RAON online School: Elastic scattering: the optical model

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Recommended bibliography

- G.R. Satchler, *Introduction to nuclear reactions*, Macmillan (1990)
- G.R. Satchler, *Direct Nuclear Reactions*, Oxford University Press (1983)
- N. Glendenning, *Direct Nuclear Reactions*, World Scientific (2004)
- I.J. Thompson and F.M. Nunes, *Nuclear Reactions for Astrophysics*, Cambridge University Press (2009)
- A.M.M., *Models for nuclear reactions with weakly bound systems*, Proceedings of the International School of Physics Enrico Fermi Course 201 "Nuclear Physics with Stable and Radioactive Ion Beams" (https://arxiv.org/abs/1807.04349).

Modelling nuclear reactions

Why reaction theory is important?

- Many physical processes occurring spontaneously in nature (e.g. stars) or artificially (e.g. nuclear reactor) involve nuclear reactions. We need theoretical tools to evaluate their rates and cross sections.

- Reaction theory provides the necessary framework to extract meaningful structure information from measured cross sections and also permits the understanding of the dynamics of nuclear collisions.

- The many-body scattering problem is not solvable in general, so specific models tailored to specific types of reactions are used (elastic, breakup, transfer, knockout...) each of them emphasizing some particular degrees of freedom.

- In particular, exotic nuclei close to driplines are usually weakly-bound and breakup (coupling to the continuum) is important and must be taken into account in the reaction model.

Spatial and time scales

- Typical size of a nucleus: $R \simeq 1.20 \times A^{1/3}$ fm$\sim$ 5 fm
- Typical length scale of the nuclear force between two nuclei: $a \sim 1$ fm
- Typical length scale of the Coulomb force between two nuclei:
  $a_0 = \frac{Z_1 Z_2 e^2}{8\pi\epsilon_o E_{CM}} \sim 10$ fm
- Reduced de Broglie wavelength associated to the motion of a particle:
  $\lambda\!\!\!^{-} = 1/k = \hbar/p$
  (for a massive particle$\lambda\!\!\!^{-} = \hbar/\sqrt{2mE}$)
- To "observe" an object of size R, we need to use radiation with $\lambda \sim R$) .

TABLE 2.1 *Reduced de Broglie wavelengths $\lambda\!\!\!^{-}$, in fm, for various particles and energies*

Energy	*Photon*	*Electron*	*Pion*	*Proton*	*α-Particles*	^{16}O	^{40}Ar	^{208}Pb
1 MeV	197	140	12	4.5	2.3	1.14	0.72	0.32
10 MeV	19.7	18.7	3.7	1.4	0.72	0.36	0.23	0.10
100 MeV	2.0	2.0	1.0	0.45	0.23	0.11	0.072	0.032
1 GeV	0.20	0.20	0.17	0.12	0.068	0.035	0.023	0.010

☞*A classical description of the scattering based on trajectories is valid when the wavelength associated to the motion is short compared to the length scale of the interaction (no significant diffraction effects). Otherwise a quantum-mechanical description is needed.*

Direct versus compound reactions

DIRECT: elastic, inelastic, transfer,...

- "fast" collisions (10^{-21} s).
- only a few modes (degrees of freedom) involved
- small momentum transfer
- angular distribution asymmetric about $\pi/2$ (forward peaked)

COMPOUND: complete, incomplete fusion.

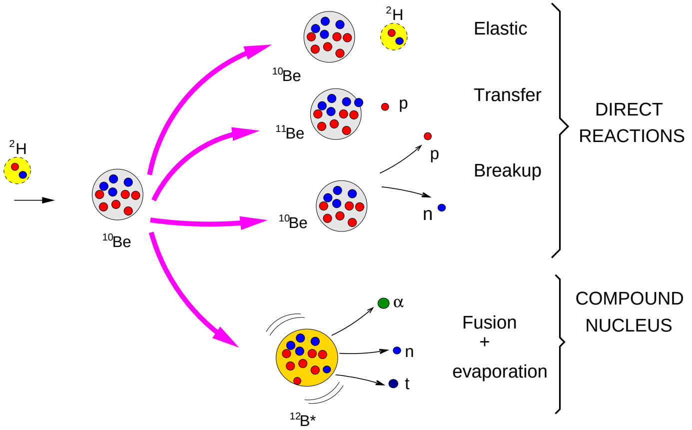
- "slow" collisions ($10^{-18} - 10^{-16}$ s).
- many degrees of freedom involved
- large amount of momentum transfer
- "loss of memory" $\Rightarrow$ dominated by statistical decay of different of emitted particles; almost symmetric distributions forward/backward (in CM)

Examples of direct and compound nucleus reactions

$\boxed{a + A \rightarrow b + B + Q}$ $\qquad Q = (M_a + M_A - M_b - M_B)c^2$ (energy released)

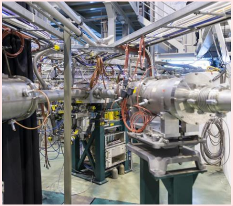
- Elastic scattering: $b = a,\ B = A \quad (Q = 0)$
  E.g.: $\alpha + {}^{197}\mathrm{Au} \rightarrow \alpha + {}^{197}\mathrm{Au}$
- Inelastic scattering: $b = a,\ B = A^* \quad (Q < 0)$
  E.g.: $\alpha + {}^{197}\mathrm{Au} \rightarrow \alpha + {}^{197}\mathrm{Au}^*$
- Rearrangement or transfer: $b \neq a,\ B \neq A \quad Q$ positive or negative
  E.g.: $d + {}^{208}\mathrm{Pb} \rightarrow p + {}^{209}\mathrm{Pb}$
- Breakup: $a = b + x \Rightarrow a + A \rightarrow b + x + A \quad (Q < 0)$
  E.g.: $d + {}^{208}\mathrm{Pb} \rightarrow p + n + {}^{208}\mathrm{Pb}$
- Fusion: reaction occurs via the formation of an intermediate compound nucleus:
  $a + B \rightarrow C^* \rightarrow b + B$
  A special case is that of capture reactions ($b = \gamma$):
  E.g.: $p + {}^{197}\mathrm{Au} \rightarrow {}^{198}\mathrm{Hg}^* \rightarrow \gamma + {}^{198}\mathrm{Hg}_{\mathrm{g.s.}}$

