INTERIOR-POINT METHODS

Inequality constrained minimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b, \end{array}$$

- f_i convex, twice continuously differentiable

- we assume problem is strictly feasible: there exists $\tilde{x}$ with

$$\tilde{x} \in \mathbf{dom}\, f_0, \qquad f_i(\tilde{x}) < 0, \quad i = 1, \ldots, m, \qquad A\tilde{x} = b$$

Inequality constrained minimization

hence, strong duality holds and dual optimum is attained

there exist dual optimal $\lambda^\star \in \mathbf{R}^m$, $\nu^\star \in \mathbf{R}^p$, which together with $x^\star$ satisfy the KKT conditions

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$$
\begin{array}{rcll}
Ax^\star = b, \quad f_i(x^\star) & \leq & 0, & i = 1, \ldots, m \\
\lambda^\star & \succeq & 0 & \\
\nabla f_0(x^\star) + \sum_{i=1}^m \lambda_i^\star \nabla f_i(x^\star) + A^T \nu^\star & = & 0 & \\
\lambda_i^\star f_i(x^\star) & = & 0, & i = 1, \ldots, m.
\end{array}
$$

Idea : solve the problem (or the KKT conditions) by applying Newton's method to a sequence equality constrained problems, or to a sequence of modified versions of the KKT conditions.

Hierarchy of convex optimizations algorithms

Linear equality constrained quadratic problems

Newton Method

Interior-point methods

Examples

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b, \end{array}$$

- ✓LPs
- ✓QPs
- ✓QCQPs
- ✓Linear constrained entropy maximization

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} x_i \log x_i \\ \text{subject to} & Fx \preceq g \\ & Ax = b, \end{array}$$

- ✓Many other problems can be reformulated in the required form.

Logarithmic barrier

Our goal is to approximately formulate the inequality constrained problem as an equality constrained problem

making the inequality constraints implicit in the objective:

Logarithmic barrier

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making the inequality constraints implicit in the objective:

$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^m I_-(f_i(x)) \\ \text{subject to} & Ax = b, \end{array}$$

$$I_-(u) = \begin{cases} 0 & u \leq 0 \\ \infty & u > 0. \end{cases}$$

Problem?

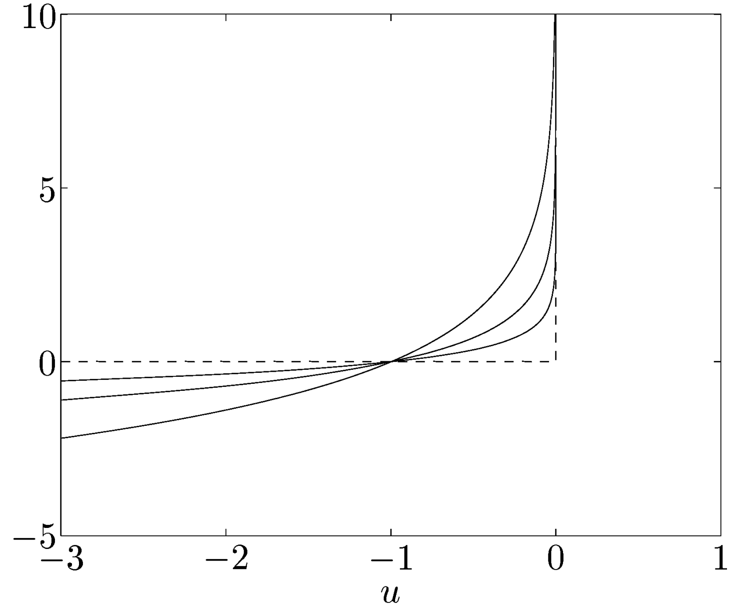
Logarithmic barrier

approximate the indicator function

$$\widehat{I}_-(u) = -(1/t)\log(-u),$$

t > 0 is a parameter that sets the accuracy of the approximation.

convex and nondecreasing
differentiable



Logarithmic barrier

$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^{m} -(1/t)\log(-f_i(x)) \\ \text{subject to} & Ax = b. \end{array}$$

Logarithmic barrier

$$
\begin{array}{ll}
\text{minimize} & f_0(x) + \sum_{i=1}^{m} -(1/t)\log(-f_i(x)) \\
\text{subject to} & Ax = b.
\end{array}
$$

log barrier

$$\phi(x) = -\sum_{i=1}^{m} \log(-f_i(x)),$$

$$\mathbf{dom}\,\phi = \{x \in \mathbf{R}^n \mid f_i(x) < 0,\ i = 1, \ldots, m\}$$

