Ahmet Hamdi Ünal
Fall 2020

PHYS 414 – Final Project

Newton (a)

We have the following functions:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho \rightarrow \frac{1}{\rho}\frac{dP}{dr} = -\frac{Gm}{r^2}$$

Note that I have replaced p with P to avoid any confusion. Differentiating the second function with respect to r gives:

$$\frac{d}{dr}\left(\frac{1}{\rho}\frac{dP}{dr}\right) = \frac{2Gm}{r^3} - \frac{G}{r^2}\frac{dm}{dr} = \frac{2}{r}\frac{Gm}{r^2} - \frac{G}{r^2}\frac{dm}{dr}$$

By using the main equations, we can simplify the RHS by replacing them with their equivalents:

$$\frac{d}{dr}\left(\frac{1}{\rho}\frac{dP}{dr}\right) = -\frac{2}{\rho r}\frac{dP}{dr} - G4\pi\rho$$

Leaving the term with ρ at the RHS, we get:

$$\frac{d}{dr}\left(\frac{1}{\rho}\frac{dP}{dr}\right) + \frac{2}{r}\frac{dP}{dr} = -4\pi G\rho$$

We can right LHS as a single derivation after multiplying both sides with r^2:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G r^2 \rho$$

Now, we divide it back to r^2 to get rid of r^2 at the RHS. This gives:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

From now on, we have to use other equalities given to us to simplify further. The following are given:

$$P = K\rho^{1+1/n}$$

Let $\rho = \rho_c\theta^n$, where ρ_c is a constant.

Using chain rule, we can calculate dP/dr that is in the equation above.

$$\frac{dP}{dr} = \frac{dP}{d\rho}\frac{d\rho}{dr} = K\left(\frac{n+1}{n}\right)\rho^{\frac{1}{n}}\left(\rho_c n\theta^{n-1}\frac{d\theta}{dr}\right)$$
$$= K(n+1)\rho_c^{\frac{n+1}{n}}\theta^n\frac{d\theta}{dr}$$

Putting this and $\rho = \rho_C\theta^n$ into the equation above, we get:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\theta^n}K(n+1)\rho_c^{\frac{1}{n}}\theta^n\frac{d\theta}{dr}\right) = -4\pi\rho_c G\theta^n$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 K(n+1)\rho_c^{\frac{1}{n}}\frac{d\theta}{dr}\right) = -4\pi\rho_c G\theta^n$$

Further simplification on the RHS leads to:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{K(n+1)\rho_c^{\frac{1}{n}-1}}{4\pi G}\frac{d\theta}{dr}\right) = -\theta^n$$

The last thing is to scale the radius r. Let $r = \alpha\varepsilon$ where $\alpha^2 = \frac{K(n+1)}{4\pi G}\rho_c^{\frac{1}{n}-1}$. Note that Noting

$$\frac{d}{dr} = \frac{d}{d\varepsilon}\frac{d\varepsilon}{dr} = \frac{1}{\alpha}\frac{d}{d\varepsilon}$$

, we get:

$$\frac{1}{\alpha^3\varepsilon^2}\frac{d}{d\varepsilon}\left(\alpha^3\varepsilon^2\frac{d\theta}{d\varepsilon}\right) = -\theta^n$$

Performing the simplifications and moving the terms at the RHS to the LHS, we get the Lane-Emden equation:

$$\frac{1}{\varepsilon^2}\frac{d}{d\varepsilon}\left(\varepsilon^2\frac{d\theta}{d\varepsilon}\right) + \theta^n = 0$$

Since it requires more than a single-line command, I will provide a Mathematica Notebook that prints the series for regular solutions at the center. The main idea is to create a series of unknown coefficients, and then determining these coefficients by using the Lane-Emden Equation. Here is the result and see the notebook fpna.nb for the details:

```
In[22]:= (series /. solution)[[1]]
Out[22]= 1 - ξ^2/6 + n ξ^4/120 + O[ξ]^5
```

For $n = 1$, the following command solves the differential equation:

$$\text{FullSimplify}[\text{DSolveValue}[\{2/x * y'[x] + y''[x] + y[x] == 0, y[0] == 1, y'[0] == 0\}, y[x], x]]$$

Note that I have expanded the equation before using Mathematica. FullSimplify command is to simplify the output result, otherwise it gives it in the form of imaginary numbers i. The result is:

$$\frac{\text{Sin}[x]}{x}$$

Put it into the main equation to check.

