

Inelastic scattering: DWBA method

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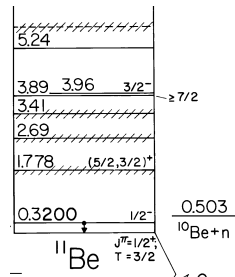
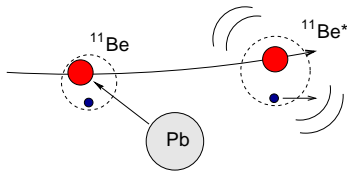


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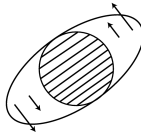
December 8, 2021

Inelastic scattering to bound states

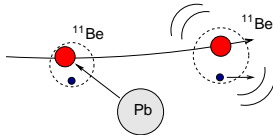
- Nuclei are not inert or *frozen* objects; they do have an internal structure of protons and neutrons that can be modified (excited) during the collision.
- Quantum systems exhibit, in general, an energy spectrum with bound and unbound levels.



- ① **COLLECTIVE:** Involve a collective motion of several nucleons which can be interpreted macroscopically as **rotations** or **surface vibrations** of the nucleus.



- ② **FEW-BODY/SINGLE-PARTICLE:** Involve the excitation of a nucleon or cluster.



Formal treatment of inelastic scattering

- **Goal:** Determine the differential cross section for an inelastic process of the form: $a + A \rightarrow a + A^*$
- The scattering wavefunction $\Psi_{\mathbf{K}_0}^{(+)}(\mathbf{R}, \xi)$ must contain, besides the elastic scattering component, additional components associated to excited states.
- Asymptotically:

$$\Psi_{\mathbf{K}_0}^{(+)}(\mathbf{R}, \xi) \xrightarrow{R \gg} \underbrace{e^{i\mathbf{K}_0 \cdot \mathbf{R}} \phi_0(\xi)}_{\text{incident}} + \underbrace{f_{0,0}(\theta) \frac{e^{iK_0 R}}{R} \phi_0(\xi)}_{\text{elastic}} + \underbrace{\sum_{n>0} f_{n,0}(\theta) \frac{e^{iK_n R}}{R} \phi_n(\xi)}_{\text{inelastic}}$$

where $\phi_n(\xi)$ are internal wfs of the nuclei being excited in some model.

- Cross sections:

$$\left(\frac{d\sigma(\theta)}{d\Omega} \right)_{0 \rightarrow n} = \frac{K_n}{K_0} |f_{n,0}(\theta)|^2$$

The coupled-channels method

We need to incorporate explicitly in the Hamiltonian the internal structure of the nucleus being excited (e.g. **projectile**).

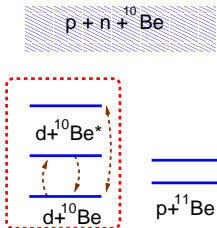
$$H = T_R + h(\xi) + V(\mathbf{R}, \xi)$$

- T_R : Kinetic energy for projectile-target relative motion.
- $\{\xi\}$: Internal degrees of freedom of the projectile (depend on the model).
- $V(\mathbf{R}, \xi)$: Projectile-target interaction.
- $h(\xi)$: Internal Hamiltonian of the projectile.

$$h(\xi)\phi_n(\xi) = \varepsilon_n\phi_n(\xi)$$

- $\phi_n(\xi)$: internal states of the projectile.

Modelspace and scattering wavefunction: $d+^{10}\text{Be} \rightarrow d+^{10}\text{Be}^*$ example



☞ The modelspace is composed by ground states (elastic channel) and some excited states (inelastic scattering)

Boundary conditions for scattering wavefunction:

$$\Psi_{\mathbf{K}_0}^{(+)}(\mathbf{R}, \xi) \xrightarrow{R \gg} \underbrace{e^{i\mathbf{K}_0 \cdot \mathbf{R}} \phi_0(\xi)}_{\text{incident}} + \underbrace{f_{0,0}(\theta) \frac{e^{iK_0 R}}{R} \phi_0(\xi)}_{\text{elastic}} + \underbrace{\sum_{n>0} f_{n,0}(\theta) \frac{e^{iK_n R}}{R} \phi_n(\xi)}_{\text{inelastic}}$$

Cross sections:

$$\left(\frac{d\sigma(\theta)}{d\Omega} \right)_{0 \rightarrow n} = \frac{K_n}{K_0} |f_{n,0}(\theta)|^2 \quad f_{n,0}(\theta) = \text{scattering amplitude}$$