- convex (follows from composition rules)
- twice continuously differentiable

Logarithmic barrier

$$\phi(x) = -\sum_{i=1}^{m} \log(-f_i(x)),$$

$$\nabla h(x) = g'(f(x))\nabla f(x) \longrightarrow \boxed{\nabla \phi(x) = \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x),}$$

Logarithmic barrier

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$$\nabla h(x) = g'(f(x))\nabla f(x) \quad \longrightarrow \quad \boxed{\nabla \phi(x) = \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x),}$$

$$\nabla^2 h(x) = g'(f(x))\nabla^2 f(x) + g''(f(x))\nabla f(x)\nabla f(x)^T$$

$$\downarrow$$

$$\boxed{\nabla^2 \phi(x) = \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla^2 f_i(x)}$$

Central path

$$\begin{array}{ll} \text{minimize} & t f_0(x) + \phi(x) \\ \text{subject to} & Ax = b, \end{array}$$

- for $t > 0$, define $x^\star(t)$ as the solution
- central path is $\{x^\star(t) \mid t > 0\}$

Central points

Central path

$$
\begin{array}{ll}
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\end{array}
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Central points

Characteristics of central points

$\boxed{Ax^\star(t) = b,}$ $\boxed{f_i(x^\star(t)) < 0}$

there exists a $\hat{\nu} \in \mathbf{R}^p$ such that

$$
\begin{aligned}
\boxed{0 = t\nabla f_0(x^\star(t)) + \nabla\phi(x^\star(t)) + A^T\hat{\nu}} \\
= t\nabla f_0(x^\star(t)) + \sum_{i=1}^{m} \frac{1}{-f_i(x^\star(t))}\nabla f_i(x^\star(t)) + A^T\hat{\nu}
\end{aligned}
$$

Example

Inequality form linear programming

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b,\end{array}$$

$$\phi(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \qquad \mathbf{dom}\,\phi = \{x \mid Ax \prec b\},$$

$a_1^T, \ldots, a_m^T$ are the rows of A

$$\nabla \phi(x) = \sum_{i=1}^{m} \frac{1}{b_i - a_i^T x} a_i, \qquad \nabla^2 \phi(x) = \sum_{i=1}^{m} \frac{1}{(b_i - a_i^T x)^2} a_i a_i^T,$$

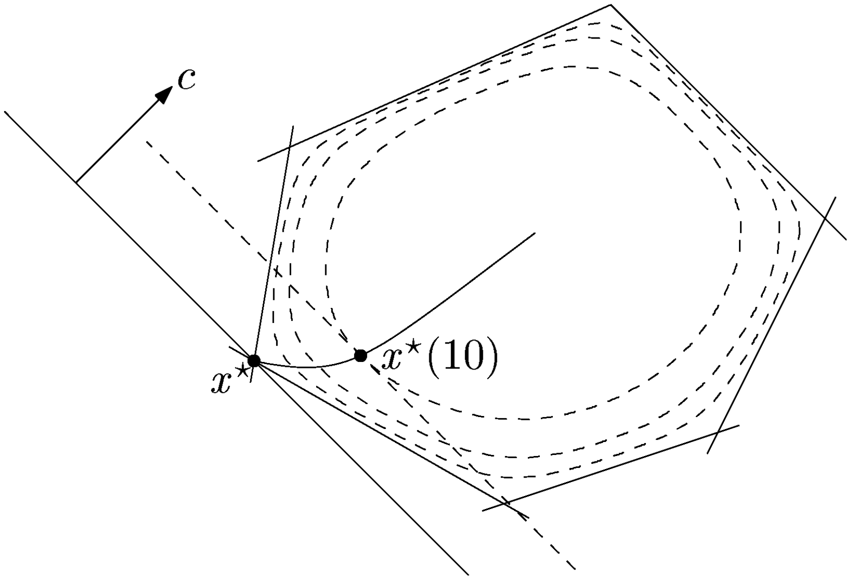
$$\nabla \phi(x) = A^T d, \qquad \nabla^2 \phi(x) = A^T \mathbf{diag}(d)^2 A,$$

$d \in \mathbf{R}^m$ are given by $d_i = 1/(b_i - a_i^T x)$

$$tc + \sum_{i=1}^{m} \frac{1}{b_i - a_i^T x} a_i = tc + A^T d = 0.$$

Example

$$tc + \sum_{i=1}^{m} \frac{1}{b_i - a_i^T x} a_i = tc + A^T d = 0.$$



Dual points from central path

Every central point yields a dual feasible point, and hence a lower bound on the optimal value p*

$$\lambda_i^\star(t) = -\frac{1}{t f_i(x^\star(t))}, \quad i = 1, \ldots, m, \qquad \nu^\star(t) = \hat{\nu}/t.$$

