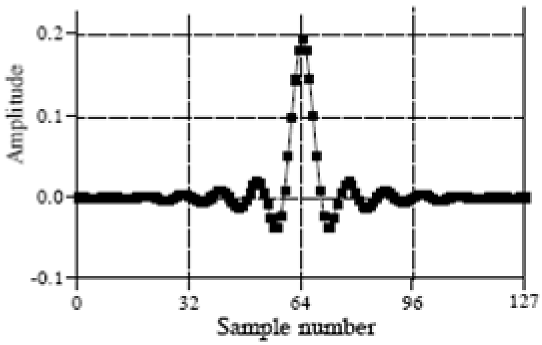
Filter Design and Analysis
Lecture Outline

1) Z-transform
2) Transfer Function
3) Filter Analysis in z-domain
4) Window Method

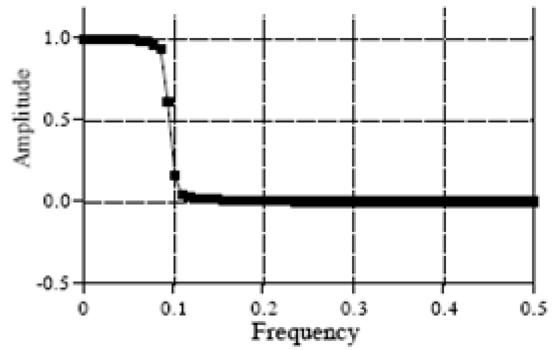
Z-transform

Impulse response



DTFT

Frequency response



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$$y[n] = 2x[n] - 0.5\,y[n-1] + 0.9\,y[n-2]$$

Z-transform

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

Z-transform

can be considered as a bridge between the difference equations and the frequency response of an LTI system.

It plays the same role for discrete-time systems as the Laplace transform does for continuous time systems: it allows to replace the difference equations with the algebraic equations that are much easier to solve

Z~transform and DTFT

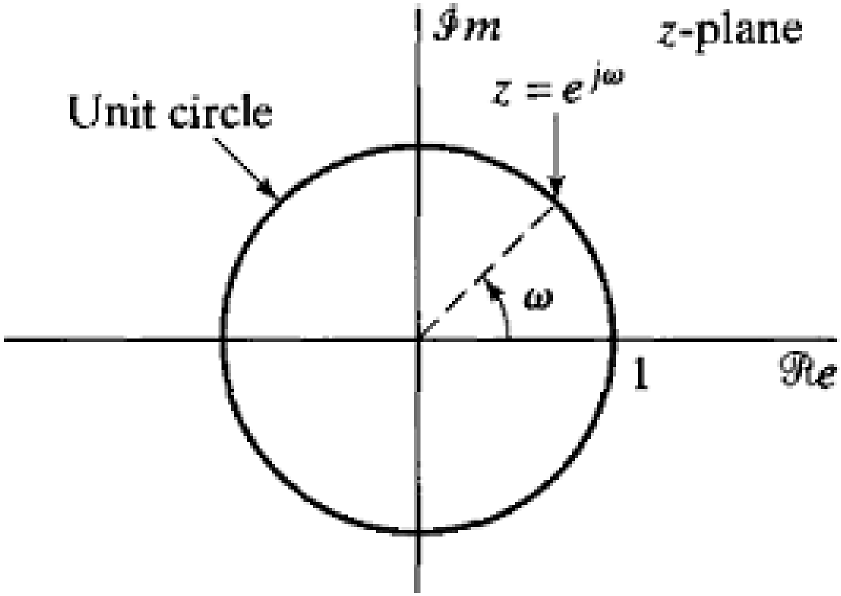
Complex z:

$$z = re^{j\omega}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{+\infty} (x[n]r^{-n})e^{-j\omega n}$$

$$|r| = 1$$

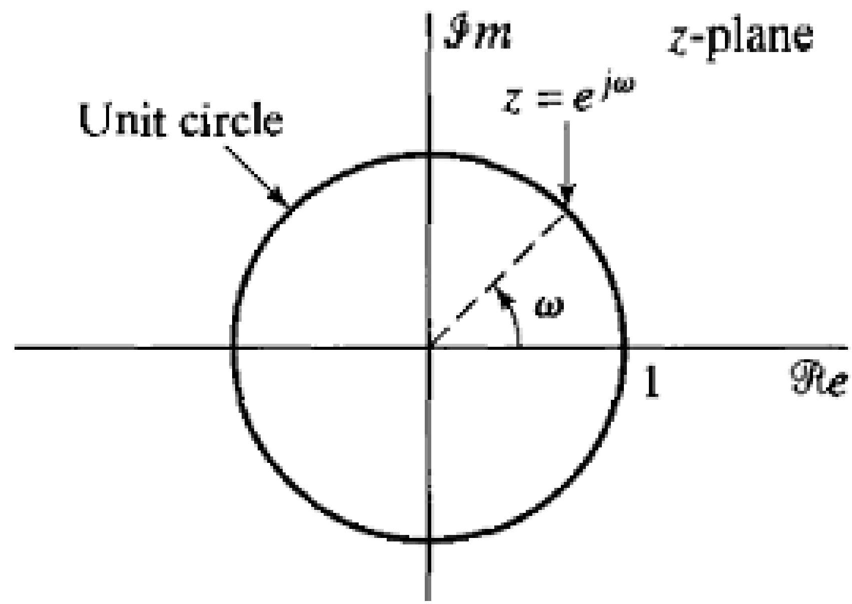
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$



