Fourier Analysis
Lecture Outline

1) **Frequency Response**
2) **Fourier Transforms**
3) **Spectral Analysis of Signals**
4) **Fast Convolution**

Exponentials and LTI Systems

$$x[n] = e^{j\omega n}$$

What happens to **exponentials** after passing them through an **LTI system**?

$$y[n] = \sum_{k=-\infty}^{+\infty} x[n-k]\,h[k] = \sum_{k=-\infty}^{+\infty} e^{j\omega(n-k)}\,h[k] = e^{j\omega n} \sum_{k=-\infty}^{+\infty} e^{-j\omega k}\,h[k]$$

Frequency Response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} e^{-j\omega k}\, h[k]$$

$$y[n] = H(e^{j\omega})\, e^{j\omega n} = H(e^{j\omega})\, x[n]$$

(angular) frequency

$$e^{j\omega n} \rightarrow H(e^{j\omega})\, e^{j\omega n}$$

Frequency Response characterizes how an LTI system responds to a signal in the frequency domain, i.e. how an LTI system modifies the frequency components of an input signal

Euler's Relation

$$\begin{cases} e^{j\omega} = \cos\omega + j\sin\omega \\ e^{-j\omega} = \cos\omega - j\sin\omega \end{cases} \qquad \begin{cases} \cos\omega = (e^{j\omega} + e^{-j\omega})/2 \\ \sin\omega = (e^{j\omega} - e^{-j\omega})/2j \end{cases}$$

Complex Frequency Response in rectangular form:

$$H(e^{j\omega}) = H_{RE}(e^{j\omega}) + jH_{IM}(e^{j\omega})$$

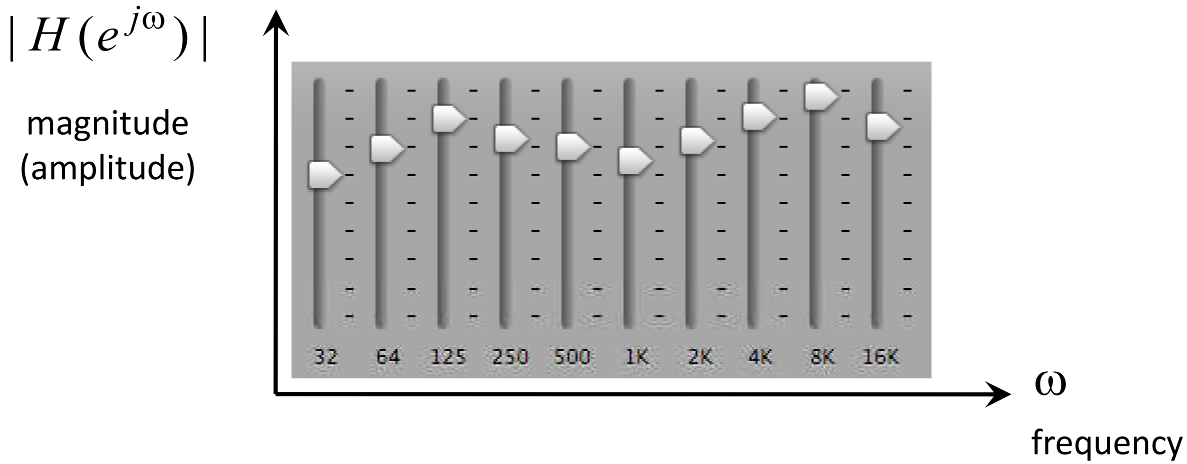
Complex Frequency Response in polar form:

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{\angle H(e^{j\omega})}$$

Phase Response

Magnitude Response

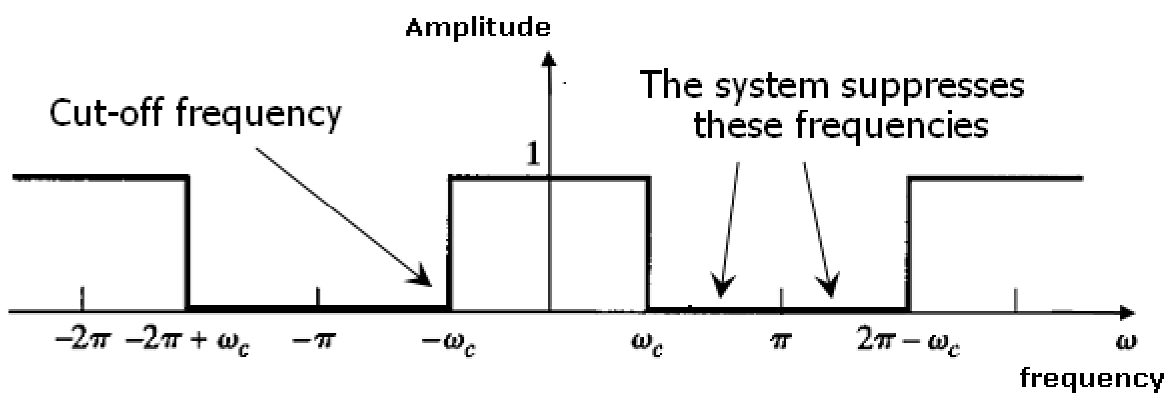
Frequency Response



Periodicity

Frequency Response is 2π-periodic:

$$H(e^{j(\omega+2\pi)}) = \sum_{k=-\infty}^{+\infty} e^{-jk(\omega+2\pi)}\, h[k] = \sum_{k=-\infty}^{+\infty} e^{-jk\omega} e^{-j2\pi k}\, h[k] = \sum_{k=-\infty}^{+\infty} e^{-jk\omega}\, h[k] = H(e^{j\omega})$$



Exercise 1

Find the frequency response of the ideal delay system defined as:

y[n]=x[n-d]

$$x[n] = e^{j\omega n} \qquad y[n] = x[n-d] = e^{j\omega(n-d)} = e^{-j\omega d} e^{j\omega n}$$

$$H(e^{j\omega}) = e^{-j\omega d}$$

$$|H(e^{j\omega})| = 1$$

$$\angle H(e^{j\omega}) = -\omega d$$

(the system doesn't change the amplitude of frequency components, but it shifts all phase components in proportion to the delay parameter *d*)

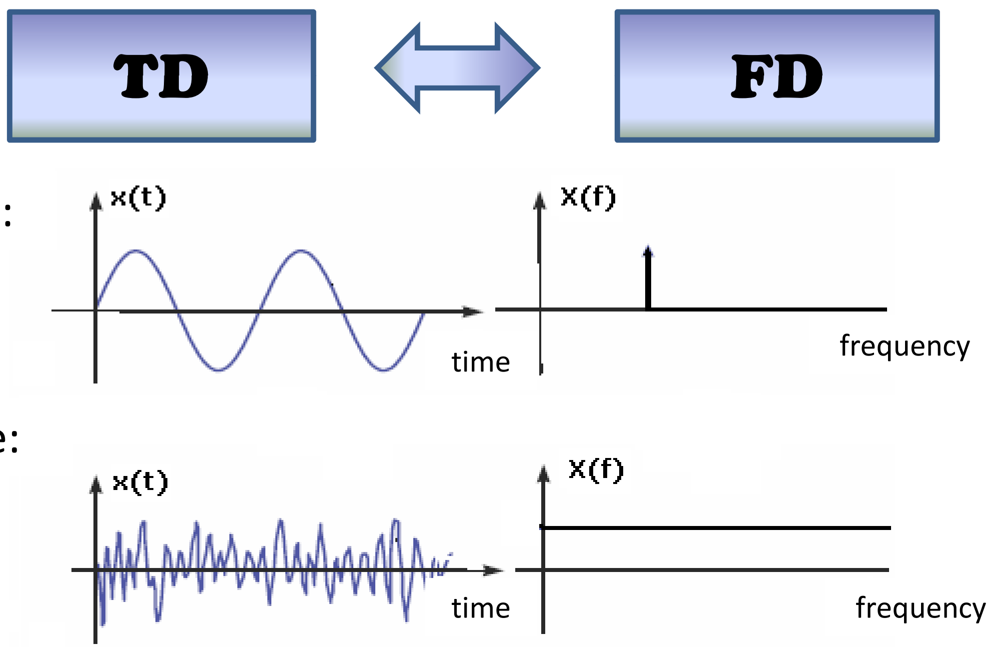
Fourier Transform

Fourier Transform

is a mathematical technique for the signal *analysis* and *synthesis* based on decomposition of the signal into a *set of sine wave components with different frequencies.*

One sinusoid:

White noise:



The Family of FTs

FOURIER SERIES Continuous signal Discrete spectrum *periodic signal*	**FOURIER TRANSFORM** Continuous signal Continuous spectrum *aperiodic signal*
periodic signal Discrete signal Discrete spectrum **DISCRETE FOURIER TRANSFORM (DFT)**	*aperiodic signal* Discrete signal Continuous spectrum **DISCRETE-TIME FOURIER TRANSFORM (DTFT)**

Fourier Series

Periodic signal x(t) with period T_0 can be written as:

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T_0}$$ - the distance between adjacent frequencies

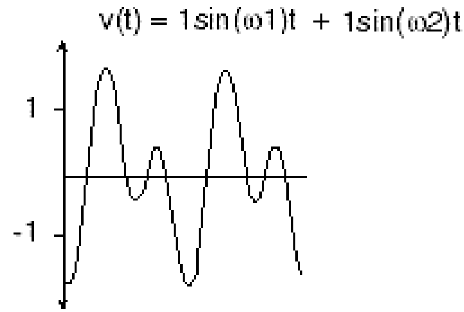
$2\omega_0, 3\omega_0, 4\omega_0, \ldots$ - *harmonics*

$$c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

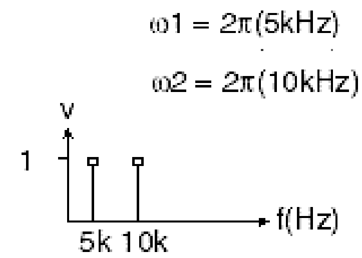
$$\begin{cases} x(t) = \displaystyle\sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t} \\ c_k = \dfrac{1}{T_0} \displaystyle\int_{T_0} x(t) e^{-jk\omega_0 t} dt \end{cases}$$