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$$f_i(x^\star(t)) < 0 \longrightarrow \boxed{\lambda^\star(t) \succ 0}$$

$$t\nabla f_0(x^\star(t)) + \sum_{i=1}^{m} \frac{1}{-f_i(x^\star(t))} \nabla f_i(x^\star(t)) + A^T\hat{\nu}$$

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$$f_i(x^\star(t)) \ < \ 0 \quad \longrightarrow \quad \boxed{\lambda^\star(t) \ \succ \ 0}$$

$$t\nabla f_0(x^\star(t)) + \sum_{i=1}^{m} \frac{1}{-f_i(x^\star(t))} \nabla f_i(x^\star(t)) + A^T \hat{\nu}$$

$$\nabla f_0(x^\star(t)) + \sum_{i=1}^{m} \lambda_i^\star(t) \nabla f_i(x^\star(t)) + A^T \nu^\star(t) = 0,$$

Dual points from central path

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \nu^T (Ax - b),$$

$x^\star(t)$ minimizes the Lagrangian

Dual points from central path

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$\lambda^\star(t)$, $\nu^\star(t)$ is a dual feasible pair

$$g(\lambda^\star(t), \nu^\star(t)) \quad = \quad f_0(x^\star(t)) + \sum_{i=1}^{m} \lambda_i^\star(t) f_i(x^\star(t)) + \nu^\star(t)^T (Ax^\star(t) - b)$$

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$$\begin{aligned} g(\lambda^\star(t), \nu^\star(t)) &= f_0(x^\star(t)) + \sum_{i=1}^{m} \lambda_i^\star(t) f_i(x^\star(t)) + \nu^\star(t)^T(Ax^\star(t) - b) \\ &= f_0(x^\star(t)) - m/t. \end{aligned}$$

$$p^\star \geq g(\lambda^\star(t), \nu^\star(t)) = f_0(x^\star(t)) - m/t \longrightarrow \boxed{f_0(x^\star(t)) - p^\star \leq m/t}$$

Interpretation via KKT conditions

Central path conditions

A point x is equal to $x^\star(t)$ if and only if there exists λ, ν such that

$$Ax = b, \quad f_i(x) \leq 0, \quad i = 1, \ldots, m$$

$$\lambda \succ 0$$

$$\nabla f_0(x) + \textstyle\sum_{i=1}^m \lambda_i \nabla f_i(x) + A^T \nu = 0$$

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$$\lambda \succ 0$$

$$\nabla f_0(x) + \sum_{i=1}^m \lambda_i \nabla f_i(x) + A^T \nu = 0$$

$$\boxed{-\lambda_i f_i(x) = 1/t, \quad i = 1, \ldots, m.}$$

Barrier method

accuracy ϵ

$$\boxed{t = m/\epsilon}$$

$$\begin{array}{ll} \text{minimize} & (m/\epsilon) f_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

Barrier method

Algorithm 11.1 *Barrier method.*

given strictly feasible x, $t := t^{(0)} > 0$, $\mu > 1$, tolerance $\epsilon > 0$.

repeat

1. *Centering step.*
   Compute $x^\star(t)$ by minimizing $tf_0 + \phi$, subject to $Ax = b$, starting at x.
2. *Update.* $x := x^\star(t)$.
3. *Stopping criterion.* **quit** if $m/t < \epsilon$.
4. *Increase* t. $t := \mu t$.

Barrier method

Algorithm 11.1 *Barrier method.*

outer iteration

given strictly fea… $\mu > 1$, tolerance $\epsilon > 0$.

inner iteration

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3. *Stopping criterion.* **quit** if $m/t < \epsilon$.
4. *Increase* t. $t := \mu t$.

- terminates with $f_0(x) - p^\star \le \epsilon$ (stopping criterion follows from $f_0(x^\star(t)) - p^\star \le m/t$)
- centering usually done using Newton's method, starting at current x
- choice of μ involves a trade-off: large μ means fewer outer iterations, more inner (Newton) iterations; typical values: $\mu = 10$–20

Barrier method

Example

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

$A \in \mathbf{R}^{100\times 50}$ $\quad p^\star = 1$

$x^{(0)}$ is on the central path, with a duality gap of 100

terminated when the duality gap is less than 10^-6 $\quad \lambda(x)^2/2 \le 10^{-5}$

