Pricing green financial products

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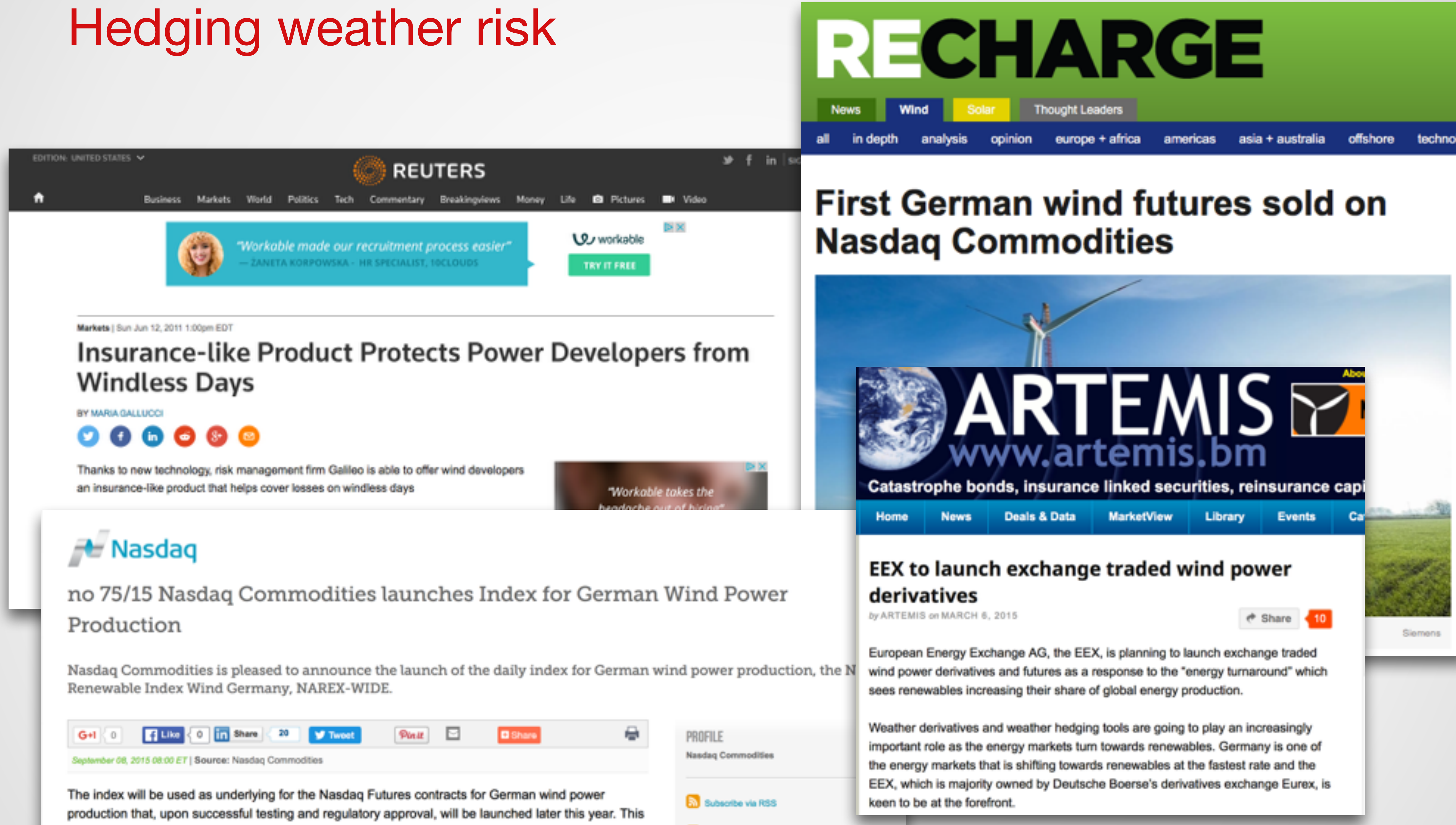
Motzen Conference 2016, July 7-9

Hedging weather risk

Hedging weather risk



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First German wind futures sold on Nasdaq Commodities

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Business Markets World Politics Tech Commentary Breakingviews Money Life Pictures Video

Markets | Sun Jun 12, 2011 1:00pm EDT

Insurance-like Product Protects Power Developers from Windless Days

BY MARIA GALLUCCI

Thanks to new technology, risk management firm Galileo is able to offer wind developers an insurance-like product that helps cover losses on windless days

Nasdaq

no 75/15 Nasdaq Commodities launches Index for German Wind Power Production

Nasdaq Commodities is pleased to announce the launch of the daily index for German wind power production, the N Renewable Index Wind Germany, NAREX-WIDE.

September 08, 2015 08:00 ET | Source: Nasdaq Commodities

The index will be used as underlying for the Nasdaq Futures contracts for German wind power production that, upon successful testing and regulatory approval, will be launched later this year. This

PROFILE

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EEX to launch exchange traded wind power derivatives

by ARTEMIS on MARCH 6, 2015

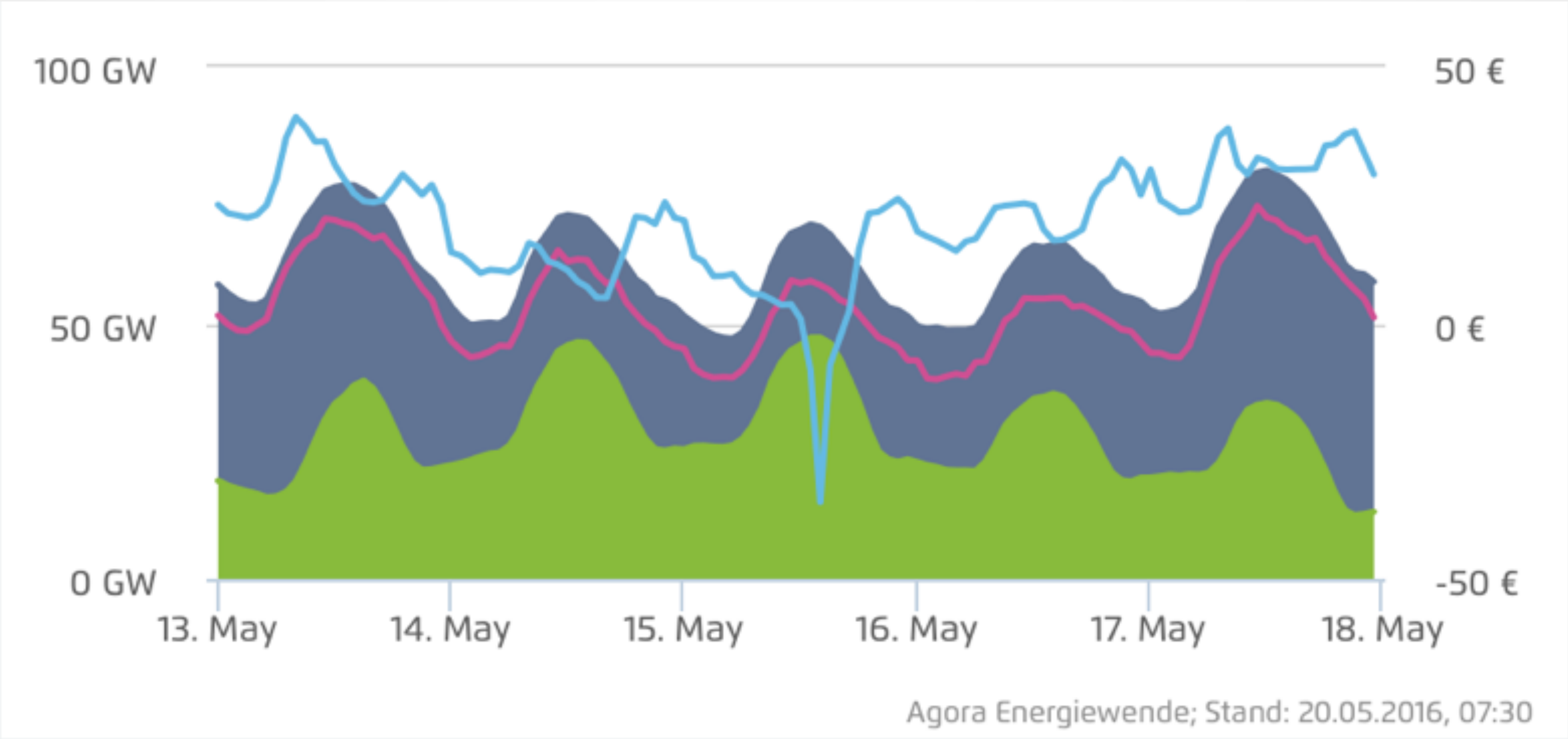
European Energy Exchange AG, the EEX, is planning to launch exchange traded wind power derivatives and futures as a response to the "energy turnaround" which sees renewables increasing their share of global energy production.

Weather derivatives and weather hedging tools are going to play an increasingly important role as the energy markets turn towards renewables. Germany is one of the energy markets that is shifting towards renewables at the fastest rate and the EEX, which is majority owned by Deutsche Boerse's derivatives exchange Eurex, is keen to be at the forefront.

