Signals and Systems
Lecture Outline

1) **Classification of Signals**
2) **Sampling and Quantization**
3) **Discrete-time signals**
4) **Discrete-time systems**
5) **Convolution**

Signals

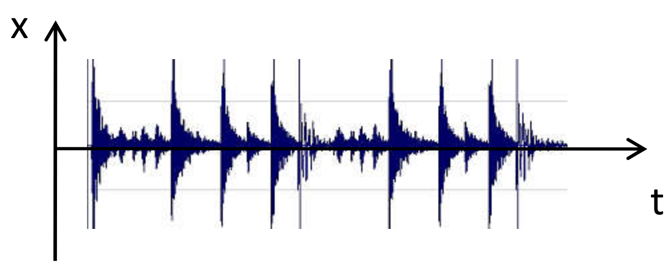
Signal is a function that conveys information about the behavior or attributes of some phenomenon

From a physical point of view,
signal is a physical quantity which varies with respect to time or space and conveys information from source to destination

From a mathematical point of view,
signal is a custom function

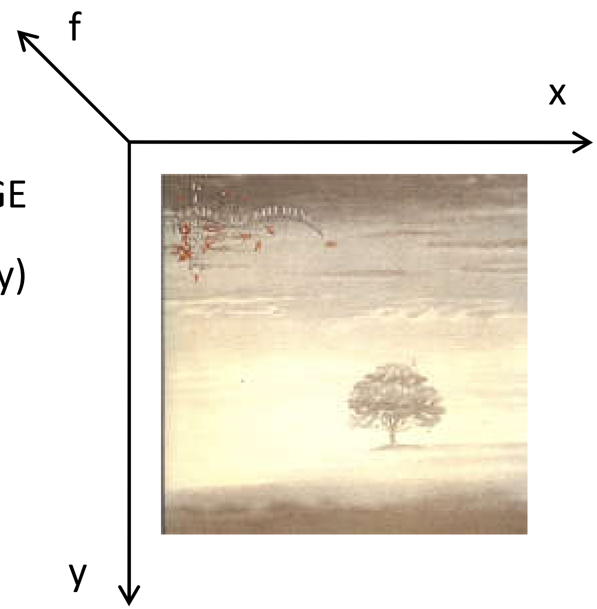
MUSIC

x(t)



IMAGE

f(x,y)

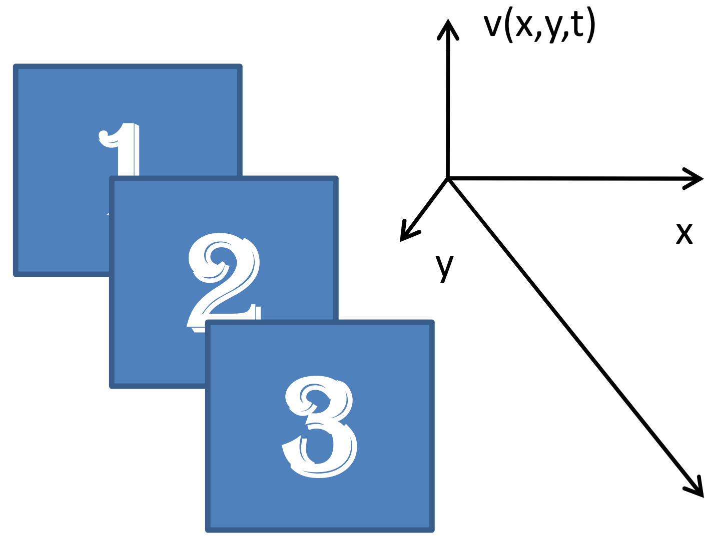


Classification of Signals

Signals differ by:

Number of independent variables | **Limitation** | **Repetition** | **Continuity** | **Certainty of description**

- **1-Dimensional Signals** x(t)
- **2-Dimensional Signals** f(x,y)
- **3-Dimensional Signals** v(x,y,t)
- **Multi-Dimensional Signals**



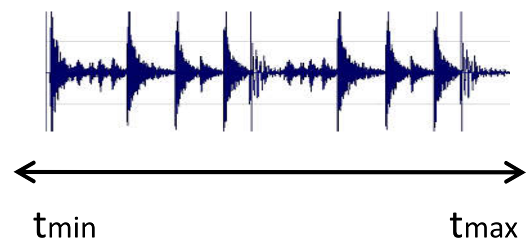
Classification of Signals

Signals differ by:

Number of independent variables | **Limitation** | **Repetition** | **Continuity** | **Certainty of description**

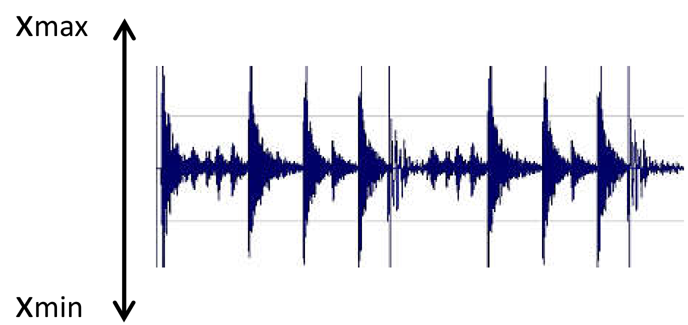
$|t_{max} - t_{min}| < \infty$

- **Finite Signals**
- **Infinite Signals**



$|x_{max} - x_{min}| < \infty$

- **Bounded Signals**
- **Unbounded Signals**



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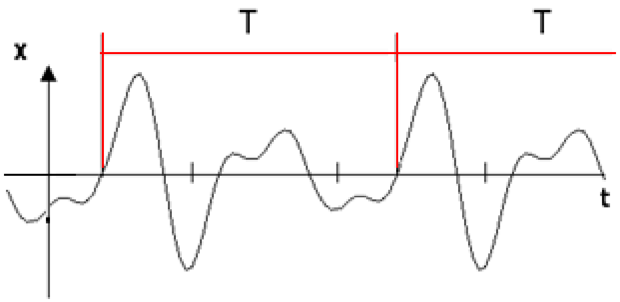
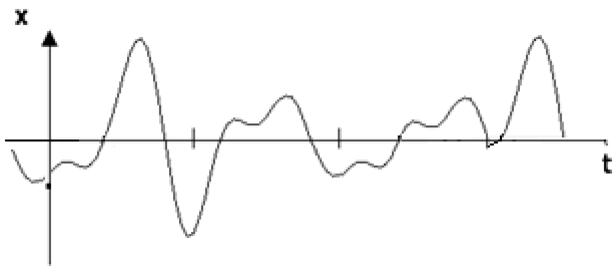


$x(t) = x(t+kT) \quad k=1,2,3,\ldots$

- **Periodic Signals**
- **Aperiodic Signals**



Classification of Signals

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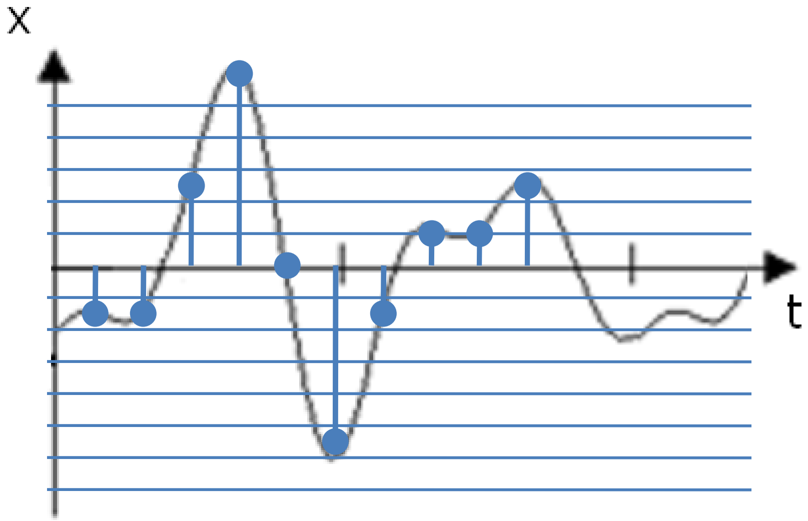


