Dice in Box as End Effector of 2R Robot

Description

This project is based on the default project. Instead of having some fictitious moving the box, the external forces (and torques) will be provided by a 2R robot and the box-dice system will be attached to the robot as an end effector. Given the motor torque on each revolute joint (frictionless and massless), I simulated the motion of the dice experiencing elastic impact due to the robot arm trajectory in the box at the end of the 2R robot. The side wall of the is designed to be rigidly attached to the robot link and the angle between the attached wall and that link is always 90 degrees.

Schematics

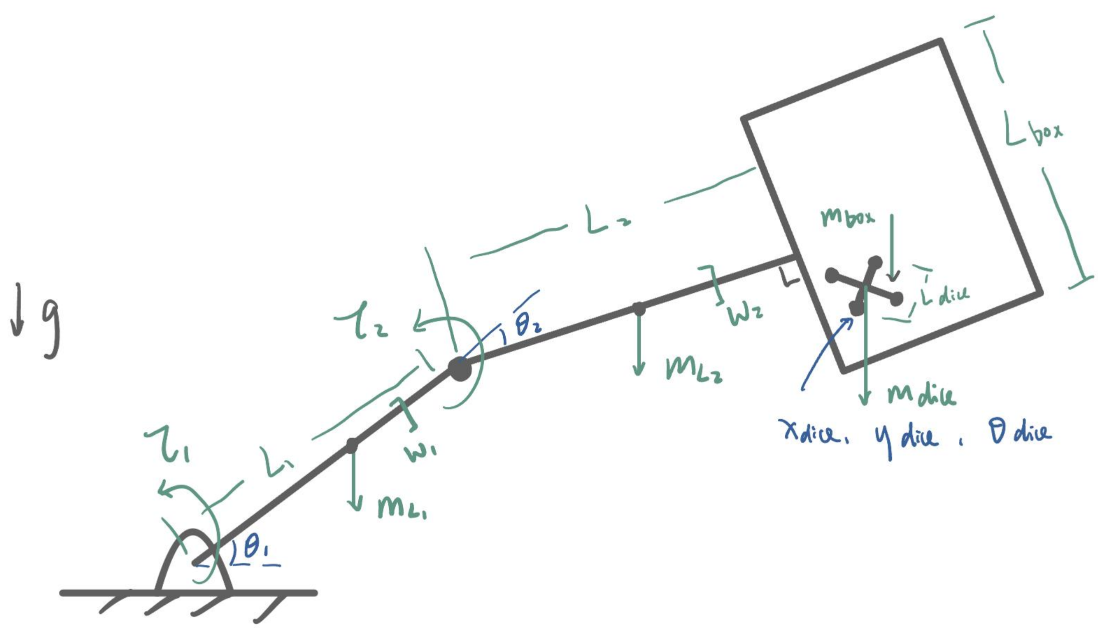


Figure 1: Model schematic of the entire system with parameters

Table 1: Parameters explained

Variable name	Description
τ_1	Torque by motor 1
τ_2	Torque by motor 2
m_{l1}	Mass of link 1
m_{l2}	Mass of link 2
m_{dice}	Mass of the dice
m_{box}	Mass of the box
L_1	Length of link 1
L_2	Length of link 2
w_1	Width of link 1
w_2	Width of link 2
L_{box}	Side length of the box
L_{dice}	Side length of the dice
θ_1	Configuration (angle between link 1 and ground)
θ_2	Configuration (angle between link 2 and link1)
x_{dice}	Configuration (dice x location)
y_{dice}	Configuration (dice y location)
θ_{dice}	Configuration (dice angle of rotation)

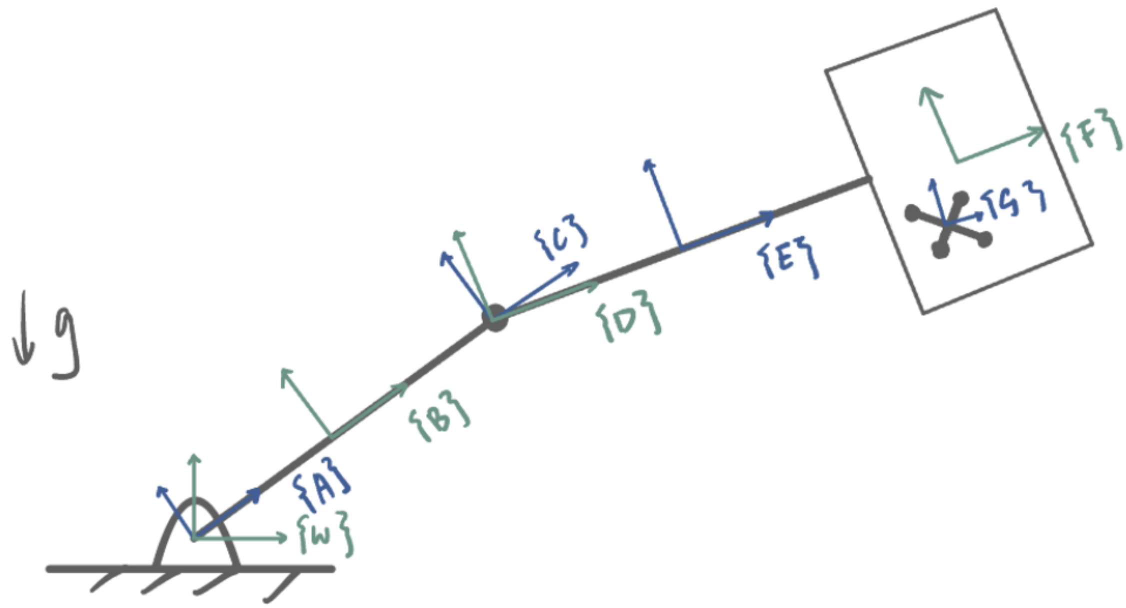


Figure 2: Rigid body transformation of the entire system

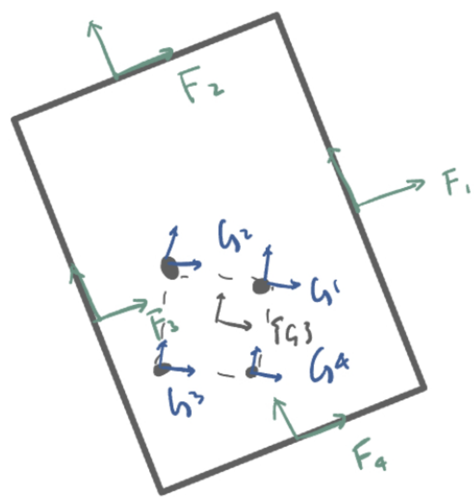


Figure 3: Box wall and dice vertices transformation

Calculation

- Compute different reference frame
  - gWA: pure rotation from world frame **{W}** to link 1 frame **{A}**
  - gAB: pure translation from link 1 frame **{A}** to link 1 COM frame **{B}**
  - gBC: pure translation from link 2 COM **{B}** to joint frame **{C}**
  - gCD: pure rotation from joint frame **{C}** to link 2 frame **{D}**
  - gDE: pure translation from link 2 frame **{D}** to link 2 COM frame **{E}**
  - gEF: pure translation from link 2 COM frame **{E}** to box frame **{F}**
  - gWG: rotation and translation from world frame **{W}** to dice frame **{G}**
- Box wall and dice vertices transformation
  - To get box wall frame **{F_i , i = 1,2,3,4}**, only move either x or y for each translation
  - To get dice vertices frame **{G_i , i = 1,2,3,4}**, move both x and y for each translation
- Calculate Lagrangian for the entire robot-box system
  - Configuration and external forces

$$q = \begin{bmatrix} \theta_1 \\ \theta_2 \\ x_D \\ y_D \\ \theta_D \end{bmatrix} \qquad F_{ext} = \begin{bmatrix} \tau_1 \\ \tau_2 + m_{box} g L_2 \\ 0 \\ m_{dice} g \\ 0 \end{bmatrix}$$