Example: the d+^{10}Be reaction



Linking theory with experiments: the cross section

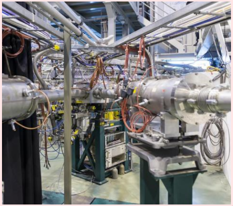
EXPERIMENT



THEORY
($H\Psi = E\Psi$)

Linking theory with experiments: the cross section

EXPERIMENT

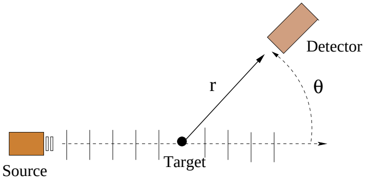


THEORY
$(H\Psi = E\Psi)$

CROSS SECTIONS

$$\frac{d\sigma}{d\Omega}, \frac{d\sigma}{dE}, etc$$

Experimental cross section



$$\Delta I = I_0 \, n_t \, \frac{d\sigma}{d\Omega} \Delta\Omega$$

- ΔI: detected particles per unit time in $\Delta\Omega$ (s^{-1})
- I_0: incident particles per unit time and unit area ($s^{-1}L^{-2}$)
- n_t: number of target nuclei within the beam
- $\Delta\Omega$: solid angle of detector ($=\Delta A/r^2$)
- $d\sigma/d\Omega$: differential cross section (L^2)

$$\frac{d\sigma}{d\Omega} = \frac{\text{flux of scattered particles through } dA = r^2 d\Omega}{\text{incident flux}}$$

Model Hamiltonian and model wavefunction

Full Hamiltonian

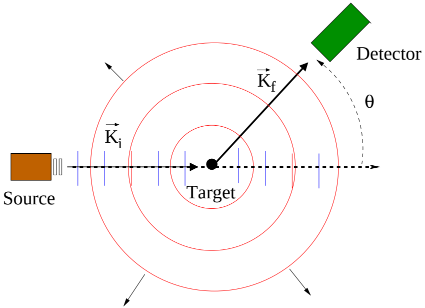
$$\boxed{H = \underbrace{H_p(\xi_p) + H_t(\xi_t)}_{\text{internal dynamics}} + \underbrace{\hat{T}_{\mathbf{R}} + V(\mathbf{R},\xi_p,\xi_t)}_{\text{relative motion}}}$$

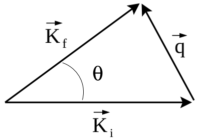
- $\hat{T}_{\mathbf{R}}$: proj.–target kinetic energy
- $H_p(\xi_p)$: projectile internal Hamiltonian
- $H_t(\xi_t)$: target internal Hamiltonian
- $V(\mathbf{R},\xi_p,\xi_t)$: projectile–target interaction

Time-independent Schrödinger equation:

$$\boxed{[H - E]\Psi(\mathbf{R},\xi_p,\xi_t) = 0}$$

The scattering wavefunction





Among the many mathematical solutions of $[H - E]\Psi = 0$ we are interested in those behaving asymptotically as:

$$\boxed{\Psi^{(+)}_{\mathbf{K}_\alpha} \to \Phi_\alpha(\xi_\alpha) e^{i\mathbf{K}_\alpha \cdot \mathbf{R}_\alpha} + (\text{outgoing spherical waves in } \alpha, \beta, \gamma, \ldots)}$$

where α denotes the incident channel and β,γ,... other (non-elastic channels)

Scattering amplitude and cross sections

Asymptotically, when the projectile and target are well far apart,

$$\Psi^{(+)}_{\mathbf{K}_\alpha} \xrightarrow{R_\alpha \gg} \Phi_\alpha(\xi_\alpha) e^{i\mathbf{K}_\alpha \cdot \mathbf{R}_\alpha} + \Phi_\alpha(\xi_\alpha) f_{\alpha,\alpha}(\theta) \frac{e^{iK_\alpha R_\alpha}}{R_\alpha} \qquad \text{(elastic)}$$

$$+ \sum_{\alpha' \neq \alpha} \Phi_{\alpha'}(\xi_\alpha) f_{\alpha',\alpha}(\theta) \frac{e^{iK_{\alpha'} R_\alpha}}{R_\alpha} \qquad \text{(inelastic)}$$

$$\Psi^{(+)}_{\mathbf{K}_\alpha} \xrightarrow{R_\beta \gg} \sum_\beta \Phi_\beta(\xi_\beta) f_{\beta,\alpha}(\theta) \frac{e^{iK_\beta R_\beta}}{R_\beta} \qquad \text{(transfer)}$$

where the function $f_{\beta,\alpha}$ modulating the outgoing waves is called scattering amplitude

Cross sections:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\alpha\to\beta} = \frac{\mu_\alpha}{\mu_\beta}\frac{K_\beta}{K_\alpha}\left|f_{\beta,\alpha}(\theta)\right|^2} \qquad E = \frac{\hbar^2 K_\alpha^2}{2\mu_\alpha} + \varepsilon_\alpha = \frac{\hbar^2 K_\beta^2}{2\mu_\beta} + \varepsilon_\beta$$

Defining our model space: Feshbach formalism

- Divide the full space into two groups: P and Q
  - ⇨ P: channels of interest
  - ⇨ Q: remaining channels
- Write $\Psi = \Psi_P + \Psi_Q$

$$\boxed{\begin{aligned}(E - H_{PP})\Psi_P &= H_{PQ}\Psi_Q \\ (E - H_{QQ})\Psi_Q &= H_{QP}\Psi_P\end{aligned}} \qquad (\, H_{PP} = PHP, H_{PQ} = PHQ, \text{etc}\,)$$

- Eliminate (formally) Ψ_Q:

$$\boxed{\underbrace{\left[H_{PP} + H_{PQ}\frac{1}{E - H_{QQ} + i\epsilon}H_{QP}\right]}_{H_{\rm eff}}\Psi_P = E\Psi_P}$$

