Hacer la transformada de Fourier de una ley de potencias para la función de correlación $\xi(r) = \left(\frac{r}{r_0}\right)^{-\mu}$

donde $\xi(r_0) = 1$ y $n = \mu - d$

$$P(k) = \int_0^\infty \xi(r) \frac{\sin kr}{kr} 4\pi r^2 dr$$

$$= \int_0^\infty \left(\frac{r}{r_0}\right)^{-\mu} \frac{\sin kr}{kr} 4\pi r^2 dr$$

$$= \frac{r_0^\mu 4\pi}{k} \int_0^\infty r^{1-\mu} \sin kr \, dr$$

$$= \frac{r_0^\mu 4\pi}{k} \int_0^\infty r^{(2-\mu)-1} \sin kr \, dr$$

y usando que $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$$P(k) = \frac{r_0^\mu 4\pi}{k} \int_0^\infty r^{(2-\mu)-1} \frac{e^{ikr} - e^{-ikr}}{2i} dr$$

$$= \frac{r_0^\mu 4\pi}{k} \left[\frac{1}{2i}\int_0^\infty r^{(2-\mu)-1} e^{ikr} dr - \frac{1}{2i}\int_0^\infty r^{(2-\mu)-1} e^{-ikr} dr\right]$$

recordando que

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad y \quad \int_0^\infty t^b e^{-at} dt = \frac{\Gamma(b+1)}{a^{b+1}}$$

donde tomamos $b = (2-\mu)-1$ y $a = ik$

tenemos

$$\frac{1}{2i} \int_0^\infty r^b e^{-ab} dn = \frac{1}{2i} \frac{\Gamma((2-\mu)-1+1)}{(ik)^{(2-\mu)-1+1}}$$

$$= \frac{1}{2i} \frac{\Gamma(2-\mu)}{(ik)^{2-\mu}}$$

$$a = -ik \;,\; b = (2-\mu)-1$$

$$\frac{1}{2i}\int_0^\infty r^{(2-\mu)-1}\, e^{ikr}\, dr = \frac{1}{2i}\int_0^\infty r^{b}\, e^{-ar}\, dr$$

$$= \frac{1}{2i}\,\frac{\Gamma(2-\mu)}{(-ik)^{2-\mu}}$$

así tenemos

$$P(k) = \frac{4\pi r_0^{\mu}}{k}\,\frac{1}{2i}\left(\frac{\Gamma(2-\mu)}{(-ik)^{2-\mu}} - \frac{\Gamma(2-\mu)}{(ik)^{2-\mu}}\right)$$

$$= \frac{4\pi r_0^{\mu}}{k}\,\frac{\Gamma(2-\mu)}{2i}\left(\frac{1}{(-ik)^{2-\mu}} - \frac{1}{(ik)^{2-\mu}}\right)$$

$$= \frac{4\pi r_0^{\mu}}{k}\,\frac{\Gamma(2-\mu)}{2i}\,\frac{1}{k^{2-\mu}}\left(\frac{1}{(-i)^{2-\mu}} - \frac{1}{(i)^{2-\mu}}\right)$$

$$= \frac{4\pi (kr_0)^{\mu}}{k^3}\,\frac{\Gamma(2-\mu)}{2i}\left(\frac{1}{(e^{-i\pi/2})^{2-\mu}} - \frac{1}{(e^{i\pi/2})^{2-\mu}}\right)$$

$$= \frac{4\pi (kr_0)^{\mu}}{k^3}\,\frac{\Gamma(2-\mu)}{2i}\left(\frac{e^{i\pi/2(2-\mu)} - e^{-i\frac{\pi}{2}(2-\mu)}}{e^{i\pi/2(2-\mu)}\, e^{-i\pi/2(2-\mu)}}\right)$$

$$\boxed{P(k) = \frac{4\pi (kr_0)^{\mu}}{k^3}\,\Gamma(2-\mu)\,\sin\left(\frac{\pi}{2}(2-\mu)\right)}$$

donde tenemos la restricción $1<\mu<3$ y $-2<n<0$

y como $n = \mu - d$, entonces $d = 3$

$$\mu = n + d = n + 3$$

$$P(k) = \frac{4\pi}{k^3}(kr_0)^{n}(kr_0)^{3}\,\Gamma(2-\mu)\,\sin\left(\frac{\pi}{2}(2-\mu)\right)$$

$$= 4\pi\, r_0^{n}\, r_0^{3}\, k^{n}\,\Gamma(2-\mu)\,\sin\left(\frac{\pi}{2}(2-\mu)\right)$$

Para expresar el espectro de potencias de la forma $P(k) = A^2 \left(\frac{k}{k_q}\right)^{n+d}$

tomamos $A^2 = 4\pi\,\Gamma(2-n)\sin\left(\frac{\pi}{2}(2-n)\right)$

y $k_q^n = \frac{1}{r_0^{n+3}}$

así $\boxed{P(k) = A^2 \left(\frac{k}{k_q}\right)^{n}}$

o bien $\boxed{P(k) = A^2 \left(\frac{k}{k_q}\right)^{n-d}}$