Inelastic scattering: the Coupled-Channels and DWBA methods

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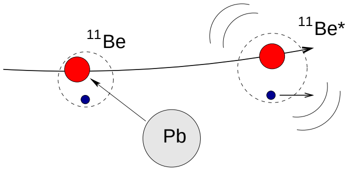
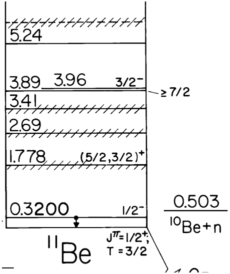
Material available at: `https://github.com/ammoro/RAON`

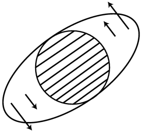
Inelastic scattering to bound states

- Nuclei are not inert or *frozen* objects; they do have an internal structure of protons and neutrons that can be modified (excited) during the collision.
- Quantum systems exhibit, in general, an energy spectrum with bound and unbound levels.

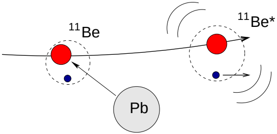




1. COLLECTIVE: Involve a collective motion of several nucleons which can be interpreted macroscopically as rotations or surface vibrations of the nucleus.



2. FEW-BODY/SINGLE-PARTICLE: Involve the excitation of a nucleon or cluster.



- **Goal:** Determine the differential cross section for an inelastic process of the form: $a + A \to a + A^*$

- The scattering wavefunction $\Psi^{(+)}_{\mathbf{K}_0}(\mathbf{R}, \xi)$ must contain, besides the elastic scattering component, additional components associated to excited states.

- Asymptotically:

$$\Psi^{(+)}_{\mathbf{K}_0}(\mathbf{R}, \xi) \xrightarrow{R\gg} \underbrace{e^{i\mathbf{K}_0\cdot\mathbf{R}}\phi_0(\xi)}_{\text{incident}} + \underbrace{f_{0,0}(\theta)\frac{e^{iK_0R}}{R}\phi_0(\xi)}_{\text{elastic}} + \underbrace{\sum_{n>0} f_{n,0}(\theta)\frac{e^{iK_nR}}{R}\phi_n(\xi)}_{\text{inelastic}}$$

where $\phi_n(\xi)$ are internal wfs of the nuclei being excited in some model.

- Cross sections:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right)_{0\to n} = \frac{K_n}{K_0}|f_{n,0}(\theta)|^2$$

The coupled-channels method

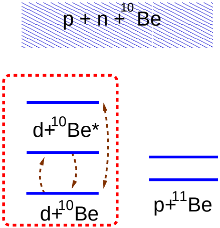
We need to incorporate explicitly in the Hamiltonian the internal structure of the nucleus being excited (e.g. projectile).

$$\boxed{H = T_R + h(\xi) + V(\mathbf{R}, \xi)}$$

- T_R: Kinetic energy for projectile-target relative motion.
- $\{\xi\}$: Internal degrees of freedom of the projectile (depend on the model).
- $V(\mathbf{R}, \xi)$: Projectile-target interaction.
- $h(\xi)$: Internal Hamiltonian of the projectile.

$$\boxed{h(\xi)\phi_n(\xi) = \varepsilon_n \phi_n(\xi)}$$

- $\phi_n(\xi)$: internal states of the projectile.



☞ The modelspace is composed by ground states (elastic channel) and some excited states (inelastic scattering)

Boundary conditions for scattering wavefunction:

$$\Psi^{(+)}_{\mathbf{K}_0}(\mathbf{R},\xi) \xrightarrow{R\gg} \underbrace{e^{i\mathbf{K}_0\cdot\mathbf{R}}\phi_0(\xi)}_{\text{incident}} + \underbrace{f_{0,0}(\theta)\frac{e^{iK_0R}}{R}\phi_0(\xi)}_{\text{elastic}} + \underbrace{\sum_{n>0} f_{n,0}(\theta)\frac{e^{iK_nR}}{R}\phi_n(\xi)}_{\text{inelastic}}$$

Cross sections:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right)_{0\to n} = \frac{K_n}{K_0}|f_{n,0}(\theta)|^2 \qquad f_{n,0}(\theta) = \text{scattering amplitude}$$