Region of Convergence (ROC)

$$ROC = \left\{ z \in C \mid \sum_{n=-\infty}^{+\infty} x[n] z^{-n} < \infty \right\} \qquad (5.1)$$

The ROC consists of all z such that the inequality (5.1) holds. This means that if the ROC contains value z=z0, then it also contains all values of z on the circle |z|=|z0|. In other words, any ROC contains a set of circles in z-plane



Exercise 1

Find z-transform of the signal x[n] = k, where k = const, n=0,1,2,…

Solution:

$$X(z) = \sum_{n=0}^{+\infty} kz^{-n} \qquad (6.1)$$

For |z|>1 the series (6.1) converges:

$$X(z) = \sum_{n=0}^{+\infty} kz^{-n} = k \cdot \frac{1}{1 - z^{-1}} = \frac{kz}{z-1}$$

For |z|≤1 the series (6.1) diverges. Therefore, ROC consists of all z: |z|>1.

Note, we've considered the signal x[n]=k for n>0. The same equation is often rewritten as:
x[n] = ku[n], where u[n] is a unit step

Z~transform Pairs

Signal	Z-transform	ROC
$\delta[n]$	1	All z
$ku[n]$	$\dfrac{k}{1-z^{-1}}$	$\lvert z\rvert>1$
$-ku[-n-1]$	$\dfrac{k}{1-z^{-1}}$	$\lvert z\rvert<1$
$\delta[n--k]$	z^{-k}	All z, except z=0 (if k>0) and z=∞ (if k<0)
$k^n u[n]$	$\dfrac{1}{1-kz^{-1}}$	$\lvert z\rvert>\lvert k\rvert$
$-k^n u[-n-1]$	$\dfrac{1}{1-kz^{-1}}$	$\lvert z\rvert<\lvert k\rvert$
$nk^n u[n]$	$\dfrac{kz^{-1}}{(1-kz^{-1})^2}$	$\lvert z\rvert>\lvert k\rvert$

Z-transform Properties

Linearity:

$$ax[n]+by[n] \xleftrightarrow{Z} aX(z)+bY(z)$$

ROC is the **intersection** of ROCs of X(z) and Y(z)

Time shifting:

$$x[n-k] \xleftrightarrow{Z} z^{-k}X(z)$$

ROC is the **same** (except for possible addition/deletion of z=0 or z=∞)

Frequency shifting:

$$z_0 x[n] \xleftrightarrow{Z} X(z/z_0)$$

ROC is **scaled** by $|z_0|$

Convolution property:

$$x[n]*y[n] \xleftrightarrow{Z} X(z)Y(z)$$

ROC is the **intersection** of ROCs of *X(z)* and *Y(z)*

Inverse Z-transform

$$x[n] = Z^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz$$

Formal analytic solution of this equation is based on the Cauchy integral theorem and can be quite difficult.

Fortunately, several informal methods are available for finding the inverse z-transform:

1. *Inspection method*
2. *Partial fraction method*
3. *Power series expansion*

Inspection Method

Find the signal whose z-transform is: $$X(z) = \frac{2}{1-0.8z^{-1}} + 5z^{-3}$$

By inspection (*the table of z-transform pairs*), we can recognize three z-transform pairs:

$$2(0.8)^n u[n] \xleftrightarrow{Z} \frac{2}{1-0.8z^{-1}}$$ for |z|>|0.8|

$$-2(0.8)^n u[n-1] \xleftrightarrow{Z} \frac{2}{1-0.8z^{-1}}$$ for |z|<|0.8|

$$5\delta[n-3] \xleftrightarrow{Z} 5z^{-3}$$ all *z*, except *z*=0

Hence the following two inverse z-transforms are possible:

1) $x[n] = 2(0.8)^n u[n] + 5\delta[n-3]$ for |z|>|0.8|

2) $x[n] = -2(0.8)^n u[-n-1] + 5\delta[n-3]$ for |z|<|0.8|, z≠0

Partial Fraction Method

The basic idea of the method is to represent a ratio of two polynomials in z as the sum of simpler individuals.

If the degree of the numerator N(z) is not greater than the degree of the denominator D(z), the partial fraction expansion is expressed as:

$$X(z) = \frac{N(z)}{D(z)} = A_0 + \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \ldots + \frac{A_N}{1 - p_N z^{-1}}$$

$$A_0 = [X(z)]_{z=0}$$

$$A_i = [(1 - p_i z^{-1}) X(z)]_{z=p_i} = \left[\frac{N(z)}{\prod_{i \neq j} 1 - p_j z^{-1}} \right]_{z=p_i}$$

Partial Fraction Method

Find the signal whose z-transform is: $$X(z) = z^{-1} + 2 + \frac{6}{1 - 1.6z^{-1} - 0.8z^{-2}}$$

After factoring denominator we get:

$$\frac{6}{1 - 1.6z^{-1} - 0.8z^{-2}} = \frac{6}{(1 - 2z^{-1})(1 + 0.4z^{-1})} = \frac{A}{1 - 2z^{-1}} + \frac{B}{1 + 0.4z^{-1}}$$

$$A = \left[\frac{6}{1 + 0.4z^{-1}}\right]_{z=2} = \frac{6}{1.2} = 5$$

$$B = \left[\frac{6}{1 - 2z^{-1}}\right]_{z=-0.4} = \frac{6}{6} = 1$$

Partial Fraction Method (contd.)