Barrier method

Example

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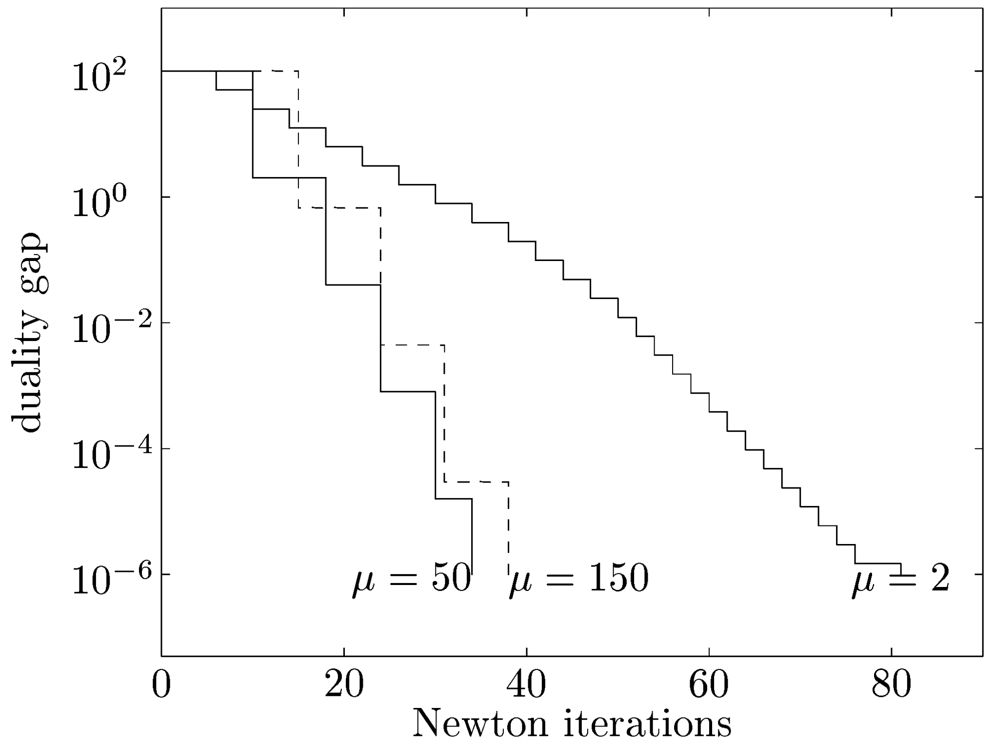
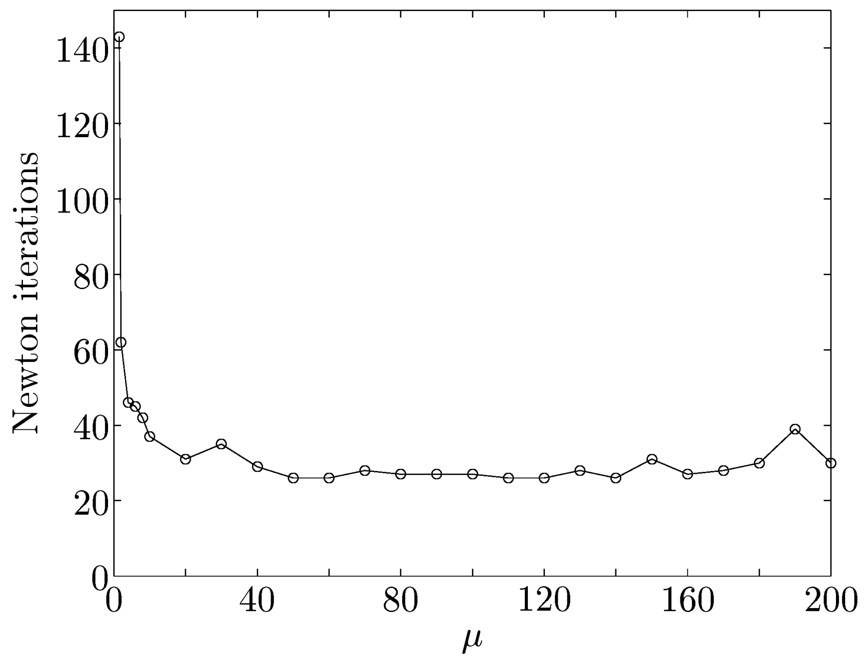