- **Continuous (Analog) Signals**
- **Discrete-time (Sampled) Signals**
- **Discrete Signals**
- **Digital (Digitized) Signals**

Classification of Signals

Signals differ by:

Number of independent variables **Limitation** **Repetition** **Continuity** **Certainty of description**

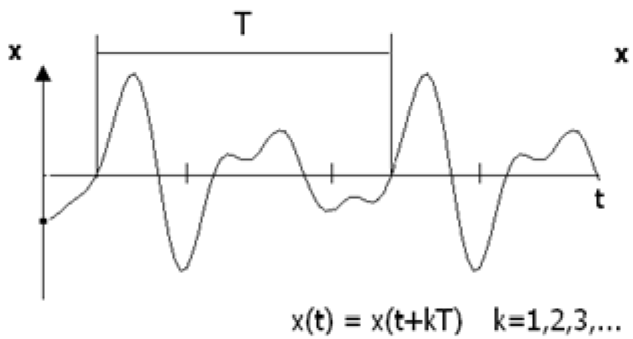
- **Deterministic Signals**

can be uniquely determined by a well-defined process such as mathematical expression, algorithm, or look-up table

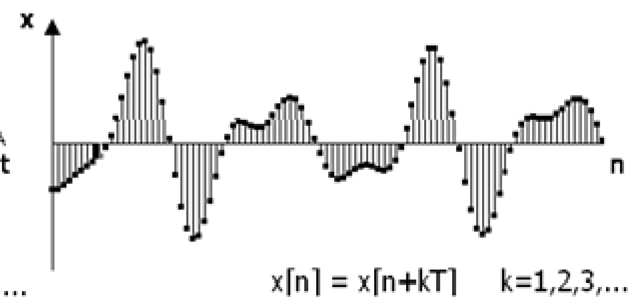
- **Random Signals**

can be expressed as a random variable; it is characterized statistically (e.g. by probability density)

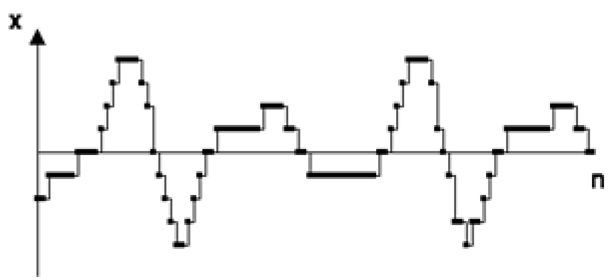
More Examples



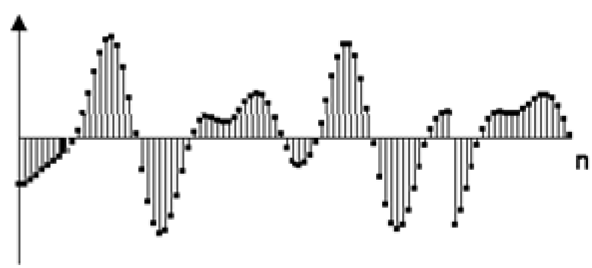
Continuous Periodic signal



Discrete-time Periodic signal



Digital Periodic signal



Discrete-time Aperiodic signal

Sinusoidal Signals

$$x(t) = A\sin(\omega t + \varphi) = A\sin(2\pi f t + \varphi)$$

Amplitude

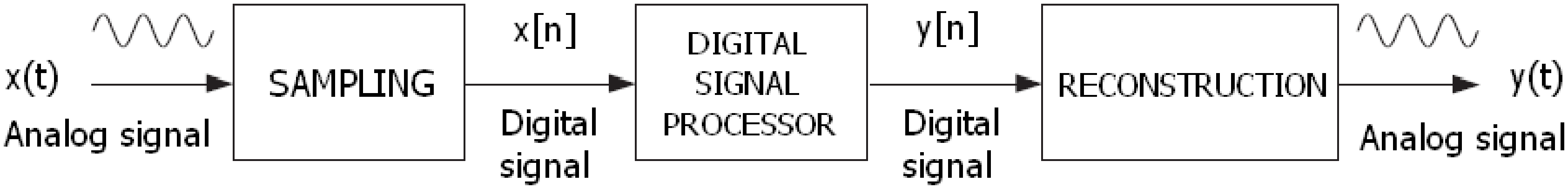
Phase (in radians)

Frequency (in cycles per second (Hz))

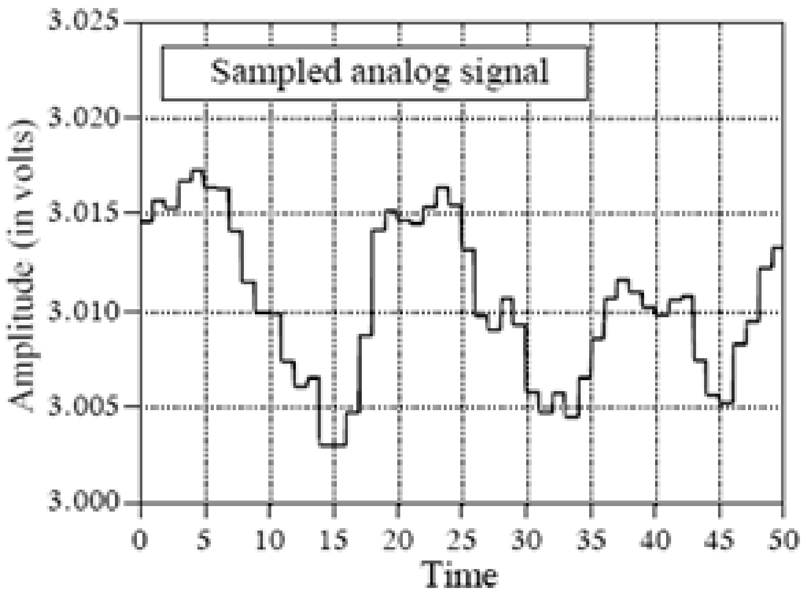
Angular frequency (in radians per second)

Sinusoidal signals play a very important role in DSP for two reasons:
1) many analog signals are inherently created from superimposed sinusoids (sound waves, for instance);
2) Fourier Transform (one of the basic transforms in DSP) decomposes signals into sine waves.

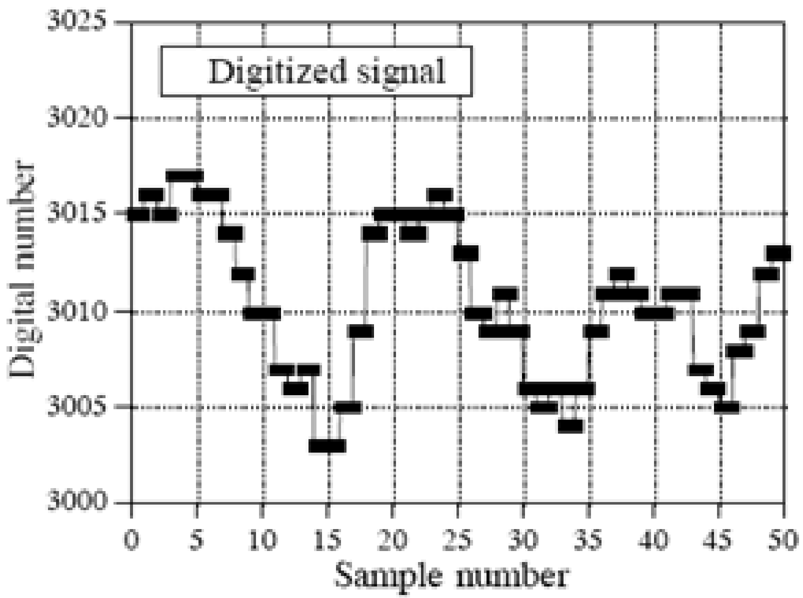
DSP Scheme


x(t)
Analog signal
SAMPLING
x[n]
Digital signal
DIGITAL SIGNAL PROCESSOR
y[n]
Digital signal
RECONSTRUCTION
y(t)
Analog signal