  - Kinetic energy

$$KE = \frac{1}{2} v_i^T \mathcal{I}_i v_i \qquad \begin{cases} v_i = \left(g_i^{-1} \cdot \dot{g}_i\right)^{\vee} \\ \mathcal{I}_i = \begin{bmatrix} m_i & 0 & 0 & 0 & 0 & 0 \\ 0 & m_i & 0 & 0 & 0 & 0 \\ 0 & 0 & m_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_i \end{bmatrix} \end{cases}$$

$$i = \{\text{link 1, link 2, box, dice}\}$$

  - Potential energy

$$V = m_i g y_i \quad i = \{\text{link } 1,\ \text{link } 2,\ \text{box},\ \text{dice}\}$$

- Lagrangian

$$L = KE - V$$

- Solve for Eluer-Lagrangian equation to get $\frac{d^2}{dt}q$

$$EL: \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F_{ext}$$

- Elastic impact and impact constraint
  - Elastic impact

$$\begin{cases} P\big|_{\tau^-}^{\tau^+} = \lambda \nabla \phi \\ H\big|_{\tau^-}^{\tau^+} = P\dot{q} - L(q, \dot{q})\big|_{\tau^-}^{\tau^+} = 0 \end{cases}$$

  - Impact constraint
    There are 16 impact constraints altogether, each corresponding to one of the four dice vertices interacting with one of the four walls of the box. In the code, the transformation is represented as g_{BiDj} (i = 1,2,3,4; j = 1,2,3,4), where i is dice vertex index and j is box wall index.
    For the walls that are vertical to link 2, which are wall 1 and wall 3, I extracted the x location as the impact constraints Φ and named them Φ_{1j} and Φ_{3j}. By the same token, I took the y location as the impact constraints for wall 2 and wall 4 that are horizontal to the link attached, and named them Φ_{2j} and Φ_{4j}.
    With this information, I was able to solve for the impact update equation as well as the impact condition to detect the impact.

Simulation

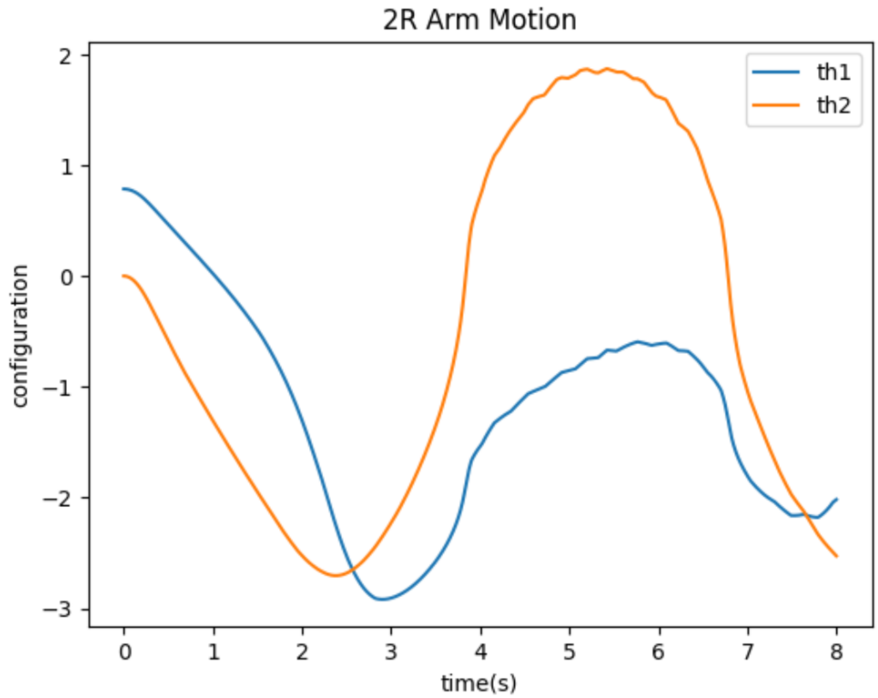


Figure 4: Robot arm motion of the 2 joint angle

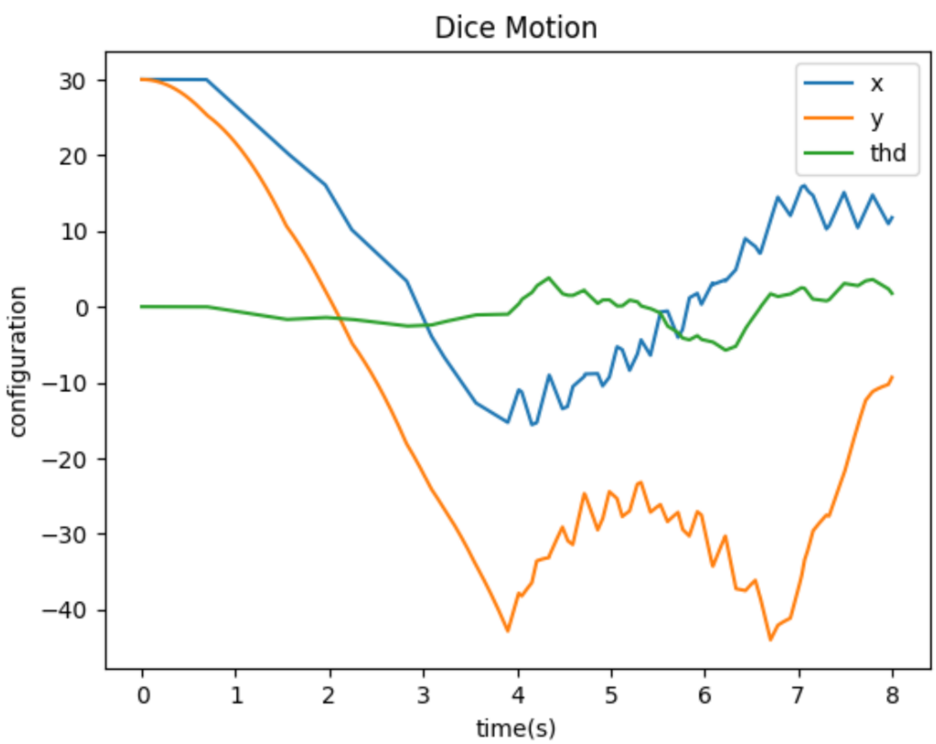


Figure 5: Dice motion of its location and rotation in space

Conclusion

The simulation looks reasonable to me. In the beginning, the dice is simply doing a free fall and the robot arm is operating with the given torque. Once the dice starts to experience the impact, the dice motion, especially the rotation angle and the x location, starts changing. As the impact gets more intense between 4s to 7s, it is worth noticing that the impacts are also subtly affecting the joint angles of the 2R robot.

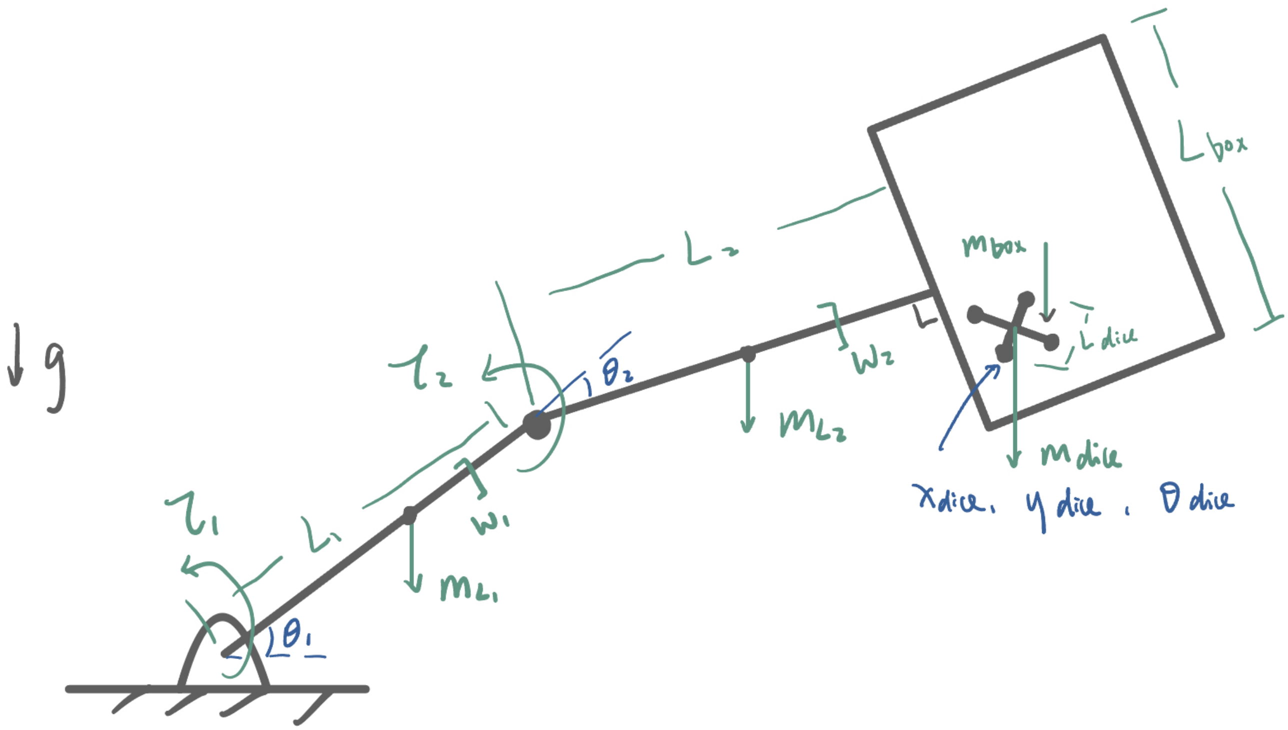
Thank you so much for guiding this amazing journey of dynamics through the quarter! I really enjoyed it and learned a lot.