Barrier method



10^{-3}

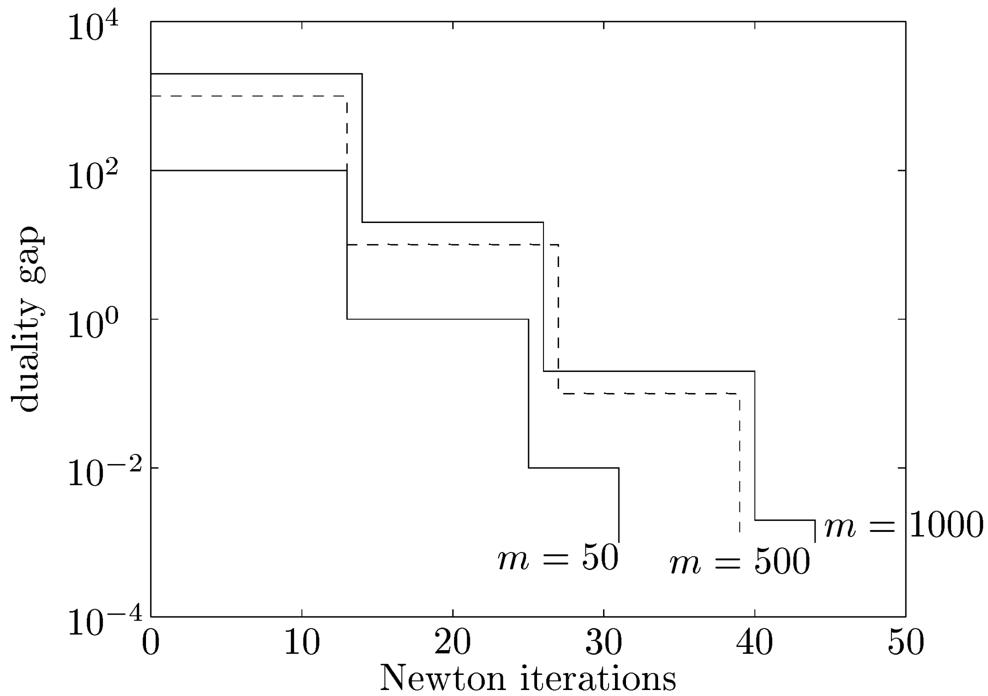
Example

Goal : examine the performance of the barrier method as a function of the problem dimensions.

$A \in \mathbf{R}^{m \times 2m}$

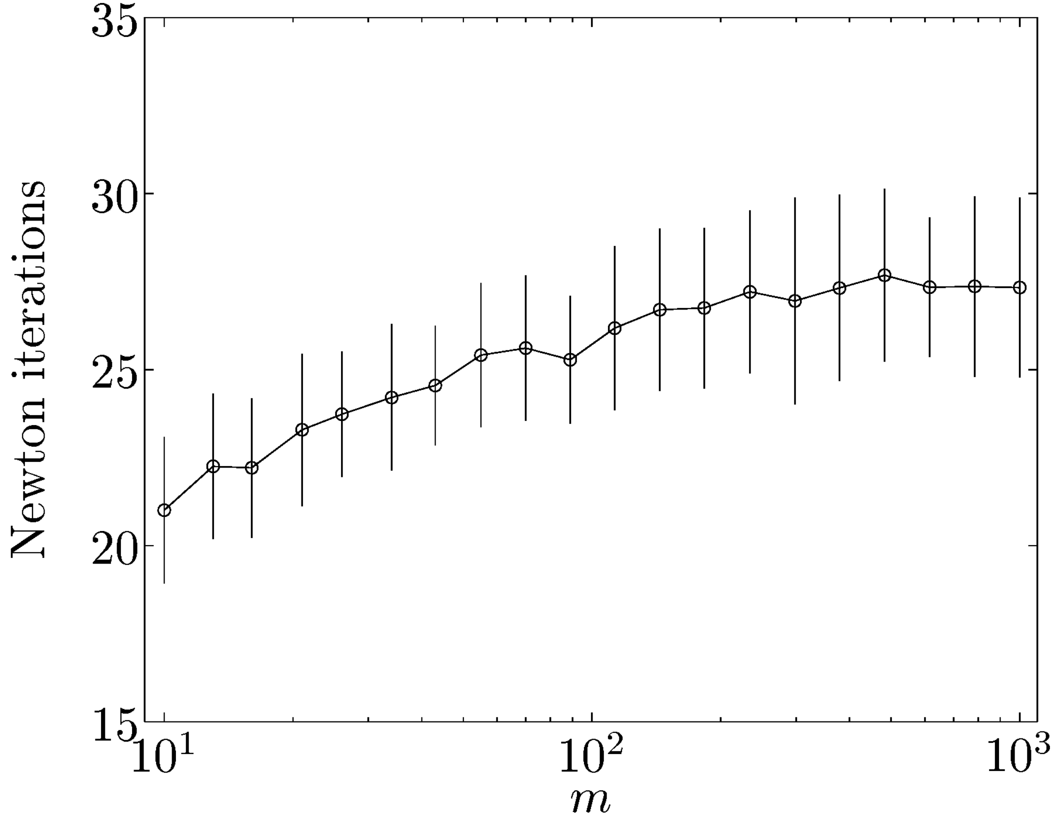
$\mu = 100$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \succeq 0 \end{array}$$



Example

100 problem instances for each of 20 values of m, ranging from m = 10 to m = 1000.