Sampling

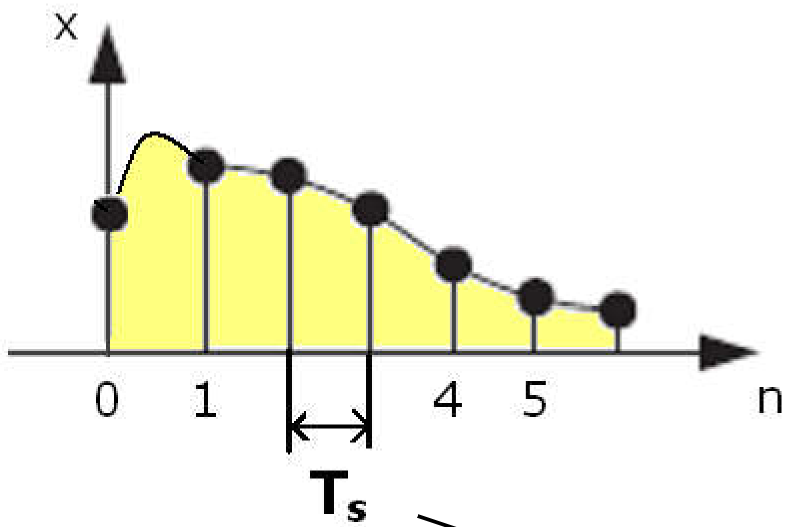


1) S/H : SAMPLE-and-HOLD



2) QUANTIZATION (ADC)

Ideal Sampler vs. Actual Sampler



Sampling period

$$f_s = \frac{1}{T_s}$$ - Sampling rate (sampling frequency)

Ideal Sampler

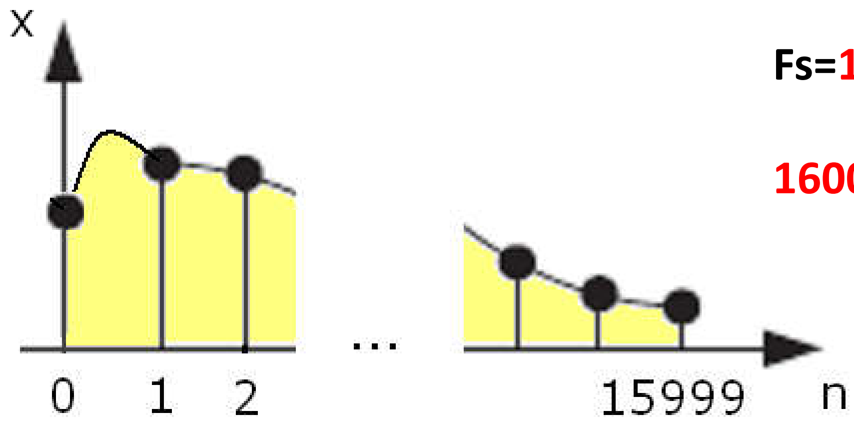
$$x[n] = x_a(nT_s)$$

Actual Sampler

$$x[n] = \frac{1}{2\Delta T} \int_{nT_s - \Delta T}^{nT_s + \Delta T} x_a(t)\,dt$$

Exercise 1

Given an analog audio signal 400 ms long sampled at sampling rate 16000 Hz. How many samples does the signal contain?



Fs=16000 Hz ⇔ 16000 samples / sec

16000 samples in 1000 ms = x samples in 400 ms

x = 6400

Discrete-time Sinusoidal Signal

$$x[n] = x_a(nT_s) = A\sin(\omega\, nT_s + \varphi) = A\sin(2\pi f\, nT_s + \varphi)$$

$$x[n] = A\sin(\Omega\, n + \varphi) = A\sin(\pi F n + \varphi)$$

$$\Omega = \omega T_s = \frac{2\pi f}{f_s}$$ - Digital angular frequency

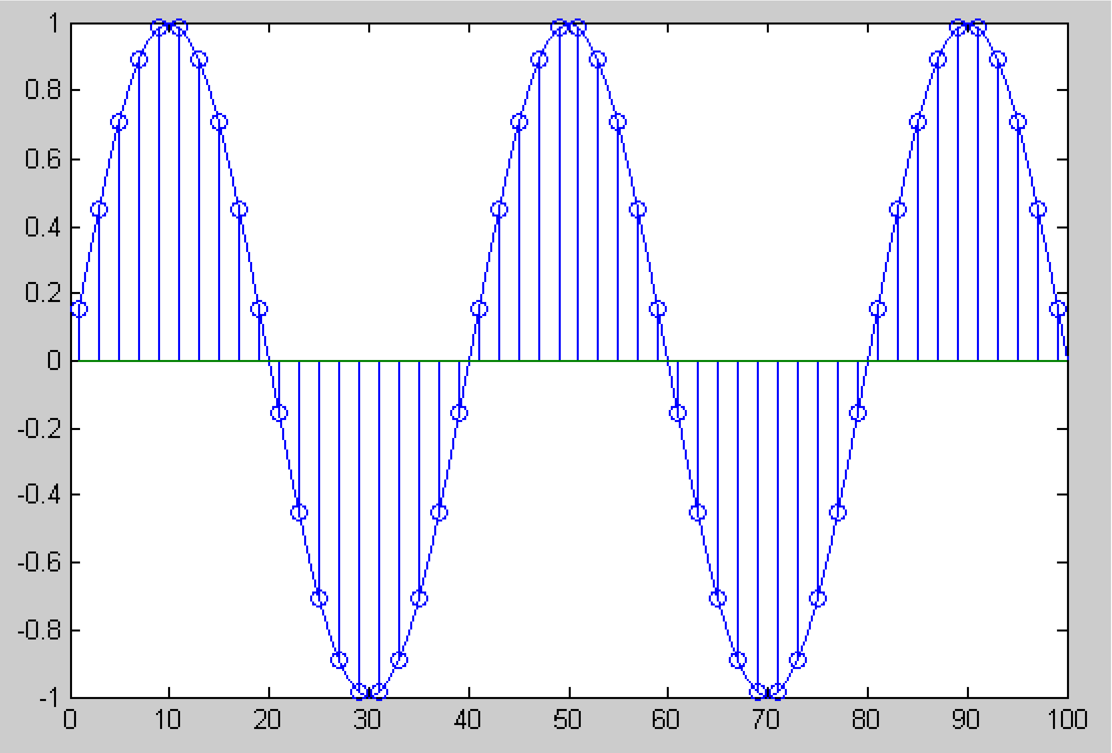
$$F = \frac{\Omega}{\pi} = \frac{f}{(f_s/2)}$$ - Normalized digital frequency

Generate Sinusoidal Signal

Generate 300 samples of sinusoidal signal with frequency 100 Hz and sampling rate 4000 Hz. Plot first 100 samples of the signal.

```
f = 100;
fs = 4000;
ang_freq = 2*pi*f
n = 1:300;
x = sin(ang_freq
plot(n(1:100),x(1

%stem(n(1:100),x(
```



Proper Sampling

"Proper sampling" means that it's possible to exactly reconstruct continuous signal from samples of its digital representation

Sampling Theorem

Continuous signal can be **properly sampled** if and only if it **does not contain** frequency components above **1/2 of the sampling rate**

$$1)\quad |f| \leq f_{max}$$

$$2)\quad f_s > 2 f_{max}$$

The minimum sampling rate f_s is called the *Nyquist rate*

$f_{max}=f_s/2$ is called the *Nyquist frequency*

Exercise 2

Suppose we have two sinusoidal signals with frequencies f_1=6000 Hz and f_2=10000 Hz. What will happen to the latter sine wave if we sample both signals at f_s=16000 Hz?

