Gödelian Explorations

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Preface

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1 For Editors' Use

This is a testing page left for editorial purposes and will not be included in the publication.

1.1 TODO: Chapter Proposal

1.1.1 Yutong: What is a … *motivic model*?

(Denef & Loeser 1999) says

$$\chi\left(\left\{\varphi := x \neq 0 \wedge (\exists y)\left(x = y^2\right)\right\}\right) = \frac{1}{2}([\mathbb{L}] - \mathbb{1})$$
$$\varphi(\mathbb{F}) \leftrightarrow \zeta\left(\frac{1}{2}([\mathbb{L}] - \mathbb{1})\right)$$

for $\mathbb{F}$ with a sufficiently large characteristics from compactness, as

> … [over] finite fields of characteristic > 2, half of the units are squares …
>
> (— Cluckers & Nicaise & Sebag 2011: 23)

1.1.2 Yutong: What is a … *motivic path integral*?

(Brown 2017) says

$$K(x, y) = \int \exp(iS(\phi))\, D\phi.$$

1.2 Format Cheatsheet

[Quarto's official docs]

Denef, J. & Loeser, F. 1999, October 22. Definable sets, motives and p-adic integrals. (http://arxiv.org/abs/math/9910107) (Accessed 2023-11-18)

Cluckers, Raf & Nicaise, Johannes & Sebag, Julien (eds.). 2011. *[I] Motivic integration and its interactions with model theory and non-Archimedean geometry* Vol. 1. Cambridge ; New York: Cambridge University Press.

Brown, Francis. 2017, February 1. Feynman Amplitudes and Cosmic Galois group. (http://arxiv.org/abs/1512.06409) (Accessed 2023-11-16)

1.2.1 Quiver CD

1.2.1.1 SVG

$$X \times T = \mathfrak{X}_{\overline{\eta}} \qquad\qquad T = \mathfrak{X}_R$$

$$K \boxtimes L = \mathbb{1}_{\dots} \qquad\qquad L = \mathbb{1}_T$$

$$\begin{tikzcd}
X \times_S T \arrow[r, "\alpha"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "p"] \\
X \arrow[r, "\beta"'] & S
\end{tikzcd}$$

$$\begin{aligned}
X &= \overline{\eta} \\
K &= \mathbb{1}_X \\
\mathrm{LHS} &= K \otimes \beta^* p_* L \\
&= H^*(\mathfrak{X})_{\overline{\eta}} \\
\mathrm{RHS} &= q_*(K \boxtimes L) \\
&= H^*(\mathfrak{X}_{\overline{\eta}})
\end{aligned}$$

$$p_* L$$

1.2.1.2 iframe

1.2.2 Indented Text

> The other metaphorical analogue to Gödel's Theorem which I find provocative suggests that ultimately, we cannot understand our own minds/brains.... All the limitative theorems of mathematics and the theory of computation suggest that once your ability to represent your own structure has reached a certain critical point, that is the kiss of death: it guarantees that you can never represent yourself totally.