Weather risk in energy production

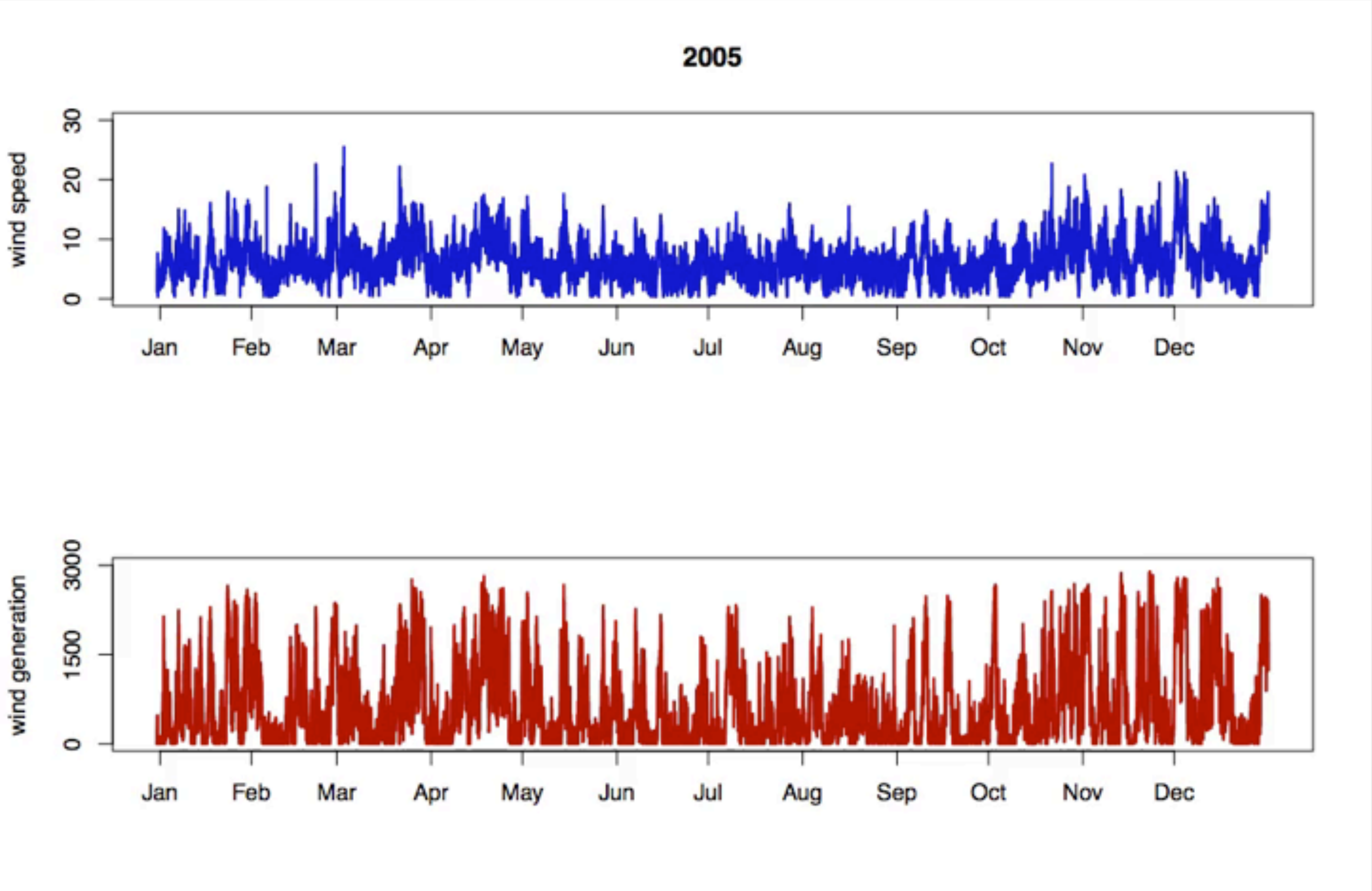
- Renewables become dominating energy source
- Energy output highly dependent on weather conditions
  - “Lack of wind” phenomenon
  - Sudden wind speed changes: strong power drop
  - Unsteady wind direction: zero wind power
  - Cloud coverage over one solar panel affects the panel output of an entire farm excessively

Pentecost 2016 in Germany: renewables cover over 80% of electricity demand - prices drop below zero



Electricity price; electricity demand; conventional power generation; renewable power generation

Having wind ≠ producing energy



Green financial products

Hedge weather related risk exposures on energy production due to climate volatility

- Payments based on weather related measurements
- Applicable to any renewable energy project
  - biomass, wave, tidal, hydropower, solar energy
  - hedging against “lack of wind”
  - prices below zero (post-feed-in)
- Underlying: indices on wind speed, sun duration, cloud coverage, etc.

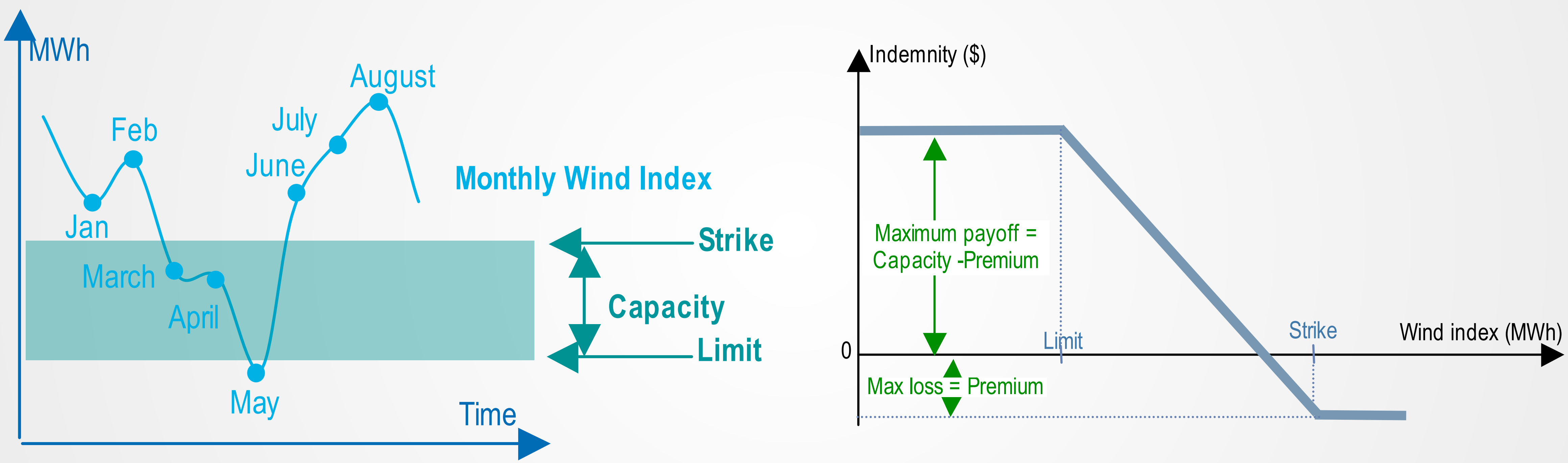
Wind futures at European Energy eXchange (EEX)

- **Contract:** A contract settling against the expected on-site power production of future delivery periods
- **Underlying:** Average wind load factor per contract period; alternatives:
    - wind speed, wind direction, wind duration indices

Example: Cumulative Average Wind Speed index:

$$CAWS(\tau_1; \tau_2) = \int_{\tau_1}^{\tau_2} W(s)ds$$

Wind futures mechanism



Left: Wind futures mechanism; Right: payoff profile

Weather Derivatives at Chicago Mercantile Exchange

CME products

- $\mathrm{HDD}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \max(18^\circ\mathrm{C} - T_t, 0) dt$
- $\mathrm{CDD}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \max(T_t - 18^\circ\mathrm{C}, 0) dt$
- $\mathrm{CAT}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} T_t dt$, where $T_t = \frac{T_{t,max} + T_{t,min}}{2}$
- $\mathrm{AAT}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \widetilde{T}_t dt$, where $\widetilde{T}_t = \frac{1}{24} \int_1^{24} T_{t_i} dt_i$ and T_{t_i} denotes the temperature of hour t_i, (also referred to as C24AT index).

Research questions

- How to model weather dynamics?
- How to estimate market price of risk?
- How to create Gaussian stochastic drivers?