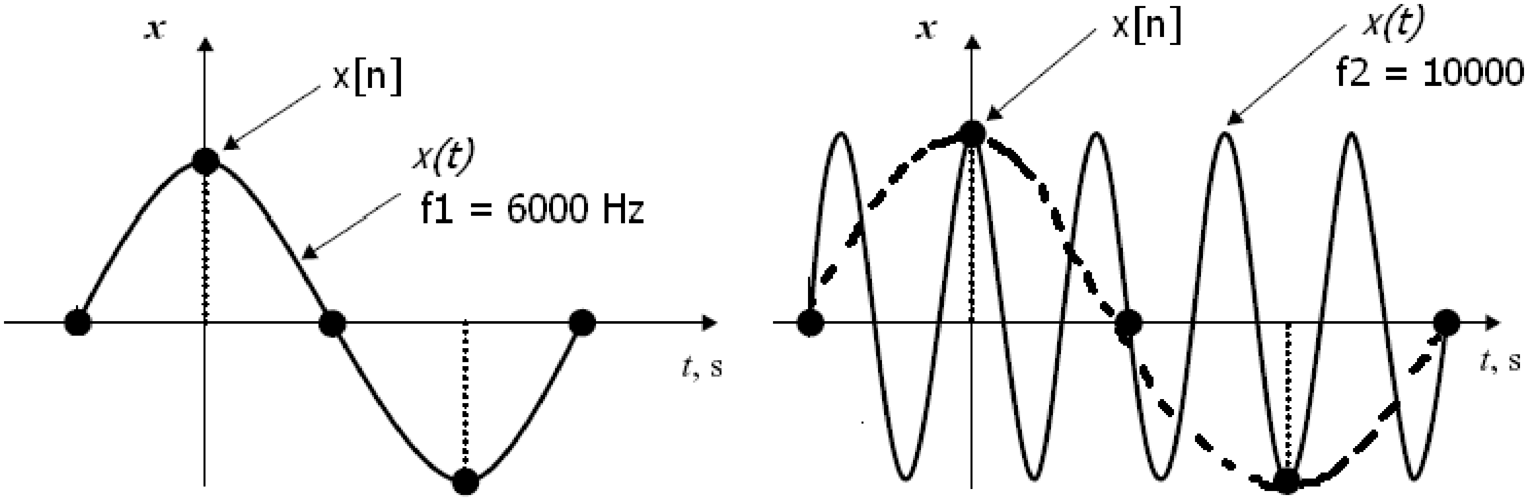
$$x_1[n] = \cos\left(2\pi \cdot 6000 \cdot n \cdot \frac{1}{16000}\right) = \cos\left(\frac{6\pi}{8} n\right)$$

$$x_2[n] = \cos\left(2\pi \cdot 10000 \cdot n \cdot \frac{1}{16000}\right) = \cos\left(\frac{10\pi}{8} n\right)$$

$$x_2[n] = \cos\left(\frac{10\pi}{8} n\right) = \cos\left(2\pi n - \frac{6\pi}{8} n\right) = \cos\left(\frac{6\pi}{8} n\right) = x_1[n]$$

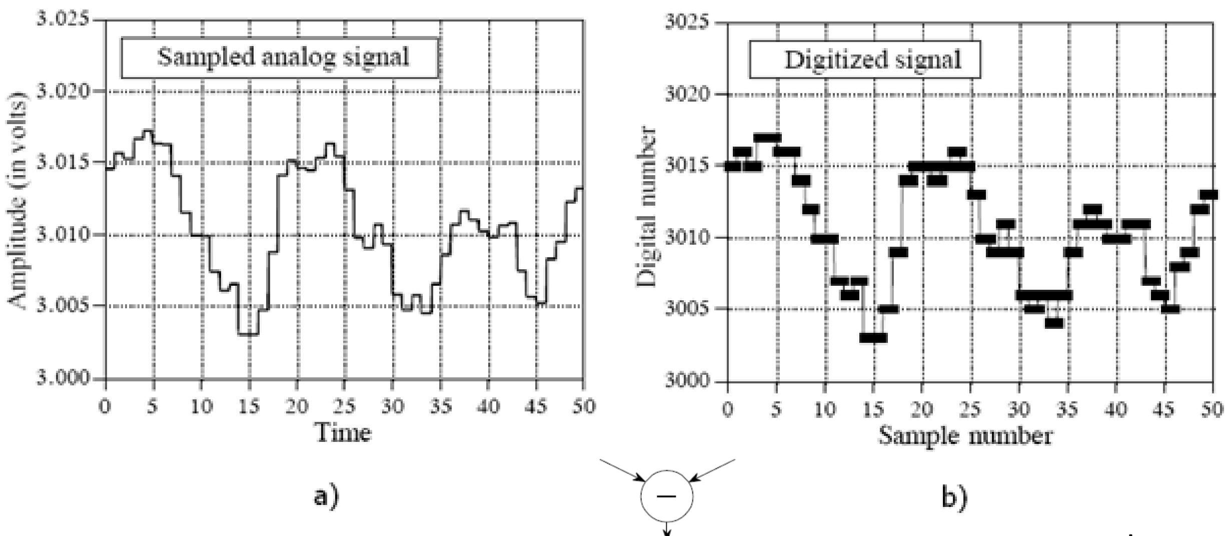
It will act just like the sine wave with frequency 6000 Hz

Aliasing

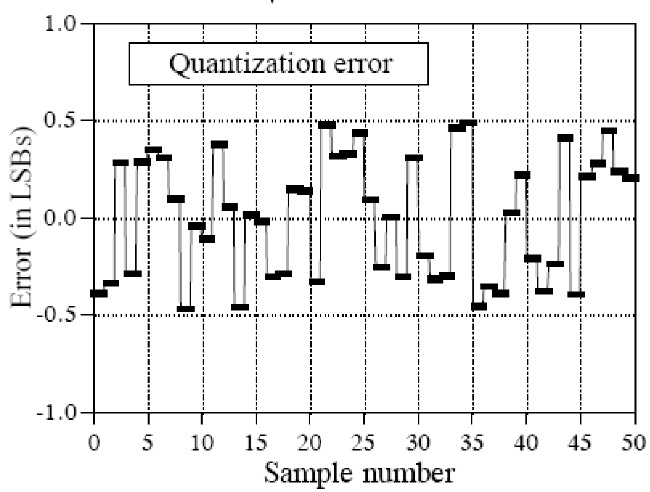


Aliasing: frequency f2 "acts" like frequency f1

Quantization



Any single sample in the digitized signal can have a maximum error of ±1/2 LSB (Least Significant Bit, the distance between adjacent quantization levels).

