Journal of Computational and Applied Mathematics 224 (2009) 66–76

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Newton's method and high-order algorithms for the *n*th root computation

François Dubeau

Département de mathématiques, Faculté des sciences, Université de Sherbrooke, 2500, boul. de l'Université, Sherbrooke (Qc), Canada, J1K 2R1

ARTICLE INFO

Article history:
Received 11 May 2007
Received in revised form 13 March 2008

MSC:
65-01
11B37
65B99
65D99
65H05

Keywords:
*n*th root
Newton's method
High-order method

ABSTRACT

Two modifications of Newton's method to accelerate the convergence of the *n*th root computation of a strictly positive real number are revisited. Both modifications lead to methods with prefixed order of convergence $p \in \mathbb{N}$, $p \geq 2$. We consider affine combinations of the two modified *p*th-order methods which lead to a family of methods of order p with arbitrarily small asymptotic constants. Moreover the methods are of order $p+1$ for some specific values of a parameter. Then we consider affine combinations of the three methods of order $p+1$ to get methods of order $p+1$ again with arbitrarily small asymptotic constants. The methods can be of order $p+2$ with arbitrarily small asymptotic constants, and also of order $p+3$ for some specific values of the parameters of the affine combination. It is shown that infinitely many *p*th-order methods exist for the *n*th root computation of a strictly positive real number for any $p \geq 3$.

1. Introduction

The computation of the *n*th root $r^{1/n}$ of a strictly positive real number r is an old problem [1,12]. Recently several authors have suggested high-order methods for the computation of $r^{1/2}$. In [8,16], continued fraction expansions are used to derive such methods. In [10], methods similar to those presented in [16] are obtained as a special case of a determinantal family of root-finding methods [9]. For the computation of the *n*th root, third- and fourth-order methods are presented in [6]. General high-order methods can be derived from the application of Newton's method to an appropriate modified function [3] or using a modification of Newton's method applied to the original function [7]. Finally, using combinations of basic functions identified for methods proposed in [3,7], new high-order methods are derived for the computation of $r^{1/2}$ in [13].

In this paper we start with a review of two extensions of Newton's method applied to the function

$$f(x) = x^n - r \tag{1.1}$$

to find the *n*th root of r. In order to accelerate the convergence, we consider the following two possibilities. Firstly, as suggested in [4,5] and developed in [3], we consider $F_p(x) = g_p(x)f(x)$ and then apply Newton's method on $F_p(x)$ to get the *p*th-order iterative method

$$x_{k+1} = \Phi_{0,p}(x_k) = x_k - \frac{F_p(x_k)}{F_p^{(1)}(x_k)}. \tag{1.2}$$

Secondly, as used in [7], we look for a good choice of $G_p(x)$ for changing the step size of the correction of Newton's method to obtain the *p*th-order iterative method

$$x_{k+1} = \Phi_{1,p}(x_k) = x_k - G_p(x_k)\frac{f(x_k)}{f^{(1)}(x_k)}. \tag{1.3}$$

E-mail address: francois.dubeau@usherbrooke.ca.

0377-0427/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2008.04.014

Both methods are revisited in Section 2 to get the higher-order derivatives needed in the next two sections. In Section 3.1 we consider affine combinations of pth-order methods revisited in Section 2 of the form

$$x_{k+1} = \Phi_{\lambda,p}(x_k) = (1-\lambda)\,\Phi_{0,p}(x_k) + \lambda\,\Phi_{1,p}(x_k). \tag{1.4}$$

We obtain new pth-order methods for any values of the parameter λ with arbitrarily small asymptotic constants, except for some specific values of the parameter for which the new methods are of order $p+1$. In Section 3.2 we consider affine combination of methods of order $p+1$ of the form

$$x_{k+1} = \Psi_{(\mu_0,\mu_1),p+1}(x_k) \tag{1.5}$$

where

$$\Psi_{(\mu_0,\mu_1),p+1}(x_k) = (1-\mu_0-\mu_1)\,\Phi_{\lambda_p,p}(x_k) + \mu_0\,\Phi_{0,p+1}(x_k) + \mu_1\,\Phi_{1,p+1}(x_k). \tag{1.6}$$

All these methods are of order $p+1$ and the asymptotic constants can be made arbitrarily small. They can be of order $p+2$ with arbitrarily small asymptotic constants, and even of order $p+3$ for some specific values of the parameters of the affine combination. Finally, in Section 4 we compare the methods with respect to their asymptotic constants and in Section 5 we present some examples.

2. High-order methods revisited

In this paper we use the following notation

$$\binom{\delta}{i} = \begin{cases} 1 & \text{for } i=0, \\ \dfrac{\delta(\delta-1)\cdots(\delta-(i-1))}{i!} & \text{for } i \geq 1. \end{cases} \tag{2.1}$$

The following lemma can be proved by mathematical induction.

Lemma 2.1. *Let $H(\cdot)$ be a regular real-valued function and $\delta \in \mathbb{R}\setminus\{0\}$. Let $h(y) = H(y^\delta)$, then for $j \geq 1$ we have*

$$\frac{\mathrm{d}^j}{\mathrm{d}y^j}h(y) = \sum_{k=1}^{j} H^{(k)}(y^\delta)\,y^{k\delta-j}w_{j,k}(\delta) \tag{2.2}$$

where $w_{1,1}(\delta) = \delta$, and

$$w_{j,1}(\delta) = (\delta-(j-1))w_{j-1,1}(\delta), \tag{2.3}$$

$$w_{j,j}(\delta) = \delta w_{j-1,j-1}(\delta), \tag{2.4}$$

for $j \geq 2$, and

$$w_{j,k}(\delta) = \delta w_{j-1,k-1}(\delta) + (k\delta-(j-1))w_{j-1,k}(\delta) \tag{2.5}$$

for $j \geq 3$ and $k = 2,\ldots,j-1$. $\square$

Moreover $w_{j,j}(\delta) = \delta^j$, $w_{j,1}(\delta) = j!\binom{\delta}{j}$, and

$$\begin{cases} w_{j,j-1}(\delta) = \alpha_{j,1}(\delta-1)\delta^{j-1} & \text{for } j\geq 2 \\ w_{j,j-2} = (\beta_{j,2}\delta+\alpha_{j,2})(\delta-1)\delta^{j-2} & \text{for } j \geq 3 \\ w_{j,j-3} = (\gamma_{j,3}\delta^2+\beta_{j,3}\delta+\alpha_{j,3})(\delta-1)\delta^{j-3} & \text{for } j\geq 4 \end{cases}$$

where $\alpha_{j,1} = \binom{j}{2}$, $\beta_{j,2} = \frac{3j-5}{4}\binom{j}{3}$, $\alpha_{j,2} = -\frac{3j-1}{4}\binom{j}{3}$, $\gamma_{j,3} = \frac{1}{2}(j-3)(j-2)\binom{j}{4}$, $\beta_{j,3} = -(j^2-3j+1)\binom{j}{4}$, and $\alpha_{j,3} = \frac{1}{2}j(j-1)\binom{j}{4}$.

One approach to get high-order methods for finding $r^{1/n}$ is to use a function $g_p(x)$ such that $F_p(x) = g_p(x)f(x) = g_p(x)\,(x^n-r)$ satisfies the assumptions of the following result about Newton's method.