The FEB Four Algorithm

Econometrics

$$
\begin{array}{c}
W_t \\
\downarrow \\
X_t = W_t - \Lambda_t \\
\downarrow \\
X_t = \beta^\top X_{t-L} + \varepsilon_t, \quad \varepsilon_t = \sigma_t e_t \\
\downarrow \\
e_t = \frac{\widehat{X}_t}{\widehat{\sigma}_t} \sim \mathrm{N}(0,1)
\end{array}
$$

Fin. Mathematics

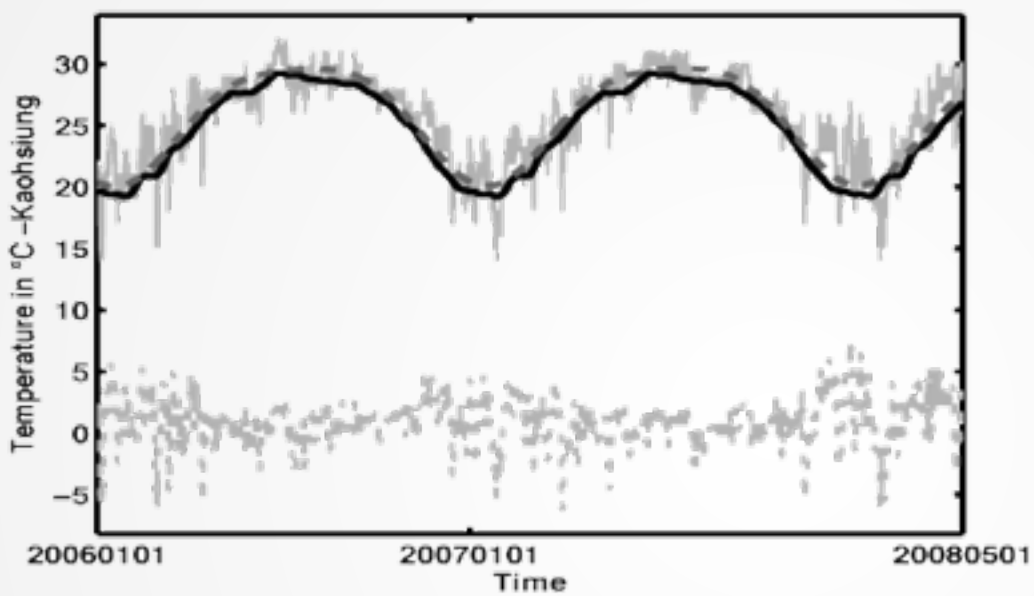
$$
\begin{array}{c}
CAR(3) \\
\downarrow \\
F_{CAWS(t,\tau_1,\tau_2)} \\
\downarrow \\
MPR
\end{array}
$$

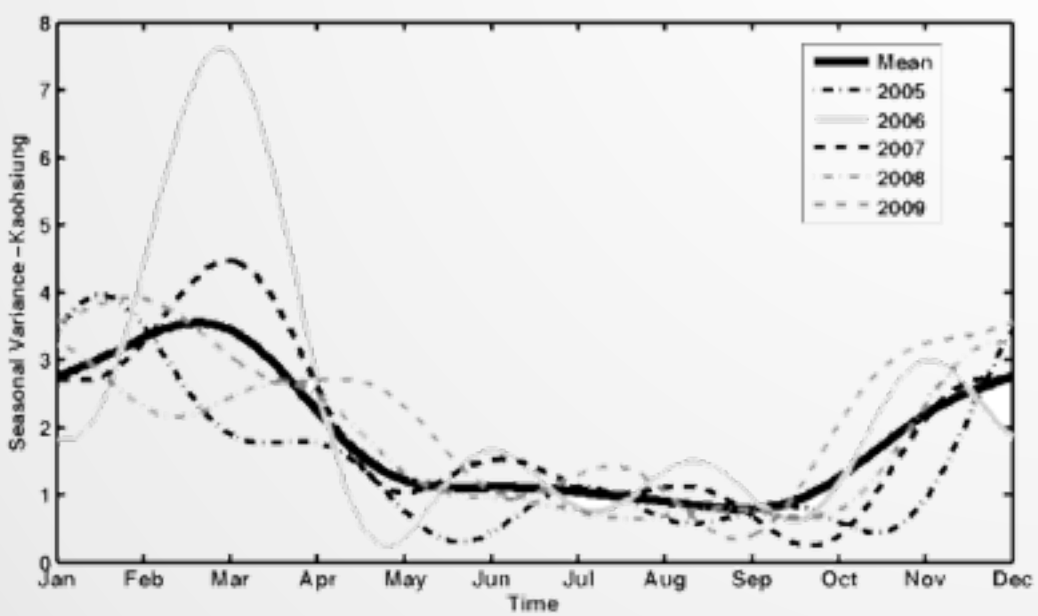
Benth et al. (2007), Härdle et al. (2011)

