تمرین سری شش

1)

a) $\sin\left(\frac{2\pi n}{3}\right)\cos\left(\frac{n\pi}{2}\right)$

$$x[n]y[n] \xrightarrow{FS} \sum_{r=\langle N\rangle} a_r b_{k-r}$$ ← راه اول: استفاده از رابطه‌ی ضرب دو سیگنال:

← راه دوم (و ساده‌تر در مورد این سوال): رابطه‌ی ضرب به جمع مثلثات.

$$\sin a \times \cos b = \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right)$$

$$\Rightarrow x[n] = \frac{1}{2}\left(\sin\left(\frac{7\pi}{6}n\right) + \sin\left(\frac{\pi}{6}n\right)\right)$$

$$= \frac{1}{2}\left(\frac{1}{2j}\left(e^{j\frac{7\pi}{6}n} - e^{-j\frac{7\pi}{6}n}\right) + \frac{1}{2j}\left(e^{j\frac{\pi}{6}n} - e^{-j\frac{\pi}{6}n}\right)\right)$$

$$\rightarrow N = 12 \Rightarrow \omega_0 = \frac{\pi}{6}$$

$$\Rightarrow \begin{cases} a_{-7} = \frac{1}{4j} & a_{-1} = \frac{1}{4j} \\ a_7 = -\frac{1}{4j} & a_1 = -\frac{1}{4j} \end{cases}$$

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b) $N_0 = 4 \Rightarrow a_K = \frac{1}{N_0} \sum_{K=\langle N \rangle} x[n] e^{-jK\omega_0 n}$

$$= \frac{1}{4} \sum_{N=-2}^{3} x[n] e^{-jK\omega_0 n}$$

$\downarrow$

[حالا باید این سیگنال رو روی یک دوره تناوب پیاده سازی کنیم

حالا این دوره تناوب می‌تونه از ۰ تا ۵ باشه، از ۱ تا ۶ باشه یا هر قسمت از سیگنال، اما معمولاً سعی می‌کنیم

یک بازه‌ی متقارن اطراف صفر رو در نظر بگیریم، تا بعد بتونیم یه رابطه جمع و جور تر بنویسیم. برای مثال و توضیح بیشتر،

رجوع بشه به تمرین سری سوم، سوال ۱ قسمت a. یک بار دوره رو (بازه انتگرال رو) از ۰ تا ۲π در نظر

گرفتیم، یک بار از π- تا π.]

$$= \frac{1}{4}\Big(-e^{-jK\omega_0(-2)} + 2e^{-jK\omega_0(-1)} + \underbrace{e^{-jK\omega_0(0)}}_{1}$$

$$+ 2e^{-jK\omega_0(1)} - e^{-jK\omega_0(2)} + \underbrace{0 \times e^{-jK\omega_0(3)}}_{0} \Big)$$

$\Rightarrow \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

$$\Rightarrow a_K = \frac{1}{4}\left(1 - 2\cos\left(\frac{2K\pi}{2}\right) + 4\cos\left(\frac{K\pi}{2}\right)\right)$$

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۳)

$$y[n]-\frac{1}{4}y[n-1]=x[n]$$

$$x[n]=\cos\left(\frac{\pi}{4}n\right)+3\cos\left(\frac{\pi}{2}n\right)$$

$$\downarrow \qquad\qquad \downarrow$$

$$N_0=8 \qquad N_0=4 \quad \Rightarrow N=8$$

$$\Rightarrow x[n]=\frac{1}{2}\left(e^{j\frac{\pi n}{4}}+e^{-j\frac{\pi n}{4}}\right)+\frac{1}{2}\times 3\left(e^{j\frac{\pi n}{2}}+e^{-j\frac{\pi n}{2}}\right) \qquad *$$

$$N=8\Rightarrow \omega_0=\frac{\pi}{4}\Rightarrow\begin{cases}a_1=a_{-1}=\frac{1}{2}\\ a_2=a_{-2}=\frac{3}{2}\\ \text{o.w.}\ ,\ a_k=0\end{cases}$$

یعنی می‌تونیم عبارت های معادله تفاضلی سیستم را برحسب ضرایب سری فوریه‌ی خودشون بنویسیم.

$$y[n]\xrightarrow{FS}b_k \quad\Rightarrow y[n-1]\xrightarrow{FS}b_k\times e^{-jk\omega_0}$$

$$y[n]-\frac{1}{4}y[n-1]=x[n]$$

$$FS\Big\downarrow \sum b_k e^{+jk\omega_0 n}-\frac{1}{4}\sum\left(b_k\times e^{-jk\omega_0}\right)\times e^{+jk\omega_0 n}=\sum a_k e^{jk\omega_0 n}$$

$$\Rightarrow \sum b_k\times\left(1-\frac{1}{4}e^{-jk\omega_0}\right)\times e^{+jk\omega_0 n}=\sum a_k e^{jk\omega_0 n}=x[n]$$