We expand the total wave function in a subset of internal states representing the adopted modelspace:

$$\Psi_{\text{model}}(\mathbf{R}, \xi) = \phi_0(\xi)\chi_0(\mathbf{K}_0, \mathbf{R}) + \sum_{n>0} \phi_n(\xi)\chi_n(\mathbf{K}_n, \mathbf{R})$$

and impose the boundary conditions for the (unknown) $\chi_n(\mathbf{R})$:

$$\begin{aligned}\chi_0^{(+)}(\mathbf{K}_0, \mathbf{R}) &\rightarrow e^{i\mathbf{K}_0 \cdot \mathbf{R}} + f_{0,0}(\theta) \frac{e^{iK_0 R}}{R} && \text{for } n=0 \text{ (elastic)} \\ \chi_n^{(+)}(\mathbf{K}_n, \mathbf{R}) &\rightarrow f_{n,0}(\theta) \frac{e^{iK_n R}}{R} && \text{for } n>0 \text{ (non-elastic)}\end{aligned}$$

Calculation of $\chi_n^{(+)}(\mathbf{R})$: the coupled equations

- The model wavefunction must satisfy the Schrödinger equation:

$$[H - E]\Psi_{\text{model}}^{(+)}(\mathbf{R}, \xi) = 0$$

- Multiply on the left by each $\phi_n^*(\xi)$, and integrate over $\xi \Rightarrow$ coupled channels equations for $\{\chi_n(\mathbf{R})\}$:

$$[E - \varepsilon_n - T_R - V_{n,n}(\mathbf{R})]\chi_n(\mathbf{R}) = \sum_{n' \neq n} V_{n,n'}(\mathbf{R})\chi_{n'}(\mathbf{R})$$

- The structure information is embedded in the coupling potentials:

$$V_{n,n'}(\mathbf{R}) = \int d\xi \phi_{n'}^*(\xi) V(\mathbf{R}, \xi) \phi_n(\xi)$$

👉 $\phi_n(\xi)$ will depend on the assumed structure model (collective, few-body, etc).

Optical Model

- The Hamiltonian:

$$H = T_R + V(\mathbf{R})$$

- Internal states: Just $\phi_0(\xi)$

- Model wavefunction:

$$\Psi_{\text{mod}}(\mathbf{R}, \xi) \equiv \chi_0(\mathbf{K}, \mathbf{R})\phi_0(\xi)$$

- Schrödinger equation:

$$[H - E]\chi_0(\mathbf{K}, \mathbf{R}) = 0$$

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- Schrödinger equation:

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Coupled-channels method

- The Hamiltonian:

$$H = T_R + h(\xi) + V(\mathbf{R}, \xi)$$

- Internal states:

$$h(\xi)\phi_n(\xi) = \varepsilon_n\phi_n(\xi)$$

- Model wavefunction:

$$\Psi_{\text{model}}(\mathbf{R}, \xi) = \phi_0(\xi)\chi_0(\mathbf{K}, \mathbf{R}) + \sum_{n>0} \phi_n(\xi)\chi_n(\mathbf{K}, \mathbf{R})$$

- Schrödinger equation:

$$[H - E]\Psi_{\text{model}}(\mathbf{R}, \xi) = 0$$

\Downarrow

$$[E - \varepsilon_n - T_R - V_{n,n}(\mathbf{R})]\chi_n(\mathbf{K}, \mathbf{R}) = \sum_{n' \neq n} V_{n,n'}(\mathbf{R})\chi_{n'}(\mathbf{K}, \mathbf{R})$$

The DWBA approximation for inelastic scattering

- Assume that we can write the p-t interaction as: $V(\mathbf{R}, \xi) = V_0(R) + \Delta V(\mathbf{R}, \xi)$
- Use central $V_0(R)$ part to calculate the (distorted) waves for p-t relative motion:

$$\begin{aligned} \left[\hat{T}_{\mathbf{R}} + V_0(R) - E_i \right] \chi_i^{(+)}(\mathbf{K}_i, \mathbf{R}) &= 0 \quad (E_i = \text{c.m. energy}) \\ \left[\hat{T}_{\mathbf{R}} + V_0(R) - E_f \right] \chi_f^{(+)}(\mathbf{K}_f, \mathbf{R}) &= 0 \quad (E_f = E_i + Q = E_i - E_x) \end{aligned}$$