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d}{dx}\frac{sinx}{x}\right)+\frac{sinx}{x}$$

$$=\frac{1}{x^2}\frac{d}{dx}(x\cos(x)-\sin(x))+\frac{sinx}{x}$$

$$=-\frac{sinx}{x}+\frac{sinx}{x}=0$$

Therefore, the solution is correct.

Using the following equation that is provided to us, we can find the total mass of the star.

$$\frac{dm}{dr}=4\pi r^2\rho \rightarrow M=\int_0^R 4\pi r^2\rho(r)dr$$

Since $r=\alpha\varepsilon$, then $dr=\alpha\, d\varepsilon$. The boundaries are also must be modified. $r=R=\alpha\varepsilon_{\uparrow}\rightarrow\varepsilon_{\uparrow}=R/\alpha$ and similarly, $\varepsilon_{\downarrow}=0$.

$$M=4\pi\alpha^3\int_0^{\frac{R}{\alpha}}\varepsilon^2\,\theta^n\,d\varepsilon$$

$$=-4\pi\alpha^3\int_0^{\frac{R}{\alpha}}\frac{d}{d\varepsilon}\left(\varepsilon^2\frac{d\theta}{d\varepsilon}\right)d\varepsilon$$

Using the fundamental theorem of Calculus, we get:

$$M=-4\pi\alpha R^2\left(\frac{d\theta}{d\varepsilon}\right)\Big|_0^{R/\alpha}$$

This is the same result in the project manual. Let us denote $\left(\frac{d\theta}{d\varepsilon}\right)\Big|_0^{R/\alpha}$ with $\theta'(\varepsilon_n)$, by letting $\varepsilon_n=\varepsilon_{\uparrow}=R/\alpha$. Also noting $\rho(r)=\rho_c\theta^n$, the equation for M we found becomes:

$$M=-4\pi\rho_c\alpha R^2\,\theta'(\varepsilon_n)$$

At last, writing αR^2 as $\frac{1}{\varepsilon_n}R^3$, which are equal, we get the version in the manual:

$$M=4\pi\rho_c R^3\frac{-\theta'(\varepsilon_n)}{\varepsilon_n}$$

Denoting the already defined α as:

$$\alpha^2=\frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1}$$

Then, as calculated above,

$$\varepsilon_n=\frac{R}{\alpha}\rightarrow R=\varepsilon_n\alpha=\varepsilon_n\left(\frac{K(n+1)}{4\pi G}\right)^{\frac{1}{2}}\rho_c^{\frac{1-n}{2n}}$$

Putting this into the equation above, we get:

$$M=-4\pi\rho_c\varepsilon_n^2\left(\frac{K(n+1)}{4\pi G}\right)^{\frac{3}{2}}\rho_c^{\frac{3-3n}{2n}}\theta'(\varepsilon_n)$$

$$=-4\pi\varepsilon_n^2\left(\frac{K(n+1)}{4\pi G}\right)^{\frac{3}{2}}\rho_c^{\frac{3-n}{2n}}\theta'(\varepsilon_n)$$

The final observation is the following two:

$$R\sim\rho_c^{\frac{1-n}{2n}}$$

$$M\sim\rho_c^{\frac{3-n}{2n}}$$

Therefore,

$$R^{\frac{3-n}{1-n}}\sim\rho_c^{\frac{3-n}{2n}}\sim M$$

More precisely,

$$R^{\frac{3-n}{1-n}}=\varepsilon_n^{\frac{3-n}{1-n}}\left(\frac{K(n+1)}{4\pi G}\right)^{\frac{3-n}{2-2n}}\rho_c^{\frac{3-n}{2n}}$$

$$M = -4\pi\varepsilon_n^2\left(\frac{K(n+1)}{4\pi G}\right)^{\frac{3}{2}}\theta'(\varepsilon_n)\rho_c^{\frac{3-n}{2n}}$$

So,

$$R^{\frac{3-n}{1-n}} = \left(\frac{\varepsilon_n^{\frac{1+n}{1-n}}\left(\frac{K(n+1)}{4\pi G}\right)^{\frac{n}{1-n}}}{4\pi\theta'(\varepsilon_n)}\right)M$$