Feasibility and phase I methods

✓barrier method requires a strictly feasible starting point $x^{(0)}$

✓a preliminary stage, called phase I, in which a strictly feasible point is computed

✓phase II stage.

$$f_i(x) \leq 0, \quad i = 1, \ldots, m, \qquad Ax = b,$$

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✓phase II stage.

$$f_i(x) \le 0, \quad i = 1, \dots, m, \qquad Ax = b, \qquad Ax^{(0)} = b.$$

$$\begin{array}{ll} \text{minimize} & s \\ \text{subject to} & f_i(x) \le s, \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

$x \in \mathbf{R}^n$, $s \in \mathbf{R}$

Feasibility and phase I methods

- ✓ barrier method requires a strictly feasible starting point $x^{(0)}$
- ✓ a preliminary stage, called phase I, in which a strictly feasible point is computed
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$x \in \mathbf{R}^n$, $s \in \mathbf{R}$

strictly feasible

apply the barrier method to solve the problem

Feasibility and phase I methods

$$\begin{array}{ll} \text{minimize (over } x,\ s) & s \\ \text{subject to} & f_i(x) \leq s, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- if x, s feasible, with $s < 0$, then x is strictly feasible

Feasibility and phase I methods

$$\begin{array}{lll} \text{minimize (over } x, s) & s & \\ \text{subject to} & f_i(x) \le s, & i = 1, \dots, m \\ & Ax = b & \end{array}$$

- if x, s feasible, with $s < 0$, then x is strictly feasible
- if optimal value $\bar{p}^\star$ is positive, then problem is infeasible
- if $\bar{p}^\star = 0$ and attained, then problem is feasible (but not strictly)

Primal-dual interior-point methods

modified KKT conditions

$$r_t(x, \lambda, \nu) = 0$$

$$r_t(x, \lambda, \nu) = \begin{bmatrix} \nabla f_0(x) + Df(x)^T \lambda + A^T \nu \\ -\mathbf{diag}(\lambda) f(x) - (1/t)\mathbf{1} \\ Ax - b \end{bmatrix}$$

$t > 0$

$$f : \mathbf{R}^n \to \mathbf{R}^m$$

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}, \qquad Df(x) = \begin{bmatrix} \nabla f_1(x)^T \\ \vdots \\ \nabla f_m(x)^T \end{bmatrix}$$

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If x, λ, ν satisfy $r_t(x, \lambda, \nu) = 0$ (and $f_i(x) < 0$), then $x = x^\star(t)$, $\lambda = \lambda^\star(t)$, and $\nu = \nu^\star(t)$.

Primal-dual interior-point methods

$$r_{\text{dual}} = \nabla f_0(x) + Df(x)^T \lambda + A^T \nu,$$

$$r_{\text{pri}} = Ax - b,$$

$$r_{\text{cent}} = -\,\mathbf{diag}(\lambda) f(x) - (1/t)\mathbf{1}$$

$$r_t(x, \lambda, \nu) = 0$$

$$y = (x, \lambda, \nu), \qquad \Delta y = (\Delta x, \Delta \lambda, \Delta \nu),$$

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$$r_t(x, \lambda, \nu) = 0$$

$$y = (x, \lambda, \nu), \qquad \Delta y = (\Delta x, \Delta \lambda, \Delta \nu),$$

$$r_t(y + \Delta y) \approx r_t(y) + Dr_t(y)\Delta y = 0,$$

$$\Delta y = -Dr_t(y)^{-1} r_t(y)$$

Primal-dual interior-point methods

$$\begin{bmatrix} \nabla^2 f_0(x) + \sum_{i=1}^m \lambda_i \nabla^2 f_i(x) & Df(x)^T & A^T \\ -\,\mathbf{diag}(\lambda) Df(x) & -\,\mathbf{diag}(f(x)) & 0 \\ A & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta \nu \end{bmatrix} = - \begin{bmatrix} r_{\text{dual}} \\ r_{\text{cent}} \\ r_{\text{pri}} \end{bmatrix}$$

primal-dual search direction

$$\Delta y_{\text{pd}} = (\Delta x_{\text{pd}}, \Delta \lambda_{\text{pd}}, \Delta \nu_{\text{pd}})$$