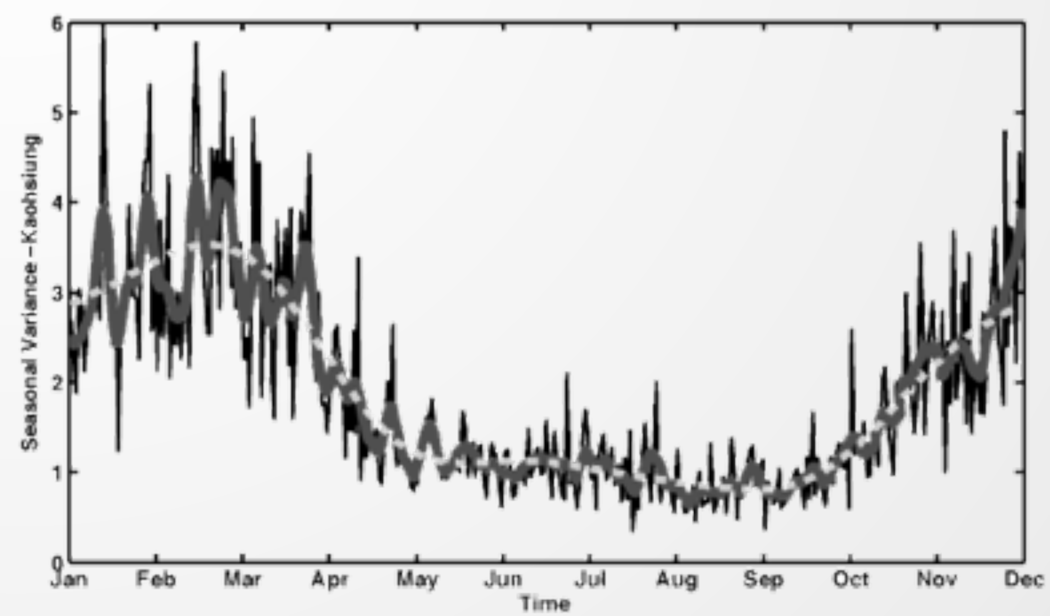
Outline

1. Motivation
2. Econometric methods and normality
3. Stochastic pricing of WD
4. Summary

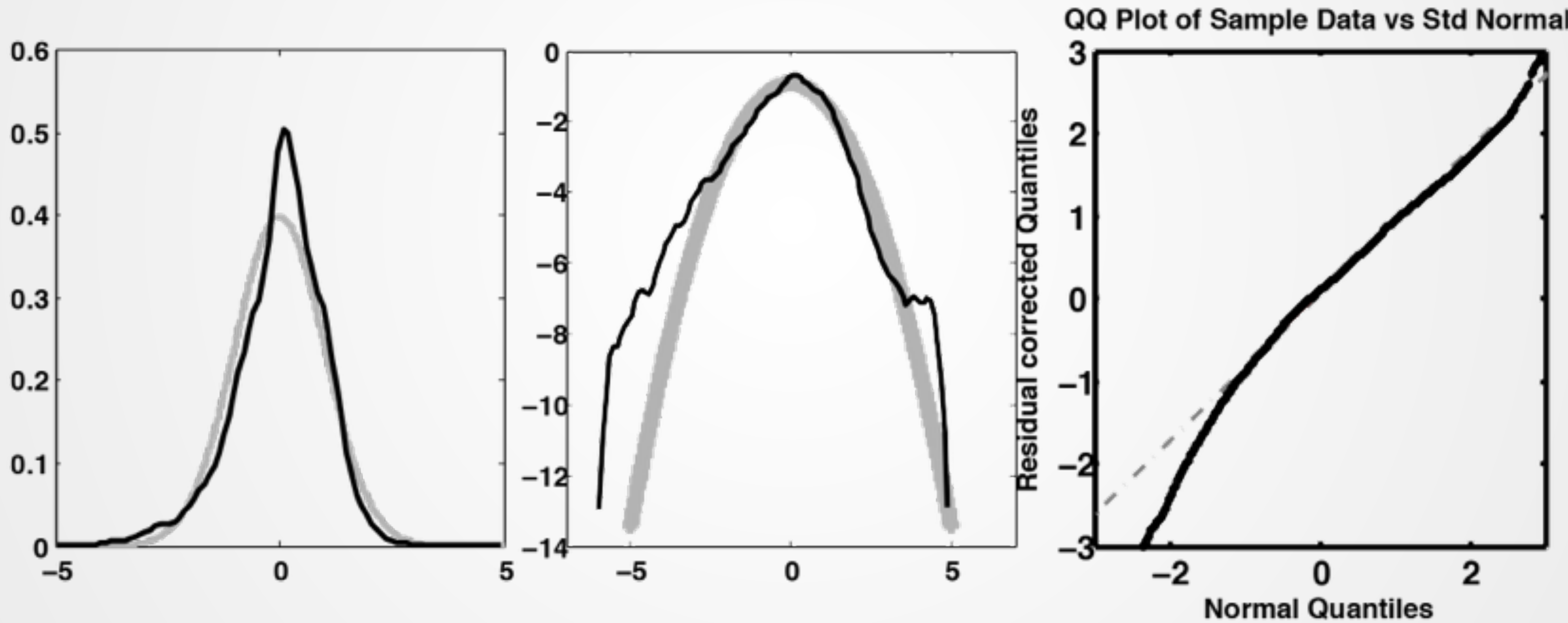
Stylised facts in temperature



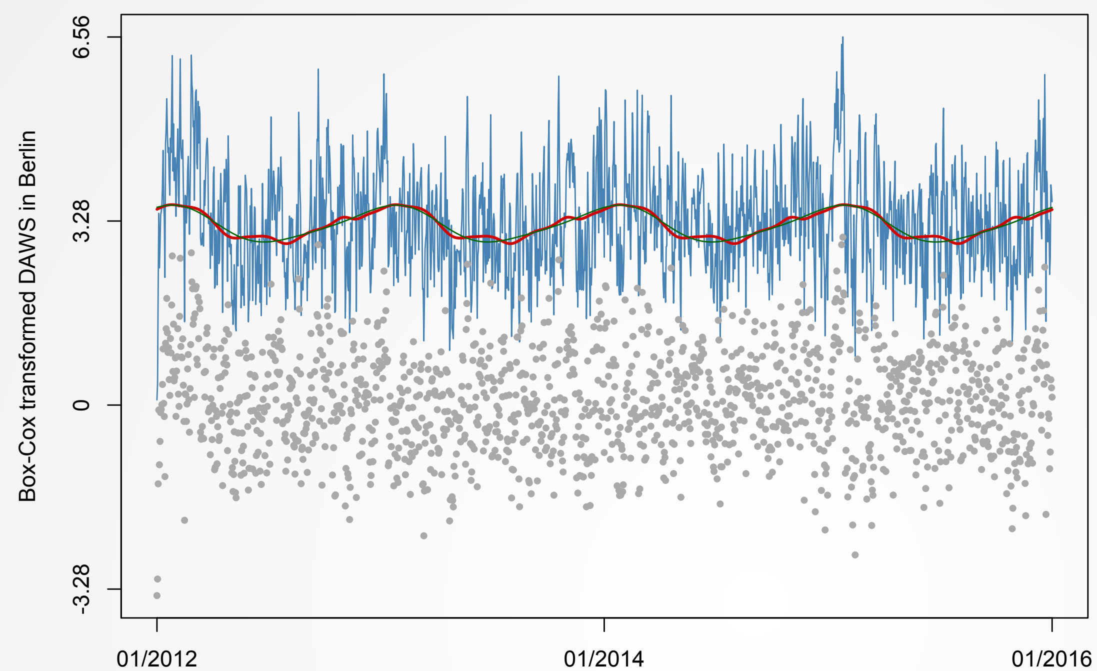




Stylised facts in temperature



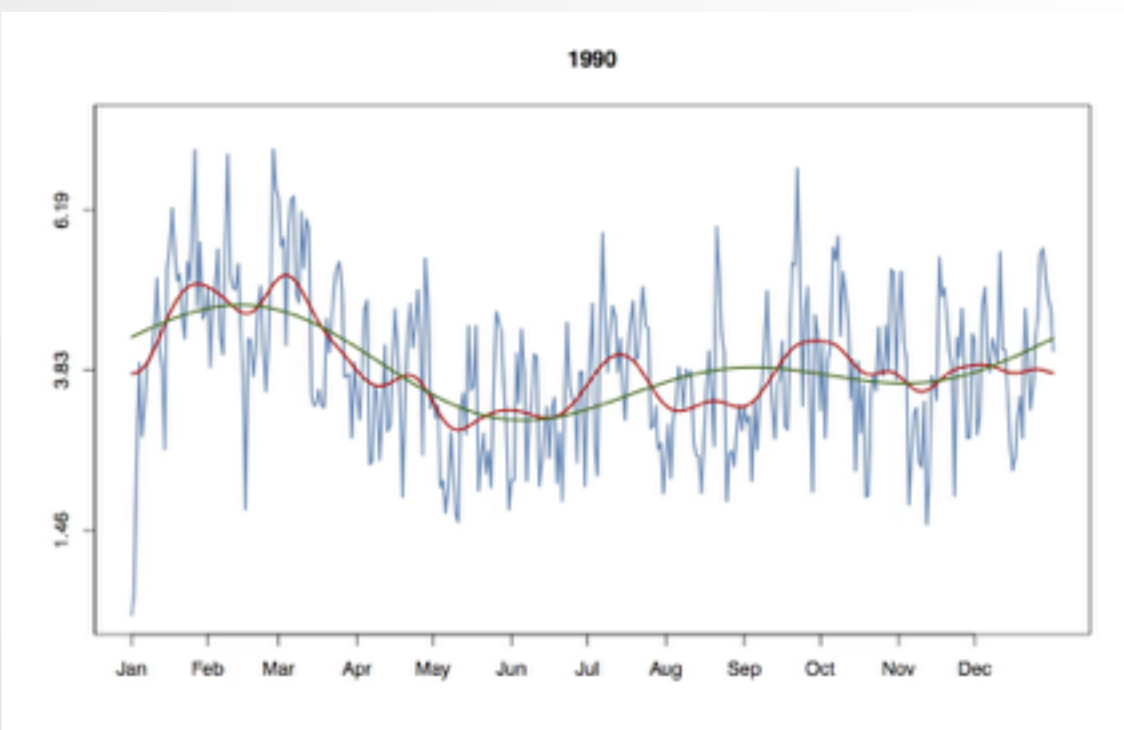
Stylised facts in wind

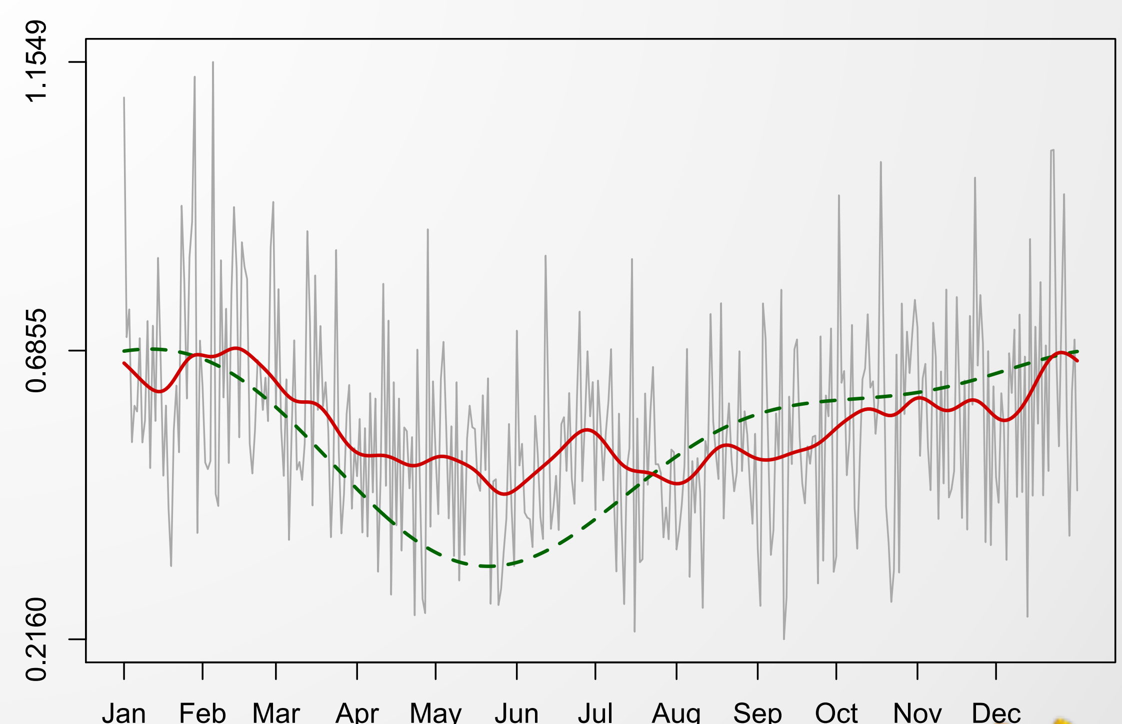


Box-Cox transformation

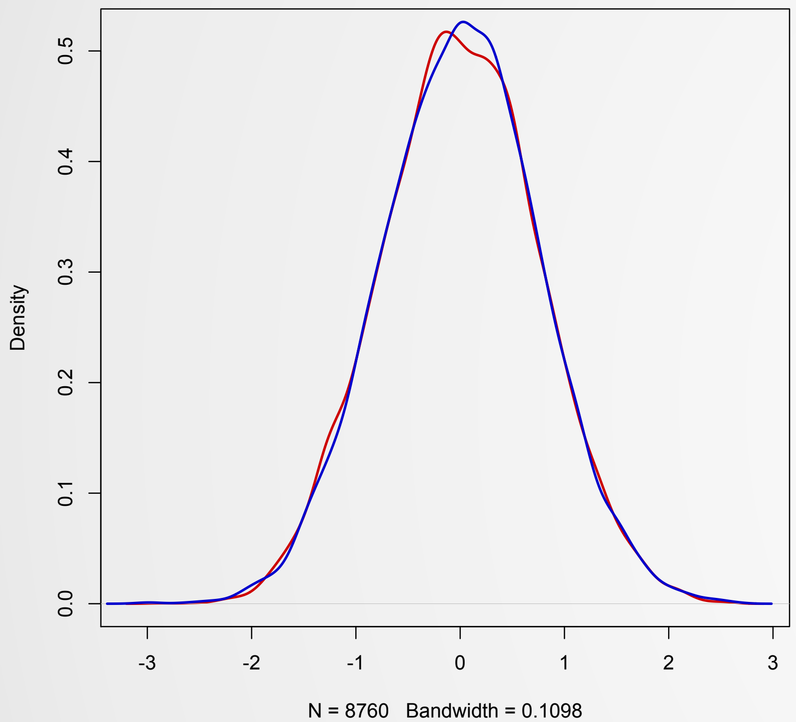
$$\widetilde{W}_t = \frac{W_t^{\lambda} - 1}{\lambda}$$

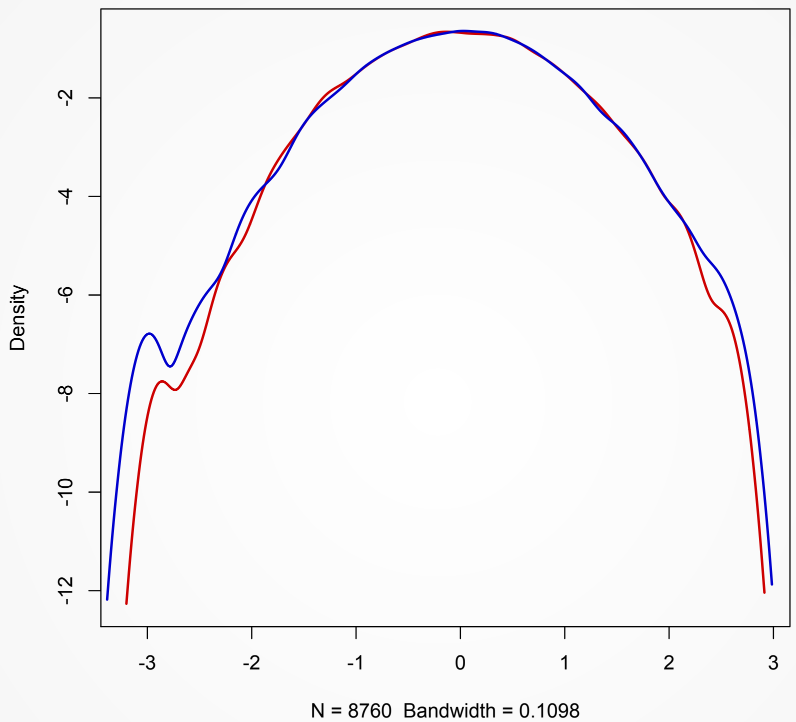
$$\widehat{\lambda}_{norm} = 0.375$$

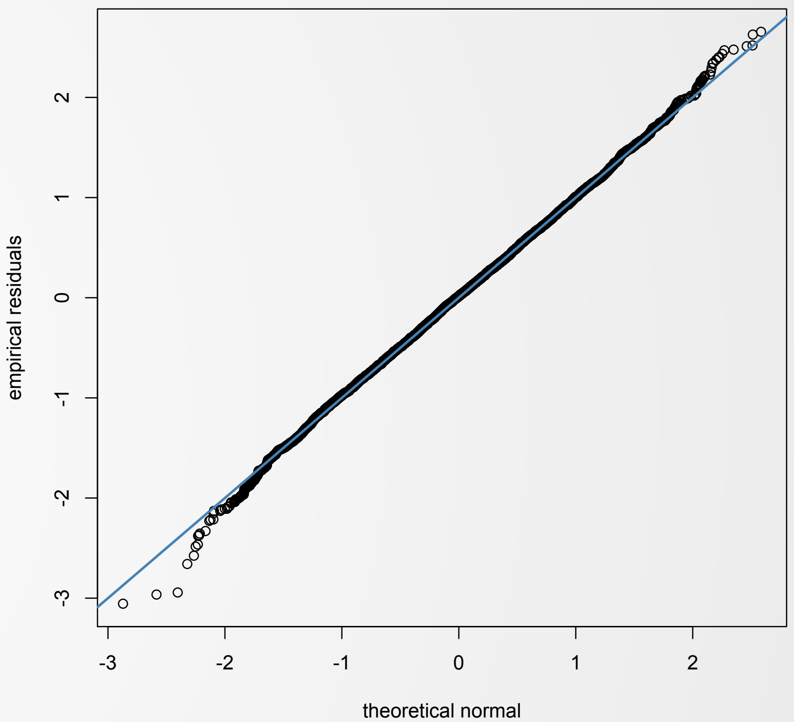




