Digital Filtering
Lecture Outline

1) **Characteristics of Filters**
2) **Types of Filters**
3) **Difference Equations**
4) **Block Convolution**

Digital Filters

(Frequency-selective) Digital Filter

is a discrete-time system whose main purpose is to modify certain frequencies relative to others

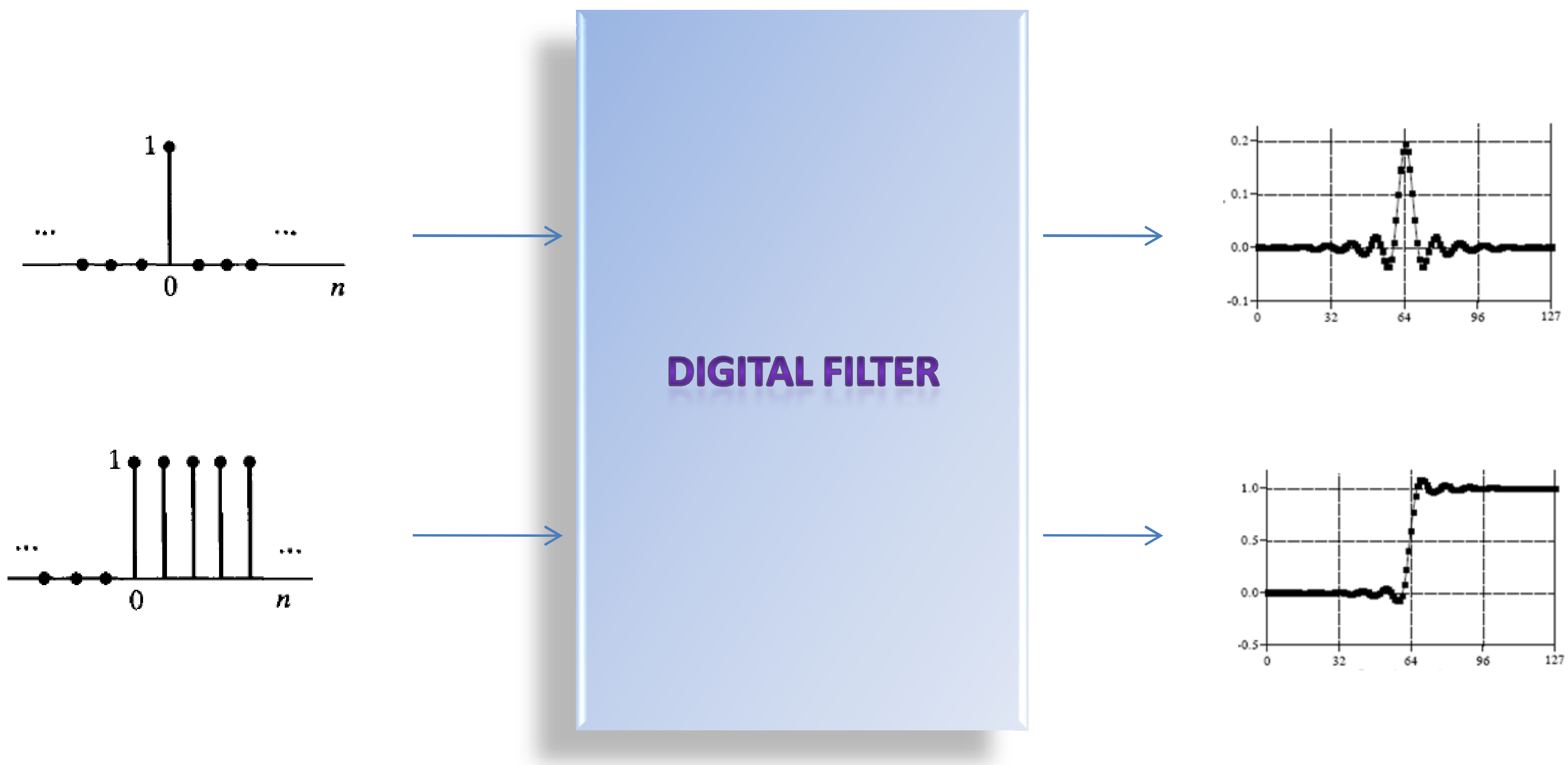
Digital filters are used for two general purposes:

1. *separation of signals* that have been combined
   Signal separation is needed when a signal has been contaminated with interference, noise, or other signals
2. *restoration of signals* that have been distorted in some way

The main characteristics of digital filters are:

- Impulse response
- Frequency response
- Step response

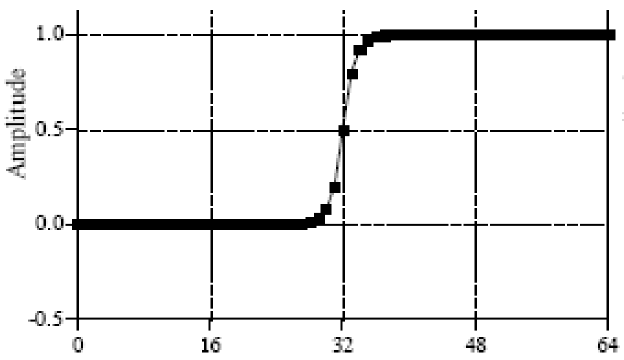
Filter Responses



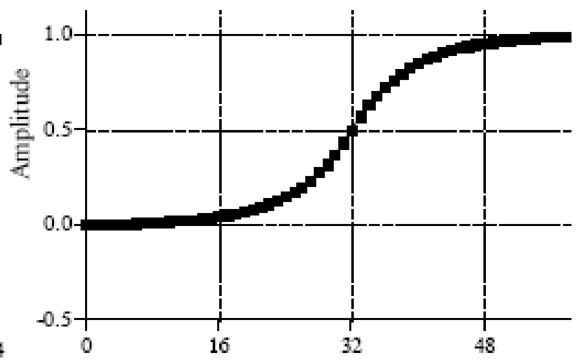
Step Response

Step Response is an output signal of a system when the input signal is a unit step