- $H_{\rm eff}$ too complicated (complex, energy dependent, non-local) so, in practice, it is usually replaced by a simpler, effective Hamiltonian:

$$H_{\rm eff} \longrightarrow H_{\rm model} \quad \text{(complex, energy dependent)}$$

Strategy for reaction calculations

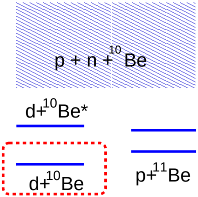
We need to make a choice for:

1. **Modelspace:** what channels are to be included?
2. **Structure model:** for projectile and target

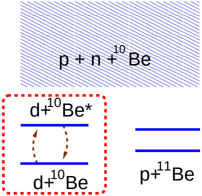
   (Microscopic, collective, cluster...)
3. **Reaction formalism**

   (will depend on the process to be studied)

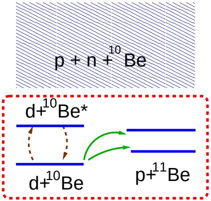
Choice of the modelspace: the d+^{10}Be example



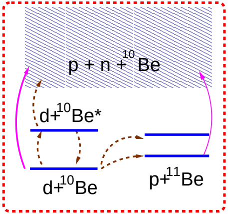
(a) 1 channel (elastic)



(b) 2 channels (elastic + inelastic)



(c) elastic + inelastic + transfer

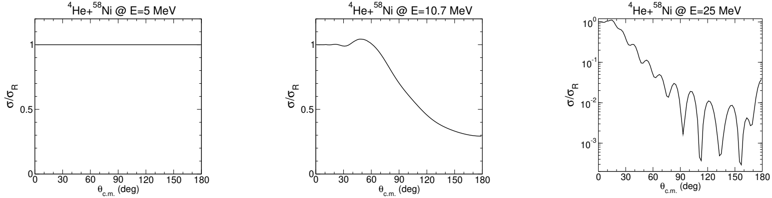


(d) elastic + inelastic + transfer + breakup

Single-channel approach to elastic scattering: the optical model

The optical model

- Elastic scattering angular distributions exhibit a large variety of patterns depending on the colliding system and energy.



- The goal of the optical model is to describe these features by using an effective potential (optical potential)

- In general, the optical potential contains an imaginary part which is meant to account for absorptive (nonelastic) processes.

Solving Schrodinger equation

- Effective Hamiltonian:

$$H = T_{\mathbf{R}} + U(\mathbf{R}) \qquad (U(\mathbf{R}) \text{ complex!})$$

- Schrödinger equation:

$$\boxed{[T_{\mathbf{R}} + U(\mathbf{R}) - E_\alpha]\chi_0^{(+)}(\mathbf{K},\mathbf{R}) = 0} \qquad (E_\alpha = \text{incident energy in CM})$$

- Boundary condition: Plane wave plus spherical wave, multiplied by the scattering amplitude $f(\theta,\phi)$:

$$\chi_0^{(+)}(\mathbf{K},\mathbf{R}) \rightarrow e^{i\mathbf{K}\cdot\mathbf{R}} + f(\theta,\phi)\frac{e^{iKR}}{R} \qquad K = \frac{\sqrt{2\mu E_\alpha}}{\hbar}$$

- Elastic differential cross section:

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$$

Partial wave decomposition

- For a central potential $[U(\mathbf{R}) = U(R)]$, the scattering wavefunction can be expanded in spherical harmonics:

$$\boxed{\chi_0^{(+)}(\mathbf{K},\mathbf{R}) = \frac{4\pi}{KR}\sum_{\ell} i^{\ell}\chi_{\ell}(K,R)\sum_{m} Y^{*}_{\ell m}(\hat{K})Y_{\ell m}(\hat{R}) = \frac{1}{KR}\sum_{\ell} i^{\ell}\chi_{\ell}(K,R)(2\ell+1)P_{\ell}(\cos\theta)}$$

- The radial wavefuntions $\chi_{\ell}(K,R)$ satisfy the equation:

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dR^2} + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{R^2} + U(R) - E_0\right]\chi_{\ell}(K,R) = 0.$$

- Asymptotic boundary condition: beyond the range of short-range potentials:

$$\begin{aligned}\chi_{\ell}(K,R) &\rightarrow F_{\ell}(KR) + T_{\ell}H_{\ell}^{(+)}(KR)\\ &= \frac{i}{2}[H_{\ell}^{(-)}(KR) - S_{\ell}H_{\ell}^{(+)}(KR)]\end{aligned}$$

where: $F_{\ell}(KR) \rightarrow \sin(KR - \ell\pi/2)$; $H_{\ell}^{(\pm)}(KR) \rightarrow e^{\pm i(KR-\ell\pi/2)}$

Asymptotic solutions of the radial wavefunctions

- For $R \gg \Rightarrow U(R) = 0 \Rightarrow \chi_\ell(K,R)$ will be a combination of F_ℓ and G_ℓ

$$\boxed{F_\ell(KR) \to \sin(KR - \ell\pi/2) \qquad G_\ell(KR) \to \cos(KR - \ell\pi/2)}$$

or their *outgoing/ingoing* combinations:

$$\boxed{H^{(\pm)}(KR) \equiv G_\ell(KR) \pm iF_\ell(KR) \to e^{\pm i(KR-\ell\pi/2)}}$$

- The physical solution is determined by the known boundary conditions:

$$\begin{array}{cccccc}
 & \chi_0^{(+)}(\mathbf{KR}) & \to & e^{i\mathbf{K}\cdot\mathbf{R}} & + & f(\theta)\dfrac{e^{iKR}}{R} \\
 & \Downarrow & & \Downarrow & & \Downarrow \\
U = 0 & \chi_\ell(KR) & \to & F_\ell(KR) & + & 0 \\
U \neq 0 & \chi_\ell(KR) & \to & F_\ell(KR) & + & T_\ell H^{(+)}(KR)
\end{array}$$

☞ The coefficients T_ℓ are to be determined by numerical integration.