$$X(z) = z^{-1} + 2 + \frac{5}{1-2z^{-1}} + \frac{1}{1+0.4z^{-1}}$$

And now we can apply the Inspection method. According to the table of z-transform pairs, three different inverse z-transforms are possible:

$x[n] = \delta[n-1] + 2\delta[n] + 5(2)^n u[n] + (-0.4)^n u[n]$ for $|z|>|2|$

$x[n] = \delta[n-1] + 2\delta[n] - 5(2)^n u[-n-1] + (-0.4)^n u[n]$ for $|-0.4|<|z|<|2|$

$x[n] = \delta[n-1] + 2\delta[n] - 5(2)^n u[-n-1] - (-0.4)^n u[-n-1]$ for $|z|<|-0.4|$, $z\neq0$

Filters in Z-domain

$$y[n] = \sum_{k=0}^{N} a_k x[n-k] - \sum_{m=1}^{M} b_m y[n-m]$$

TD

$$Y(z) = X(z)\sum_{k=0}^{N} a_k z^{-k} - Y(z)\sum_{m=1}^{M} b_m z^{-m}$$

ZD

Transfer Function

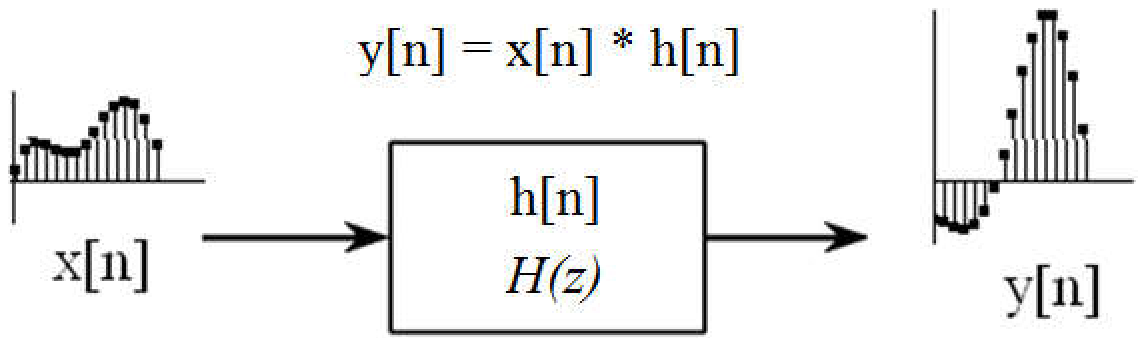
Transfer Function

is the ratio of the z-transform of the system's output to the z-transform of the system's input.

According to the convolution property of z-transform, the transfer function is the z-transform of the impulse response of a system

$$H(z) = \frac{Y(z)}{X(z)}$$

$$y[n] = x[n] * h[n]$$



$$Y(z) = X(z)\, H(z)$$

Transfer Function

$$H(z) = \frac{Y(z)}{X(z)}$$

$$Y(z) = X(z)\sum_{k=0}^{N} a_k z^{-k} - Y(z)\sum_{m=1}^{M} b_m z^{-m}$$

$$H(z) = \frac{\sum_{k=0}^{N} a_k z^{-k}}{1 + \sum_{m=1}^{M} b_m z^{-m}}$$

Zeros and Poles

zeros

$$H(z) = \frac{a_0\,(1 - z_1 z^{-1})(1 - z_2 z^{-1})\ldots(1 - z_N z^{-1})}{b_0\,(1 - p_1 z^{-1})(1 - p_2 z^{-1})\ldots(1 - p_M z^{-1})} = \frac{a_0}{b_0}\,\frac{\prod_{k=1}^{N}(1 - z_k z^{-1})}{\prod_{m=1}^{M}(1 - z_m z^{-1})}$$

poles

zeros

$$H(z) = \frac{G(z - z_1)(z - z_2)\ldots(z - z_N)}{(z - p_1)(z - p_2)\ldots(z - p_M)} = G\,\frac{\prod_{k=1}^{N}(z - z_k)}{\prod_{m=1}^{M}(z - z_m)}$$

