Calculus

Gate Notes

CALCULUS

Basics (pre-calculus):

Logorithms: If $\log_b a = m$ then $a = b^m$

1. $\log_c(ab) = \log_c a + \log_c b$
2. $\log_c(a/b) = \log_c a - \log_c b$
3. $\log_{b^n} a^m = \frac{m}{n} \log_b a$
4. $\log_b a = \frac{1}{\log_a b} = \log_c a \cdot \log_b c = \frac{\log_c a}{\log_c b}$
5. $a^x = e^{\log_e a^x} = e^{x \log_e a}$ $\qquad \log_e \rightarrow \ln$
6. $\log_a a = 1$, $\log_a 1 = 0$, $\log_c(1/a) = -\log_c a$

Ex: $\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} = 2.322$

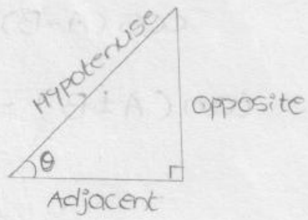
GATE IN(08): The expression $e^{-\ln x}$ for $x > 0$ is equal to

sol: $e^{-\ln x} = e^{-\log_e x}$

$= e^{\log_e(1/x)} = 1/x = x^{-1}$

Trignometry:

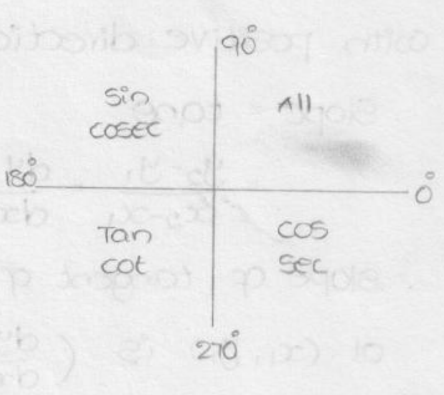
$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\text{cosec}\theta}$



$\cos\theta = \frac{\text{Adjacent}}{\text{hypotenuse}} = \frac{1}{\sec\theta}$

$\tan\theta = \frac{\text{opposite}}{\text{Adjacent}} = \frac{\sin\theta}{\cos\theta} = \frac{1}{\cot\theta}$

	0°	30°	45°	60°	90°
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞



(i) If the angle is $(90 \pm \theta)$, $(270 \pm \theta)$ i.e about vertical line, then co-function of initial function will apply

(ii) If the angle is $(180 \pm \theta), (360 \pm \theta)$ i.e about horizontal line then same function will apply.

(iii) The positivity or negativity of the resulting function depends on the quadrant in which initial function lies.

Ex: $\sin(270+\theta) = -\cos\theta$ $\quad\quad$ $\cot(270-\theta) = \tan\theta$

$\cos(90+\theta) = -\sin\theta$ $\quad\quad$ $\sec(180+\theta) = -\sec\theta$

Formulas:

1. $\sin^2\theta + \cos^2\theta = 1$; $\csc^2\theta - \cot^2\theta = 1$; $1+\tan^2\theta = \sec^2\theta$.

2. $\sin(A \pm B) = \sin A \cdot \cos B \pm \cos A \cdot \overset{\sin}{\cancel{\cos}} B$

put $B = A \Rightarrow \sin 2A = 2\sin A \cos A$

$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

3. $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

put $B = A$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= 1 - 2\sin^2 A = 2\cos^2 A - 1$

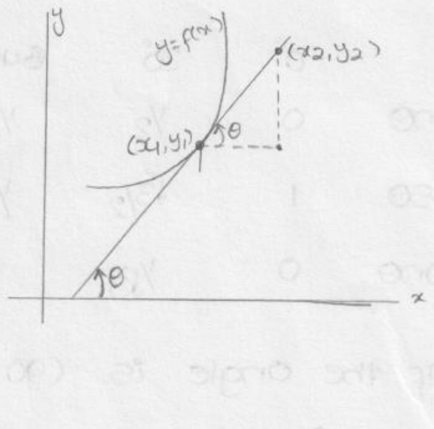
$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$

4. $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \cdot \tan B}$

put $B = A \Rightarrow \tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Slope of a line: It is the tangent of angle made by the line with positive direction of x-axis.



