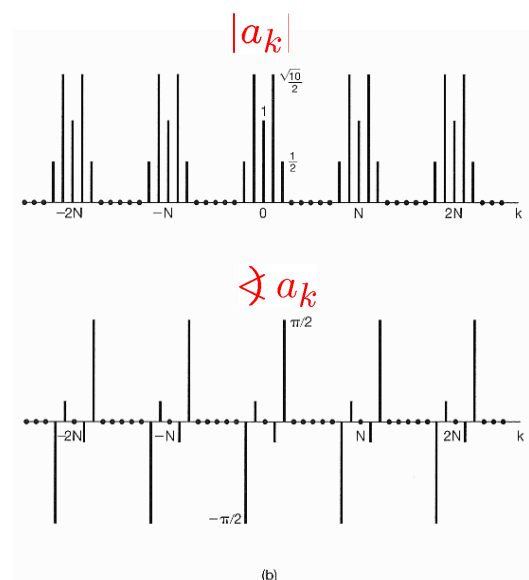
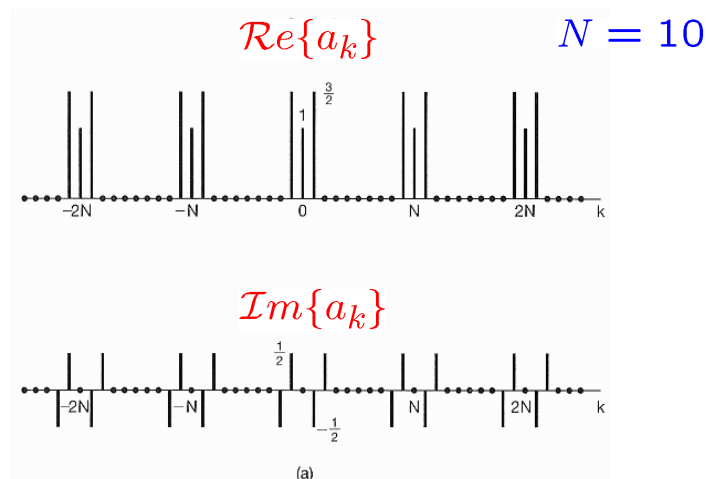
■ Example 3.11:

$$a = |a| e^{j\angle a}$$

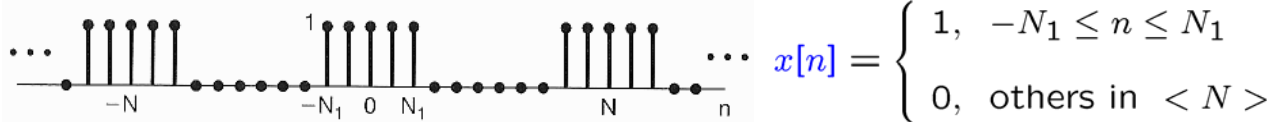
$$a = |a| [\cos(\angle a) + j \sin(\angle a)]$$

$$a = b + jc = \sqrt{b^2 + c^2} \left[\frac{b}{\sqrt{b^2 + c^2}} + j \frac{c}{\sqrt{b^2 + c^2}} \right]$$

$$\Rightarrow \begin{cases} a_0 = 1 \\ a_1 = \left(\frac{3}{2} + \frac{1}{2j}\right) = \frac{3}{2} - \frac{1}{2}j \\ a_{-1} = \left(\frac{3}{2} - \frac{1}{2j}\right) = \frac{3}{2} + \frac{1}{2}j \\ a_2 = \frac{1}{2}j \\ a_{-2} = -\frac{1}{2}j \\ a_k = 0, \text{ others in } \langle N \rangle \end{cases}$$



■ Example 3.12: $a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(\frac{2\pi}{N})n}$



$$\begin{aligned}
 a_k &= \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 \cdot e^{-jk(\frac{2\pi}{N})n} = \frac{1}{N} \sum_{n=-N_1}^{N_1} \left(e^{-jk(\frac{2\pi}{N})} \right)^n \\
 &= \frac{1}{N} \left[(\cdot)^{-N_1} + (\cdot)^{-N_1+1} + \dots + (\cdot)^{N_1} \right] \\
 &= \frac{1}{N} (\cdot)^{-N_1} \left[\frac{1 - (\cdot)^{(2N_1+1)}}{1 - (\cdot)} \right] \quad (\cdot) \neq 1 \\
 &= \frac{1}{N} (\cdot)^{-N_1} \left[1 + (\cdot)^1 + \dots + (\cdot)^{2N_1} \right]
 \end{aligned}$$

• Let $m = n + N_1$ or $n = m - N_1$

$$a_k = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk(\frac{2\pi}{N})(m-N_1)} = \frac{1}{N} e^{jk(\frac{2\pi}{N})N_1} \sum_{m=0}^{2N_1} e^{-jk(\frac{2\pi}{N})m}$$

■ Example 3.12:

• $k = 0, \pm N, \pm 2N, \dots$

$$a_k = \frac{2N_1 + 1}{N}$$

$$\begin{aligned}
 &1 - e^{-j\theta} \\
 &= e^{-j\theta/2} e^{j\theta/2} - e^{-j\theta/2} e^{-j\theta/2} \\
 &= e^{-j\theta/2} (e^{j\theta/2} - e^{-j\theta/2})
 \end{aligned}$$

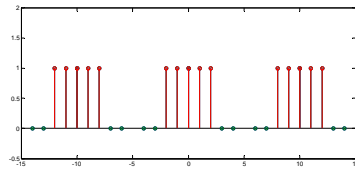
• $k \neq 0, \pm N, \pm 2N, \dots$

$$\begin{aligned}
 a_k &= \frac{1}{N} e^{jk(\frac{2\pi}{N})N_1} \left(\frac{1 - e^{-jk(\frac{2\pi}{N})(2N_1+1)}}{1 - e^{-jk(\frac{2\pi}{N})}} \right) \quad \left(N_1 + \frac{1}{2} \right) \\
 &= \frac{1}{N} \frac{e^{-jk(\frac{2\pi}{N})} \left[e^{jk(\frac{2\pi}{N})(2N_1+1)} - e^{-jk(\frac{2\pi}{N})(2N_1+1)} \right]}{e^{-jk(\frac{2\pi}{N})} \left[e^{jk(\frac{2\pi}{N})} - e^{-jk(\frac{2\pi}{N})} \right]} \\
 &= \frac{1}{N} \frac{\sin \left[\left(\frac{2\pi}{N} \right) k \left(N_1 + \frac{1}{2} \right) \right]}{\sin \left[\left(\frac{\pi}{N} \right) k \right]}
 \end{aligned}$$

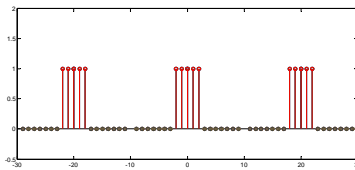
Example 3.12:

- $2N_1 + 1 = 5$

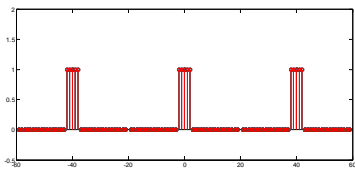
- $N = 10$



- $N = 20$

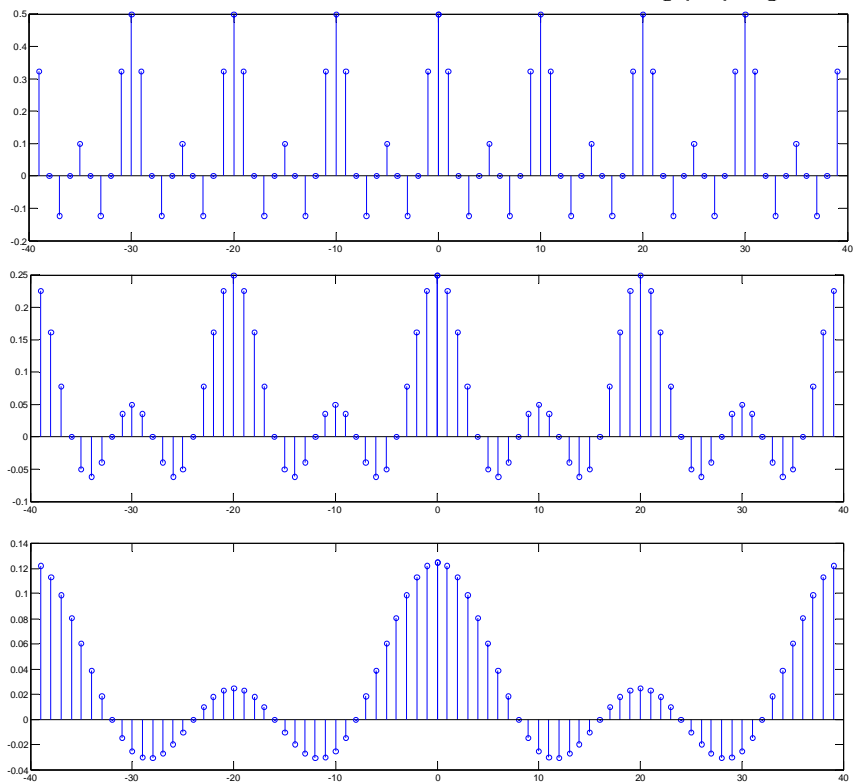


- $N = 40$



a_k

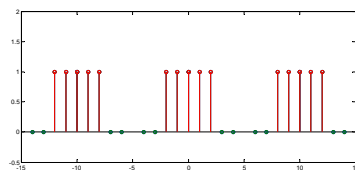
$$a_k = \frac{1}{N} \frac{\sin \left[\left(\frac{2\pi}{N} \right) k \left(N_1 + \frac{1}{2} \right) \right]}{\sin \left[\left(\frac{\pi}{N} \right) k \right]}$$



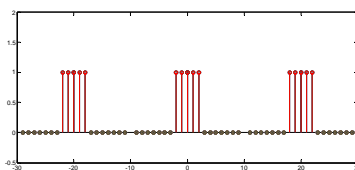
Example 3.12:

- $2N_1 + 1 = 5$

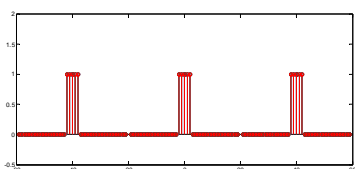
- $N = 10$



- $N = 20$

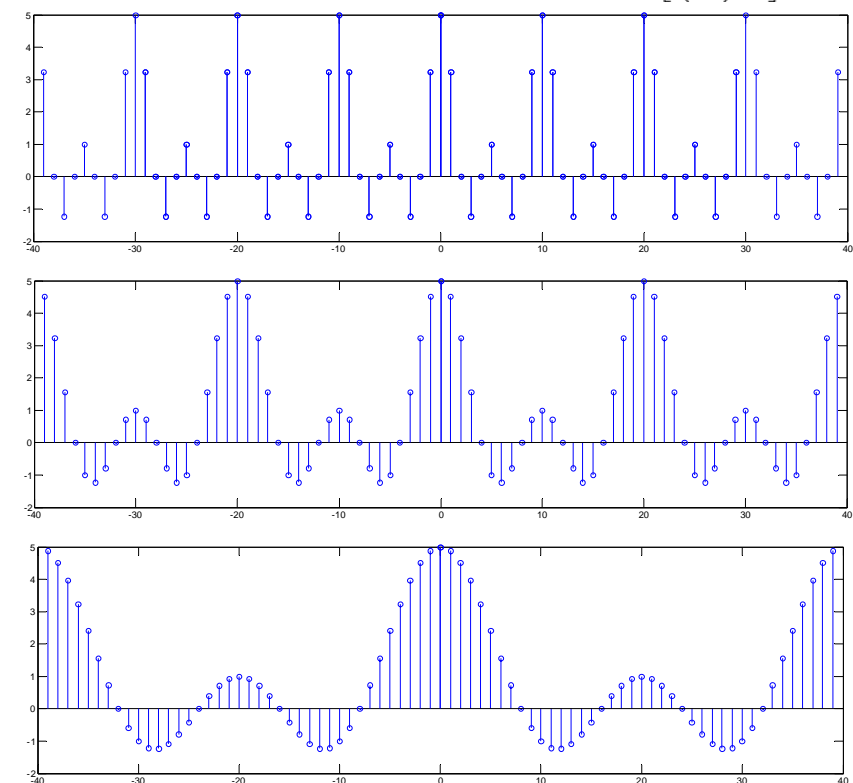


- $N = 40$

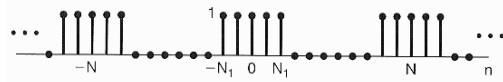


Na_k

$$a_k = \frac{1}{N} \frac{\sin \left[\left(\frac{2\pi}{N} \right) k \left(N_1 + \frac{1}{2} \right) \right]}{\sin \left[\left(\frac{\pi}{N} \right) k \right]}$$

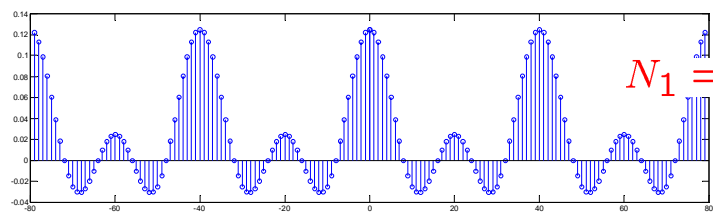
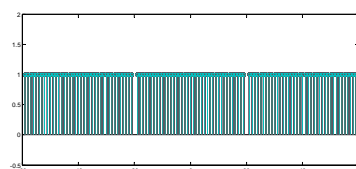
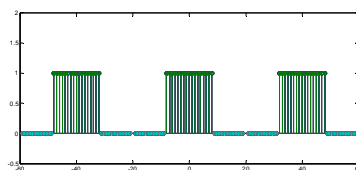
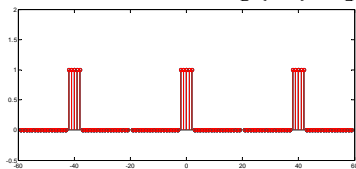


Example 3.12:

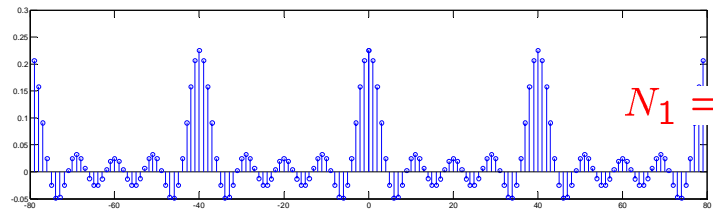


• $N = 40$

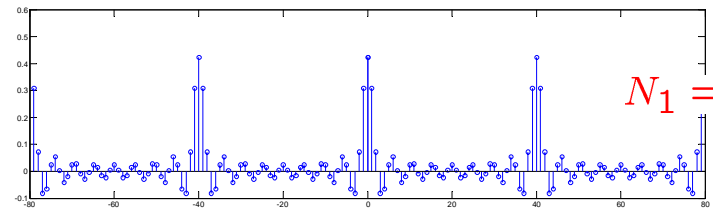
$$a_k = \frac{1}{N} \frac{\sin \left[\left(\frac{2\pi}{N} \right) k \left(N_1 + \frac{1}{2} \right) \right]}{\sin \left[\left(\frac{\pi}{N} \right) k \right]}$$



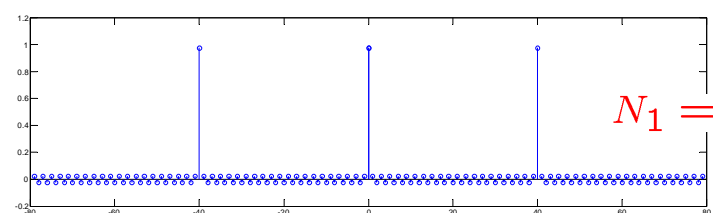
$N_1 = 2$



$N_1 = 4$



$N_1 = 8$

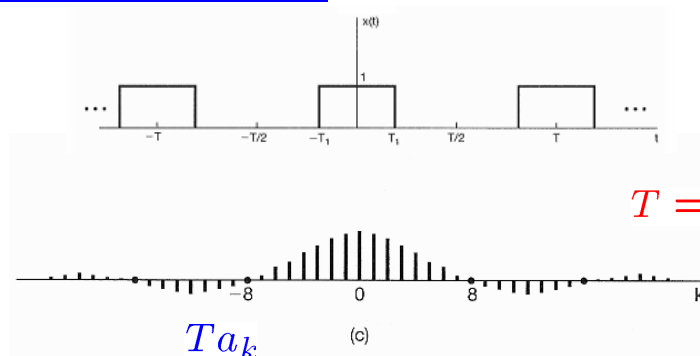


$N_1 = 19$

Fourier Series Representation of CT Periodic Signals

Examples 3.5 (CT) & 3.12 (DT):

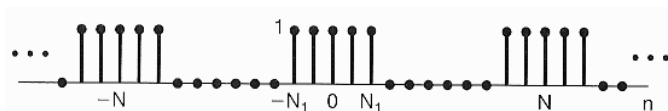
$$T a_k = T \frac{\sin(k\frac{\pi}{8})}{k\pi}$$



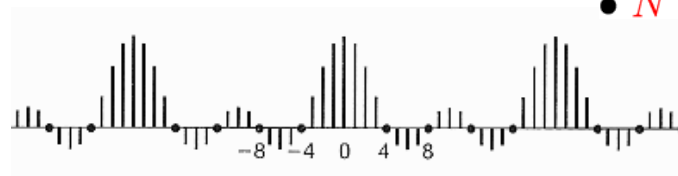
$T = 16T_1$

$$a_k = \frac{1}{N} \frac{\sin \left[\left(\frac{2\pi}{N} \right) k \left(N_1 + \frac{1}{2} \right) \right]}{\sin \left[\left(\frac{\pi}{N} \right) k \right]}$$

$$a_k = \frac{2N_1 + 1}{N}$$



• $N = 20$



Partial Sum:

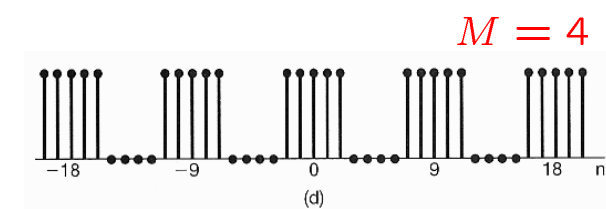
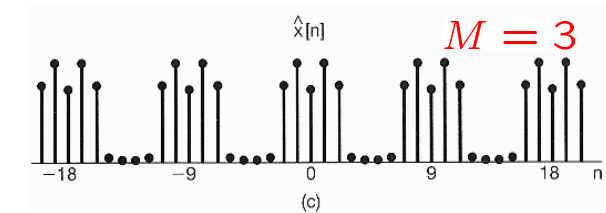
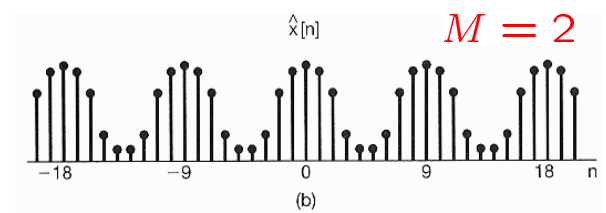
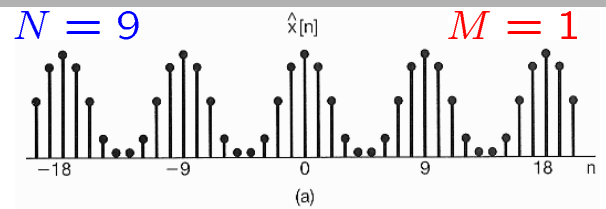
$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

- If N is odd

$$\hat{x}[n] = \sum_{k=-M}^M a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

- If N is even

$$\hat{x}[n] = \sum_{k=-M+1}^M a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$



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- Filtering & Examples of CT & DT Filters

Section	Property
	Linearity
	Time Shifting
	Frequency Shifting
	Conjugation
	Time Reversal
	Time Scaling
	Periodic Convolution
3.7.1	Multiplication
3.7.2	First Difference
	Running Sum
	Conjugate Symmetry for Real Signals
	Symmetry for Real and Even Signals
	Symmetry for Real and Odd Signals
	Even-Odd Decomposition for Real Signals
3.7.3	Parseval's Relation for Periodic Signals

Properties of DT Fourier Series

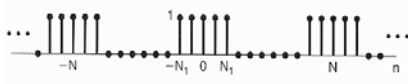
TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$x[n]$ } Periodic with period N and $y[n]$ } fundamental frequency $\omega_0 = 2\pi/N$	a_k } Periodic with b_k } period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a_{-k}^*
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k$ (viewed as periodic) (with period mN)
Periodic Convolution	$\sum_{r=(N)} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=(N)} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^n x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})} \right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a_{-k}^* \\ \Re\{a_k\} = \Re\{a_{-k}\} \\ \Im\{a_k\} = -\Im\{a_{-k}\} \\ a_k = a_{-k} \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\begin{cases} \Re\{a_k\} \\ j\Im\{a_k\} \end{cases}$
Parseval's Relation for Periodic Signals		
$\frac{1}{N} \sum_{n=(N)} x[n] ^2 = \sum_{k=(N)} a_k ^2$		

■ In Summary:

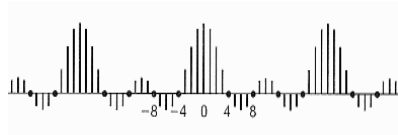
- The **synthesis** equation:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$



- The **analysis** equation:

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$



$$a_k = a_{k+N}$$

- $x[n] \xleftrightarrow{\mathcal{FS}} a_k$: DT Fourier series pair

Properties of DT Fourier Series

■ Linearity:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

- $x[n], y[n]$: periodic signals with period N

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k$$

$$y[n] \xleftrightarrow{\mathcal{FS}} b_k$$

$$\Rightarrow z[n] = Ax[n] + By[n] \xleftrightarrow{\mathcal{FS}} c_k = Aa_k + Bb_k$$

■ Time Shifting:

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k$$

$$\Rightarrow x[n - n_0] \xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 n_0} a_k = e^{-jk\left(\frac{2\pi}{N}\right)n_0} a_k$$

■ Multiplication:

- $x[n], y[n]$: periodic signals with period N

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k \qquad x[n] = \sum_{l=\langle N \rangle} a_l e^{jlw_0 n}$$

$$y[n] \xleftrightarrow{\mathcal{FS}} b_k \qquad y[n] = \sum_{m=\langle N \rangle} b_m e^{jm w_0 n}$$

$\Rightarrow x[n]y[n]$: also periodic with N

$$c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

$$x[n]y[n] \xleftrightarrow{\mathcal{FS}} d_k = \sum_{l=\langle N \rangle} a_l b_{k-l}$$

\Rightarrow a periodic convolution

Add

■ First Difference:

$$x[n] = \sum_{l=\langle N \rangle} a_l e^{jk w_0 n}$$

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k$$

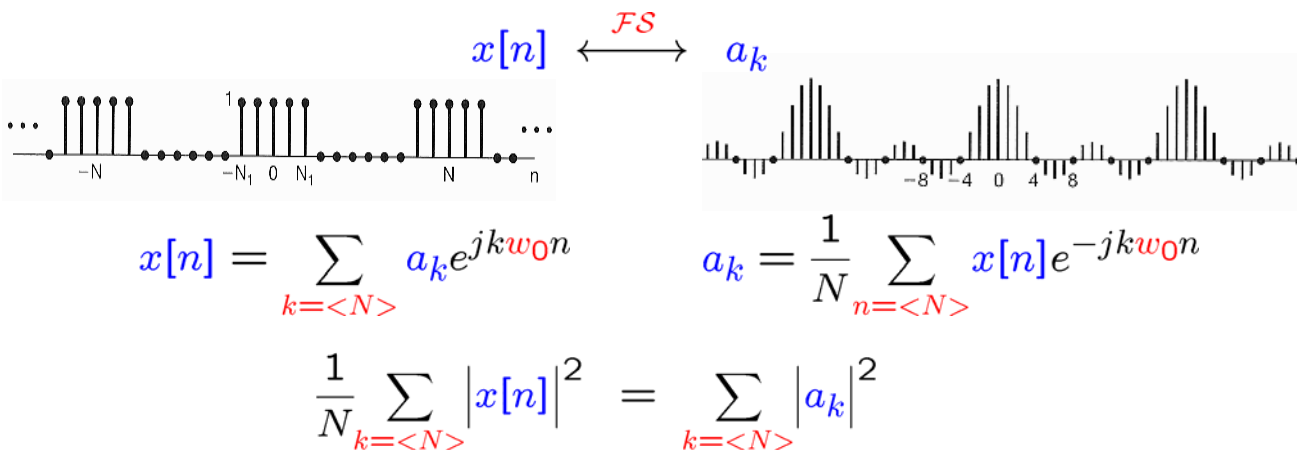
$$\Rightarrow x[n - n_0] \xleftrightarrow{\mathcal{FS}} e^{-jk w_0 n_0} a_k = e^{-jk \left(\frac{2\pi}{N}\right) n_0} a_k$$

$$\Rightarrow x[n - 1] \xleftrightarrow{\mathcal{FS}} e^{-jk w_0} a_k = e^{-jk \left(\frac{2\pi}{N}\right)} a_k$$

$$x[n] - x[n - 1] \xleftrightarrow{\mathcal{FS}} \left(1 - e^{-jk \left(\frac{2\pi}{N}\right)}\right) a_k$$

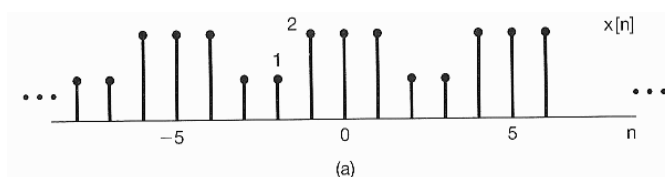
Parseval's relation for DT periodic signals:

- As shown in Problem 3.57:



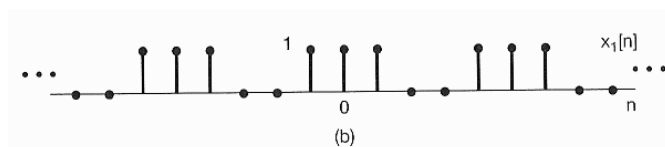
- Parseval's relation** states that the **total average power** in a periodic signal equals the **sum of the average powers** in **all** of its **harmonic components** (only **N** distinct harmonic components in DT)

Example 3.13:

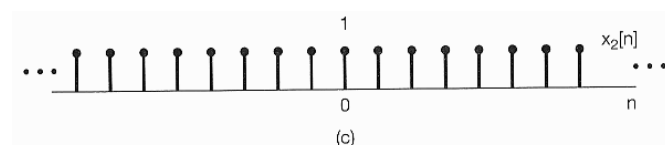


$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n}$$

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k$$



$$x_1[n] \xleftrightarrow{\mathcal{FS}} b_k$$



$$x_2[n] \xleftrightarrow{\mathcal{FS}} c_k$$

$$\Rightarrow b_k = \begin{cases} \frac{1}{5} \frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \dots \\ \frac{3}{5}, & \text{for } k = 0, \pm 5, \pm 10, \dots \end{cases}$$

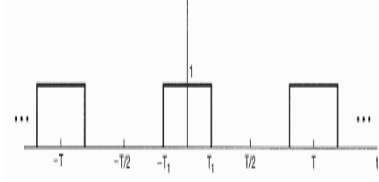
$$x[n] = x_1[n] + x_2[n]$$

$$\Rightarrow c_k = \begin{cases} 0, & \text{for } k \neq 0, \pm 5, \pm 10, \dots \\ 1, & \text{for } k = 0, \pm 5, \pm 10, \dots \end{cases} \Rightarrow a_k = b_k + c_k$$

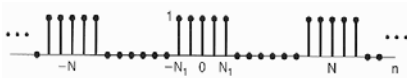
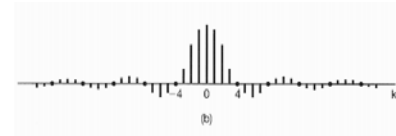
CT & DT Fourier Series Representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

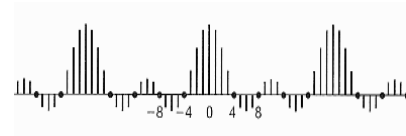
$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$



$$x(t) \xleftrightarrow{CTFS} a_k$$



$$x[n] \xleftrightarrow{DTFS} a_k$$



$$x[n] = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 n}$$

$$a_k = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk\omega_0 n}$$

Outline

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

■ The Response of an LTI System:

$$in \rightarrow \boxed{\text{LTI}} \rightarrow out \quad \left\{ \begin{array}{l} \text{CT: } e^{st} \rightarrow H(s)e^{st} \\ \text{DT: } z^n \rightarrow H(z)z^n \end{array} \right.$$

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st} dt \quad \Rightarrow \text{the impulse response}$$

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k} \quad \Rightarrow \text{the system functions}$$

- If $s = jw$ or $z = e^{jw}$:

$$H(jw) = \int_{-\infty}^{+\infty} h(t)e^{-jw t} dt \quad \Rightarrow \text{the frequency response}$$

$$H(e^{jw}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-jwn}$$

■ In Summary:

$$a = |a|e^{j\angle a}$$

$$H = |H|e^{j\angle H}$$

$$in \rightarrow \boxed{\begin{array}{c} \text{LTI} \\ H(s/z/w) \end{array}} \rightarrow out \quad \left\{ \begin{array}{l} \text{CT: } e^{s_i t} \rightarrow H(s_i)e^{s_i t} \\ \text{DT: } z_i^n \rightarrow H(z_i)z_i^n \end{array} \right.$$

($s_i = jw_i$ or $z_i = e^{jw_i}$)

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \rightarrow y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n} \rightarrow y[n] = \sum_{k=\langle N \rangle} a_k H(e^{j(\frac{2\pi}{N})k}) e^{jk(\frac{2\pi}{N})n}$$

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- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- **Filtering** & Examples of CT & DT Filters

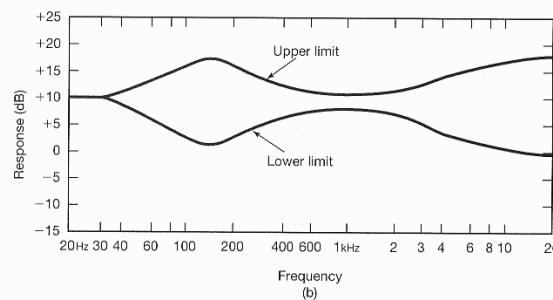
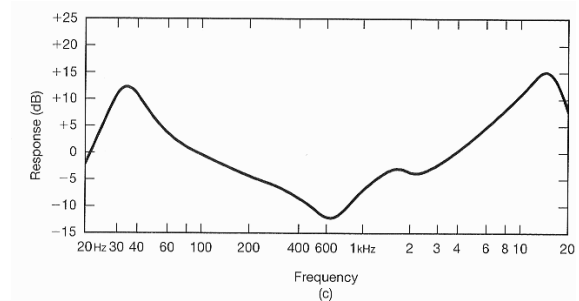
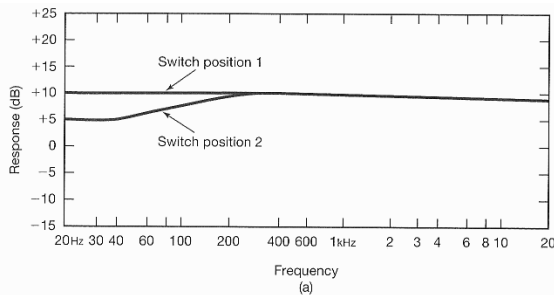
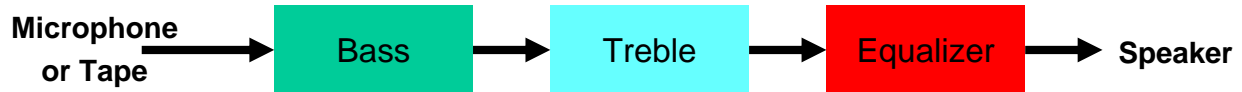
- Filtering:



- **Change** the relative amplitudes of the frequency components in a signal,
 - **Frequency-shaping filters**
- OR, significantly **attenuate** or **eliminate** some frequency components entirely
 - **Frequency-selective filters**

Frequency-Shaping Filters:

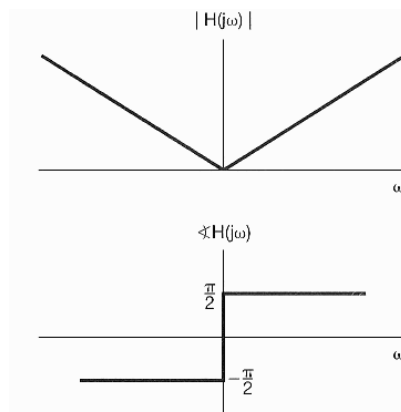
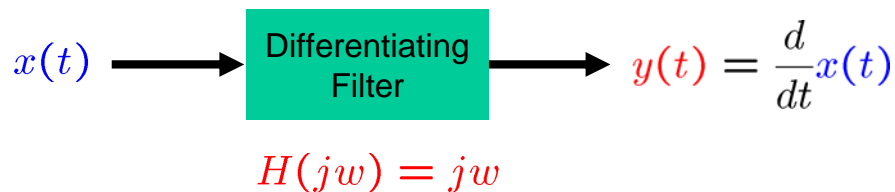
- Audio System:



Filtering

Frequency-Shaping Filters:

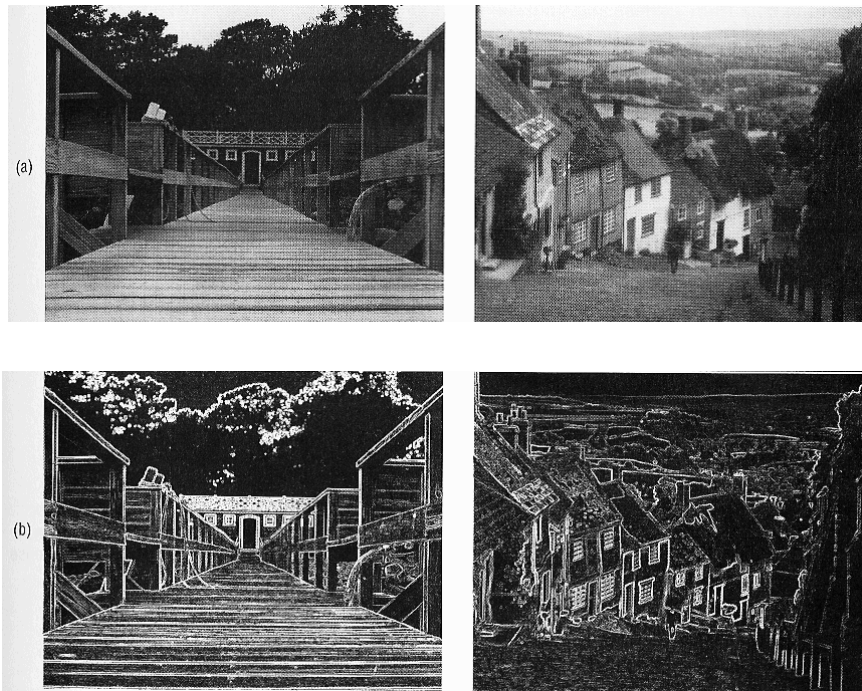
- Differentiating filter:



Frequency-Shaping Filters:

- Differentiating filter on enhancing edges: $H(j\omega) = j\omega$

$$x(t) \longrightarrow \text{Differentiating Filter} \longrightarrow y(t) = \frac{d}{dt}x(t)$$



Frequency-Shaping Filters:

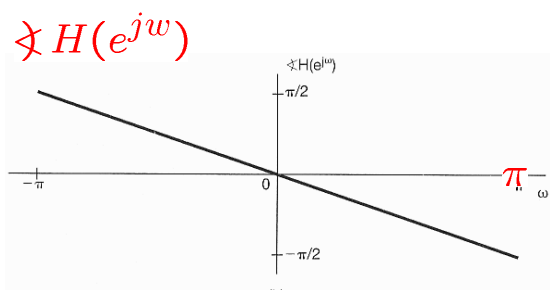
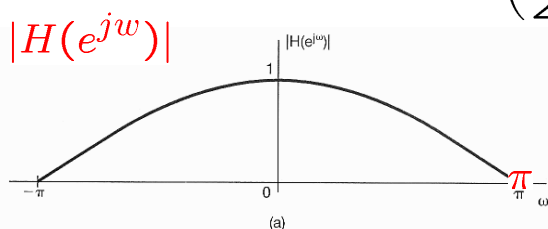
- A simple DT filter: Two-point average

$$y[n] = \frac{1}{2} (x[n] + x[n-1]) =$$

$$x[n] = H(e^{j\omega}) x[n]$$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{2} [1 + e^{-j\omega}] = \frac{1}{2} e^{-j(\frac{\omega}{2})} [e^{j(\frac{\omega}{2})} + e^{-j(\frac{\omega}{2})}]$$

$$= e^{-j(\frac{\omega}{2})} \cos\left(\frac{\omega}{2}\right)$$

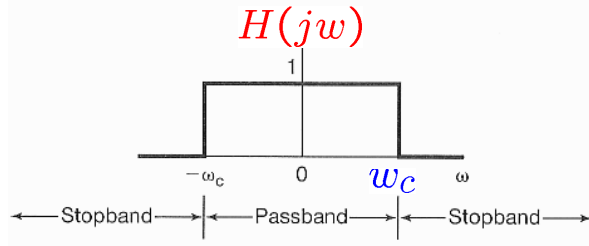


$$\text{if } x[n] = K e^{j(\frac{\pi}{2}) \cdot n}$$

$$\text{then } y[n] = H\left(e^{j(\frac{\pi}{2})}\right) K e^{j(\frac{\pi}{2}) \cdot n}$$

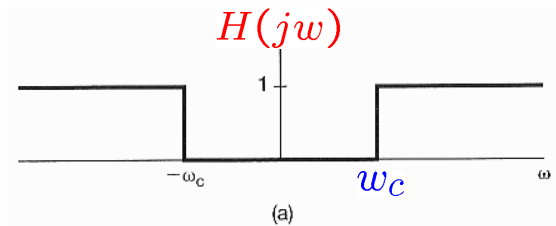
Frequency-Selective Filters:

- Select some bands of frequencies and reject others



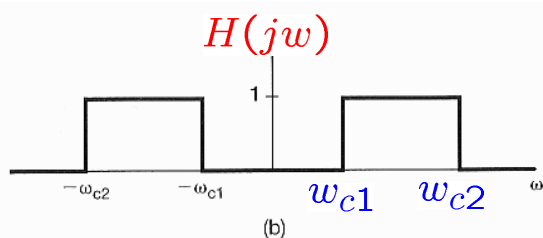
CT ideal lowpass filter

$$H(jw) = \begin{cases} 1, & |w| \leq w_c \\ 0, & |w| > w_c \end{cases}$$



CT ideal highpass filter

$$H(jw) = \begin{cases} 0, & |w| < w_c \\ 1, & |w| \geq w_c \end{cases}$$

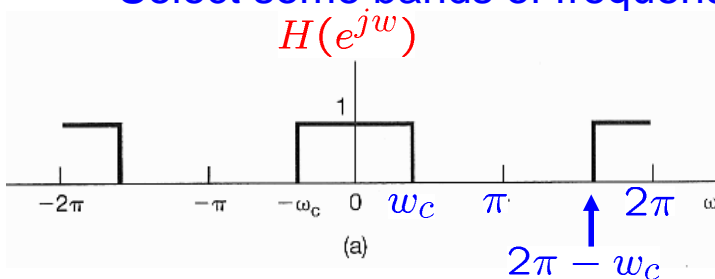


CT ideal bandpass filter

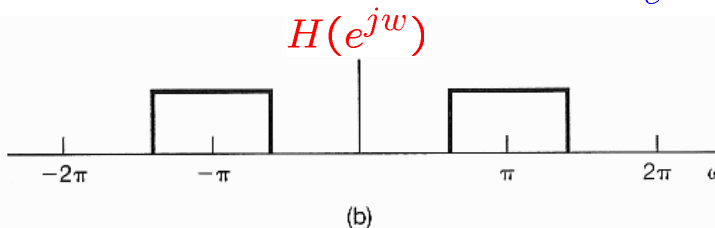
$$H(jw) = \begin{cases} 1, & w_{c1} \leq |w| \leq w_{c2} \\ 0, & \text{otherwise} \end{cases}$$

Frequency-Selective Filters:

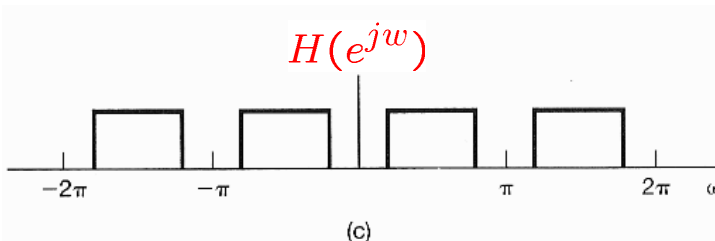
- Select some bands of frequencies and reject others



DT ideal lowpass filter



DT ideal highpass filter

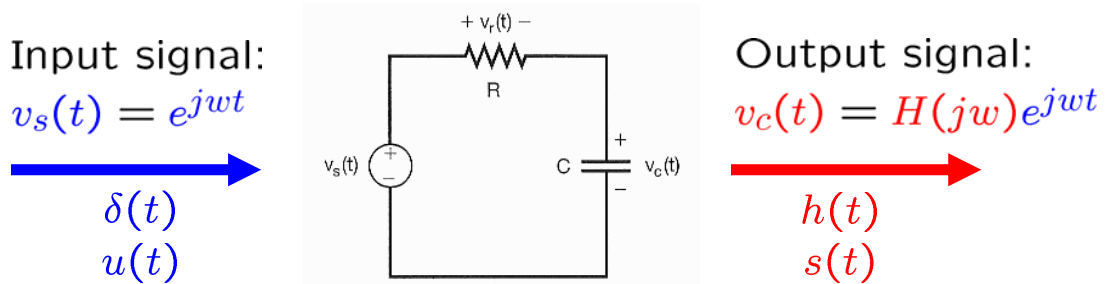


DT ideal bandpass filter

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CT Filters by Differential Equations

▪ A Simple RC Lowpass Filter:



$$\Rightarrow RC \frac{d}{dt} v_c(t) + v_c(t) = v_s(t)$$

$$\Rightarrow RC \frac{d}{dt} [H(j\omega)e^{j\omega t}] + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

$$\Rightarrow RC j\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

$$\Rightarrow H(j\omega)e^{j\omega t} = \frac{1}{1 + RCj\omega} e^{j\omega t}$$

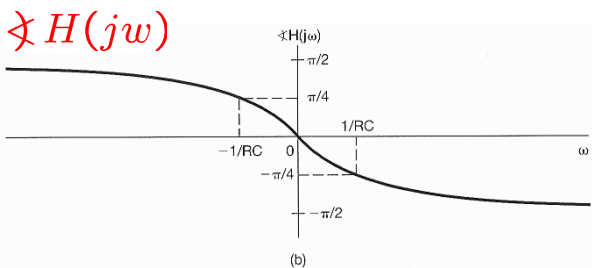
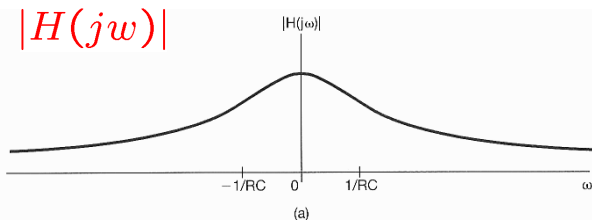
- A Simple RC Lowpass Filter: $H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt$

$$\Rightarrow H(j\omega) = \frac{1}{1 + RCj\omega}$$

$$\Rightarrow h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$$\Rightarrow s(t) = [1 - e^{-t/RC}] u(t)$$

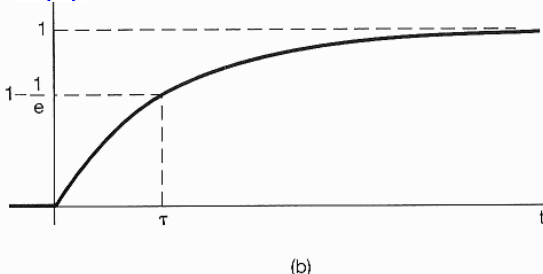
$$H = |H| e^{j\angle H}$$



$h(t)$ impulse response



$s(t)$ step response



- A Simple RC Highpass Filter: $\begin{matrix} h(t) \\ \uparrow \\ s(t) \end{matrix}$ Output signal: $v_r(t) = G(j\omega) e^{j\omega t}$

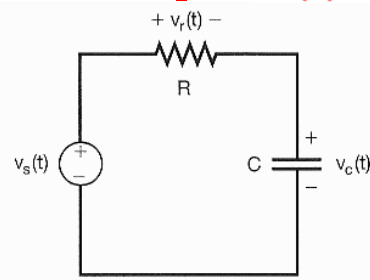
Input signal:

$$v_s(t) = e^{j\omega t}$$



$$\delta(t)$$

$$u(t)$$



$$\Rightarrow RC \frac{d}{dt} v_r(t) + v_r(t) = RC \frac{d}{dt} v_s(t)$$

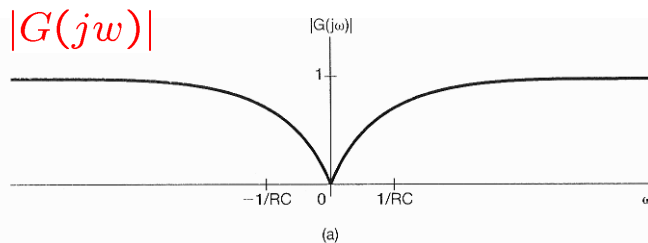
$$\Rightarrow RC \frac{d}{dt} [G(j\omega) e^{j\omega t}] + G(j\omega) e^{j\omega t} = RC \frac{d}{dt} e^{j\omega t}$$

$$\Rightarrow RC j\omega G(j\omega) e^{j\omega t} + G(j\omega) e^{j\omega t} = RC j\omega e^{j\omega t}$$

$$\Rightarrow G(j\omega) e^{j\omega t} = \frac{j\omega RC}{1 + j\omega RC} e^{j\omega t}$$

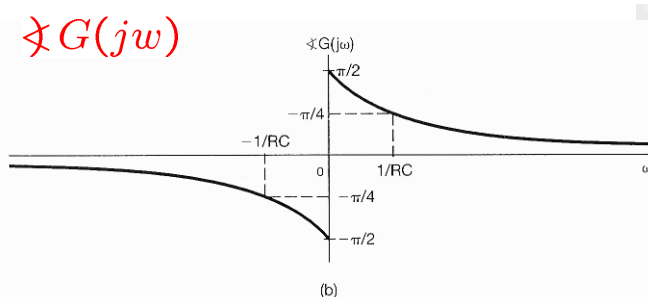
■ A Simple RC Highpass Filter:

$$\Rightarrow G(j\omega) = \frac{j\omega RC}{1 + j\omega RC}$$



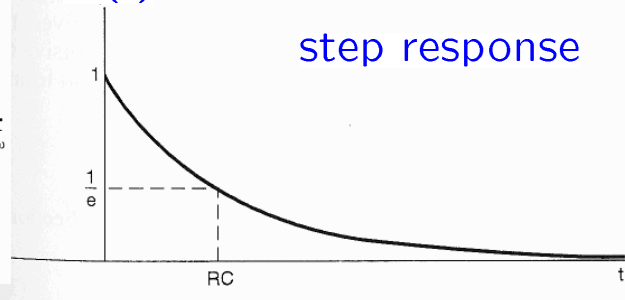
$$v_r(t) = v_s(t) - v_c(t)$$

$$\Rightarrow v_r(t) = e^{-t/RC} u(t)$$



$$v_r(t)$$

step response



DT Filters by Difference Equations

■ First-Order Recursive DT Filters:

$$y[n] - ay[n-1] = x[n]$$

- If $x[n] = e^{j\omega n}$, then $y[n] = H(e^{j\omega})e^{j\omega n}$

where $H(e^{j\omega})$: the frequency response

$$\Rightarrow H(e^{j\omega})e^{j\omega n} - a H(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}$$

$$\Rightarrow [1 - a e^{-j\omega}] H(e^{j\omega})e^{j\omega n} = e^{j\omega n}$$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

First-Order Recursive DT Filters:

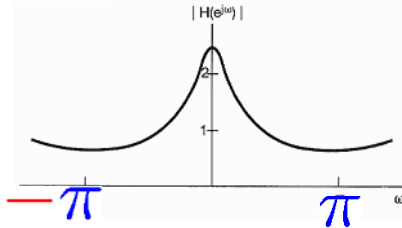
$$H(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

$$y[n] = a y[n-1] + x[n]$$

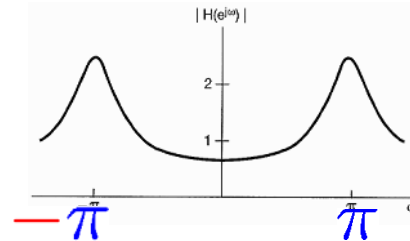
lowpass filter: $0 < a < 1$

highpass filter: $-1 < a < 0$

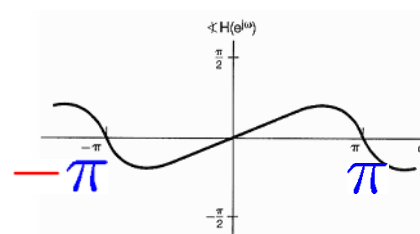
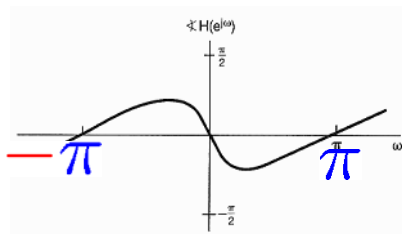
$|H(e^{j\omega})|$ $a = 0.6$



$a = -0.6$



$\angle H(e^{j\omega})$

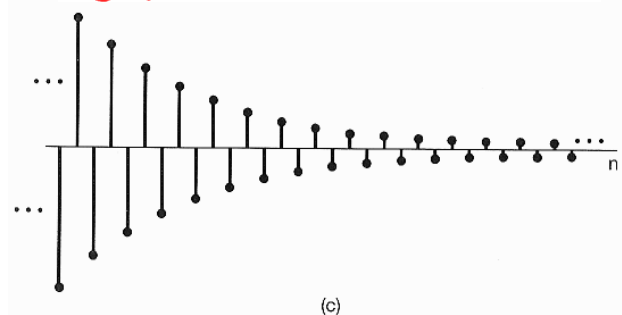
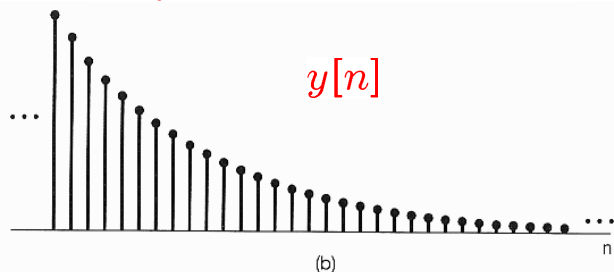


First-Order Recursive DT Filters:

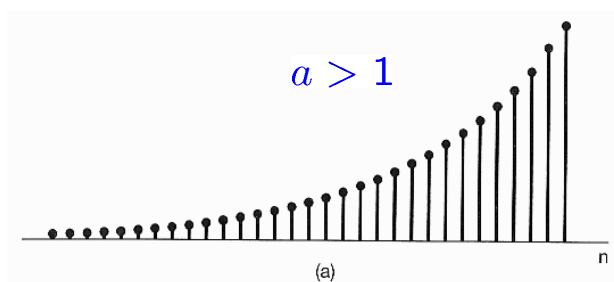
$$y[n] = a y[n-1] + x[n]$$

lowpass filter: $0 < a < 1$

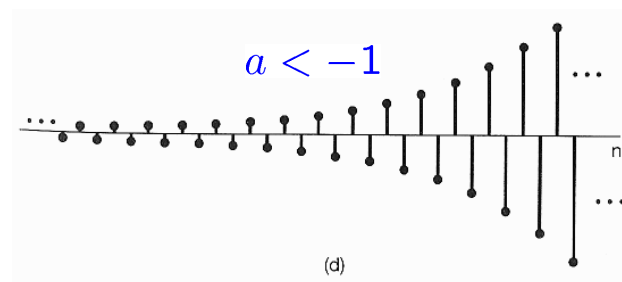
highpass filter: $-1 < a < 0$



$a > 1$



$a < -1$



■ Nonrecursive DT Filters:

- An FIR nonrecursive difference equation:

$$\begin{aligned}
 y[n] &= \sum_{k=-N}^M b_k x[n-k] \\
 &= b_{-N} x[n+N] + b_{-N+1} x[n+N-1] + \cdots \\
 &\quad + b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]
 \end{aligned}$$

$$b_k =$$

$$b_k =$$

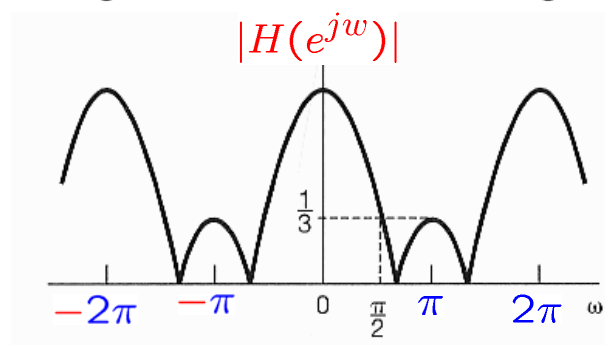
■ Nonrecursive DT Filters:

- Three-point moving average (lowpass) filter:

$$y[n] = \frac{1}{3} (x[n+1] + x[n] + x[n-1])$$

$$\Rightarrow h[n] = \frac{1}{3} (\delta[n+1] + \delta[n] + \delta[n-1])$$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{3} (e^{j\omega} + 1 + e^{-j\omega}) = \frac{1}{3} (1 + 2 \cos \omega)$$



Nonrecursive DT Filters:

- N+M+1 moving average (lowpass) filter:

$$y[n] = \frac{1}{N + M + 1} \sum_{k=-N}^M x[n - k]$$

$$\Rightarrow H(e^{jw}) = \frac{1}{N + M + 1} \sum_{k=-N}^M e^{-jwk}$$

$$\Rightarrow H(e^{jw}) = \frac{1}{N + M + 1} e^{jw\left(\frac{N-M}{2}\right)} \frac{\sin\left((M + N + 1)\frac{w}{2}\right)}{\sin\left(\frac{w}{2}\right)}$$

$$\frac{1 - e^{-ja}}{1 - e^{-jb}} = \frac{e^{-ja/2} (e^{ja/2} - e^{-ja/2})}{e^{-jb/2} (e^{jb/2} - e^{-jb/2})}$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

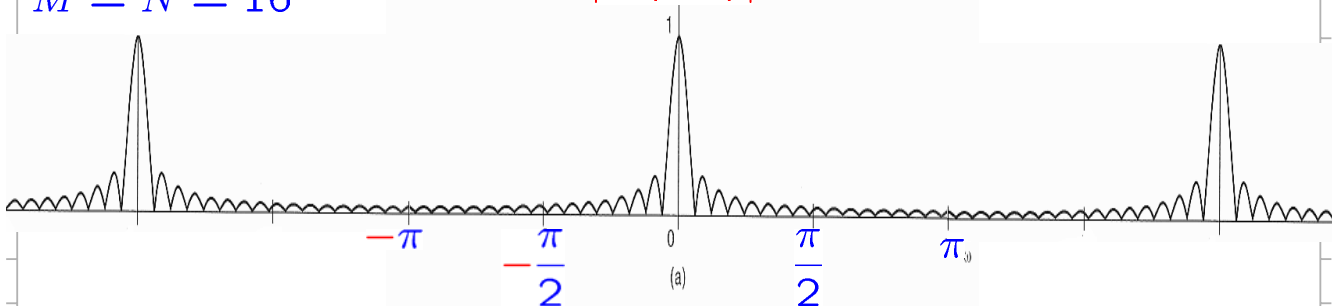
$$\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

Nonrecursive DT Filters:

- N+M+1 moving average (lowpass) filter:

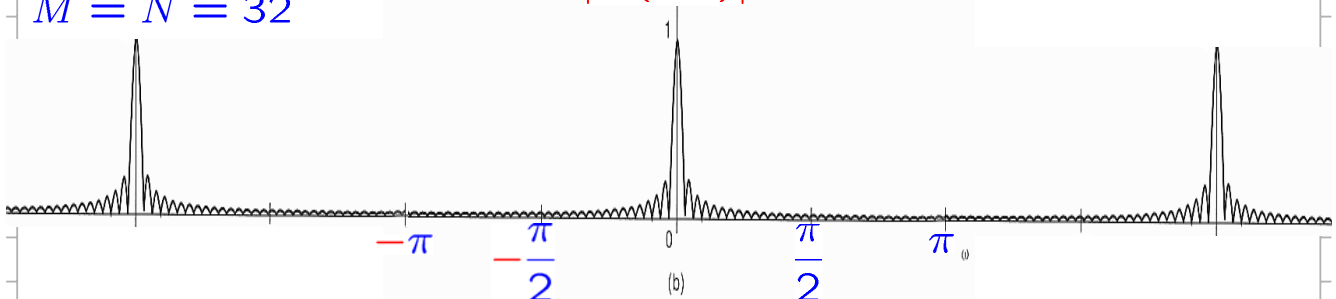
$$M = N = 16$$

$$|H(e^{jw})|$$

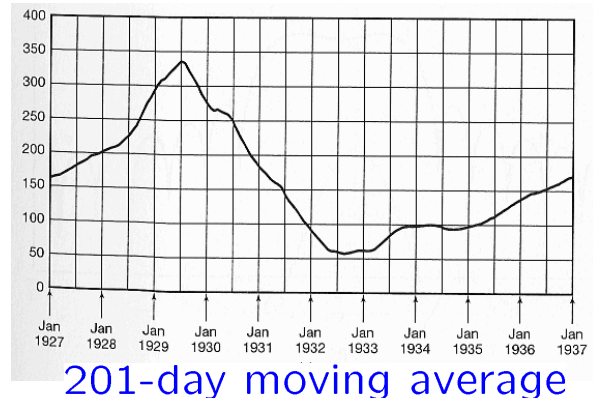
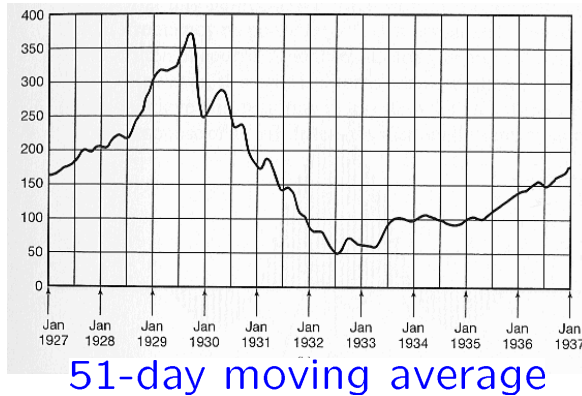
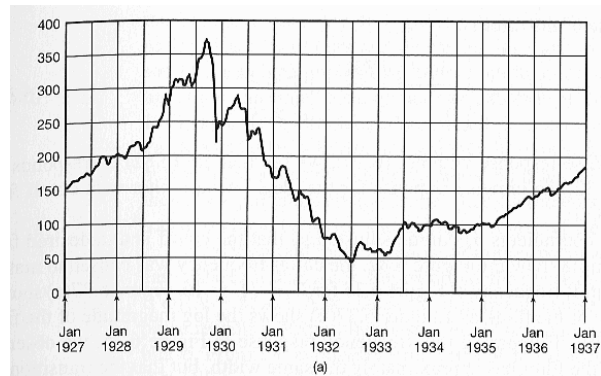


$$M = N = 32$$

$$|H(e^{jw})|$$



Lowpass Filtering on Dow Jones Weekly Stock Market Index:



Nonrecursive DT Filters:

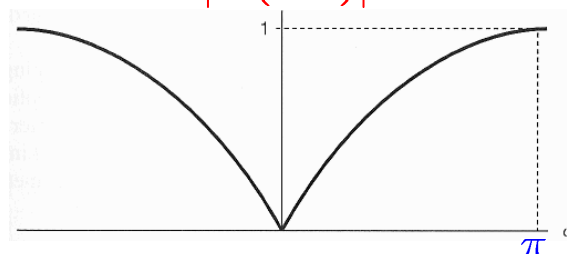
- Highpass filters:

$$y[n] = \frac{x[n] - x[n-1]}{2}$$

$$\Rightarrow h[n] = \frac{1}{2} \{ \delta[n] - \delta[n-1] \}$$

$$\begin{aligned} \Rightarrow H(e^{j\omega}) &= \frac{1}{2} [1 - e^{-j\omega}] = \frac{1}{2} e^{-j(\frac{\omega}{2})} [e^{j(\frac{\omega}{2})} - e^{-j(\frac{\omega}{2})}] \\ &= j e^{-j(\frac{\omega}{2})} \sin\left(\frac{\omega}{2}\right) \end{aligned}$$

$$|H(e^{j\omega})|$$



$$1 \pm e^{-j\theta} = e^{-j\theta/2} (e^{j\theta/2} \pm e^{-j\theta/2})$$

Correction

- On page 235, Eq. 3.139

$$1 \pm e^{-j\theta} = e^{-j\theta/2} (e^{j\theta/2} \pm e^{-j\theta/2})$$

$$\begin{aligned}\Rightarrow H(e^{jw}) &= \frac{1}{2} [1 + e^{-jw}] = \frac{1}{2} e^{-j(\frac{w}{2})} [e^{j(\frac{w}{2})} + e^{-j(\frac{w}{2})}] \\ &= e^{-j(\frac{w}{2})} \cos\left(\frac{w}{2}\right)\end{aligned}$$

- On page 249, Eq. 3.164

$$\begin{aligned}\Rightarrow H(e^{jw}) &= \frac{1}{2} [1 - e^{-jw}] = \frac{1}{2} e^{-j(\frac{w}{2})} [e^{j(\frac{w}{2})} - e^{-j(\frac{w}{2})}] \\ &= j e^{-j(\frac{w}{2})} \sin\left(\frac{w}{2}\right)\end{aligned}$$

Chapter 3: Fourier Series Representation of Periodic Signals

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- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- FS Representation of CT Periodic Signals
- Convergence of the FS
- Properties of CT FS
 - Linearity
 - Time Reversal
 - Differentiation
 - Symmetry for Real and Even Signals
 - Even-Odd Decomposition for Real Signals
 - Time Shifting
 - Time Scaling
 - Integration
 - Frequency Shifting
 - Periodic Convolution
 - Conjugate Symmetry for Real Signals
 - Symmetry for Real and Odd Signals
 - Parseval's Relation for Periodic Signals
 - Conjugation
 - Multiplication
- FS Representation of DT Periodic Signals
- Properties of DT FS
 - Multiplication
 - First Difference
 - Running Sum
- FS & LTI Systems
- Filtering
 - Frequency-shaping filters & Frequency-selective filters
- Examples of CT & DT Filters

Signals & Systems [\(Chap 1\)](#)LTI & Convolution [\(Chap 2\)](#)**Bounded/Convergent****Periodic****FS**[\(Chap 3\)](#)

– CT

– DT

Aperiodic**FT**

– CT

[\(Chap 4\)](#)

– DT

[\(Chap 5\)](#)**Unbounded/Non-convergent****LT**

– CT

[\(Chap 9\)](#)**zT**

– DT

[\(Chap 10\)](#)Time-Frequency [\(Chap 6\)](#)Communication [\(Chap 8\)](#)CT-DT [\(Chap 7\)](#)Control [\(Chap 11\)](#)