- In first order of $\Delta V(\mathbf{R}, \xi)$ (DWBA) :

$$f_{i \rightarrow f}^{\text{DWBA}}(\theta) = -\frac{\mu}{2\pi\hbar^2} \int \chi_f^{(-)*}(\mathbf{K}_f, \mathbf{R}) \Delta V_{if}(\mathbf{R}) \chi_i^{(+)}(\mathbf{K}_i, \mathbf{R}) d\mathbf{R}$$

with the transition potential:

$$\Delta V_{if}(\mathbf{R}) \equiv \int \phi_f^*(\xi) \Delta V(\mathbf{R}, \xi) \phi_i(\xi) d\xi$$

Multipole expansion of the interaction: reduced matrix elements

- In actual calculations, the internal states will have definite spin/parity:

$$\phi_i(\xi) = |I_i M_i\rangle \quad \text{and} \quad \phi_f(\xi) = |I_f M_f\rangle$$

- The projectile-target interaction can be expanded in multipoles:

$$V(\mathbf{R}, \xi) = \sum_{\lambda, \mu} V_{\lambda\mu}(R, \xi) Y_{\lambda\mu}(\hat{R}) \equiv V_0(\mathbf{R}) + \Delta V(\mathbf{R}, \xi)$$

- In many practical (and important) situations:

$$\Delta V(\mathbf{R}, \xi) = \sum_{\lambda > 0} \underbrace{\mathcal{F}_\lambda(R)}_{\text{formfactor}} \sum_{\mu} \mathcal{T}_{\lambda, \mu}(\xi) Y_{\lambda\mu}(\hat{R})$$

- DWBA and CC calculations require the coupling potentials

$$\langle I_f M_f | \Delta V(\mathbf{R}, \xi) | I_i M_i \rangle = \sum_{\lambda > 0} \mathcal{F}_\lambda(R) \langle I_f M_f | \mathcal{T}_{\lambda\mu}(\xi) | I_i M_i \rangle Y_{\lambda\mu}(\hat{R})$$

- Wigner-Eckart theorem \rightarrow **reduced matrix elements** (r.m.e.):

$$\langle I_f M_f | \mathcal{T}_{\lambda\mu}(\xi) | I_i M_i \rangle = (2I_f + 1)^{-1/2} \langle I_f M_f | I_i M_i \lambda \mu \rangle \underbrace{\langle I_f || \mathcal{T}_\lambda(\xi) || I_i \rangle_{\text{BM}}}_{\text{r.m.e.}}$$

Coupling potentials for Coulomb excitation

- Transition potentials:

$$\Delta V_{if}(\mathbf{R}) \equiv \langle f; I_f M_f | \Delta V | i; I_i M_i \rangle = \sum_{\lambda > 0, \mu} \frac{4\pi\kappa}{2\lambda + 1} \frac{Z_i e}{R^{\lambda+1}} \langle f; I_f M_f | \mathcal{M}(E\lambda, \mu) | i; I_i M_i \rangle Y_{\lambda\mu}(\hat{R})$$

- Wigner-Eckart theorem \Rightarrow reduced matrix elements (BM convention):

$$\langle f; I_f M_f | \mathcal{M}(E\lambda, \mu) | i; I_i M_i \rangle = (2I_f + 1)^{-1/2} \langle I_i M_i \lambda \mu | I_f M_f \rangle \langle f; I_f || \mathcal{M}(E\lambda, \mu) || i; I_i \rangle$$

- Relation to physical quantities (Coulomb case)

$$B(E\lambda; I_i \rightarrow I_f) = (2I_i + 1)^{-1} |\langle f; I_f || \mathcal{M}(E\lambda, \mu) || i; I_i \rangle|^2 \quad (I_i \neq I_f)$$

$$Q_2 = \sqrt{16\pi/5} (2I + 1)^{-1/2} \langle II20 | II \rangle \langle I || \mathcal{M}(E2) || I \rangle \quad (I_i = I_f \equiv I)$$

DWBA SCATTERING AMPLITUDE FOR A TRANSITION OF MULTIPOLARITY λ :

$$f(\theta)_{iM_i \rightarrow fM_f} = -\frac{\mu}{2\pi\hbar^2} \frac{4\pi\kappa Z_i e}{2\lambda + 1} \langle f; I_f M_f | \mathcal{M}(E\lambda, \mu) | i; I_i M_i \rangle \int d\mathbf{R} \chi_f^{(-)*}(\mathbf{K}_f, \mathbf{R}) \frac{Y_{\lambda\mu}(\hat{R})}{R^{\lambda+1}} \chi_i^{(+)}(\mathbf{K}_i, \mathbf{R})$$