Appendix: Code

In [1]:
```python
#Import cell
import sympy as sym
import numpy as np
import matplotlib.pyplot as plt
import math

def custom_latex_printer(exp,**options):
    from google.colab.output._publish import javascript
    url = "https://cdnjs.cloudflare.com/ajax/libs/mathjax/3.1.1/latest.js?config=TeX-AMS_HTML"
    javascript(url=url)
    return sym.printing.latex(exp,**options)
sym.init_printing(use_latex="mathjax",latex_printer=custom_latex_printer)
```

In [2]:
```python
from sympy import *
```



In [3]:
```python
"""
L1: length of link 1
w1: width of link 1
L2: length of link 2
w2: width of link 2
LB: side length of box
LD: side length of dice
mL1: mass of link 1
mL2: mass of link 2
mB: mass of box
mD: mass of dice
"""
L1 = 20
w1 = 2
L2 = 20
w2 = 2
LB = 8
LD = 2
mL1 = 10
mL2 = 10
mB = 50
mD = 10
grav = 9.8

t = symbols("t")
th1 = Function(r"\theta_1")(t)
th2 = Function(r"\theta_2")(t)
xD = Function(r"x_D")(t)
yD = Function(r"y_D")(t)
thD = Function(r"\theta_D")(t)

# External forces
FD = -mD*grav
FB = -mB*grav
Tor1 = 8000
Tor2 = 1000
TB = -L2*mB*grav
F_ext = Matrix([Tor1,Tor2+TB,0,FD,0])

q = Matrix([th1,th2,xD,yD,thD])
qd = diff(q,t)
qdd = diff(qd,t)
```

In [4]:
```python
def inv(M): # M 4x4
    R = M[:3,:3]# R 3x3
```

```python
    p = M[:3,-1]# p 3x1
    Rinv = R.T
    pinv = -R.T @ p
    M_inv = Matrix([[Rinv[0,0],Rinv[0,1],Rinv[0,2],pinv[0,0]],
                    [Rinv[1,0],Rinv[1,1],Rinv[1,2],pinv[1,0]],
                    [Rinv[2,0],Rinv[2,1],Rinv[2,2],pinv[2,0]],
                    [0,0,0,1]])
    return M_inv

def unhat(M):
    return Matrix([M[0,3],M[1,3],M[2,3],M[2,1],M[0,2],M[1,0]])
```

In [5]:

```python
gWA = Matrix([[cos(th1),-sin(th1),0,0],[sin(th1),cos(th1),0,0],[0,0,1,0],[0,0,0,1]])
gAB = Matrix([[1,0,0,L1/2],[0,1,0,0],[0,0,1,0],[0,0,0,1]])
gBC = Matrix([[1,0,0,L1/2],[0,1,0,0],[0,0,1,0],[0,0,0,1]])
gCD = Matrix([[cos(2*pi-th2),-sin(2*pi-th2),0,0],[sin(2*pi-th2),cos(2*pi-th2),0,0],[0,0,1,0],[0,0,0,1]])
gDE = Matrix([[1,0,0,L2/2],[0,1,0,0],[0,0,1,0],[0,0,0,1]])
gEF = Matrix([[1,0,0,L2/2+LB/2],[0,1,0,0],[0,0,1,0],[0,0,0,1]])

gWG = Matrix([[cos(thD),-sin(thD),0,xD],[sin(thD),cos(thD),0,yD],[0,0,1,0],[0,0,0,1]])

gWB = gWA @ gAB
gWC = gWA @ gAB @ gBC
gWE = gWA @ gAB @ gBC @ gCD @ gDE
gWF = gWA @ gAB @ gBC @ gCD @ gDE @ gEF

print("W -> B")
display(gWB)
print("W -> C")
display(gWC)
print("W -> E")
display(gWE)
print("W -> F (box)")
display(gWF)
print("W -> G (dice)")
display(gWG)
```

```
W -> B
```

$$\begin{bmatrix} \cos(\theta_1(t)) & -\sin(\theta_1(t)) & 0 & 10.0\cos(\theta_1(t)) \\ \sin(\theta_1(t)) & \cos(\theta_1(t)) & 0 & 10.0\sin(\theta_1(t)) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
W -> C
```

$$\begin{bmatrix} \cos(\theta_1(t)) & -\sin(\theta_1(t)) & 0 & 20.0\cos(\theta_1(t)) \\ \sin(\theta_1(t)) & \cos(\theta_1(t)) & 0 & 20.0\sin(\theta_1(t)) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
W -> E
```

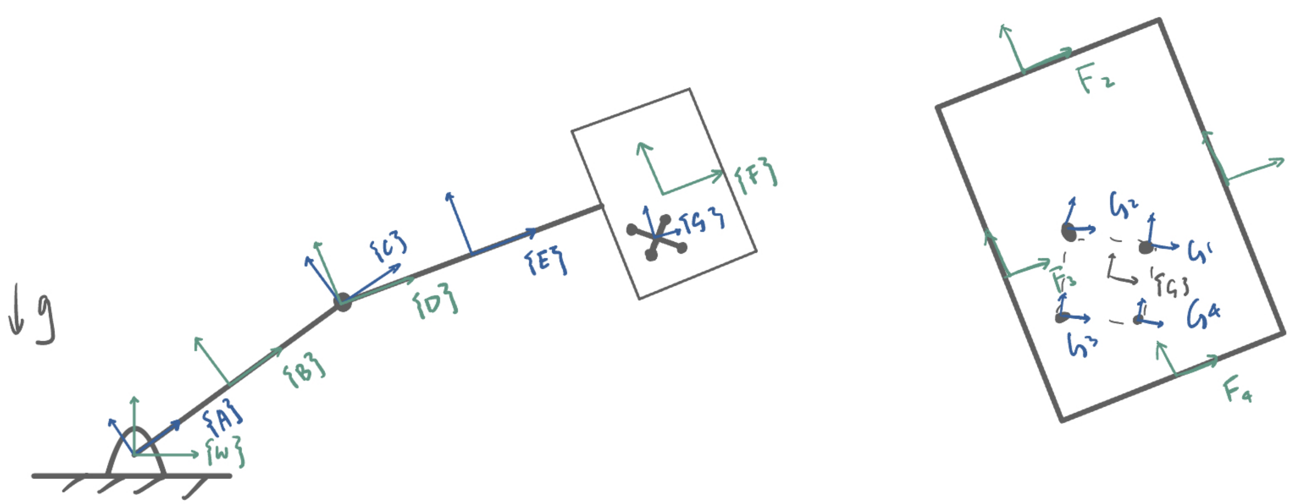
$$\begin{bmatrix} \sin(\theta_1(t))\sin(\theta_2(t)) + \cos(\theta_1(t))\cos(\theta_2(t)) & -\sin(\theta_1(t))\cos(\theta_2(t)) + \sin(\theta_2(t))\cos(\theta_1(t)) & 0 & 10.0\sin(\theta_1(t))\sin(\theta_2(t)) + 10.0\cos(\theta_1(t))\cos(\theta_2(t)) + 20.0\cos(\theta_1(t)) \\ \sin(\theta_1(t))\cos(\theta_2(t)) - \sin(\theta_2(t))\cos(\theta_1(t)) & \sin(\theta_1(t))\sin(\theta_2(t)) + \cos(\theta_1(t))\cos(\theta_2(t)) & 0 & 10.0\sin(\theta_1(t))\cos(\theta_2(t)) + 20.0\sin(\theta_1(t)) - 10.0\sin(\theta_2(t))\cos(\theta_1(t)) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
W -> F (box)
```

$$\begin{bmatrix} \sin(\theta_1(t))\sin(\theta_2(t)) + \cos(\theta_1(t))\cos(\theta_2(t)) & -\sin(\theta_1(t))\cos(\theta_2(t)) + \sin(\theta_2(t))\cos(\theta_1(t)) & 0 & 24.0\sin(\theta_1(t))\sin(\theta_2(t)) + 24.0\cos(\theta_1(t))\cos(\theta_2(t)) + 20.0\cos(\theta_1(t)) \\ \sin(\theta_1(t))\cos(\theta_2(t)) - \sin(\theta_2(t))\cos(\theta_1(t)) & \sin(\theta_1(t))\sin(\theta_2(t)) + \cos(\theta_1(t))\cos(\theta_2(t)) & 0 & 24.0\sin(\theta_1(t))\cos(\theta_2(t)) + 20.0\sin(\theta_1(t)) - 24.0\sin(\theta_2(t))\cos(\theta_1(t)) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
W -> G (dice)
```