Slope $= \tan\theta$

$= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{dy}{dx}$

slope of tangent of curve ~~at~~ $y = f(x)$ at (x_1, y_1) is $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}$

Equation of straight line:

1. point slope form: point is (x_1, y_1) & slope m

$$y - y_1 = m(x - x_1)$$

2. Two point form:
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

3. slope intercept form: slope is 'm', y-intercept is 'C'

$$y = mx + C$$

4. Intercept form: 'x' intercept is 'a', 'y' intercept is 'b'

$$\frac{x}{a} + \frac{y}{b} = 1$$

5. General equation of straight line $ax + by + c = 0$

$$y = \left(\frac{-a}{b}\right)x + \left(\frac{-c}{b}\right)$$

slope $= -a/b$ y-intercept $-c/b$

$$ax + by = -c$$

$$\frac{x}{(-c/a)} + \frac{y}{(-c/b)} = 1 \Rightarrow \text{x-intercept} = -c/a$$

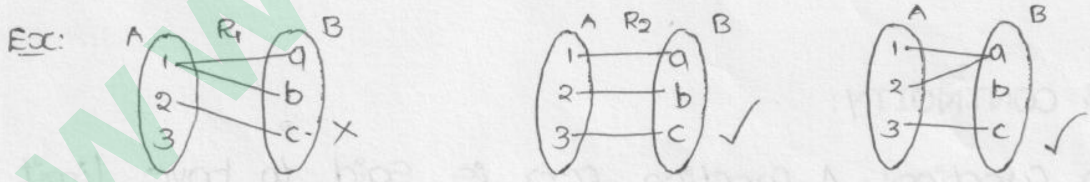
Function: A function from set A to set B is a relation from A to B satisfying the condition

"To each element in A, there exist a unique element in B"

i.e (i) every element in A must be associated

(ii) It must be associated with a unique element in B.

Ex:



Explicit function: $z = f(x_1, x_2, \ldots x_n)$

Dependent variable (z); Independent variable ($x_1, x_2, \ldots x_n$)

Implicit function: $\phi(z, x_1, x_2, \ldots x_n) = C$

composit function: If $z = f(x, y)$ where $x = \phi(t)$ & $y = \psi(t)$ i.e z is a function of some function.

Some Special functions:

1. Even function: $f(-x) = f(x)$ Ex: x^2, $\cos x$ ...

2. odd function: $f(-x) = -f(x)$ Ex: x, $\sin x$ --

3. Modulus function:

$$f(x) = |x| = \begin{cases} x & ; x>0 \\ -x & ; x<0 \\ 0 & ; x=0 \end{cases}$$

GATE-99: $f(x) = e^x$ Neither even nor odd

4. Step / Greatest integer function:

$$f(x) = [x] = n \in \mathbb{Z} \quad \text{where } n \le x \le n+1$$

Ex: $[7.2] = 7$; $[7.999] = 7$; $[-1.2] = -2$

Symmetric properties of the curve: Let $f(x,y) = c$ be the eqn of a curve

(i) If $f(x,y)$ contains only even powers of x i.e $f(-x,y) = f(x,y)$ then it is symmetric about y-axis

(ii) If $f(x,y)$ contains only even powers of y i.e $f(x,-y) = f(x,y)$ then it is symmetric about x-axis

(iii) If $f(x,y) = f(y,x)$ then it is symmetric about $y=x$.

GATE (97): The curve given by the equation $x^2+y^2 = 3axy$ is

(a) Symmetrical about x-axis (b) y-axis (c) line $y=x$

(d) tangential to $x=y=a/3$

Sol: $f(x,y) = f(y,x) \Rightarrow x^2+y^2 = 3axy$

$\therefore$ Symmetrical about line $y=x$.