The scattering amplitude

- The scattering amplitude is the coefficient of e^{iKR}/R.

$$\boxed{\begin{aligned} f(\theta) &= \frac{1}{K}\sum_{\ell}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta) \\ &= \frac{1}{2iK}\sum_{\ell}(2\ell+1)(S_\ell-1)P_\ell(\cos\theta). \end{aligned}}$$

- Elastic cross section:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2.$$

Numerical integration of Schrodinger equation

1. Fix a *matching radius*, R_m, such that $U(R_m) \approx 0$
2. Integrate $\chi_\ell(R)$ from $R = 0$ up to R_m, starting with the condition:

$$\lim_{R\to 0} \chi_\ell(K,R) = 0$$

3. At $R = R_m$ impose the boundary condition:

$$\begin{aligned}\chi_\ell(K,R) &\to F_\ell(KR) + T_\ell H_\ell^{(+)}(KR) \\ &= \frac{i}{2}[H_\ell^{(-)}(KR) - S_\ell H_\ell^{(+)}(KR)]\end{aligned}$$

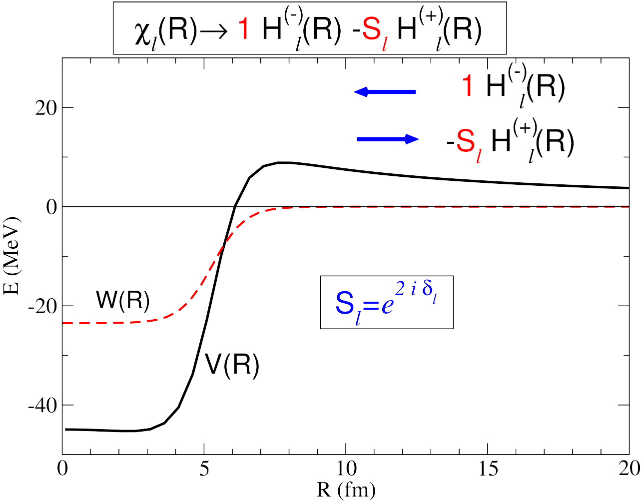
☞ $S_\ell = 1 + 2iT_\ell$ = **S-matrix**

4. Phase-shifts:

$$\boxed{S_\ell \equiv e^{i2\delta_\ell}} \qquad \boxed{T_\ell = e^{i\delta_\ell}\sin(\delta_\ell)}$$

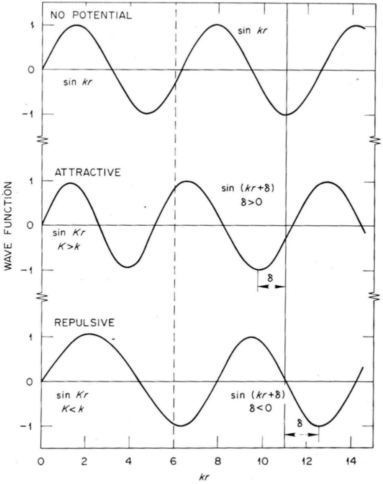
$$\boxed{\chi_\ell(K,R) \to e^{i\delta_\ell}\sin(KR + \delta_\ell - \ell\pi/2)}$$

Interpretation of the S-matrix (single-channel case)



Interpretation of the S-matrix (single-channel case)

- S_ℓ =coefficient of the outgoing wave for partial wave ℓ.
- $|S_\ell|^2$ is the *survival* probability for the partial wave ℓ:
  - U real $\Rightarrow |S_\ell| = 1 \Rightarrow \delta_\ell$ real
  - U complex $\Rightarrow |S_\ell| < 1 \Rightarrow \delta_\ell$ complex
- For $\ell \gg \Rightarrow S_\ell \to 1$
- Sign of $Re[\delta]$:
  - $Re[\delta] > 0 \Rightarrow$ attractive potential
  - $Re[\delta] < 0 \Rightarrow$ repulsive potential
  - $Re[\delta] = 0$ ($S_\ell = 1$) $\Rightarrow$ no potential ($U(R) = 0$)

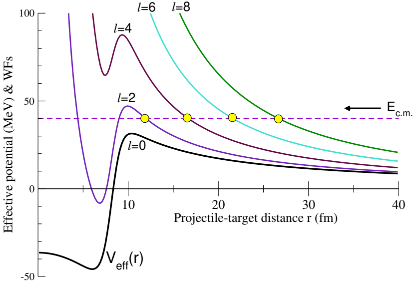


Interpretation of the S-matrix (single-channel case)

Effective potential:

$$V_{\text{eff}}(r) = V_N(r) + V_C(r) + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}$$

As the ℓ value increases, so does the centrifugal potential, preventing the projectile from approaching the target and hence reducing the effect of the nuclear (real and imaginary) potentials. Thus, for $\ell \gg \Rightarrow S_\ell \to 1$



Coulomb plus nuclear case

Radial equation:

$$\boxed{\left[\frac{d^2}{dR^2} + K^2 - \frac{2\eta K}{R} + \frac{2\mu}{\hbar^2}U(R) + \frac{\ell(\ell+1)}{R^2}\right]\chi_\ell(K,R) = 0}$$

$$\boxed{\eta = \frac{Z_p Z_t e^2}{4\pi\epsilon_0 \hbar v} = \frac{Z_p Z_t e^2 \mu}{4\pi\epsilon_0 \hbar^2 K}}$$

(Sommerfeld parameter)

Asymptotic condition:

$$\chi^{(+)}(\mathbf{K},\mathbf{R}) \to e^{i[\mathbf{K}\cdot\mathbf{R}+\eta\log(kR-\mathbf{K}\cdot\mathbf{R})]} + f(\theta)\frac{e^{i(KR-\eta\log 2KR)}}{R}$$

$$\boxed{\begin{aligned}\chi_\ell(K,R) &\to e^{i\sigma_\ell}\left[F_\ell(\eta,KR) + T_\ell H_\ell^{(+)}(\eta,KR)\right] \\ &= \frac{i}{2}e^{i\sigma_\ell}\left[H_\ell^{(-)}(\eta,KR) - S_\ell H_\ell^{(+)}(\eta,KR)\right]\end{aligned}}$$

- ☞ $\sigma_\ell(\eta)$=Coulomb phase shift
- ☞ $F_\ell(\eta,KR)$=regular Coulomb wave
- ☞ $H_\ell^{(\pm)}(\eta,KR)$=outgoing/ingoing Coulomb wave

