ENUMERATION OF SMIRNOV WORDS OVER THREE-LETTER AND FOUR-LETTER ALPHABETS

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1. Smirnov Words

Smirnov words are defined to be words without consecutive equal letters. The importance of these words can easily be explained [1, p. 205]:

> Start from a Smirnov word and substitute for any letter a_j that appears in it an arbitrary nonempty sequence of letters a_j. When this operation is done at all places of a Smirnov word, it gives rise to an unconstrained word. Conversely, any word can be associated to a unique Smirnov word by collapsing into single letters maximal groups of contiguous equal letters.

2. Ternary Smirnov Words

Remark 1 [2, Rem. A.1]: *In the theoretical framework of the breakthrough article* [3]*, the numbers $\mathfrak{T}(\theta,\mathfrak{s};k,i,j)$ of distinct ternary Smirnov words that start with the letter θ, end with a letter $\mathfrak{s}\in\{\theta,\beta\}$, and contain k letters θ, i letters α, and j letters β, can be calculated in the following way:*

(i)

$$\mathfrak{T}(\theta,\theta;k,i,j)=\begin{cases}\binom{k-1}{\frac{k+i-j-1}{2}}\binom{\frac{k+i+j-3}{2}}{k-2}\ , & \textit{if } k+i+j \ \textit{odd}\ ,\\[2ex] (k+i-j)\cdot\binom{k-1}{\frac{k+i-j}{2}}\binom{\frac{k+i+j}{2}-2}{k-2}\ , & \textit{if } k+i+j \ \textit{even}\ .\end{cases} \tag{2.1}$$

(ii)

$$\mathfrak{T}(\theta,\beta;k,i,j)=\begin{cases}(k+j-i)\cdot\binom{k-1}{\frac{k+j-i-1}{2}}\binom{\frac{k+i+j-3}{2}}{k-1}\ , & \textit{if } k+i+j \ \textit{odd}\ ,\\[2ex] \binom{k-1}{\frac{k+j-i}{2}-1}\binom{\frac{k+i+j}{2}-1}{k-1}+(k+j-i)\cdot\binom{k-1}{\frac{k+j-i}{2}}\binom{\frac{k+i+j}{2}-2}{k-1}\ , & \textit{if } k+i+j \ \textit{even}\end{cases}$$

$$=\mathfrak{T}(\beta,\theta;k,i,j)\ . \tag{2.2}$$

3. Smirnov Words over a Four-letter Alphabet

Remark 2 [2, Rem. A.2]: *In the framework of* [3], *the numbers* $\mathfrak{F}(\theta, \mathfrak{s}; k, i, j, h)$ *of distinct Smirnov words, over the four-letter alphabet* $(\theta, \alpha, \beta, \gamma)$ *and with the Parikh vector* (k, i, j, h)*, that start with the letter* θ *and end with a letter* $\mathfrak{s} \in \{\theta, \alpha\}$*, can be calculated in the following way:*

(i)

$$\begin{aligned}
&\mathfrak{F}(\theta, \theta; k, i, j, h) = \\
&\sum_{\substack{0 \le \mathrm{p} \le k-1\,, \\ 0 \le \mathrm{r} \le \lfloor \frac{1}{2}(i+j+h-k+1) \rfloor}} \ \sum_{\substack{\mathrm{p} \le \mathrm{s} \le k-1\,, \\ \mathrm{r} \le \mathrm{t} \le \lfloor \frac{1}{2}(i+j+h-k+1) \rfloor}} \binom{k+\mathrm{t}-1}{\mathrm{p},\ \ \mathrm{r},\ \ \mathrm{s}-\mathrm{p},\ \ \mathrm{t}-\mathrm{r},\ \ k-\mathrm{s}-1} \\
&\qquad \times 2^{-i-j-h+3k-2\mathrm{s}-2\mathrm{r}+4\mathrm{t}-3} \cdot 3^{i+j+h-2k+\mathrm{s}+\mathrm{r}-3\mathrm{t}+2} \\
&\times \Bigg(\binom{k-\mathrm{s}-1}{-i-j-h+3k-2\mathrm{s}-\mathrm{r}+3\mathrm{t}-3} \binom{-i-j-h+3k-2\mathrm{s}+3\mathrm{t}-3}{-i+k+\mathrm{p}-\mathrm{s}+\mathrm{t}-1} \binom{-j-h+2k-2\mathrm{p}+2\mathrm{t}-2}{-j+k-\mathrm{p}+\mathrm{t}-1} \\
&- \frac{4}{9} \cdot \binom{k-\mathrm{s}-1}{-i-j-h+3k-2\mathrm{s}-\mathrm{r}+3\mathrm{t}-1} \binom{-i-j-h+3k-2\mathrm{s}+3\mathrm{t}}{-i+k+\mathrm{p}-\mathrm{s}+\mathrm{t}} \binom{-j-h+2k-2\mathrm{p}+2\mathrm{t}}{-j+k-\mathrm{p}+\mathrm{t}} \Bigg) \ . \qquad (3.1)
\end{aligned}$$

