

ENUMERATION OF SMIRNOV WORDS OVER THREE-LETTER AND FOUR-LETTER ALPHABETS

ANDREY O. MATVEEV

1. SMIRNOV WORDS

Smirnov words are defined to be words without consecutive equal letters. The importance of these words can easily be explained [1, p. 205]:

Start from a Smirnov word and substitute for any letter a_j that appears in it an arbitrary nonempty sequence of letters a_j . When this operation is done at all places of a Smirnov word, it gives rise to an unconstrained word. Conversely, any word can be associated to a unique Smirnov word by collapsing into single letters maximal groups of contiguous equal letters.

2. TERNARY SMIRNOV WORDS

Remark 1 [3, Rem. A.1]: *In the theoretical framework of the breakthrough article [2], the numbers $\mathfrak{T}(\theta, \mathfrak{s}; k, i, j)$ of distinct ternary Smirnov words that start with the letter θ , end with a letter $\mathfrak{s} \in \{\theta, \beta\}$, and contain k letters θ , i letters α , and j letters β , can be calculated in the following way:*

(i)

$$\mathfrak{T}(\theta, \theta; k, i, j) = \begin{cases} \left(\binom{k-1}{\frac{k+i-j-1}{2}} \binom{\frac{k+i+j-3}{2}}{k-2} \right), & \text{if } k+i+j \text{ odd} , \\ (k+i-j) \cdot \left(\binom{k-1}{\frac{k+i-j}{2}} \binom{\frac{k+i+j}{2}-2}{k-2} \right), & \text{if } k+i+j \text{ even} . \end{cases} \quad (2.1)$$

(ii)

$$\begin{aligned} \mathfrak{T}(\theta, \beta; k, i, j) = & \begin{cases} (k+j-i) \cdot \left(\binom{k-1}{\frac{k+j-i-1}{2}} \binom{\frac{k+i+j}{2}}{k-1} \right), & \text{if } k+i+j \text{ odd} , \\ \left(\binom{k-1}{\frac{k+j-i}{2}-1} \binom{\frac{k+i+j}{2}-1}{k-1} \right) + (k+j-i) \cdot \left(\binom{k-1}{\frac{k+j-i}{2}} \binom{\frac{k+i+j}{2}-2}{k-1} \right), & \text{if } k+i+j \text{ even} \end{cases} \\ & = \mathfrak{T}(\beta, \theta; k, i, j) . \quad (2.2) \end{aligned}$$

3. SMIRNOV WORDS OVER A FOUR-LETTER ALPHABET

Remark 2 [3, Rem. A.2]: *In the framework of [2], the numbers $\mathfrak{F}(\theta, \mathfrak{s}; k, i, j, h)$ of distinct Smirnov words, over the four-letter alphabet $(\theta, \alpha, \beta, \gamma)$ and with the Parikh vector (k, i, j, h) , that start with the letter θ and end with a letter $\mathfrak{s} \in \{\theta, \alpha\}$, can be calculated in the following way:*

(i)

$$\begin{aligned} \mathfrak{F}(\theta, \theta; k, i, j, h) = & \sum_{\substack{0 \leq p \leq k-1, \\ 0 \leq r \leq \lfloor \frac{1}{2}(i+j+h-k+1) \rfloor}} \sum_{\substack{p \leq s \leq k-1, \\ r \leq t \leq \lfloor \frac{1}{2}(i+j+h-k+1) \rfloor}} \binom{k+t-1}{p, r, s-p, t-r, k-s-1} \\ & \times 2^{-i-j-h+3k-2s-2r+4t-3} \cdot 3^{i+j+h-2k+s+r-3t+2} \\ & \times \left(\binom{k-s-1}{-i-j-h+3k-2s-r+3t-3} \binom{-i-j-h+3k-2s+3t-3}{-i+k+p-s+t-1} \binom{-j-h+2k-2p+2t-2}{-j+k-p+t-1} \right. \\ & \left. - \frac{4}{9} \cdot \binom{k-s-1}{-i-j-h+3k-2s-r+3t-1} \binom{-i-j-h+3k-2s+3t}{-i+k+p-s+t} \binom{-j-h+2k-2p+2t}{-j+k-p+t} \right). \quad (3.1) \end{aligned}$$

(ii)

$$\begin{aligned} \mathfrak{F}(\theta, \alpha; k, i, j, h) = & \sum_{\substack{0 \leq p \leq k-1, \\ 0 \leq r \leq \lfloor \frac{1}{2}(i+j+h-k) \rfloor}} \sum_{\substack{p \leq s \leq k-1, \\ r \leq t \leq \lfloor \frac{1}{2}(i+j+h-k) \rfloor}} \binom{k+t-1}{p, r, s-p, t-r, k-s-1} \\ & \times 2^{-i-j-h+3k-2s-2r+4t-2} \cdot 3^{i+j+h-2k+s+r-3t+1} \\ & \times \left(\binom{k-s-1}{-i-j-h+3k-2s-r+3t-2} \binom{-i-j-h+3k-2s+3t-2}{-i+k+p-s+t} \binom{-j-h+2k-2p+2t-2}{-j+k-p+t-1} \right. \\ & \quad + \frac{2}{3} \cdot \left(\binom{k-s-1}{-i-j-h+3k-2s-r+3t-1} \binom{-i-j-h+3k-2s+3t-1}{-i+k+p-s+t} \right. \\ & \quad \left. \left. + \frac{2}{3} \cdot \binom{k-s-1}{-i-j-h+3k-2s-r+3t} \binom{-i-j-h+3k-2s+3t}{-i+k+p-s+t} \right) \cdot \binom{-j-h+2k-2p+2t}{-j+k-p+t} \right) \\ & = \mathfrak{F}(\alpha, \theta; k, i, j, h). \quad (3.2) \end{aligned}$$

REFERENCES

- [1] *Flajolet P., Sedgewick R.* Analytic combinatorics. Cambridge: Cambridge University Press, 2009.
- [2] *Prodinger H.* Ternary Smirnov words and generating functions. Integers, 2018, 18, Paper A69.
- [3] *Matveev A.O.* Symmetric cycles. Singapore: Jenny Stanford Publishing, 2023.