جایگذاری $\Rightarrow b_1=b_{-1}=\dfrac{a_1}{1-\frac{1}{4}e^{-jk\omega_0}}\ ,\ b_2=b_{-2}=\dfrac{a_2}{1-\frac{1}{4}e^{-jk\omega_0}}$

*

به جای $x[n]$

4)

$$N = 8 \Rightarrow \omega_0 = \frac{2\pi}{8} = \frac{\pi}{4} \quad (1)$$

$$x[n] = \sum a_k e^{jk\omega_0 n} \underset{\downarrow \text{و}}{\overset{(1)}{=}} \sum (-a_{k-4}) \underbrace{e^{jk\frac{\pi}{4}n}}$$

$a_k = -a_{k-4}$

سعی می‌کنیم طوری بنویسیم که در K توان هم K-4 شود

$$? \times e^{jk\frac{\pi}{4}n} = e^{j(k-4)\frac{\pi}{4}n}$$

$$\downarrow e^{jk\frac{\pi}{4}n + j\frac{\pi}{4}n \times (-4)} \Rightarrow ? = e^{-j\pi n}$$

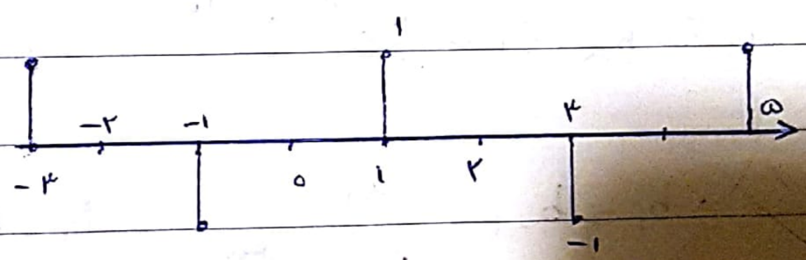
$$\Rightarrow x[n] = -e^{-j\pi n} \sum a_{k-4} e^{j(k-4)\frac{\pi}{4}n} \overset{k-4=k'}{=\!=} e^{-j\pi n} \underbrace{\sum a_{k'} e^{jk'\frac{\pi}{4}n}}_{x[n]}$$

$$\Rightarrow x[n] = e^{-j\pi n} x[n]$$

$$\Rightarrow x[n]\big(1 - \underbrace{e^{-j\pi n}}_{(-1)^n}\big) = 0$$

$$\Rightarrow \begin{cases} \text{زوج } n, \; 2x[n] = 0 \Rightarrow x[n] = 0 \text{ for } n = 2k \\ \text{فرد } n, \; x[2k+1] = (-1) \end{cases}$$

$\Rightarrow x[n]:$



تمرین های تبدیل فوریه

1)

a) $x[n] = u[n+1] - u[n-5]$

$\rightarrow \alpha^n u[n] \xrightarrow{FT} \dfrac{1}{1-\alpha e^{-jw}}$

$\rightarrow x[n-n_0] \xrightarrow{FT} e^{-jwn_0} X(e^{jw})$

$$\Rightarrow \begin{cases} u[n+1] \rightarrow \dfrac{e^{jw}}{1-e^{-jw}} \\ u[n-5] \rightarrow \dfrac{e^{-5jw}}{1-e^{-jw}} \end{cases}$$

$$\Rightarrow X(e^{jw}) = \frac{e^{jw} - e^{-5jw}}{1-e^{-jw}}$$

b) $x[n] = \sin(\frac{\pi}{2} n) + \cos(n)$

$$\rightarrow \sin[w_0 n] \xrightarrow{FT} \frac{\pi}{j} \sum_{-\infty}^{+\infty} \delta(w-w_0-2k\pi) - \delta(w+w_0-2k\pi)$$

$$\rightarrow \cos[w_0 n] \xrightarrow{FT} \pi \sum_{-\infty}^{+\infty} \delta(w-w_0-2k\pi) + \delta(w+w_0-2k\pi)$$

$$\Rightarrow X(e^{jw}) = \frac{\pi}{j} \sum \delta(w-\frac{\pi}{2}-2k\pi) - \delta(w+\frac{\pi}{2}-2k\pi)$$

$$+ \pi \sum \delta(w-1-2k\pi) + \delta(w+1-2k\pi)$$

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c) $x[n] = (n-1)\left(\frac{1}{4}\right)^{|n|} = \overbrace{n\left(\frac{1}{4}\right)^{|n|}}^{X_2(e^{jw})} - \overbrace{\left(\frac{1}{4}\right)^{|n|}}^{X_1(e^{jw})}$