LIMITS & CONTINUITY:

Limit of a function: A function $f(x)$ is said to have limit value 'l' as x tend to 'a' if

$$\lim_{x\to a} f(x) = l$$

Left limit: when $x<a$, $x\to a$ $\quad \lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$

Right limit: when $x>a$, $x\to a$ $\quad \lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$

A limit exists if LHL = RHL

Ex: If $f(x) = \begin{cases} 2x+3 & \text{for } x \geq 2 \\ x+9 & \text{for } x<2 \end{cases}$ then $\lt_{x \to 2} f(x) = ?$

sol: $\lt_{x\to 2^+} 2x+3 = 7 \neq \lt_{x \to 2^-} x+9 = 11$

$\therefore$ Limit doesnot exist.

Indeterminate form: $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$

Standard limits:

(i) $\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$

(ii) $\lt_{x\to 0} \frac{e^{mx}-1}{x} = e$

(iii) $\lt_{x\to 0} \frac{a^x-1}{x} = \log_e a$

(iv) $\lt_{x\to 0} [1+ax]^{1/x} = e^a$

(v) $\lt_{x\to 0} [1+\frac{a}{x}]^x = e^a$

(vi) $\lt_{x\to 0} \frac{\sin x}{x} = 1$ or $\lt_{x\to 0} \frac{\tan x}{x} = 1$

(vii) $\lt_{x\to\infty} \frac{\sin x}{x} = 0$

(viii) $\lt_{x\to 0} \frac{\sin mx}{x} = m$

(ix) $\lt_{x\to 0} \left[\frac{a^x+b^x}{2}\right]^{1/x} = \sqrt{ab}$

(x) $\lt_{x\to 0} [\cos x + a\sin bx]^{1/x} = e^{ab}$

(xi) $\lt_{x\to 0} \left[\frac{1-\cos ax}{x^2}\right] = \frac{a^2}{2}$

(xii) $\lt_{x\to\infty} \frac{\log_e x}{x} = 0$

Limit properties:

1. $\lt_{x\to a} [c\,f(x)] = c \lt_{x\to a} f(x)$
2. $\lt_{x\to a} [f(x) \pm g(x)] = \lt_{x\to a} f(x) \pm \lt_{x\to a} g(x)$
3. $\lt_{x\to a} [f(x) \cdot g(x)] = \lt_{x\to a} f(x) \cdot \lt_{x\to a} g(x)$
4. $\lt_{x\to a} \frac{f(x)}{g(x)} = \frac{\lt_{x\to a} f(x)}{\lt_{x\to a} g(x)}$ provided $\lt_{x\to a} g(x) \neq 0$
5. $\lt_{x\to a} [f(x)]^n = \left[\lt_{x\to a} f(x)\right]^n$ where n is a real number.
6. $\lt_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lt_{x\to a} f(x)}$
7. $\lt_{x\to a} c = c$; c is a real number
8. If $P(x)$ is a polynomial then $\lt_{x\to a} P(x) = P(a)$

(GATE-95): $\lim\limits_{x\to 0} x\sin(1/x) = $ ____.

sol: Let $x = t \Rightarrow 1/x = 1/t$, $t\to\infty$

$$\lim_{x\to 0} x\sin(1/x) = \lim_{t\to\infty} \frac{\sin t}{t} = 0$$

(GATE-99): $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+n}} =$

sol: $$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{n}{n\sqrt{1+1/n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}} = \frac{1}{\sqrt{1+0}} = 1$$

(GATE-01 IN): $\lim\limits_{x\to\pi/4} \frac{\sin 2(x-\pi/4)}{x-\pi/4}$

sol: Let $x-\pi/4 = t \Rightarrow t\to 0$

$$\lim_{x\to\pi/4} \frac{\sin 2(x-\pi/4)}{x-\pi/4} = \lim_{t\to 0} \frac{\sin 2t}{t} = 2$$