Synthesis (Inverse FT)

Analysis (Direct FT)



signal
sum of 2 sinusoids



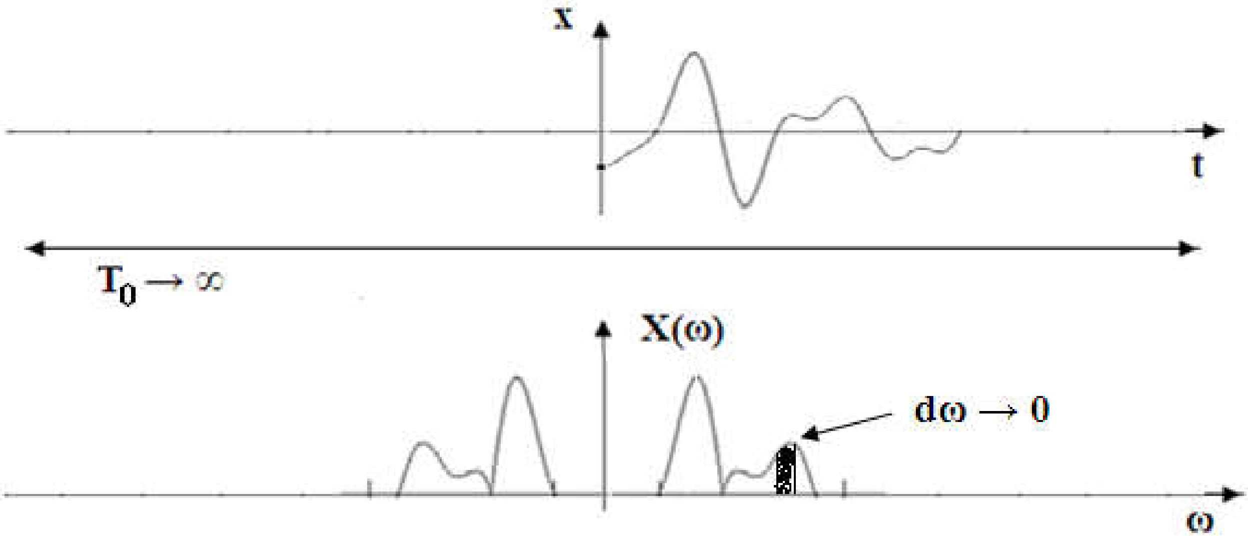
Fourier series
2 coefficients

Fourier Transform

$T_0 \to \infty$

$d\omega \to 0$

$$\begin{cases} x(t) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega \\ X(\omega) = \displaystyle\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \end{cases}$$



Discrete-Time Fourier Transform

Inverse DTFT

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

DTFT

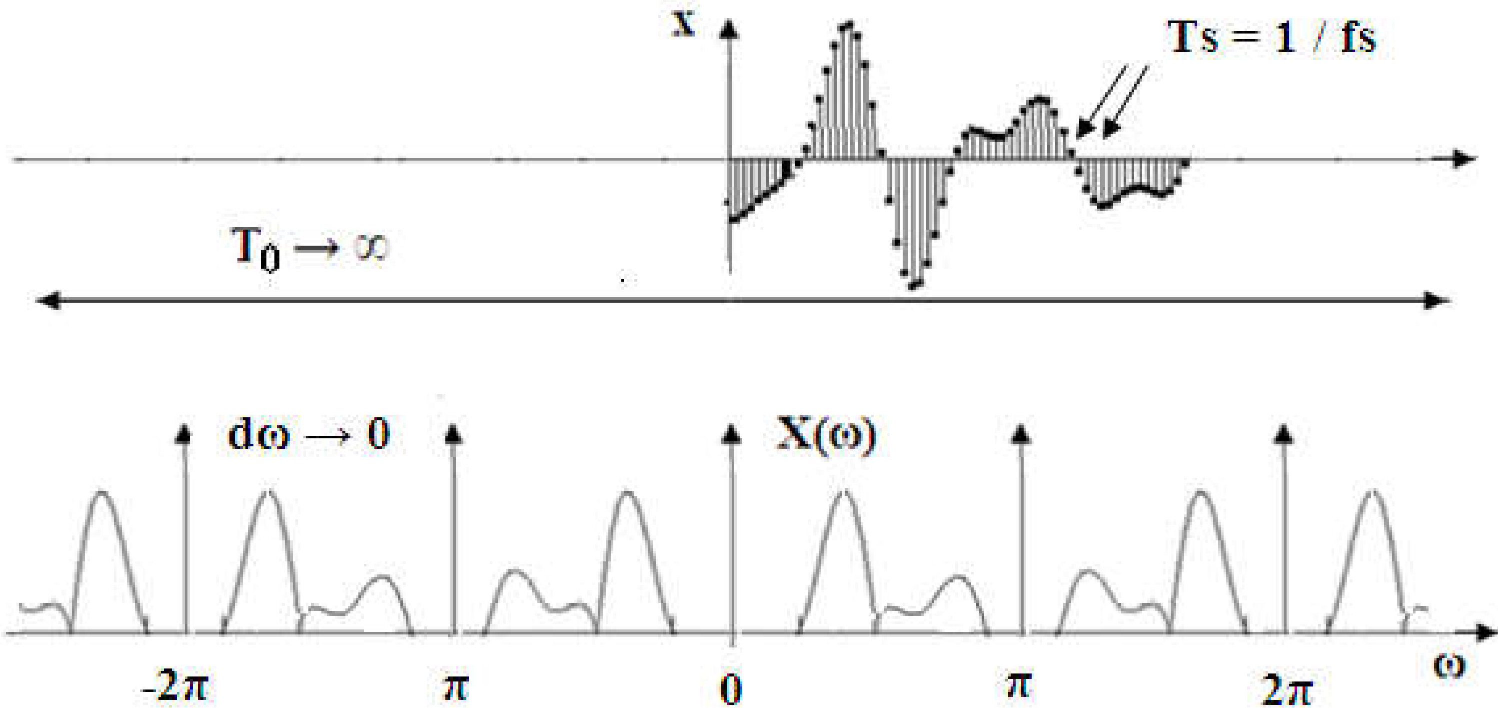
$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n}$$

$$\Omega = \omega T_s = \frac{2\pi f}{f_s}$$

Like the frequency response, and unlike the continuous FT-spectrum, the DTFT-spectrum is periodic with period 2π:

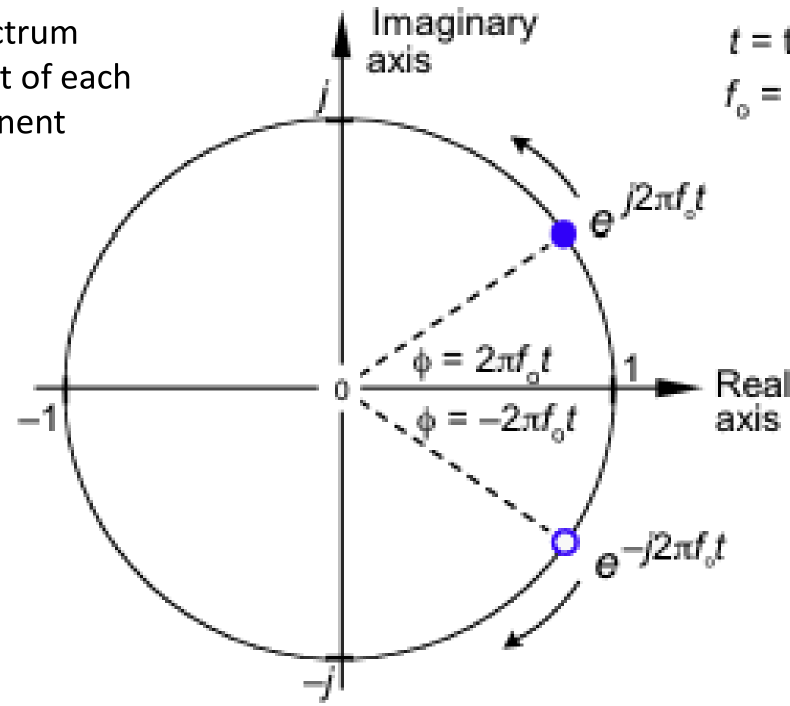
$$X(\Omega + 2\pi k) = \sum_{k=-\infty}^{+\infty} x[n] e^{-j\Omega n + 2\pi k n j} = \sum_{k=-\infty}^{+\infty} x[n] e^{-jn\Omega} = X(\Omega)$$

Discrete-Time Fourier Transform



Frequency Axis

Notation $X(e^{j\omega})$ reflects the same idea: each value of the spectrum determines the relative amount of each complex sinusoidal component



t = time in seconds,

f_o = frequency in Hz

Magnitude and Phase Spectrum

The magnitude of DTFT is called the ***Magnitude spectrum*** $|X(e^{j\omega})|$

The phase of DTFT is called the ***Phase spectrum*** $\angle X(e^{j\omega})$

The ***Power spectrum*** is defined as $|X(e^{j\omega})|^2$

$X(e^{j\omega})$ is sometimes referred to as the ***Fourier image*** of signal

Discrete Fourier Transform

The DFT can be derived from the DTFT in two steps:

1) signal truncating
2) frequency sampling

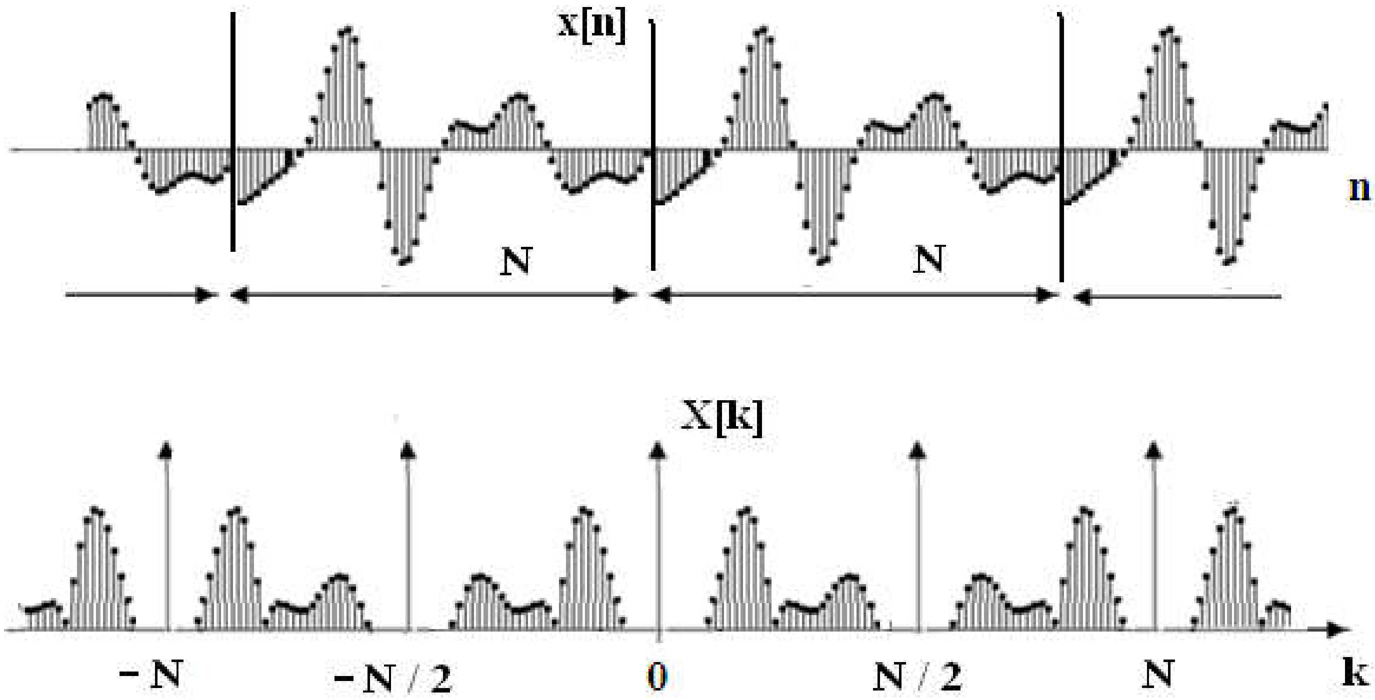
$$\omega_k = \frac{2\pi k}{N}, \qquad k = 0,..,N-1$$

$$X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} = \sum_{k=-\infty}^{+\infty} x[n]e^{-j\omega n}\Big|_{\omega=\frac{2\pi k}{N}} = \sum_{k=0}^{N-1} x[n]e^{-jnk\frac{2\pi}{N}} = X[k]$$

Inverse DFT
$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jnk\frac{2\pi}{N}}$$

DFT
$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-jnk\frac{2\pi}{N}}$$

Discrete Fourier Transform

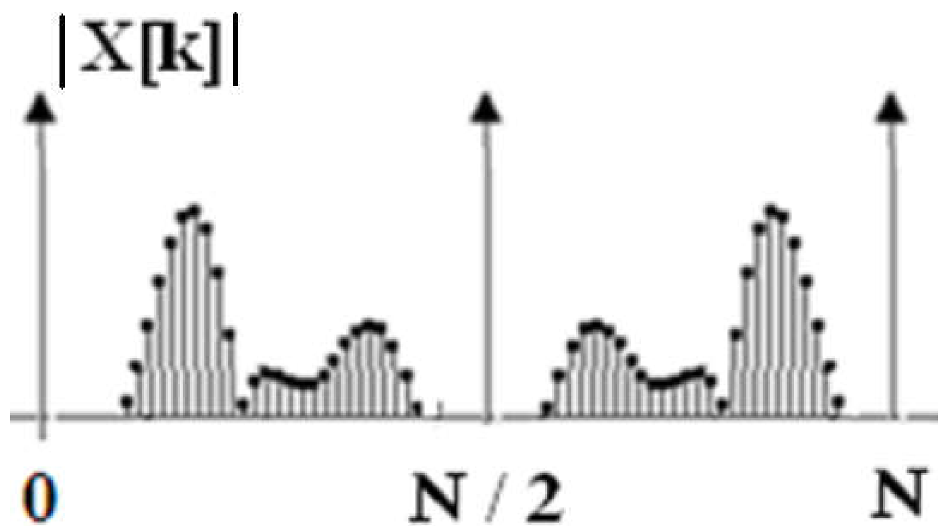


Frequency Resolution

$$\Delta = \frac{f_s}{N}$$ - Frequency resolution

The k^{th} element in the DFT-coefficients array stands for frequency f_k whose value is

$$f_k = k\Delta = \frac{kf_s}{N}, \qquad k = 0,..,N-1$$



Frequency Resolution

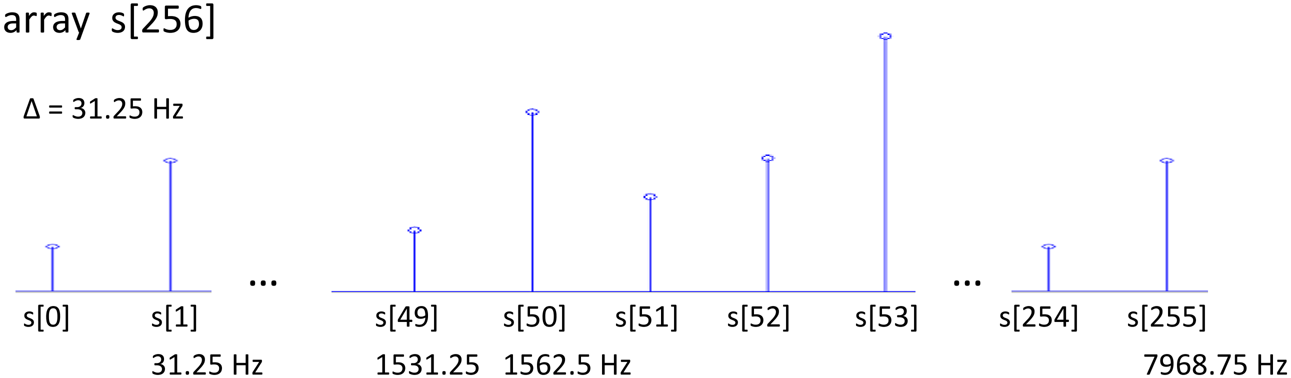
Given a signal sampled at frequency 8000 Hz. What is the frequency resolution of 256-point DFT of the signal? What frequency (in Hz) does the 50^{th} spectral sample represent?

The frequency resolution is: Δ = 8000 / 256 = 31.25 (Hz).

Therefore, the 50^{th} spectral sample represents the frequency:

$$f_{50} = 50 * 31.25 = 1562.5 \text{ (Hz)}$$

Spectrum: array s[256]



Fast Fourier Transform

DFT was not used in engineering for a long time because of its computational expensiveness. An algorithm based on a straightforward implementation of the DFT formulae has the time complexity of **$O(N^2)$** and is too slow for real-time applications.

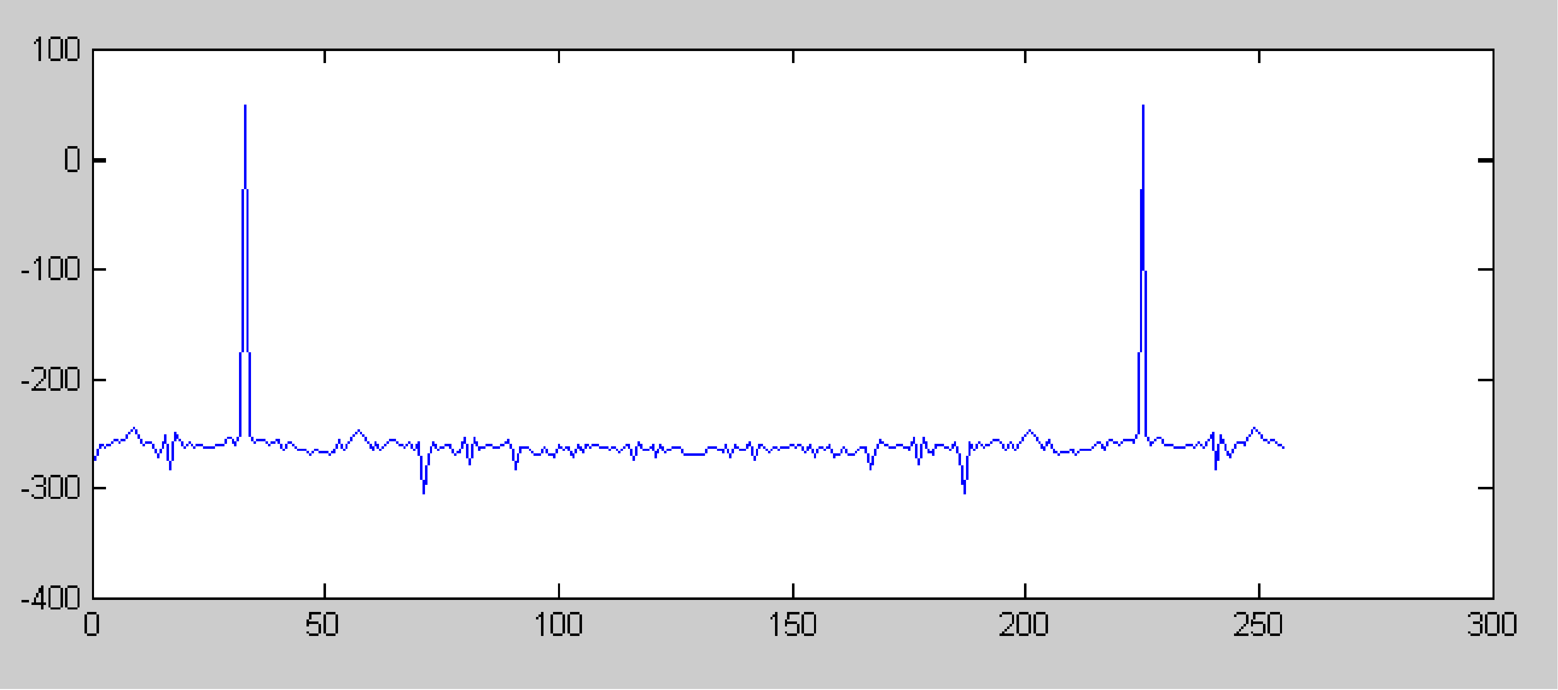
A true revolution was made in 1965, with the inventing of *Fast Fourier transform algorithm (FFT)* by J.W.Cooley and J.Tukey.
This is radix-2 algorithm that has the time complexity of **$O(N*\log_2 N)$**.
The term “radix-2” means that the algorithm performs N-point FFT for N a power of 2.

