CONVEX OPTIMIZATION PROBLEMS

Optimization problem in standard form

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

- $x \in \mathbf{R}^n$ is the optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$, are the inequality constraint functions
- $h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions

domain of the optimization problem

$$
\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \;\cap\; \bigcap_{i=1}^{p} \mathbf{dom}\, h_i
$$

Optimization problem in standard form

A point $x \in \mathcal{D}$ is *feasible* if it satisfies the constraints

The problem is feasible if there exists at least one feasible point, and infeasible otherwise.

The set of all feasible points is called the feasible set or the constraint set.

optimal value:

$$p^\star = \inf\{f_0(x) \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}$$

- $p^\star = \infty$ if problem is infeasible (no x satisfies the constraints)
- $p^\star = -\infty$ if problem is unbounded below

Optimal and locally optimal points

a feasible x is **optimal** if $f_0(x) = p^\star$

X_{opt} is the set of optimal points

$$X_{\text{opt}} = \{x \mid f_i(x) \le 0,\ i = 1, \dots, m,\ h_i(x) = 0,\ i = 1, \dots, p,\ f_0(x) = p^\star\}.$$

x is *locally optimal* if there is an $R > 0$ such that

$$\begin{aligned} f(x) = \inf\{f_0(z) \mid\ & f_i(z) \le 0,\ i = 1, \dots, m, \\ & h_i(z) = 0,\ i = 1, \dots, p,\ \|z - x\|_2 \le R\}, \end{aligned}$$

or

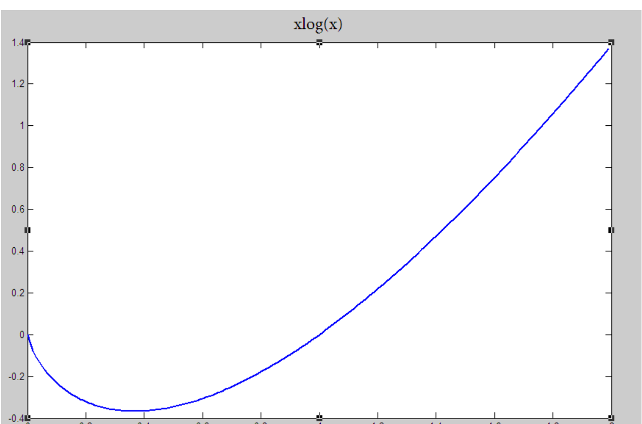
x solves the optimization problem

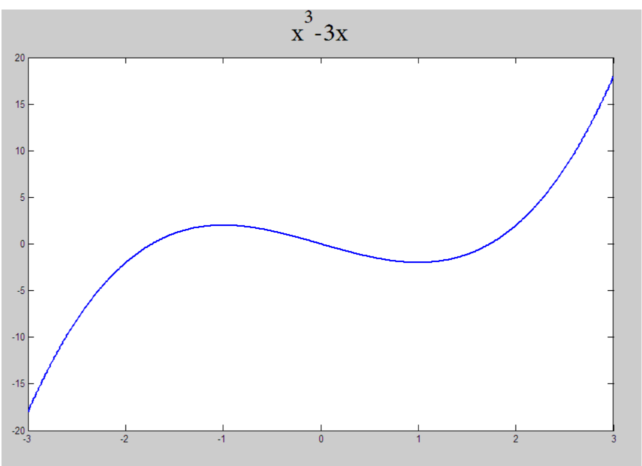
$$\begin{array}{ll} \text{minimize} & f_0(z) \\ \text{subject to} & f_i(z) \le 0, \quad i = 1, \dots, m \\ & h_i(z) = 0, \quad i = 1, \dots, p \\ & \|z - x\|_2 \le R \end{array}$$

Optimal and locally optimal points

Examples:

- $f_0(x) = 1/x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$ ➔ $p^\star = 0$, no optimal point
- $f_0(x) = -\log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$ ➔ $p^\star = -\infty$
- $f_0(x) = x \log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$ ➔ $p^\star = -1/e$, $x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x$ ➔ $p^\star = -\infty$ local optimum at $x = 1$





Implicit constraints

the standard form optimization problem has an **implicit constraint**

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \;\cap\; \bigcap_{i=1}^{p} \mathbf{dom}\, h_i,$$