Stylised facts in wind







PGFPwindforpricing

Gaussian stochastic drivers?

Common methods for estimation of $\widehat{\sigma}_t$

- Truncated Fourier Series
- Local Linear Smoothing

do not reach normality

Example: Berlin

	TFS	LL
Jarque-Bera-test	10.591	10.413
p-value	0.005	0.005
AD-test	0.761	0.585
p-value	0.048	0.052

Use: Inter Expectile Factor Range

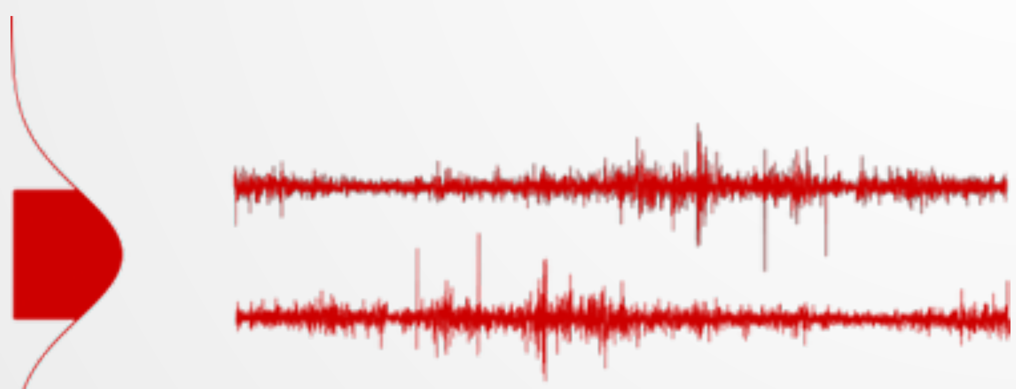
Have a closer look at the tail dynamics

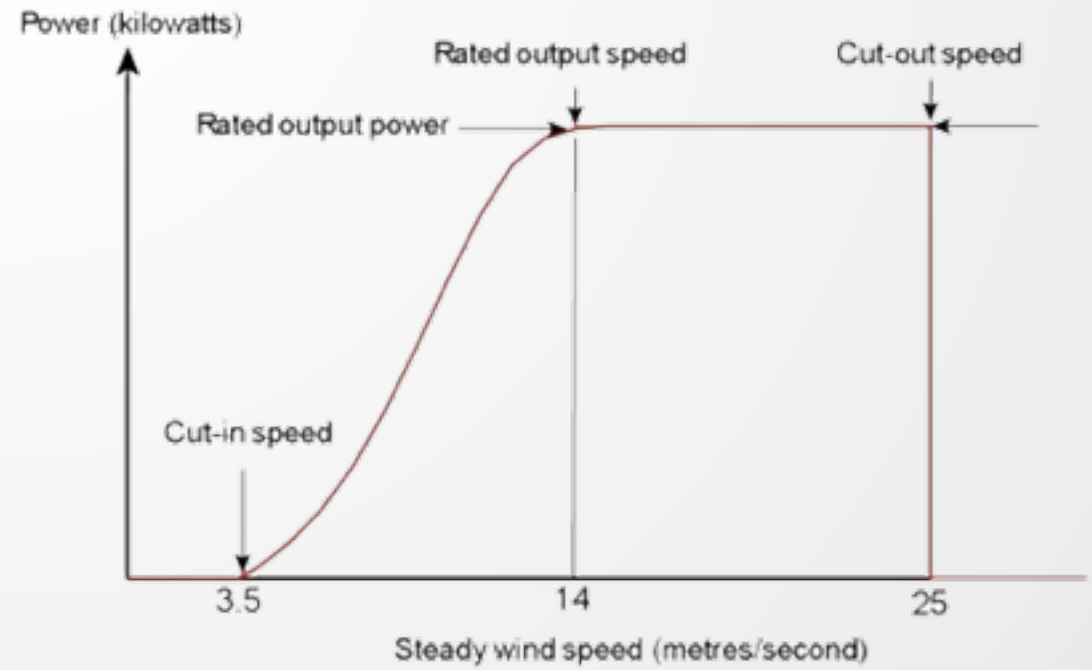
Typically: $\mu_t + \sigma_t$

Non-Gaussian world: tail event variability related to cut-in and cut-out speed

Employ FASTEC methodology

(FActorisable Sparse Tail Event Curves)