Theorem 2.2 ([2,5,15]). *Let p be an integer ≥ 2 and let $F_p(x)$ be a regular function such that $F_p(\alpha)=0$, $F_p^{(1)}(\alpha)\neq 0$, $F_p^{(j)}(\alpha)=0$ for $j=2,\ldots,p-1$, and $F_p^{(p)}(\alpha)\neq 0$. Then Newton's method applied to the equation $F_p(x)=0$ generates a sequence $\{x_k\}_{k=0}^{+\infty}$ where*

$$x_{k+1} = \Phi_{0,p}(x_k) = x_k - \frac{F_p(x_k)}{F_p^{(1)}(x_k)} \quad (k=0,1,2,\ldots)$$

which converges to α for a given x_0 sufficiently close to α. Moreover, the convergence is of order p and the asymptotic constant is

$$K_{0,p}(\alpha) = \lim_{k\to+\infty}\frac{x_{k+1}-\alpha}{(x_k-\alpha)^p} = \frac{p-1}{p!}\frac{F_p^{(p)}(\alpha)}{F_p^{(1)}(\alpha)}. \quad \square$$

One such function $g_p(x)$ suggested in [3] is

$$g_p(x) = \sum_{i=1}^{p-1} a_i (x^n - r)^{i-1} = \sum_{i=1}^{p-1} \binom{1/n}{i} \frac{(x^n - r)^{i-1}}{r^i}. \tag{2.6}$$

If we set $a_i = \frac{1}{r^i}\binom{1/n}{i}$ for $i = 1, \ldots, p-1$, then

$$F_p(x) = \sum_{i=1}^{p-1} a_i (x^n - r)^i. \tag{2.7}$$

In the next theorem we give expressions for all the derivatives of $F_p(x)$.

Theorem 2.3 (*[3]*). *Let n and p be integers ≥ 2 and $F_p(x)$ be given by (2.7). Then*

$$F_p^{(j)}(r^{1/n}) = u_{p,j}(n) n^j \binom{1/n}{j} j! r^{-\frac{j}{n}} \tag{2.8}$$

where

$$u_{p,0}(n) = 0, \tag{2.9}$$

$$u_{p,1}(n) = 1, \tag{2.10}$$

$$u_{p,j}(n) = 0 \qquad \text{for } j = 2, \ldots, p-1, \tag{2.11}$$

$$u_{p,p}(n) = -1, \tag{2.12}$$

and for $j \geq p+1$

$$u_{p,j}(n) = -\left[1 + \sum_{k=p}^{j-1} w_{j,k}(1/n) n^k \frac{\binom{1/n}{k} k!}{\binom{1/n}{j} j!} u_{p,k}(n) \right]. \tag{2.13}$$

Proof. We have the following identity

$$F_p((y + r)^{1/n}) = \sum_{i=1}^{p-1} a_i y^i = Q_p(y). \tag{2.14}$$

Then we use Lemma 2.1 with $\delta = \frac{1}{n}$ and $H(\xi^{1/n}) = F_p(\xi^{1/n})$ to get

$$\begin{aligned} \frac{d^j}{dy^j} F_p((r + y)^{1/n}) &= \frac{d^j}{d\xi^j} F_p(\xi^{1/n})|_{\xi = r+y} \\ &= \sum_{k=1}^{j} F_p^{(k)}((r + y)^{1/n})(r + y)^{\frac{k}{n} - j} w_{j,k}(1/n). \end{aligned} \tag{2.15}$$

From the fact that

$$Q_p^{(j)}(y) = \begin{cases} \displaystyle\sum_{i=j}^{p-1} a_i \frac{i!}{(i-j)!} y^{i-j} & \text{for } j = 1, \ldots, p-1, \\ 0 & \text{for } j \geq p, \end{cases}$$

we obtain the result by setting $y = 0$ recursively for $j = 1, 2, 3, \ldots$. □

It follows that the sequence $\{x_k\}_{k=0}^{+\infty}$ generated by: x_0 given sufficiently close to $r^{1/n}$, and

$$x_{k+1} = \Phi_{0,p}(x_k) = x_k - \frac{(x_k^n - r) \sum_{i=1}^{p-1} \binom{1/n}{i} \left(\frac{x_k^n}{r} - 1\right)^{i-1}}{n x_k^{n-1} \sum_{i=1}^{p-1} i \binom{1/n}{i} \left(\frac{x_k^n}{r} - 1\right)^{i-1}} \tag{2.16}$$

for $k = 0, 1, 2, \ldots$, converges to $r^{1/n}$. Moreover, the convergence is of order p, and the asymptotic constant is

$$K_{0,p}(r^{1/n}) = \lim_{k \to +\infty} \frac{x_{k+1} - r^{1/n}}{(x_k - r^{1/n})^p} = -(p-1) n^p \binom{1/n}{p} r^{-\frac{p-1}{n}}. \tag{2.17}$$

Another approach to get high-order methods for finding $r^{1/n}$ is to use a function $G_p(x)$ such that the modified Newton's method given by $x_{k+1} = \Phi_{1,p}(x_k) = x_k - G_p(x_k)\frac{f(x_k)}{f^{(1)}(x_k)}$ satisfies the assumptions of the following result about fixed-point methods.

Theorem 2.4 (*[11,14]*). *Let p be an integer ≥ 2 and let $\Phi_{1,p}(x)$ be a regular function such that $\Phi_{1,p}(\alpha) = \alpha$, $\Phi_{1,p}^{(j)}(\alpha) = 0$ for $j = 1, \ldots, p-1$, and $\Phi_{1,p}^{(p)}(\alpha) \neq 0$. Then the sequence $\{x_k\}_{k=0}^{+\infty}$ generated by $x_{k+1} = \Phi_{1,p}(x_k)$ for $k = 0, 1, 2, \ldots$, converges to α for a given x_0 sufficiently close to α. Moreover, the convergence is of order p, and the asymptotic constant is*

$$K_p(\alpha) = \lim_{k\to+\infty} \frac{x_{k+1} - \alpha}{(x_k - \alpha)^p} = \frac{\Phi_{1,p}^{(p)}(\alpha)}{p!}. \quad \square$$

As established in [7], for $p \geq 2$ we take

$$G_p(x) = \frac{\sum_{i=0}^{p-1} \binom{1/n}{i} \left(\frac{r}{x^n} - 1\right)^i - 1}{\frac{1}{n}\left(\frac{r}{x^n} - 1\right)}, \tag{2.18}$$

and define

$$\Phi_{1,p}(x) = x - G_p(x)\frac{x}{n}\left(1 - \frac{r}{x^n}\right) = x \sum_{i=0}^{p-1} \binom{1/n}{i} \left(\frac{r}{x^n} - 1\right)^i. \tag{2.19}$$

The next theorem presents a formula for all the derivatives of $\Phi_{1,p}(x)$.