- the constraints $f_i(x) \leq 0$, $h_i(x) = 0$ are the explicit constraints
- a problem is **unconstrained** if it has no explicit constraints ($m = p = 0$)

Example:

$$\text{minimize} \quad f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

Feasibility problem

$$
\begin{array}{ll}
\text{find} & x \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\
& h_i(x) = 0, \quad i = 1, \dots, p.
\end{array}
$$

a special case of the general problem

$$
\begin{array}{ll}
\text{minimize} & 0 \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\
& h_i(x) = 0, \quad i = 1, \dots, p
\end{array}
$$

- $p^\star = 0$ if constraints are feasible; any feasible x is optimal
- $p^\star = \infty$ if constraints are infeasible

Convex optimization problem

standard form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & a_i^T x = b_i, \quad i = 1, \dots, p \end{array}$$

$f_0, f_1, \dots, f_m$ are convex

equality constraints are affine

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

Concave maximization

feasible set of a convex optimization problem is convex

problem is *quasiconvex* if f_0 is quasiconvex (and $f_1, \dots, f_m$ convex)

Convex optimization problem

Example:

$$\begin{array}{ll} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1+x_2^2) \leq 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0 \end{array}$$

not convex

not affine

feasible set $\{(x_1, x_2) \mid x_1 = -x_2 \leq 0\}$ is convex

Not a convex problem

Two problems are equivalent if the solution of one is readily obtained from the solution of the other, and vice-versa

Equivalent (but not identical) convex problem:

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 + x_2 = 0 \end{array}$$

Convex optimization problem

any locally optimal point of a convex problem is (globally) optimal

proof: suppose x is locally optimal and y is optimal with $f_0(y) < f_0(x)$

x locally optimal means there is an $R > 0$ such that

$$z \text{ feasible}, \quad \|z - x\|_2 \leq R \quad \Longrightarrow \quad f_0(z) \geq f_0(x)$$

consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$

- $\|y - x\|_2 > R$, so $0 < \theta < 1/2$
- z is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_2 = R/2$

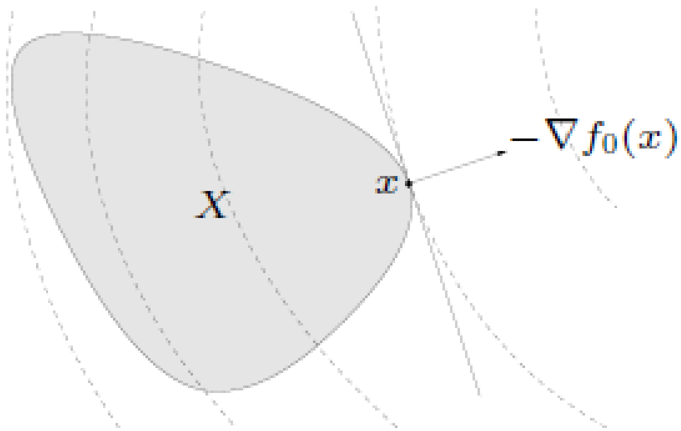
$$f_0(z) \leq \theta f_0(x) + (1 - \theta) f_0(y) < f_0(x)$$

Optimality criterion for differentiable f_0

Feasible set

x is optimal if and only if $x \in X$ and

$$\nabla f_0(x)^T (y - x) \geq 0 \text{ for all } y \in X.$$



$a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

$$\{x \mid a^T x = a^T x_0\}$$

$-\nabla f_0(x)$ defines a supporting hyperplane to the feasible set at x

Optimality criterion for differentiable f_0

- **unconstrained problem:** x is optimal if and only if

$$x \in \mathbf{dom}\, f_0, \qquad \nabla f_0(x) = 0$$

Example:

$$\boxed{\text{min} \quad f_0(x) = (1/2)x^T P x + q^T x + r,}$$

$P \in \mathbf{S}^n_+$

$$\nabla f_0(x) = Px + q = 0.$$

Optimality criterion for differentiable f_0

- **equality constrained problem**

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \quad Ax = b$$

x is optimal if and only if there exists a ν such that

$$x \in \mathbf{dom}\, f_0, \qquad Ax = b, \qquad \nabla f_0(x) + A^T \nu = 0$$

Equivalent convex problems

Some common transformations that preserve convexity:

eliminating equality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

finding a particular solution x_0 $Ax = b$, and a matrix F whose range is the nullspace of A

$$\begin{array}{ll} \text{minimize} & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m, \end{array}$$