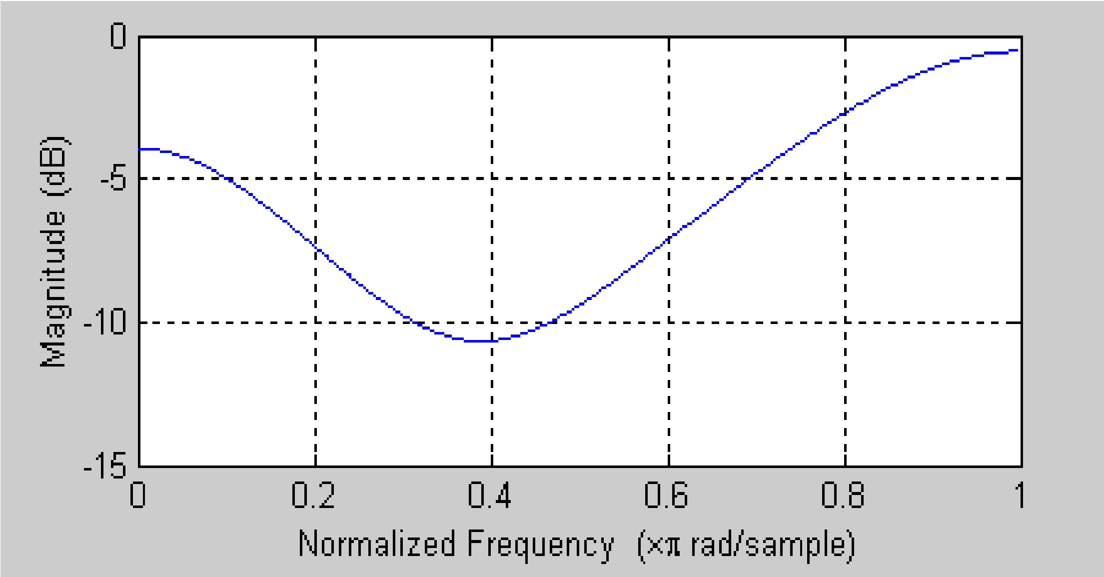
poles

Zeros and Poles

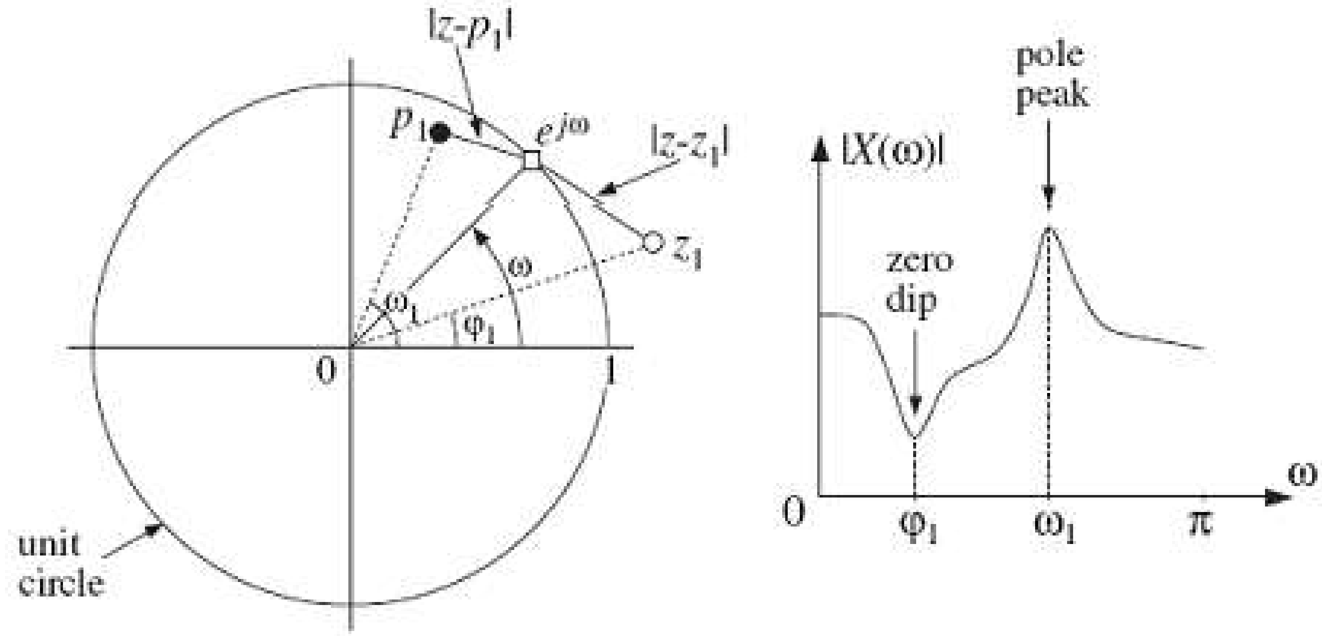
$$X(z) = \frac{1 - 0.4z^{-1} + 0.29z^{-2}}{1 - 1.6z^{-1} - 0.8z^{-2}} = \frac{(1 - [0.2 - 0.5j]z^{-1})(1 + [0.2 - 0.5j]z^{-1})}{(1 - 2z^{-1})(1 + 0.4z^{-1})}$$

