

Ragax: Ragalur Expressions

Using derivatives to validate Indian Classical Music

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Key Takeaways

After this talk you should be able to:

- implement a regular expression matcher function and
- invent your own regular expression operators,

because **derivatives are that simple**.

You will also learn about:

- the most basic concept of a Raga;
- laziness, memoization and least fixed point;
- derivatives for context free grammars; and
- derivatives for trees.

Regular Expressions

Matching a string of characters

Using a regex we can validate a string.

$$a(a|b)^*$$

ab ✓

aabbba ✓

ac ✗

ba ✗

$$a \cdot (a \mid b)^*$$

Derivatives

The Brzozowski derivative is the **easiest way to evaluate a regular expression**.

Three functions:

- Nullable
- Simplification (optional)
- Derivative

What is a Derivative

The derivative of an expression is the expression that is left to match after the given character has been matched [1].

For example:

$$\begin{aligned}\partial_a a \cdot b \cdot c &= b \cdot c \\ \partial_a (a \cdot b \mid a \cdot c) &= (b \mid c) \\ \partial_b b \cdot c &= c \\ \partial_c c &= \epsilon \\ \partial_a a^* &= a^*\end{aligned}$$

Basic Operators

empty set	\emptyset
empty string	ε
character	a
concatenation	$r \cdot s$
zero or more	r^*
logical or	$r \mid s$

Basic Operators

```
data Regex = EmptySet
  | EmptyString
  | Character Char
  | Concat Regex Regex
  | ZeroOrMore Regex
  | Or Regex Regex
```

Does the expression match the empty string.

$$\nu(\emptyset) = \text{false}$$

$$\nu(\varepsilon) = \text{true}$$

$$\nu(a) = \text{false}$$

$$\nu(r \cdot s) = \nu(r) \text{ and } \nu(s)$$

$$\nu(r^*) = \text{true}$$

$$\nu(r \mid s) = \nu(r) \text{ or } \nu(s)$$

Does the expression match the empty string.

```
nullable :: Regex -> Bool
nullable EmptySet = False
nullable EmptyString = True
nullable (Character _) = False
nullable (Concat a b) = nullable a && nullable b
nullable (ZeroOrMore _) = True
nullable (Or a b) = nullable a || nullable b
```

Nullable Examples

$$\nu(a \cdot b \cdot c) = \text{X}$$

$$\nu(\varepsilon) = \checkmark$$

$$\nu(a \mid b) = \text{X}$$

$$\nu(\varepsilon \mid a) = \checkmark$$

$$\nu(a \cdot \varepsilon) = \text{X}$$

$$\nu((a \cdot b)^*) = \checkmark$$

$$\nu(c \cdot (a \cdot b)^*) = \text{X}$$

Derivative Rules

$$\partial_a \emptyset = \emptyset$$

$$\partial_a \epsilon = \emptyset$$

$$\partial_a a = \epsilon$$

$$\partial_a b = \emptyset \quad \text{for } b \neq a$$

$$\partial_a (r \cdot s) = \partial_a r \cdot s \mid j(r) \cdot \partial_a s$$

$$\partial_a (r^*) = \partial_a r \cdot r^*$$

$$\partial_a (r \mid s) = \partial_a r \mid \partial_a s$$

$$\begin{aligned} j(r) &= \epsilon && \text{if } \nu(r) \\ &= \emptyset && \text{otherwise} \end{aligned}$$

Derivative Rules

```
deriv :: Expr -> Char -> Expr
deriv EmptyString c = EmptySet
deriv EmptySet c = EmptySet
deriv (Character a) c = if a == c
    then EmptyString else EmptySet
deriv (Concat r s) c =
    let left = deriv r c
        right = deriv s c
    in if nullable r
        then Or (Concat left s) right
        else Concat left s
deriv (ZeroOrMore r) c =
    Concat (deriv r c) (ZeroOrMore r)
deriv (Or r s) c =
    Or (deriv r c) (deriv s c)
```