Wind index data

- Wind speed data scattered over a region
- Wind speed at turbine height (extrapolation methods)

- MERRA hourly reanalysis data for Germany
  - Latitude: ϕ : 61 levels at $0.125^\circ \approx$ 10km resolution
  - Longitude: λ : 75 levels at $0.125^\circ \approx$ 10km resolution
  - Time: 8760 hours p.a.
  - Period: 1990-2014: 25 years

Measures of tail events

- Loss function, Breckling and Chambers (1988)

  $\rho_{\tau,\gamma}(u) = |\tau - \boldsymbol{I}\{u < 0\}|\,|u|^{\gamma}, \quad \gamma \geq 1$

  $z_\tau = \arg\min_{\theta} \mathrm{E}\,\rho_{\tau,\gamma}(Y - \theta)$

  - Quantile - ALD location estimate

    $q_\tau = \arg\min_{\theta} \mathrm{E}\,\rho_{\tau,1}(Y - \theta)$

  - Expectile - AND location estimate

    $e_\tau = \arg\min_{\theta} \mathrm{E}\,\rho_{\tau,2}(Y - \theta)$

LQRcheck

FASTEC construction

- Data: $\{(\mathbf{X}_i, \mathbf{Y}_i)\}_{i=1}^{n}$ in $\mathbb{R}^{p+m}$ i.i.d.
- Linear model for τ-expectile curve of Y_j, $j = 1, ..., m, 0 < \tau < 1$:

$$e_j(\tau|\mathbf{X}_i) = \mathbf{X}_i^\top \Gamma_{*j}(\tau),$$

where coefficients for j response: $\Gamma_{*j}(\tau) \in \mathbb{R}^p$
- Sparse factorisation: $f_k^\tau(\mathbf{X_i}) = \varphi_{\mathbf{k}}^\top(\tau)\mathbf{X_i}$ factors

$$e_j(\tau|\mathbf{X}_i) = \sum_{k=1}^{r} \psi_{j,k}(\tau) f_k^\tau(\mathbf{X}_i),$$

where r : number of factors;
$\Gamma_{*j}(\tau) = (\sum_{k=1}^r \psi_{j,k}(\tau)\varphi_{k,1}(\tau), ..., \sum_{k=1}^r \psi_{j,k}(\tau)\varphi_{k,p}(\tau))$

FASTEC_with_Expectiles

Multivariate Expectile Regression with penalised loss

$$\widehat{\Gamma}_\lambda(\tau) = \arg\min_{\Gamma \in \mathbb{R}^{p\times m}} \left\{ (mn)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} \rho_\tau \left(Y_{ij} - \mathbf{X}_i^\top \Gamma_{*j} \right) + \lambda \|\Gamma\|_* \right\},$$

$\|\Gamma\|_* = \sum_{j=1}^{\min(p,m)} \sigma_j(\Gamma)$ nuclear norm of Γ

$\mathbf{X}_i$: B-splines

$\mathbf{Y}_i$: wind power density; $(n \times m)$-matrix

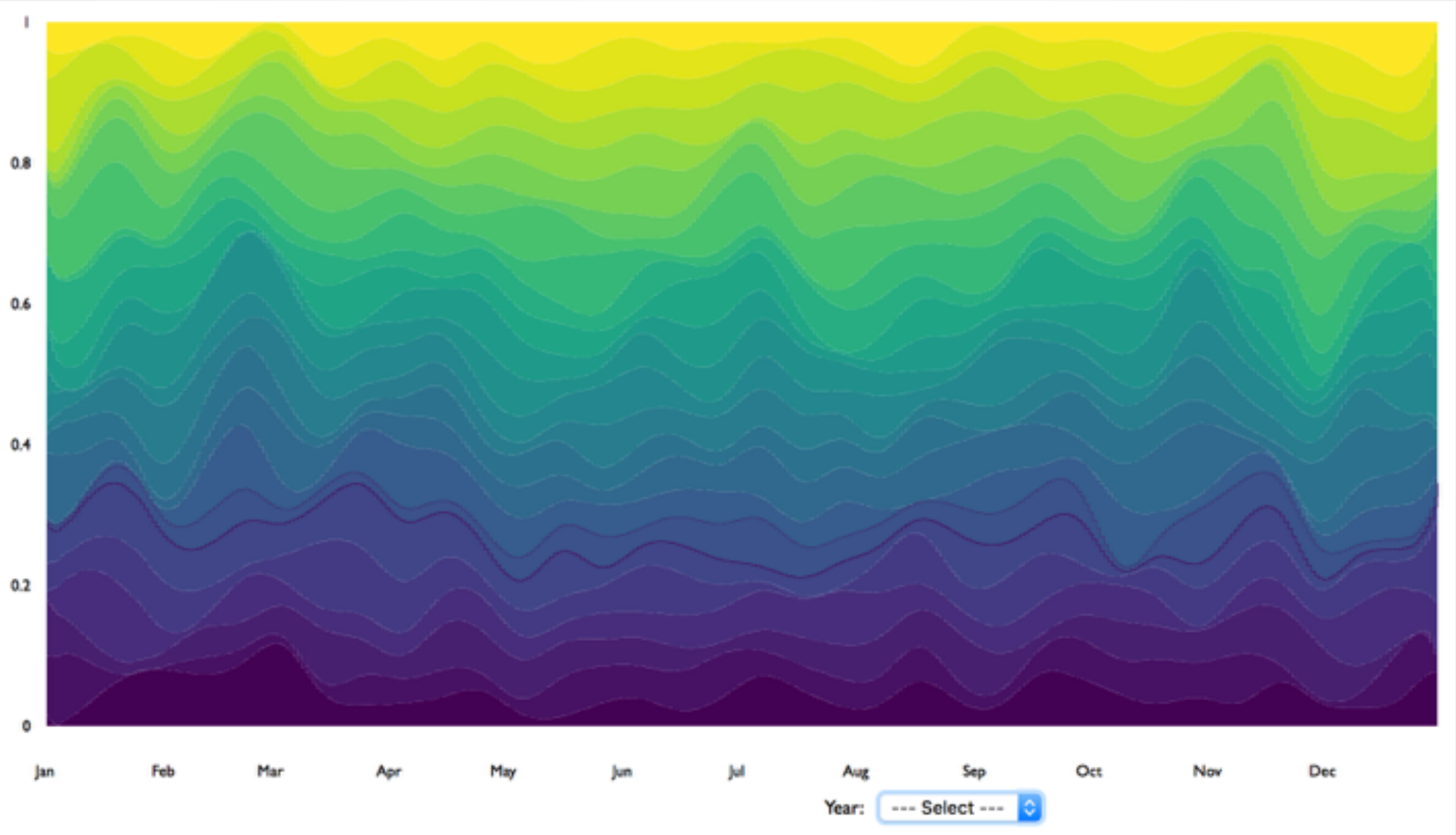
Γ : factor matrix

λ : penalisation parameter ▸ Optimal λ

▸ FISTA algorithm

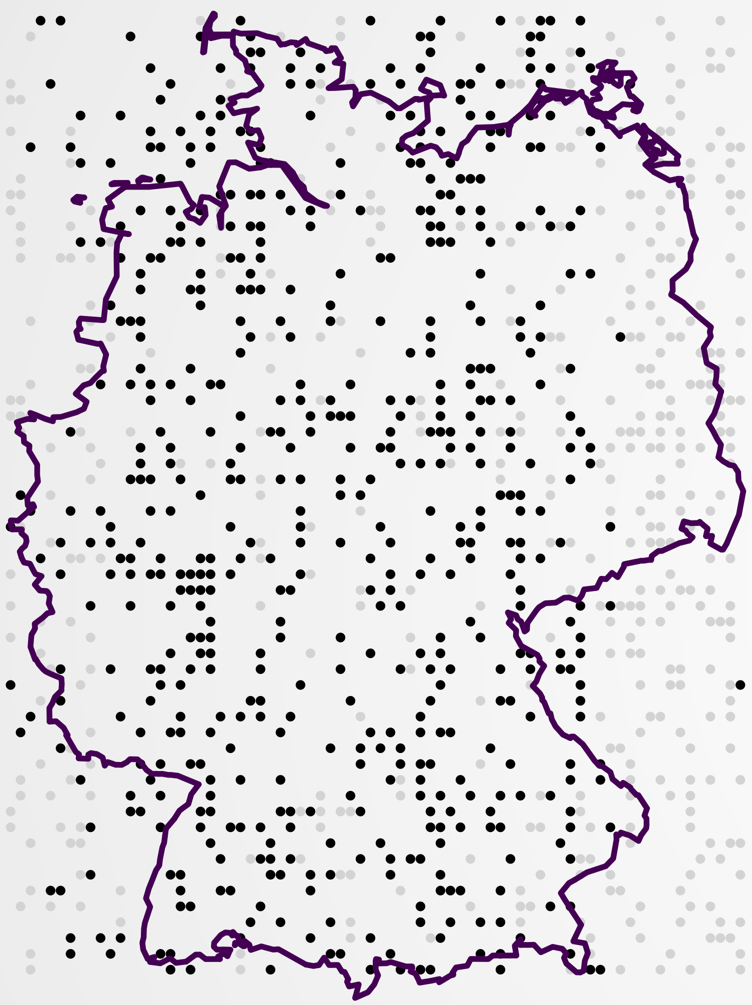
Inter Expectile Factor Range

$$\text{IEFR} = |\varphi_{\mathbf{1}}^{\top}(\tau = \mathbf{0.99})\mathrm{X}_{\mathbf{i}} - \varphi_{\mathbf{1}}^{\top}(\tau = \mathbf{0.25})\mathrm{X}_{\mathbf{i}}|$$