Equivalent convex problems

introducing equality constraints

$$
\begin{array}{ll}
\text{minimize} & f_0(A_0 x + b_0) \\
\text{subject to} & f_i(A_i x + b_i) \leq 0, \quad i = 1, \ldots, m
\end{array}
$$

is equivalent to

$$
\begin{array}{ll}
\text{minimize (over } x,\ y_i\text{)} & f_0(y_0) \\
\text{subject to} & f_i(y_i) \leq 0, \quad i = 1, \ldots, m \\
 & y_i = A_i x + b_i, \quad i = 0, 1, \ldots, m
\end{array}
$$

Slack variables

Slack variables

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

idea

$f_i(x) \leq 0$ if and only if there is an $s_i \geq 0$ that satisfies $f_i(x) + s_i = 0$.

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & s_i \geq 0, \quad i = 1, \ldots, m \\ & f_i(x) + s_i = 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p, \end{array}$$

$x \in \mathbf{R}^n$ and $s \in \mathbf{R}^m$

if (x, s) is feasible for the problem $\longrightarrow$ x is feasible for the original

Equivalent convex problems

introducing slack variables for linear inequalities

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m
\end{array}
$$

is equivalent to

$$
\begin{array}{ll}
\text{minimize (over } x, s) & f_0(x) \\
\text{subject to} & a_i^T x + s_i = b_i, \quad i = 1, \ldots, m \\
& s_i \geq 0, \quad i = 1, \ldots m
\end{array}
$$

Epigraph problem form

Epigraph form

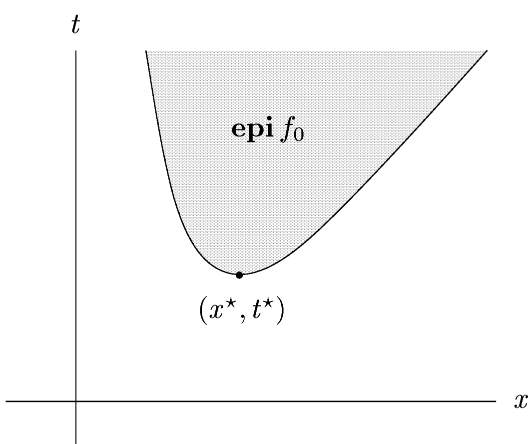
$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

The epigraph form is the problem

Equivalent convex problems

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & f_0(x) - t \le 0 \\ & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p, \end{array}$$

$x \in \mathbf{R}^n$ and $t \in \mathbf{R}$



Minimizing over some variables

$$\inf_{x,y} f(x,y) = \inf_x \tilde{f}(x) \qquad \tilde{f}(x) = \inf_y f(x,y).$$

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_i(x_1) \le 0, \quad i = 1, \dots, m_1 \\ & \tilde{f}_i(x_2) \le 0, \quad i = 1, \dots, m_2, \end{array}$$

$x = (x_1, x_2)$.

Equivalent convex problems

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x_1) \\ \text{subject to} & f_i(x_1) \le 0, \quad i = 1, \dots, m_1. \end{array}$$

$$\tilde{f}_0(x_1) = \inf\{f_0(x_1, z) \mid \tilde{f}_i(z) \le 0, \ i = 1, \dots, m_2\}.$$

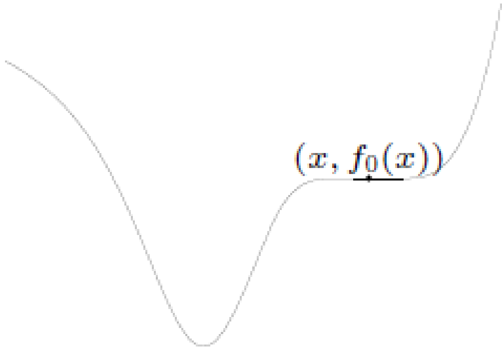
Quasiconvex optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