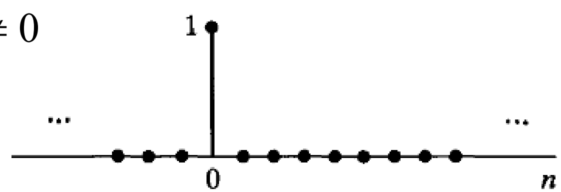


The amplitude of each discrete-time sample is quantized into one of the 2^B levels, where *B* is the number of bits that the ADC has to represent for each sample (*ADC rate*). For example, 8-bit ADC will provide only 2^8=256 possible values for quantized signal

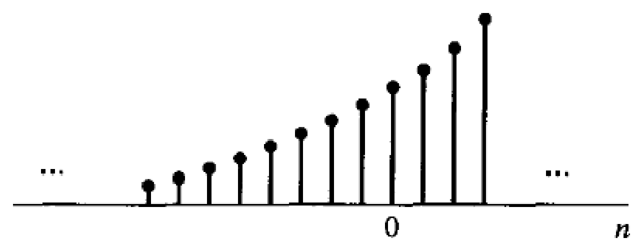
Useful Discrete-time Signals

Unit impulse (Kronecker's δ -function)

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



Exponential signals:

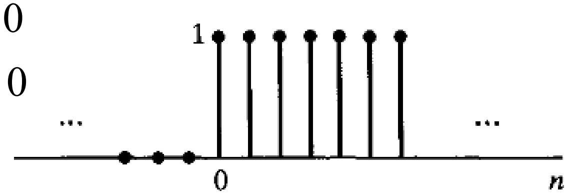


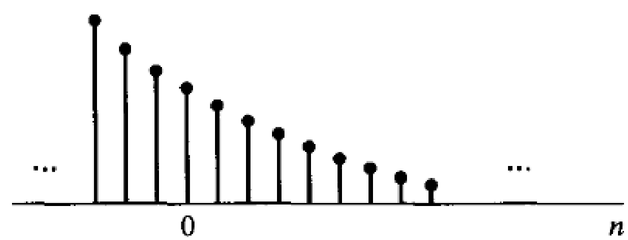
$$x[n] = A\alpha^n$$

$\alpha > 1$

Unit step

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$





$$x[n] = A\alpha^n$$

$0 < \alpha < 1$

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Random Signals

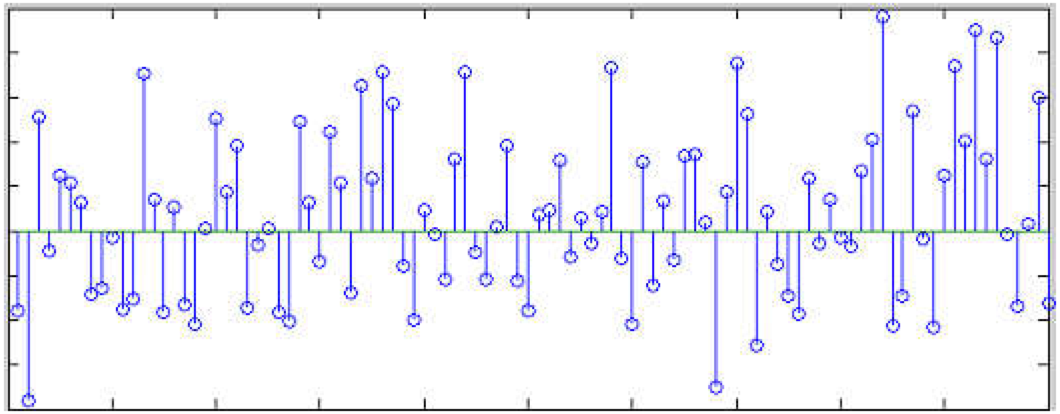
Random signals (such as speech, music, various kinds of noise, etc.) are modeled in terms of stochastic signals. In other words, each sample *x[n]* of a random signal is assumed to be an outcome of some underlying random variable $\mathbf{x}_n$.

Therefore, probability distributions must be specified, in order to describe the random process.

The noise is an obvious example of a random signal. For instance, if we need to describe **Gaussian noise**, we don't have to specify the particular values of samples. Instead, we generate a sample set with normal (Gaussian) distribution:

$$P(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2}$$

$$\mu = \frac{1}{N}\sum_{i=0}^{N-1} x_i \qquad \sigma^2 = \frac{1}{N}\sum_{i=0}^{N-1}(x_i-\mu)^2$$



Stationary Signals

Random signals can be either *stationary*, or *non-stationary*.

Stationarity in a strict sense (SSS) means that *all* of the distribution functions of random process do not change over time.

A weaker form of stationarity commonly used in DSP is known as ***wide-sense stationarity (WSS)***. WSS random processes only require that *mean* and *covariance* do not vary with respect to time.

Most of the real-world signals are **non-stationary**. The examples of stationary signals are stationary noises (e.g. white noise).

Signal-to-Noise Ratio

Signal-to-Noise ratio (SNR) is the measure that compares the level of a desired signal to the level of a background noise:

$$SNR = \frac{P_{Signal}}{P_{Noise}} = \frac{A_{Signal}^2}{A_{Noise}^2}$$

There is also an alternative definition of SNR (used mostly in image processing), according to which SNR is the ratio of mean to standard deviation of a signal:

$$SNR = \frac{\mu}{\sigma}$$

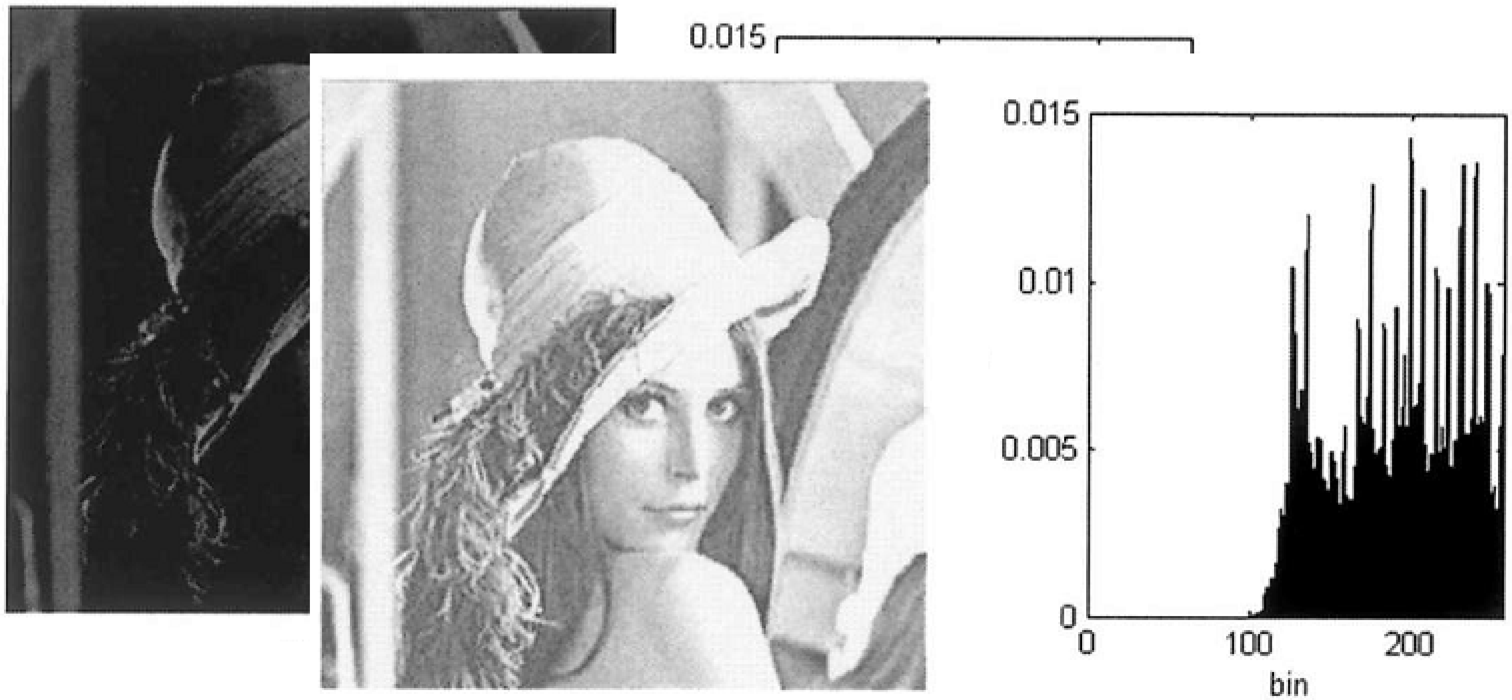
Amplitude-Time Parameters

If the signal's independent variable is time or sample number, the signal is said to be represented in *time domain*.