(GATE-02 CE): $\lim\limits_{n\to\infty} n^{1/n} =$

sol: Let $y = \lim\limits_{n\to\infty} n^{1/n}$ Apply log on b.s

$$\log_e y = \lim_{n\to\infty} \log_e n^{1/n} = \lim_{n\to\infty} \frac{1}{n}\log_e n = 0$$

$$\log_e y = 0 \Rightarrow y = e^0 = 1$$

(GATE-03): $\lim\limits_{x\to 0} \frac{\sin^2 x}{x} = \underline{0}$

sol: $$\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \times \sin x$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \times \lim_{x\to 0} \sin x$$

$$= 1\times 0 = 0$$

(GATE-04): $\lim\limits_{x\to 0} \frac{x^3+x^2}{2x^3-7x^2} =$

sol: $$\lim_{x\to 0} \frac{x^3+x^2}{2x^3-7x^2} = \lim_{x\to 0} \frac{x^2(x+1)}{x^2(2x-7)}$$

$$= \lim_{x\to 0} \frac{x+1}{2x-7} = \frac{1}{-7} = -\frac{1}{7}$$

(GATE-07 ME): $\lim\limits_{x\to 0} \frac{e^x-(1+x+x^2/2)}{x^3}$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

by substituting we will get

$$\lim_{x\to 0} \frac{\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\infty}{x^3} = \lim_{x\to 0}\left(\frac{1}{3!}+\frac{x}{4!}+\frac{x^2}{5!}+\cdots+\infty\right)$$

$$= \frac{1}{3!}+0 = \frac{1}{6}$$

(GATE-07 EC): $\lim_{\theta\to 0} \frac{\sin(\theta/2)}{\theta}$ is

Sol: $\lim_{x\to 0} \frac{\sin mx}{x} = m \Rightarrow \lim_{\theta\to 0} \frac{\sin(\frac{1}{2}\cdot\theta)}{\theta} = \frac{1}{2} = 0.5$

(GATE-08 CS): $\lim_{x\to\infty} \frac{x-\sin x}{x+\cos x} =$

Sol: $= \lim_{x\to\infty} \frac{1-\frac{\sin x}{x}}{1+\frac{\cos x}{x}} = \frac{1-\lim_{x\to\infty}\frac{\sin x}{x}}{1+\lim_{x\to\infty}\frac{\cos x}{x}} = \frac{1-0}{1+0} = 1$

(GATE-08 ME): $\lim_{x\to 8} \frac{x^{1/3}-2}{x-8}$

Sol: $= \lim_{x\to 8} \frac{x^{1/3}-8^{1/3}}{x-8}$ $\qquad \lim_{x\to a}\frac{x^n-a^n}{x-a} = n\cdot a^{n-1}$

$$= \frac{1}{3}\cdot 8^{\frac{1}{3}-1} = \frac{1}{3}\times 8^{-2/3} = \frac{1}{3}\times\frac{1}{4} = \frac{1}{12}$$

(GATE-08 PI): $\lim_{x\to 0} \frac{\sin x}{e^x\cdot x} =$

Sol: $\lim_{x\to 0} \frac{\sin x}{e^x\cdot x} = \lim_{x\to 0}\frac{\sin x}{x} \times \lim_{x\to 0}\frac{1}{e^x}$

$$= 1\times\frac{1}{e^0} = 1\times 1 = 1$$

(GATE-10 CS): $\lim_{n\to\infty} (1-1/n)^{2n} =$

Sol: $\lim_{n\to\infty}(1-1/n)^{2n} = \lim_{n\to\infty}\left[\left(1+\frac{-1}{n}\right)^n\right]^2$

$$= \left[\lim_{n\to\infty}\left(1+\frac{(-1)}{n}\right)^n\right]^2 = (e^{-1})^2 = e^{-2}$$

(GATE-14 CE): $\lim_{x\to\infty} \frac{x+\sin x}{x} =$

Sol: $\lim_{x\to\infty}\frac{x}{x} + \underbrace{\lim_{x\to\infty}\frac{\sin x}{x}}_{0} = \lim_{x\to\infty} 1 + 0 = 1+0 = 1$

Pblm: $\lim_{x\to 0} \frac{|x|}{x} = ?$

Sol: $LHL = \lim_{x\to 0^-}\frac{|x|}{x} = \lim_{h\to 0}\frac{|-h|}{-h} = -1$

$RHL = \lim_{x\to 0^+}\frac{|x|}{x} = \lim_{h\to 0}\frac{|h|}{h} = 1$

$LHL \neq RHL$ $\therefore$ limit doesn't exist.