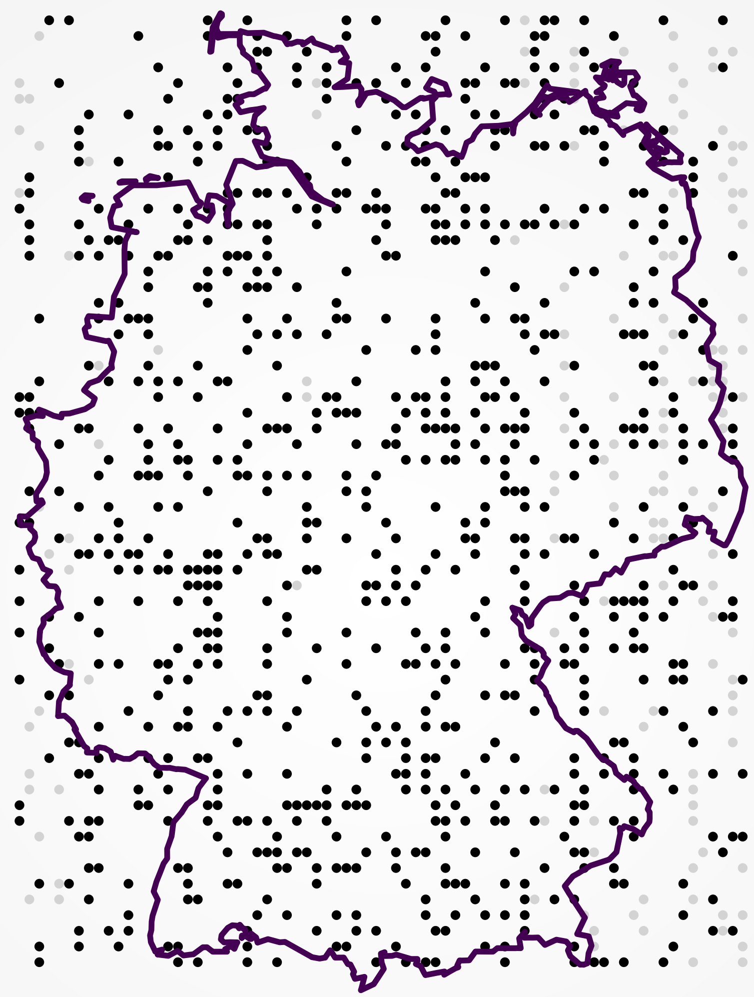


PGFPIEFRstreamgraph

Normalisation with spread of factors



$\widehat{\sigma}_t$ employing LL



$\widehat{\sigma}_t$ employing IEFR

Normality $\alpha > 0.1$

1000 random sites in across Germany, tested with Anderson Darling, (Jarque Bera)

LL: 42% (57%) not rejected

IEFR: 73% (84%) not rejected

Weather Derivatives

CME products

- $HDD(\tau_1,\tau_2)=\int_{\tau_1}^{\tau_2}\max(18°C - T_t, 0)dt$
- $CDD(\tau_1,\tau_2)=\int_{\tau_1}^{\tau_2}\max(T_t - 18°C, 0)dt$
- $CAT(\tau_1,\tau_2)=\int_{\tau_1}^{\tau_2} T_t dt$, where $T_t = \frac{T_{t,max}+T_{t,min}}{2}$
- $AAT(\tau_1,\tau_2)=\int_{\tau_1}^{\tau_2} \widetilde{T}_t dt$, where $\widetilde{T}_t = \frac{1}{24}\int_1^{24} T_{t_i} dt_i$ and T_{t_i} denotes the temperature of hour t_i, (also referred to as C24AT index).

Stochastic pricing

Ornstein-Uhlenbeck process $\mathbf{X}_t \in \mathbb{R}^p$:

$$d\mathbf{X}_t = \mathbf{A}\mathbf{X}_t dt + \mathbf{e}_p \sigma_t dB_t$$

$\mathbf{e}_k$: kth unit vector in $\mathbb{R}^p$ for $k = 1, ...p$, $\sigma_t > 0$, A: $(p \times p)$-matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & \dots & 0 & 1 \\ -\alpha_p & -\alpha_{p-1} & \dots & & -\alpha_1 \end{pmatrix}$$

Proof

CAWS Futures

For $0 \leq t \leq \tau_1 < \tau_2$:

$$F_{CAWS(t,\tau_1,\tau_2)} = \mathsf{E}^{Q_\lambda}\left[\int_{\tau_1}^{\tau_2} W_s ds | \mathcal{F}_t\right]$$

$$= \int_{\tau_1}^{\tau_2} \Lambda_u du + \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{X}_t + \int_t^{\tau_1} \lambda_u \sigma_u \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{e}_p du$$

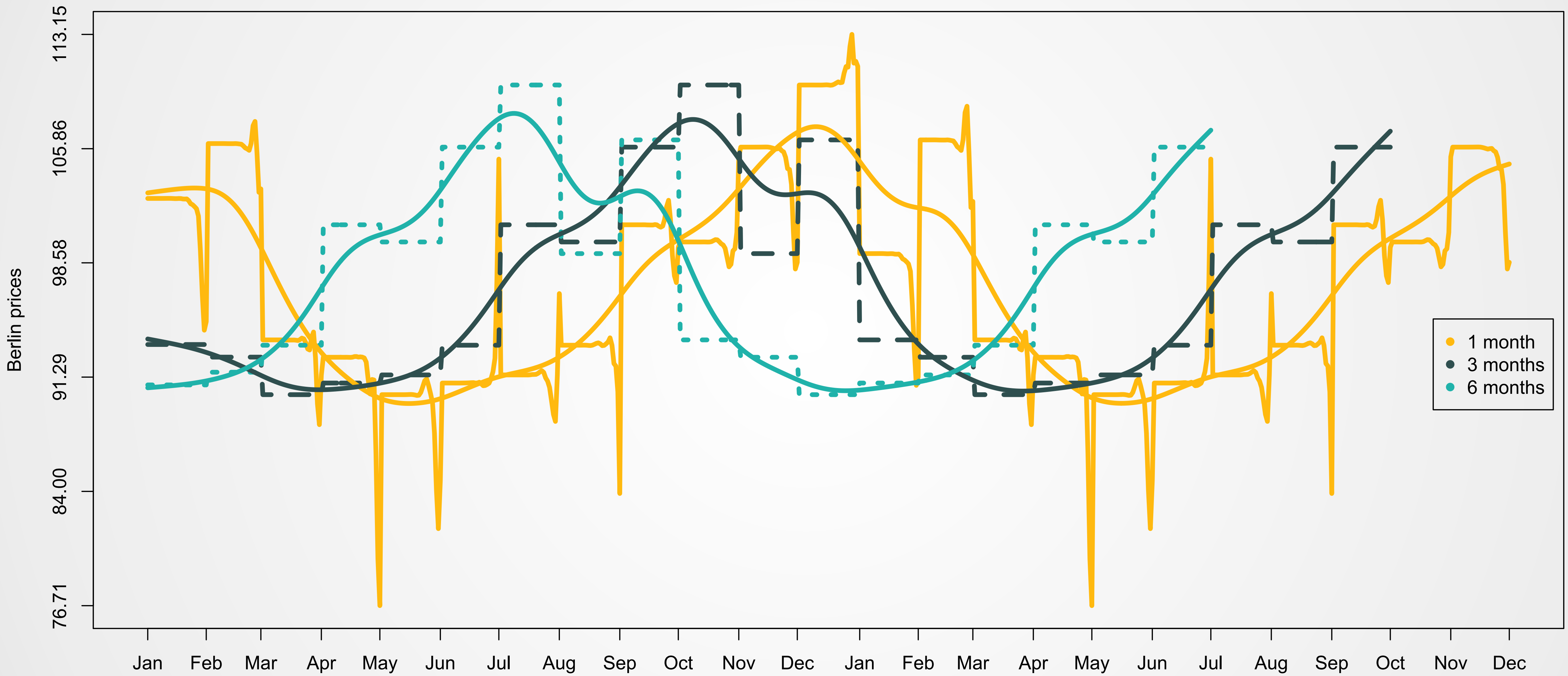
$$+ \int_{\tau_1}^{\tau_2} \lambda_u \sigma_u \mathbf{e}_1^\top \mathbf{A}^{-1}\left[\exp\{\mathbf{A}(\tau_2 - u)\} - I_p\right]\mathbf{e}_p du \quad (2)$$

with $\mathbf{a}_{t,\tau_1,\tau_2} = \mathbf{e}_1^\top \mathbf{A}^{-1}\left[\exp\{\mathbf{A}(\tau_2 - t)\} - \exp\{\mathbf{A}(\tau_1 - t)\}\right]$, $I_p : p \times p$ identity matrix