Energy of a signal segment $$E(k,N)=\sum_{i=k}^{k+N-1}|x_i|^2$$

Zero-crossing rate $$ZCR(k,N)=\frac{1}{N}\sum_{i=k}^{k+N-1}\frac{|sign(x_{i+1})-sign(x_i)|}{2}$$

Histograms

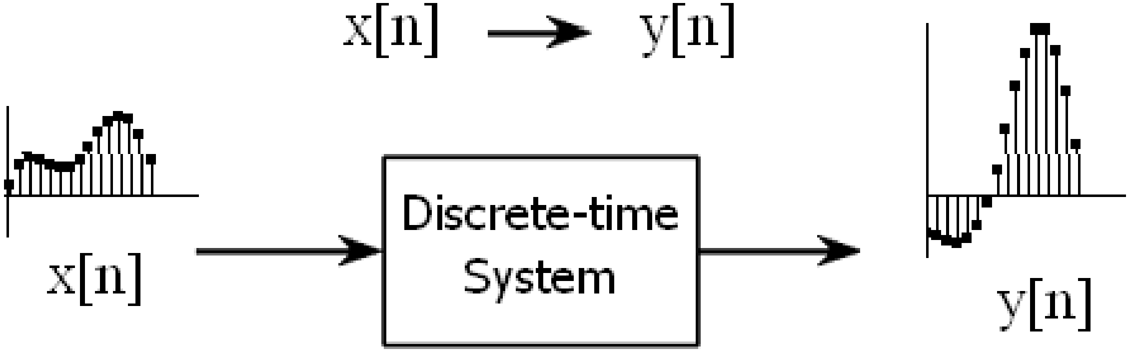


$$\text{histogram} = p(k) = \frac{n_k}{N}$$

Discrete-time Systems

Discrete-time System is a transformation or operator that maps an input discrete-time signal into an output discrete-time signal

$$x[n] \longrightarrow y[n]$$



$$y[n] = T\{x[n]\}$$

LTI Systems

Homogeneity property:

IF $x[n] \rightarrow y[n]$ THEN

$\boldsymbol{k}\, x[n] \rightarrow \boldsymbol{k}\, y[n]$ for any ***k=const***

Additivity property:

IF $(x_1[n] \rightarrow y_1[n])$ AND $(x_2[n] \rightarrow y_2[n])$ THEN

$(x_1[n] + x_2[n]) \rightarrow (y_1[n] + y_2[n])$

Time Invariance:

IF $x[n] \rightarrow y[n]$ THEN

$x[n-\mathbf{d}] \rightarrow y[n-\mathbf{d}]$ for any **d** = ***const***

Useful Properties of LTI Systems

Sinusoidal Fidelity:

If the input signal is sinusoidal, then the output signal is also sinusoidal, with the same frequency:

$$A_1 \sin(\omega n + \varphi_1) \rightarrow A_2 \sin(\omega n + \varphi_2)$$

Static Linearity:

If the input signal is constant, then the output signal is also constant:

$$c_1[n] \rightarrow c_2[n], \qquad c_1, c_2 = const$$

Sinusoidal Fidelity

Given a system transforming input signal in the following way:

5sin(20πn + π/3) + 2sin(5πn) → cos(5πn)

Does the system have a sinusoidal fidelity property?

Yes, it does. The component “2sin(5πn)” is transformed into “cos(5πn)”, so only the phase and the amplitude are changing here, while the frequency is the same.
The component “5sin(20πn+π/3)” doesn’t have a corresponding output component. We may assume that the amplitude of this component is reducing to 0.

Causality, Stability

A discrete-time system is *causal*, if every output signal sample depends only on the preceding input signal samples

A discrete-time system is *stable*, if and only if it transforms a bounded input signal x[n] into a bounded output signal y[n]:

$$|\, x[n] \,| \le B_x < \infty, \ \text{ and } \ |\, y[n] \,| \le B_y < \infty, \quad \text{for all } n$$

Examples

NON-CAUSAL

the ***forward difference system*** defined by the relationship:

$$y[n] = x[n+1] - x[n]$$

CAUSAL

the ***backward difference system*** defined by the relationship:

$$y[n] = x[n] - x[n-1]$$

Superposition Principle

The combination of Homogeneity and Additivity properties leads to the ***Superposition Principle***, one of the fundamental concepts of DSP:

IF ($x_1[n] \rightarrow y_1[n]$) AND ($x_2[n] \rightarrow y_2[n]$) THEN

($a x_1[n] + b x_2[n]$) $\rightarrow$ ($a y_1[n] + b y_2[n]$) for any *a, b=const*

Combining with Time Invariance property:

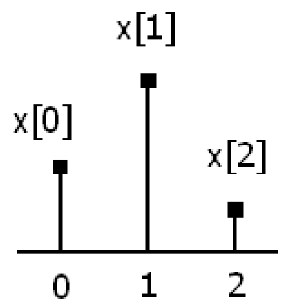
$$\sum_{k=0}^{N-1} a_k x_k[n-d_k] \rightarrow \sum_{k=0}^{N-1} a_k y_k[n-d_k]$$

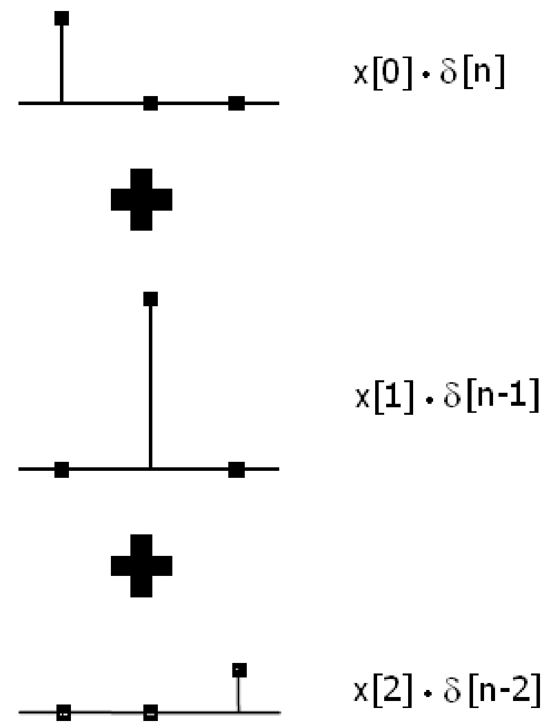
Impulse Decomposition

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

signal x[n]