Continuity of $f(x)$: A function $f(x)$ is said to be continous at $x=a$, when $\lim_{x\to a} f(x) = f(a)$

Ex: $f(x) = \begin{cases} 2x+3 & \text{for } x>2 \\ x+5 & \text{for } x<2 \\ 15 & \text{for } x=2 \end{cases}$

$$\lim_{x\to 2^+} f(x) = 7 = \lim_{x\to 2^-} f(x) = 7 \neq f(2) = 15$$

Limit exists but not continous.

(GATE-97): If $y=|x|$ for $x<0$ and $y=x$ for $x\geq 0$ then

√(a) $\frac{dy}{dx}$ is discontinous at $x=0$

(b) y is discontinous at $x=0$

(c) y is not defined at $x=0$

(d) Both y & dy/dx are discontinous at $x=0$

Sol: $y=|x|$ is continous everywhere

$\frac{dy}{dx} = \frac{|x|}{x}$ is not continous at $x=0$ because $\lim_{x\to 0} \frac{|x|}{x}$ = not exist

∴ Ans: (a)

(GATE-2013 CS): which of the following function is continous at $x=3$

√(a) $f(x) = \begin{cases} 2 & \text{if } x=3 \\ x-1 & \text{if } x>3 \\ \frac{x+3}{3} & \text{if } x<3 \end{cases}$

(b) $f(x) = \begin{cases} 4 & \text{if } x=3 \\ 8-x & \text{if } x\neq 3 \end{cases}$

(c) $f(x) = \begin{cases} x+3 & \text{if } x\leq 3 \\ x-4 & \text{if } x>3 \end{cases}$

(d) $f(x) = \frac{1}{x^3-27}$ if $x\neq 3$

Sol:

(a) $\lim_{x\to 3^+} f(x) = 3-1 = 2$ $\quad \lim_{x\to 3^-} f(x) = \frac{3+3}{3} = 2$ $\quad f(3) = 2$

∴ $f(x)$ is continous at $x=3$

Ans: (a)

Note: If a function is continous at $x=a$ then limit of the function also exists at $x=a$ and is equal to $f(a)$.

Differentiability: A function $f(x)$ is said to be differentiable at $x=a$ if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. and is denoted $f'(a)$

In general the derivative function is given by

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Ex: If $f(x) = x^2$ then $f'(x) = 2x$ & $f'(x) = 2a$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a} x+a = 2a$$

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{x^2+\Delta x^2+2\cdot x\cdot\Delta x - x^2}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x(\Delta x+2x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \Delta x+2x = 0+2x = 2x$$

* Left hand derivative: $LHD = \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h}$

Right hand derivative: $RHD = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$

Necessary condition for function to be differential is

$$LHD = RHD$$

* Every differentiable function is a continous function. But every continous function is not differentiable

Note:

(i) $|x|$ is not differentiable at $x=0$

$$LHD = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{|-h|-0}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

$$RHD = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|-0}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

$LHD \neq RHD \Rightarrow |x|$ not differentiable.

(ii) $|x-a|$ is not differentiable at $x=a$

(iii) $|ax+b|$ is not differentiable at $x=-b/a$

(GATE-95): The function $f(x) = |x+1|$ on the interval $[-2,0]$

Sol: $|x+a|$ is continous everywhere.

$|x+a|$ is not differentiable at $x=-a$.

$\therefore\ |x+1|$ is not differentiable at $x=-1 \in [-2,0]$

Ans: continous, but not differentiable.