Coulomb plus nuclear case: scattering amplitude

Total scattering amplitude:

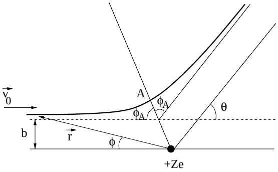
$$\boxed{f(\theta) = f_C(\theta) + \frac{1}{2iK}\sum_{\ell}(2\ell+1)e^{2i\sigma_\ell}(S_\ell - 1)P_\ell(\cos\theta)}$$

☞ $f_C(\theta)$ is the amplitude for pure Coulomb:

$$\boxed{\frac{d\sigma_R}{d\Omega} = |f_C(\theta)|^2 = \frac{\eta^2}{4K^2\sin^4(\frac{1}{2}\theta)} = \left(\frac{Z_p Z_t e^2}{16\pi\epsilon_0 E}\right)^2 \frac{1}{\sin^4(\frac{1}{2}\theta)}}$$

Classical interpretation of elastic scattering

Equations of classical trajectories



- A classical trajectory can be characterized by the polar variables (r, ϕ). The equation of the trajectory is determined by the following key concepts:
- Energy conservation:

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2\mu r^2} + V(r) = \frac{1}{2}\mu v_0^2$$

- Angular momentum conservation:

$$L = \mu r^2 \frac{d\phi}{dt} = \mu \cdot v_0 \cdot b$$

- Impact parameter b: Is the distance of the target to the straight line of the projectile. Determines the angular momentum.

- Time independent differential equation for the trajectory.

$$\boxed{\frac{d\phi}{dr} = \frac{L}{r^2\sqrt{2\mu\left[E - \frac{L^2}{2\mu r^2} - V(r)\right]}}}$$

- Effective potential: Includes the centrigugal term.

$$V_{\mathrm{ef},L}(r) \equiv \frac{L^2}{2\mu r^2} + V(r) = E\left(\frac{b}{r}\right)^2 + V(r)$$

- Distance of closest approach: In it the radial velocity vanishes, and the trajectory inverts its motion (Turning point).

$$\frac{dr}{dt} = 0 \quad \Rightarrow \quad E = \frac{L^2}{2\mu r_{CA}^2} + V(r_{CA})$$

- Trajectory:

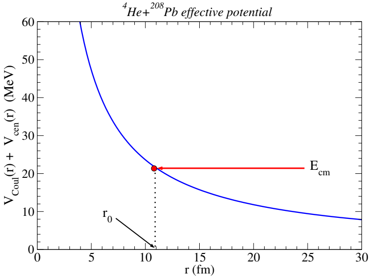
$$\boxed{\phi(r) = \int_{r_{CA}}^{r} du \frac{L}{u^2\sqrt{2\mu\left[E - V_{\mathrm{ef},L}(u)\right]}}}$$

- Asymptotic angle: $\phi_A = \phi(\infty)$

Distance of closest approach

☞ For $b = 0$ (head-on collision) $\Rightarrow E = V(r_{CA})$. For Coulomb:

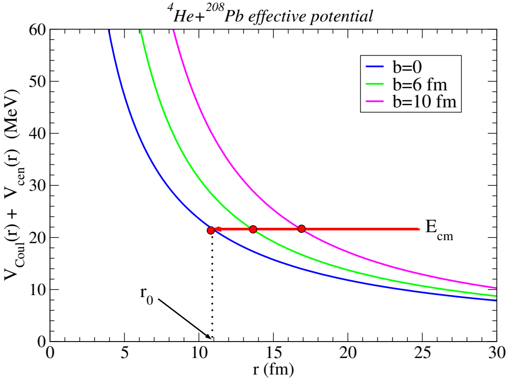
$$r_{CA} = \frac{e^2}{4\pi\epsilon_0}\frac{Z_1 Z_2}{E} = 2a_0$$



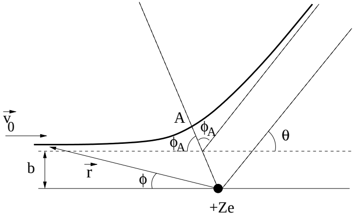
☞ a_0 is the length scale of the Coulomb interaction, given by half the distance of closest approach.

☞ For $b > 0 \Rightarrow r_{CA} > 2a_0$. For example, for pure Coulomb:

$$r_{CA} = a_0 + \sqrt{a_0^2 + b^2} \qquad (Coulomb)$$



Deflection function



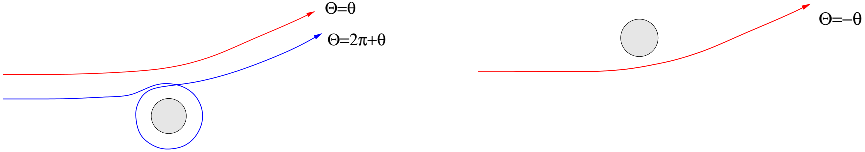
$$\boxed{2\varphi_A + \Theta = \pi}$$

Θ= deflexion angle

Deflection function: is the expression $\Theta(b)$ of the deflection angle Θ as a function of the impact parameter b.

$$\boxed{\Theta(b) = \pi - 2\int_{r_0}^{\infty} \frac{b}{r^2} \frac{dr}{\sqrt{1 - \frac{V(r)}{E} - \frac{b^2}{r^2}}}}$$

- Scattering angle: Angle formed by the final direction and the initial direction. $0 \leq \theta \leq \pi$. It is the quantity observed experimentally in a scattering experiment.
- Deflection angle: Angle which is covered by the trajectory $\Theta = \pm\theta + 2n\pi$. Several deflection angles can correspond to the same scattering angle.