Theorem 2.5 (*[7]*). *Let $f(x)$ be given by (1.1). Let n and p be integers ≥ 2, and let $\Phi_{1,p}(x)$ be given by (2.19). Then*

$$\Phi_{1,p}^{(j)}(r^{1/n}) = (-1)^j v_{p,j}(n) n^j \binom{1/n}{j} j! r^{\frac{1-j}{n}} \tag{2.20}$$

where

$$v_{p,0}(n) = 1, \tag{2.21}$$

$$v_{p,j}(n) = 0 \quad \text{for } j = 1, \ldots, p-1, \tag{2.22}$$

$$v_{p,p}(n) = -1, \tag{2.23}$$

and for $j \geq p+1$

$$v_{p,j}(n) = \sum_{k=p}^{j-1} (-1)^k v_{p,k}(n) n^{k-1} \frac{\binom{1/n}{k} k!}{\binom{1/n}{j} j!} w_{j-1,k-1}(-1/n). \tag{2.24}$$

Proof. We clearly have $\Phi_{1,p}(r^{1/n}) = r^{1/n}$. Also, using (2.19) and the identity

$$\binom{1/n}{i+1} = \binom{1/n}{i} \frac{\frac{1}{n} - i}{i+1},$$

we get

$$\begin{aligned}\Phi_{1,p}^{(1)}(x) &= \sum_{i=0}^{p-1} \binom{1/n}{i} \left(\frac{r}{x^n} - 1\right)^i - \sum_{i=1}^{p-1} \binom{1/n}{i} i \left(\frac{r}{x^n} - 1\right)^{i-1} \left(\frac{nr}{x^n}\right) \\ &= np \binom{1/n}{p} \left(\frac{r}{x^n} - 1\right)^{p-1}.\end{aligned}$$

Since

$$\Phi_{1,p}^{(1)}\left(\left(\frac{1+y}{r}\right)^{-1/n}\right) = np \binom{1/n}{p} y^{p-1} \tag{2.25}$$

we use Lemma 2.1 with $\delta = -\frac{1}{n}$ and $H(\xi^{-1/n}) = \Phi^{(1)}_{1,p}(\xi^{-1/n})$ to obtain

$$\begin{aligned}\frac{\mathrm{d}^j}{\mathrm{d}y^j}\Phi^{(1)}_{1,p}\left(\left(\frac{1+y}{r}\right)^{-1/n}\right) &= \frac{\mathrm{d}^j}{\mathrm{d}\xi^j}\Phi^{(1)}_{1,p}(\xi^{1/n})|_{\xi=\frac{1+y}{r}}\left[\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1+y}{r}\right)\right]^j \\ &= \frac{1}{r^j}\sum_{k=1}^{j}\Phi^{(1+k)}_{1,p}\left(\left(\frac{1+y}{r}\right)^{-1/n}\right)\left(\frac{1+y}{r}\right)^{-\frac{k}{n}-j} w_{j,k}(-1/n) \\ &= \sum_{k=1}^{j}\Phi^{(1+k)}_{1,p}\left(\left(\frac{1+y}{r}\right)^{-1/n}\right)(1+y)^{-\frac{k}{n}-j}r^{\frac{k}{n}}w_{j,k}(-1/n).\end{aligned}$$

Since

$$\frac{\mathrm{d}^j}{\mathrm{d}y^j}np\binom{1/n}{p}y^{p-1} = \begin{cases} np\binom{1/n}{p}\binom{p-1}{j}y^{p-1-j} & \text{for } j=0,\ldots,p-1,\\ 0 & \text{for } j\geq p,\end{cases}$$

the result follows by setting $y = 0$ recursively for $j = 2, 3, 4, \ldots$. □

Thus the sequence $\{x_k\}_{k=0}^{+\infty}$ generated by: x_0 given sufficiently close to $r^{1/n}$, and

$$x_{k+1} = \Phi_{1,p}(x_k) = x_k\sum_{i=0}^{p-1}\binom{1/n}{i}\left(\frac{r}{x_k^n}-1\right)^i \tag{2.26}$$

for $k = 0, 1, 2, \ldots$, converges to $r^{1/n}$. The convergence is of order p, and the asymptotic constant is

$$K_{1,p}(r^{1/n}) = \frac{\Phi^{(p)}_{1,p}(\alpha)}{p!} = (-1)^{p-1}n^p\binom{1/n}{p}r^{-\frac{p-1}{n}}. \tag{2.27}$$

It is possible to show that for any $x_0 > r^{1/n}$ the sequence $\{x_k\}_{k=0}^{+\infty}$ is monotonically decreasing and converges to $r^{1/n}$ [7].

3. More high-order methods

In the preceding section we have revisited two families of methods for the computation of $r^{1/n}$. In this section we combine the two families of methods to get new high-order methods. We consider $p \geq 3$ since $\Phi_{0,2}(x) = \Phi_{1,2}(x)$ and it corresponds to Newton's method of order 2.

3.1. Combination of pth-order methods

We start with the two pth-order methods given by (2.16) and (2.26) and consider an affine combination

$$\Phi_{\lambda,p}(x) = (1-\lambda)\,\Phi_{0,p}(x) + \lambda\,\Phi_{1,p}(x) \tag{3.1}$$

where the parameter $\lambda \in \mathbb{R}$. Since $\Phi_{\lambda,p}(r^{1/n}) = r^{1/n}$ and $\Phi^{(1)}_{\lambda,p}(r^{1/n}) = (1-\lambda)\,\Phi^{(1)}_{0,p}(r^{1/n}) + \lambda\,\Phi^{(1)}_{1,p}(r^{1/n}) = 0$, the sequence $\{x_k\}_{k=0}^{+\infty}$ generated by: x_0 given sufficiently close to $r^{1/n}$, and

$$x_{k+1} = \Phi_{\lambda,p}(x_k) \tag{3.2}$$

for $k = 0, 1, 2, \ldots$, converges to $r^{1/n}$. Moreover, the convergence is of order p, and the asymptotic constant is given by

$$K_{\lambda,p}(r^{1/n}) = (1-\lambda)K_{0,p}(r^{1/n}) + \lambda K_{1,p}(r^{1/n}).$$

From (2.17) and (2.27), we get

$$K_{\lambda,p}(r^{1/n}) = B(\lambda;p)n^p\binom{1/n}{p}r^{-\frac{p-1}{n}} \tag{3.3}$$

where

$$B(\lambda;p) = \lambda\left((p-1)+(-1)^{p-1}\right) - (p-1). \tag{3.4}$$

This asymptotic constant can be made arbitrarily small for $p \geq 3$, and is 0 for the value of λ which is a solution of

$$B(\lambda;p) = 0, \tag{3.5}$$

namely for

$$\lambda = \lambda_p = \frac{(p-1)}{(p-1)+(-1)^{p-1}}.$$

In that case the method is of order $p+1$ and its asymptotic constant is given by

$$\begin{aligned} K^{+}_{\lambda_p,p+1}(r^{1/n}) &= \lim_{k\to+\infty} \frac{x_{k+1}-r^{1/n}}{(x_k-r^{1/n})^{p+1}} \\ &= \lim_{k\to+\infty} \frac{\Phi_{\lambda_p,p}(x_k)-\Phi_{\lambda_p,p}(r^{1/n})}{(x_k-r^{1/n})^{p+1}} \\ &= (1-\lambda_p)\frac{pF_p^{(p+1)}(r^{1/n})}{(p+1)!F_p^{(1)}(r^{1/n})} + \lambda_p \frac{\Phi_{1,p}^{(p+1)}(r^{1/n})}{(p+1)!}. \end{aligned}$$