Simplification

$$\emptyset \cdot r \approx \emptyset$$

$$r \cdot \emptyset \approx \emptyset$$

$$\varepsilon \cdot r \approx r$$

$$r \cdot \varepsilon \approx r$$

$$r \mid r \approx r$$

$$\emptyset \mid r \approx r$$

$$(r^*)^* \approx r^*$$

$$\varepsilon^* \approx \varepsilon$$

$$\emptyset^* \approx \varepsilon$$

$$\nu(\text{foldl}(\partial, r, \text{str}))$$

$$\nu(\text{foldl}(\text{simp} \cdot \partial, r, \text{str}))$$

```
nullable (foldl (simplify . deriv) expr string)
```

```
func matches(r *expr, str string) bool {  
    for _, c := range str {  
        r = simplify(deriv(r, c))  
    }  
    return nullable(r)  
}
```


Example: Matching a sequence of notes

Using a regex we can validate the C Major Pentatonic Scale.

$$c \cdot (c|d|e|g|a)^*$$

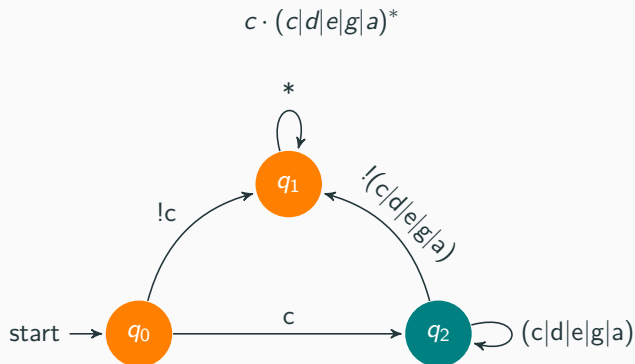
ceg ✓

$$\begin{aligned}\partial_c c \cdot (c|d|e|g|a)^* &= \varepsilon \cdot (c|d|e|g|a)^* \\ \partial_e \varepsilon \cdot (c|d|e|g|a)^* &= (\emptyset \cdot (c|d|e|g|a)^*) \mid (\emptyset|\emptyset|\varepsilon|\emptyset|\emptyset) \cdot (c|d|e|g|a)^* \\ &= (c|d|e|g|a)^* \\ \partial_g (c|d|e|g|a)^* &= (\emptyset|\emptyset|\emptyset|\varepsilon|\emptyset) \cdot (c|d|e|g|a)^* \\ &= (c|d|e|g|a)^*\end{aligned}$$

$$\nu((c|d|e|g|a)^*) = \checkmark$$

Questions?

Deterministic Finite Automata

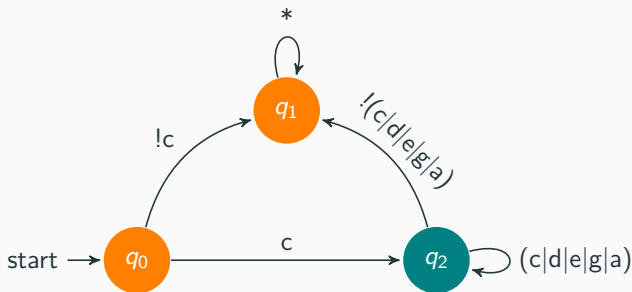


Memoization and Simplification

$$q_0 = c \cdot (c|d|e|g|a)^*$$

$$q_1 = \emptyset$$

$$q_2 = (c|d|e|g|a)^*$$



- Memoizing deriv = transition function
- Memoizing nullable = accept function
- Simplification = minimization [4]

Ragas - Indian Classical Music
Thank you: Ryan Lemmer and
Antoine Van Gelder

Video

<https://www.youtube.com/watch?v=iElMWziZ62A>

- Ragas are indian version of western scales [5].
- Stricter than western scales.
- Possible next note depends on current note.
- Notes labeled differently and relative to root note.

Raga	S	r	R	g	G	m	M	P	d	D	n	N
Western	c	c♯	d	d♯	e	f	f♯	g	g♯	a	a♯	b

I will skip over microtones and all the other theory, just because its not relevant to this talk.

Example Raga

- Raag Bhupali (a type of Pentatonic scale)
- Ascent: S R G P D S'
- Descent: S' D P G R S

- Western Pentatonic scale
- Ascent: c d e g a c¹
- Descent: c¹ a g e d c

Given the current note you must choose the next ascending or descending note.

For example given G (e) you can choose P (g) if you want to ascend or R (d) if you want to descend.

Play Raag Bhupali

Questions?