1.2.3 Color CSS

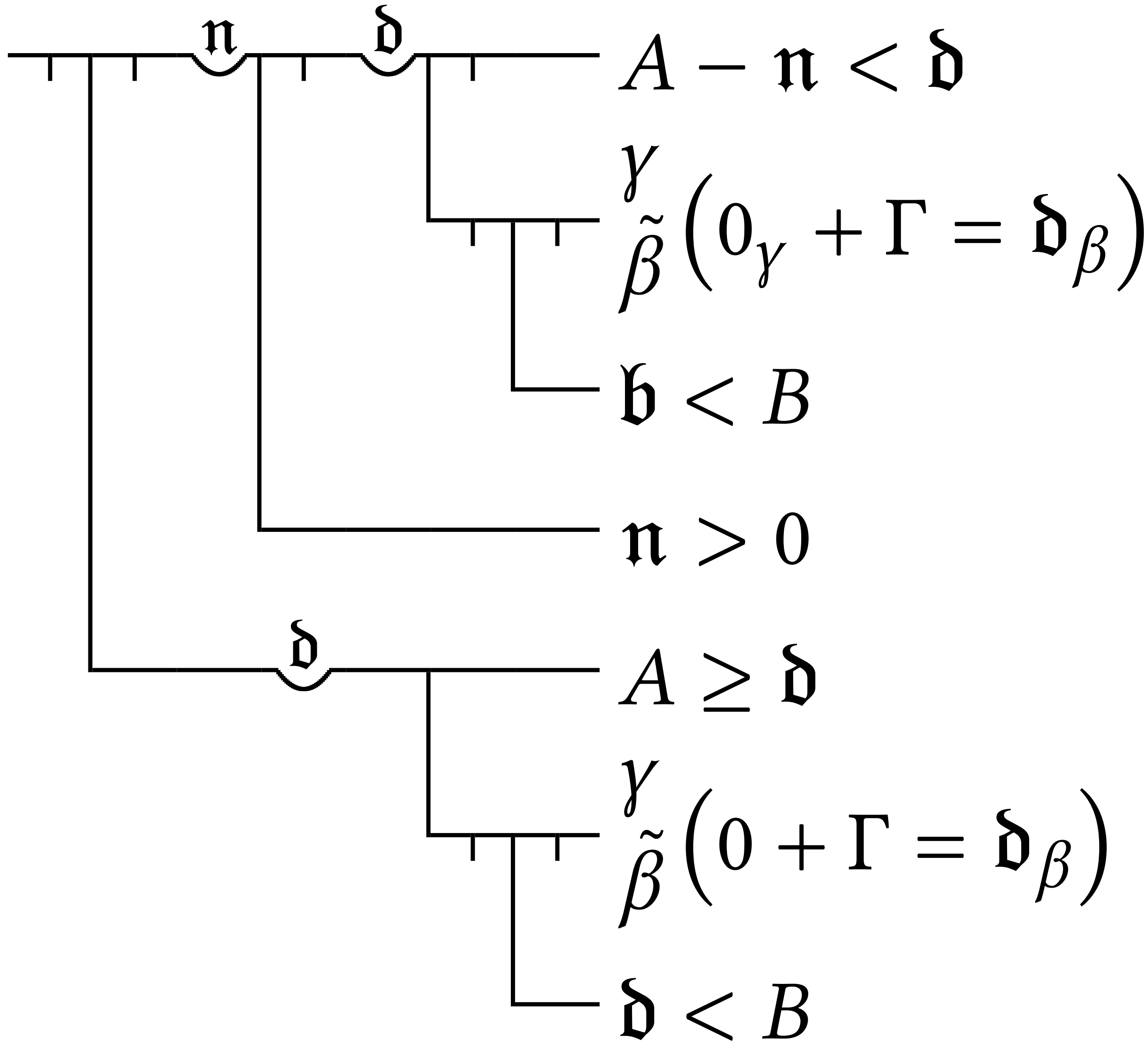
Prof. Mar's favorite color

1.2.4 TODO: Conditional

Some text in PDF.

Will only appear in PDF.

1.2.5 Frege



1.2.6 Tufte

- twoside class
- tufte.qmd
- auto break sidenotes

1.2.7 Graphviz with MathJax

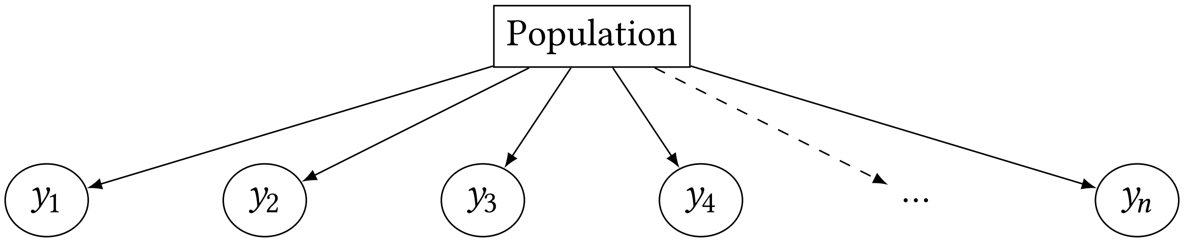
Actually, this is horrible as it requires manual coding of size and position.