$$\left(\frac{1}{4}\right)^{|n|} \xrightarrow{FT} \sum_{n=-\infty}^{+\infty} \left(\frac{1}{4}\right)^{|n|} e^{-jwn}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{4}\right)^{-n} e^{-jwn} + \sum_{n=1}^{+\infty} \left(\frac{1}{4}\right)^{n} e^{-jwn}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{4} e^{jw}\right)^{n} + \sum_{n=0}^{+\infty} \left(\frac{1}{4}\right)^{n+1} e^{-jw(n+1)}$$

$$= \quad '' \quad + \frac{e^{-jw}}{4} \sum_{n=0}^{+\infty} \left(\frac{1}{4} e^{-jw}\right)^{n}$$

$$= \frac{1}{1 - \frac{1}{4} e^{jw}} + \frac{e^{-jw}}{4}\left(\frac{1}{1 - \frac{1}{4} e^{-jw}}\right) = X_1(e^{jw})$$

$x_2[n] = n\, x_1[n]$

$$\Rightarrow X_2(e^{jw}) = j \frac{dX_1(e^{jw})}{dw}$$

$$\Rightarrow X(e^{jw}) = j \frac{dX_1(e^{jw})}{dw} - X_1(e^{jw})$$

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۲)

a) $X(e^{j\omega}) = \cos^{۲}\omega + \sin^{۲}۳\omega$

$$= \left(\frac{1}{۲}(e^{j\omega}+e^{-j\omega})\right)^{۲} + \left(\frac{1}{۲j}(e^{۳j\omega}-e^{-۳j\omega})\right)^{۲}$$

$$= \frac{1}{۴}\left(e^{۲j\omega}+e^{-۲j\omega}+۲\right) + \frac{-1}{۴}\left(e^{۶j\omega}+e^{-۶j\omega}-۲\right)$$

$$= 1 + \frac{1}{۴}e^{۲j\omega} + \frac{1}{۴}e^{-۲j\omega} - \frac{1}{۴}e^{۶j\omega} - \frac{1}{۴}e^{-۶j\omega}$$

$$\delta[n] \xrightarrow{FT} 1$$

$$x[n-n_0] \xrightarrow{FT} e^{-j\omega n_0} X(e^{j\omega})$$

$$\Rightarrow x[n] = \delta[n] + \frac{1}{۴}\delta[n+۲] + \frac{1}{۴}\delta[n-۲]$$

$$- \frac{1}{۴}\delta[n+۶] - \frac{1}{۴}\delta[n-۶]$$

b) $X(e^{j\omega}) = e^{-j\omega/۲}$ for $-\pi \leq \omega \leq \pi$

$$\Rightarrow x[n] = \frac{1}{۲\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{۲\pi}\int_{-\pi}^{\pi} e^{-j\frac{\omega}{۲}} e^{j\omega n}\, d\omega$$

$$= \frac{1}{۲\pi}\int_{-\pi}^{\pi} e^{\omega j(n-\frac{1}{۲})} = \frac{1}{۲\pi}\left(\frac{1}{j(n+\frac{1}{۲})} e^{\omega \cdot j(n-\frac{1}{۲})}\Big|_{-\pi}^{\pi}\right)$$

$$= \frac{1}{۲\pi j(n-\frac{1}{۲})} \times e^{\pi \times j(n-\frac{1}{۲})} - e^{-\pi j(n-\frac{1}{۲})} = \frac{\sin \pi(n-\frac{1}{۲})}{\pi(n-\frac{1}{۲})}$$

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$$c)\ X(e^{j\omega}) = \frac{e^{-j\omega} - 0.2}{1-0.2e^{-j\omega}} = \frac{e^{-j\omega}}{1-0.2e^{-j\omega}} - \frac{0.2}{1-0.2e^{-j\omega}}$$

$$\alpha^n u[n] \xrightarrow{FT} \frac{1}{1-\alpha e^{-j\omega}} \Rightarrow \frac{1}{1-0.2e^{-j\omega}} \xrightarrow{F^{-1}} (0.2)^n u[n]$$

$$x[n-n_0] \xrightarrow{FT} e^{-j\omega n_0} X(e^{j\omega})$$

$$\Rightarrow x[n] = (0.2)^{n-1} u[n-1] - (0.2)^{n+1} u[n]$$

3)

$$a)\ x[1-n] + x[-1-n] = x_1[n]$$

$$x[-n] \xrightarrow{FT} X(e^{-j\omega})$$

$$x[n-n_0] \xrightarrow{FT} e^{-j\omega n_0} X(e^{j\omega})$$

$$\Rightarrow X_1(e^{j\omega}) = e^{-j\omega} X(e^{-j\omega}) + e^{j\omega} X(e^{-j\omega})$$