CROSS SECTIONS:

$$\left(\frac{d\sigma}{d\Omega} \right)_{iM_i \rightarrow fM_f} = \frac{K_f}{K_i} |f(\theta)_{iM_i \rightarrow fM_f}|^2$$

UNPOLARIZED CROSS SECTION:

$$\left(\frac{d\sigma}{d\Omega} \right)_{I_i \rightarrow I_f} = \frac{1}{(2I_i + 1)} \frac{K_f}{K_i} \sum_{M_i, M_f} |f(\theta)_{iM_i \rightarrow fM_f}|^2$$

What can we learn measuring Coulomb excitation?

- ☞ For a inelastic excitation $i \rightarrow f$ of multipolarity λ the differential cross section is proportional to the **electric transition probability** $B(E\lambda; I_i \rightarrow I_f)$ because

$$B(E\lambda; i \rightarrow f) = \frac{1}{2I_i + 1} |\langle f I_f | \mathcal{M}(E\lambda) | i I_i \rangle_{\text{BM}}|^2$$



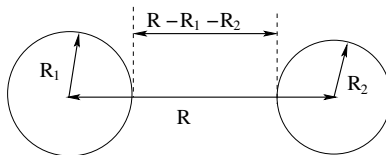
$$\frac{d\sigma}{d\Omega} \propto |\langle f I_f | \mathcal{M}(E\lambda) | i I_i \rangle|^2 \propto B(E\lambda; I_i \rightarrow I_f)$$

- ☞ *If the approximations involved in the derivation of the DWBA approximation are valid, the transition probabilities $B(E\lambda; I_f \rightarrow I_f)$ can be obtained comparing the magnitude of the inelastic cross sections with DWBA calculations.*

- **Central potential:** Typically $U_{\text{nuc}}(\mathbf{R}) = V(R - R_0)$, $R_0 = R_1 + R_2$.

☞ Eg: Woods-Saxon parametrization

$$U_{\text{nuc}}(R) = -\frac{V_0}{1 + \exp\left(\frac{R-R_0}{a_r}\right)} - i\frac{W_0}{1 + \exp\left(\frac{R-R_i}{a_i}\right)}$$




- Deformed radius: $r(\theta, \phi) = R_0 + \sum_{\lambda, \mu} \delta_{\lambda, \mu} Y_{\lambda, \mu}(\theta, \phi)$
- Deformed potential: $V(R - R_0) \rightarrow V(R - r(\theta, \phi)) \equiv V(\mathbf{R}, \xi)$
- Multipole expansion of the potential:

$$V(\mathbf{R}, \xi) = V(R - R_0) - \sum_{\lambda, \mu} \hat{\delta}_{\lambda, \mu} \frac{dV(R - R_0)}{dR} Y_{\lambda, \mu}(\theta, \phi) + \dots$$

($\hat{\delta}_{\lambda}$ =deformation length operators)

- Transition potentials for a multipole λ :

$$V_{if}(\mathbf{R}) \equiv \langle f | V | i \rangle = - \frac{dV(R - R_0)}{dR} \langle f; I_f M_f | \hat{\delta}_{\lambda, \mu} | i; I_i M_i \rangle Y_{\lambda, \mu}(\hat{R})$$

 *The nuclear transition potentials are proportional to the matrix element of the deformation length operator.*

DWBA SCATTERING AMPLITUDE:

$$f(\mathbf{K}', \mathbf{K})_{iM_i \rightarrow fM_f} = -\frac{\mu}{2\pi\hbar^2} \langle f; I_f M_f | \hat{\delta}_{\lambda\mu} | i; I_i M_i \rangle \int d\mathbf{R} \chi_f^{(-)*}(\mathbf{K}', \mathbf{R}) \frac{dV}{dR} Y_{\lambda\mu}(\hat{\mathbf{R}}) \chi_i^{(+)}(\mathbf{K}, \mathbf{R})$$