(GATE-07 IN): The function $f(x) = |x|^3$, at $x=0$ is = (where x is real)

(a) continous but not differentiable

(b) once differentiable but not twice

✓(c) twice " " thrice (d) thrice differentiable

Sol:

$$f(x) = |x|^3 = \begin{cases} x^3 ; & x>0 \\ -x^3 ; & x<0 \\ 0 ; & x=0 \end{cases}$$

$f(0)=0$, LHL = RHL = 0 $\Rightarrow$ continous.

$$LHD = \lim_{h\to 0} \frac{f(0-h)-f(0)}{h} = \lim_{h\to 0} \frac{|-h|^3-0}{-h} = \lim_{h\to 0} \frac{-h^3}{-h} = \lim_{h\to 0} h^2 = 0$$

$$RHD = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|^3-0}{-h} = \lim_{h\to 0} -h^2 = 0$$

LHD = RHD $\Rightarrow$ differentiable.

$$f'(x) = \begin{cases} 3x^2 ; & x>0 \\ -3x^2 ; & x<0 \\ 0 ; & x=0 \end{cases} \qquad f''(x) = \begin{cases} 6x ; & x>0 \\ -6x ; & x<0 \\ 0 ; & x=0 \end{cases} \qquad f'''(x) = \begin{cases} 6 ; & x>0 \\ -6 ; & x<0 \\ \text{Not diff'ble} ; & x=0 \end{cases}$$

$\therefore$ f(x) is twice differentiable but not thrice.

(GATE-10 ME): The function $y = |2-3x|$

Sol:

$y = |2-3x|$ continous $\forall\ x \in R$

$y = |2-3x|$ not differentiable at $x = 2/3$ $\qquad 2-3x=0$, $x = 2/3$

Derivatives of Some functions:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	e^x	e^x	$\sec x$	$\sec x \cdot \tan x$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\log_e^x$	$1/x$	$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cdot \cot x$
$\frac{1}{x^n}$	$\frac{-n}{x^{n+1}}$	$\sin x$	$\cos x$	$\sin^{-1}x$ (or) $-\cos^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
		$\cos x$	$-\sin x$	$\tan^{-1}x$ (or) $-\cot^{-1}x$	$\frac{1}{1+x^2}$
$\frac{1}{\sqrt{x}}$	$\frac{-1}{2x\sqrt{x}}$	$\tan x$	$\sec^2 x$		
a^x	$a^x \log_e a$	$\cot x$	$-\mathrm{cosec}^2 x$	$\sec^{-1}x$ (or) $-\mathrm{cosec}^{-1}x$	$\frac{1}{x\sqrt{x^2-1}}$

Note: (1) $(uv)' = uv' + u'v$

(2) $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$

(3) $\frac{d}{dx}[f(g(x))] = f'(g(x)).g'(x)$

(4) Leibniz formula: $(f.g)^n = nc_0 f^n g + nc_1 f^{n-1}.g' + \dots + nc_n f.g^n$

where 'n' is order of differentiation.

L'Hospital rule: If $\underset{x\to a}{lt} \frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\underset{x\to a}{lt} \frac{f(x)}{g(x)} = \underset{x\to a}{lt} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

Note: 1. In case of other interminate (de) forms $0.\infty$, 0^0, ∞^0, 1^∞, $\infty-\infty$, first we have to reduce it to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and use this rule

2. For the forms 0^0, ∞^0, 1^∞ take logarithm and then take limits

3. $0^\infty = 0$, $\infty^\infty = \infty$, $\infty.\infty = \infty$, $\infty+\infty = \infty$, $\infty^{-\infty} = 0$ are not indeterminate

Ex: $\underset{x\to 0}{lt} \frac{\ln(1+x)}{\sin x} = \frac{0}{0}$ form

Apply L'H rule $= \underset{x\to 0}{lt} \frac{1/(1+x)}{\cos x} = \frac{1}{1} = 1$