From Theorems 2.3 and 2.5 we have

$$F_p^{(p+1)}(r^{1/n}) = \frac{(np-(p+2))}{2(np-1)}(p-1)n^{p+1}(p+1)!\binom{1/n}{p+1} r^{-\frac{p+1}{n}} \tag{3.6}$$

and

$$\Phi_{1,p}^{(p+1)}(r^{1/n}) = (-1)^{p+1}\frac{(n+1)}{2(np-1)}p(p-1)n^{p+1}(p+1)!\binom{1/n}{p+1} r^{-\frac{p+1}{n}}. \tag{3.7}$$

We finally obtain

$$K^{+}_{\lambda_p,p+1}(r^{1/n}) = (-1)^{p+1}C(n,p)n^{p+1}\binom{1/n}{p+1} r^{-p/n}. \tag{3.8}$$

where

$$C(n,p) = \frac{p(p-1)}{2(np-1)}\left[\frac{2np-(n+3)}{(p-1)+(-1)^{p-1}}\right]. \tag{3.9}$$

3.2. Combination of $(p+1)$th-order methods

We consider the affine combination of the three $(p+1)$th-order methods

$$\Psi_{(\mu_0,\mu_1),p+1}(x) = (1-\mu_0-\mu_1)\,\Phi_{\lambda_p,p}(x) + \mu_0\,\Phi_{0,p+1}(x) + \mu_1\,\Phi_{1,p+1}(x) \tag{3.10}$$

where μ_0 and $\mu_1 \in \mathbb{R}$. Let us observe that for $\mu_0+\mu_1 = 1$ we have $\Psi_{(\mu_0,\mu_1),p+1}(x) = \Phi_{\mu_1,p+1}(x)$. Since $\Psi_{(\mu_0,\mu_1),p+1}(r^{1/n}) = r^{1/n}$ and $\Psi^{(1)}_{(\mu_0,\mu_1),p+1}(r^{1/n}) = 0$, the sequence $\{x_k\}_{k=0}^{+\infty}$ generated by: x_0 given sufficiently close to $r^{1/n}$, and

$$x_{k+1} = \Psi_{(\mu_0,\mu_1),p+1}(x_k) \tag{3.11}$$

for $k = 0, 1, 2, \ldots,$ converges to $r^{1/n}$. Moreover, the convergence is of order $p+1$ and the asymptotic constant is given by

$$K_{(\mu_0,\mu_1),p+1}(r^{1/n}) = (1-\mu_0-\mu_1)K^{+}_{\lambda_p,p+1}(r^{1/n}) + \mu_0 K_{0,p+1}(r^{1/n}) + \mu_1 K_{1,p+1}(r^{1/n}). \tag{3.12}$$

From (3.12), (2.17) and (2.27), we get

$$K_{(\mu_0,\mu_1),p+1}(r^{1/n}) = (-1)^{p+1}R(\mu_0,\mu_1;n,p)n^{p+1}\binom{1/n}{p+1} r^{-\frac{p}{n}} \tag{3.13}$$

where

$$R(\mu_0,\mu_1;n,p) = C(n,p)(1-\mu_0-\mu_1) + (-1)^p p\mu_0 - \mu_1. \tag{3.14}$$

This asymptotic constant can be made arbitrarily small, and is 0 for the values of (μ_0,μ_1) which are solutions of

$$R(\mu_0,\mu_1;n,p) = 0, \tag{3.15}$$

which is equivalent to

$$\left[1+(-1)^{p+1}\frac{p}{C(n,p)}\right]\mu_0 + \left[1+\frac{1}{C(n,p)}\right]\mu_1 = 1. \tag{3.16}$$

If (μ_0^*,μ_1^*) is any solution of (3.15), or (3.16), the method is of order $p+2$. Its asymptotic constant is given by

$$\begin{aligned} K^{*}_{(\mu_0^*,\mu_1^*),p+2}(r^{1/n}) &= \lim_{k\to+\infty} \frac{x_{k+1}-r^{1/n}}{(x_k-r^{1/n})^{p+2}} \\ &= \lim_{k\to+\infty} \frac{\Psi_{(\mu_0^*,\mu_1^*),p+1}(x_k)-\Psi_{(\mu_0^*,\mu_1^*),p+1}(r^{1/n})}{(x_k-r^{1/n})^{p+2}} \\ &= (1-\mu_0^*-\mu_1^*)\left[(1-\lambda_p)\left(\frac{(p+1)F_p^{(p+2)}(r^{1/n})}{(p+2)!F_p^{(1)}(r^{1/n})} - \frac{p-1}{2p!}F_p^{(p)}(r^{1/n})\frac{F_p^{(3)}(r^{1/n})}{[F_p^{(1)}(r^{1/n})]^2}\right) + \lambda_p\frac{\Phi_{1,p}^{(p+2)}(r^{1/n})}{(p+2)!}\right] \\ &\quad + \mu_0^*\frac{(p+1)F_{p+1}^{(p+2)}(r^{1/n})}{(p+2)!F_{p+1}^{(1)}(r^{1/n})} + \mu_1^*\frac{\Phi_{1,p+1}^{(p+2)}(r^{1/n})}{(p+2)!}, \end{aligned} \tag{3.17}$$

and from Theorems 2.3 and 2.5 we obtain

$$K^{*}_{(\mu_0^*,\mu_1^*),p+2}(r^{1/n}) = (-1)^{p+2}S(\mu_0^*,\mu_1^*;n,p)n^{p+2}\binom{1/n}{p+2}r^{-\frac{p+1}{n}} \tag{3.18}$$

where

$$\begin{aligned} S(\mu_0^*,\mu_1^*;n,p) \;=\; &(1-\mu_0^*-\mu_1^*)\Big[(-1)^{p+2}(1-\lambda_p)\Big((p+1)u_{p,p+2}(n) \\ &+\;\frac{(p+2)(p+1)(p-1)(n-1)(2n-1)}{2(np-1)(n(p+1)-1)}u_{p,3}(n)\Big)+\lambda_p v_{p,p+2}(n)\Big] \\ &+(-1)^{p+2}\mu_0^*(p+1)u_{p+1,p+2}(n)+\mu_1^* v_{p+1,p+2}(n). \end{aligned} \tag{3.19}$$