- HTML: Graphviz + MathJax
- TeX

```
//| code-fold: true

d3 = require("d3@7", "d3-graphviz@2");
mathjaxBlob = (await fetch('https://cdn.jsdelivr.net/npm/mathjax@3/es5/tex-svg.js')).blob();
mj = {
  delete window.MathJax
  delete window.ContextMenu
  window.MathJax = {
    loader: {load: ['output/svg']},
  }
  const MathJax = await import(URL.createObjectURL(await mathjaxBlob))
    .then(() => window.MathJax );
  return MathJax;
};

seir_graph = {
  let graph = d3.create('div').style('width', `${width}px`);

  // Here's the source code describing the graph to graphviz.
  // Note that nodes and edge labels contain LaTeX code that
  // will be passed to MathJax. I guess it gets piped through
  // a couple of things; hence, the double escape leading to
  // quadruple backslashes \\\\.
  let source_code = `digraph {
      S [pos="0,0!"]
      E [pos="2.7,0!"]
      I_1 [pos="4,0!"]
      I_2 [pos="6,0.5!"]
      I_3 [pos="8,0!"]
      D [pos="10,0!"]
      R [pos="6,-1.5!"]
      S -> E [label="\\\\beta_1 I_1 S + \\\\beta_2 I_2 S + \\\\beta_3 I_3"]
      E -> I_1 [label="\\\\alpha E"]
      I_1 -> I_2 [label="p_1 I_1"]
      I_2 -> I_3 [label="p_2 I_2"]
      I_3-> D [label="\\\\mu I_3"]
      I_1 -> R [label="\\\\gamma_1 I_1"]
      I_2 -> R [label="\\\\gamma_2 I_2"]
      I_3 -> R [label="\\\\gamma_3 I_3"]
    }`;
  d3.graphviz(graph.node())
    .width(width)
    .fit(true) // Doesn't quite work; see transform in penultimate line.
    .zoom(false) // Re-transform for fit breaks the zoom.
    .engine('neato')
    .renderDot(source_code);

  // The image is completely contained in a top level group,
  // which we're going to manipulate
  let main_group = graph.select('g');

  // Don't really want the title
  main_group.select('title').remove();

  // Typeset the nodes
  main_group.selectAll('.node').each(function(e, i) {
    let text = d3.select(this).select('text');
```

```
    if (text.node() != null) {
      let x = parseFloat(text.attr('x'));
      let y = parseFloat(text.attr('y'));
      let tex_group = main_group
        .append('g')
        .attr('transform', `translate(${x - 8} ${y - 10})`)
        .append(() =>
          mj.tex2svg(String.raw`${text.text()}`).querySelector("svg")
        );
      text.remove();
    }
  });

  // Placement of the typeset edge labels is a bit trickier. The following
  // list of shifts adjusts the placement of the labels from the location
  // specified by graphviz.
  let shifts = [
    [60, -30],
    [-9, -27],
    [-37, -15],
    [10, 5],
    [-20, -10],
    [10, 0],
    [-25, -12],
    [44, 5]
  ];
  main_group.selectAll('.edge').each(function(e, i) {
    let text = d3.select(this).select('text');
    if (text.node() != null) {
      let x = parseFloat(text.attr('x'));
      let y = parseFloat(text.attr('y'));
      let tex_group = main_group
        .append('g')
        .attr(
          'transform',
          `translate(${x + shifts[i][0]} ${y + shifts[i][1]}) scale(0.75)`
        )
        .append(() =>
          mj.tex2svg(String.raw`${text.text()}`).querySelector("svg")
        );
      text.remove();
    }
  });

  // There's far more space to the left of the graph than I'd expect;
  // I guess the reason is that the first, pre-shifted edge label extends
  // quite far to the left. A hacky fix is to redefine the main transform
  // to fit it a bit better. Unfortunately, this breaks zoom.
  main_group.attr('transform', `translate(-130, 204) scale(1.15)`);
  return graph.node();
}
```