The total wave function is expanded in a subset of internal states representing the adopted modelspace:

$$\boxed{\Psi_{\rm model}(\mathbf{R},\xi) = \phi_0(\xi)\chi_0(\mathbf{K}_0,\mathbf{R}) + \sum_{n>0}\phi_n(\xi)\chi_n(\mathbf{K}_n,\mathbf{R})}$$

and impose the boundary conditions for the (unknown) $\chi_n(\mathbf{R})$:

$$\chi_0^{(+)}(\mathbf{K}_0,\mathbf{R}) \to e^{i\mathbf{K}_0\cdot\mathbf{R}} + f_{0,0}(\theta)\frac{e^{iK_0R}}{R} \qquad \text{for n=0 (elastic)}$$

$$\chi_n^{(+)}(\mathbf{K}_n,\mathbf{R}) \to f_{n,0}(\theta)\frac{e^{iK_nR}}{R} \qquad \text{for n>0 (non-elastic)}$$

Calculation of $\chi_n^{(+)}(\mathbf{R})$: the coupled equations

- The model wavefunction must satisfy the Schrödinger equation:

$$[H - E]\Psi^{(+)}_{\text{model}}(\mathbf{R}, \xi) = 0$$

- Multiply on the left by each $\phi_n^*(\xi)$, and integrate over $\xi \Rightarrow$ coupled channels equations for $\{\chi_n(\mathbf{R})\}$:

$$\boxed{[E - \varepsilon_n - T_R - V_{n,n}(\mathbf{R})]\chi_n(\mathbf{R}) = \sum_{n' \neq n} V_{n,n'}(\mathbf{R})\chi_{n'}(\mathbf{R})}$$

- The structure information is embedded in the coupling potentials:

$$\boxed{V_{n,n'}(\mathbf{R}) = \int d\xi \phi^*_{n'}(\xi) V(\mathbf{R}, \xi)\phi_n(\xi)}$$

☞ *$\phi_n(\xi)$ will depend on the assumed structure model (collective, few-body, etc).*

Optical Model

- The Hamiltonian:
  $H = T_R + V(\mathbf{R})$

- Internal states: Just $\phi_0(\xi)$

- Model wavefunction:
  $\Psi_{\rm mod}(\mathbf{R}, \xi) \equiv \chi_0(\mathbf{K}, \mathbf{R})\phi_0(\xi)$

- Schrödinger equation:
  $[H - E]\chi_0(\mathbf{K}, \mathbf{R}) = 0$

Optical Model

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Coupled-channels method

- The Hamiltonian:
  $H = T_R + h(\xi) + V(\mathbf{R},\xi)$
- Internal states:
  $h(\xi)\phi_n(\xi) = \varepsilon_n\phi_n(\xi)$
- Model wavefunction:
  $\Psi_{\rm model}(\mathbf{R},\xi) = \phi_0(\xi)\chi_0(\mathbf{K},\mathbf{R}) + \sum_{n>0}\phi_n(\xi)\chi_n(\mathbf{K},\mathbf{R})$
- Schrödinger equation:

$$[H-E]\Psi_{\rm model}(\mathbf{R},\xi) = 0$$

$$\Downarrow$$

$$[E-\varepsilon_n - T_R - V_{n,n}(\mathbf{R})]\chi_n(\mathbf{K},\mathbf{R}) = \sum_{n'\neq n} V_{n,n'}(\mathbf{R})\chi_{n'}(\mathbf{K},\mathbf{R})$$

- Assume that we can write the p-t interaction as: $V(\mathbf{R},\xi) = V_0(R) + \Delta V(\mathbf{R},\xi)$
- Use central $V_0(R)$ part to calculate the (distorted) waves for p-t relative motion:

$$\left[\hat{T}_{\mathbf{R}} + V_0(R) - E_i\right]\chi_i^{(+)}(\mathbf{K}_i,\mathbf{R}) = 0 \qquad (E_i = \text{c.m. energy})$$

$$\left[\hat{T}_{\mathbf{R}} + V_0(R) - E_f\right]\chi_f^{(+)}(\mathbf{K}_f,\mathbf{R}) = 0 \qquad (E_f = E_i + Q = E_i - E_x)$$