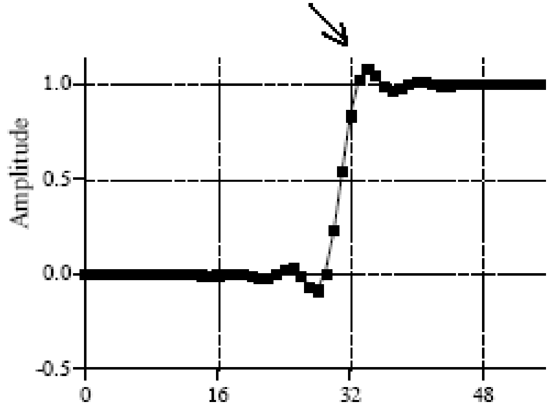
good



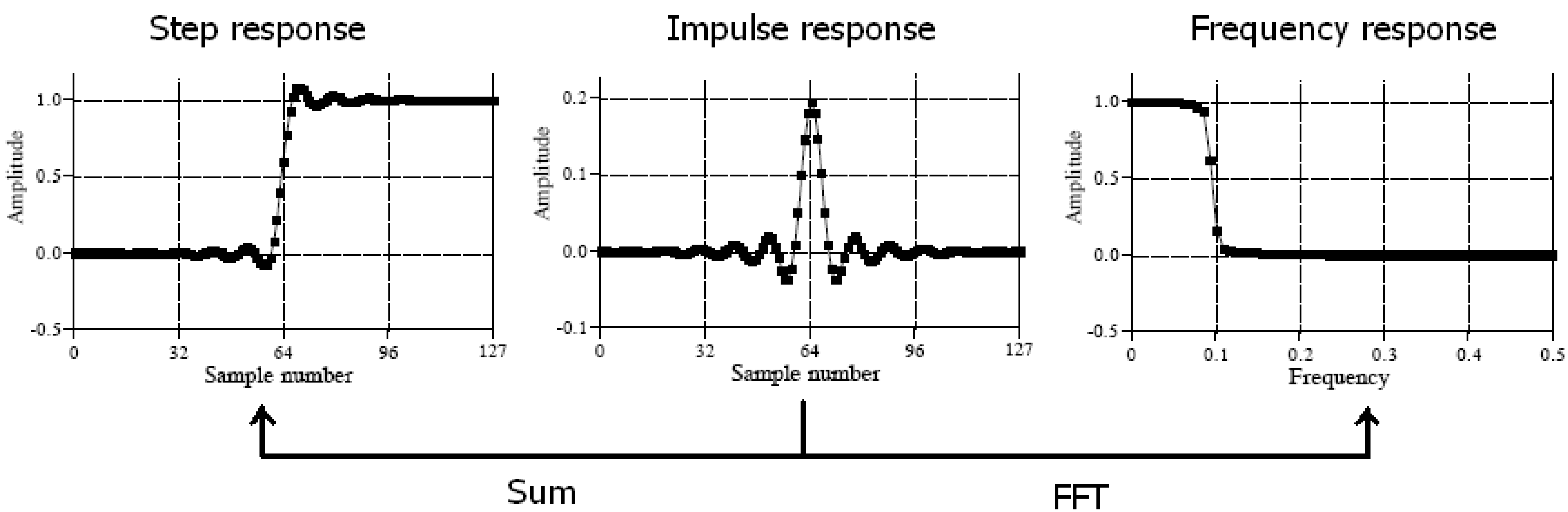
long



overshoot



Characteristics of Filters



Filter Kernel

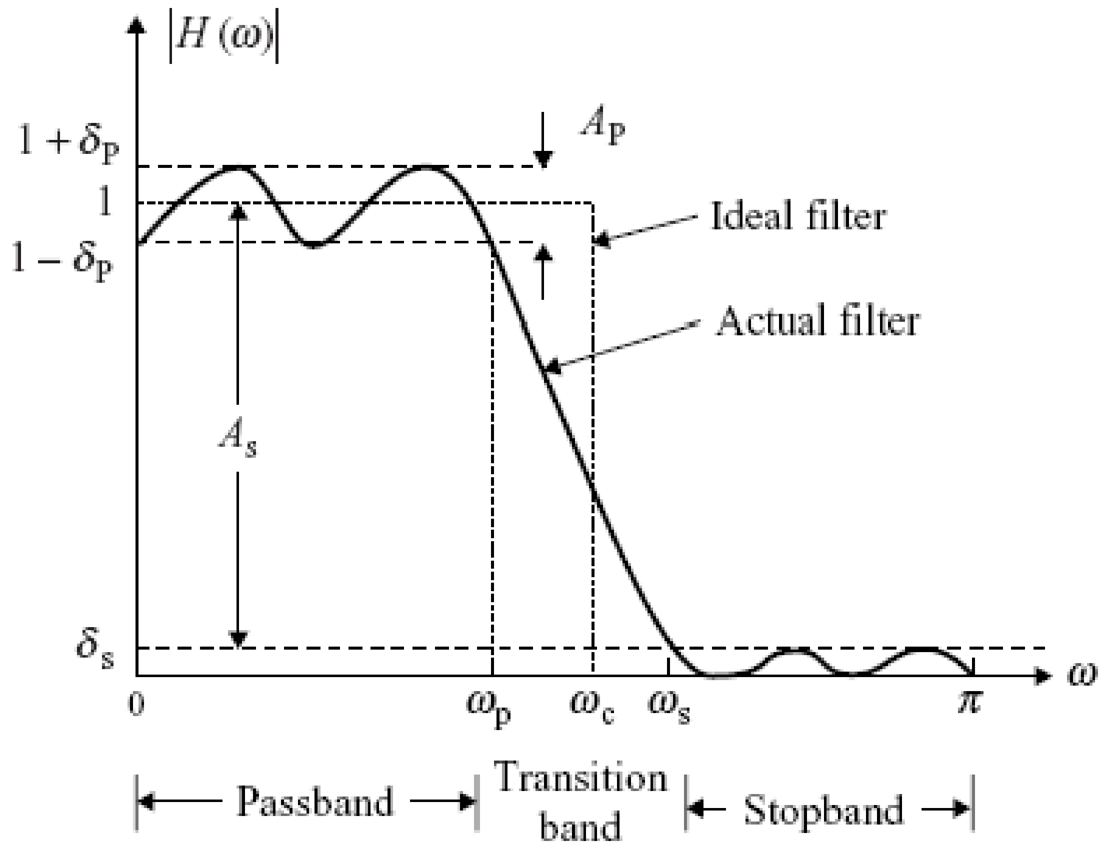
Filter Kernel **is the impulse response of a digital filter**

If we implement a real-time DSP system, we can operate only with the data we already have, and therefore, the ***causal*** filters are the only option.
On the other hand, if the full amount of data is available (e.g., fully loaded image), we may use non-causal filters for signal processing.