An expression for a Raga

Recursive Regular Expressions

- One extra concept: Reference
- Two operations
- Define a reference: $\#myref = (a \cdot b)^*$
- Use a reference: $(@myref \mid c)$

$$\begin{aligned}\partial_a @q &= \partial_a \#q \\ \nu(@q) &= \nu(\#q)\end{aligned}$$

A Grammar for a Raga

- Raag Bhupali (a type of Pentatonic scale)
- Ascent: S R G P D S'
- Descent: S' D P G R S

$$\#S = (S \cdot (@R \mid @D))^*$$

$$\#R = R \cdot (@G \mid \varepsilon)$$

$$\#G = G \cdot (@P \mid @R)$$

$$\#P = P \cdot (@D \mid @G)$$

$$\#D = D \cdot (\varepsilon \mid @P)$$

Demo

Context Free Grammars

$$\begin{aligned}\#S &= (S \cdot (@R \mid @D))^* = @S \cdot (S \cdot (@R \mid @D)) \mid \varepsilon \\ \#R &= R \cdot (@G \mid \varepsilon) \\ \#G &= G \cdot (@P \mid @R) \\ \#P &= P \cdot (@D \mid @G) \\ \#D &= D \cdot (\varepsilon \mid @P)\end{aligned}$$

nullable and derivative each have infinite recursion.

$$\nu(\#S) = (\nu(@S) \text{ and } \nu(S \cdot (@R \mid @D))) \text{ or } \nu(\varepsilon)$$

This has been solved using [3] functional concepts:

- laziness - The Brake
- memoization - The Handbrake
- least fixed point - The Gas

Instead of evaluating a function and returning a value,

```
func eval(params ...) value {  
    ...  
    return value(...)  
}
```

we defer the evaluation and return a function that will return the value.

```
func lazyEval(params ...) func() value {  
    return func() value {  
        return eval(params)  
    }  
}
```

A function's results are cached for the given inputs.

When a function call results in calling the same function with the same inputs, we return a chosen fixed point.

Instead of storing the field values of **concat**, **or** and **zero or more** we rather store functions that when called will return the value of the field.

This allows us to avoid any recursion until the value is needed.

$$\partial_a(r|s) = \lambda(\partial_a r) \mid \lambda(\partial_a s)$$

$$\partial_a(r^*) = \lambda(\partial_a r) \cdot r^*$$

$$\partial_a(r \cdot s) = \lambda(\lambda(\partial_a r) \cdot s) \mid \lambda(\lambda(j(r)) \cdot \lambda(\partial_a s))$$

$$\partial_n \# S = \lambda(\partial_n (@S \cdot (S \cdot (@R \mid @D)))) \mid \lambda(\partial_n \varepsilon)$$

Memoization - The Handbrake

Eventually nullable is going to be called.

$$\begin{aligned}\nu(\partial_n \# S) &= \nu(\lambda(\partial_n(@S \cdot (S \cdot (@R \mid @D)))) \mid \lambda(\partial_n \varepsilon)) \\ &= \nu(\lambda(\partial_n(@S \cdot (S \cdot (@R \mid @D)))) \mid \nu(\lambda(\partial_n \varepsilon)))\end{aligned}$$

Which will result in the execution of a lazy derivative function.

$$\begin{aligned}\lambda(\partial_n(@S \cdot (S \cdot (@R \mid @D)))) &= \partial_n(@S \cdot (S \cdot (@R \mid @D))) \\ &= \lambda(\lambda(\partial_n @S) \cdot \lambda((S \cdot (@R \mid @D)))) \mid \\ &\quad \lambda(\lambda(j(@S)) \cdot \lambda(\partial_n(S \cdot (@R \mid @D)))) \\ \lambda(\partial_n @S) &= \partial_n @S\end{aligned}$$

Which can result in infinite recursion.

$$\partial_n @S = \partial_n \# S$$

Memoizing helps by closing the loop.

$$\partial_n @S = \lambda(\partial_n(@S \cdot (S \cdot (@R \mid @D)))) \mid \lambda(\partial_n \varepsilon)$$

Least Fixed Point - The Gas

BUT Nullable is relentless and still needs to return a true or a false.

Laziness and memoization are not enough.

A least fixed point returns a bottom when an input is recursively revisited.