1.2.7.1 Let's try the TikZJax or just plain tikz



1.2.8 TODO: mathnote

```
\def\mathnote#1{%
  \tag*{\rlap{\hspace\marginparsep{\parbox[t]{\marginparwidth}{\footnotesize#1}}}}%
}
\def\mathnotes#1{%
  \tag*{\rlap{\hspace\marginparsep\smash{\parbox[t]{\marginparwidth}{\footnotesize#1}}}}%
}
\def\mathnoteps#1{%
  \tag*{\rlap{\hspace\marginparsep{\parbox[t]{\marginparwidth}{\footnotesize#1}}}}%
}
```

https://tex.stackexchange.com/questions/120104/collectively-aligning-multiple-align-environments

macro: intertext

1.2.9 TODO: BibLaTeX with Zotero Group Library Sync?

- The responsive layout on mobile portrait might not be as polished; if content is overlapped thus unintelligible, please rotate device to landscape. **You might also wish to try access the pdf version of this book.**

2 'Must' and 'Might' — The Modal Logic of Necessity and Possibility

2.1 Modes of Truth and Modal Logics

Historically, notions like *necessity*, *possibility*, *impossibility*, and *contingency* were thought of as modes of truth or ways in which a proposition could be true or false. *Modal logic* began as the study of the logic of these modes of truth.

Aristotle, in Chapter 9 of *De Interpretatione*, discusses modality in his famous example of the sea battle. Suppose the sea battle will be fought tomorrow. Then it was true yesterday that it would be fought tomorrow. So if all past truths are necessarily true, then it is necessarily true now that the battle will be fought tomorrow. A similar argument holds on the supposition that the sea battle will not be fought tomorrow. Aristotle proposed solving this problem of *logical fatalism* by denying that future contingent propositions have definite truth-values.

Using the '□' for 'it is necessary that', the principle that all necessary truths are in fact

$$\Box P \to P, \tag{T}$$

but adding its converse that all truths are necessary truths:

$$P \to \Box P \tag{V}$$

collapses the notions of truth and necessary truth.

Medieval philosophers, concerned with such theological issues as articulating the nature of the Trinity, appealed to such modal notions as essence and accident, contingency and necessity in their labyrinthine theological reflections.[2] Akin to the problem is logical fatalism is the problem of *theological fatalism*: the problem of reconciling divine foreknowledge and human freedom. Saint Augustine (354 - 430) in his treatise *On the Free Choice of Will* considers an argument for *theological fatalism* proposed by Evodius. Evodius argued that "God foreknew that man would sin, that which God foreknew must necessarily come to pass." We may set forth this argument for theological fatalism for a particular case as follows:

> If God knew that Adam would sin, then, necessarily Adam sinned.
> God knew that Adam would sin (because God is omniscient).
> Therefore, Adam necessarily sinned.

St. Thomas Aquinas (1225–1274) in his *Summa Contra Gentiles* (part I, chapter 67) criticized this kind of argument as resting on an amphiboly. The critical first premise "if God knew Adam would sin, then, *necessarily*, Adam sinned" is ambiguous between

> (1*a*) It is necessarily the case that if God knows that Adam will sin then Adam will sin.
> (1*b*) If God knows that Adam will sin, then it is necessary that Adam will sin.

Aquinas called (1*a*) the *necessity of the consequence* contrasting it with (1*b*) the *necessity of the consequent*. Using the '□' to abbreviate 'it is necessary that', the difference between these two can be made more perspicuous with symbols:

$$\Box(P \to Q) \tag{1a}$$

$$(P \to \Box Q) \tag{1b}$$

Solving the famous theological problem of reconciling divine foreknowledge with human freedom may turn on exposing ambiguities of this sort.

Perhaps the most famous theological application of modal logic is Saint Anselm's modal ontological argument. According to Saint Anselm (1033–1109), it follows from God's nature that it is necessary that God exists if God exists at all. Moreover, this conditional itself, being a conceptual truth, is itself necessarily true. We then have the following argument:

> Necessarily, if God exists, then God necessarily exists.
> It is possible that God exists.
> Therefore, God (actually) exists.

Using '$\Diamond$' for 'it is possible that', the above argument can be symbolized:

$$\Box(G \to \Box G). \quad \Diamond G \quad \therefore G \tag{2}$$

The question of whether Anselm's argument is valid became a precise question when various systems of modal logic were proposed and developed in the 1960s.