(ii)

$$\begin{aligned}
&\mathfrak{F}(\theta, \alpha; k, i, j, h) = \\
&\sum_{\substack{0 \le \mathrm{p} \le k-1\,, \\ 0 \le \mathrm{r} \le \lfloor \frac{1}{2}(i+j+h-k) \rfloor}} \ \sum_{\substack{\mathrm{p} \le \mathrm{s} \le k-1\,, \\ \mathrm{r} \le \mathrm{t} \le \lfloor \frac{1}{2}(i+j+h-k) \rfloor}} \binom{k+\mathrm{t}-1}{\mathrm{p},\ \ \mathrm{r},\ \ \mathrm{s}-\mathrm{p},\ \ \mathrm{t}-\mathrm{r},\ \ k-\mathrm{s}-1} \\
&\qquad \times 2^{-i-j-h+3k-2\mathrm{s}-2\mathrm{r}+4\mathrm{t}-2} \cdot 3^{i+j+h-2k+\mathrm{s}+\mathrm{r}-3\mathrm{t}+1} \\
&\times \Bigg(\binom{k-\mathrm{s}-1}{-i-j-h+3k-2\mathrm{s}-\mathrm{r}+3\mathrm{t}-2} \binom{-i-j-h+3k-2\mathrm{s}+3\mathrm{t}-2}{-i+k+\mathrm{p}-\mathrm{s}+\mathrm{t}} \binom{-j-h+2k-2\mathrm{p}+2\mathrm{t}-2}{-j+k-\mathrm{p}+\mathrm{t}-1} \\
&\qquad + \frac{2}{3} \cdot \Bigg(\binom{k-\mathrm{s}-1}{-i-j-h+3k-2\mathrm{s}-\mathrm{r}+3\mathrm{t}-1} \binom{-i-j-h+3k-2\mathrm{s}+3\mathrm{t}-1}{-i+k+\mathrm{p}-\mathrm{s}+\mathrm{t}} \\
&+ \frac{2}{3} \cdot \binom{k-\mathrm{s}-1}{-i-j-h+3k-2\mathrm{s}-\mathrm{r}+3\mathrm{t}} \binom{-i-j-h+3k-2\mathrm{s}+3\mathrm{t}}{-i+k+\mathrm{p}-\mathrm{s}+\mathrm{t}} \Bigg) \cdot \binom{-j-h+2k-2\mathrm{p}+2\mathrm{t}}{-j+k-\mathrm{p}+\mathrm{t}} \Bigg) \\
&\qquad\qquad = \mathfrak{F}(\alpha, \theta; k, i, j, h) \ . \qquad (3.2)
\end{aligned}$$

References

[1] *Flajolet P.*, *Sedgewick R.* Analytic combinatorics. Cambridge: Cambridge University Press, 2009.

[2] *Matveev A.O.* Symmetric cycles. Singapore: Jenny Stanford Publishing, 2023.

[3] *Prodinger H.* Ternary Smirnov words and generating functions. Integers, 2018, 18, Paper A69.