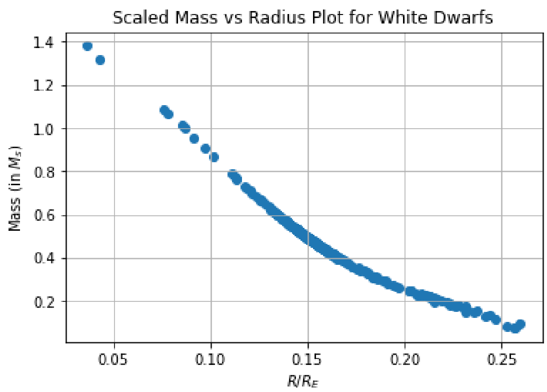
(The term before M is the constant of proportionality.)

Newton (b)

Using the following identities, one can extract R from g:

$$R = \sqrt{\frac{Gm_p}{g}}$$

, where m_p is the mass of the white dwarf. However, first, conversions must be done since our gs are in $\log_{10}$ format, m_ps are in sun masses and R must be scaled with R_E for plotting. When all these are performed, the resulting plot becomes:



Newton (c)

Series expansion of the given equation (Eq. 8) around $x = 0$ since $x \ll 1$ gives the following results:

$$\text{Series}[C*(x*(2*x^2-3)*(x^2+1)^{1/2} + 3*\text{ArcSinh}[x]),\{x,0,10\}]$$

$$\rightarrow \frac{8Cx^5}{5} - \frac{4Cx^7}{7} + \frac{Cx^9}{3} + O[x]^{11}$$

The leading term in this series expansion is $8Cx^5/5$. It is also given that:

$$x = \left(\frac{\rho}{D}\right)^{1/q} \rightarrow x^5 = \rho^{5/q}\frac{1}{D^{5/q}}$$

Substituting this for x^5 in the leading term gives:

$$\frac{8Cx^5}{5} = \frac{8C}{5D^{5/q}}\rho^{\frac{5}{q}} = K_*\rho^{1+\frac{1}{n_*}}$$

This directly gives:

$$K_* = \frac{8C}{5D^{5/q}}$$

$$\frac{5}{q} = 1 + \frac{1}{n_*} \rightarrow n_* = \frac{q}{5-q}$$

First, let us modify the following equation we got in the preceding question:

$$R^{\frac{3-n}{1-n}} = \left(\frac{\varepsilon_n^{\frac{1+n}{1-n}}\left(\frac{K(n+1)}{4\pi G}\right)^{\frac{n}{1-n}}}{4\pi\theta'(\varepsilon_n)}\right)M$$

Then, exponentiating both sides with $\frac{1-n}{n}$:

$$R^{\frac{3-n}{n}} = \frac{K}{G}\left(\frac{(n+1)\,\varepsilon_n^{\frac{1+n}{n}}}{(4\pi)^{\frac{1}{n}}(\theta'(\varepsilon_n))^{\frac{1-n}{n}}}\right)M^{\frac{1-n}{n}}$$

or equivalently:

An important observation can be that when $n = 3$, M is independent of R and when $n = 1$, R is independent of M. Therefore, it may be reasonable to expect $1 < n < 3$.

Now, putting this equation in a much simpler form (Chandrasekhar, 1958, p. 98), we get:

$$\frac{K}{G} = N_n M^{\frac{n-1}{n}} R^{\frac{3-n}{n}}$$

by letting $N_n = \frac{1}{n+1}\left(\frac{4\pi}{{}_0\omega_n}\right)^{\frac{1}{n}}$ where ${}_0\omega_n \coloneqq -\varepsilon_1^{\frac{n+1}{n-1}}\left(\frac{d\theta_n}{d\varepsilon}\right)_{\varepsilon=\varepsilon_1}$. The nice part of this is for the values of n between $(1,3)$, $N_n \in$

$[0.42, 0.35]$ (Chandrasekhar, 1958, p. 96). Therefore, equation simplifies to:

$$\frac{K}{(0.4)G} \approx M^{\frac{n-1}{n}} R^{\frac{3-n}{n}}$$

for $1 < n < 3$.