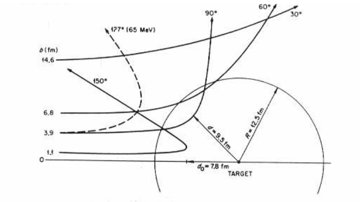


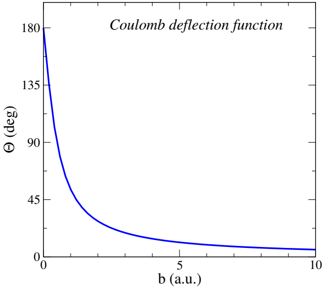
- ☞ For each impact parameter b there is a single value of the deflection angle Θ and of the scattering angle $\theta(b)$.
- ☞ For a given scattering angle θ there may be several trajectories, corresponding to different values of b.
- ☞ $\Theta = \theta > 0$ is a near-side trajectory (the projectile bypasses the target "near" the detector).
- ☞ $\Theta = -\theta < 0$ is a far-side trajectory (the projectile bypasses the target "far" from the detector).
- ☞ $\Theta = \pm\theta + 2\pi n$ are orbiting trajectories (the projectile "orbits" around the target).

Classical deflection function for point Coulomb case

- For a point Coulomb potential, the deflection function is given analytically by:

$$\boxed{b = a_0 \cot\left(\frac{\theta}{2}\right)}$$



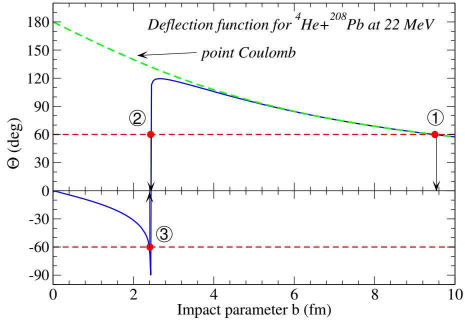


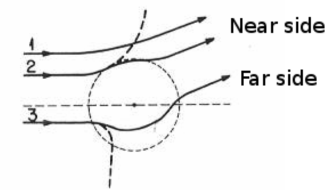
☞ When b increases, for a given energy E, r_{CA} increases and θ decreases.

☞ When E increases, for a given b, r_{CA} decreases and θ decreases.

Coulomb + nuclear scattering: deflection function

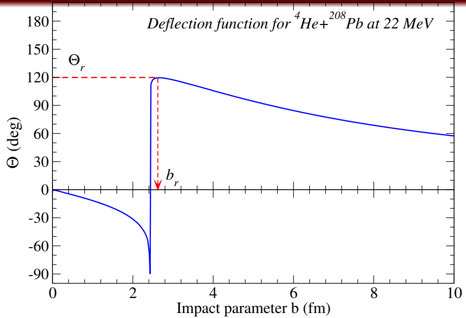
☞ For large values of b, the scattering is Coulombic (the projectile does not feel the nuclear potential).





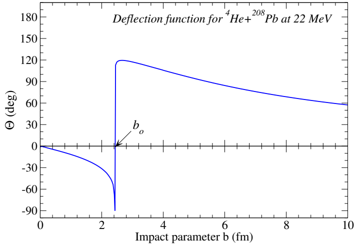
☞ For a given scattering angle θ there are in general 3 values of b contributing to this angle. (1) is the Coulomb trajectory, (2) is the nuclear near-side trajectory and (3) is the nuclear far-side trajectory.

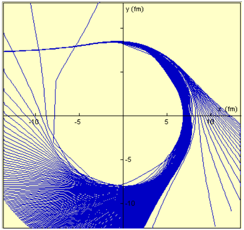
Coulomb + nuclear scattering: Rainbow



- The deflection function has a maximum at $b = b_r \to \Theta_r$ (rainbow angle)
- For $b = b_r$: $\frac{d\Theta}{db} = 0 \Rightarrow \frac{db}{d\Theta} = 0 \Rightarrow \frac{d\sigma}{d\Omega} \to \infty$
- In the vicinity of b_r, many trajectories give approximately the same scattering angle (Θ_r)
- For angles greater than the rainbow, ($\theta > \Theta_r$), the Coulomb trajectories and nuclear nearside trajectories do not contribute to the cross section. So, classically, there is a sharp decrease in the differential cross section for ($\theta > \Theta_r$). This is the "shadow region", to be discussed later.

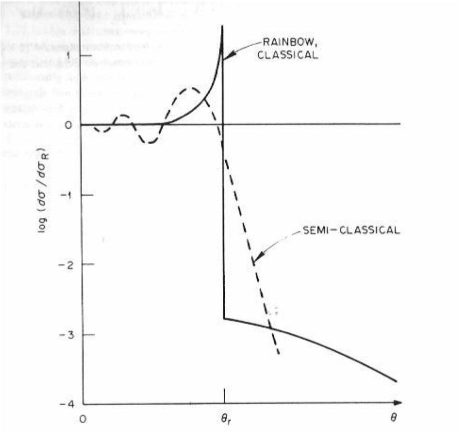
Coulomb and nuclear scattering: Orbiting





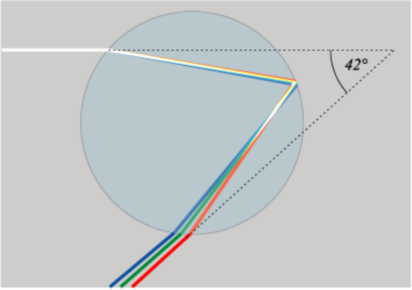
- Orbiting: For each scattering energy, above the Coulomb barrier, there is a certain impact parameter b_o which makes the effective barrier exactly equal to the scattering energy $E = V_B(b_o)$, $R_{CA} = R_B(b_0)$. This trajectory spends a long time at distances $r \simeq R_B(b_0)$, making the deflection angle very large and negative. This is the "orbiting".
- Impact parameters slightly larger than b_0 produce very different scattering angles. Thus, $db/d\theta$ is very small, so orbiting has a very small impact on the cross sections. Trajectories with $b < b_o$ imply overlap of the nuclei, so they should not contribute to elastic scattering. However, they will be crucial for fusion.

Coulomb + nuclear scattering: undulatory effects



☞ In a treatment beyond the classical limit, several trajectories may interfere, and the divergence at the rainbow is smoothed.

