Convex Analysis and Optimization

http://www.mia.uni-saarland.de/Teaching/cao15.shtml

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— Theoretical Assignment 2 —

Submission deadline:
19/11/2015
end of the lecture

Exercise 1: (5 points)
Prove the following statements:

(a) The intersection of an arbitrary collection of convex sets is convex, i.e., for an arbitrary index set I and convex sets $C_i \subset \mathbb{R}^N$ it holds that $\bigcap_{i\in I} C_i$ is convex.

(b) The closed and the open half-space are convex sets.

(c) The open unit ball $\{x \in \mathbb{R}^N |\, \|x\| < r\}$ and the closed unit ball $\{x \in \mathbb{R}^N |\, \|x\| \leq r\}$ are convex sets.

(d) The unit simplex $\{x \in \mathbb{R}^N |\, \sum_{i=1}^N x_i = 1,\ \forall i\colon x_i \geq 0\}$ is a convex set.

(e) The epigraph of the function $f\colon \mathbb{R}^N \to \mathbb{R}$ given by $x \mapsto \|x\|_2$ is a convex set.

Exercise 2: (2 points)
Prove or disprove.

(a) There is a closed convex set $C \subset \mathbb{R}^N$ such that its convex hull $\mathrm{conv}(C)$ is not closed.

(b) There is a closed set $C \subset \mathbb{R}^N$ such that its convex hull $\mathrm{conv}(C)$ is not closed.

Exercise 3: (3 points)
Compute the projection $\mathcal{P}_C(\bar{x})$ of a general point $\bar{x} \in \mathbb{R}^2_+ := \{x \in \mathbb{R}^2 |\, x_i \geq 0,\ \forall i = 1, 2\}$ onto the unit simplex

$$\{x \in \mathbb{R}^2 |\, \sum_{i=1}^2 x_i = 1,\ \forall i\colon x_i \geq 0\}\,.$$

— Programming Assignment 2 —

Submission deadline:
19/11/2015
end of the lecture

Submission of Programming Results:

- Compress your filled files (zip or tar.gz) e.g. `ex02_NAME1_NAME2_NAME3.tar.gz`
- Send the compressed file to `ochs@mia.uni-saarland.de` with subject: `[CAO15] Submission ex02`

- Download `ex02.tar.gz` from `www.mia.uni-saarland.de/Teaching/cao15.shtml` and extract the files, e.g., `tar -xzvf ex02.tar.gz` .

Let $C, D \subset \mathbb{R}^2$ be two convex sets in $\mathbb{R}^2$ with non-empty intersection. We consider the problem

$$\text{find } x^* \in C \cap D\,.$$

In general, the projection of any $x^{(0)} \in \mathbb{R}^2$ onto $C \cap D$ would solve the problem. Unfortunately, the set $C \cap D$ can have a structure onto which projecting is difficult. Often, the projection onto the individual sets is simple and can be computed efficiently. Therefore, the idea to solve the problem is to iteratively combine projections onto C and projections onto D. In the following, the goal is to implement several such methods. Let $\mathcal{P}_C(\bar{x})$ denote the projection of a point $\bar{x} \in \mathbb{R}^2$ onto the convex set C, i.e.

$$\mathcal{P}_C(\bar{x}) := \arg\min_{x \in C} \frac{1}{2}\|x - \bar{x}\|_2^2\,.$$

The same for the set D.

Algorithm 1: Alternating projection method (POCS: projection onto convex sets)

- **Iterations** ($n \geq 0$): Update ($x^{(0)} \in \mathbb{R}^2$)

$$x^{(n+1)} = \mathcal{P}_C(\mathcal{P}_D(x^{(n)}))$$

Algorithm 2: Averaged projections method

- **Iterations** ($n \geq 0$): Update ($x^{(0)} \in \mathbb{R}^2$)

$$x^{(n+1)} = \frac{1}{2}(\mathcal{P}_C(x^{(n)}) + \mathcal{P}_D(x^{(n)}))$$

Algorithm 3: Dykstra's projection algorithm

- **Iterations** ($n \geq 0$): Update ($x^{(0)} \in \mathbb{R}^2$, $p^{(0)} = 0$, $q^{(0)} = 0$)

$$\begin{aligned}
y^{(n)} &= \mathcal{P}_D(x^{(n)} + p^{(n)}) \\
p^{(n+1)} &= x^{(n)} + p^{(n)} - y^{(n)} \\
x^{(n+1)} &= \mathcal{P}_D(y^{(n)} + q^{(n)}) \\
q^{(n+1)} &= y^{(n)} + q^{(n)} - x^{(n+1)}
\end{aligned}$$

Exercise 4: (5 points)
Let the sets C and D be given by

$$C := \{x \in \mathbb{R}^2 |\, \|x - c\|_2 \leq r\}$$

with $c := (\frac{3}{2}, \frac{3}{2})^\top$ and $r = \frac{3}{2}$, and

$$D := \{x \in \mathbb{R}^2 |\, \langle x, b \rangle \leq \beta\}$$

with $b = (1, 1)^\top$ and $\beta = 1$.

(a) Implement the general projection onto a circle parametrized like C in the file `proj_circle.m` and the projection onto the half space as given by the set D in `proj_halfspace.m`.

(b) Implement

- Algorithm 1 in the file `pocs.m`,
- Algorithm 2 in the file `avrg_proj.m`,
- Algorithm 3 in the file `dykstra.m`.

(Use an appropriate stopping criterion.)
(*Hint: Use* `feval` *to evaluate the projection functions. See the documentation for details.*)

(c) Show that the averaged projection method is actually a special case of the POCS method in an appropriate product space.

Exercise 5: (5 points)
We want to prove convergence of the alternating projection method (Algorithm 1). Assume C and D are closed convex sets and there exists $\bar{x} \in C \cap D$ and define the sequence of intermediate points $y^{(n)} := \mathcal{P}_D(x^{(n)})$. Then $x^{(n+1)} = \mathcal{P}_C(y^{(n)})$.

(a) Show that $D \subset \{z \in \mathbb{R}^2 |\, \langle x^{(n)} - y^{(n)}, z - y^{(n)} \rangle \leq 0\}$.

(b) Show that for any $\bar{x} \in C \cap D$, it holds that

$$\|y^{(n)} - \bar{x}\|_2^2 \leq \|x^{(n)} - \bar{x}\|_2^2 - \|y^{(n)} - x^{(n)}\|_2^2 .$$

Similarly, we can show (*not to be proved!*)

$$\|x^{(n+1)} - \bar{x}\|_2^2 \leq \|y^{(n)} - \bar{x}\|_2^2 - \|x^{(n+1)} - y^{(n)}\|_2^2 .$$

This shows that $x^{(n+1)}$ is closer to $\bar{x}$ than $y^{(n)}$ and $y^{(n)}$ is closer to $\bar{x}$ than $x^{(n)}$.

(c) Show that $(x^{(n)})_{n\in\mathbb{N}}$ has an accumulation point $x^* \in C$.

(d) Show that x^* also lies in D, thus $x^* \in C \cap D$.