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35898139 HW4

1a) $$\min \left\| \underbrace{\begin{bmatrix} A \\ \sqrt{\lambda} I \end{bmatrix}}_{\bar{A}} x - \underbrace{\begin{bmatrix} b \\ 0 \end{bmatrix}}_{\bar{b}} \right\|_2^2$$

$$\bar{A}'\bar{A}x = \bar{A}'\bar{b}$$

$$\begin{bmatrix} A^T & \sqrt{\lambda} I \end{bmatrix} \begin{bmatrix} A \\ \sqrt{\lambda} I \end{bmatrix} x = \begin{bmatrix} A^T & \sqrt{\lambda} I \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$(A^T A - \lambda I)x = A^T b$$

$$x = (A^T A + \lambda I)^{-1} A^T b$$

$$\|x\|_2^2 = \|(A^T A + \lambda I)^{-1} A^T b\|_2^2$$

$$\|(QDQ^T + \lambda QQ^T)^{-1} A^T b\|^2$$

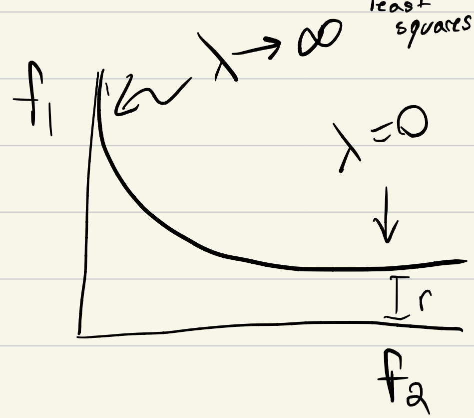
side note: $(QAQ^T)^{-1} = Q A^{-1} Q^T$

$$\|\cancel{Q}^{\,1} (D + \lambda I)^{-1} Q^T A^T b\|^2$$

$$\|(D + \lambda I)^{-1} \underbrace{Q^T A^T b}_{g}\|^2$$

$$\boxed{\|x\|_2^2 = \sum_i^n \frac{1}{d_i + \lambda} g_i^2}$$

- As $\lambda \to \infty$, $\|x\|_2^2 = 0$
- As $\lambda = 0$, $\|Ax - b\|_2^2 = r$ ← left over from least squares



2a) $$\sum_{i=1}^{n} \left(\|x - c_i\|_2^2 - d_i^2 \right)^2 \qquad c_i \in R^n$$

$$\frac{\partial}{\partial x_j} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{m} (x_j - c_{ij})^2 - d_i^2 \right)^2 \right)$$

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_j} \left(\sum_{j=1}^{m} (x_j - c_{ij})^2 - d_i^2 \right)^2$$

$$\frac{\partial}{\partial x_j} = \sum_{i=1}^{m} 2 \left(\|x - c_i\|_2^2 - d_i^2 \right) \cdot 2 \cdot (x_j - c_{ij})$$

$$\nabla f = \sum_{i=1}^{m} 4 \left(\|x - c_i\|_2^2 - d_i^2 \right) \cdot (x - c_i)$$

Jacobian derivation

2b)

In the form that textbook puts them in

$$\sum_{i=1}^{m} \left(\|x-c_i\|_2^2 - d_i^2 \right)^2 \qquad f_i = \|x-c_i\|_2^2$$

1. $\sum_{i=1}^{m} \left(f_i - d_i^2 \right) \qquad F = \begin{matrix} f_i - d_i^2 \\ \vdots \\ f_m - d_m^2 \end{matrix}$

2. $\min \|F(x)\|^2$

3. $x_{k+1} = \operatorname{argmin} \left\{ \sum_{i=1}^{m} \left(f_i(x_k) + \nabla f_i(x_k)^T (x - x_k) - d_i^2 \right)^2 \right\}$

$$r_i(x) = \|x-c_i\|_2^2 - d_i^2$$

$$r_i(x) = \|x-c_i\|_2^2 - d_i^2$$

$$\frac{\partial r}{\partial x_j} = \frac{\partial}{\partial x_j} \sum_{k}^{n} (x_k - c_{ik})^2$$

$$\frac{\partial r}{\partial x_j} = 2(x_j - c_{ij})$$

$$\nabla r_i = 2(x - c_i)$$

$$\boxed{J(x) = \begin{bmatrix} \nabla r_1(x)^T \\ \vdots \\ \nabla r_m(x)^T \end{bmatrix}}$$

LS $\min \|A_k x - b_k\|$

$$A_k = \begin{pmatrix} \nabla f_1(x_k)^T \\ \vdots \\ \nabla f_m(x_k)^T \end{pmatrix} = J(x_k)$$

$$b_k = \begin{pmatrix} \nabla f_1(x_k)^T x_k - f_1(x_k) - d_1^2 \\ \vdots \\ \nabla f_m(x_k)^T x_k - f_m(x_k) - d_m^2 \end{pmatrix}$$

2c) Contra-positive proof:

if A^TA is rank deficient then $c_1 \dots c_m$ lie on the same line

$$A = J(x_k) = 2\begin{bmatrix}(x_k - c_1)^T \\ \vdots \\ (x_k - c_m)^T\end{bmatrix}$$

$\text{rank}(A^TA) = \text{rank}(A)$ ~ proof of $N(A) = N(A^TA)$ in past HW

$A \in \mathbb{R}^{m\times 2}$

$\exists y \neq 0$ s.t. $\begin{bmatrix}(x_k - c_1)^T \\ \vdots \\ (x_k - c_m)^T\end{bmatrix} y = 0$ ~ rank deficient

$$\begin{bmatrix}c_1^T \\ \vdots \\ c_m^T\end{bmatrix} y = \begin{bmatrix}x_k \\ \vdots \\ x_k\end{bmatrix} y \Rightarrow$$

All on same line from def.

$\alpha^T x = \beta \quad \alpha \neq 0 \quad \alpha \in \mathbb{R}$

$\beta \in \mathbb{R} \quad x \in \mathbb{R}^n$

in our scenario

$\alpha = y \quad \beta = y^T x_k$