OPTIMIZATION

Mathematical optimization Problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \ldots, m \end{array}$$

limits or bounds

- $x = (x_1, \ldots, x_n)$: optimization variables
- $f_0 : \mathbf{R}^n \to \mathbf{R}$: objective function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$: constraint functions

optimal solution $x^\star$ has smallest value of f_0 among all vectors that satisfy the constraints

for any z with $f_1(z) \leq b_1, \ldots, f_m(z) \leq b_m$, we have

$$f_0(z) \geq f_0(x^\star).$$

optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \ldots, m \end{array}$$

✓making the best possible choice of a vector in $\mathbf{R}^n$ from a set of candidate choices.

✓The variable x represents the choice made

✓the constraints represent requirements or specifications that limit the possible choices

✓the objective value $f_o(x)$ represents the cost of choosing x

solution of the optimization problem:

a choice that has minimum cost, among all choices that meet the firm requirements

Applications: portfolio optimization

the best way to invest some capital in a set of n assets

variable x_i $\quad x \in \mathbf{R}^n$

Constraints: a limit on the budget, investments nonnegative, and a minimum acceptable value

objective or cost function: a measure of the overall risk

Applications : Data Fitting

find a model, from a family of potential models, best fits some observed data

- variables: model parameters
- constraints: prior information, parameter limits
- objective: measure of misfit or prediction error

An amazing variety of practical problems involving decision making can be cast in the form of a mathematical optimization problem

civil, chemical, mechanical, computer and aerospace engineering in network design and operation, finance, …

Solving optimization problems

general optimization problem

- very difficult to solve
- methods involve some compromise, *e.g.*, very long computation time, or not always finding the solution

exceptions: certain problem classes can be solved efficiently and reliably

- least-squares problems
- linear programming problems
- convex optimization problems

Least-squares problems

$$\boxed{\text{minimize} \quad f_0(x) = \|Ax - b\|_2^2 = \textstyle\sum_{i=1}^k (a_i^T x - b_i)^2.}$$

$x \in \mathbf{R}^n$

$A \in \mathbf{R}^{k \times n}$

a_i^T are the rows of A

$b \in \mathcal{R}(A)$

$b \notin \mathcal{R}(A)$

Approximation interpretation

$a_1, \ldots, a_n \in \mathbf{R}^m$ are the columns of A

regressors

By expressing Ax as

$$Ax = x_1 a_1 + \cdots + x_n a_n,$$

regression of b

The approximation problem is also called the regression problem.

Least-squares problems

Estimation interpretation

Linear measurement model

unknown, but presumed to be small

$$y = Ax + v$$

Smaller values of v are more plausible than larger values

v has the value $y - A\hat{x}$

$$\hat{x} = \text{argmin}_z ||Az - y||.$$

Least-squares problems

Solving LS Problem

$$(A^T A)x = A^T b,$$

analytical solution: $x^\star = (A^T A)^{-1} A^T b$

reliable and efficient algorithms and software

weighted least-squares $$\sum_{i=1}^{k} w_i (a_i^T x - b_i)^2,$$

regularization $$\sum_{i=1}^{k} (a_i^T x - b_i)^2 + \rho \sum_{i=1}^{n} x_i^2$$

Linear programming

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1, \ldots, m \end{array}$$

Nonlinear program

particular forms of the objective and constraint functions

For example :linear Program

The objective and the constraints functions are linear

Linearity of a function

for all $x,\ y \in \mathbf{R}^n$ and all $\alpha,\ \beta \in \mathbf{R}$.

$$f_i(\alpha x + \beta y) = \alpha f_i(x) + \beta f_i(y)$$

Linear programming

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m.
\end{array}
$$

$c, a_1, \ldots, a_m \in \mathbf{R}^n$

$b_1, \ldots, b_m \in \mathbf{R}$

solving linear programs

- no analytical formula for solution
- reliable and efficient algorithms and software

Some applications lead directly to linear programs in the above form

- not as easy to recognize as least-squares problems
- a few standard tricks used to convert problems into linear programs

example : using linear programming

Chebyshev approximation problem

$$\text{minimize} \quad \max_{i=1,\ldots,k} |a_i^T x - b_i|.$$

$x \in \mathbf{R}^n$

$a_1, \ldots, a_k \in \mathbf{R}^n,\ b_1, \ldots, b_k \in \mathbf{R}$

$$\text{minimize} \quad \|Ax - b\|_\infty$$

✓Approximation Interpretation
✓Estimation Interpretation

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x - t \leq b_i, \quad i = 1, \ldots, k \\ & -a_i^T x - t \leq -b_i, \quad i = 1, \ldots, k, \end{array}$$

$t \in \mathbf{R}$

Convex optimization

convex optimization

The objective and the constraints functions are convex

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1, \ldots, m \end{array}$$

$f_0, \ldots, f_m : \mathbf{R}^n \to \mathbf{R}$ are convex:

$$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y)$$

if $\alpha + \beta = 1$, $\alpha \ge 0$, $\beta \ge 0$

Convex optimization

✓Include least square as a special case
✓any linear program is a convex optimization problem
✓convex optimization a generalization of linear programming.

solving convex optimization problems

- no analytical solution
- reliable and efficient algorithms

using convex optimization

- often difficult to recognize
- many tricks for transforming problems into convex form
- surprisingly many problems can be solved via convex optimization

Nonlinear Optimization

an optimization problem when

- objective or constraint functions are not linear
- not known to be convex.

- No effective methods for solving the general nonlinear programming problem
- Methods involve some compromise.

local optimization methods (nonlinear programming)

- find a point that minimizes f_0 among feasible points near it
- fast, can handle large problems
- require initial guess
- provide no information about distance to (global) optimum

Nonlinear Optimization

global optimization methods

- find the (global) solution
- worst-case complexity grows exponentially with problem size

these algorithms are often based on solving convex subproblems