$$\begin{bmatrix} \cos(\theta_D(t)) & -\sin(\theta_D(t)) & 0 & x_D(t) \\ \sin(\theta_D(t)) & \cos(\theta_D(t)) & 0 & y_D(t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



In [6]:

```python
# Dice vertex
gd1 = Matrix([[1,0,0,LD/2],[0,1,0,LD/2],[0,0,1,0],[0,0,0,1]])
gd2 = Matrix([[1,0,0,-LD/2],[0,1,0,LD/2],[0,0,1,0],[0,0,0,1]])
gd3 = Matrix([[1,0,0,-LD/2],[0,1,0,-LD/2],[0,0,1,0],[0,0,0,1]])
gd4 = Matrix([[1,0,0,LD/2],[0,1,0,-LD/2],[0,0,1,0],[0,0,0,1]])

# W to dice
```

```
gwd1 = gWG @ gd1
gwd2 = gWG @ gd2
gwd3 = gWG @ gd3
gwd4 = gWG @ gd4

# Box wall midpoint
gb1 = Matrix([[1,0,0,LB/2],[0,1,0,0],[0,0,1,0],[0,0,0,1]])
gb2 = Matrix([[1,0,0,0],[0,1,0,LB/2],[0,0,1,0],[0,0,0,1]])
gb3 = Matrix([[1,0,0,-LB/2],[0,1,0,0],[0,0,1,0],[0,0,0,1]])
gb4 = Matrix([[1,0,0,0],[0,1,0,-LB/2],[0,0,1,0],[0,0,0,1]])

# W to box
gwb1 = gWF @ gb1
gwb2 = gWF @ gb2
gwb3 = gWF @ gb3
gwb4 = gWF @ gb4

# box-dice
gb1d1 = inv(gwb1) @ gwd1
gb1d2 = inv(gwb1) @ gwd2
gb1d3 = inv(gwb1) @ gwd3
gb1d4 = inv(gwb1) @ gwd4

gb2d1 = inv(gwb2) @ gwd1
gb2d2 = inv(gwb2) @ gwd2
gb2d3 = inv(gwb2) @ gwd3
gb2d4 = inv(gwb2) @ gwd4

gb3d1 = inv(gwb3) @ gwd1
gb3d2 = inv(gwb3) @ gwd2
gb3d3 = inv(gwb3) @ gwd3
gb3d4 = inv(gwb3) @ gwd4

gb4d1 = inv(gwb4) @ gwd1
gb4d2 = inv(gwb4) @ gwd2
gb4d3 = inv(gwb4) @ gwd3
gb4d4 = inv(gwb4) @ gwd4
```

In [7]:

```
J_box = 1/6*mB*LB**2
J_dice = 1/6*mB*LD**2
J_L1 = 1/6*mL1*L1*w1
J_L2 = 1/6*mL2*L2*w2


I_box = Matrix([[mB,0,0,0,0,0],[0,mB,0,0,0,0],[0,0,mB,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,J_box]])
I_dice = Matrix([[mD,0,0,0,0,0],[0,mD,0,0,0,0],[0,0,mD,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,J_dice]])
I_L1 = Matrix([[mL1,0,0,0,0,0],[0,mL1,0,0,0,0],[0,0,mL1,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,J_L1]])
I_L2 = Matrix([[mL2,0,0,0,0,0],[0,mL2,0,0,0,0],[0,0,mL2,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,0],[0,0,0,0,0,J_L2]])

# display(I_box)
# display(I_dice)
# display(I_L1)
# display(I_L2)
```

In [8]:

```
# Lagrangian
# KE
VelL1 = unhat(inv(gWB) @ diff(gWB,t))
VelL2 = unhat(inv(gWE) @ diff(gWE,t))
VelB = unhat(inv(gWF) @ diff(gWF,t))
VelD = unhat(inv(gWG) @ diff(gWG,t))
KEL1 = 1/2 * VelL1.T @ I_L1 @ VelL1
KEL2 = 1/2 * VelL2.T @ I_L2 @ VelL2
KEB = 1/2 * VelB.T @ I_box @ VelB
KED = 1/2 * VelD.T @ I_dice @ VelD
KE = simplify(KEL1 + KEL2 + KEB + KED)

# V
yL1 = Matrix([0,1,0,0]).T*gWB*Matrix([0,0,0,1])
yL2 = Matrix([0,1,0,0]).T*gWE*Matrix([0,0,0,1])
yB = Matrix([0,1,0,0]).T*gWF*Matrix([0,0,0,1])

V = simplify(grav*(mL1*yL1[0] + mL2*yL2[0] + mB*yB[0] + mD*yD))

L = simplify(KE[0] - V)
print("Lagrangian:")
display(L)
```

Lagrangian:

$$-98.0y_D(t) - 12740.0\sin\left(\theta_1(t) - \theta_2(t)\right) - 12740.0\sin\left(\theta_1(t)\right) - 1.81898940354586\cdot 10^{-12}\sin^2\left(\theta_2(t)\right)\left(\frac{d}{dt}\theta_1(t)\right)^2 + 26000.0\cos\left(\theta_2(t)\right)\left(\frac{d}{dt}\theta_1(t)\right)^2 - 26000.0\cos\left(\theta_2(t)\right)\frac{d}{dt}\theta_1(t)\frac{d}{dt}\theta_2(t) + 27733.3333333333\left(\frac{d}{dt}\theta_1(t)\right)^2 - 30400.0\frac{d}{dt}\theta_1(t)\frac{d}{dt}\theta_2(t) + 15200.0\left(\frac{d}{dt}\theta_2(t)\right)^2 + 16.6666666666667\left(\frac{d}{dt}\theta_D(t)\right)^2 + 5.0\left(\frac{d}{dt}x_D(t)\right)^2 + 5.0\left(\frac{d}{dt}y_D(t)\right)^2$$

In [9]:

```
L_mat = Matrix([L])
dLdq = L_mat.jacobian(q)
dLdqd = L_mat.jacobian(qd)
ddtdLdqd = diff(dLdqd,t)
lhs = simplify(ddtdLdqd - dLdq).T
rhs = F_ext
EL = Eq(lhs,rhs)
sol = solve(EL,qdd)
print("EL:")
display(EL)
```