Gottlob Frege (1848–1925), the inventor (or discoverer) of modern predicate-quantifier logic, relegated modality to autobiographical information about the speaker, and for many years logicians only investigated extensional logic.

One of the most puzzling validities, at least to the beginning logical students is known as *Lewis's Dilemma*:

$$P \land \neg P \to Q,$$

which states *"a contradiction implies anything"*[1]. This implication follows from the inference rules of simplification, addition, and *modus tollendo ponens*[2], which are themselves not particularly puzzling.

[1] Or *"ex falso quodlibet"*, in Latin.

[2] *"mode that denies by affirming"*

1.	~~Show~~	$P \land \neg P \to Q$	6, CD
2.		$P \land \neg P$	Assume (CD)
3.		P	2, S
4.		$\neg P$	2, S
5.		$P \lor Q$	3, ADD
6.		Q	5, 4 MTP

The following theorems are known as the *paradoxes of material implication*:

$$\neg P \to (P \to Q) \tag{T18}$$
← law of denying the antecedent

$$Q \to (P \to Q) \tag{T2}$$
← law of affirming the consequent

$$(P \to Q) \lor (Q \to \neg Q) \tag{T58}$$
← conditional excluded middle

$$(\neg P \to P) \to P \tag{T114}$$
← Consequentia Mirabilis, "admirable consequence" [Cantor's Δ?]

C. I. Lewis investigated modal logic in order to find a stricter form of the conditional which would not result in such paradoxes. Lewis defined *strict implication* $P \Rightarrow Q$ (read "P *strictly implies* Q") by combining modality with the truth-functional conditional:

$$P \Rightarrow Q \quad := \quad \Box(P \land \neg Q),$$

or alternatively,

$$P \Rightarrow Q \quad := \quad \Box(P \to Q).$$

The notion of strict implication was characterized by such axioms as:

The philosopher Leibniz (1646–1716) explicitly invoked that language of possible worlds to explain the difference between necessary and contingent truths. What is logically or necessarily true are those truths truth in *all* possible worlds, whereas a contingent truth is one that is true in *some* possible worlds.

Drawing upon this logical connection between universal and existential quantification and the modal notions of necessity and possibility, we obtain a modal version of the classical Aristotelian Square of Opposition and the duality of modal laws known as the laws of modal negation.

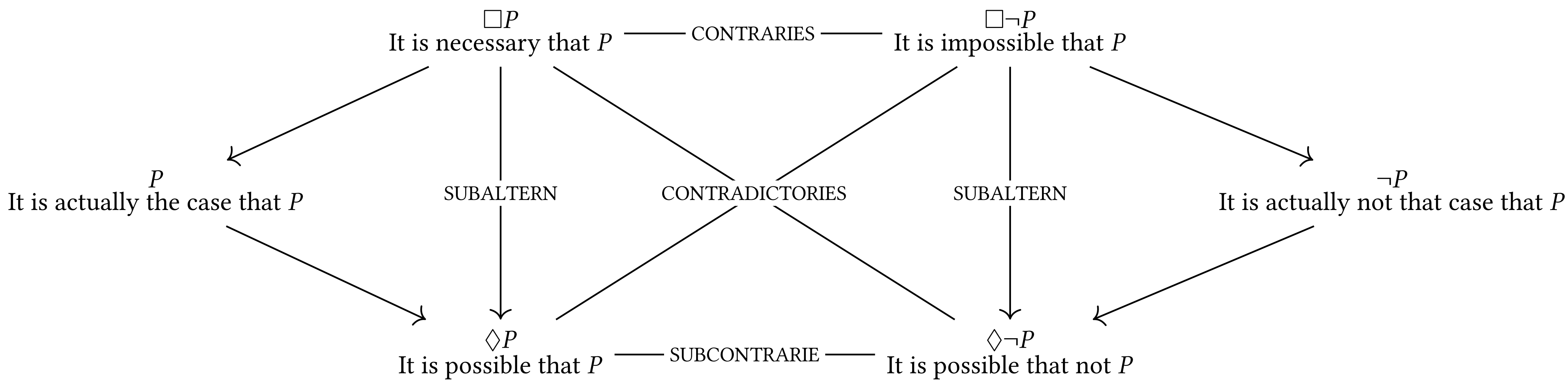


Figure 2.1: An Aristotelian Diamond of Opposition

In the modern development of modal logic, logicians noticed that a host other phenomena—such as deontic notions of obligation and permissibility, epistemic notions of knowledge and belief, as well as temporal operators—share these logical relations and hence can be represented as modal logics.