Primal-dual interior-point methods

$$\begin{bmatrix} \nabla^2 f_0(x) + \sum_{i=1}^m \lambda_i \nabla^2 f_i(x) & Df(x)^T & A^T \\ -\,\mathbf{diag}(\lambda) Df(x) & -\,\mathbf{diag}(f(x)) & 0 \\ A & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta \nu \end{bmatrix} = - \begin{bmatrix} r_{\mathrm{dual}} \\ r_{\mathrm{cent}} \\ r_{\mathrm{pri}} \end{bmatrix}$$

primal-dual search direction

$$\Delta y_{\mathrm{pd}} = (\Delta x_{\mathrm{pd}}, \Delta \lambda_{\mathrm{pd}}, \Delta \nu_{\mathrm{pd}})$$

Comparison with barrier method search directions

$$\Delta \lambda_{\mathrm{pd}} = -\,\mathbf{diag}(f(x))^{-1}\,\mathbf{diag}(\lambda) Df(x) \Delta x_{\mathrm{pd}} + \mathbf{diag}(f(x))^{-1} r_{\mathrm{cent}},$$

$$\begin{bmatrix} H_{\mathrm{pd}} & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\mathrm{pd}} \\ \Delta \nu_{\mathrm{pd}} \end{bmatrix} = - \begin{bmatrix} r_{\mathrm{dual}} + Df(x)^T\,\mathbf{diag}(f(x))^{-1} r_{\mathrm{cent}} \\ r_{\mathrm{pri}} \end{bmatrix}$$

$$H_{\mathrm{pd}} = \nabla^2 f_0(x) + \sum_{i=1}^m \lambda_i \nabla^2 f_i(x) + \sum_{i=1}^m \frac{\lambda_i}{-f_i(x)} \nabla f_i(x) \nabla f_i(x)^T.$$

Comparison with barrier method search directions

$$\begin{bmatrix} t\nabla^2 f_0(x) + \nabla^2\phi(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\rm nt} \\ \nu_{\rm nt} \end{bmatrix} = - \begin{bmatrix} t\nabla f_0(x) + \nabla\phi(x) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} H_{\rm bar} & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\rm bar} \\ \nu_{\rm bar} \end{bmatrix} = - \begin{bmatrix} t\nabla f_0(x) + \nabla\phi(x) \\ r_{\rm pri} \end{bmatrix}$$

$$= - \begin{bmatrix} t\nabla f_0(x) + \sum_{i=1}^m \frac{1}{-f_i(x)}\nabla f_i(x) \\ r_{\rm pri} \end{bmatrix},$$

$$H_{\rm bar} = t\nabla^2 f_0(x) + \sum_{i=1}^m \frac{1}{-f_i(x)}\nabla^2 f_i(x) + \sum_{i=1}^m \frac{1}{f_i(x)^2}\nabla f_i(x)\nabla f_i(x)^T$$

Comparison with barrier method search directions

$$\begin{bmatrix} t\nabla^2 f_0(x) + \nabla^2\phi(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\rm nt} \\ \nu_{\rm nt} \end{bmatrix} = - \begin{bmatrix} t\nabla f_0(x) + \nabla\phi(x) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} H_{\rm bar} & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\rm bar} \\ \nu_{\rm bar} \end{bmatrix} = - \begin{bmatrix} t\nabla f_0(x) + \nabla\phi(x) \\ r_{\rm pri} \end{bmatrix}$$

$$= - \begin{bmatrix} t\nabla f_0(x) + \sum_{i=1}^m \frac{1}{-f_i(x)} \nabla f_i(x) \\ r_{\rm pri} \end{bmatrix},$$

$$H_{\rm bar} = t\nabla^2 f_0(x) + \sum_{i=1}^m \frac{1}{-f_i(x)} \nabla^2 f_i(x) + \sum_{i=1}^m \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T$$

$$H_{\rm pd} = \nabla^2 f_0(x) + \sum_{i=1}^m \lambda_i \nabla^2 f_i(x) + \sum_{i=1}^m \frac{\lambda_i}{-f_i(x)} \nabla f_i(x) \nabla f_i(x)^T,$$

$$-f_i(x)\lambda_i = 1/t,$$

Primal-dual interior-point

Algorithm 11.2 *Primal-dual interior-point method.*

given x that satisfies $f_1(x) < 0, \ldots, f_m(x) < 0$, $\lambda \succ 0$, $\mu > 1$, $\epsilon_{\rm feas} > 0$, $\epsilon > 0$.

repeat

1. *Determine* t. Set $t := \mu m / \hat{\eta}$.
2. Compute primal-dual search direction $\Delta y_{\rm pd}$.
3. *Line search and update.*
   Determine step length $s > 0$ and set $y := y + s \Delta y_{\rm pd}$.

until $\|r_{\rm pri}\|_2 \leq \epsilon_{\rm feas}$, $\|r_{\rm dual}\|_2 \leq \epsilon_{\rm feas}$, and $\hat{\eta} \leq \epsilon$.