In terms of impulse response, a digital system is **causal** if and only if:

$$h[n]=0, \quad n<0$$

Frequency Response



Classification of Filters

Shape of the phase response:

- Zero phase filters
- Linear phase filters
- Nonlinear phase filters

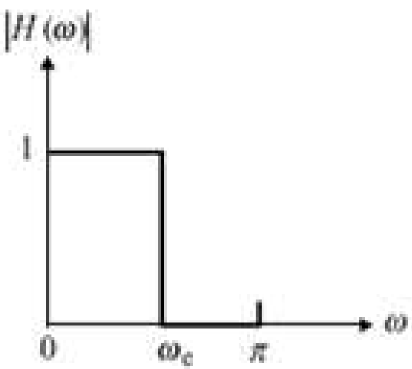
Shape of the frequency response:

- Low-pass filters
- High-pass filters
- Band-pass filters
- Band-reject filters
- Custom filters

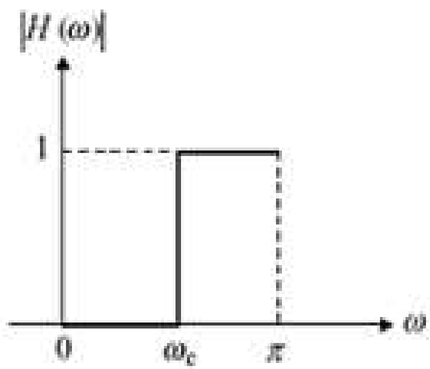
Type of the impulse response:

- FIR (Finite Impulse Response) filters
- IIR (Infinite Impulse Response) filters

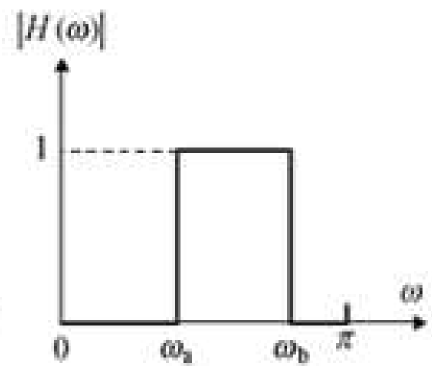
The Shape of Frequency Response



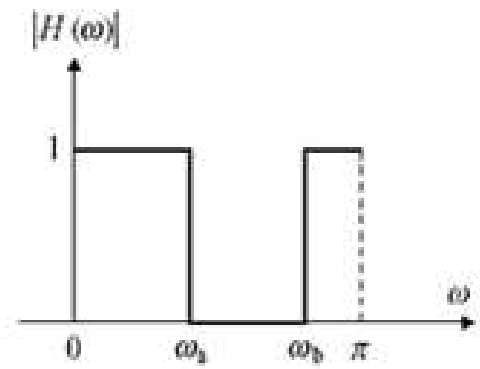
Low-pass



High-pass



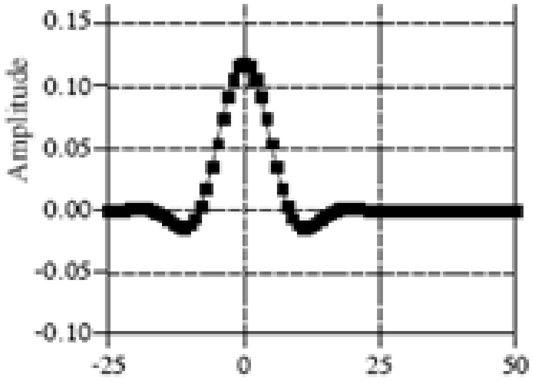
Band-pass



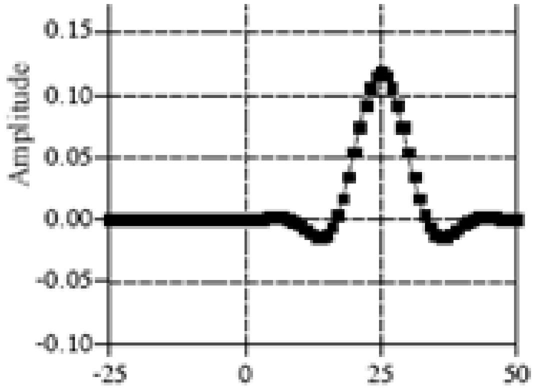
Band-reject

Phase Response

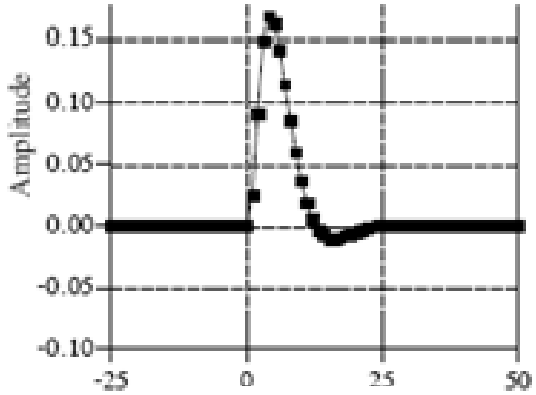
Kernels of filters:



Zero-phase



Linear phase



Nonlinear phase

Difference Equations

Just as continuous-time systems are described by the differential equations, discrete-time systems are described by the *difference equations:*