Currently, there are many different FFT algorithms involving a wide range of mathematics:

- ✓*Prime-factor FFT* algorithm
- ✓*Bruun's FFT* algorithm
- ✓*Rader's FFT* algorithm
- ✓*Bluestein's FFT* algorithm.

Fast Fourier Transform

generate 256 samples of a sine wave with frequency 1000 Hz, A=2, and sampling rate of 8000 Hz. Plot the magnitude and phase spectrums



Decibels (dB)

$$L_{dB} = 10\log\frac{P}{P_0} = 20\log\frac{A}{A_0}$$

where P and A are ***the power and amplitude*** of signal, respectively;
P_0 and A_0 are the ***reference power and reference amplitude*** of signal, respectively.

The values of A_0 and P_0 are usually chosen in terms of particular practical purposes.

Since decibels are a way of expressing the ratio between two signals, they are ideal for describing the gain of a system, i.e. the ratio between the output and the input signal. However, engineers also use decibels to specify the amplitude of a single signal, by referencing it to some standard. E.g., the term: dBV means that the signal is being referenced to a 1 volt rms signal.

Fourier Transform Properties

Linearity: $z[n] = ax[n] + by[n] \overset{F}{\longleftrightarrow} Z(e^{j\omega}) = aX(e^{j\omega}) + bY(e^{j\omega})$

Time shifting: $y[n] = x[n-d] \overset{F}{\longleftrightarrow} Y(e^{j\omega}) = e^{-j\omega d} X(e^{j\omega})$

Frequency shifting: $y[n] = e^{j\theta n} x[n] \overset{F}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j(\omega-\theta)})$

Symmetry:

$$X_{RE}(e^{j\omega}) = X_{RE}(e^{-j\omega})$$

$$X_{IM}(e^{j\omega}) = -X_{IM}(e^{-j\omega})$$

Fourier Transform Properties

Convolution Theorem: $z[n] = x[n] * y[n] \xleftrightarrow{F} Z(e^{j\omega}) = X(e^{j\omega})Y(e^{j\omega})$

Modulation Theorem: $y[n] = x[n]w[n] \xleftrightarrow{F} Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\, W(e^{j(\omega-\theta)})d\theta$

Parseval's Theorem: $E = \sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2\, d\omega$

Impulse Response – FT – Frequency Response

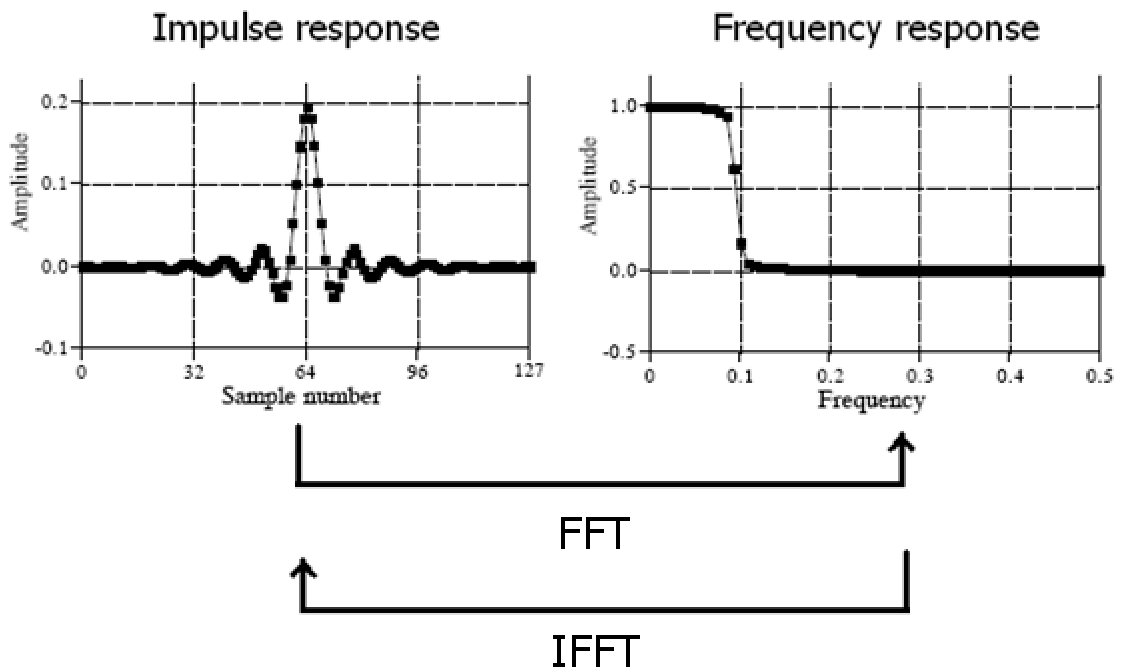
Frequency Response of an LTI system is the **Fourier Transform** of its **Impulse Response**

$$h^*[n]=\frac{1}{2\pi}\int_{2\pi}H(e^{j\omega})e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{2\pi}\sum_{k=-\infty}^{+\infty}h[k]e^{-j\omega k}\ e^{j\omega n}d\omega$$

$$h^*[n]=\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}h[k]\int_{2\pi}e^{j\omega(n-k)}d\omega=h[n]$$ since $$\int_{2\pi}e^{j\omega(n-k)}d\omega=\begin{cases}2\pi, & n=k\\ 0, & n\neq k\end{cases}$$

Thus, $h[n]\overset{F}{\longleftrightarrow}H(e^{j\omega})$

Impulse Response – FT – Frequency Response



Sampling in FD

The *continuous* output signal $x_s(t)$ of the sampler can be considered as the signal $x_c(t)$ modulated with the periodic *impulse train* $\delta(t–nT_s)$:

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\,\delta(t - nT_s)$$

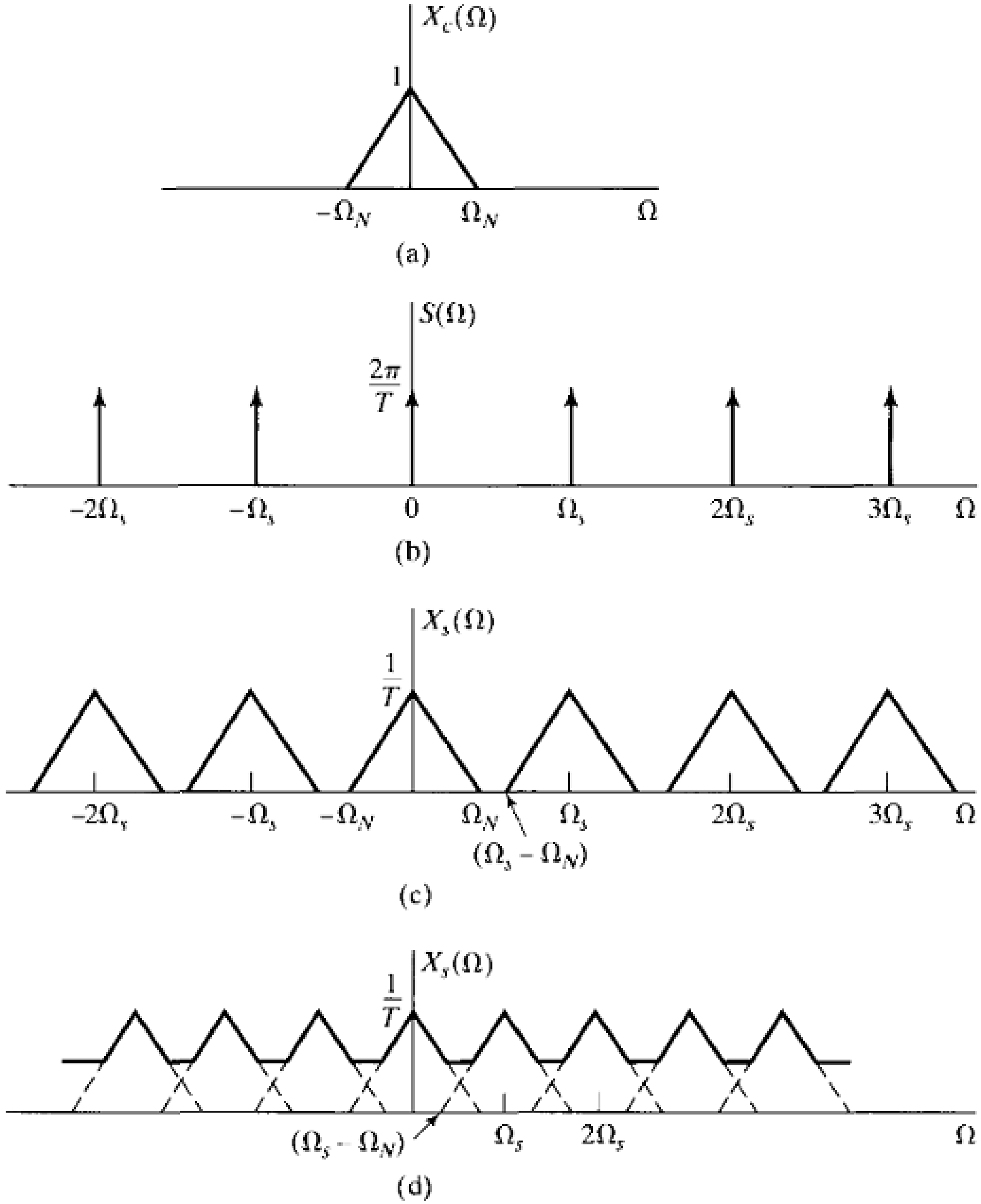
Sampling in FD

The Fourier transform of a periodic impulse train is also periodic impulse train:

$$S(\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\Omega - k\Omega)$$

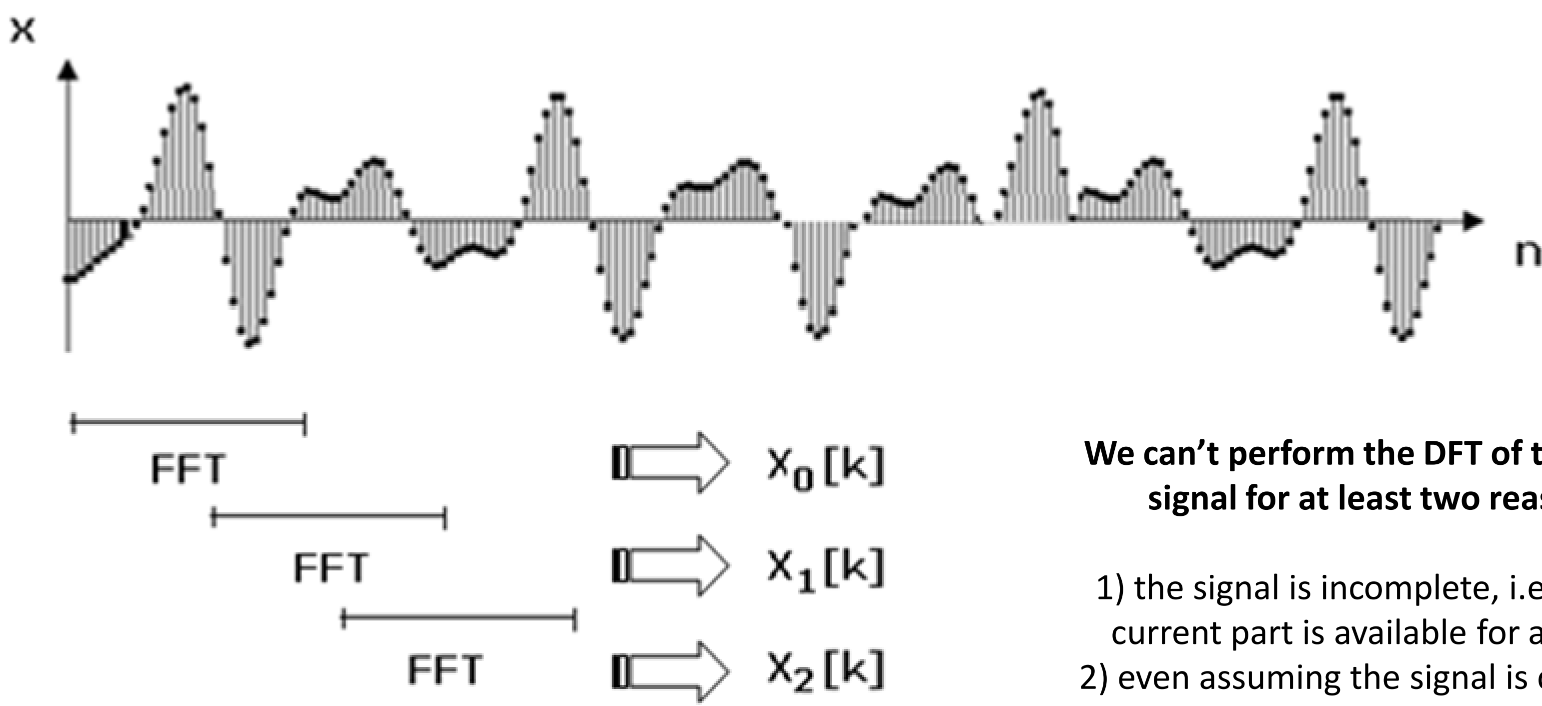
Since, $x_s(t)$ is a multiplication of the signal $x_c(t)$ by $\delta(t-nT_s)$, according to the Modulation Theorem, the spectrum of signal $x_s(t)$ is the convolution of spectra:

$$X_s(\Omega) = \frac{1}{2\pi} X_c(\Omega) * S(\Omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c(\Omega - k\Omega_s)$$



$X_c(\Omega)$
1
$-\Omega_N$
Ω_N
Ω
(a)
$S(\Omega)$
$\frac{2\pi}{T}$
$-2\Omega_s$
$-\Omega_s$
0
Ω_s
$2\Omega_s$
$3\Omega_s$
Ω
(b)
$X_s(\Omega)$
$\frac{1}{T}$
$-2\Omega_s$
$-\Omega_s$
$-\Omega_N$
Ω_N
Ω_s
$2\Omega_s$
$3\Omega_s$
Ω
$(\Omega_s - \Omega_N)$
(c)
$X_s(\Omega)$
$\frac{1}{T}$
$(\Omega_s - \Omega_N)$
Ω_s
$2\Omega_s$
Ω
(d)

Spectral Analysis



We can’t perform the DFT of the entire signal for at least two reasons:

1) the signal is incomplete, i.e. only its current part is available for analysis;
2) even assuming the signal is complete, the resulting spectrum will be noisy, since it will contain all signal frequency components, including those that capture the slowly varying changes of signal

Windowing

Extracting each *N-point* block from the signal is equivalent to multiplication of the signal by the *rectangular window function,* or *rectangular pulse:*

$$w[n] = \begin{cases} 1, & n \le N \\ 0, & n > N \end{cases}$$

According to the Modulation Theorem, windowing leads to the periodic convolution of signal spectrum and rectangular pulse spectrum:

$$\underbrace{y[n] = x[n]w[n]}_{\text{windowing}} \quad \xleftrightarrow{F} \quad Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\, W(e^{j(\omega-\theta)})d\theta$$

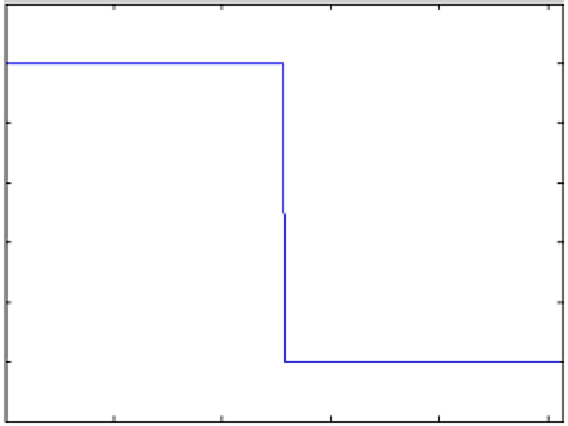
Fourier Image of the Rectangular Pulse

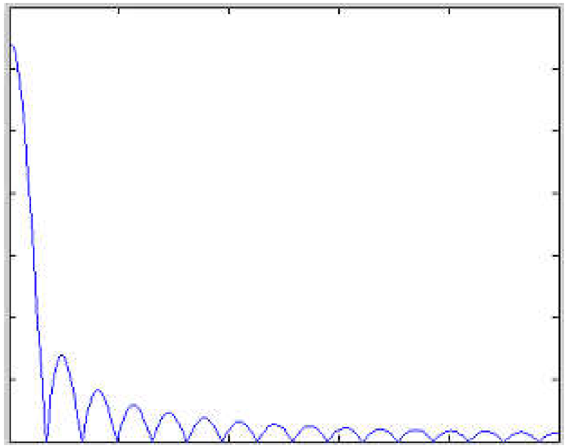
$$W(e^{j\omega}) = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1-e^{-j\omega(N-1)}}{1-e^{-j\omega}}$$

$$\sin(\omega/2) = \frac{e^{j\omega/2} - e^{-j\omega/2}}{2j} = \frac{1-e^{-j\omega}}{2j\, e^{j\omega/2}}$$

$$W(e^{j\omega}) = e^{-j\omega(N-1)/2}\frac{\sin(\omega N/2)}{\sin(\omega/2)} \approx Ne^{-j\omega(N-1)/2} sinc(fN)$$

$$sinc(x) = \frac{\sin(\pi x)}{\pi x}$$



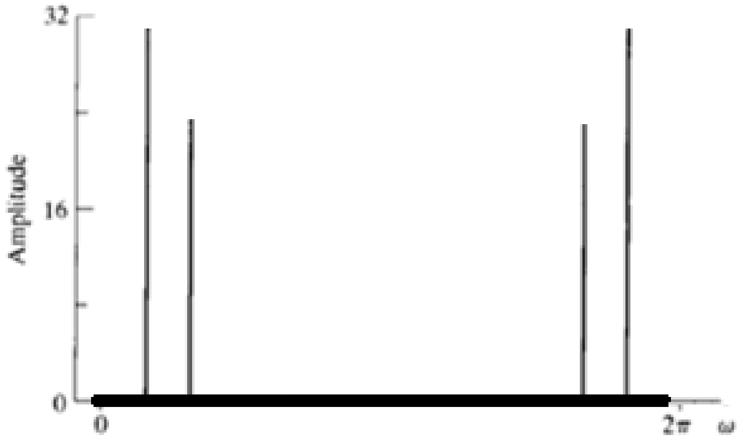


Spectral Leakage

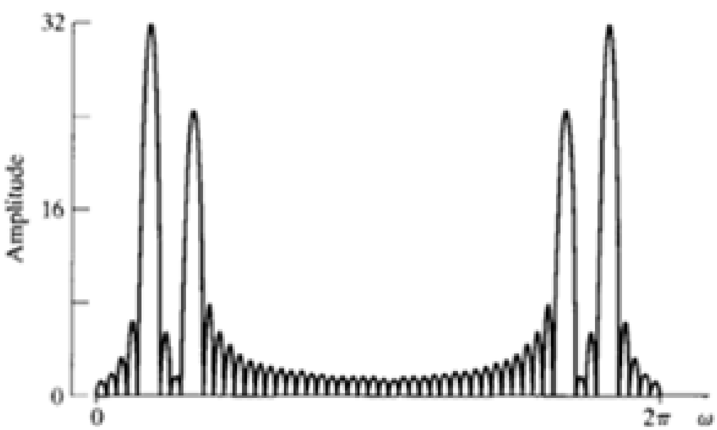
example

$$x[n] = \cos(\frac{2\pi}{16}n) + 0.75\cos(\frac{2\pi}{8}n)$$

DTFT of x[n]