EL:

$$\left[\begin{matrix}-3.63797880709171 \cdot 10^{-12} \sin^{2}\left(\theta_{2}(t)\right) \frac{d^{2}}{dt^{2}} \theta_{1}(t) - 52000.0 \sin\left(\theta_{2}(t)\right) \frac{d}{dt} \theta_{1}(t) \frac{d}{dt} \theta_{2}(t) + 26000.0 \sin\left(\theta_{2}(t)\right) \left(\frac{d}{dt} \theta_{2}(t)\right)^{2} - 3.63797880709171 \cdot 10^{-12} \sin\left(2 \theta_{2}(t)\right) \frac{d}{dt} \theta_{1}(t) \frac{d}{dt} \theta_{2}(t) \\ + 12740.0 \cos\left(\theta_{1}(t) - \theta_{2}(t)\right) + 12740.0 \cos\left(\theta_{1}(t)\right) + 52000.0 \cos\left(\theta_{2}(t)\right) \frac{d^{2}}{dt^{2}} \theta_{1}(t) - 26000.0 \cos\left(\theta_{2}(t)\right) \frac{d^{2}}{dt^{2}} \theta_{2}(t) + 55466.6666666667 \frac{d^{2}}{dt^{2}} \theta_{1}(t) - 30400.0 \frac{d^{2}}{dt^{2}} \theta_{2}(t) \\ 26000.0 \sin\left(\theta_{2}(t)\right) \left(\frac{d}{dt} \theta_{1}(t)\right)^{2} + 1.81898940354586 \cdot 10^{-12} \sin\left(2 \theta_{2}(t)\right) \left(\frac{d}{dt} \theta_{1}(t)\right)^{2} - 12740.0 \cos\left(\theta_{1}(t) - \theta_{2}(t)\right) - 26000.0 \cos\left(\theta_{2}(t)\right) \frac{d^{2}}{dt^{2}} \theta_{1}(t) - 30400.0 \frac{d^{2}}{dt^{2}} \theta_{1}(t) \\ + 30400.0 \frac{d^{2}}{dt^{2}} \theta_{2}(t) \\ 10.0 \frac{d^{2}}{dt^{2}} x_{D}(t) \\ 10.0 \frac{d^{2}}{dt^{2}} y_{D}(t) + 98.0 \\ 33.3333333333333 \frac{d^{2}}{dt^{2}} \theta_{D}(t)\end{matrix}\right] = \left[\begin{matrix}8000 \\ -8800.0 \\ 0 \\ -98.0 \\ 0\end{matrix}\right]$$

In [10]:

```
sol_th1dd = sol[qdd[0]]
sol_th2dd = sol[qdd[1]]
sol_xDdd = sol[qdd[2]]
sol_yDdd = sol[qdd[3]]
sol_thDdd = sol[qdd[4]]
var_ls = [*q,*qd,t]
f_th1dd = lambdify(var_ls,sol_th1dd)
f_th2dd = lambdify(var_ls,sol_th2dd)
f_xDdd = lambdify(var_ls,sol_xDdd)
f_yDdd = lambdify(var_ls,sol_yDdd)
f_thDdd = lambdify(var_ls,sol_thDdd)
```

In [11]:

```
# Dummy for q and qd
# q = Matrix([th1,th2,xD,yD,thD])
dum_th1 = symbols(r"\theta_d1")
dum_th2 = symbols(r"\theta_d2")
dum_xD = symbols(r"x_dD")
dum_yD = symbols(r"y_dD")
dum_thD = symbols(r"\theta_dD")

dum_th1d = symbols(r"\dot{\theta_{d1}}")
dum_th2d = symbols(r"\dot{\theta_{d2}}")
dum_xDd = symbols(r"\dot{x_{dD}}")
dum_yDd = symbols(r"\dot{y_{dD}}")
dum_thDd = symbols(r"\dot{\theta_{dD}}")

dummy_dict = {
    q[0]:dum_th1,q[1]:dum_th2,q[2]:dum_xD,q[3]:dum_yD,q[4]:dum_thD,
    qd[0]:dum_th1d,qd[1]:dum_th2d,qd[2]:dum_xDd,qd[3]:dum_yDd,qd[4]:dum_thDd
}

# for k in dummy_dict.values():
#     display(k)
```

In [12]:

```
# Hamiltonian
H = simplify((dLdqd*qd - L_mat)[0])
print("Hamiltonian")
H_sub = H.subs(dummy_dict)
display(H_sub)
```

Hamiltonian

$$-1.81898940354586 \cdot 10^{-12} \dot{\theta_{d1}}^{2} \sin^{2}\left(\theta_{d2}\right) + 26000.0 \dot{\theta_{d1}}^{2} \cos\left(\theta_{d2}\right) + 27733.3333333333 \dot{\theta_{d1}}^{2} - 26000.0 \dot{\theta_{d1}} \dot{\theta_{d2}} \cos\left(\theta_{d2}\right) - 30400.0 \dot{\theta_{d1}} \dot{\theta_{d2}} + 15200.0 \dot{\theta_{d2}}^{2} + 16.6666666666667 \dot{\theta_{dD}}^{2} \\ + 5.0 \dot{x_{dD}}^{2} + 5.0 \dot{y_{dD}}^{2} + 98.0 y_{dD} + 12740.0 \sin\left(\theta_{d1}\right) + 12740.0 \sin\left(\theta_{d1} - \theta_{d2}\right)$$

In [13]:

```
# Impact condition (4 verts * 4 walls)
# x of wall 1 & 3 [0,-1]
# y of wall 2 & 4 [1,-1]

# phiBD
phi11 = gb1d1[0,-1].subs(dummy_dict)
phi12 = gb1d2[0,-1].subs(dummy_dict)
phi13 = gb1d3[0,-1].subs(dummy_dict)
phi14 = gb1d4[0,-1].subs(dummy_dict)

phi21 = gb2d1[1,-1].subs(dummy_dict)
phi22 = gb2d2[1,-1].subs(dummy_dict)
phi23 = gb2d3[1,-1].subs(dummy_dict)
phi24 = gb2d4[1,-1].subs(dummy_dict)

phi31 = gb3d1[0,-1].subs(dummy_dict)
phi32 = gb3d2[0,-1].subs(dummy_dict)
phi33 = gb3d3[0,-1].subs(dummy_dict)
phi34 = gb3d4[0,-1].subs(dummy_dict)

phi41 = gb4d1[1,-1].subs(dummy_dict)
phi42 = gb4d2[1,-1].subs(dummy_dict)
phi43 = gb4d3[1,-1].subs(dummy_dict)
phi44 = gb4d4[1,-1].subs(dummy_dict)


phi = Matrix([phi11,phi12,phi13,phi14,phi21,phi22,phi23,phi24,phi31,phi32,phi33,phi34,phi41,phi42,phi43,phi44])
# phi = Matrix([phi11,phi21,phi31,phi41,phi12,phi22,phi32,phi42,phi13,phi23,phi33,phi43,phi14,phi24,phi34,phi44])
```

In [14]:

```
# Impact update equation
dLdqd_sub = dLdqd.subs(dummy_dict)
dphidq = phi.jacobian([dum_th1,dum_th2,dum_xD,dum_yD,dum_thD])
lam = symbols("\lambda")
dum_th1dp = symbols(r"\dot{\theta_{d1}^{+}}")
dum_th2dp = symbols(r"\dot{\theta_{d2}^{+}}")
dum_xDdp = symbols(r"\dot{x_{dD}^{+}}")
dum_yDdp = symbols(r"\dot{y_{dD}^{+}}")
dum_thDdp = symbols(r"\dot{\theta_{dD}^{+}}")

impact_dict = {dum_th1d:dum_th1dp,
               dum_th2d:dum_th2dp,
               dum_xDd:dum_xDdp,
               dum_yDd:dum_yDdp,
               dum_thDd:dum_thDdp}

# at t+
```

```
lam = symbols(r"\lambda")
dLdqd_p = dLdqd_sub.subs(impact_dict)
dphidq_p = dphidq.subs(impact_dict)
H_p = H_sub.subs(impact_dict)

momt = simplify(dLdqd_p - dLdqd_sub)
lhs = Matrix([momt[0],momt[1],momt[2],momt[3],momt[4],simplify(H_p - H_sub)])

display(lhs)
```