$$b)\ (n-1)^2 x[n] = n^2 x[n] - 2n x[n] + x[n] = x_2[n]$$

$$n x[n] \xrightarrow{FT} j \frac{dX(e^{j\omega})}{d\omega}$$

$$\Rightarrow n^2 x[n] = j^2 \frac{d^2 X(e^{j\omega})}{d\omega} \Rightarrow X_2(e^{j\omega}) = j^2 \frac{d^2 X(e^{j\omega})}{d\omega} - 2j \frac{X(e^{j\omega})}{d\omega} + X(e^{j\omega})$$

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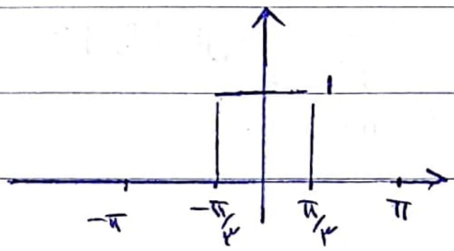
F)

$$h[n] = \frac{\sin(\frac{\pi n}{4})}{\pi n} \xrightarrow{FT} H(e^{jw}) = \begin{cases} 1 & 0 \leq |w| < \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |w| \leq \pi \end{cases}$$



a: $x[n] = \delta[n+1] + \delta[n-1]$

$$\delta[n] \xrightarrow{FT} 1$$

$$x[n-n_0] \xrightarrow{FT} e^{-jwn_0} X(e^{jw})$$

$$\Rightarrow X(e^{jw}) = e^{jw} + e^{-jw} = 2\cos w$$

$$\Rightarrow Y(e^{jw}) = X(e^{jw}) \times H(e^{jw}) = \begin{cases} 2\cos w & |w| < \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \leq |w| < \pi \end{cases}$$

$$\xrightarrow{L} y[n] = x[n] * h[n] = \frac{\sin \frac{\pi}{4} n}{\pi n} * \delta[n+1]$$

$$+ \frac{\sin \pi/_4 n}{\pi n} * \delta[n-1]$$

$$= \frac{\sin \frac{\pi}{4}(n+1)}{\pi(n+1)} + \frac{\sin \pi/_4 (n-1)}{\pi(n-1)}$$

a) $y[n] + 0.5\, y[n-1] = x[n]$

الف) $\xrightarrow{F.T} \quad Y(e^{jw}) + \frac{1}{2} e^{-jw} Y(e^{jw}) = X(e^{jw})$

$$\Rightarrow H(e^{jw}) = \frac{1}{1+\frac{1}{2}e^{-jw}}$$

ب)

a) $x[n] = (0.5)^n u[n] \rightsquigarrow \alpha^n u[n] \xrightarrow{FT} \dfrac{1}{1-\alpha e^{-jw}}$

$$\Rightarrow X(e^{jw}) = \frac{1}{1-\frac{1}{2}e^{-jw}}$$

$$\Rightarrow Y(e^{jw}) = X(e^{jw})\, H(e^{jw}) = \frac{1}{1-\frac{1}{2}e^{-jw}} \times \frac{1}{1+\frac{1}{2}e^{-jw}}$$

$$= \frac{0.5}{1-\frac{1}{2}e^{-jw}} + \frac{0.5}{1+\frac{1}{2}e^{-jw}} \xrightarrow{F^{-1}} \frac{1}{2}\left(\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{2}\right)^n u[n]\right)$$

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b) $x[n] = (-۰٫۵)^n u[n]$

$$\Rightarrow X(e^{jw}) = \frac{1}{1+\frac{1}{۲}e^{-jw}}$$

$$\Rightarrow Y(e^{jw}) = \frac{1}{(1+\frac{1}{۲}e^{-jw})^۲}$$

$$\rightsquigarrow (n+1)\alpha^n u[n] \xrightarrow{FT} \frac{1}{(1-\alpha e^{-jw})^۲}$$

$$\Rightarrow y[n] = (n+1)\left(-\frac{1}{۲}\right)^n u[n]$$

c) $X(e^{jw}) = 1 + ۲e^{-۳jw}$

$$\Rightarrow Y(e^{jw}) = \underbrace{\frac{1}{1+\frac{1}{۲}e^{-jw}}}_{\downarrow F^{-1}} + \frac{۲e^{-۳jw}}{1+\frac{1}{۲}e^{-jw}} \; \downarrow F^{-1}$$

$$\left(-\frac{1}{۲}\right)^n u[n] \qquad ۲\times\left(-\frac{1}{۲}\right)^{n-۳} u[n-۳]$$

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