We are to make a curve fit, so let it write in the following form by letting $K' = K/(0.4)G$:

$$K' R^{\frac{n-3}{n}} \approx M^{\frac{n-1}{n}}$$

Now, there is still a problem because this equation depends on both K and n. To be able to perform curve fit, I will first find a reasonable value for n and then find K from the curve fit.

Using the condition I have imposed above:

$$1 < n < 3 \qquad \leftrightarrow \qquad 1 < \frac{q}{5-q} < 3$$

, which is equal to (as long as $q \neq 5$):

$$5 - q < q < 15 - 3q$$

Due to $5 - q < q$, $q = 3, 4, \ldots$

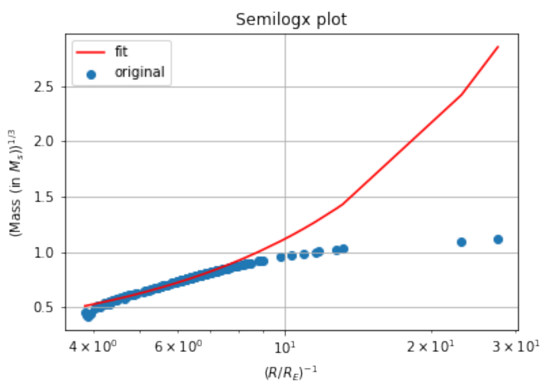
Due to $q < 15 - 3q$, $q = \cdots, 2, 3$

Their intersection allows us that only $q = 3$ is possible (note that I have used the hint in the manual that restricts q to be an integer). If $q = 3$, then $n_* = 3/2$.

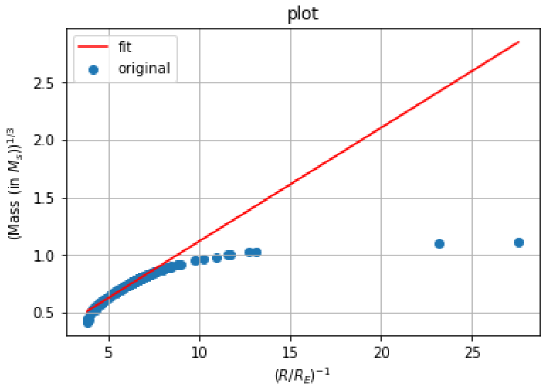
Therefore, equation above becomes:

$$K' R^{-1} \approx M^{1/3}$$

The last step is to define $M' \coloneqq M^{1/3}$ and $R = R^{-1}$ and fit a line by discarding high mass stars. Below, I present the semilogx plot for 30 heavy stars discarded:



, which is a good fit for low-mass stars. It is actually linear, as expected:



The slope is found to be 0.1, and the corresponding K value is $(2.6)10^{-12}$.

Recalling:

$$R = \varepsilon_n \alpha = \varepsilon_n \left(\frac{K(n+1)}{4\pi G} \right)^{\frac{1}{2}} \rho_c^{\frac{1-n}{2n}}$$

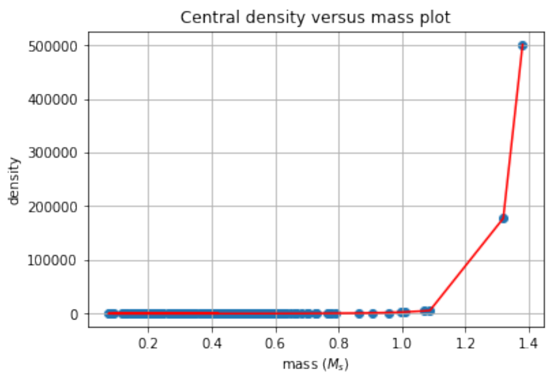
, we can calculate ρ_c as

$$\rho_c = \left(\frac{K(n+1)}{4\pi G} \right)^{\frac{n}{n-1}} \left(\frac{R}{\varepsilon_n} \right)^{\frac{2n}{1-n}}$$