$$z1 = 0.2 - 0.5j$$
$$z2 = 0.2 + 0.5j$$

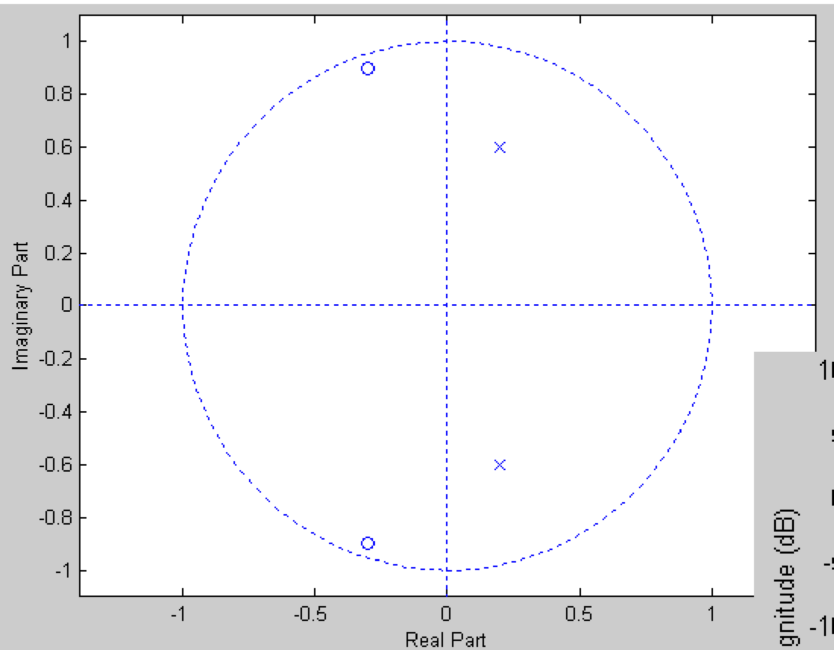
$$p1 = 2$$
$$p2 = -0.4$$



Zeros and Poles

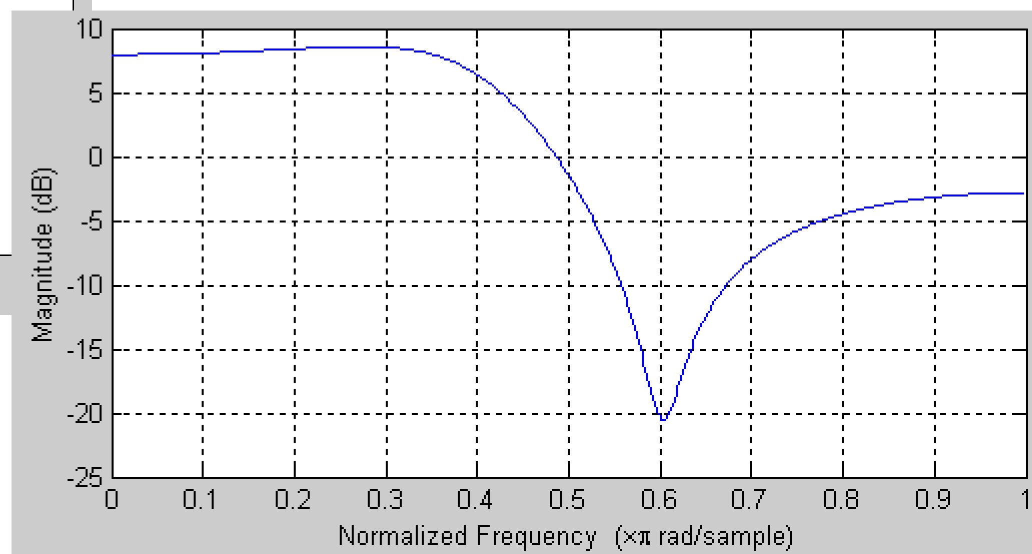


Exercise 2

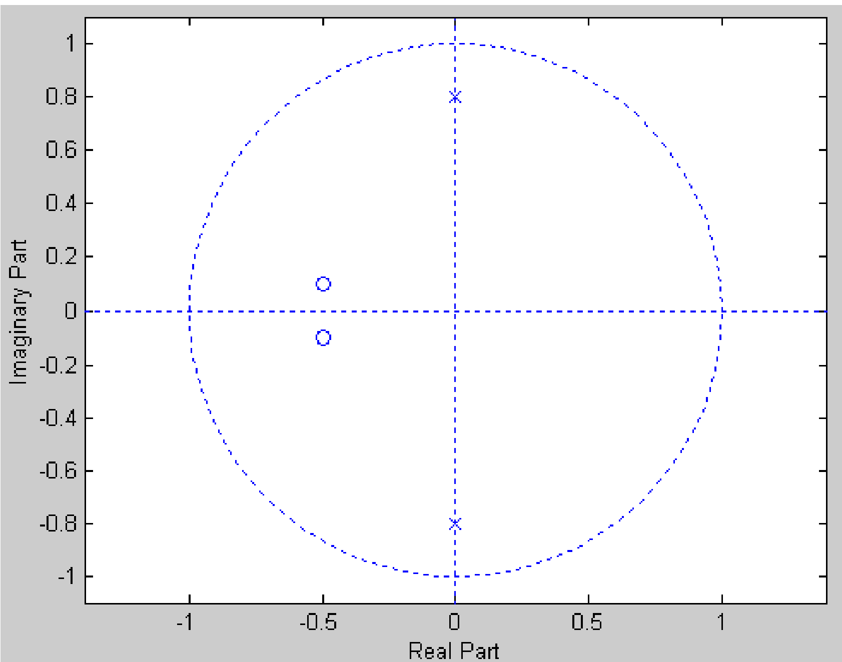



```
z = [-0.3+0.9j; -0.3-0.9j];
p = [ 0.2+0.6j;  0.2-0.6j];

zplane( z, p );
[b, a] = zp2tf( z, p, 1 );
freqz( b, a );
```