DTFT of x[n]w[n]



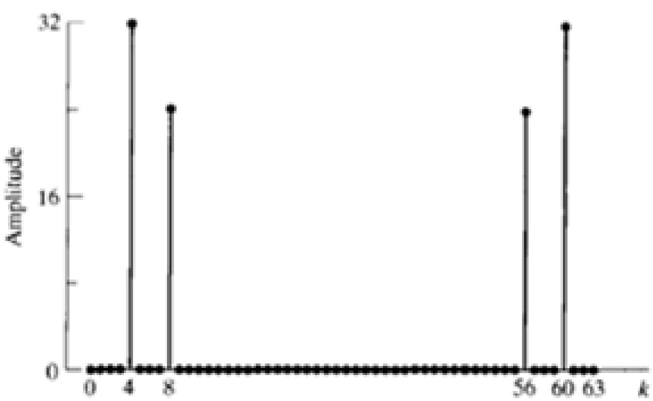
Spectral Sampling

$$x[n] = \cos(\frac{2\pi}{16}n) + 0.75\cos(\frac{2\pi}{8}n)$$

$$Y[k] = Y(e^{j\omega})|_{\omega=\frac{2\pi k}{N}}$$

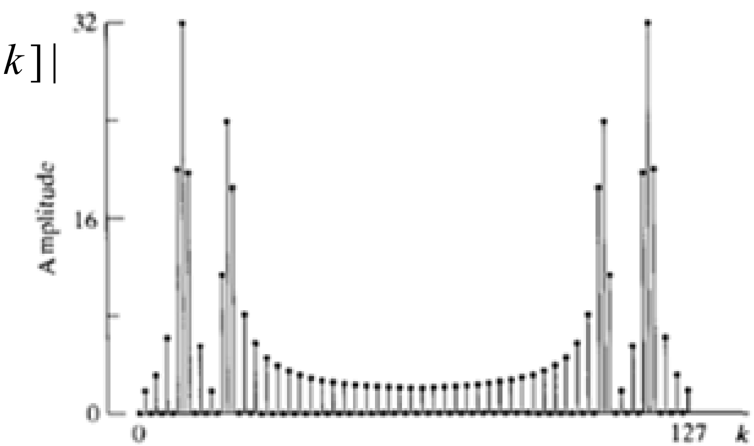
64-point DFT

$|Y[k]|$

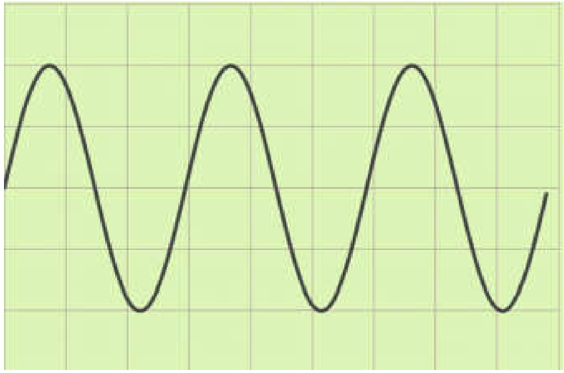


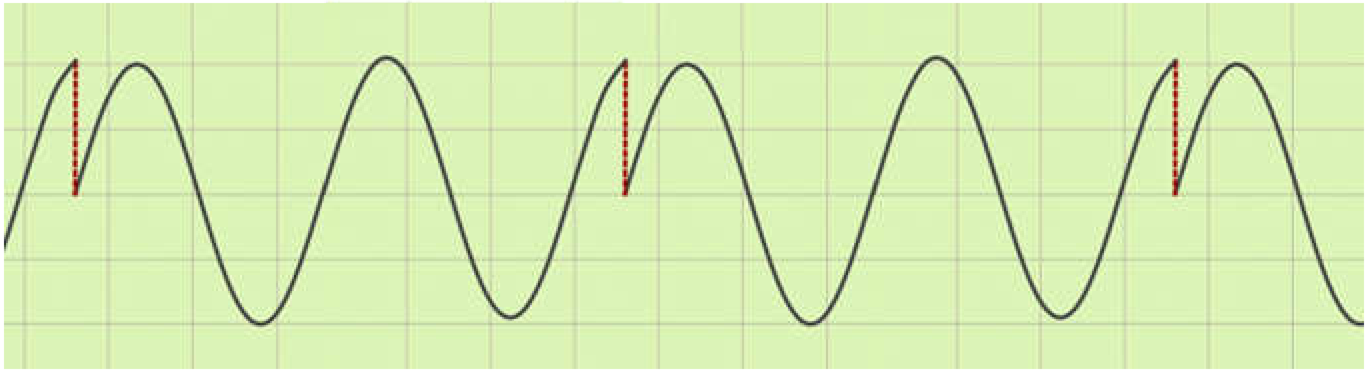
128-point DFT (zero-padding)

$|Y[k]|$



Discontinuity Issues





Convergence of Fourier Transform

The sufficient condition for the ***existence*** and ***uniform convergence*** of FT is the ***absolute summability***:

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

There is also a “relaxed” condition: ***square*** **summability** (the convergence in ***mean-square sense***):

$$\lim_{m\to\infty} \int_{-\pi}^{\pi} |X(e^{j\omega}) - X_M(e^{j\omega})|^2 \, d\omega = 0$$

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n] e^{-j\omega n}$$

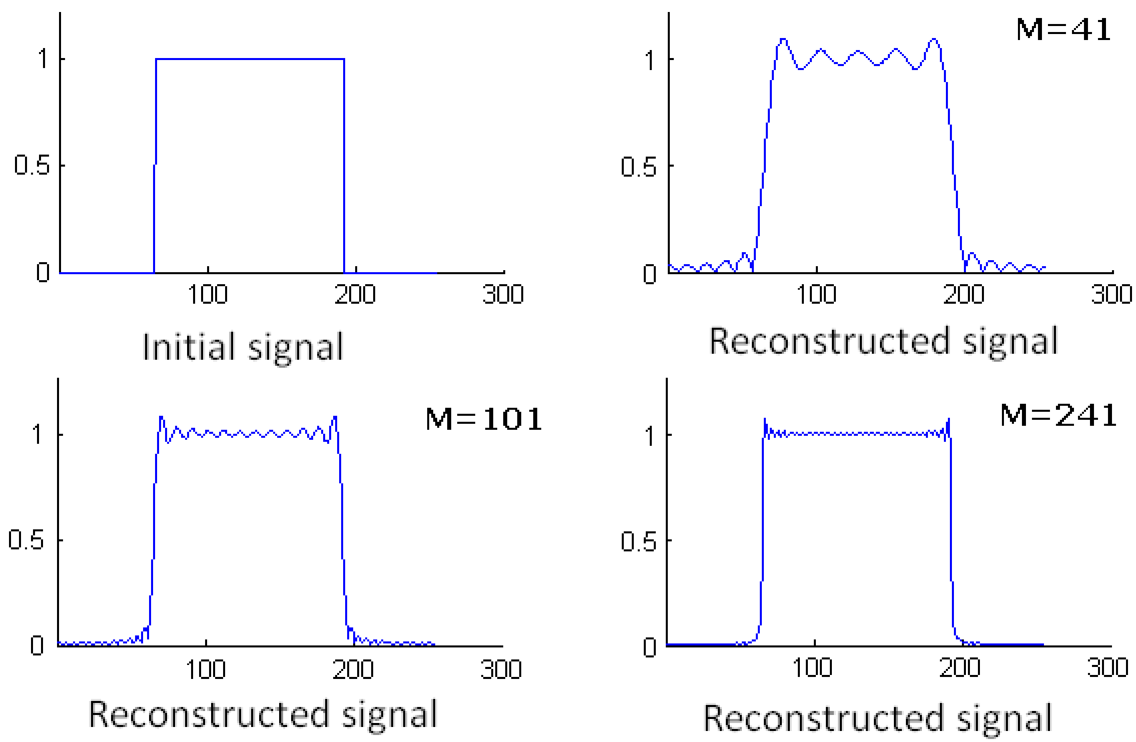
Gibbs Effect

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}$$

Gibbs phenomenon:

oscillations at the discontinuity points.

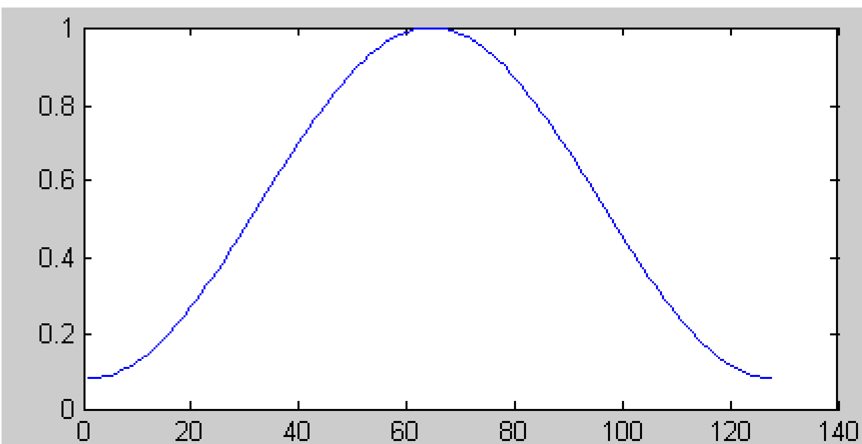
As *M* increases the oscillations become more frequent and the width of the overshoot decreases. However, the size of the ripples does not decrease (their amplitude is equal to approximately 9% of the total amplitude)