$f_0 : \mathbf{R}^n \to \mathbf{R}$ quasiconvex

$f_1, \ldots, f_m$ convex

can have locally optimal points that are not (globally) optimal



Quasiconvex optimization

first-order condition for quasiconvexity

x is optimal if

$$x \in X, \qquad \nabla f_0(x)^T(y - x) > 0 \text{ for all } y \in X.$$

Differences with convex case

Quasiconvex optimization

if f_0 is quasiconvex, there exists a family of functions ϕ_t such that:

- $\phi_t(x)$ is convex in x for fixed t
- t-sublevel set of f_0 is 0-sublevel set of ϕ_t, *i.e.*,

$$f_0(x) \leq t \quad \Longleftrightarrow \quad \phi_t(x) \leq 0$$

---

the feasibility problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & \phi_t(x) \leq 0 \\ & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b, \end{array}$$

- for fixed t, a convex feasibility problem in x
- if feasible, we can conclude that $t \geq p^\star$; if infeasible, $t \leq p^\star$

Quasiconvex optimization

Start with an interval [l,u] known to contain the optimal value p*.

Algorithm 4.1 *Bisection method for quasiconvex optimization.*

given $l \leq p^\star$, $u \geq p^\star$, tolerance $\epsilon > 0$.

repeat

1. $t := (l + u)/2$.
2. Solve the convex feasibility problem (4.26).
3. **if** (4.26) is feasible, $u := t$; **else** $l := t$.

until $u - l \leq \epsilon$.

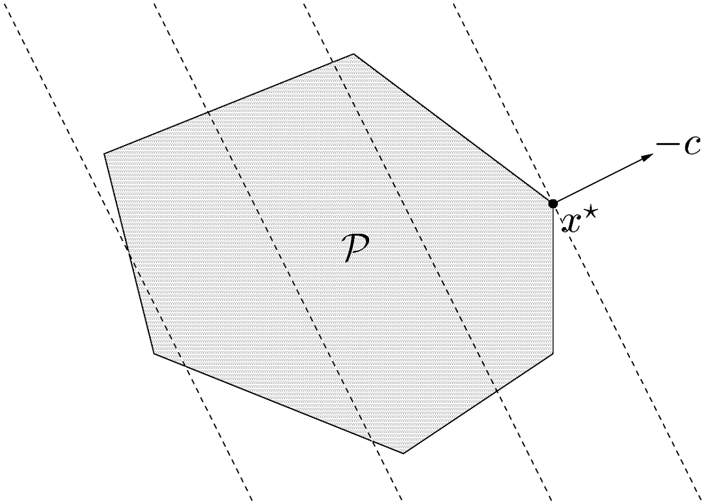
Linear program(LP)

convex problem with affine objective and constraint functions

$$
\begin{array}{ll}
\text{minimize} & c^T x + d \\
\text{subject to} & Gx \preceq h \\
& Ax = b,
\end{array}
$$

$G \in \mathbf{R}^{m\times n}$ and $A \in \mathbf{R}^{p\times n}$

feasible set is a polyhedron



Linear program(LP)

Examples: $\boxed{\text{diet problem:}}$ choose quantities $x_1, \ldots, x_n$ of n foods

- one unit of food j costs c_j, contains amount a_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i

find cheapest healthy diet

$$\boxed{\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \succeq b \\ & x \succeq 0. \end{array}}$$

Linear program(LP)

Examples:

piecewise-linear minimization

$$f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i)$$

$$\text{minimize} \quad \max_{i=1,\ldots,m}(a_i^T x + b_i)$$

Epigraph problem:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \max_{i=1,\ldots,m}(a_i^T x + b_i) \leq t, \end{array}$$

equivalent to an LP

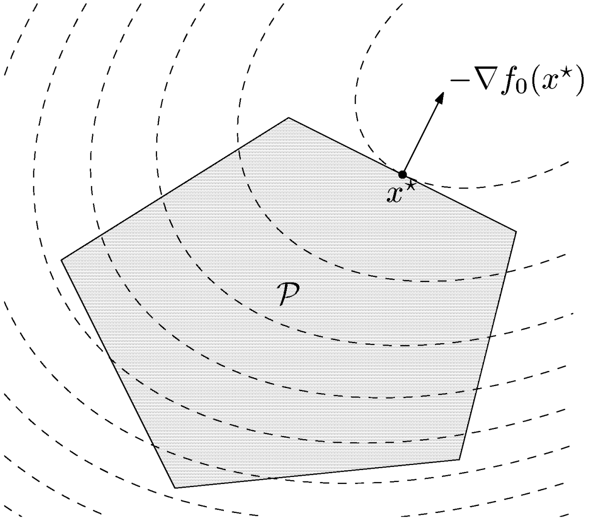
$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m. \end{array}$$