Here again, this asymptotic constant can be arbitrarily small, and 0 for the values of (μ_0^*,μ_1^*) which are solutions of

$$S(\mu_0^*,\mu_1^*;n,p) = 0. \tag{3.20}$$

If (μ_0^{**},μ_1^{**}) is any solution of the system (3.15) and (3.20), the method is of order $p+3$. Its asymptotic constant is given by

$$\begin{aligned} K^{**}_{(\mu_0^{**},\mu_1^{**}),p+3}(r^{1/n}) \;&=\; \lim_{k\to+\infty}\frac{x_{k+1}-r^{1/n}}{(x_k-r^{1/n})^{p+3}} \\ &=\; \lim_{k\to+\infty}\frac{\Psi_{(\mu_0^{**},\mu_1^{**}),p+1}(x_k)-\Psi_{(\mu_0^{**},\mu_1^{**}),p+1}(r^{1/n})}{(x_k-r^{1/n})^{p+3}} \\ &=\;(1-\mu_0^{**}-\mu_1^{**})\Bigg[(1-\lambda_p)\Bigg(\frac{(p+2)F_p^{(p+3)}(r^{1/n})}{(p+3)!F_p^{(1)}(r^{1/n})}-\frac{pF_p^{(p+1)}(r^{1/n})F_p^{(3)}(r^{1/n})}{2!\,(p+1)![F_p^{(1)}(r^{1/n})]^2} \\ &\quad-\;\frac{(p-1)F_p^{(p)}(r^{1/n})F_p^{(4)}(r^{1/n})}{3!\,p![F_p^{(1)}(r^{1/n})]^2}\Bigg)+\lambda_p\frac{\Phi_{1,p}^{(p+3)}(r^{1/n})}{(p+3)!}\Bigg] \\ &\quad+\mu_0^{**}\frac{(p+2)F_{p+1}^{(p+3)}(r^{1/n})}{(p+3)!F_{p+1}^{(1)}(r^{1/n})}+\mu_1^{**}\frac{\Phi_{1,p+1}^{(p+3)}(r^{1/n})}{(p+3)!}, \end{aligned} \tag{3.21}$$

and from Theorems 2.3 and 2.5 we obtain

$$K^{**}_{(\mu_0^{**},\mu_1^{**}),p+3}(r^{1/n}) = (-1)^{p+3}W(n,p)n^{p+3}\binom{1/n}{p+3}r^{-\frac{p+2}{n}} \tag{3.22}$$

where

$$\begin{aligned} W(n,p) \;=\; &(1-\mu_0^{**}-\mu_1^{**})\Big[(-1)^{p+3}(1-\lambda_p)\Big((p+2)u_{p,p+3}(n)-\frac{(p+3)(p+2)p(n-1)(2n-1)}{2(n(p+1)-1)(n(p+2)-1)}u_{p,p+1}(n)u_{p,3}(n) \\ &+\;\frac{(p+3)(p+2)(p+1)(n-1)(2n-1)(3n-1)}{6(np-1)(n(p+1)-1)(n(p+2)-1)}u_{p,4}(n)\Big)+\lambda_p v_{p,p+3}(n)\Big] \\ &+(-1)^{p+3}\mu_0^{**}(p+2)u_{p+1,p+3}(n)+\mu_1^{**}v_{p+1,p+3}(n). \end{aligned} \tag{3.23}$$

4. Comparison of asymptotic constants

To compare methods of the same order we compare their asymptotic constants. For methods of order p, from (2.17) and (2.27), we have

$$K_{0,p}(r^{1/n}) = (-1)^p(p-1)K_{1,p}(r^{1/n}),$$

then $|K_{0,p}(r^{1/n})| > |K_{1,p}(r^{1/n})|$ for $p > 2$. Using (3.3) we have

$$K_{\lambda,p}(r^{1/n}) = \begin{cases} -\dfrac{1}{(p-1)}B(\lambda;p)K_{0,p}(r^{1/n}), \\ (-1)^{p-1}B(\lambda;p)K_{1,p}(r^{1/n}). \end{cases} \tag{4.1}$$

If we set $\lambda = \lambda_p + \Delta\lambda$ and $B_p = \left[(p-1)+(-1)^{p-1}\right]^{-1}$, then

$$|K_{\lambda,p}(r^{1/n})| < |K_{0,p}(r^{1/n})| \quad \text{iff } |\Delta\lambda| < (p-1)B_p,$$

and

$$|K_{\lambda,p}(r^{1/n})| < |K_{1,p}(r^{1/n})| \quad \text{iff } |\Delta\lambda| < B_p.$$

Table 1
Values of λ_p and B_p

p	3	4	5	6
λ_p	2/3	3/2	4/5	5/4
B_p	1/3	1/2	1/5	1/4

Table 2
Values of $C(n, p)$

p	3	4	5	6
$C(2, p)$	7/5	33/7	10/3	285/44
$C(5, p)$	11/7	96/19	7/2	195/29

For methods of order $p+1$, we have

$$K^{+}_{\lambda_p,p+1}(r^{1/n}) = \begin{cases} -\dfrac{1}{p}C(n,p)K_{0,p+1}(r^{1/n}), \\ (-1)^p C(n,p)K_{1,p+1}(r^{1/n}), \end{cases} \tag{4.2}$$

and

$$K_{(\mu_0,\mu_1),p+1}(r^{1/n}) = \begin{cases} \dfrac{(-1)^p}{p}R(\mu_0,\mu_1;n,p)K_{0,p+1}(r^{1/n}), \\ -R(\mu_0,\mu_1;n,p)K_{1,p+1}(r^{1/n}). \end{cases} \tag{4.3}$$

For methods of order $p+2$, we have

$$K^{*}_{(\mu_0,\mu_1),p+1}(r^{1/n}) = \begin{cases} \dfrac{(-1)^{p+1}}{p+1}S(\mu_0,\mu_1;n,p)K_{0,p+2}(r^{1/n}), \\ -S(\mu_0,\mu_1;n,p)K_{1,p+2}(r^{1/n}). \end{cases} \tag{4.4}$$

Finally, for methods of order $p+3$, we have

$$K^{**}_{(\mu_0^{**},\mu_1^{**}),p+1}(r^{1/n}) = \begin{cases} \dfrac{(-1)^{p+2}}{p+2}W(n,p)K_{0,p+3}(r^{1/n}), \\ -W(n,p)K_{1,p+3}(r^{1/n}). \end{cases} \tag{4.5}$$

Even if we have the preceding expressions to compare the asymptotic constants, it is not easy to find a general conclusion as the examples of the next section suggest.

5. Examples

We have computed $35^{1/n}$ for $n = 2, 5$. The results for the different iterative methods are given in Appendix A for $n = 2$ with $x_0 = 95/16$, and in Appendix B for $n = 5$ with $x_0 = 131/64$. The asymptotic constants are estimated by the formula

$$|K_{\bullet,p}(r^{1/n})| = \frac{|x_4 - r^{1/n}|}{|x_3 - r^{1/n}|^p}$$

for a method of order p.

We have considered the iterative methods $\Phi_{0,p}$ and $\Phi_{1,p}$ for $p = 3, 4, 5, 6, 7$. Also we have illustrated, with $\Phi_{7/12,3}$, $\Phi_{5/6,3}$ and $\Phi_{25/16,4}$, methods of the family $\Phi_{\lambda,p}$ which are of order p with asymptotic constants less then $K_{1,p}(35^{1/n})$, and also $K_{0,p}(35^{1/n})$, for $n = 2, 5$ and $p = 3, 4$. The values of $B(\lambda; p)$ are respectively $B(7/12; 3) = -1/4$, $B(5/6; 3) = 1/2$, and $B(25/16; 4) = 1/8$. The results for these methods appear in Tables 3–7 for $n = 2$ and in Tables 8–12 for $n = 5$. We have also given $\Phi_{\lambda_p,p}$ for $p = 3, 4, 5, 6$. The corresponding values of λ_p and B_p are given in Table 1, and those of $C(n, p)$ are given in Table 2.