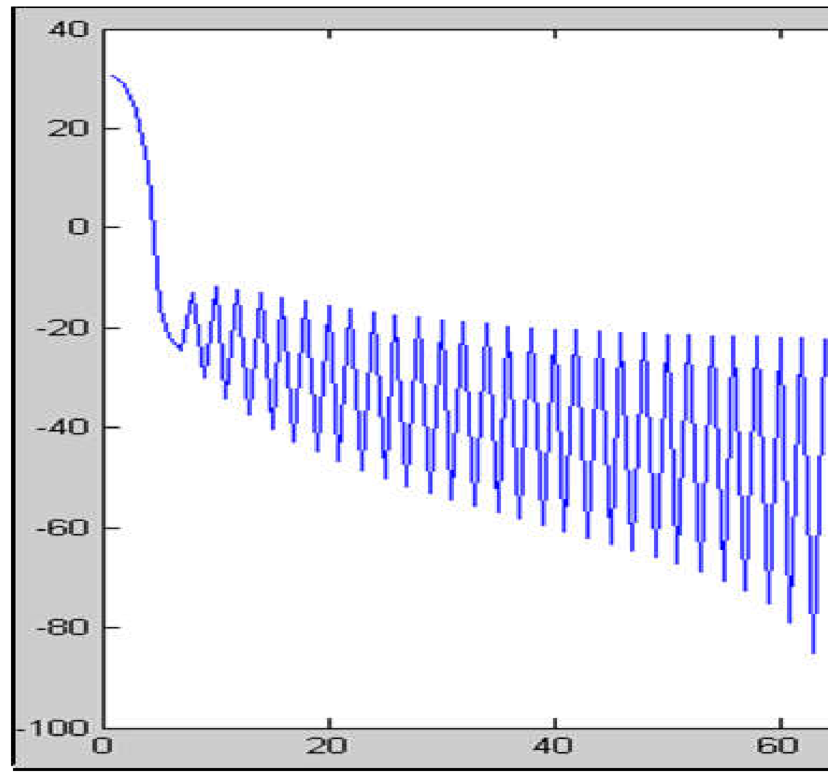


Hamming Window

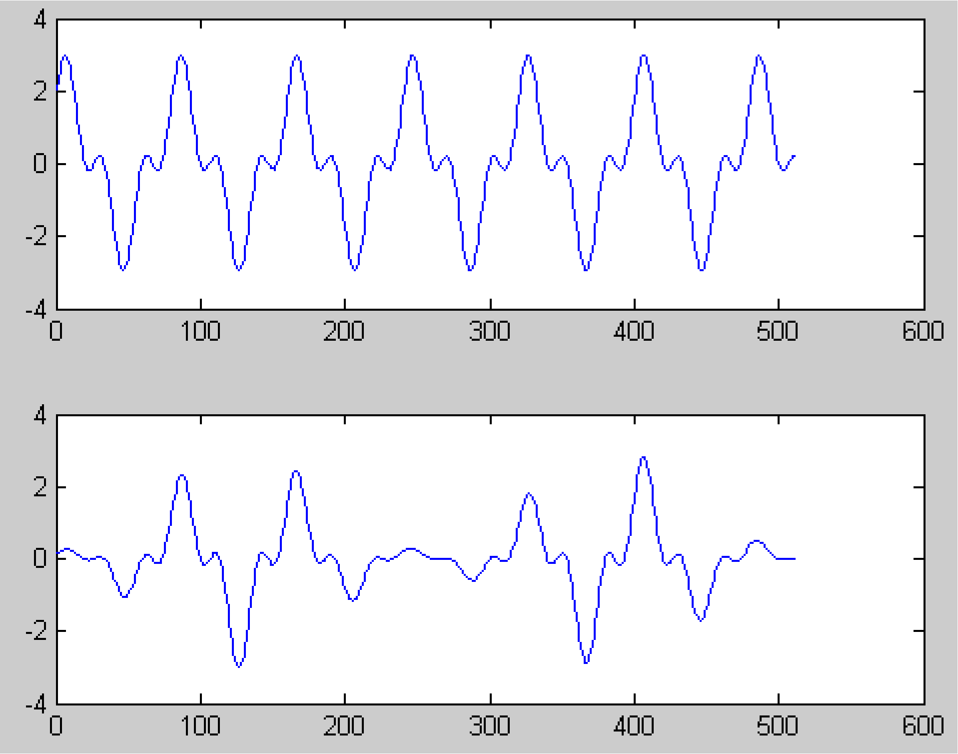
Fourier image of Hamming window

Hamming window

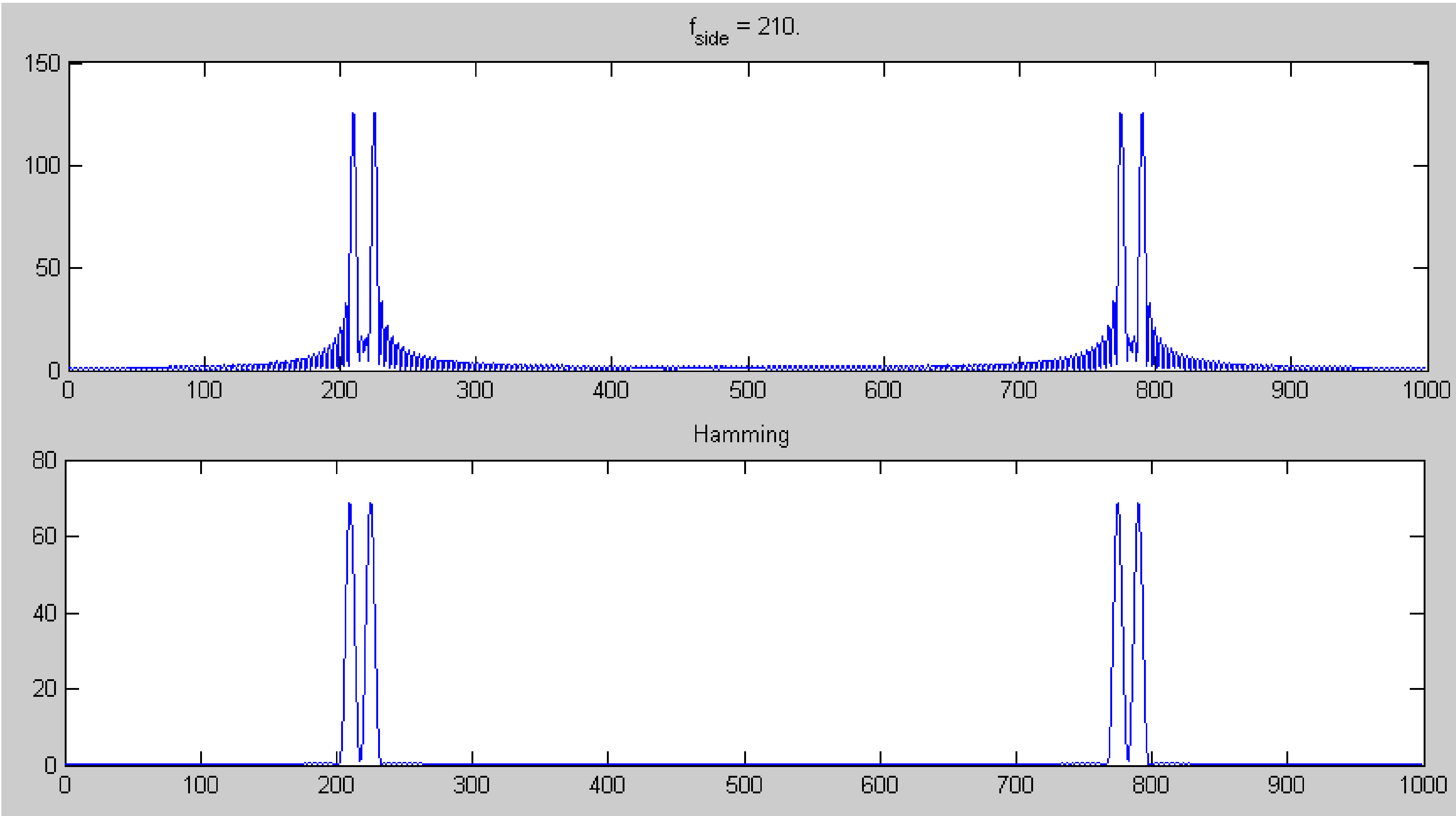




Windowing



Hamming Window vs. Rectangular window



Window Functions

	Formula for *w[n], n = 0..M–1*	Width of main lobe	Peak side-lobe amplitude
Rectangular window	$w[n]=1$	4π / (M+1)	-13 dB
Bartlett window	$w[n]=\begin{cases} 2n/M, & 0\le n\le M/2 \\ 2-2n/M, & M/2<n<M \end{cases}$	8π / M	-25 dB
Hamming window	$w[n]=0.54-0.46\cos(\frac{2\pi}{M}n)$	8π / M	-31 dB
Blackman window	$w[n]=0.42-0.5\cos(\frac{2\pi}{M}n)+0.08\cos(\frac{4\pi}{M}n)$	12π / M	-57 dB
Hanning window	$w[n]=0.5-0.5\cos(\frac{2\pi}{M}n)$	8π / M	-41 dB

Short-Time Fourier Transform

Short-Time Fourier Transform (STFT)

is a Fourier-related transform used to determine the frequency and phase content of local sections of a signal as it changes over time

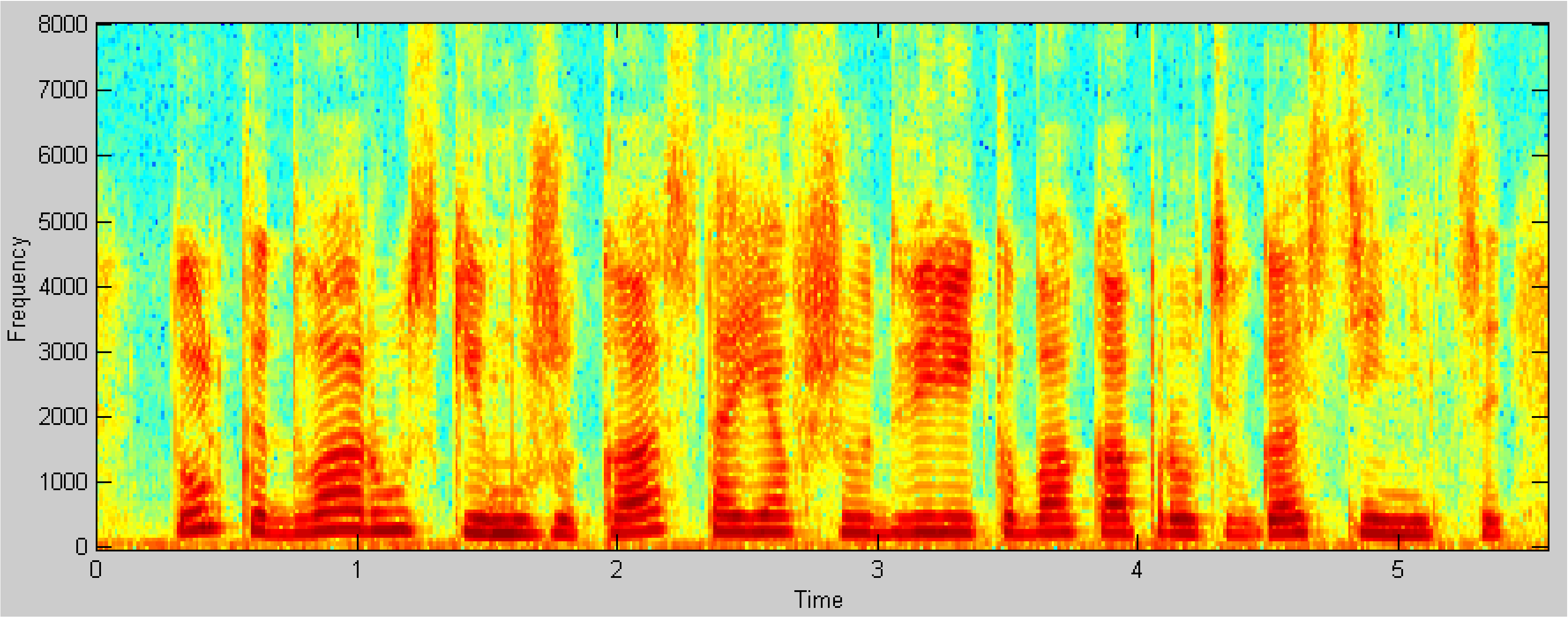
$$X(m,\omega) = \sum_{n=-\infty}^{+\infty} x[n+m]\,w[m]\,e^{-j\omega m}$$

The more rapidly signal characteristics change, the shorter the window should be.
However, as the window ***becomes shorter***, frequency resolution ***decreases***.
On the other hand, as the window length ***decreases***, the ability to resolve changes in time ***increases***.

Consequently, the choice of window length is ***trade-off between frequency resolution and time resolution***

Spectrogram

Plot the spectrogram of the signal contained in file “d:\1.wav”

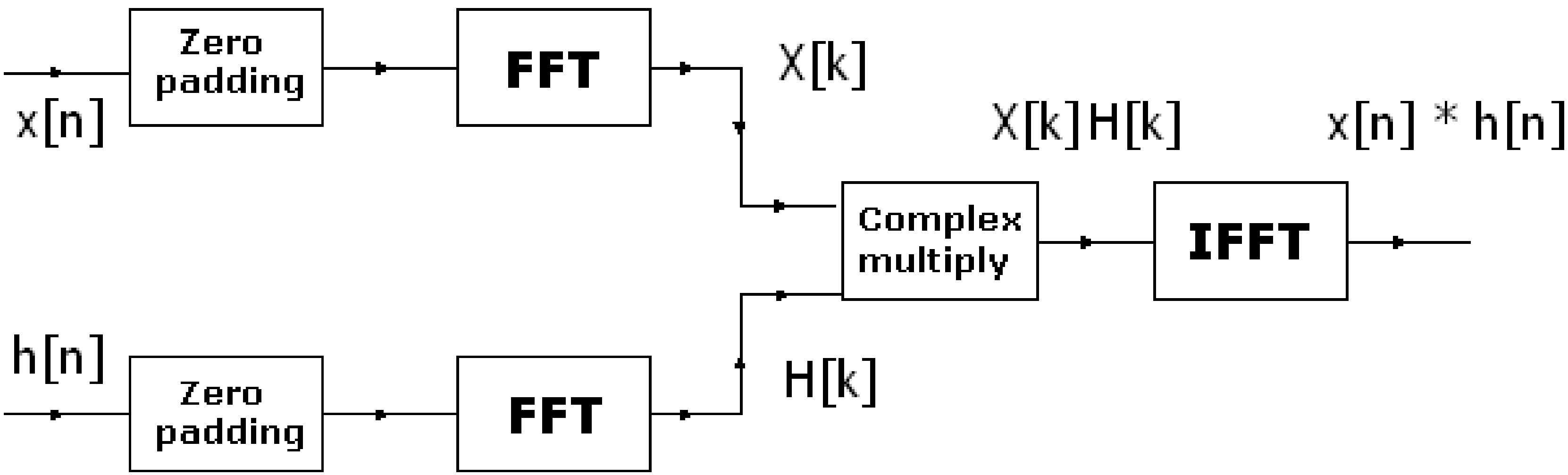


Fast Convolution

$$y[n] = \sum_{k=0}^{M-1} x[n-k]\, h[k]$$

Convolution Theorem:

$$y[n] = x[n] * h[n] \quad \xleftrightarrow{F} \quad Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$



Fast Convolution

DIRECT CONVOLUTION

$$y[n] = \sum_{k=0}^{M-1} x[n-k]\, h[k]$$

FAST CONVOLUTION

$$y[n] = x[n] * h[n] \xleftrightarrow{F} Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

example

256-point signal x[n]

64-point signal h[n]

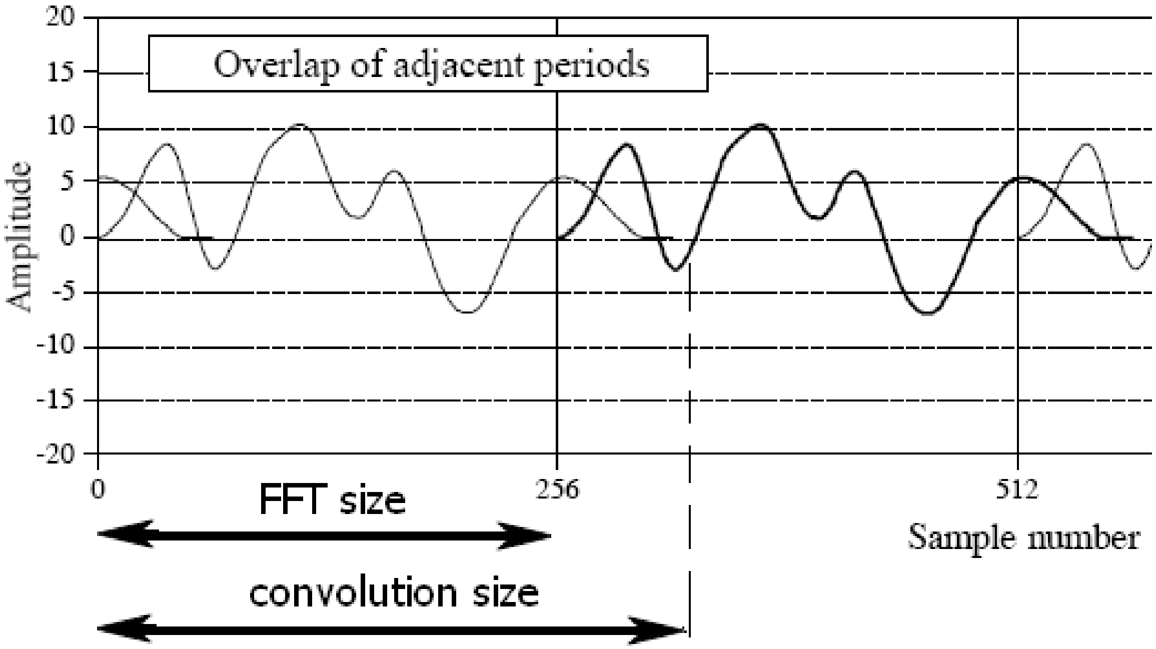
319-point signal y[n] = x[n] * h[n]

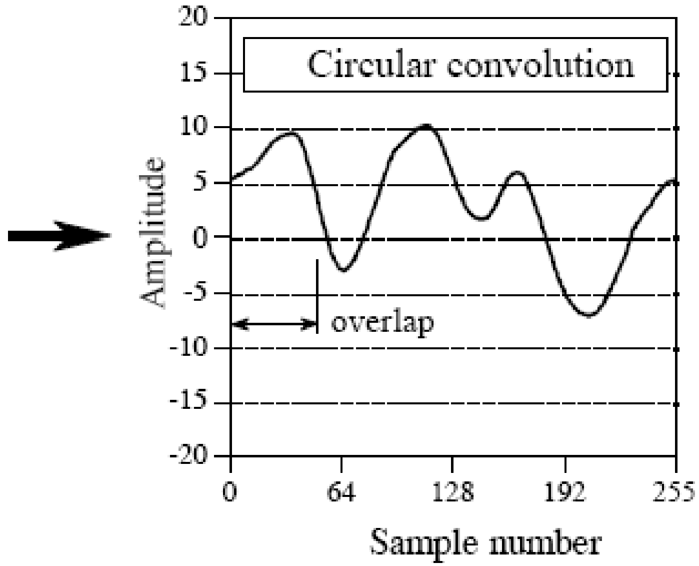
256-point DFT X[k]

256-point DFT H[k]

256-point Inverse DFT y[n]

Circular Convolution





Fast Convolution (Correct)

DIRECT CONVOLUTION

$$y[n] = \sum_{k=0}^{M-1} x[n-k]\, h[k]$$

FAST CONVOLUTION

$$y[n] = x[n] * h[n] \overset{F}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

example

ZERO-PADDING

256-point signal x[n]

64-point signal h[n]

319-point signal y[n] = x[n] * h[n]

512-point DFT X[k]

512-point DFT H[k]

512-point Inverse DFT y[n]

DeConvolution

$y[n] = x[n] * h[n]$ $h[n] - ?$