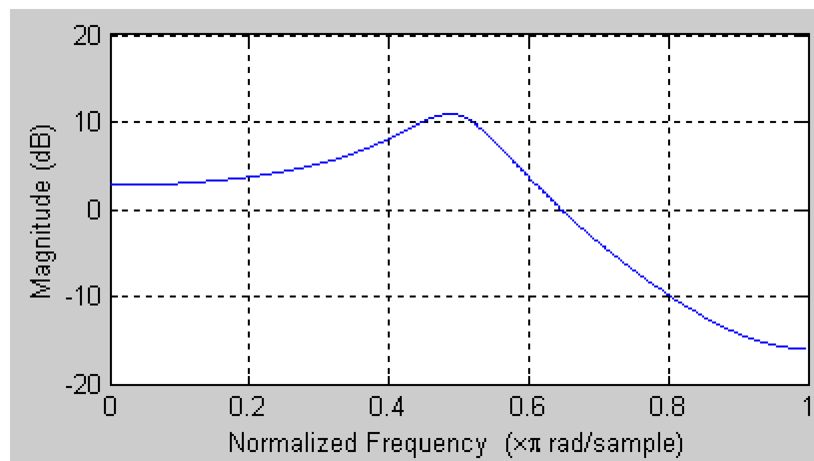
Exercise 3




```
z = [-0.5+0.1j; -0.5-0.1j];
p = [0.8j; -0.8j];

figure(1);
zplane( z, p );

figure(2)
[b, a] = zp2tf( z, p, 1 );
freqz( b, a );
```



Exercise 3

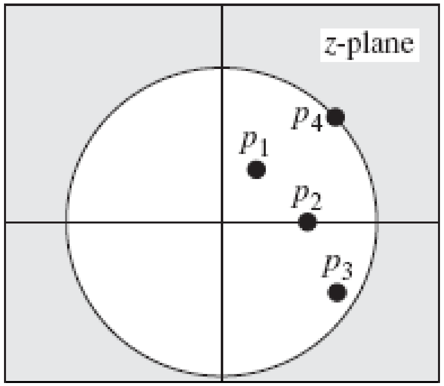
```
z = [-0.5+0.1j; -0.5-0.1j];
p = [0.8j; -0.8j];
```

$$H(z) = \frac{(z+0.5-0.1j)(z+0.5+0.1j)}{(z-0.8j)(z+0.8j)} = \frac{(z+0.5)^2+0.01}{z^2+0.64} = \frac{z^2+z+0.26}{z^2+0.64} = \frac{1+z^{-1}+0.26z^{-2}}{1+0.64z^{-2}}$$

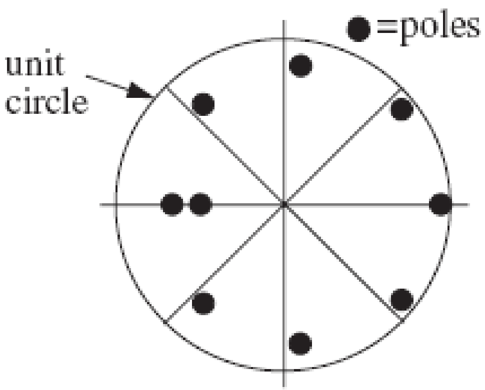
$$y[n] = x[n] + x[n-1] + 0.26x[n-2] - 0.64y[n-2]$$

Stability and Causality

AN LTI SYSTEM IS BOTH STABLE AND CAUSAL
if all its poles lie strictly inside the unit circle in the z-plane.