For the method $\Psi_{(\mu_0,\mu_1),p+1}$, the expressions for $R(\mu_0, \mu_1; n, p)$ and $S(\mu_0, \mu_1; n, p)$ are given below:

$$R(\mu_0,\mu_1;n,p) = \begin{cases} \dfrac{-1}{5}(22\mu_0 + 12\mu_1 - 7) & \text{for } n = 2, p = 3, \\ \dfrac{-1}{7}(5\mu_0 + 40\mu_1 - 33) & \text{for } n = 2, p = 4, \\ \dfrac{-1}{7}(32\mu_0 + 18\mu_1 - 11) & \text{for } n = 5, p = 3, \\ \dfrac{-1}{19}(20\mu_0 + 115\mu_1 - 96) & \text{for } n = 5, p = 3, \end{cases}$$

Table 3
Methods of order 3: computation of $|x_k - 35^{1/2}|$ and estimated asymptotic constant $|K_{\bullet,3}(35^{1/2})|$

k	$\Phi_{0,3}$	$\Phi_{1,3}$	$\Phi_{7/12,3}$	$\Phi_{5/6,3}$
1	$2.81\ldots \times 10^{-7}$	$1.39\ldots \times 10^{-7}$	$3.59\ldots \times 10^{-8}$	$6.91\ldots \times 10^{-8}$
2	$6.35\ldots \times 10^{-22}$	$3.85\ldots \times 10^{-23}$	$1.65\ldots \times 10^{-25}$	$2.36\ldots \times 10^{-24}$
3	$7.32\ldots \times 10^{-66}$	$8.20\ldots \times 10^{-70}$	$1.62\ldots \times 10^{-77}$	$9.45\ldots \times 10^{-74}$
4	$1.12\ldots \times 10^{-197}$	$7.89\ldots \times 10^{-210}$	$1.52\ldots \times 10^{-233}$	$6.04\ldots \times 10^{-222}$
	Estimated asymptotic constant $|K_{\bullet,3}(35^{1/2})|$			
	$|K_{0,3}(35^{1/2})|$	$|K_{1,3}(35^{1/2})|$	$|K_{7/12,3}(35^{1/2})|$	$|K_{5/6,3}(35^{1/2})|$
	$2.85\ldots \times 10^{-2}$	$1.42\ldots \times 10^{-2}$	$3.57\ldots \times 10^{-3}$	$7.14\ldots \times 10^{-3}$

Table 4
Methods of order 4: computation of $|x_k - 35^{1/2}|$ and estimated asymptotic constant $|K_{\bullet,4}(35^{1/2})|$

k	$\Phi_{0,4}$	$\Phi_{1,4}$	$\Phi_{25/16,4}$	$\Phi_{2/3,3}$	$\Psi_{(1/5,3/10),4}$
1	$1.91\ldots \times 10^{-9}$	$6.27\ldots \times 10^{-10}$	$9.53\ldots \times 10^{-11}$	$8.86\ldots \times 10^{-10}$	$1.27\ldots \times 10^{-10}$
2	$1.20\ldots \times 10^{-37}$	$4.67\ldots \times 10^{-40}$	$3.12\ldots \times 10^{-44}$	$2.61\ldots \times 10^{-39}$	$1.57\ldots \times 10^{-43}$
3	$1.93\ldots \times 10^{-150}$	$1.43\ldots \times 10^{-160}$	$3.58\ldots \times 10^{-178}$	$1.97\ldots \times 10^{-157}$	$3.71\ldots \times 10^{-175}$
4	$1.28\ldots \times 10^{-601}$	$1.28\ldots \times 10^{-642}$	$6.20\ldots \times 10^{-714}$	$6.41\ldots \times 10^{-630}$	$1.15\ldots \times 10^{-701}$
	Estimated asymptotic constant $|K_{\bullet,4}(35^{1/2})|$				
	$|K_{0,4}(35^{1/2})|$	$|K_{1,4}(35^{1/2})|$	$|K_{25/16,4}(35^{1/2})|$	$|K^{+}_{2/3,4}(35^{1/2})|$	$|K_{(1/5,3/10),4}(35^{1/2})|$
	$9.05\ldots \times 10^{-3}$	$3.01\ldots \times 10^{-3}$	$3.77\ldots \times 10^{-4}$	$4.22\ldots \times 10^{-3}$	$6.03\ldots \times 10^{-4}$

Table 5
Methods of order 5: computation of $|x_k - 35^{1/2}|$ and estimated asymptotic constant $|K_{\bullet,5}(35^{1/2})|$

k	$\Phi_{0,5}$	$\Phi_{1,5}$	$\Phi_{3/2,4}$	$\Psi_{(1/5,13/60),4}$
1	$1.29\ldots \times 10^{-11}$	$3.16\ldots \times 10^{-12}$	$1.58\ldots \times 10^{-11}$	$9.19\ldots \times 10^{-13}$
2	$1.03\ldots \times 10^{-57}$	$2.26\ldots \times 10^{-61}$	$2.62\ldots \times 10^{-57}$	$1.34\ldots \times 10^{-64}$
3	$3.43\ldots \times 10^{-288}$	$4.23\ldots \times 10^{-307}$	$4.21\ldots \times 10^{-286}$	$8.82\ldots \times 10^{-324}$
4	$1.36\ldots \times 10^{-1440}$	$9.73\ldots \times 10^{-1536}$	$4.47\ldots \times 10^{-1430}$	$1.09\ldots \times 10^{-1619}$
	Estimated asymptotic constant $|K_{\bullet,5}(35^{1/2})|$			
	$|K_{0,5}(35^{1/2})|$	$|K_{1,5}(35^{1/2})|$	$|K^{+}_{3/2,5}(35^{1/2})|$	$|K^{*}_{(1/5,13/60),5}(35^{1/2})|$
	$2.85\ldots \times 10^{-3}$	$7.14\ldots \times 10^{-4}$	$3.36\ldots \times 10^{-3}$	$2.04\ldots \times 10^{-4}$

Table 6
Methods of order 6: computation of $|x_k - 35^{1/2}|$ and estimated asymptotic constant $|K_{\bullet,6}(35^{1/2})|$

k	$\Phi_{0,6}$	$\Phi_{1,6}$	$\Phi_{4/5,5}$	$\Psi_{(4/25,29/100),4}$
1	$8.80\ldots \times 10^{-14}$	$1.70\ldots \times 10^{-14}$	$5.78\ldots \times 10^{-14}$	$5.02\ldots \times 10^{-15}$
2	$4.20\ldots \times 10^{-82}$	$4.51\ldots \times 10^{-87}$	$2.25\ldots \times 10^{-83}$	$8.65\ldots \times 10^{-91}$
3	$5.03\ldots \times 10^{-492}$	$1.53\ldots \times 10^{-522}$	$7.92\ldots \times 10^{-500}$	$2.24\ldots \times 10^{-545}$
4	$1.46\ldots \times 10^{-2951}$	$2.35\ldots \times 10^{-3135}$	$1.49\ldots \times 10^{-2998}$	$6.81\ldots \times 10^{-3273}$
	Estimated asymptotic constant $|K_{\bullet,6}(35^{1/2})|$			
	$|K_{0,6}(35^{1/2})|$	$|K_{1,6}(35^{1/2})|$	$|K^{+}_{4/5,6}(35^{1/2})|$	$|K^{**}_{(4/25,29/100),6}(35^{1/2})|$
	$9.05\ldots \times 10^{-4}$	$1.81\ldots \times 10^{-4}$	$6.03\ldots \times 10^{-4}$	$5.34\ldots \times 10^{-5}$