For the nullable function our bottom is false.

$$\begin{aligned}\nu(\lambda(\partial_n \# S)) &= \dots \\ \nu(\lambda(\partial_n (@S \cdot (S \cdot (@R \mid @D)))) \mid \nu(\lambda(\partial_n \varepsilon))) &= \dots \\ \nu(\lambda(\lambda(\partial_n @S) \cdot \lambda(\dots))) &= \dots \\ \nu(\lambda(\partial_n @S)) &= \text{bottom} \\ \nu(\lambda(\lambda(\partial_n @S) \cdot \lambda(\dots))) &= \text{bottom} \ \& \ \text{false} \\ &= \text{false} \\ \nu(\partial_n (@S \cdot (S \cdot (@R \mid @D)))) \mid \nu(\lambda(\partial_n \varepsilon)) &= \text{false} \mid \text{false} \\ &= \text{false} \\ \nu(\lambda(\partial_n \# S)) &= \text{false}\end{aligned}$$

Least Fixed Point (without Laziness)

$$\begin{aligned}\nu(\#S) &= \nu(@S \cdot (S \cdot (@R \mid @D)) \mid \varepsilon) \\ &= (\nu(@S) \text{ and } \nu(S \cdot (@R \mid @D))) \text{ or } \nu(\varepsilon) \\ &= (\text{bottom} \text{ and } \text{false}) \text{ or } \text{true} \\ &= (\text{false} \text{ and } \text{false}) \text{ or } \text{true} \\ &= \text{true} \\ &= \nu((S \cdot (@R \mid @D))^*)\end{aligned}$$

Demo

Now we have a fully general Context Free Grammar validator.

Yacc, Antlr, Flex, Bison, etc. definitely still perform better, especially in worst case.

But derivatives:

- are a lot easier to implement and understand than LR(1), LL(1), LALR parsers;
- have a lot in common with the functional Parser pattern;
- can validate the full set of Context Free Grammars, not just a subset.

Trees

<http://relaxng.org/> [2] - RELAX NG is a schema language for XML, like XSchema and DTD.

Implementation and specification are done using derivatives.

XMLNodes instead of Characters.

New Operators: Not, Interleave and Optional

$$\partial_a(r \&\& s) = (\partial_a r \&\& s) \mid (\partial_a s \&\& r)$$

$$\nu(r \&\& s) = \nu(r) \text{ and } \nu(s)$$

$$\emptyset \&\& r \approx \emptyset$$

$$\varepsilon \&\& r \approx r$$

$$\partial_a!(r) = !(\partial_a r)$$

$$\nu(! (r)) = \text{not}(\nu(r))$$

$$(r)? \approx r \mid \varepsilon$$

```
deriv :: Expr -> Tree -> Expr
deriv (TreeNode nameExpr childExpr)
  (Node name children) =
  if nameExpr == name then
    let childDeriv = foldl deriv childExpr children
    in if nullable childDeriv
       then Empty
       else EmptySet
  else EmptySet

nullable (TreeNode _ _) = False
```

Video

<https://www.youtube.com/watch?v=SvjSP2xYZm>

katydid.github.io - Relapse: Tree Validation Language.

Uses derivatives and memoization to build Visual Pushdown Automata.

JSON, Protobufs, Reflected Go Structures and XML

New Operators: And, Contains and ZAny

$$\begin{aligned}\partial_a(r \& s) &= (\partial_a r \& \partial_a s) \\ \nu(r \& s) &= \nu(r) \text{ and } \nu(s) \\ \emptyset \& r &\approx \emptyset \\ r \& r &\approx r \\ * &\approx !(\emptyset) \\ .r &\approx * \cdot r \cdot *\end{aligned}$$

Playground and Tour



J. A. Brzozowski.

Derivatives of regular expressions.

Journal of the ACM (JACM), 11(4):481–494, 1964.



M. Makoto and J. Clark.

RELAX NG home page.

<http://relaxng.org>.



M. Might, D. Darais, and D. Spiewak.

Parsing with derivatives: a functional pearl.

In *Acm sigplan notices*. ACM, 2011.



S. Owens, J. Reppy, and A. Turon.

Regular-expression derivatives re-examined.

Journal of Functional Programming, 19(02):173–190, 2009.



Sādhana.

Hindustani Classical Music.

<http://raag-hindustani.com/>.