Quadratic program(QP)

objective function is convex quadratic and the constraint functions are affine.

$$\begin{array}{ll} \text{minimize} & (1/2)x^T Px + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b, \end{array}$$

$P \in \mathbf{S}^n_+$, $G \in \mathbf{R}^{m \times n}$, and $A \in \mathbf{R}^{p \times n}$



minimize a convex quadratic function over a polyhedron

QPs include LPs as a special case

Quadratic program(QP)

Examples: least-squares

$$\|Ax - b\|_2^2 = x^T A^T A x - 2b^T A x + b^T b$$

minimize $\quad \|Ax - b\|_2^2$

$$(A^T A)x = A^T b,$$

- can add linear constraints, *e.g.*, $l \preceq x \preceq u$

constrained regression or constrained least-squares,

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & l_i \le x_i \le u_i, \quad i = 1, \ldots, n, \end{array}$$

Quadratic program(QP)

Linear program with random cost

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b, \end{array}$$

$c \in \mathbf{R}^n$ is *random*.

mean value $\bar{c}$ and covariance $\mathbf{E}(c - \bar{c})(c - \bar{c})^T = \Sigma.$

Quadratic program(QP)

the cost $c^T x$ is a (scalar) random variable with mean $\mathbf{E}\, c^T x = \bar{c}^T x$ and variance

$$\mathbf{var}(c^T x) = \mathbf{E}(c^T x - \mathbf{E}\, c^T x)^2 = x^T \Sigma x.$$

$$\begin{array}{ll} \text{minimize} & \bar{c}^T x + \gamma x^T \Sigma x \\ \text{subject to} & Gx \preceq h \\ & Ax = b. \end{array}$$

risk aversion parameter; controls the trade-off between expected cost and variance

Quadratically constrained quadratic program (QCQP)

the objective and the inequality constraint functions are convex quadratic

$$\begin{array}{ll}\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\ & Ax = b,\end{array}$$

$$P_i \in \mathbf{S}_+^n, \; i = 0, 1 \ldots, m$$

QCQPs include QPs (and therefore also LPs) as a special case

Multiobjective optimization

general vector optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p. \end{array}$$

vector objective $f_0 : \mathbf{R}^n \to \mathbf{R}^q$

$$f_0(x) = (F_1(x), \dots, F_q(x))$$

A Multi-objective optimization problem is convex if

$f_1, \dots, f_m$ are convex

$h_1, \dots, h_p$ are affine

the objectives $F_1, \dots, F_q$ are convex

Multi-objective optimization (Multicriterion)

$$f_0(x) = (F_1(x), \ldots, F_q(x))$$

- q different objectives F_i

roughly speaking we want all F_i's to be small

- feasible $x^\star$ is optimal if

$$y \text{ feasible} \quad \Longrightarrow \quad f_0(x^\star) \preceq f_0(y)$$

In other words, $x^\star$ is simultaneously optimal for each of the scalar problems

$$\begin{array}{ll} \text{minimize} & F_j(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p, \end{array} \qquad \text{for } j = 1, \ldots, q$$