$$\hat{\eta}(x, \lambda) = -f(x)^T \lambda.$$

Backtracking

$$x^+ = x + s\Delta x_{\rm pd}, \qquad \lambda^+ = \lambda + s\Delta\lambda_{\rm pd}, \qquad \nu^+ = \nu + s\Delta\nu_{\rm pd}$$

Example

Example

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

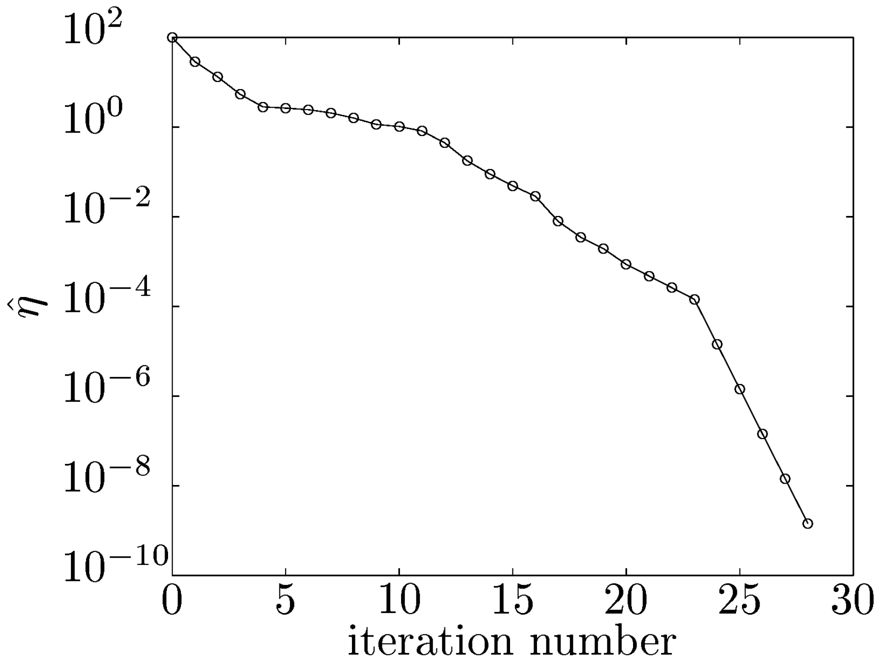
$A \in \mathbf{R}^{100\times 50}$ $\qquad p^\star = 1$

$$\lambda_i^{(0)} = -1/f_i(x^{(0)})$$

$\hat{\eta} = 100$

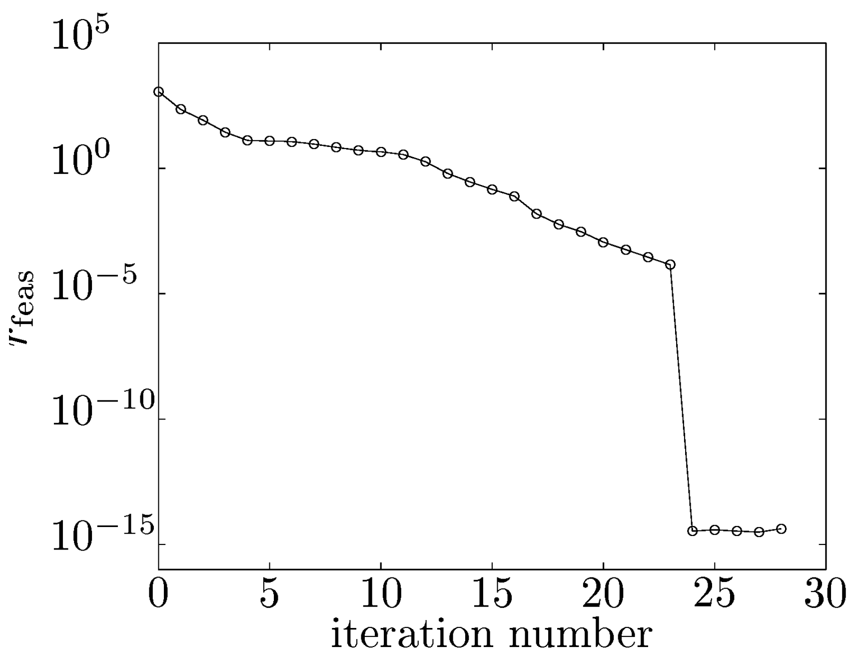
$\mu = 10,$

$\epsilon = 10^{-8}$



Example

$$r_{\text{feas}} = \left(\|r_{\text{pri}}\|_2^2 + \|r_{\text{dual}}\|_2^2\right)^{1/2}$$



example

$A \in \mathbf{R}^{m \times 2m}$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \succeq 0 \end{array}$$