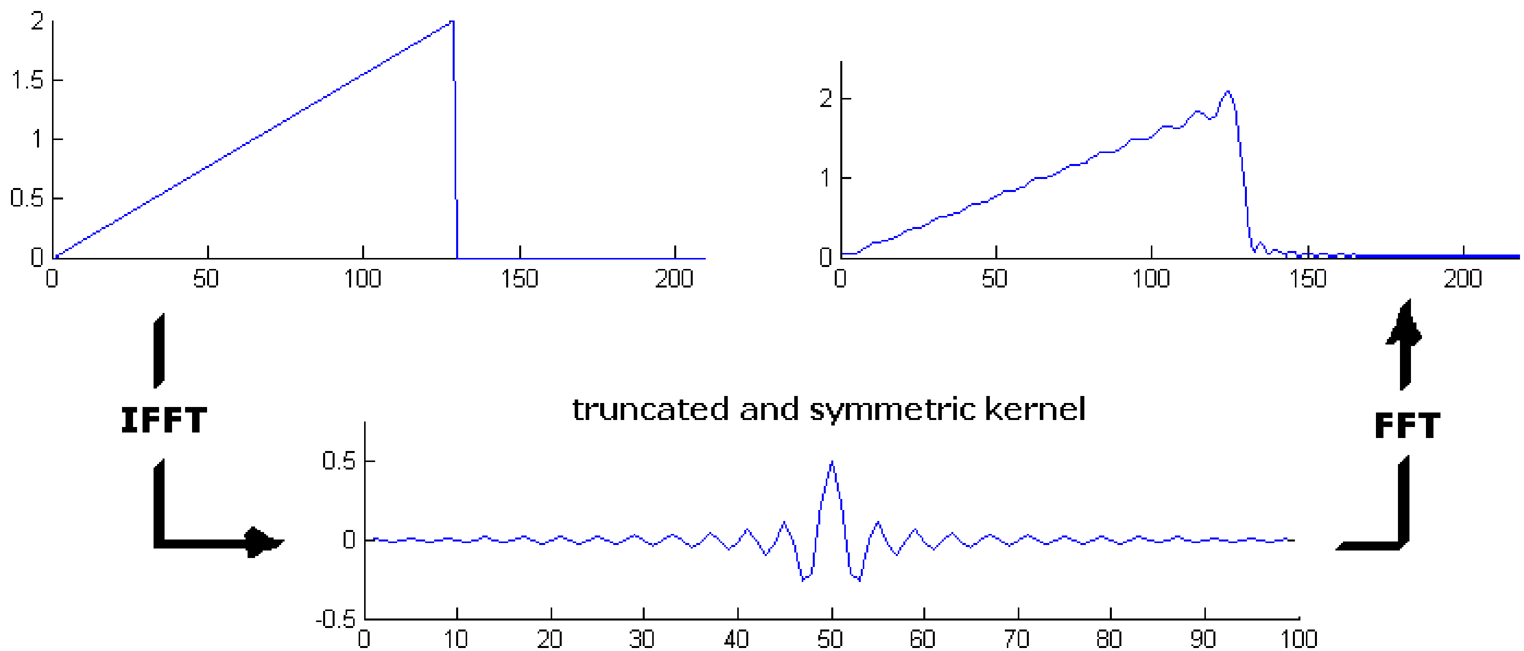
Causal filter



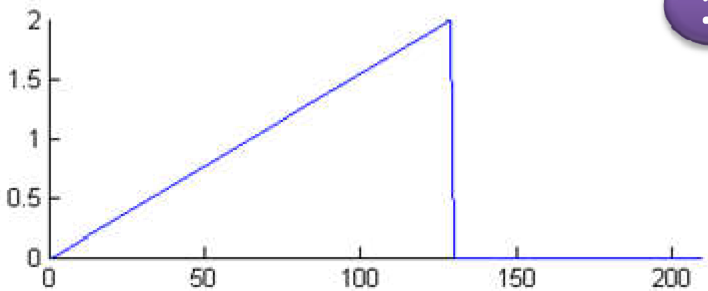
Stable and causal filter

$$|z| > \max_i \{|p_i|\}$$

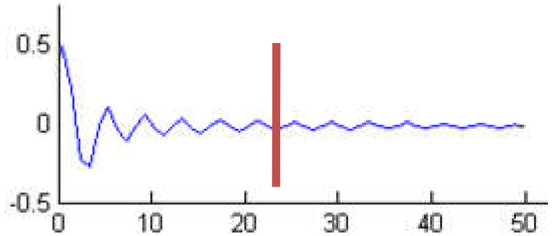
Filter Design by Windowing



Filter Design by Windowing

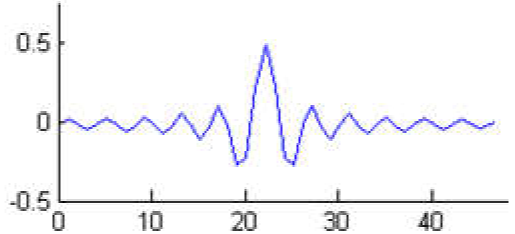


1 *Specify the Frequency Response in N-point arrays of real and imaginary parts (set imaginary parts to zero)*

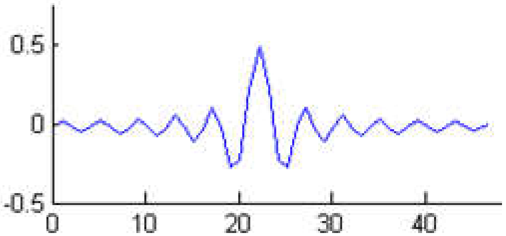


2 *Obtain the filter kernel by taking IFFT of the Frequency Response*

2a *Truncate & shift the filter kernel*



Filter Design by Windowing



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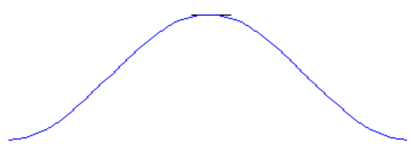

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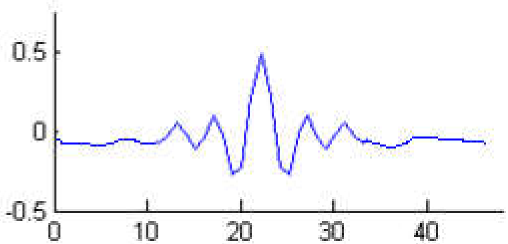
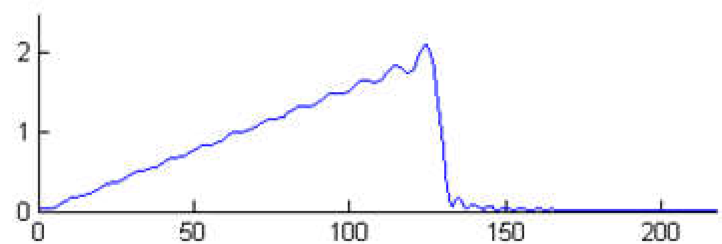