Table 7
Methods of order 7: computation of $|x_k - 35^{1/2}|$ and estimated asymptotic constant $|K_{\bullet,7}(35^{1/2})|$

k	$\Phi_{0,7}$	$\Phi_{1,7}$	$\Phi_{5/4,6}$	$\Psi_{(-41/195,166/195),5}$
1	$6.01\ldots \times 10^{-16}$	$9.17\ldots \times 10^{-17}$	$6.38\ldots \times 10^{-16}$	$3.40\ldots \times 10^{-16}$
2	$8.26\ldots \times 10^{-111}$	$3.81\ldots \times 10^{-117}$	$1.34\ldots \times 10^{-110}$	$8.86\ldots \times 10^{-113}$
3	$7.59\ldots \times 10^{-775}$	$5.64\ldots \times 10^{-820}$	$2.44\ldots \times 10^{-773}$	$7.15\ldots \times 10^{-789}$
	Estimated asymptotic constant $|K_{\bullet,7}(35^{1/2})|$			
	$|K_{0,7}(35^{1/2})|$	$|K_{1,7}(35^{1/2})|$	$|K^{+}_{5/4,7}(35^{1/2})|$	$|K^{**}_{(-41/195,166/195),7}(35^{1/2})|$
	$2.88\ldots \times 10^{-4}$	$4.81\ldots \times 10^{-5}$	$3.11\ldots \times 10^{-4}$	$1.66\ldots \times 10^{-4}$

Table 8
Methods of order 3: computation of $|x_k - 35^{1/5}|$ and estimated asymptotic constant $|K_{\bullet,3}(35^{1/5})|$

k	$\Phi_{0,3}$	$\Phi_{1,3}$	$\Phi_{7/12,3}$	$\Phi_{5/6,3}$
1	$3.62\ldots \times 10^{-6}$	$1.73\ldots \times 10^{-6}$	$4.98\ldots \times 10^{-7}$	$8.41\ldots \times 10^{-7}$
2	$1.37\ldots \times 10^{-16}$	$7.55\ldots \times 10^{-18}$	$4.48\ldots \times 10^{-20}$	$4.31\ldots \times 10^{-19}$
3	$7.58\ldots \times 10^{-48}$	$6.24\ldots \times 10^{-52}$	$3.25\ldots \times 10^{-59}$	$5.80\ldots \times 10^{-56}$
4	$1.26\ldots \times 10^{-141}$	$3.52\ldots \times 10^{-154}$	$1.24\ldots \times 10^{-176}$	$1.41\ldots \times 10^{-166}$
	Estimated asymptotic constant $\|K_{\bullet,3}(35^{1/5})\|$			
	$\|K_{0,3}(35^{1/5})\|$	$\|K_{1,3}(35^{1/5})\|$	$\|K_{7/12,3}(35^{1/5})\|$	$\|K_{5/6,3}(35^{1/5})\|$
	2.89…	1.44…	0.361…	0.723…

Table 9
Methods of order 4: computation of $|x_k - 35^{1/5}|$ and estimated asymptotic constant $|K_{\bullet,4}(35^{1/5})|$

k	$\Phi_{0,4}$	$\Phi_{1,4}$	$\Phi_{25/16,4}$	$\Phi_{2/3,3}$	$\Psi_{(1/5,2/7),4}$
1	$1.01\ldots \times 10^{-7}$	$3.14\ldots \times 10^{-8}$	$7.62\ldots \times 10^{-9}$	$5.17\ldots \times 10^{-8}$	$2.57\ldots \times 10^{-9}$
2	$7.76\ldots \times 10^{-28}$	$2.44\ldots \times 10^{-30}$	$1.05\ldots \times 10^{-33}$	$2.80\ldots \times 10^{-29}$	$8.52\ldots \times 10^{-36}$
3	$2.71\ldots \times 10^{-108}$	$8.87\ldots \times 10^{-119}$	$3.78\ldots \times 10^{-133}$	$2.42\ldots \times 10^{-114}$	$1.01\ldots \times 10^{-141}$
4	$4.06\ldots \times 10^{-430}$	$1.54\ldots \times 10^{-472}$	$6.37\ldots \times 10^{-531}$	$1.34\ldots \times 10^{-454}$	$2.08\ldots \times 10^{-565}$
	Estimated asymptotic constant $\|K_{\bullet,4}(35^{1/5})\|$				
	$\|K_{0,4}(35^{1/5})\|$	$\|K_{1,4}(35^{1/5})\|$	$\|K_{25/16,4}(35^{1/5})\|$	$\|K^{+}_{2/3,5}(35^{1/5})\|$	$\|K_{(1/5,2/7),4}(35^{1/5})\|$
	7.46…	2.48…	0.31…	3.90…	0.192…

Table 10
Methods of order 5: computation of $|x_k - 35^{1/5}|$ and estimated asymptotic constant $|K_{\bullet,5}(35^{1/5})|$

k	$\Phi_{0,5}$	$\Phi_{1,5}$	$\Phi_{3/2,4}$	$\Psi_{(1/5,23/90),4}$
1	$2.71\ldots \times 10^{-9}$	$6.19\ldots \times 10^{-10}$	$3.27\ldots \times 10^{-9}$	$6.79\ldots \times 10^{-11}$
2	$2.75\ldots \times 10^{-42}$	$4.25\ldots \times 10^{-46}$	$8.88\ldots \times 10^{-42}$	$7.06\ldots \times 10^{-52}$
3	$2.94\ldots \times 10^{-207}$	$6.44\ldots \times 10^{-227}$	$1.29\ldots \times 10^{-204}$	$8.59\ldots \times 10^{-257}$
4	$4.13\ldots \times 10^{-1032}$	$5.16\ldots \times 10^{-1131}$	$8.63\ldots \times 10^{-1019}$	$2.28\ldots \times 10^{-1281}$
	Estimated asymptotic constant $\|K_{\bullet,5}(35^{1/5})\|$			
	$\|K_{0,5}(35^{1/5})\|$	$\|K_{1,5}(35^{1/5})\|$	$\|K^{+}_{3/2,5}(35^{1/5})\|$	$\|K^{*}_{(1/5,23/90),5}(35^{1/5})\|$
	18.56…	4.64…	23.45…	0.48…