objectives are noncompeting

no compromises have to be made among the objectives

Optimal and Pareto optimal points

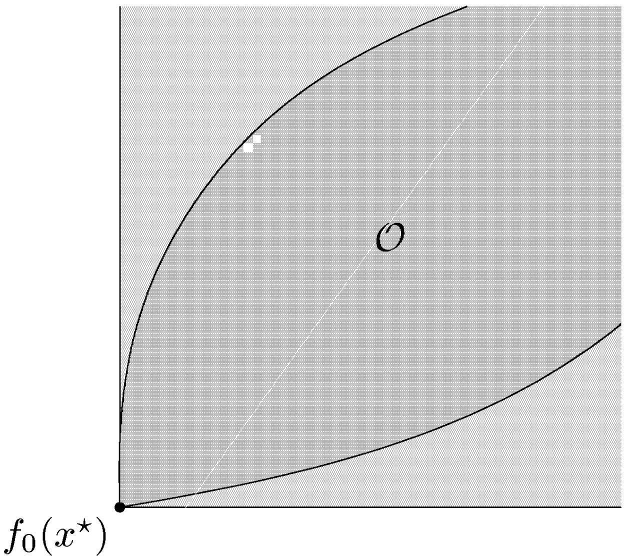
set of achievable objective values

$$\mathcal{O} = \{f_0(x) \mid x \text{ feasible}\}$$

- feasible x is **optimal** if $f_0(x)$ is a minimum value of $\mathcal{O}$

$$\mathcal{O} \subseteq f_0(x^\star) + K$$

$$K = \mathbf{R}^2_+$$



optimal point

Multi-objective optimization

the problem does not have an optimal point or optimal value.

a point is Pareto optimal if and only if it is feasible and there is no better feasible point.

- feasible x^{po} is Pareto optimal if

$$y \text{ feasible}, \quad f_0(y) \preceq f_0(x^{\mathrm{po}}) \quad \Longrightarrow \quad f_0(x^{\mathrm{po}}) = f_0(y)$$

if a feasible point is not Pareto optimal, there is at least one other feasible point that is better.

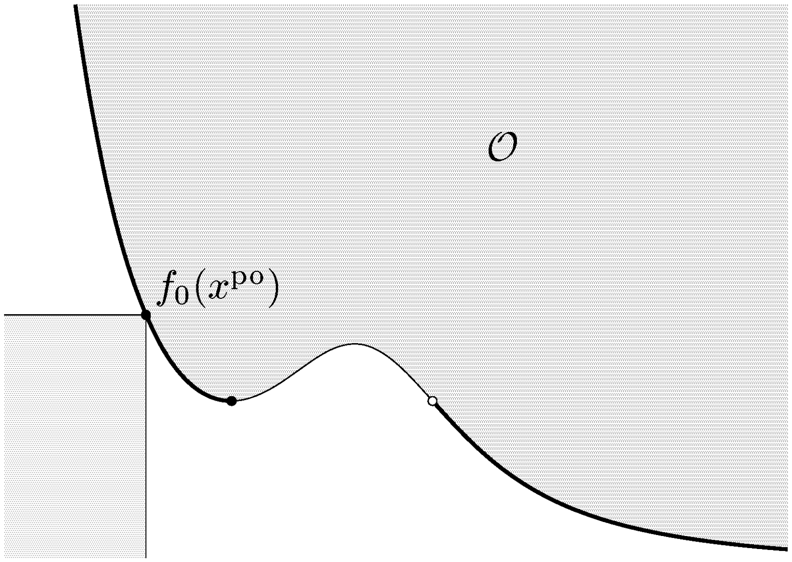
In searching for good points, then, we can clearly limit our search to Pareto optimal points.

Optimal and Pareto optimal points

- feasible x is **Pareto optimal** if $f_0(x)$ is a minimal value of $\mathcal{O}$

$$(f_0(x) - K) \cap \mathcal{O} = \{f_0(x)\}$$

Pareto optimal points



Multi-objective optimization

Example :

Regularized least-squares

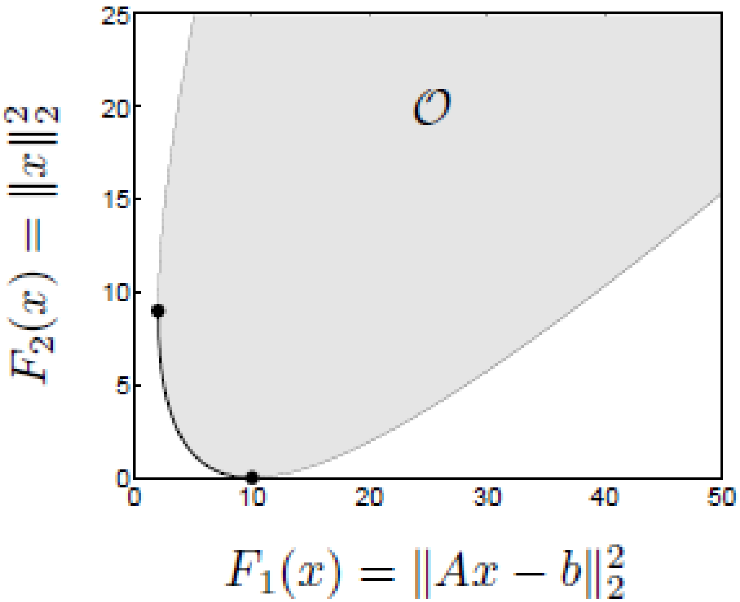
$A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^{m}$