Atmospheric rainbow



Elastic scattering phenomenology

Nucleus-nucleus scattering: Optical Potential

Optical potential: $\mathcal{V} \approx U(r) = U_{\rm nuc}(r) + V_{\rm coul}(r)$

- Coulomb potential: charge sphere distribution

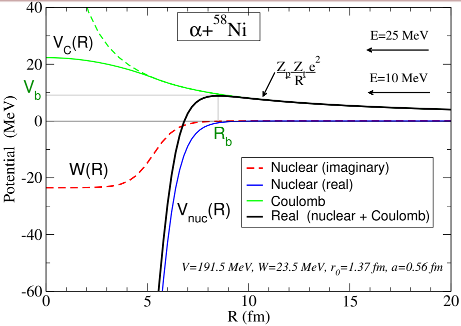
$$\boxed{V_{\rm coul}(r) = \begin{cases} \frac{Z_1 Z_2 e^2}{2R_c}\left(3 - \frac{r^2}{R_c^2}\right) & \text{if } r \leq R_c \\ \frac{Z_1 Z_2 e^2}{r} & \text{if } r \geq R_c \end{cases}}$$

- Nuclear potential (complex): Eg. Woods-Saxon parametrization

$$\boxed{U_{\rm nuc}(r) = V(r) + iW(r) = -\frac{V_0(E)}{1+\exp\left(\frac{r-R_V}{a_V}\right)} - i\,\frac{W_0(E)}{1+\exp\left(\frac{r-R_W}{a_W}\right)}}$$

- Potential parameters: 6, fitted to reproduce the elastic differential cross sections.
  - Depths $V_0(E), W_0(E)$;
  - Radii $R_{V,W} = r_{V,W}(A_p^{1/3} + A_t^{1/3})$. $r_V \approx r_W \sim 1.1 - 1.4$ fm.
  - Difuseness $a_V \approx a_W \sim 0.5 - 0.7$ fm

Nucleus-nucleus scattering: The Coulomb barrier



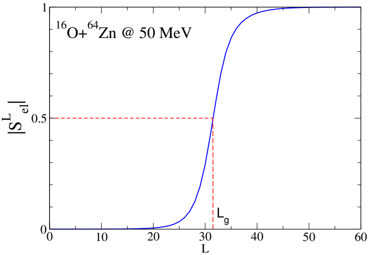
- The maximum of $V_N(r) + V_C(r)$ defines the Coulomb barrier. The radius of the barrier is R_b. The height of the barrier is $V_b = V_N(R_b) + V_C(R_b)$
- As a **rough approximation,**

$$\boxed{R_b \simeq 1.44(A_p^{1/3} + A_t^{1/3})\ \text{fm}}$$

$$\boxed{V_b \simeq \frac{Z_p Z_t e^2}{4\pi\epsilon_0 R_b} \approx \frac{Z_p Z_t}{(A_p^{1/3} + A_t^{1/3})}\ [\text{MeV}]}$$

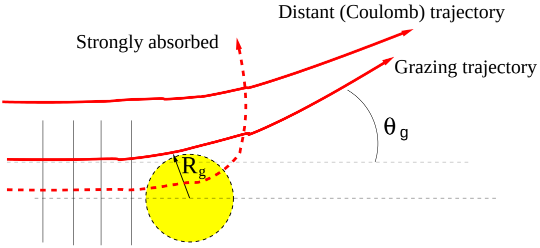
Nucleus-Nucleus Elastic scattering: Strong absorption

- The nuclear attraction is determined by the real part of the optical potential $V(r)$. Together with the Coulomb potential, determines the Coulomb barrier.
- The absorption, which corresponds to the removal of flux from the elastic channel, is determined by the imaginary part of the optical potential $W(r)$.
- Elastic scattering of heavy nuclei (beyond He) display strong absorption. One can define a grazing angular momentum (ℓ_g), such that:
  - $|S_\ell| \approx 0$ when $\ell \ll \ell_g$ and $|S_\ell| \to 1$ when $\ell \gg \ell_g$.
  - A convenient quantitative definition of the grazing angular momentum (ℓ_g) is provided by the condition $|S(\ell_g)| \simeq \frac{1}{2}$



Strong absorption: Classical interpretation

- The grazing angular momentum ℓ_g is associated to a grazing distance R_g, which is its distance of closest approach $R_g = a_0 + \sqrt{a_0^2 + (\ell_g + 1/2)^2/k^2}$.
- When Coulomb is weak (or absent): $kR_g \approx (\ell_g + 1/2)$
- The grazing distance $R_g \simeq (1.4 - 1.5)(A_p^{1/3} + A_t^{1/3})$ is approximately independent of the energy, so ℓ_g increases with energy.
- Angular momenta with $\ell < \ell_g$ are associated with trajectories which come inside R_g, and are strongly absorbed.



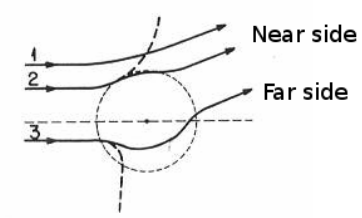
Near-side and far-side decomposition

For $\ell \gg 1$ and $\frac{1}{\ell+\frac{1}{2}} \lesssim \theta \lesssim \pi - \frac{1}{\ell+\frac{1}{2}}$

$$P_\ell(\cos\theta) \simeq \frac{e^{i\left((\ell+\frac{1}{2})\theta-\frac{\pi}{4}\right)} - e^{-i\left((\ell+\frac{1}{2})\theta-\frac{\pi}{4}\right)}}{\sqrt{2\pi\left(\ell+\frac{1}{2}\right)\cos\theta}} \quad \Rightarrow \quad f(\theta) = f^{\rm far}(\theta) + f^{\rm near}(\theta)$$

Classically, the contributions would correspond to near-side and far-side trajectories:

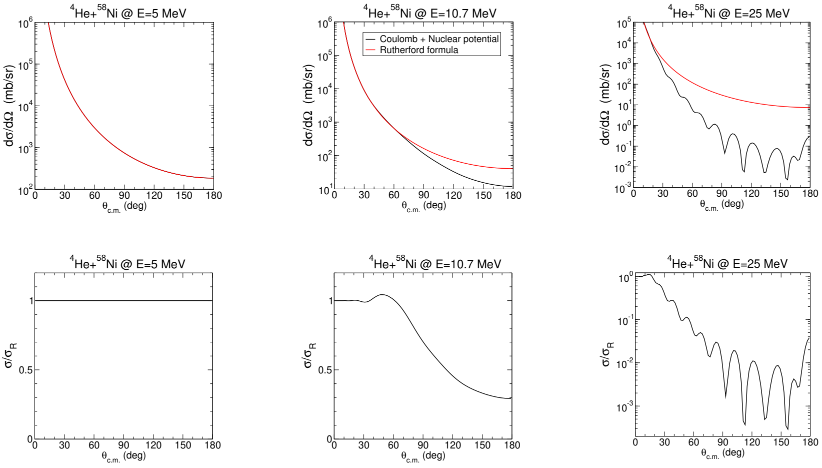
- The repulsive Coulomb potential tend to deflect the trajectories outward from the target (near-side trajectories).
- Nuclear attraction tends to bend the trajectories inwards (far-side trajectories).
- Near- and far-side trajectories may give rise to the same scattering angle so, if their amplitudes are similar, interference effects will occur.