Ex: $\underset{x\to 0}{lt} x^x = 0^0$ form   Let $y = x^x$

Apply logarithm first $\ln y = \ln x^x = x \ln x$

$$\underset{x\to 0}{lt} \ln y = \underset{x\to 0}{lt} x.\ln x$$

$$\ln\left[\underset{x\to 0}{lt} y\right] = \underset{x\to 0}{lt} \frac{\ln x}{1/x} = \frac{\infty}{\infty} \text{ Apply L'H}$$

$$= \underset{x\to 0}{lt} \frac{1/x}{-1/x^2} = -\underset{x\to 0}{lt} x = 0$$

$$\underset{x\to 0}{lt} y = \underset{x\to 0}{lt} x^x = e^0 = 1$$

(GATE-93 ME): $\underset{x\to 0}{lt} \frac{x(e^x-1) + 2(\cos x - 1)}{x(1-\cos x)}$ $\frac{0}{0}$

sol: using L'H rule

$$\underset{x\to 0}{lt} \frac{(e^x-1) + x.e^x + 2(-\sin x)}{(1-\cos x) + x(\sin x)} \quad \frac{0}{0}$$

Again apply L'H rule.

$$= \lim_{x \to 0} \frac{e^x + e^x + xe^x - 2\cos x}{\sin x + \sin x + x\cos x} \quad \frac{0}{0}$$

Again apply L'H rule

$$= \lim_{x \to 0} \frac{e^x + e^x + e^x + xe^x + 2\sin x}{\cos x + \cos x + \cos x - x\sin x} = \frac{1+1+1+0}{1+1+1-0} = 1$$

(GATE-95 CS): $\lim_{x \to \infty} \frac{x^3 - \cos x}{x^2 + (\sin x)^2} \quad \frac{\infty}{\infty}$

Sol: Apply L' Hospital

$$\lim_{x \to \infty} \frac{3x^2 + \sin x}{2x + 2\sin x \cdot \cos x} = \lim_{x \to \infty} \frac{3x + \frac{\sin x}{x}}{2 + \frac{\sin 2x}{x}} = \frac{3 \times \infty + 0}{2 + 0} = \infty$$

(GATE-99 IN): $\lim_{x \to 0} \frac{1}{10} \cdot \frac{1 - e^{-j5x}}{1 - e^{-jx}} \quad \frac{0}{0}$

Sol: Apply L' Hospital

$$\lim_{x \to 0} \frac{1}{10} \cdot \frac{-(-5j)e^{-j5x}}{-(-j) \cdot e^{-jx}} = \frac{5j}{10j} = 1/2$$

(GATE-99): $\lim_{x \to a} (x-a)^{x-a} \quad 0^0$

Sol: Let $y = (x-a)^{(x-a)} \Rightarrow \log y = (x-a)\log(x-a)$

$$\lim_{x \to a} \log y = \lim_{x \to a} [(x-a)\log(x-a)]$$

$$\log\left[\lim_{x \to a} y\right] = \lim_{x \to a} \frac{\log(x-a)}{1/(x-a)} \quad \frac{\infty}{\infty}$$

Apply L'H rule $= \lim_{x \to a} \frac{1/x-a}{-1/(x-a)^2} = \lim_{x \to a} -(x-a) = 0$

$$\lim_{x \to a} y = \lim_{x \to a} (x-a)^{(x-a)} = e^0 = 1$$

(GATE-07 PI): $\lim_{x \to \pi/4} \frac{\cos x - \sin x}{x - \pi/4} \quad \frac{0}{0}$

Sol: Apply L'H rule $= \lim_{x \to \pi/4} \frac{-\sin x - \sin x}{1} = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$

(GATE-12 ME, PI): $\lim_{x \to 0} \left(\frac{1 - \cos x}{x^2}\right) \quad \frac{0}{0}$