N=3

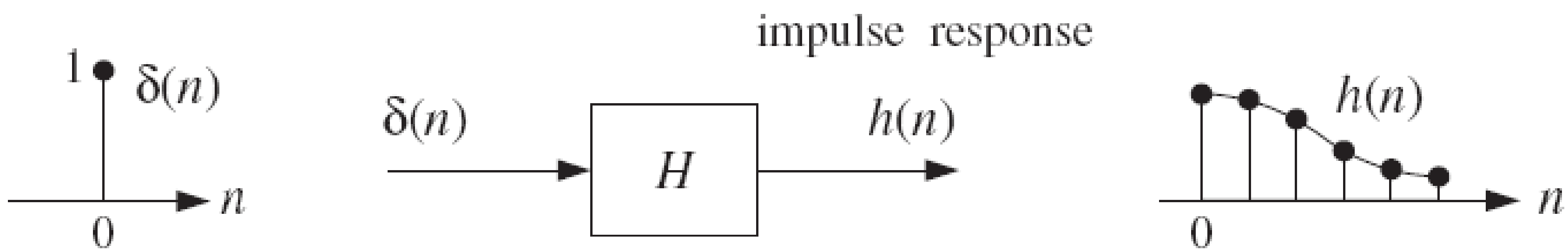




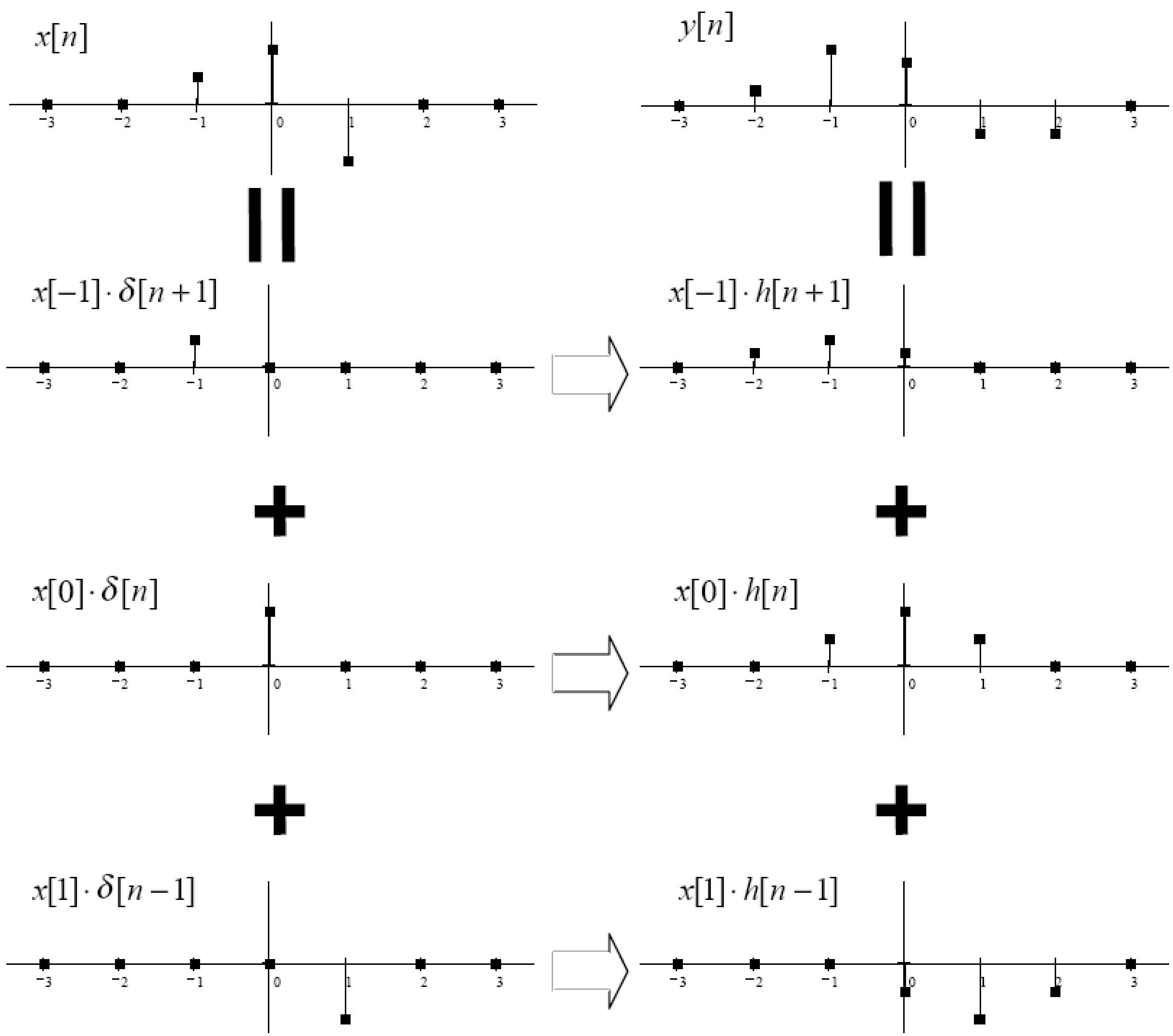
x[n] is the weighted sum
of 3 signals δ[n] with different delays

Impulse Response

Impulse Response is the output signal of a system when the input signal is a unit impulse



$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k] \longrightarrow y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]$$



$x[n]$
$y[n]$
$x[-1]\cdot\delta[n+1]$
$x[-1]\cdot h[n+1]$
$x[0]\cdot\delta[n]$
$x[0]\cdot h[n]$
$x[1]\cdot\delta[n-1]$
$x[1]\cdot h[n-1]$

Convolution

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

reflects the idea of the so called ***input-side algorithm*** for calculating convolution.
It analyzes how *each* sample in the input signal affects *many* samples in the output signal.

$$y[n] = x[n] * h[n] = \sum_{k=0}^{M-1} x[n-k]\, h[k]$$

according to an ***output-side algorithm*** *each* output signal sample is some combination of *many* values of the input signal and impulse response

h[k] is often referred to as the ***convolution machine***. It “slides” along the input signal samples.

Exercise 3

Convolve signal x = {1,5,3,2,6} with signal h = {2,3,1}

1	3	2

1	5	3	2	6

The size **N=5**. The size **M=3**.

```
y[0] = x[0]h[0]                                   |         |
y[1] = x[1]h[0] + x[0]h[1]                        | M-1     |
y[2] = x[2]h[0] + x[1]h[1] + x[0]h[2]                       | N
y[3] = x[3]h[0] + x[2]h[1] + x[1]h[2]                       |
y[4] = x[4]h[0] + x[3]h[1] + x[2]h[2]                       |
y[5] =            x[4]h[1] + x[3]h[2]             | M-1
y[6] =                       x[4]h[2]             |
```

The resulting signal (convolution) contains **N+M–1** samples.
N*M multiplications are performed.

Exercise 3 (Contd.)

$$
\begin{aligned}
y[0] &= \mathbf{1}\cdot 2 &&= 2\\
y[1] &= \mathbf{5}\cdot 2 + \mathbf{1}\cdot 3 &&= 13\\
y[2] &= \mathbf{3}\cdot 2 + \mathbf{5}\cdot 3 + \mathbf{1}\cdot 1 &&= 22\\
y[3] &= \mathbf{2}\cdot 2 + \mathbf{3}\cdot 3 + \mathbf{5}\cdot 1 &&= 18\\
y[4] &= \mathbf{6}\cdot 2 + \mathbf{2}\cdot 3 + \mathbf{3}\cdot 1 &&= 21\\
y[5] &= \mathbf{6}\cdot 3 + \mathbf{2}\cdot 1 &&= 20\\
y[6] &= \mathbf{6}\cdot 1 &&= 6
\end{aligned}
$$

```
x = [1 5 3 2 6];      % 5-point signal x
h = [2 3 1];          % 3-point signal h
y = conv(x, h)        % convolution
stem(y);              % plot resulting signal
```

Common Impulse Responses

h[n]= δ[n]

$$x[n] \longrightarrow \boxed{H} \longrightarrow x[n]$$

In this case, the LTI system simply passes signals without a change

h[n]= δ[n–d]

$$x[n] \longrightarrow \boxed{H} \longrightarrow x[n-d]$$

In this case, the LTI system shifts the input signal (making signal *delay*, if *d* > 0, or *advance*, if *d* < 0)

h[n]= kδ[n]

$$x[n] \longrightarrow \boxed{H} \longrightarrow kx[n]$$

The LTI systems characterized by this impulse response are called *amplifiers* (if *k*>1), or *attenuators* (if 0<*k*<1)