$$\begin{bmatrix} -3.63797880709171 \cdot 10^{-12}\dot{\theta}^{+}_{d1} \sin^{2}(\theta_{d2}) + 52000.0\dot{\theta}^{+}_{d1} \cos(\theta_{d2}) + 55466.6666666667\dot{\theta}^{+}_{d1} + 3.63797880709171 \cdot 10^{-12}\dot{\theta}_{d1} \sin^{2}(\theta_{d2}) - 52000.0\dot{\theta}_{d1} \cos(\theta_{d2}) - 55466.6666666667\dot{\theta}_{d1} - 26000.0\dot{\theta}^{+}_{d2} \cos(\theta_{d2}) - 30400.0\dot{\theta}^{+}_{d2} + 26000.0\dot{\theta}_{d2} \cos(\theta_{d2}) + 30400.0\dot{\theta}_{d2} \\ -26000.0\dot{\theta}^{+}_{d1} \cos(\theta_{d2}) - 30400.0\dot{\theta}^{+}_{d1} + 26000.0\dot{\theta}_{d1} \cos(\theta_{d2}) + 30400.0\dot{\theta}_{d1} + 30400.0\dot{\theta}^{+}_{d2} - 30400.0\dot{\theta}_{d2} \\ 10.0\dot{x}^{+}_{dD} - 10.0\dot{x}_{dD} \\ 10.0\dot{y}^{+}_{dD} - 10.0\dot{y}_{dD} \\ 33.3333333333333\dot{\theta}^{+}_{dD} - 33.3333333333333\dot{\theta}_{dD} \\ -1.81898940354586 \cdot 10^{-12}\left(\dot{\theta}^{+}_{d1}\right)^{2} \sin^{2}(\theta_{d2}) + 26000.0\left(\dot{\theta}^{+}_{d1}\right)^{2} \cos(\theta_{d2}) + 27733.3333333333\left(\dot{\theta}^{+}_{d1}\right)^{2} - 26000.0\dot{\theta}^{+}_{d1}\dot{\theta}^{+}_{d2} \cos(\theta_{d2}) - 30400.0\dot{\theta}^{+}_{d1}\dot{\theta}^{+}_{d2} + 1.81898940354586 \cdot 10^{-12}\dot{\theta}_{d1}^{2} \sin^{2}(\theta_{d2}) - 26000.0\dot{\theta}_{d1}^{2} \cos(\theta_{d2}) - 27733.3333333333\dot{\theta}_{d1}^{2} + 26000.0\dot{\theta}_{d1}\dot{\theta}_{d2} \cos(\theta_{d2}) + 30400.0\dot{\theta}_{d1}\dot{\theta}_{d2} + 15200.0\left(\dot{\theta}^{+}_{d2}\right)^{2} - 15200.0\dot{\theta}_{d2}^{2} + 16.6666666666667\left(\dot{\theta}^{+}_{dD}\right)^{2} - 16.6666666666667\dot{\theta}_{dD}^{2} + 5.0\left(\dot{x}^{+}_{dD}\right)^{2} - 5.0\dot{x}_{dD}^{2} + 5.0\left(\dot{y}^{+}_{dD}\right)^{2} - 5.0\dot{y}_{dD}^{2} \end{bmatrix}$$

In [15]:
```
# lhs.shape
# dphidq.shape
# 0-4: momemtum; 5: hamiltonian

impact_eqs = []
for i in range(phi.shape[0]):
    p1 = lam*dphidq[i,0]
    p2 = lam*dphidq[i,1]
    p3 = lam*dphidq[i,2]
    p4 = lam*dphidq[i,3]
    p5 = lam*dphidq[i,4]
    rhs = Matrix([p1,p2,p3,p4,p5,0])
    impact_eqs.append(Eq(lhs,rhs))

f_phi = lambdify(list(dummy_dict.values()),phi) #phi shape: 16x1
```

In [16]:
```
#Impact!

def impact_cond(s,f_phi,thres):
    val = f_phi(*s)
    for i in range(val.shape[0]):
        if abs(val[i][0]) < thres:
            return (True,i,val)
    return (False,None,val)

# def impact_cond(s,f_phi,prev):
#     val = f_phi(*s)
#     product = val*prev
#     print(product)
#     for i in range(val.shape[0]):
#         if product[i] < 0:
#             return (True,i,val)
#     return (False,None,val)


subs_ls = list(impact_dict.values())
def impact_update(s,impact_eqs,subs_ls):
    sub_d = {dum_th1:s[0],dum_th2:s[1],dum_xD:s[2],dum_yD:s[3],dum_thD:s[4],dum_th1d:s[5],dum_th2d:s[6],dum_xDd:s[7],dum_yDd:s[8],dum_thDd:s[9]}
    update = impact_eqs.subs(sub_d)
    sol_var = [dum_th1dp,dum_th2dp,dum_xDdp,dum_yDdp,dum_thDdp,lam]
    sol_imp = solve(update,sol_var,dict = True)
    if len(sol_imp) == 1:
        print("ERROR!")
    else:
        for sol in sol_imp:
            sol_lam = sol[lam]
            #if not math.isclose(abs(sol_lam),0,abs_tol = 1e-6):
            if abs(sol_lam) > 1e-09:
                sol0 = float(N(sol[subs_ls[0]]))
                sol1 = float(N(sol[subs_ls[1]]))
                sol2 = float(N(sol[subs_ls[2]]))
                sol3 = float(N(sol[subs_ls[3]]))
                sol4 = float(N(sol[subs_ls[4]]))
                res = np.array([s[0],s[1],s[2],s[3],s[4],sol0,sol1,sol2,sol3,sol4])
                return res
```

In [17]:
```
# Simulate
def dyn(s,ts):
    # print(s)
    func_th1dd = f_th1dd(*s,ts)
    func_th2dd = f_th2dd(*s,ts)
    func_xDdd = f_xDdd(*s,ts)
    func_yDdd = f_yDdd(*s,ts)
    func_thDdd = f_thDdd(*s,ts)
    return np.array([s[-5],s[-4],s[-3],s[-2],s[-1],func_th1dd,func_th2dd,func_xDdd,func_yDdd,func_thDdd])

def integrate(f, ts, xt, dt):
    k1 = dt * f(xt,ts)
    k2 = dt * f(xt+k1/2.,ts+dt/2.)
    k3 = dt * f(xt+k2/2.,ts+dt/2.)
    k4 = dt * f(xt+k3,ts+dt/2.)
    new_xt = xt + (1/6.) * (k1+2.0*k2+2.0*k3+k4)
    return new_xt

def impact_simulate(f, x0, tspan, dt, integrate):
    n = 0
    N = int((max(tspan)-min(tspan))/dt)
    x = np.copy(x0)
    tvec = np.linspace(min(tspan),max(tspan),N)
    xtraj = np.zeros((len(x0),N))
    ts = 0
    threshold = .1
    for i in range(N):
        ts += dt
        impact = impact_cond(x,f_phi,threshold)[0]
        loc = impact_cond(x,f_phi,threshold)[1]
```

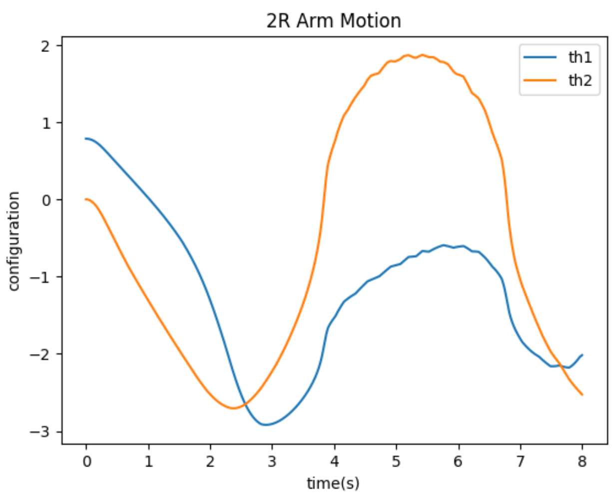
```
        phi_val = impact_cond(x,f_phi,threshold)[2]
        if impact:
            x = impact_update(x,impact_eqs[loc],subs_ls)
            # print(loc)
            # print(ts)
            xtraj[:,i]=integrate(f,ts,x,dt)
            n += 1
        else:
            xtraj[:,i]=integrate(f,ts,x,dt)
        x = np.copy(xtraj[:,i])
    print(f"number of impacts: {n}")
    return xtraj
```

In [18]:
```
s0 = np.array([np.pi/4,0,30,30,0,0,0,0,0,0])
traj = impact_simulate(dyn,s0,[0,8],.002,integrate)
```

```
number of impacts: 103
```