Patterns of elastic scattering: Energy dependence

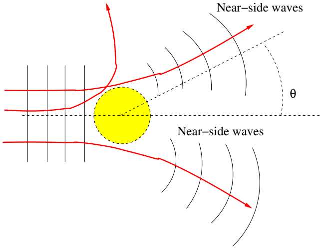
- The semi-classical vs quantum character of the scattering can be given in terms of the Sommerfeld parameter: $\eta = \frac{Z_p Z_t e^2}{4\pi\epsilon_0 \hbar v}$
- The Coulomb vs nuclear relevance, in terms of the energy of the Coulomb barrier: $V_b \simeq \frac{Z_p Z_t}{A_p^{1/3}+A_t^{1/3}}$ [MeV]
- Three distinct patterns appear for the elastic cross sections
  - Nuclear relevant $E > V_b$, quantum $\eta \lesssim 1 \Rightarrow$ Fraunhofer scattering
  - Nuclear relevant $E > V_b$, semiclassical $\eta \gg 1 \Rightarrow$ Fresnel scattering
  - Coulomb-dominated $E < V_b \Rightarrow$ Rutherford scattering

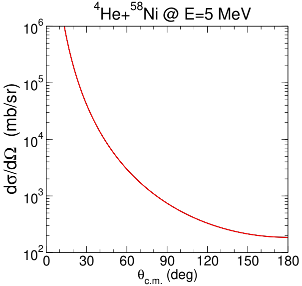
Patterns of elastic scattering: ^{4}He+^{58}Ni example



Rutherford scattering **Fresnel Scattering** **Fraunhöfer Scattering**

Rutherford scattering



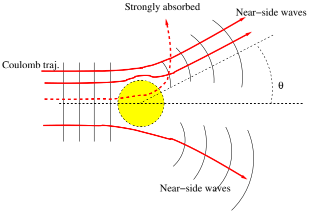


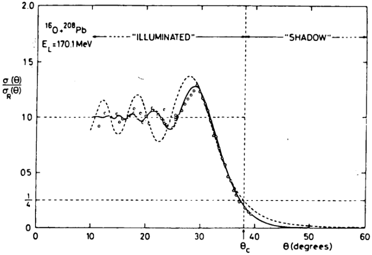
- Centre of mass energy below the Coulomb barrier($E < V_b$): Nuclear potential does not affect the scattering.
- Analytical differential cross sections (same for classical and quantum!)

$$\frac{d\sigma}{d\Omega} = \left(\kappa \frac{Z_p Z_t e^2}{2E}\right)^2 \frac{1}{\sin^4(\theta/2)}$$

Fresnel scattering

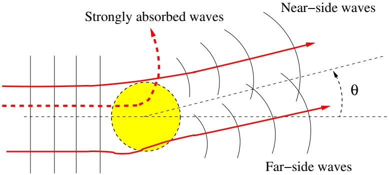
- Analogous to light scattering from an object with size $R_g \gg \lambda$. Leads to $\eta \gg 1$.
- The grazing angular momentum (ℓ_g) determines a grazing angle (θ_g), such that $\ell_g = \eta \cot(\theta_g/2)$, and a grazing distance $R_g = \frac{a_0}{2}\left(1 + \sin(\theta_g/2)^{-1}\right)$.
- Quarter-point recipe: $|S(\ell_g)| = 1/2$ implies $\sigma/\sigma_R(\theta_g) = 1/4$.
- Angular pattern divided in *illuminated* ($\theta < \theta_g$) and *shadow* ($\theta > \theta_g$) regions. Interference between pure Coulomb and near-side trajectories produce oscillations.

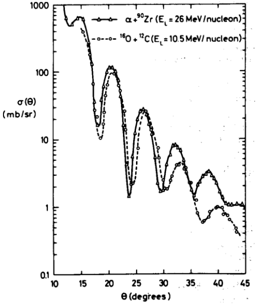




Fraunhofer scattering

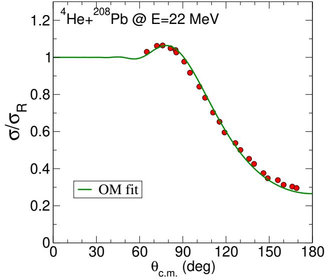
- Analogous to the scattering of light by an object which has a size $R_g \simeq \lambda$
- Waves scattering from the the two sides interfere constructively or destructively, giving rise to a diffraction pattern of maxima and minima spaced by $\boxed{\Delta\theta = \pi/\ell_g \approx \pi/kR_g}$
- Since $\Delta\theta \sim 1/\sqrt{E}$, as energy increases, oscillating pattern compresses and more oscillations appear.

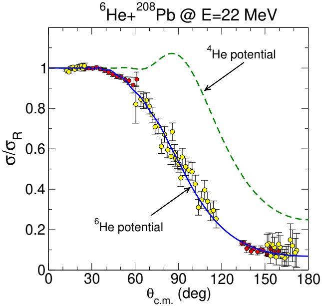




Elastic scattering of halo nuclei

How does the halo structure affect the elastic scattering?





- ^{4}He+^{208}Pb shows typical Fresnel pattern and "standard" optical model parameters
- ^{6}He+^{208}Pb shows a prominent reduction in the elastic cross section, suggesting that part of the incident flux goes to non-elastic channels (e.g. breakup, neutron transfer)