In *deontic logic*, $\Box$ is read "it is morally obligatory that" and $\Diamond$ is read "it is morally permissible that". Kant's maxim that "ought implies can", that is, whatever is obligatory is permissible, is captured by modal axiom **(D)**:

$$\Box P \to \Diamond P. \qquad \textbf{(D)}$$

In *epistemic logic*, $\Box$ is read for some subject S "it is known that" and $\Diamond$ is read "it is believed that". Some modal axioms for epistemic logic that have been considered include:

← logical omniscience

$$\Box(P \to Q) \to (\Box P \to \Box Q) \qquad \textbf{(K)}$$

← law of affirming the consequent

$$\Box P \to P \qquad \textbf{(T)}$$

← law of affirming the consequent

$$\Box P \to \Box\Box P \qquad \textbf{(4)}$$

← law of affirming the consequent

$$\neg\Box P \to \Box\neg\Box P \qquad \textbf{(E)}$$

The axiom **(K)** expresses logical omniscience insofar as this axiom requires that the knowledge of an agent is closed under *modus ponens*; and hence such knowers know all the logical consequences of their knowledge.

Axiom **(T)** states the truism that whatever is known is true. Notice that this axiom would be too strong for deontic logic insofar as an action's being obligatory doesn't imply that the agent actually performs that action.

Axiom **(4)** expresses a high degree of *positive introspective knowledge*: if someone knows P, then she knows that she knows that P. Axiom **(E)** on the other hand, expresses a high degree of *negative* introspective knowledge: if someone doesn't know that P, then he knows he doesn't know P. This axiom is contrary to the experience of Socrates: as the gadfly of Athens, Socrates found through his questioning that many of his fellow Athenians did not know what they were talking about but also didn't know that they didn't know. The gadfly of Athens believed his vocation was to sting his fellow Athenians into the awareness they were own ignorance, a service for which they did not always show adequate appreciation.

The *temporal logic* or Diodorian temporal logic was studied by the logician A. N. Prior (1914–1969). To model temporal language, we introduce a pair of modal operators for the future and a pair of modal operators for the past.

$\Box$ It *is always going* [i.e., in *all* futures] *to be the case that*
$\Diamond$ It *will* [i.e., in *some* future] be the case that

$\Box$ It *has always been* [i.e., in *all* pasts] the case that
$\Diamond$ It *was once* [i.e., in *some* past or *"once upon a time"*] the case that

The axioms for minimal tense logic include version of Axiom **(K)** for the two necessity operators:

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \qquad \text{(K}\Box\text{)}$$

Whatever has always followed from what always has been, always has been.

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \qquad \text{(K}\Box\text{)}$$

Whatever will always follow from what always will be, always will be.

It also contains two axioms concerning the interaction of the past and future that has the form of the so-called Brouwersche axiom with alternating valences:

$$\varphi \to \Box\Diamond\varphi \qquad \text{(B}\Box\Diamond\text{)}$$

What is, will always have been

$$\varphi \to \Box\Diamond\varphi \qquad \text{(B}\Box\Diamond\text{)}$$

What is, has always been going to happen

The Brouwersche **(B)** axiom was so-named by the logician Oskar Becker (*Zur Logik der Modalitäten* [1930] translated into English in 2022) after the charismatic Dutch mathematician L. E. J. Brouwer (1881–1966), who championed a philosophy of mathematics known as *intuitionism.* It happens that when the $\Diamond$ can be paraphrased as $\Diamond\neg\Diamond$, the resulting axiom has the form of the acceptable form of double negation in intuitionistic logic:

$$\varphi \to \neg\Diamond\neg\Diamond\varphi \qquad \textbf{(B)}$$

According to intuitionism, mathematical objects do not exist as eternal Platonic objects but are constructions in intuition. Intuitionists read the propositional connectives as involving not merely truth, but proof, and so they rejected such classical forms of reasoning as *reductio ad absurdum* and theorems such as the *law of excluded middle.* Intuitionists reject the following mathematical proof.