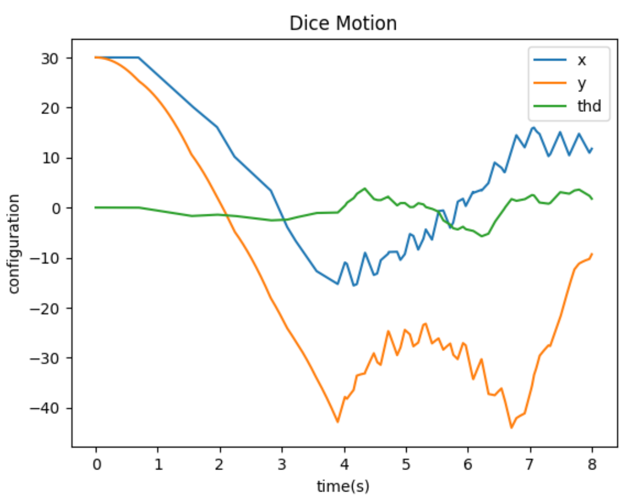
In [19]:
```
timespan = np.linspace(0,8,4000)
plt.plot(timespan,traj[0])
plt.plot(timespan,traj[1])
plt.xlabel("time(s)")
plt.ylabel("configuration")
plt.title("2R Arm Motion")
plt.legend(["th1","th2"])
```

Out[19]:
```
<matplotlib.legend.Legend at 0x7fcfa7861810>
```



In [20]:
```
plt.plot(timespan,traj[2])
plt.plot(timespan,traj[3])
plt.plot(timespan,traj[4])
plt.xlabel("time(s)")
plt.ylabel("configuration")
plt.title("Dice Motion")
plt.legend(["x","y","thd"])
```

Out[20]:
```
<matplotlib.legend.Legend at 0x7fcfa75cac50>
```



In [21]:
```
def animate_2R(arr,L1,L2,W1,W2,LB,LD,T):

    ################################
    # Imports required for animation.
    from plotly.offline import init_notebook_mode, iplot
    from IPython.display import display, HTML
    import plotly.graph_objects as go

    #######################
    # Browser configuration.
    def configure_plotly_browser_state():
        import IPython
        display(IPython.core.display.HTML('''
            <script src="/static/components/requirejs/require.js"></script>
            <script>
              requirejs.config({
                paths: {
                  base: '/static/base',
                  plotly: 'https://cdn.plot.ly/plotly-latest.min.js?noext',
                },
              });
            </script>
```

```python
        '''))
    configure_plotly_browser_state()
    init_notebook_mode(connected=False)

    ###############################################
    # Getting data from pendulum angle trajectories.
    tth1 = arr[0]
    tth2 = arr[1]
    xxD = arr[2]
    yyD = arr[3]
    tthD = arr[4]

    N = len(arr[0]) # Need this for specifying length of simulation

    # Box Vertices
    vertB1 = np.array([LB/2,LB/2,0,1])
    vertB2 = np.array([-LB/2,LB/2,0,1])
    vertB3 = np.array([-LB/2,-LB/2,0,1])
    vertB4 = np.array([LB/2,-LB/2,0,1])

    # Dice Vertices
    vertD1 = np.array([LD/2,LD/2,0,1])
    vertD2 = np.array([-LD/2,LD/2,0,1])
    vertD3 = np.array([-LD/2,-LD/2,0,1])
    vertD4 = np.array([LD/2,-LD/2,0,1])

    # Link1 Vertices
    vertL11 = np.array([L1/2,W1/2,0,1])
    vertL12 = np.array([-L1/2,W1/2,0,1])
    vertL13 = np.array([-L1/2,-W1/2,0,1])
    vertL14 = np.array([L1/2,-W1/2,0,1])

    # Link2 Vertices
    vertL21 = np.array([L2/2,W2/2,0,1])
    vertL22 = np.array([-L2/2,W2/2,0,1])
    vertL23 = np.array([-L2/2,-W2/2,0,1])
    vertL24 = np.array([L2/2,-W2/2,0,1])

    ###############################################
    # Define arrays containing data for frame axes
    # In each frame, the x and y axis are always fixed
    x_axis = np.array([0.3, 0.0])
    y_axis = np.array([0.0, 0.3])

    # Use homogeneous tranformation to transfer these two axes/points
    # back to the fixed frame
    frame_b_1_axis = np.zeros((2,N))
    frame_b_2_axis = np.zeros((2,N))
    frame_b_3_axis = np.zeros((2,N))
    frame_b_4_axis = np.zeros((2,N))
    frame_d_1_axis = np.zeros((2,N))
    frame_d_2_axis = np.zeros((2,N))
    frame_d_3_axis = np.zeros((2,N))
    frame_d_4_axis = np.zeros((2,N))
    frame_l1_1_axis = np.zeros((2,N))
    frame_l1_2_axis = np.zeros((2,N))
    frame_l1_3_axis = np.zeros((2,N))
    frame_l1_4_axis = np.zeros((2,N))
    frame_l2_1_axis = np.zeros((2,N))
    frame_l2_2_axis = np.zeros((2,N))
    frame_l2_3_axis = np.zeros((2,N))
    frame_l2_4_axis = np.zeros((2,N))

    for i in range(N): # iteration through each time step
        # evaluate homogeneous transformation
        tWA = np.array([[np.cos(tth1[i]),-np.sin(tth1[i]),0,0],[np.sin(tth1[i]),np.cos(tth1[i]),0,0],[0,0,1,0],[0,0,0,1]])
        tAB = np.array([[1,0,0,L1/2],[0,1,0,0],[0,0,1,0],[0,0,0,1]])
        tBC = np.array([[1,0,0,L1/2],[0,1,0,0],[0,0,1,0],[0,0,0,1]])
        tCD = np.array([[np.cos(2*np.pi-tth2[i]),-np.sin(2*np.pi-tth2[i]),0,0],[np.sin(2*np.pi-tth2[i]),np.cos(2*np.pi-tth2[i]),0,0],[0,0,1,0],[0,0,0,1]])
        tDE = np.array([[1,0,0,L2/2],[0,1,0,0],[0,0,1,0],[0,0,0,1]])
        tEF = np.array([[1,0,0,L2/2+LB/2],[0,1,0,0],[0,0,1,0],[0,0,0,1]])

        tWB = tWA.dot(tAB)
        tWE = tWA.dot(tAB.dot(tBC.dot(tCD.dot(tDE))))
        tWF = tWA.dot(tAB.dot(tBC.dot(tCD.dot(tDE.dot(tEF)))))
        tWG = np.array([[np.cos(tthD[i]),-np.sin(tthD[i]),0,xxD[i]],[np.sin(tthD[i]),np.cos(tthD[i]),0,yyD[i]],[0,0,1,0],[0,0,0,1]])

        # transfer the x and y axes in body frame back to fixed frame at
        # the current time step
        frame_b_1_axis[:,i] = tWF.dot(np.array([vertB1[0],vertB1[1],0,1]))[0:2]
        frame_b_2_axis[:,i] = tWF.dot(np.array([vertB2[0],vertB2[1],0,1]))[0:2]
        frame_b_3_axis[:,i] = tWF.dot(np.array([vertB3[0],vertB3[1],0,1]))[0:2]
        frame_b_4_axis[:,i] = tWF.dot(np.array([vertB4[0],vertB4[1],0,1]))[0:2]

        frame_d_1_axis[:,i] = tWG.dot(np.array([vertD1[0],vertD1[1],0,1]))[0:2]
        frame_d_2_axis[:,i] = tWG.dot(np.array([vertD2[0],vertD2[1],0,1]))[0:2]
        frame_d_3_axis[:,i] = tWG.dot(np.array([vertD3[0],vertD3[1],0,1]))[0:2]
        frame_d_4_axis[:,i] = tWG.dot(np.array([vertD4[0],vertD4[1],0,1]))[0:2]

        frame_l1_1_axis[:,i] = tWB.dot(np.array([vertL11[0],vertL11[1],0,1]))[0:2]
        frame_l1_2_axis[:,i] = tWB.dot(np.array([vertL12[0],vertL12[1],0,1]))[0:2]
        frame_l1_3_axis[:,i] = tWB.dot(np.array([vertL13[0],vertL13[1],0,1]))[0:2]
        frame_l1_4_axis[:,i] = tWB.dot(np.array([vertL14[0],vertL14[1],0,1]))[0:2]

        frame_l2_1_axis[:,i] = tWE.dot(np.array([vertL21[0],vertL21[1],0,1]))[0:2]
        frame_l2_2_axis[:,i] = tWE.dot(np.array([vertL22[0],vertL22[1],0,1]))[0:2]
        frame_l2_3_axis[:,i] = tWE.dot(np.array([vertL23[0],vertL23[1],0,1]))[0:2]
        frame_l2_4_axis[:,i] = tWE.dot(np.array([vertL24[0],vertL24[1],0,1]))[0:2]

    ####################################
    # Using these to specify axis limits.
    xm = -50 #np.min(xx1)-0.5
    xM = 50 #np.max(xx1)+0.5
    ym = -50 #np.min(yy1)-2.5
    yM = 50 #np.max(yy1)+1.5

    ###########################
    # Defining data dictionary.
    # Trajectories are here.
    data=[
        # note that except for the trajectory (which you don't need this time),
        # you don't need to define entries other than "name". The items defined
        # in this list will be related to the items defined in the "frames" list
        # later in the same order. Therefore, these entries can be considered as
        # labels for the components in each animation frame
        dict(name='Dice'),
        dict(name='Dice Vertex'),
        dict(name='Box'),
```

```python
        dict(name='Link 1'),
        dict(name='Link 2'),
        ]

    ################################
    # Preparing simulation layout.
    # Title and axis ranges are here.
    layout=dict(autosize=False, width=1000, height=1000,
                xaxis=dict(range=[xm, xM], autorange=False, zeroline=False,dtick=1),
                yaxis=dict(range=[ym, yM], autorange=False, zeroline=False,scaleanchor = "x",dtick=1),
                title='2R Simulation',
                hovermode='closest',
                updatemenus= [{'type': 'buttons',
                               'buttons': [{'label': 'Play','method': 'animate',
                                            'args': [None, {'frame': {'duration': T, 'redraw': False}}]},
                                           {'args': [[None], {'frame': {'duration': T, 'redraw': False}, 'mode': 'immediate',
                                            'transition': {'duration': 0}}],'label': 'Pause','method': 'animate'}
                                          ]
                              }]
               )

    ########################################
    # Defining the frames of the simulation.
    # This is what draws the lines from
    # joint to joint of the pendulum.
    frames=[dict(data=[# first three objects correspond to the arms and two masses,
                       # same order as in the "data" variable defined above (thus
                       # they will be labeled in the same order)
                    #    dict(x=[-1,1],
                    #         y=[0,0],
                    #         mode='lines',
                    #         line=dict(color='orange', width=3),
                    #         ),

                       dict(x=[frame_d_1_axis[0][k],frame_d_2_axis[0][k],frame_d_3_axis[0][k],frame_d_4_axis[0][k],frame_d_1_axis[0][k]],
                            y=[frame_d_1_axis[1][k],frame_d_2_axis[1][k],frame_d_3_axis[1][k],frame_d_4_axis[1][k],frame_d_1_axis[1][k]],
                            mode='lines',
                            line=dict(color='orange', width=3),
                            ),
                       go.Scatter(
                            x=[frame_d_1_axis[0][k],frame_d_2_axis[0][k],frame_d_3_axis[0][k],frame_d_4_axis[0][k],frame_d_1_axis[0][k]],
                            y=[frame_d_1_axis[1][k],frame_d_2_axis[1][k],frame_d_3_axis[1][k],frame_d_4_axis[1][k],frame_d_1_axis[1][k]],
                            mode="markers",
                            marker=dict(color="red", size=3)),
                       dict(x=[frame_b_1_axis[0][k],frame_b_2_axis[0][k],frame_b_3_axis[0][k],frame_b_4_axis[0][k],frame_b_1_axis[0][k]],
                            y=[frame_b_1_axis[1][k],frame_b_2_axis[1][k],frame_b_3_axis[1][k],frame_b_4_axis[1][k],frame_b_1_axis[1][k]],
                            mode='lines',
                            line=dict(color='green', width=3),
                            ),
                       dict(x=[frame_l1_1_axis[0][k],frame_l1_2_axis[0][k],frame_l1_3_axis[0][k],frame_l1_4_axis[0][k],frame_l1_1_axis[0][k]],
                            y=[frame_l1_1_axis[1][k],frame_l1_2_axis[1][k],frame_l1_3_axis[1][k],frame_l1_4_axis[1][k],frame_l1_1_axis[1][k]],
                            mode='lines',
                            line=dict(color='blue', width=3),
                            ),
                       dict(x=[frame_l2_1_axis[0][k],frame_l2_2_axis[0][k],frame_l2_3_axis[0][k],frame_l2_4_axis[0][k],frame_l2_1_axis[0][k]],
                            y=[frame_l2_1_axis[1][k],frame_l2_2_axis[1][k],frame_l2_3_axis[1][k],frame_l2_4_axis[1][k],frame_l2_1_axis[1][k]],
                            mode='lines',
                            line=dict(color='blue', width=3),
                            ),

                      ]) for k in range(N)]

    ########################################
    # Putting it all together and plotting.
    figure1=dict(data=data, layout=layout, frames=frames)
    iplot(figure1)
```