Table 11
Methods of order 6: computation of $|x_k - 35^{1/5}|$ and estimated asymptotic constant $|K_{\bullet,6}(35^{1/5})|$

k	$\Phi_{0,6}$	$\Phi_{1,6}$	$\Phi_{4/5,5}$	$\Psi_{(13/70,59/210),4}$
1	$7.21\ldots \times 10^{-11}$	$1.28\ldots \times 10^{-11}$	$4.76\ldots \times 10^{-11}$	$3.72\ldots \times 10^{-13}$
2	$6.45\ldots \times 10^{-60}$	$4.09\ldots \times 10^{-65}$	$3.74\ldots \times 10^{-61}$	$5.06\ldots \times 10^{-76}$
3	$3.30\ldots \times 10^{-354}$	$4.32\ldots \times 10^{-386}$	$8.83\ldots \times 10^{-362}$	$3.18\ldots \times 10^{-453}$
4	$5.90\ldots \times 10^{-2120}$	$5.94\ldots \times 10^{-2312}$	$1.51\ldots \times 10^{-2165}$	$1.95\ldots \times 10^{-2716}$
	Estimated asymptotic constant $\|K_{\bullet,6}(35^{1/5})\|$			
	$\|K_{0,6}(35^{1/5})\|$	$\|K_{1,6}(35^{1/5})\|$	$\|K^{+}_{4/5,6}(35^{1/5})\|$	$\|K^{**}_{(13/70,59/210),6}(35^{1/5})\|$
	45.6…	9.12…	31.92…	0.18…

Table 12
Methods of order 7: computation of $|x_k - 35^{1/5}|$ and estimated asymptotic constant $|K_{\bullet,7}(35^{1/5})|$

k	$\Phi_{0,7}$	$\Phi_{1,7}$	$\Phi_{5/4,6}$	$\Psi_{(-167/775,676/775),5}$
1	$1.90\ldots \times 10^{-12}$	$2.75\ldots \times 10^{-13}$	$1.99\ldots \times 10^{-12}$	$1.22\ldots \times 10^{-12}$
2	$1.02\ldots \times 10^{-80}$	$2.24\ldots \times 10^{-87}$	$1.54\ldots \times 10^{-80}$	$3.12\ldots \times 10^{-82}$
3	$1.27\ldots \times 10^{-558}$	$5.30\ldots \times 10^{-606}$	$2.64\ldots \times 10^{-557}$	$2.21\ldots \times 10^{-569}$
	Estimated asymptotic constant $\|K_{\bullet,7}(35^{1/5})\|$			
	$\|K_{0,7}(35^{1/5})\|$	$\|K_{1,7}(35^{1/5})\|$	$\|K^{+}_{5/4,7}(35^{1/5})\|$	$\|K^{**}_{(-167/775,676/775),7}(35^{1/5})\|$
	111.33…	18.55…	124.72…	75.85…

and

$$S(\mu_0, \mu_1; n, p) = \begin{cases} \dfrac{6}{49}(-\mu_0 + 4\mu_1 - 1) & \text{for } n = 2, p = 3, \\ \dfrac{1}{21}(39\mu_0 + 39\mu_1 - 25) & \text{for } n = 2, p = 4, \\ \dfrac{12}{133}(-55\mu_0 + 15\mu_1 + 6) & \text{for } n = 5, p = 3, \\ \dfrac{1}{76}(139\mu_0 + 63\mu_1 - 25) & \text{for } n = 5, p = 3. \end{cases}$$

For $n = 2$: the method $\Psi_{(1/5,3/10),4}$ is of order 4 and $R(1/5, 3/10; 2, 3) = -1/5$; the method $\Psi_{(1/5,13/60),4}$ is of order 5 because $R(1/5, 13/60; 2, 3) = 0$ but $S(1/5, 13/60; 2, 3) = -2/7$; the method $\Psi_{(4/25,29/100),4}$ is of order 6 because $R(4/25, 29/100; 2, 3) = 0$ and $S(4/25, 29/100; 2, 3) = 0$, and $W(2, 3) = -31/105$; and the method $\Psi_{(-41/195,166/195),5}$ is of order 7 because $R(-41/195, 166/195; 2, 4) = 0$ and $S(-41/195, 166/195; 2, 4) = 0$, and $W(2, 4) = -404/117$. See Tables 4–7.

For $n = 5$: the method $\Psi_{(1/5,2/7),4}$ is of order 4 and $R(1/5, 2/7; 5, 3) = -19/245$; the method $\Psi_{(1/5,23/90),4}$ is of order 5 because $R(1/5, 23/90; 5, 3) = 0$ but $S(1/5, 23/90; 5, 3) = -10/57$; the method $\Psi_{(13/70,59/210),4}$ is of order 6 because $R(13/70, 59/210; 5, 3) = 0$ and $S(13/70, 59/210; 5, 3) = 0$, and $W(5, 3) = -11/532$; and the method $\Psi_{(-167/775,676/775),5}$ is of order 7 because $R(-167/775, 676/775; 5, 4) = 0$ and $S(-167/775, 676/775; 2, 4) = 0$, and $W(5, 4) = -18\,367/4495$. See Tables 9–12.

Acknowledgements

The author would like to thank the referees for their suggestions that improved the presentation of the paper. This work has been financially supported by an individual discovery grant from NSERC (Natural Sciences and Engineering Research Council of Canada).

Appendix A. Results for $35^{1/2}$

See Tables 3–7.

Appendix B. Results for $35^{1/5}$

See Tables 8–12.

References

[1] D.F. Bateman, A historical survey of solution by functional iteration, Mathematics Magazine 62 (1989) 155–166.
[2] J.E. Dennis, R.B. Schnabel, Numerical Methods for Unconstrained Optimization and Nonlinear Equations, Prentice-Hall, Englewood Cliffs, NJ, 1983.
[3] F. Dubeau, Algorithms for *n*th root approximation, Computing 57 (1996) 365–369.
[4] W.F. Ford, J.A. Pennline, Accelerated convergence in Newton's method, SIAM Review 38 (1996) 658–659.
[5] J. Gerlach, Accelerated convergence in Newton's method, SIAM Review 36 (1994) 272–276.
[6] J.M. Gutierrez, M.A. Hernandez, M.A. Solanova, Calculus of *n*th roots and third order iterative methods, Nonlinear Analysis 47 (2001) 2875–2880.
[7] M.A. Hernandez, N. Romero, High order algorithms for approximating *n*th roots, International Journal of Computer Mathematics 81 (2004) 1001–1014.
[8] K.R. Johnson, An iterative method for approximating square roots, Mathematics Magazine 62 (1989) 253–259.
[9] B. Kalantari, On the order of convergence of determinantal family of root-finding methods, BIT 39 (1999) 96–109.
[10] B. Kalantari, I. Kalantari, High order methods for approximating square roots, BIT 36 (1996) 395–399.
[11] D. Kincaid, W. Cheney, Numerical Analysis, Brooks/Cole Pub. Co, California, 1991.
[12] R.J. Knill, A modified Babylonian algorithm, American Mathematical Monthly 99 (1992) 734–737.
[13] A.Y. Özban, New methods for approximating square roots, Applied Mathematics and Computation 175 (2006) 532–540.
[14] J. Stoer, R. Bulirsch, Introduction to Numerical Analysis, 2nd ed., Springer-Verlag, NY, 1993.
[15] J.F. Traub, Iterative Methods for the Solution of Equations, Prentice-Hall, Englewood Cliffs, NJ, 1964.
[16] A.K. Yeyios, On two sequences of algorithms for approximating square roots, Journal of Computational and Applied Mathematics 40 (1992) 63–72.