Since $n_* = 3/2$,

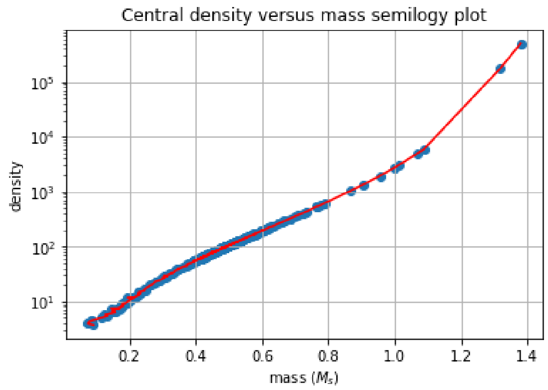
$$\rho_c = \left(\frac{5K}{8\pi G} \right)^3 \left(\frac{\varepsilon_n}{R} \right)^6$$

, where for $n = 3/2$, ε is calculated as 3.65 by Chandrasekhar (1958, p. 96). The resulting plot ρ_c vs M is as follow



As mass increases, density also increases. For visibility, semiology plot below can be more accurate to observe low-mass star central densities.

Ahmet Hamdi Ünal
Fall 2020



The shape and the overall behavior of the plot matches with the ones in the literature.

Newton (e)

Recalling my discussion in part (c) where I have talked about some specific value of n that gives strange results for the relation between R and M, I rewrite the relation between those two:

$$\frac{K}{G} = N_n M^{\frac{n-1}{n}} R^{\frac{3-n}{n}}$$

Letting $n = 3$, we get:

$$\frac{K}{G} = N_3 M^{\frac{2}{3}} \rightarrow M = \left(\frac{K}{GN_3}\right)^{3/2}$$

Again, referring to Chandrasekhar for N_3, it becomes:

$$M = \left(\frac{K}{G(0.36394)}\right)^{3/2}$$

This K is different than what we have calculated previously because it was calculated under the assumption that $x \ll 1$. Contrarily, here the assumption is $x \gg 1$. However, by starting with the assumption that $n = 3$ and $P = K\rho^{1+\frac{1}{n}}$, then solving $eq.8$ in the manual gives us the K value we want. I have no time left, but according to the literature (Ambrosino, 2020), this gives:

$$K = \frac{3^{1/3}}{\pi} \frac{hc}{8(2m_H)^{4/3}}$$

Putting this in the place of K in the preceding equation in return gives:

$$M_{ch} \approx 1.43\ (M_s)$$

, where M_s is the sun mass.

Einstein (e)

$$\nu' = \frac{2M}{r(r-2M)}, (r > R)$$

This is an integration problem, where M is real (mass), and $r > R > 0$. Imposing these conditions (*assuming*, in the syntax of Wolfram Mathematica), Mathematica can take this integral with Integrate command. Running the following command calculates the result of the integration with the imposed conditions on the variables:

$$\text{Assuming}\Big[(r > R)\&\&(R > 0)\&\&(M \in \text{Reals})\&\&(M > 0), \text{Integrate}\big[2 * M/\big(t * (t - 2 * M)\big), \{t, R, r\}\big]\Big]$$

, where I have changed the parameter r in the equation to r (since it is a dummy variable, it is not important). This command gives the result:

$$\text{Log}\left[1 - \frac{2M}{r}\right] - \text{Log}\left[1 - \frac{2M}{R}\right]$$

with the condition $2M < R$. This condition comes from the logarithm at the right to avoid negative values inside the logarithm (natural). Therefore:

$$\nu(r > R) = \text{Log}\left[1 - \frac{2M}{r}\right] - \text{Log}\left[1 - \frac{2M}{R}\right]$$

, which is a direct Mathematica integration result. At last, we have to also impose the following condition for continuity:

$$\nu(r = R) = \nu(r > R)$$

This adds an additional term to the result. Note that this is not a direct consequence of the integration, so that integrating with Mathematica does not provide us this term. Rather, it is an additional physical interpretation relying on our additional knowledge on the problem. The result becomes, then, becomes:

$$\nu(r > R) = \text{Log}\left[1 - \frac{2M}{r}\right] - \text{Log}\left[1 - \frac{2M}{R}\right] + \nu(r = R)$$

Ahmet Hamdi Ünal
Fall 2020

References

S Chandrasekhar, *An introduction to the study of stellar structure* (Univ. of Chicago Press, Chicago, 1958)

F Ambrosino, White Dwarf mass-radius relation arXiv:2012.01242v2 [astro-ph.HE] Dec 2020