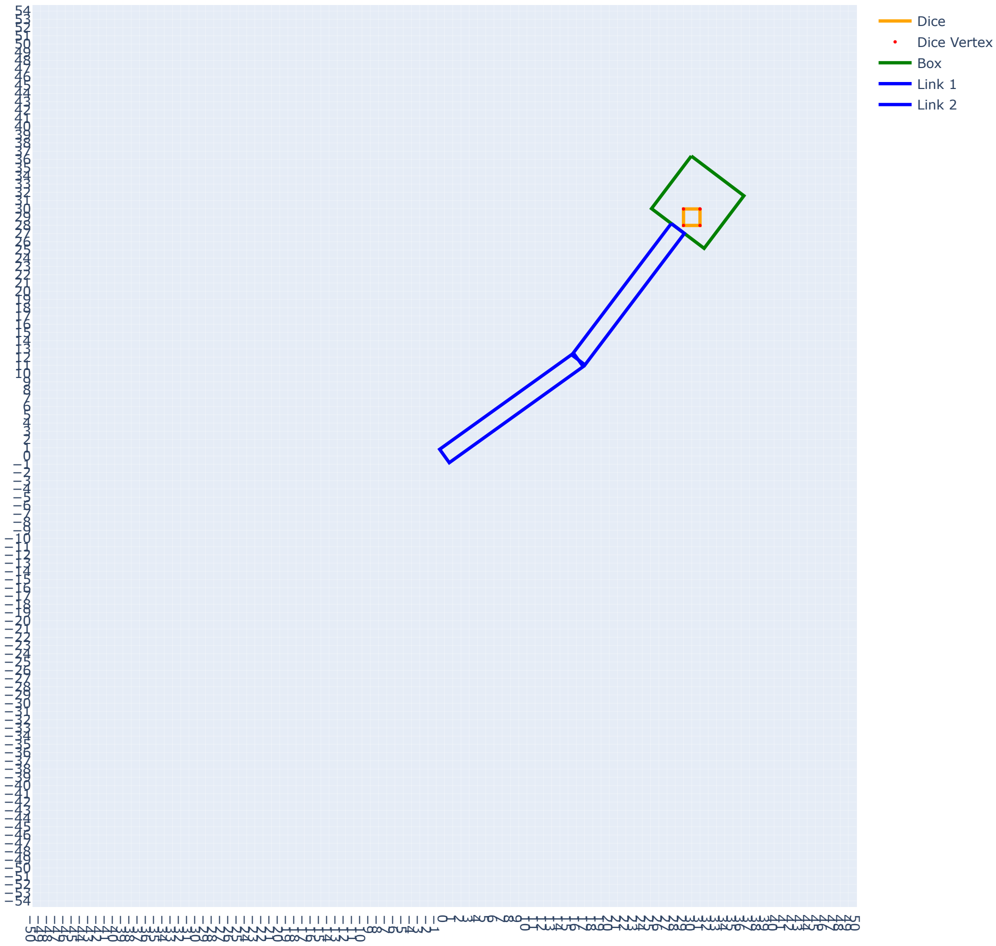
In [24]:

```python
animate_2R(traj,L1,L2,w1,w2,LB,LD,8)
```

2R Simulation

Play

Pause



In [22]: