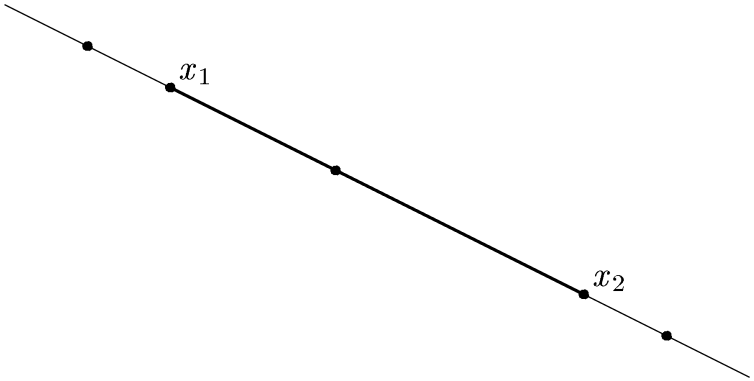
CONVEX SETS

affine and convex sets

Lines



$$y = x_2 + \theta(x_1 - x_2) \qquad \theta \in \mathbf{R}$$

$$y = \theta x_1 + (1 - \theta) x_2$$

Affine Set

A set $C \subseteq \mathbf{R}^n$ is *affine* if the line through any two distinct points in C lies in C

$x_1,\ x_2 \in C$ and $\theta \in \mathbf{R}$ $\Rightarrow$ $\theta x_1 + (1-\theta)x_2 \in C$

Example :

Solution set of linear equations.

$$C = \{x \mid Ax = b\}, \text{ where } A \in \mathbf{R}^{m\times n} \text{ and } b \in \mathbf{R}^m$$

$x_1,\ x_2 \in C$, *i.e.*, $Ax_1 = b,\ Ax_2 = b$

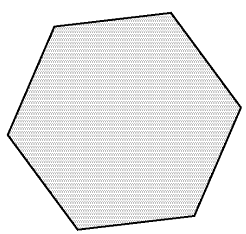
$$\begin{aligned} A(\theta x_1 + (1-\theta)x_2) &= \theta Ax_1 + (1-\theta)Ax_2 \\ &= \theta b + (1-\theta)b \\ &= b, \end{aligned}$$

Convex Set

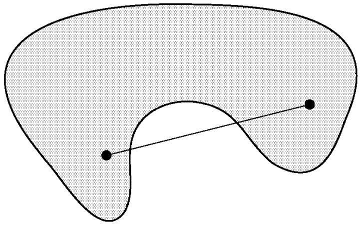
A set C is *convex* if the line segment between any two points in C lies in C

$x_1,\ x_2 \in C$ and any θ with $0 \le \theta \le 1$ $\Rightarrow$ $\theta x_1 + (1-\theta)x_2 \in C$

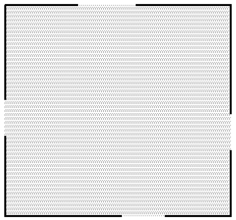
Every affine set is also convex



convex



nonconvex



nonconvex

Convex combination and convex hull

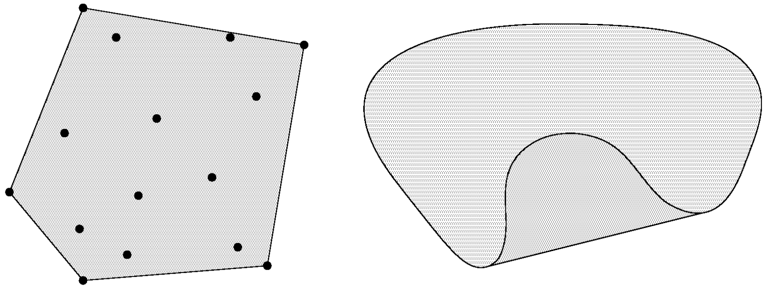
convex combination of $x_1, \ldots, x_k$: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

convex hull $\mathbf{conv}\, S$: set of all convex combinations of points in S

- ✓convex hull is always convex
- ✓It is the smallest convex set that contains



Convex cone

A set C is called a *cone*, or *nonnegative homogeneous*, if for every $x \in C$ and $\theta \geq 0$

$$\theta x \in C.$$

A set C is a *convex cone* if it is convex and a cone.

conic (nonnegative) combination of x_1 and x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0$, $\theta_2 \geq 0$

Some Important examples

Some Important examples

any single point (*i.e.*, singleton) $\{x_0\}$ affine

$\mathbf{R}^n$ affine

Any line affine — If it passes through zero, also a convex cone.

A line segment — Convex but not affine

A *ray* — Convex but not affine

Any subspace — Affine and convex cone

Hyperplanes

A *hyperplane* is a set of the form

$$\{x \mid a^T x = b\},$$

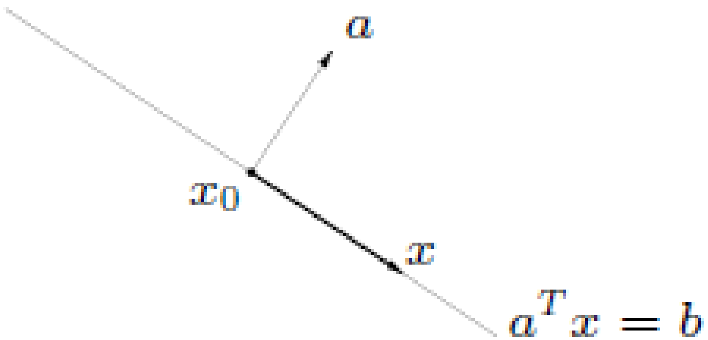
hyperplanes are affine and convex

$a \in \mathbf{R}^n$, $a \neq 0$, and $b \in \mathbf{R}$.

a is the normal vector

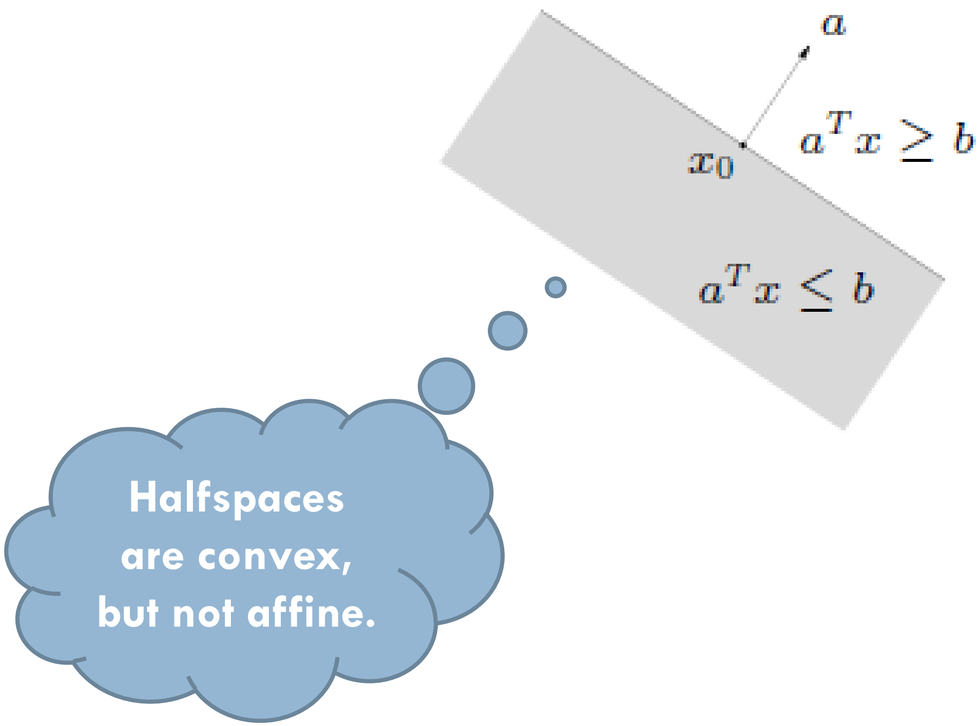
Geometrical interpretation

x_0 is any point in the hyperplane $\quad a^T x_0 = b \quad \{x \mid a^T(x - x_0) = 0\},$



halfspaces

halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



Norm balls and norm cones

norm: a function $\|\cdot\|$ that satisfies

- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t|\,\|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is partic

Norm balls are convex sets

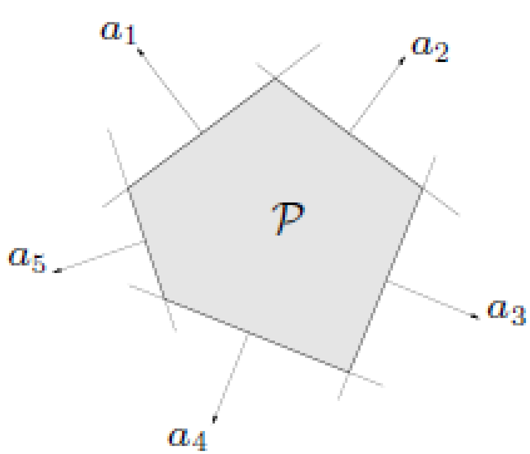
norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$

Polyhedra

solution set of finitely many linear inequalities and equalities

$$\mathcal{P} = \{x \mid a_j^T x \leq b_j,\ j = 1, \dots, m,\ c_j^T x = d_j,\ j = 1, \dots, p\}.$$

polyhedron is intersection of finite number of halfspaces and hyperplanes



Operations that preserve convexity

Operations that preserve convexity

Intersection

Convexity is preserved under intersection

Example:
a polyhedron is the intersection of halfspaces and hyperplanes (which are convex), and therefore is convex.

Affine functions

sum of a linear function and a constant

$f : \mathbf{R}^n \to \mathbf{R}^m$ is affine $\quad f(x) = Ax + b$ with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$

- the image of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

Operations that preserve convexity

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

Examples:

✓scaling and translation $S \subseteq \mathbf{R}^n$ is convex, $\alpha \in \mathbf{R}$, $a \in \mathbf{R}^n$

$$\alpha S = \{\alpha x \mid x \in S\}, \qquad S + a = \{x + a \mid x \in S\}.$$

✓Projection $S \subseteq \mathbf{R}^m \times \mathbf{R}^n$ is convex,

$$T = \{x_1 \in \mathbf{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbf{R}^n\}$$

is convex.

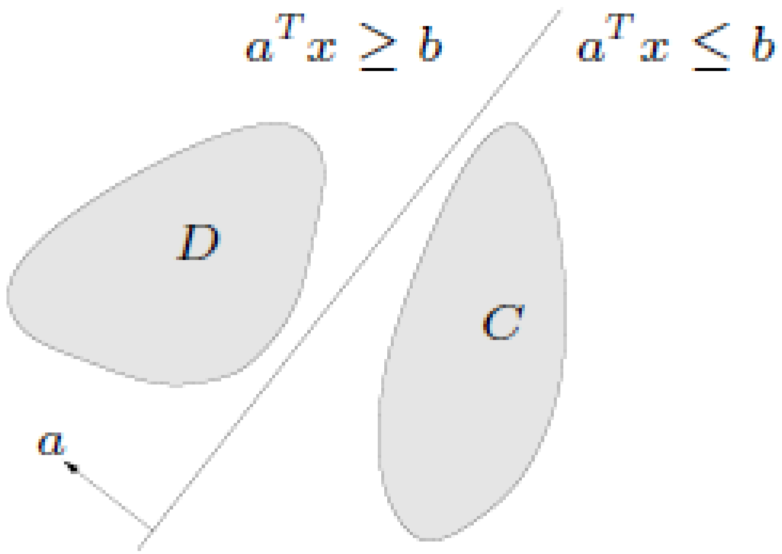
Separating hyperplane theorem

if C and D are disjoint convex sets, there exists $a \neq 0$, b such that

the hyperplane $\{x \mid a^T x = b\}$ separates C and D

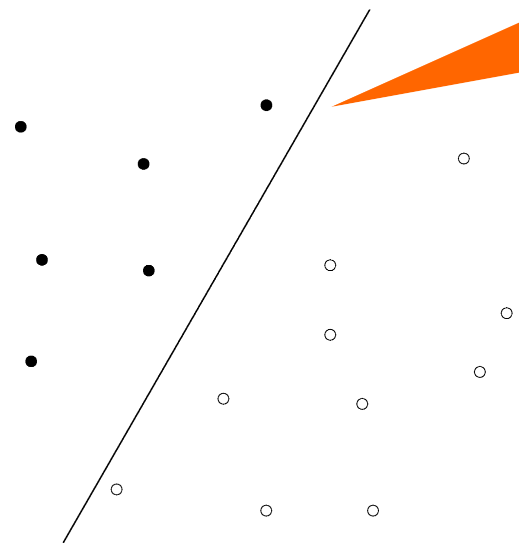
$a^T x \leq b$ for $x \in C$,

$a^T x \geq b$ for $x \in D$



Linear Discrimination

seek a hyperplane that separates
the two sets of points.



seek an affine function $f(x) = a^T x - b$ that classifies the points, *i.e.*,

$$a^T x_i - b > 0, \quad i = 1, \ldots, N, \qquad a^T y_i - b < 0, \quad i = 1, \ldots, M.$$

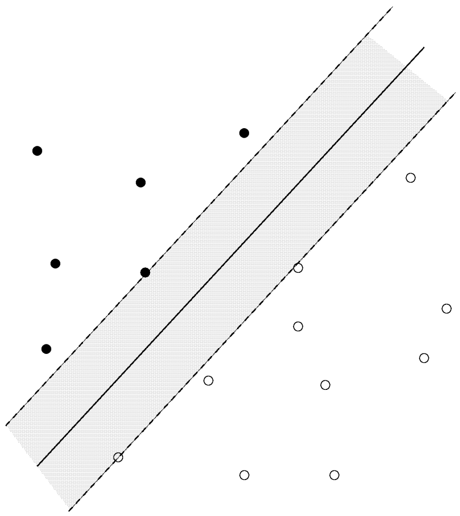
two sets of points can be linearly discriminated
if and only if their convex hulls do not intersect.

Linear Discrimination

seek the function that gives the maximum possible 'gap' between the (positive) values at the points xi and the (negative) values at the points yi.

$$\begin{array}{ll} \text{maximize} & t \\ \text{subject to} & a^T x_i - b \geq t, \quad i = 1, \ldots, N \\ & a^T y_i - b \leq -t, \quad i = 1, \ldots, M \\ & \|a\|_2 \leq 1, \end{array}$$

variables a, b, and t.

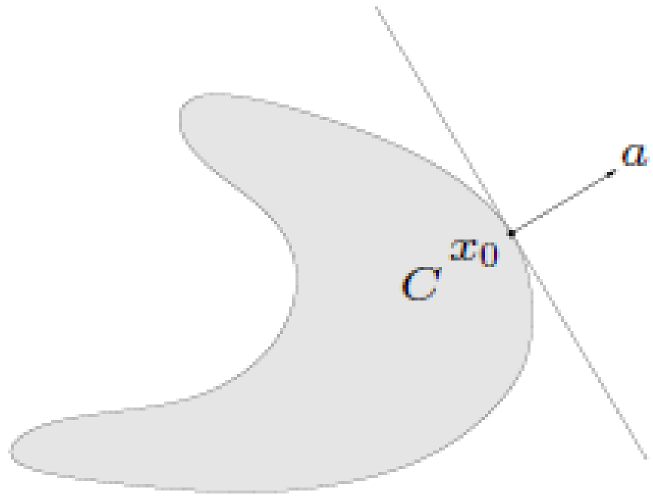


Supporting hyperplane theorem

Suppose $C \subseteq \mathbf{R}^n$, and x_0 is a point in its boundary $\mathbf{bd}\, C$

$a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$ $\{x \mid a^T x = a^T x_0\}$

Supporting hyperplane



supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C