- $F_1(x) = ||Ax - b||_2^2 = x^T A^T Ax - 2b^T Ax + b^T b$ is a measure of the misfit between Ax and b,
- $F_2(x) = ||x||_2^2 = x^T x$ is a measure of the size of x.

vector optimization problem

$$\text{minimize (w.r.t. } \mathbf{R}_+^2) \quad f_0(x) = (F_1(x), F_2(x))$$

Multi-objective optimization



example for $A \in \mathbf{R}^{100\times 10}$; heavy line is formed by Pareto optimal points

Scalarization

Scalarization is a standard technique for finding Pareto optimal (or optimal) points for a vector optimization problem

to find Pareto optimal points: choose $\lambda \succ 0$ and solve scalar problem

$$\begin{array}{ll} \text{minimize} & \lambda^T f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p, \end{array}$$

if x is optimal for scalar problem, then it is Pareto-optimal for vector optimization problem

Scalarization

$$\begin{array}{ll} \text{minimize} & \lambda^T f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p, \end{array}$$

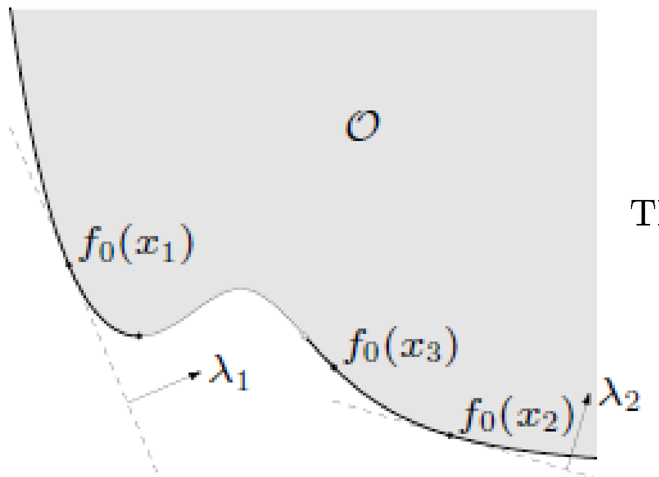
$$f_0(y) \preceq f_0(x) \Rightarrow f_0(x) - f_0(y) \succeq 0 \Rightarrow \lambda^T (f_0(x) - f_0(y)) > 0$$

Geometrical interpretation

$\lambda^T (f_0(y) - f_0(x)) \geq 0$ for all feasible y

$\{u \mid -\lambda^T (u - f_0(x)) = 0\}$ is a supporting hyperplane

Scalarization



Three Pareto optimal values $f_0(x_1)$, $f_0(x_2)$, $f_0(x_3)$ are shown.

$f_0(x_1)$ minimizes $\lambda_1^T u$ over all $u \in \mathcal{O}$

$f_0(x_2)$ minimizes $\lambda_2^T u$

The value $f_0(x_3)$ is Pareto optimal, but cannot be found by scalarization.

for convex vector optimization problems, can find (almost) all Pareto optimal points by varying $\lambda \succ_{K^*} 0$

Scalarization

to find Pareto optimal points, minimize positive weighted sum

$$\lambda^T f_0(x) = \lambda_1 F_1(x) + \cdots + \lambda_q F_q(x)$$

Example :

Regularized least-squares

$$\text{minimize (w.r.t. } \mathbf{R}^2_+) \quad f_0(x) = (F_1(x), F_2(x))$$

$$\lambda^T f_0(x) \quad = \quad \lambda_1 F_1(x) + \lambda_2 F_2(x)$$

take $\lambda = (1, \gamma)$ with $\gamma > 0$ $\longrightarrow$ minimize $\|Ax - b\|_2^2 + \gamma \|x\|_2^2$ $\longrightarrow$ for fixed γ, a LS problem