Properties of Convolution

Commutative property

$$x[n] * h[n] = h[n] * x[n]$$

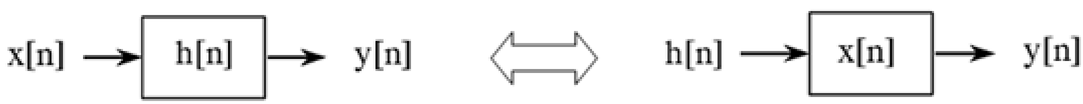
Associative property

$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$$

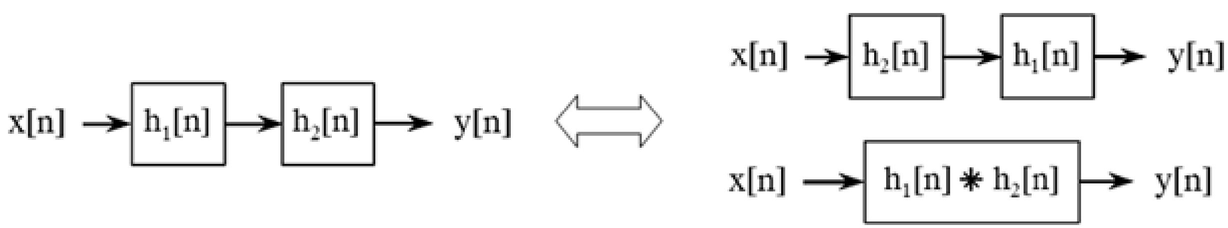
Distributive property

$$x[n] * h_1[n] + x[n] * h_2[n] = x[n] * (h_1[n] + h_2[n])$$

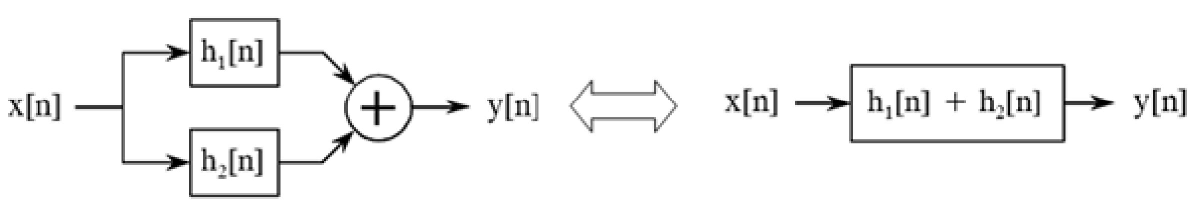
Properties of Convolution



Commutative property



Associative property



Distributive property

Cross-Correlation

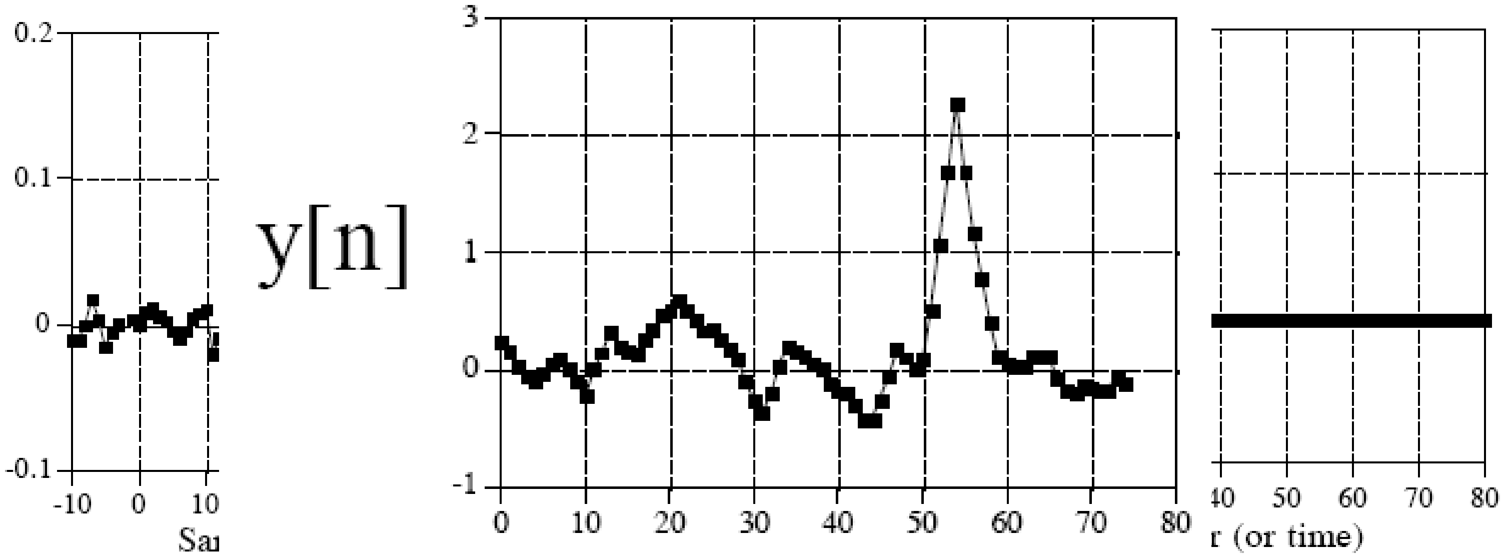
Cross-correlation is a measure of similarity of two signals.
It is commonly used for searching a long-signal for a shorter, known feature

$$y[n] = \sum_{k=-\infty}^{+\infty} x[n+k]\, g[k]$$

The difference between *convolution* and *cross-correlation* is in their convolution machines:

h[k] = g[-k]

Cross-Correlation



Cross-correlation finds application in pattern recognition, cryptanalysis, and neurophysiology

Nonlinear Systems

There are also important cases of *nonlinear* discrete-time systems, in which signals are *multiplied*, *convolved*, etc. For example, an amplitude modulation of signals is, basically, a multiplication of signal by the carrier wave. The problem is that an analysis of nonlinear systems is very complicated task. The most preferable way of dealing with non-linear systems is to apply a *linearizing transform*, such as *homomorphic transform (Lecture 9).*

Examples of linear systems are:

- digital filters and amplifiers
- signal effects (reverberation, resonance, blurring, etc.)
- wave propagation

Examples of nonlinear systems are:

- amplitude modulation
- systems for peak detection
- clipping and crossover distortion
- systems with a threshold

Accumulator

The *accumulator* system is defined by the following equation:

$$y[n]=\sum_{k=-\infty}^{n}x[k]$$

Prove that the accumulator system is LTI system

First, we'll show that the system satisfies the requirements for linearity (i.e. superposition principle). Let's define input signals $x_1[n]$ and $x_2[n]$. Their outputs are (respectively):

$$y_1[n]=\sum_{k=-\infty}^{n}x_1[k] \qquad\qquad y_2[n]=\sum_{k=-\infty}^{n}x_2[k]$$

Let's define a signal $x_3[n] = ax_1[n] + bx_2[n]$.
We must show that the corresponding output is $y_3[n] = ay_1[n] + by_2[n]$:

$$y_3[n]=\sum_{k=-\infty}^{n}x_3[k]=\sum_{k=-\infty}^{n}(ax_1[k]+bx_2[k])=a\sum_{k=-\infty}^{n}x_1[k]+b\sum_{k=-\infty}^{n}x_2[k]$$

Accumulator (Contd.)

The *accumulator* system is defined by the following equation:

$$y[n]=\sum_{k=-\infty}^{n} x[k]$$

Prove that the accumulator system is LTI system

Thus, $y_3[n] = ay_1[n] + by_2[n]$, and the system satisfies the superposition principle. Next, we'll show that the system is time-invariant. Let's see how the system will respond to a signal $x_d=x[n-n_0]$:

$$y_d[n]=\sum_{k=-\infty}^{n} x[k-n_0]$$

Substituting the change of variables $m=k-n_0$ into summation leads to:

$$y_d[n]=\sum_{m=-\infty}^{n-n0} x[m]=y[n-n_0]$$

That is, the system transforms an input signal $x[n-n_0]$ into an output signal $y[n-n_0]$.
Therefore, it's an LTI system.

Nonlinear System

Given a system transforming input signal in the following way:

$x \rightarrow 2x + \sin(x)$

Prove that the system is nonlinear

In order to prove that the system is nonlinear, we just need to show at least one example of the input-output pair that violates the superposition principle.
Let's consider input signal $x[n]=\pi/2$. The system's response is:

$$y[n] = 2x[n] + \sin(x[n]) = \pi + 1$$

According to homogeneity property, if the system is linear then $2x \rightarrow 2y$.
However, in our case the output corresponding to input $x'[n]=2x[n]$ is:

$$y'[n] = 2*2x[n] + \sin(2x[n]) = 2\pi \neq 2\pi + 2 = 2y[n]$$

Therefore, the system is nonlinear