3 *Apply a window function to the filter kernel*

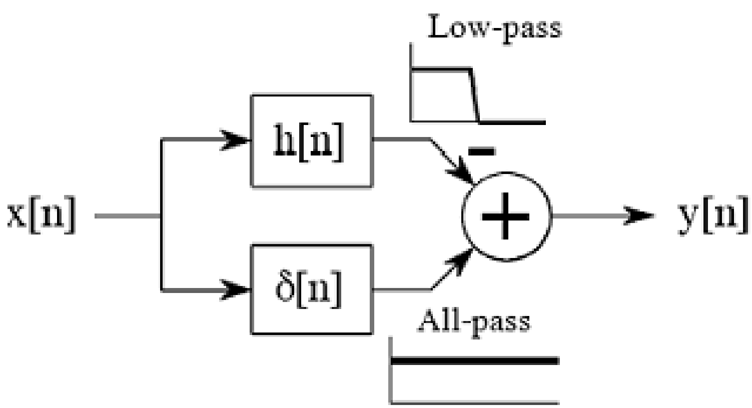
4 *Take the FFT of the windowed filter kernel.*
If the results are not satisfying repeat steps 1-3

FFT



HP Filter from a LP filter

1 *Spectral inversion*



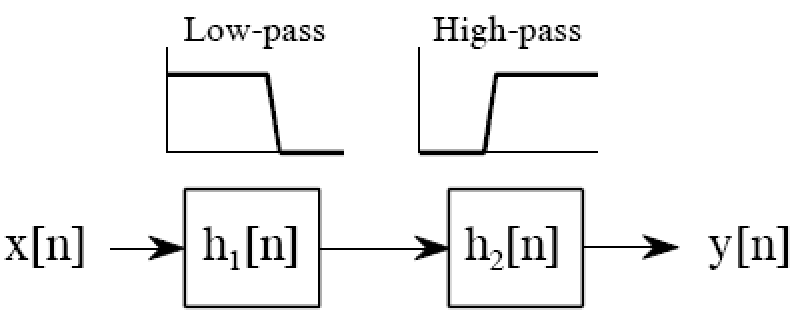
1) Change the sign of each sample in the filter kernel
2) Add 1 to the middle sample

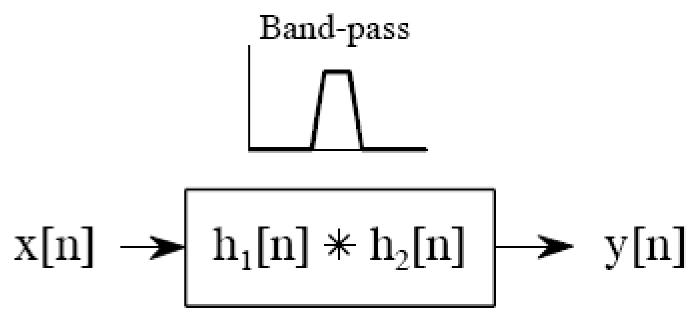
2 *Spectral reversal*

1) Change the sign of every other sample in the filter kernel

Changing the sign of every other sample is equivalent to multiplying the filter kernel by a sinusoid with a frequency of 0.5. This has the effect of shifting the frequency domain by 0.5

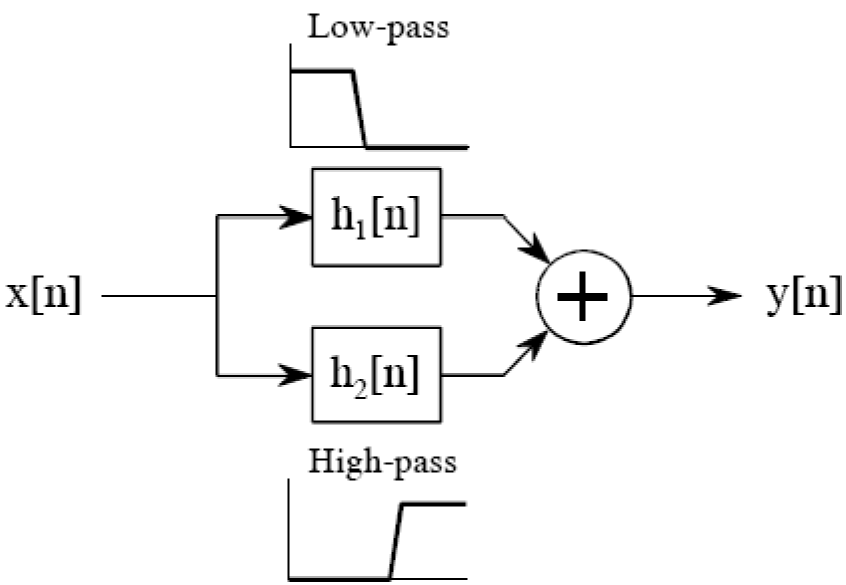
BP Filter from a LP filter

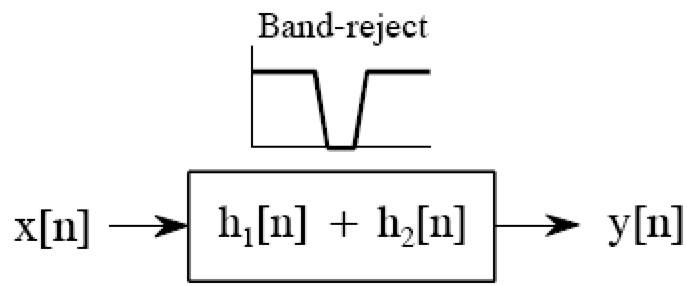




$$H_1(z)H_2(z)$$

BR Filter from a LP filter





$$H_1(z) + H_2(z)$$