CROSS SECTIONS:

$$\left(\frac{d\sigma(\theta)}{d\Omega} \right)_{iM_i \rightarrow fM_f} = \frac{K_f}{K_i} \left(\frac{\mu}{2\pi\hbar^2} \right)^2 |\langle f; I_f M_f | \hat{\delta}_{\lambda\mu} | i; I_i M_i \rangle|^2 \times \left| \int d\mathbf{R} \chi_f^{(-)*}(\mathbf{K}', \mathbf{R}) \frac{dV}{dR} Y_{\lambda\mu}(\hat{\mathbf{R}}) \chi_i^{(+)}(\mathbf{K}, \mathbf{R}) \right|^2$$

- ☞ *The differential cross section is proportional to the deformation parameters*
- ☞ *If the approximations are valid, the deformation parameters can be obtained comparing the magnitude of the inelastic cross sections with DWBA calculations.*

In general, we have both Coulomb and nuclear couplings

$$V_{if}(\mathbf{R}) = V_{if}^C(\mathbf{R}) + V_{if}^N(\mathbf{R})$$

❶ **Coulomb excitation** → electric reduced matrix elements

$$V_{if}^C(\mathbf{R}) = \sum_{\lambda > 0} \frac{4\pi\kappa}{2\lambda + 1} \frac{Z_i e}{R^{\lambda+1}} \langle f; I_f M_f | \mathcal{M}(E\lambda, \mu) | i; I_i M_i \rangle Y_{\lambda\mu}(\hat{R})$$

❷ **Nuclear excitation (collective model)** → reduced deformation lengths

$$V_{if}^N(\mathbf{R}) = -\frac{dV_0}{dR} \sum_{\lambda} \langle f; I_f M_f | \hat{\delta}_{\lambda\mu} | i; I_i M_i \rangle Y_{\lambda\mu}(\hat{R})$$

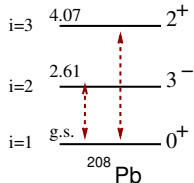
- We expect the **Coulomb** excitation to be more important when:
 - The projectile and/or target charges are large (i.e. large $Z_1 Z_2 \gg 1$)
 - At energies below the Coulomb barrier (where nuclear effects are less important).
 - At very forward angles (large impact parameters).
- If both **Coulomb** and **nuclear** contributions are important the scattering *amplitudes* for both processes should be added:

$$\left(\frac{d\sigma}{d\Omega} \right)_{i \rightarrow f} = \frac{K_f}{K_i} |f_{if}^{\text{coul}} + f_{if}^{\text{nuc}}|^2$$

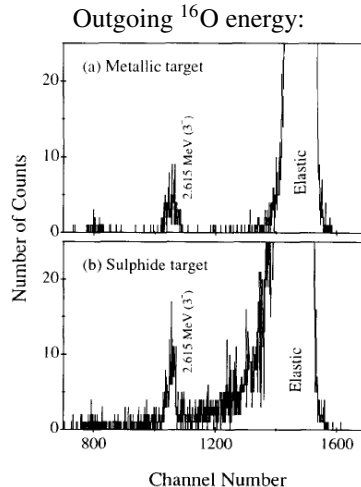
☞ *In this case, interferences effects will appear!*

Inelastic scattering example: collective excitations

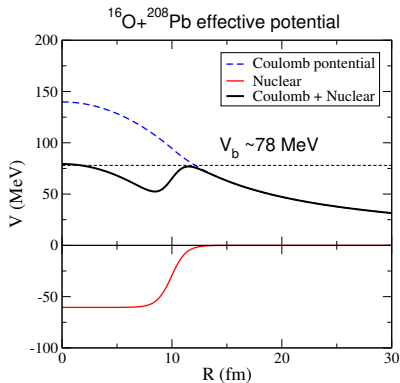
Physical example: $^{16}\text{O} + ^{208}\text{Pb} \rightarrow ^{16}\text{O} + ^{208}\text{Pb}(3^-, 2^+)$



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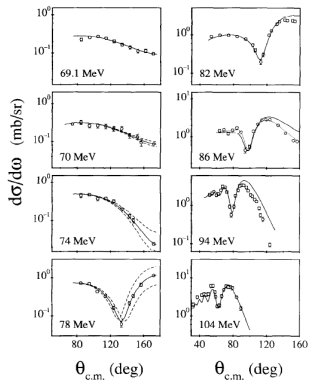
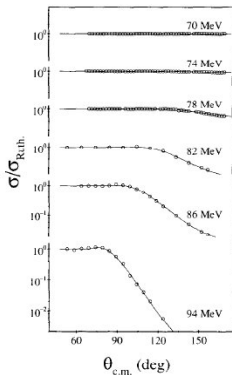
$^{16}\text{O} + ^{208}\text{Pb}$ effective interaction