IIR FILTER:

$$y[n]=\sum_{k=0}^{N}a_k x[n-k]-\underbrace{\sum_{m=1}^{M}b_m y[n-m]}_{\text{Recursive part (with feedback coefficients)}}$$

FIR FILTER:

$$y[n]=\sum_{k=0}^{N}a_k x[n-k]$$

Exercise 1

Find the impulse response of the following filter (assuming it's causal, and all coefficients are nonzero):

$$y[n] = a_0 x[n] - \sum_{m=1}^{M} b_m y[n-m]$$

Let's feed a unit impulse to the filter ($x[n] = \delta[n]$) and see what the output result will be:

$h[0] = a_0x[0] – b_1h[-1] = a_0$
$h[1] = a_0x[1] – (b_1h[0] + b_0h[-1]) = −b_1h[0]$
$h[2] = a_0x[2] – (b_2h[1] + b_1h[0] + b_0h[-1]) = −(b_2h[1] + b_1h[0])$
…

$$h[n] = a_0 \delta[n] - \sum_{m=1}^{M} b_m h[n-m]$$

Consequently, this filter has ***infinite impulse response*** h[n].
Furthermore, any filter that requires the feedback coefficients is IIR, since each sample of its impulse response is nonzero, and computed from the previous samples of its impulse response

Moving Average Filter

Find the impulse response of the *L*-point moving average filter:

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$

$$h[n] = \frac{1}{L}\sum_{k=0}^{L-1} \delta[n-k] = \begin{cases} \frac{1}{L} & , n = 0,1,2,..,L-1 \\ 0 & , n \geq L \end{cases}$$

The same output signal can be obtained by the following recursive formula

$$y[n] = y[n-1] + \frac{1}{L}(x[n] - x[n-L])$$

Moving Average Filter

Find the impulse response of 5-point moving average filter:

$$y[n] = \frac{1}{5}\left(x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]\right)$$

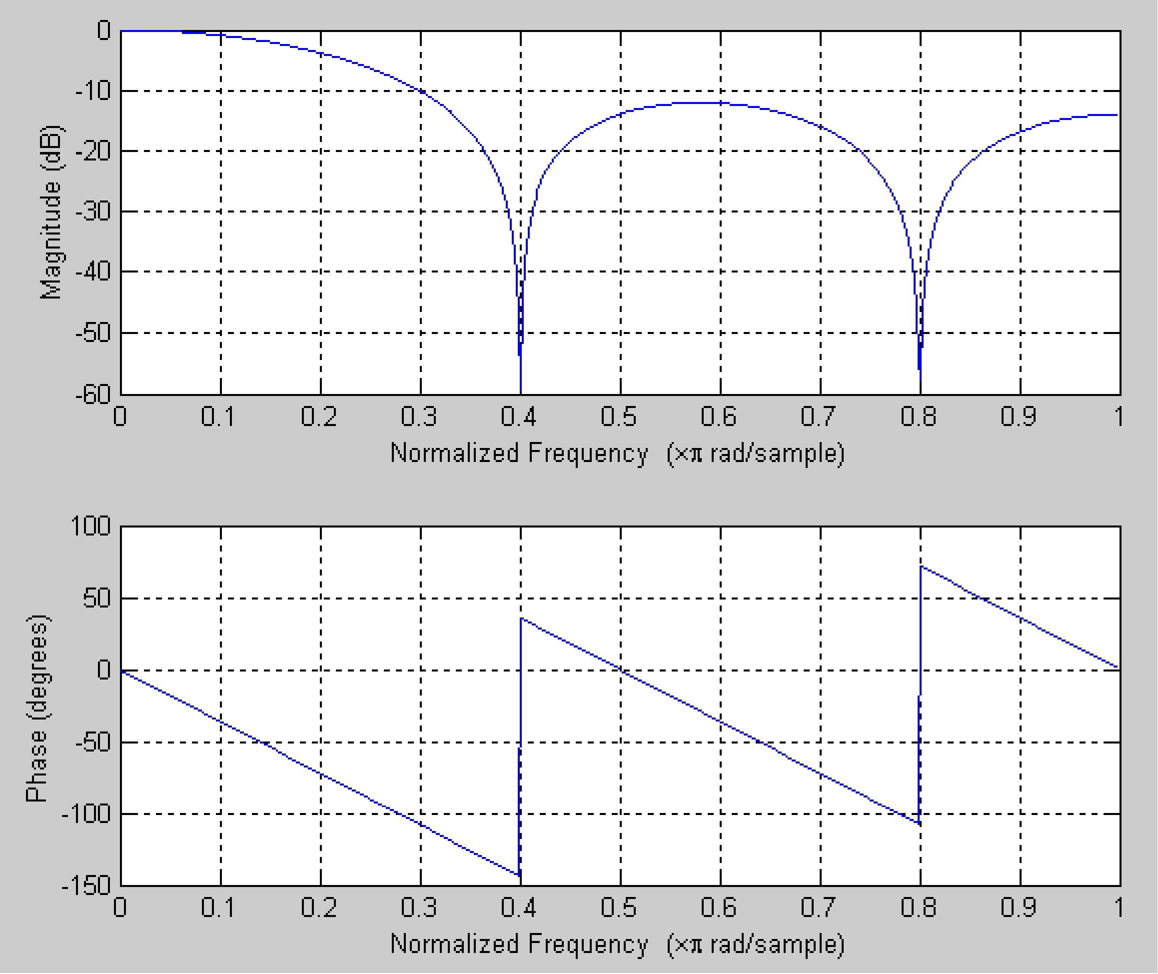
$$h[n] = [1/5 \quad 1/5 \quad 1/5 \quad 1/5 \quad 1/5]$$

$$y[n+1] = \frac{1}{5}\left(x[n+1] + x[n] + x[n-1] + x[n-2] + x[n-3]\right)$$

$$y[n+1] = y[n] + \frac{1}{5}\left(x[n+1] - x[n-4]\right)$$

This formula is much faster and looks like a description of an IIR filter, but don't be confused! Although the moving average filter *can be* implemented recursively, it is characterized by the ***finite impulse response***!

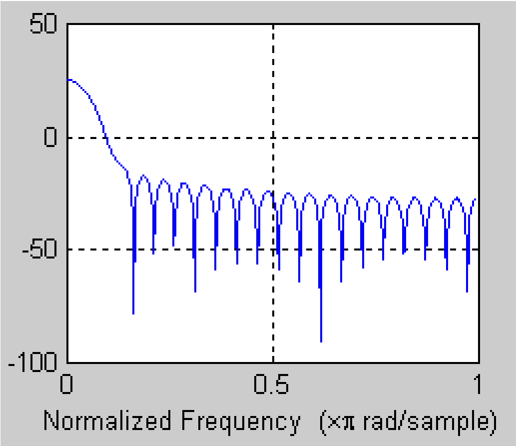
Moving Average Filter



Relatives of Moving Average

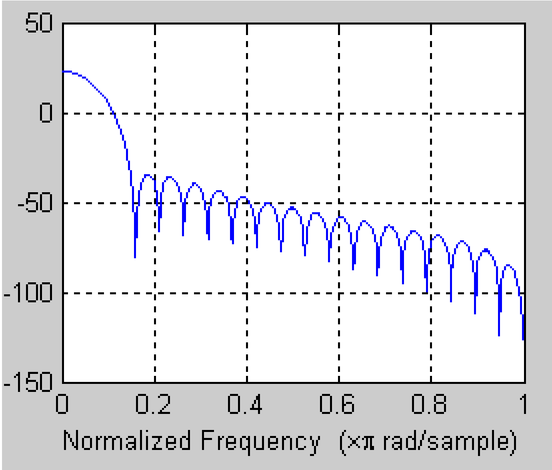
Gaussian filter

$$h[n] = \frac{1}{\sqrt{2\pi\sigma}} e^{-n^2/2\sigma^2}$$



Blackman filter

$$h[n] = 0.42 - 0.5\cos(\frac{2\pi}{M}n) + 0.08\cos(\frac{4\pi}{M}n)$$



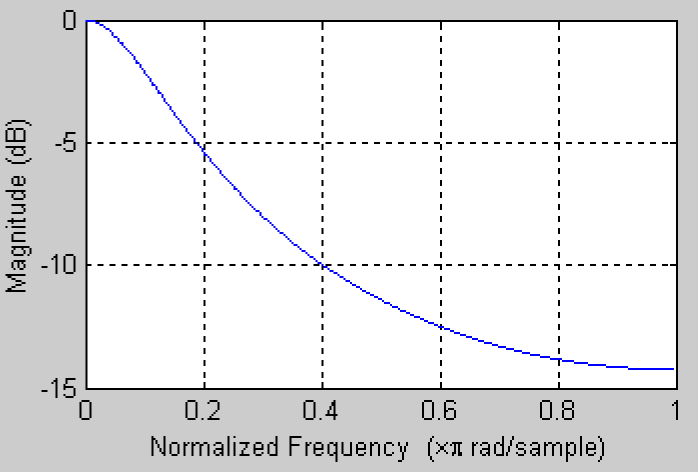
Neither MA filter nor its relatives have good frequency selective properties!

Single-Pole Low-pass Filter

$$y[n] = a_0 x[n] - b_1 y[n-1]$$

$$a_0 = 1 - e^{-\omega_c} = 1 - e^{-2\pi f_c / f_s}$$

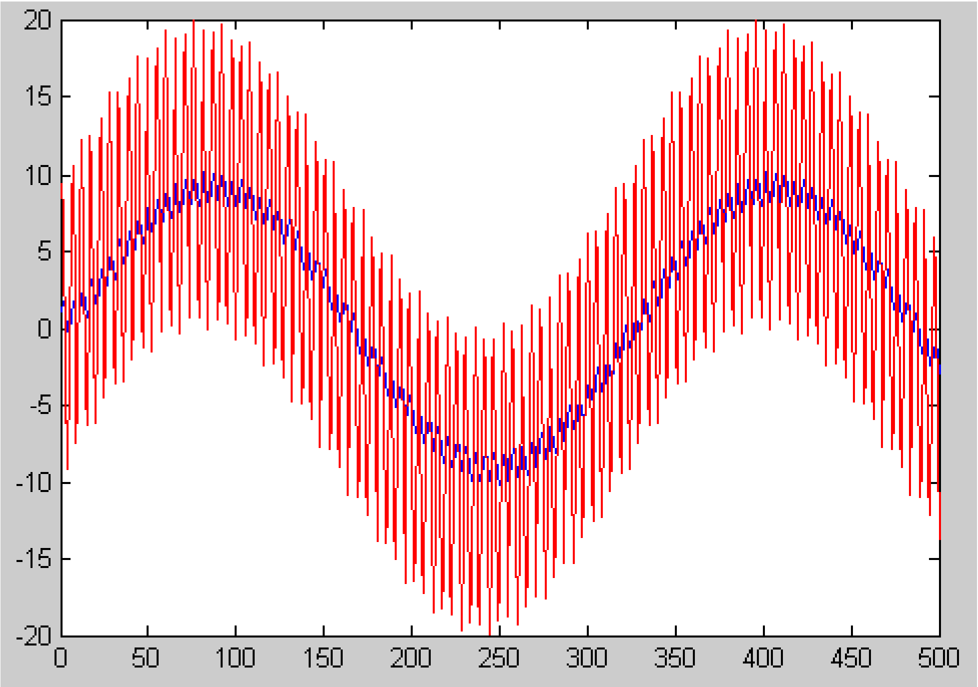
$$b_1 = -e^{-\omega_c} = -e^{-2\pi f_c / f_s}$$



Single-Pole Low-pass Filter

$$y[n] = 0.1148\,x[n] + 0.8752\,y[n-1]$$

x = 10*sin(2*pi*3000*n/16000) +
10*sin(2*pi*50*n/16000)



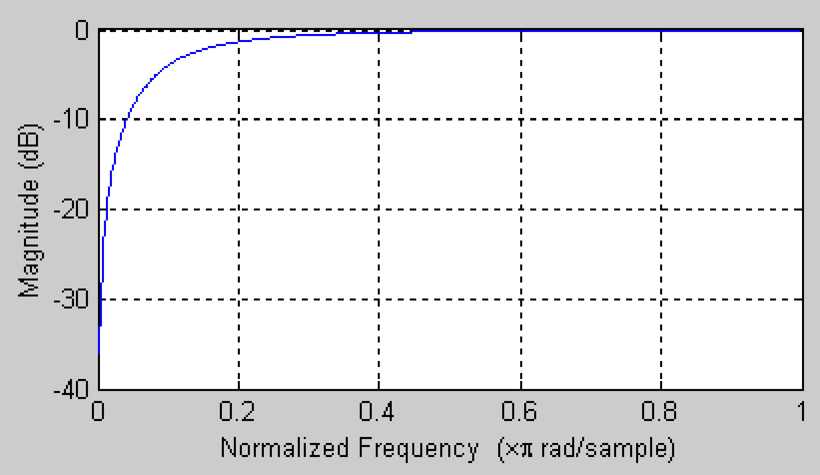
Single-Pole High-pass Filter

$$y[n] = a_0 x[n] + a_1 x[n-1] - b_1 y[n-1]$$

$$a_0 = (1 + e^{-\omega_c}) / 2$$

$$a_1 = -(1 + e^{-\omega_c}) / 2$$

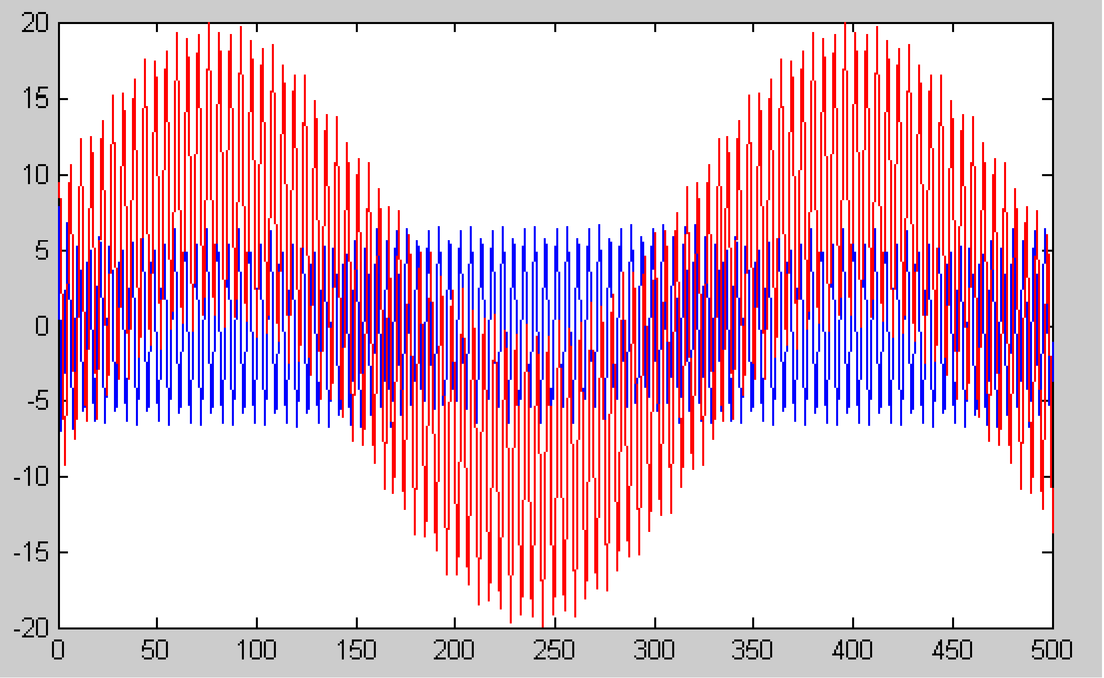
$$b_1 = -e^{-\omega_c}$$



Single-Pole High-pass Filter

$$y[n] = 0.8376\,x[n] - 0.8376\,x[n-1] + 0.6752\,y[n-1]$$

x = 10*sin(2*pi*3000*n/16000) +
10*sin(2*pi*50*n/16000)



Four Stage Low-pass Filter

$$y[n] = a_0 x[n] - b_1 y[n-1] - b_2 y[n-2] - b_3 y[n-3] - b_4 y[n-4]$$

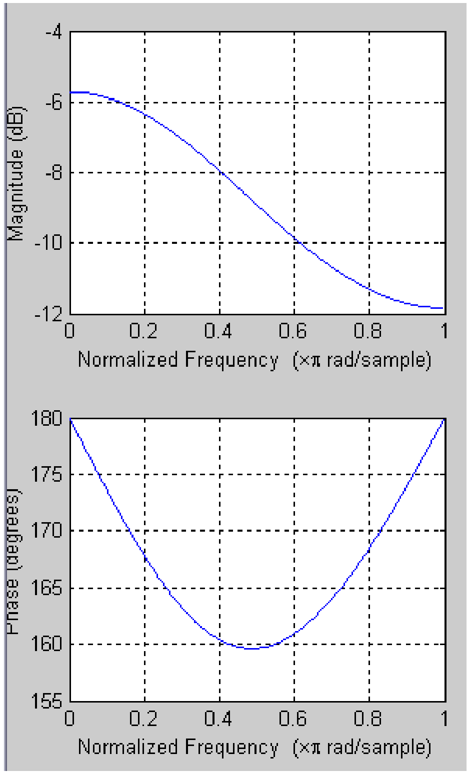
$$a_0 = (1 - e^{-14.445 f_c / f_s})^4$$

$$b_1 = -4e^{-14.445 f_c / f_s}$$

$$b_2 = 6e^{-14.445*2 f_c / f_s}$$

$$b_3 = -4e^{-14.445*3 f_c / f_s}$$

$$b_4 = e^{-14.445*4 f_c / f_s}$$



$$f_c / f_s = 0.1$$

Band-pass Filter

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] - b_1 y[n-1] - b_2 y[n-2]$$

$$a_0 = 1 - K$$

$$a_1 = 2(K - R)\cos(\omega_c)$$

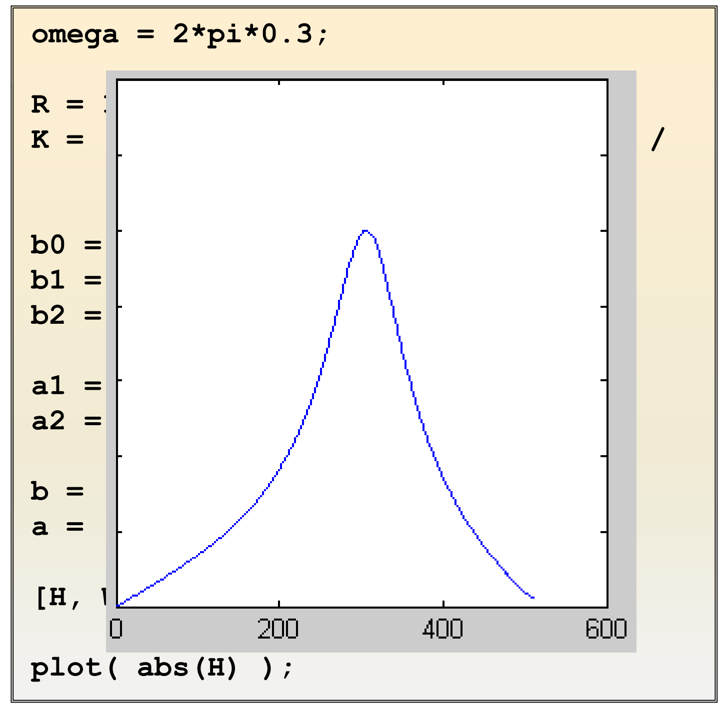
$$a_2 = R^2 - K$$

$$b_1 = -2R\cos(\omega_c)$$

$$b_2 = R^2$$

$$K = \frac{1 - 2R\cos(\omega_c) + R^2}{2 - 2\cos(\omega_c)}$$

$$R = 1 - 3B_W / f_s$$



Band-reject Filter

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] - b_1 y[n-1] - b_2 y[n-2]$$

$$a_0 = K$$

$$a_1 = -2K\cos(\omega_c)$$

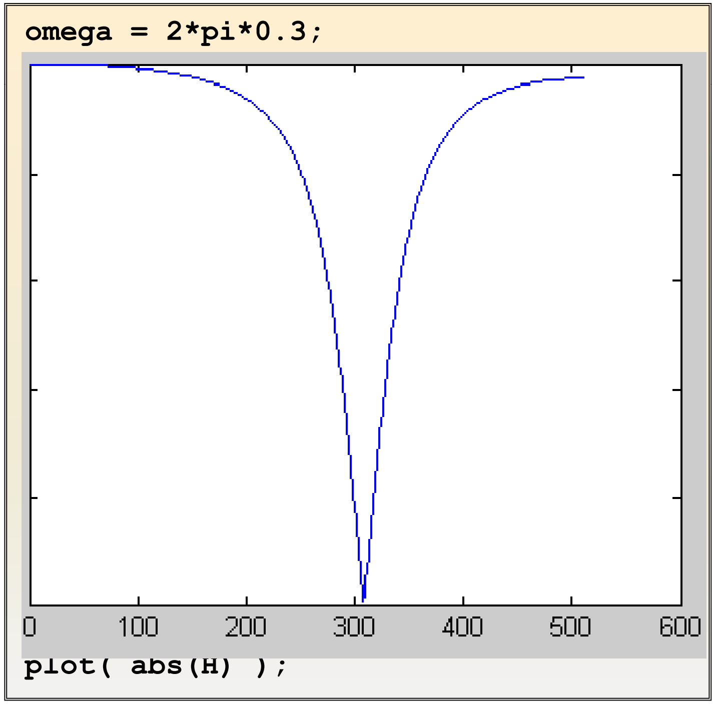
$$a_2 = K$$

$$b_1 = -2R\cos(\omega_c)$$

$$b_2 = R^2$$

$$K = \frac{1 - 2R\cos(\omega_c) + R^2}{2 - 2\cos(\omega_c)}$$

$$R = 1 - 3B_W / f_s$$



How to apply a filter?

Basically, there are three ways to carry out digital filtering:

1. Using the ***difference equations*** to obtain the output signal

   TIME DOMAIN

2. ***Convolution*** of the signal with the filter kernel

   TIME DOMAIN

3. ***Fast Convolution*** of the signal with the filter kernel

   FREQUENCY DOMAIN

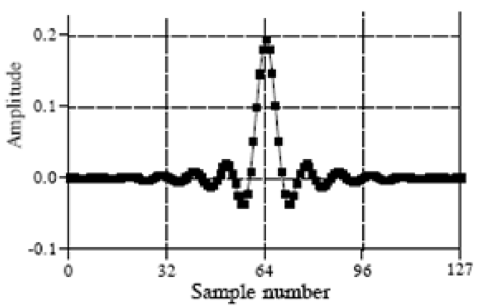
via Block Convolution

How to apply a filter?

FIR FILTER:

1. $$y[n] = \sum_{k=0}^{N} a_k x[n-k]$$

2. Impulse response

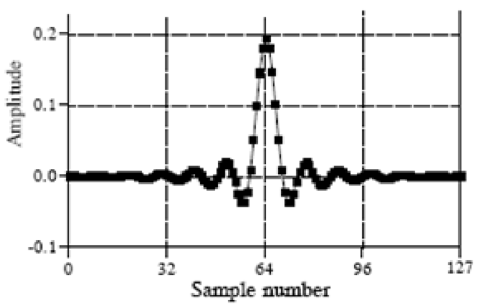


$$h = \{a_0, a_1, a_2, ..., a_N\}$$

IIR FILTER:

1. $$y[n] = \sum_{k=0}^{N} a_k x[n-k] - \sum_{m=1}^{M} b_m y[n-m]$$

Impulse response



$$h = \{h_0, h_1, h_3, ...\}$$

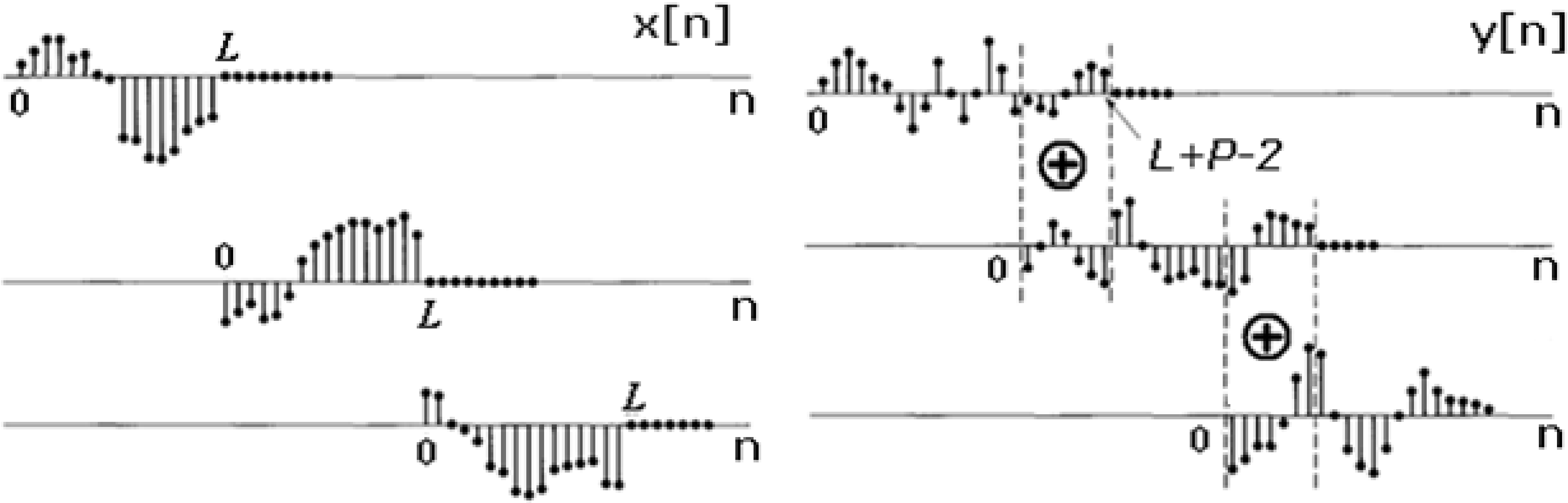
Block Convolution

The input signal is segmented into sections of length ***L***.
Each section is convolved with the finite-length ***P***-point filter kernel and the filtered sections are fitted together in an appropriate specific way.

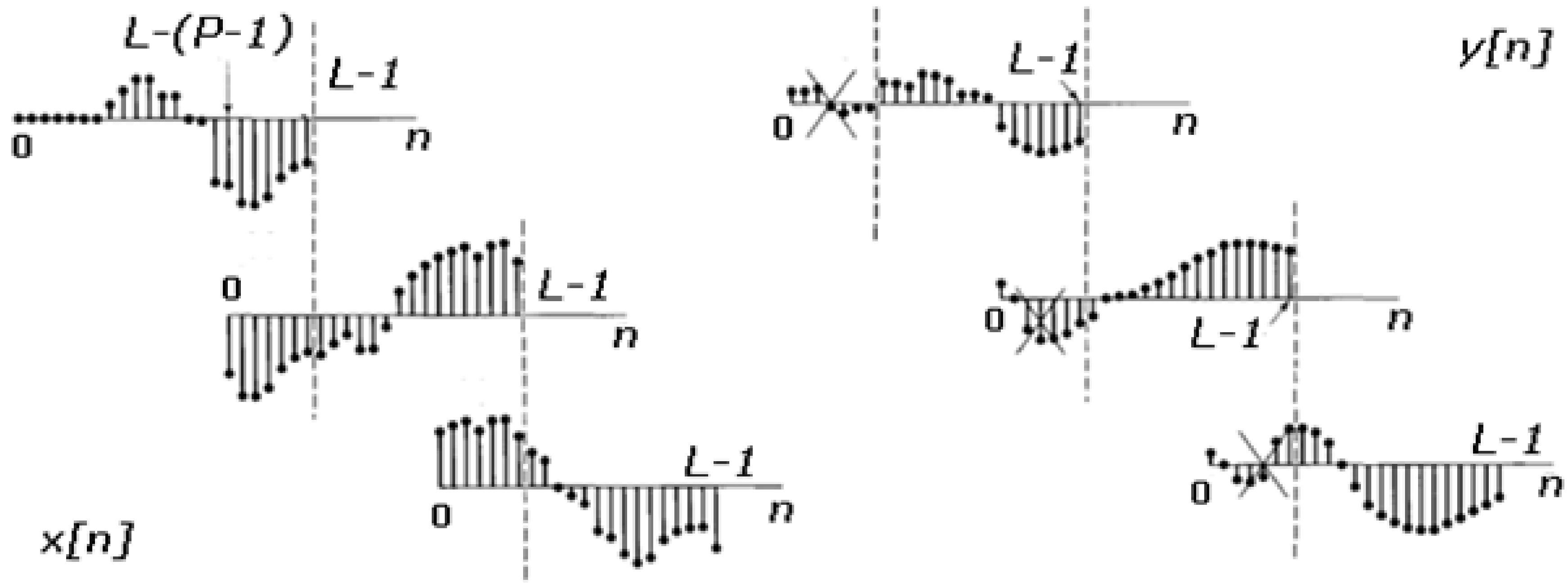
For example, if we need to apply the filter with the kernel of length ***P*=351**,
we may choose the length of each section ***L*=162**
so that the FFT size is a power of 2: $L+P-1=351+162-1=512$.

Or we may choose the length of each section ***L*=674**
so that the FFT size is 1024, etc.

Overlap-Add Method



Overlap-Save Method



Overlap-Add Filtering

Design a band-reject FIR filter with edge frequencies of 1200 Hz and 2400 Hz, and filter the noisy signal x[n]