Around the 1970s it was noticed that the famous incompleteness theorems of Gödel (1931) were propositional in character and that their logic could be captured in propositional modal logic. These modal provability logics added to Axiom **(K)** the following axiom, known as the Gödel-Löb axiom or also the well-ordering axiom.

$$\Box(\Box\varphi \to \varphi) \to \Box\varphi \qquad \textbf{(W)}$$

Modal Provability Logics proliferated from the 1950s–1970s, but the genesis of the idea goes back to a short note of Gödel's [1933] in which he noted that intuitionistic truth defined in terms of proof since provability is a kind of necessity. The above axiom can be read as a kind of soundness theorem.

if it is provable that φ being provable implies it is true, then φ is provable.

Later we will show how to use a modal provability logic to exhibit the propositional logic of key parts of Gödel's First and Second Incompleteness Theorems.

In contemporary logic, modal logic has grown beyond these philosophical origins and is at the interface of a number of disciplines including the studies information flow and dynamics, game theory, and computability.

Intuitionism

Classical Theorem. There exist two irrational numbers x and y such that x^y is irrational.

Proof. Consider

$$\sqrt{2}^{\left(\sqrt{2}^{\sqrt{2}}\right)} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^{2} = 2,$$

which is rational.
The number $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational.

- If it's rational, then $x = y = \sqrt{2}$ are both irrational yet x^y is rational.
- If $\sqrt{2}^{\sqrt{2}}$ is irrational, then $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$ are both irrational and x^y is rational.

Either way there exist x and y such that x^y is rational. $\Box$ $\Box$

Intuitionists reject this classical argument by separation of cases because it does not actually construct the numbers x and y such that xy is irrational. The idea behind intuitionistic logic is that the connectives are reinterpreted as involving a kind of provability.

Exercises

1. Symbolize the following modal arguments.
   (A) Eratosthenes must either be in Syene or Alexandria. Eratosthenes cannot be in Syene. Therefore, Eratosthenes must be in Alexandria.
   (B) Assume that justice can be defined as paying your debts and telling the truth. Then it is *morally obligatory* for Cephalus to comply to a madman's request that Cephalus return a borrowed sword and that Cephalus tell the truth about the whereabouts of a friend whom the madman wants to kill with the sword. However, if this act is *morally obligatory*, then it is *morally permissible*. However, it is *morally impermissible* (or *morally forbidden*) for Cephalus to comply. So it isn't *morally obligatory* for Cephalus to comply. Justice, therefore, cannot be defined as paying your debts and telling the truth.
   (C) It is conceivable that I am having experiences qualitatively identical to those I am having now on the supposition that I am being deceived by an evil genius. If that is conceivable, then I do not have indubitable knowledge that the external world exists.

Van Benthem, Johan. 2010. *Modal logic for open minds.* Stanford, CA: Centre for the Study of Language & Information.

(1) Johan van Benthem (Van Benthem 2010: 12) was asked to symbolize the philosophical claim that "nothing is absolutely relative". He came up with the following:

$$\neg\Box(\Diamond\varphi \wedge \Diamond\neg\varphi).$$

Use familiar equivalences from propositional logic and the modal negation laws to show that this symbolization is equivalent to

- *McKinsey's Axiom*

$$\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi \qquad \textbf{(M)}$$

(2) Match the following symbolizations with the best corresponding translation below.

symbolization	translation
$\Diamond\Box P$	*It was always the case that it will sometime be the case that* P.
$\Diamond P \rightarrow \Diamond P$	*It will sometime be the case that it was once the case that* P.
$\Box\Diamond P$	*Whatever will always be, will be.*
$\Diamond\Diamond P$	*Once upon a time, it was always the case that* P.