- In first order of $\Delta V(\mathbf{R},\xi)$ (DWBA) :

$$\boxed{f_{i\to f}^{\text{DWBA}}(\theta) = -\frac{\mu}{2\pi\hbar^2}\int \chi_f^{(-)*}(\mathbf{K}_f,\mathbf{R})\,\Delta V_{if}(\mathbf{R})\,\chi_i^{(+)}(\mathbf{K}_i,\mathbf{R})\,d\mathbf{R}}$$

with the transition potential:

$$\boxed{\Delta V_{if}(\mathbf{R}) \equiv \int \phi_f^*(\xi)\,\Delta V(\mathbf{R},\xi)\,\phi_i(\xi)\,d\xi}$$

Multipole expansion of the interaction: reduced matrix elements

- In actual calculations, the internal states will have definite spin/parity:

$$\phi_i(\xi) = |I_i M_i\rangle \qquad \text{and} \qquad \phi_f(\xi) = |I_f M_f\rangle$$

- The projectile-target interaction can be expanded in multipoles:

$$V(\mathbf{R}, \xi) = \sum_{\lambda,\mu} V_{\lambda\mu}(R, \xi) Y_{\lambda\mu}(\hat{R}) \equiv V_0(\mathbf{R}) + \Delta V(\mathbf{R}, \xi)$$

- In many practical (and important) situations:

$$\Delta V(\mathbf{R}, \xi) = \sum_{\lambda>0} \underbrace{\mathcal{F}_\lambda(R)}_{\text{formfactor}} \sum_{\mu} \underbrace{\mathcal{T}_{\lambda,\mu}(\xi)}_{\text{structure}} Y_{\lambda\mu}(\hat{R})$$

- DWBA and CC calculations require the coupling potentials

$$\langle I_f M_f | \Delta V(\mathbf{R}, \xi) | I_i M_i \rangle = \sum_{\lambda>0} \mathcal{F}_\lambda(R) \langle I_f M_f | \mathcal{T}_{\lambda\mu}(\xi) | I_i M_i \rangle Y_{\lambda\mu}(\hat{R})$$

- Wigner-Eckart theorem → reduced matrix elements (r.m.e.)*:

$$\boxed{\langle I_f M_f | \mathcal{T}_{\lambda\mu}(\xi) | I_i M_i \rangle = (2I_f + 1)^{-1/2} \langle I_f M_f | I_i M_i \lambda\mu \rangle \underbrace{\langle I_f \| \mathcal{T}_\lambda(\xi) \| I_i \rangle_{\text{BM}}}_{\text{r.m.e}}}$$

Coupling potentials for Coulomb excitation

- Transition potentials:

$$\Delta V_{if}(\mathbf{R}) \equiv \langle f; I_f M_f | \Delta V | i; I_i M_i \rangle = \sum_{\lambda>0,\mu} \frac{4\pi\kappa}{2\lambda+1} \frac{Z_t e}{R^{\lambda+1}} \langle f; I_f M_f | \mathcal{M}(E\lambda,\mu) | i; I_i M_i \rangle Y_{\lambda\mu}(\hat{R})$$

- Wigner-Eckart theorem⇒ reduced matrix elements (BM convention):

$$\langle f; I_f M_f | \mathcal{M}(E\lambda,\mu) | i; I_i M_i \rangle = (2I_f+1)^{-1/2} \langle I_i M_i \lambda\mu | I_f M_f \rangle \langle f; I_f \| \mathcal{M}(E\lambda,\mu) \| i; I_i \rangle$$

- Relation to physical quantities (Coulomb case)

$$B(E\lambda; I_i \to I_f) = (2I_i+1)^{-1} \, |\langle f; I_f \| \mathcal{M}(E\lambda,\mu) \| i; I_i \rangle|^2 \qquad (I_i \neq I_f)$$

$$Q_2 = \sqrt{16\pi/5}\,(2I+1)^{-1/2}\,\langle II20|II\rangle\,\langle I \| M(E2 \| I \rangle \qquad (I_i = I_f \equiv I)$$

DWBA scattering amplitude for a transition of multipolarity λ:

$$\boxed{f(\theta)_{iM_i\to fM_f} = -\frac{\mu}{2\pi\hbar^2}\frac{4\pi\kappa Z_t e}{2\lambda+1}\langle f; I_f M_f|\mathcal{M}(E\lambda,\mu)|i; I_i M_i\rangle \int d\mathbf{R}\,\chi_f^{(-)*}(\mathbf{K}_f,\mathbf{R})\frac{Y_{\lambda\mu}(\hat{R})}{R^{\lambda+1}}\chi_i^{(+)}(\mathbf{K}_i,\mathbf{R})}$$

Cross sections:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{iM_i\to fM_f} = \frac{K_f}{K_i}\left|f(\theta)_{iM_i\to fM_f}\right|^2}$$

Unpolarized cross section:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{I_i\to I_f} = \frac{1}{(2I_i+1)}\frac{K_f}{K_i}\sum_{M_i,M_f}\left|f(\theta)_{iM_i\to fM_f}\right|^2}$$

- For a inelastic excitation $i \to f$ of multipolarity λ the differential cross section is proportional to the electric transition probability $B(E\lambda; I_i \to I_f)$ because

$$\boxed{B(E\lambda; i \to f) = \frac{1}{2I_i + 1} |\langle f\ I_f \| \mathcal{M}(E\lambda) \| i\ I_i \rangle_{\rm BM}|^2}$$

$$\Downarrow$$

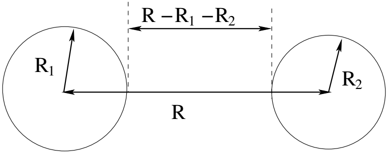
$$\boxed{\frac{d\sigma}{d\Omega} \propto |\langle f\ I_f \| \mathcal{M}(E\lambda) \| i\ I_i \rangle|^2 \propto B(E\lambda; I_i \to I_f)}$$

- *If the approximations involved in the derivation of the DWBA approximation are valid, the transition probabilities $B(E\lambda; I_f \to I_f)$ can be obtained comparing the magnitude of the inelastic cross sections with DWBA calculations.*

- Central potential: Typically $U_{\rm nuc}(\mathbf{R}) = V(R - R_0), \quad R_0 = R_1 + R_2.$

☞ Eg: Woods-Saxon parametrization

$$U_{\rm nuc}(R) = -\frac{V_0}{1+\exp\left(\frac{R-R_0}{a_r}\right)} - i\frac{W_0}{1+\exp\left(\frac{R-R_i}{a_i}\right)}$$



Deformed potential

- Deformed radius: $r(\theta,\phi) = R_0 + \sum_{\lambda,\mu} \delta_{\lambda,\mu} Y_{\lambda,\mu}(\theta,\phi)$
- Deformed potential: $V(R-R_0) \rightarrow V(R-r(\theta,\phi)) \equiv V(\mathbf{R},\xi)$
- Multipole expansion of the potential:

$$V(\mathbf{R},\xi) = V(R-R_0) - \sum_{\lambda,\mu} \hat{\delta}_{\lambda\mu} \frac{dV(R-R_0)}{dR} Y_{\lambda\mu}(\theta,\phi) + \ldots$$

($\hat{\delta}_\lambda$=deformation length operators)

- Transition potentials for a multipole λ:

$$\boxed{V_{if}(\mathbf{R}) \equiv \langle f|V|i\rangle = -\frac{dV(R-R_0)}{dR} \langle f; I_f M_f | \hat{\delta}_{\lambda\mu} | i; I_i M_i \rangle Y_{\lambda\mu}(\hat{R})}$$

☞*The nuclear transition potentials are proportional to the matrix element of the deformation length operator*.

DWBA scattering amplitude:

$$\boxed{f(\mathbf{K}',\mathbf{K})_{iM_i\to fM_f} = -\frac{\mu}{2\pi\hbar^2}\langle f; I_f M_f|\hat{\delta}_{\lambda\mu}|i; I_i M_i\rangle \int d\mathbf{R}\,\chi_f^{(-)*}(\mathbf{K}',\mathbf{R})\frac{dV}{dR}Y_{\lambda\mu}(\hat{\mathbf{R}})\chi_i^{(+)}(\mathbf{K},\mathbf{R})}$$

Cross sections:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right)_{iM_i\to fM_f} = \frac{K_f}{K_i}\left(\frac{\mu}{2\pi\hbar^2}\right)^2 \left|\langle f; I_f M_f|\hat{\delta}_{\lambda\mu}|i; I_i M_i\rangle\right|^2$$
$$\times\left|\int d\mathbf{R}\,\chi_f^{(-)*}(\mathbf{K}',\mathbf{R})\frac{dV}{dR}Y_{\lambda\mu}(\hat{\mathbf{R}})\chi_i^{(+)}(\mathbf{K},\mathbf{R})\right|^2$$

- ☞ *The differential cross section is proportional to the deformation parameters*
- ☞ *If the approximations are valid, the deformation parameters can be obtained comparing the magnitude of the inelastic cross sections with DWBA calculations.*

In general, we have both Coulomb and nuclear couplings

$$\boxed{V_{if}(\mathbf{R}) = V^C_{if}(\mathbf{R}) + V^N_{if}(\mathbf{R})}$$

1. Coulomb excitation → electric reduced matrix elements

$$V^C_{if}(\mathbf{R}) = \sum_{\lambda>0} \frac{4\pi\kappa}{2\lambda+1} \frac{Z_t e}{R^{\lambda+1}} \langle f; I_f M_f | \mathcal{M}(E\lambda,\mu) | i; I_i M_i \rangle Y_{\lambda\mu}(\hat{R})$$

2. Nuclear excitation (collective model) → reduced deformation lengths

$$V^N_{if}(\mathbf{R}) = -\frac{dV_0}{dR} \sum_{\lambda} \langle f; I_f M_f | \hat{\delta}_{\lambda\mu} | i; I_i M_i \rangle Y_{\lambda\mu}(\hat{R})$$

- We expect the Coulomb excitation to be more important when:
  - The projectile and/or target charges are large (i.e. large $Z_1 Z_2 \gg 1$)
  - At energies below the Coulomb barrier (where nuclear effects are less important).
  - At very forward angles (large impact parameters).

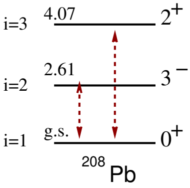
- If both Coulomb and nuclear contributions are important the scattering *amplitudes* for both processes should be added:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{i\to f} = \frac{K_f}{K_i}\left|f_{if}^{\mathrm{coul}} + f_{if}^{\mathrm{nucl}}\right|^2}$$

☞ *In this case, interferences effects will appear!*

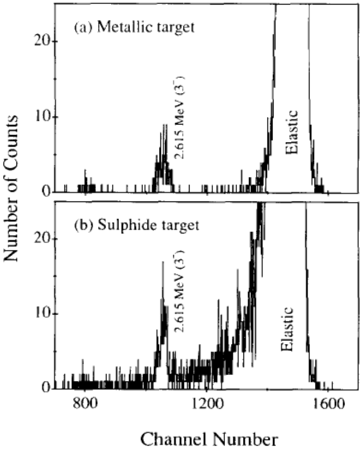
Inelastic scattering example: collective excitations

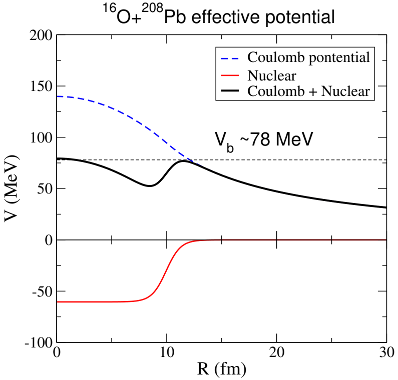
Physical example: ^{16}O + ^{208}Pb → ^{16}O+ ^{208}Pb(3^-,2^+)



Nucl. Phys. A517 (1990) 193

Outgoing ^{16}O energy:

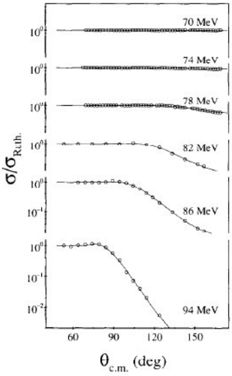


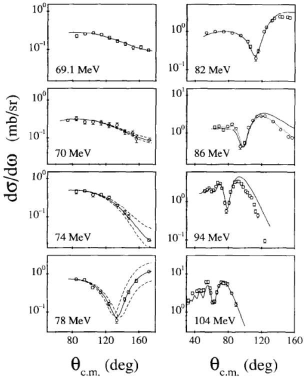


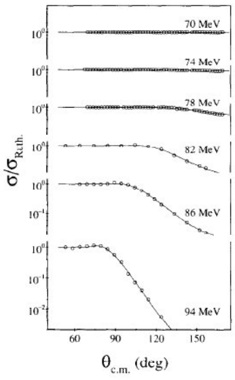
☞ Coulomb barrier:

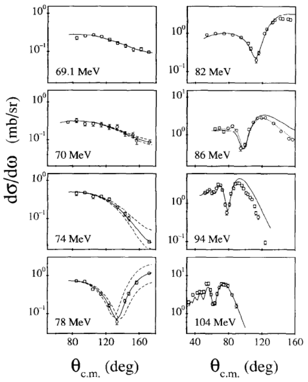
$$V_{\text{barrier}} \approx \kappa \frac{Z_p Z_t e^2}{1.44(A_p^{1/3} + A_t^{1/3})} \simeq 78 \text{ MeV}$$

Collective excitations: example





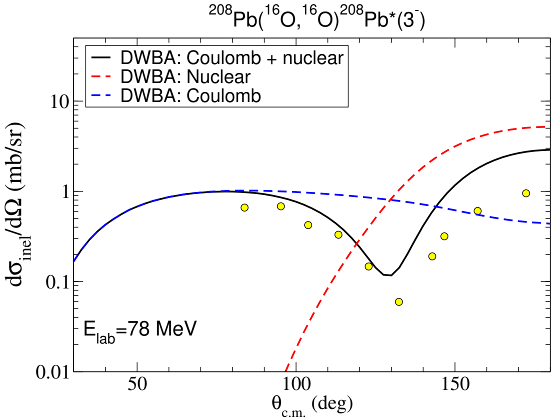




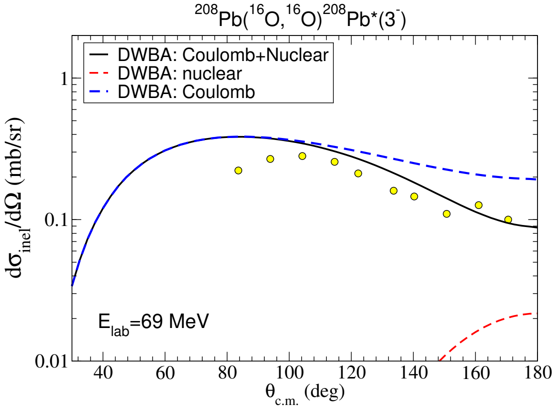
☞ Coulomb barrier:

$$V_{\text{barrier}} = \frac{Z_p Z_t e^2}{R_b} \approx \frac{Z_p Z_t e^2}{1.44(A_p^{1/3} + A_t^{1/3})} \simeq 78 \text{ MeV}$$

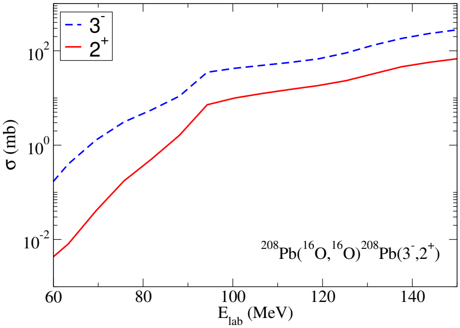
Coulomb and Nuclear excitations can produce constructive or destructive interference:



Below the barrier, the Coulomb excitation is dominant, and the interference is smaller:

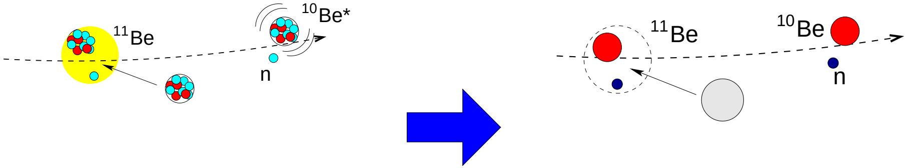


Effect of the incident energy:



Single-particle and cluster excitations

Many-body to few-body reduction



$$\mathcal{V}_{pt} = \sum_{ij} V_{ij}(\mathbf{r}_{ij})$$

$$\mathcal{V}_{pt} = U_{ct}(\mathbf{r}_{ct}) + U_{nt}(\mathbf{r}_{nt})$$

- Effective three-body Hamiltonian:

$$\boxed{H = T_{\mathbf{R}} + h_r(\mathbf{r}) + U_{ct}(\mathbf{r}_{ct}) + U_{nt}(\mathbf{r}_{nt})}$$

- $U_{ct}(\mathbf{r}_{ct})$, $U_{nt}(\mathbf{r}_{nt})$ are optical potentials describing fragment-target elastic scattering (eg. target excitation is treated effectively, through absorption)

- Some nuclei allow a description in terms of two or more clusters: d=p+n, ^{6}Li=α+d, ^{7}Li=α+^{3}H.
- Projectile-target interaction:

$$V(\mathbf{R}, \xi) \equiv V(\mathbf{R}, \mathbf{r}) = U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2)$$

- Transition potentials:

$$\boxed{V_{n,n'}(\mathbf{R}) = \int d\mathbf{r} \phi_n^*(\mathbf{r}) \left[U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2)\right] \phi_{n'}(\mathbf{r})}$$

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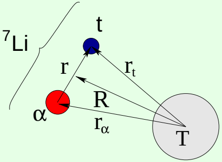
- Transition potentials:

$$\boxed{V_{n,n'}(\mathbf{R}) = \int d\mathbf{r}\phi_n^*(\mathbf{r})\left[U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2)\right]\phi_{n'}(\mathbf{r})}$$

Example: ^{7}Li=α+t

$$\mathbf{r}_\alpha = \mathbf{R} - \frac{m_t}{m_\alpha + m_t}\mathbf{r}; \quad \mathbf{r}_t = \mathbf{R} + \frac{m_\alpha}{m_\alpha + m_t}\mathbf{r}$$

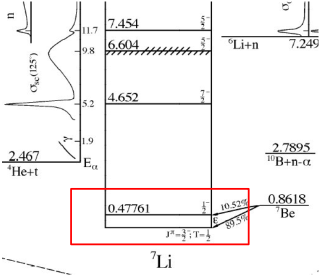


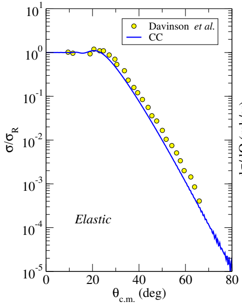
Internal states: (two-body cluster model)

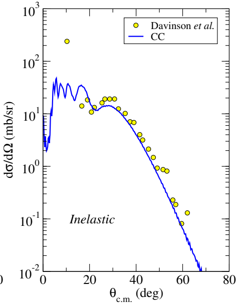
$$[T_\mathbf{r} + V_{\alpha-t}(\mathbf{r}) - \varepsilon_n]\phi_n(\mathbf{r}) = 0$$

Example: ^{7}Li(α+t) +^{208}Pb at 68 MeV

➪ CC calculation with 2 channels ($3/2^-$, $1/2^-$):





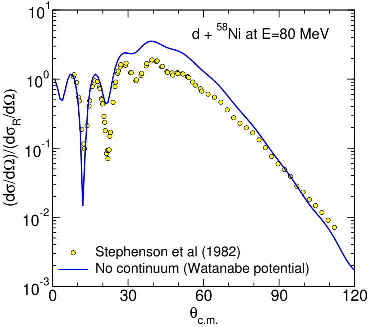


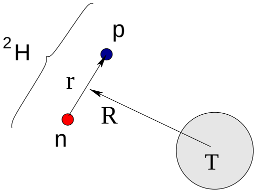
Data from Davinson et al, Phys. Lett. 139B (1984) 150)

☞ Fresco input available at `https://github.com/ammoro/RAON`

Example: Three-body calculation (p+n+^{58}Ni) with Watanabe potential:

$$V_{dt}(\mathbf{R}) = \int d\mathbf{r}\phi^*_{\rm gs}(\mathbf{r}) \left\{V_{pt}(\mathbf{r}_{pt}) + V_{nt}(\mathbf{r}_{nt})\right\} \phi_{\rm gs}(\mathbf{r})$$





☞*Three-body calculations omitting breakup channels fail to describe the experimental data.*