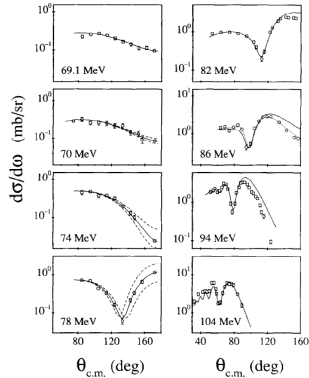
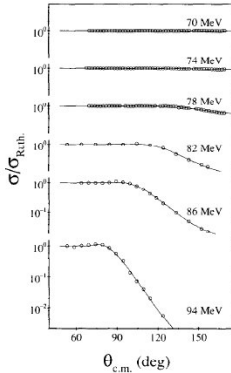
☞ Coulomb barrier:

$$V_{\text{barrier}} \approx \kappa \frac{Z_p Z_t e^2}{1.44(A_p^{1/3} + A_t^{1/3})} \simeq 78 \text{ MeV}$$

Collective excitations: example



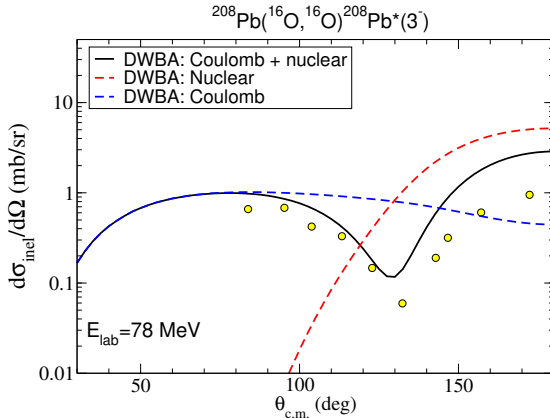
Collective excitations: example



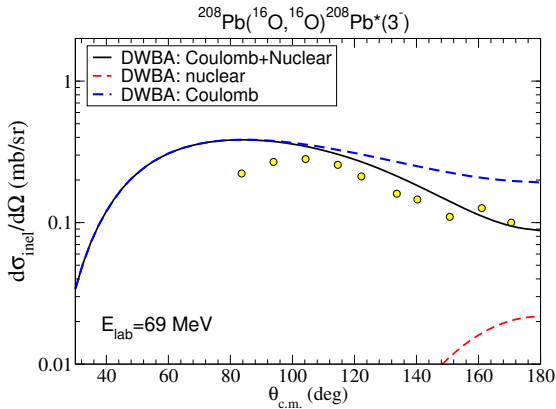
☞ Coulomb barrier:

$$V_{\text{barrier}} = \frac{Z_p Z_t e^2}{R_b} \approx \frac{Z_p Z_t e^2}{1.44(A_p^{1/3} + A_t^{1/3})} \simeq 78 \text{ MeV}$$

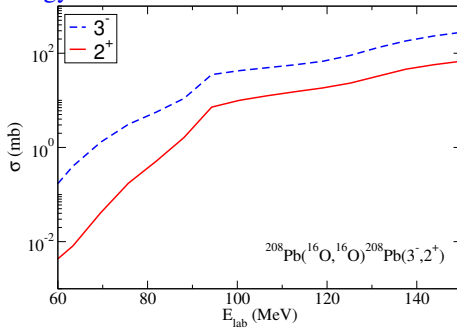
Coulomb and Nuclear excitations can produce constructive or destructive interference:



Below the barrier, the Coulomb excitation is dominant, and the interference is smaller:

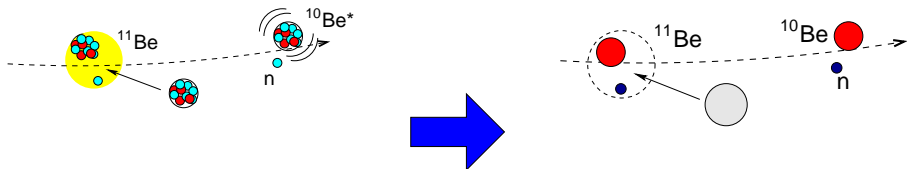


Effect of the incident energy:



Single-particle and cluster excitations

Many-body to few-body reduction



$$\mathcal{V}_{pt} = \sum_{ij} V_{ij}(\mathbf{r}_{ij})$$

$$\mathcal{V}_{pt} = U_{ct}(\mathbf{r}_{ct}) + U_{nt}(\mathbf{r}_{nt})$$

- Effective **three-body** Hamiltonian:

$$H = T_{\mathbf{R}} + h_r(\mathbf{r}) + U_{ct}(\mathbf{r}_{ct}) + U_{nt}(\mathbf{r}_{nt})$$

- $U_{ct}(\mathbf{r}_{ct})$, $U_{nt}(\mathbf{r}_{nt})$ are optical potentials describing fragment-target elastic scattering (eg. target excitation is treated effectively, through absorption)

Inelastic scattering in a few-body model

- Some nuclei allow a description in terms of two or more clusters:
 $d=p+n$, ${}^6\text{Li}=\alpha+d$, ${}^7\text{Li}=\alpha+{}^3\text{H}$.
- Projectile-target interaction:

$$V(\mathbf{R}, \xi) \equiv V(\mathbf{R}, \mathbf{r}) = U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2)$$

- Transition potentials:

$$V_{n,n'}(\mathbf{R}) = \int d\mathbf{r} \phi_n^*(\mathbf{r}) [U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2)] \phi_{n'}(\mathbf{r})$$

Inelastic scattering in a few-body model

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- Transition potentials:

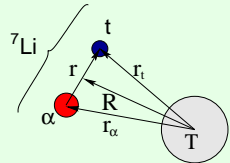
$$V_{n,n'}(\mathbf{R}) = \int d\mathbf{r} \phi_n^*(\mathbf{r}) [U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2)] \phi_{n'}(\mathbf{r})$$

Example: ${}^7\text{Li}=\alpha+t$

$$\mathbf{r}_\alpha = \mathbf{R} - \frac{m_t}{m_\alpha + m_t} \mathbf{r}; \quad \mathbf{r}_t = \mathbf{R} + \frac{m_\alpha}{m_\alpha + m_t} \mathbf{r}$$

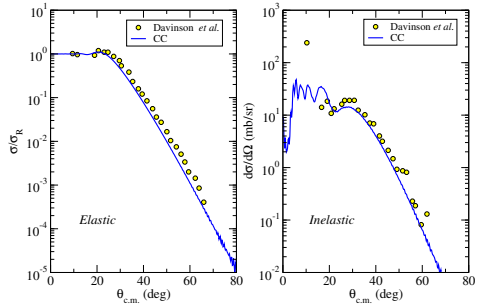
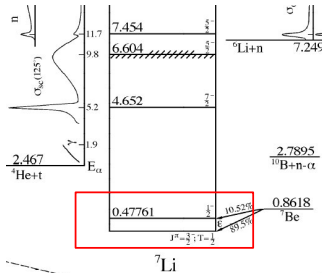
Internal states: (two-body cluster model)

$$[T_{\mathbf{r}} + V_{\alpha-t}(\mathbf{r}) - \varepsilon_n] \phi_n(\mathbf{r}) = 0$$



Example: ${}^7\text{Li}(\alpha+t) + {}^{208}\text{Pb}$ at 68 MeV

⇒ CC calculation with 2 channels ($3/2^-$, $1/2^-$):

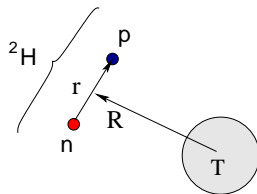
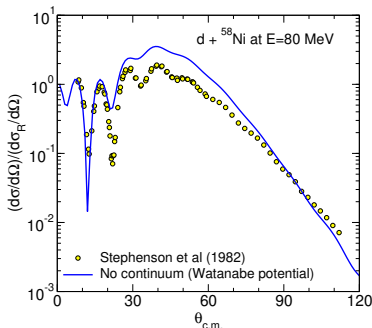


Data from Davinson et al, Phys. Lett. 139B (1984) 150

⇒ Fresco input available at <https://github.com/ammoro/IUAC>

Example: Three-body calculation (p+n+ ^{58}Ni) with Watanabe potential:

$$V_{dt}(\mathbf{R}) = \int d\mathbf{r} \phi_{gs}^*(\mathbf{r}) \{V_{pt}(\mathbf{r}_{pt}) + V_{nt}(\mathbf{r}_{nt})\} \phi_{gs}(\mathbf{r})$$



☞ *Three-body calculations omitting breakup channels fail to describe the experimental data.*