Benth et al. (2007)

fCAWSpricing

CAWS Futures



Estimated wind future prices for contract length of 1, 3 and 6 months, in Berlin 2012-2013

fCAWSpricing

Conclusion & further research

- IEFR increases percentage of non-rejection in residual normalisation
-  - procedure performs well & applicable to many subjects
- MPR estimation in wind energy: Nasdaq, EEX prices required
- Adaptive optimality for bandwidth selection and transformation
- Extend to spatial wind pricing

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Motzen Conference 2016, July 7-9

$\mathbf{X}_t$ can be written as a Continuous-time AR(p) (CAR(p)):

For $p = 1$,

$$dX_{1t} = -\alpha_1 X_{1t} dt + \sigma_t dB_t$$

For $p = 2$,

$$\begin{aligned} X_{1(t+2)} &\approx (2-\alpha_1) X_{1(t+1)} \\ &+ (\alpha_1 - \alpha_2 - 1) X_{1t} + \sigma_t (B_{t-1} - B_t) \end{aligned}$$

For $p = 3$,

$$\begin{aligned} X_{1(t+3)} &\approx (3-\alpha_1) X_{1(t+2)} + (2\alpha_1 - \alpha_2 - 3) X_{1(t+1)} \\ &+ (-\alpha_1 + \alpha_2 - \alpha_3 + 1) X_{1t} + \sigma_t (B_{t-1} - B_t) \end{aligned}$$

Proof $CAR(3) \approx AR(3)$

Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{pmatrix}$$

- use $B_{t+1} - B_t = \varepsilon_t$
- assume a time step of length one $dt = 1$
- substitute iteratively into X_1 dynamics

Proof $CAR(3) \approx AR(3)$:

$$
\begin{aligned}
X_{1(t+1)} - X_{1(t)} &= X_{2(t)}dt \\
X_{2(t+1)} - X_{2(t)} &= X_{3(t)}dt \\
X_{3(t+1)} - X_{3(t)} &= -\alpha_1 X_{1(t)}dt - \alpha_2 X_{2(t)}dt - \alpha_3 X_{3(t)}dt + \sigma_t \varepsilon_t \\
X_{1(t+2)} - X_{1(t+1)} &= X_{2(t+1)}dt \\
X_{2(t+2)} - X_{2(t+1)} &= X_{3(t+1)}dt \\
X_{3(t+2)} - X_{3(t+1)} &= -\alpha_1 X_{1(t+1)}dt - \alpha_2 X_{2(t+1)}dt \\
&\quad -\alpha_3 X_{3(t+1)}dt + \sigma_{t+1}\varepsilon_{t+1} \\
X_{1(t+3)} - X_{1(t+2)} &= X_{2(t+2)}dt \\
X_{2(t+3)} - X_{2(t+2)} &= X_{3(t+2)}dt \\
X_{3(t+3)} - X_{3(t+2)} &= -\alpha_1 X_{1(t+2)}dt - \alpha_2 X_{2(t+2)}dt \\
&\quad -\alpha_3 X_{3(t+2)}dt + \sigma_{t+2}\varepsilon_{t+2}
\end{aligned}
$$

Return

Seasonal variance: LLE - mirroring observations

To avoid the boundary problem, use mirrored observations:

Assume $h_K < 365/2$, then the observations look like $\hat{\varepsilon}^2_{-364}, \hat{\varepsilon}^2_{-363}, \ldots, \hat{\varepsilon}^2_0, \hat{\varepsilon}^2_1, \ldots, \hat{\varepsilon}^2_{730}$, where

$$
\begin{aligned}
\hat{\varepsilon}^2_t &\stackrel{\mathrm{def}}{=} \hat{\varepsilon}^2_{365+t}, -364 \leq t \leq 0 \\
\hat{\varepsilon}^2_t &\stackrel{\mathrm{def}}{=} \hat{\varepsilon}^2_{t-365}, 366 \leq t \leq 730
\end{aligned}
$$

Fast Iterative Shrinkage-Thresholding Algorithm

- Objective: $\min_\Gamma \left\{ F(\Gamma) \stackrel{def}{=} g(\Gamma) + h(\Gamma) \right\}$
- g: smooth convex function with Lipschitz continuous gradient

$$\| \bigtriangledown g(\Gamma_1) - \bigtriangledown g(\Gamma_2) \|_F \leq L_{\bigtriangledown g} \| \Gamma_1 - \Gamma_2 \|_F, \quad \forall \Gamma_1, \Gamma_2$$

  where $L_{\bigtriangledown g} = 2(mn)^{-1} \max(\tau, 1-\tau) \|X\|_F^2$ is the Lipschitz constant of $\bigtriangledown g$
- h: continuous convex function, possibly nonsmooth
- $|F(\Gamma_t) - F(\Gamma^*)| \leq \frac{2L_{\bigtriangledown g} \|\Gamma_0 - \Gamma^*\|_F^2}{(t+1)^2}$

FISTA Algorithm

1 Initialise: $\Gamma_0 = 0, \Omega_1 = 0$, step size $\delta_1 = 1$

2 For $t = 1, 2, \ldots, T$

- $\Gamma_t = \arg\min_\Gamma \left[\frac{g(\Gamma)}{L_{\triangledown g}} + \frac{1}{2} \left\| \Gamma - \left\{ \Omega_t - \frac{1}{L_{\triangledown g}} \triangledown g(\Omega_t) \right\} \right\|^2 \right]$
- when penalising nuclear norm $\Gamma_t = \mathbf{P} \left(\mathbf{R} - \frac{\lambda}{L_{\triangledown g}} \mathbf{I}_{p \times m} \right) \mathbf{Q}^\top$, and $\Omega_t - \frac{1}{L_{\triangledown g}} \triangledown g(\Omega_t) = \mathbf{PRQ}^\top$ with ALS-SVD (Hastie et al. (2014)) or sparse SVD (Yang et al. (2011))
- $\delta_{t+1} = \frac{1 + \sqrt{1 + 4\delta_t^2}}{2}$
- $\Omega_{t+1} = \Gamma_t + \frac{\delta_{t-1}}{t+1} (\Gamma_t - \Gamma_{t-1})$

3 $\widehat{\Gamma} = \Gamma_T$

Return

Assumptions

- $\{(\mathbf{X}_i, \mathbf{Y}_i)\}_{i=1}^n \in \mathbb{R}^{p+m}$ are i.i.d., $\{X_i\}_{i=1}^n \in \mathbb{R}^p \sim \mathsf{N}(0, \Sigma)$
- Conditional on $\mathbf{X}_i$, $u_{ij} = \{Y_{ij} - \mathbf{X}_i^\top \Gamma_{\cdot j}\}_{j=1}^m$ are cross-sectional independent over j
- $u_{i1}, \ldots, u_{im}$ are sub-gaussian: $\exists C > 0$ s.t. $\mathsf{P}(|u_{ij}| > s) \leq \exp\{1 - (\frac{s}{C})^2\}, j \in \{1, \ldots, m\}$
- $K_u \overset{def}{=} \max_{1 \leq j \leq m} \|U_{ij}\|_{\psi_2} = \max_{1 \leq j \leq m} \sup_{p \leq 1} p^{-\frac{1}{2}} (\mathsf{E}\,|u_{ij}|^p)^{\frac{1}{p}}$

Optimal choice of tuning parameter λ

- Under the assumptions of sample setting, selecting

$$\lambda = \frac{2}{m} S \max(\tau, 1-\tau) \sqrt{K_u^2 \|\Sigma\| \frac{p+m}{n}}, \quad \text{for } n \leq 2\min(m,p)$$

  any optimal solution $\widehat{\Gamma}_\lambda$ satisfies error bound conditions (Huang et al. (2016))
- Then

$$P\left\{ \| \bigtriangledown g(\Gamma) \| \leq m^{-1} S \max(\tau, 1-\tau) \sqrt{K_u^2 \|\Sigma\| \frac{p+m}{n}} \right\} \leq 1 - 3 \cdot 8^{-(p+m)} - 4\exp(-n/2),$$

  where S is an absolute constant.

Return