Sol: Apply L'H rule $= \lim_{x \to 0} \frac{\sin x}{2x} \quad \frac{0}{0}$

Again apply L'H $= \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$

(GATE-14 ME): $\lim_{x \to 0} \frac{x - \sin x}{1 - \cos x} \quad \frac{0}{0}$

Sol: Apply L'H rule

$= \lim_{x\to 0} \frac{1-\cos x}{\sin x} \quad \frac{0}{0}$.

again apply L'H rule $= \lim_{x\to 0} \frac{\sin x}{\cos x} = 0$

(GATE-14 ME): $\lim_{x\to 0} \left(\frac{e^{2x}-1}{\sin(4x)} \right) \quad \frac{0}{0}$

Sol: Apply L'H $= \lim_{x\to 0} \frac{2.e^{2x}}{4.\cos 4x} = \frac{2}{4} = \frac{1}{2}$

** (GATE-14 CE): $\lim_{a\to 0} \frac{x^a - 1}{a} = \frac{0}{0}$

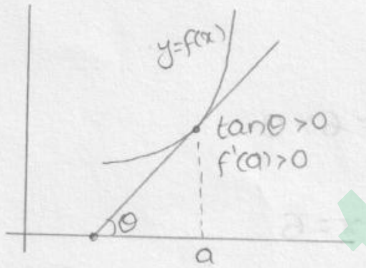
Sol: Apply L'H differentiate w.r.to a

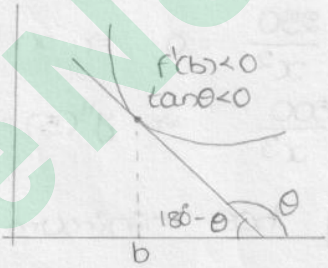
$= \lim_{a\to 0} \frac{x^a.\log x - 0}{1} = x^0.\log x = \log x$.

Increasing (or) decreasing nature of function:

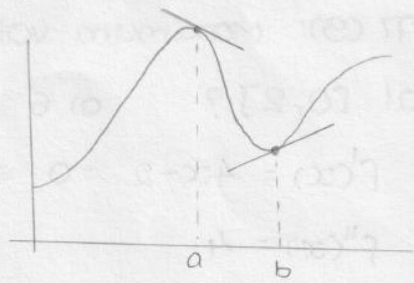
A function $y = f(x)$ will have increasing nature at $x=a$, if $f'(a)>0$

A function $y = f(x)$ will have decreasing nature at $x=b$ if $f'(b)<0$





Local
Maxima and minima:



To obtain the maxima or minima

1. Find $f'(x)$ and solve the equation $f'(x)=0$ to obtain the stationary points say $x = a, b, c$.
2. Find $f''(x)$
3. If $f''(a)<0$, then at $x=a$. $f(x)$ will have maximum value
   If $f''(b)>0$, then at $x=b$. $f(x)$ will have minimum value
   If $f''(c)=0$, then at $x=c$. $f(x)$ won't have minimum/maximum. The point is called saddle point.

Absolute maxima/minima: If $f(x)$ defined in interval $[a,b]$ then

Absolute minimum value = min { f(a), f(b), all local minimum values }

Absolute maximum value = max { f(a), f(b), all local maximum values }

If stationary points are out of given interval then don't consider them

Ex: The function $f(x) = x^3 - 9x^2 + 24x - 12$ has

Sol: $f'(x) = 3x^2 - 18x + 24 = 0$ $\quad f''(x) = 6x - 18$

$x = 2, 4$ $\quad f''(2) = -6 < 0$

$f''(4) = 6 > 0$

$\therefore f(x)$ has Max. at $x = 2$, Min at $x = 4$.

(GATE-07 EC)

Ex: The maximum value of $f(x) = x^2 - x - 2$ in the interval $[-4, 4]$ is

✓(a) 18 (b) 10 (c) -2.25 (d) indeterminate

Sol: $f'(x) = 2x - 1 = 0 \Rightarrow x = 1/2$

$f''(x) = 2$ can't find

$f(-4) = 18$ $\quad f(4) = 10$ $\quad f(1/2) = -2.25$

$\therefore f(x)$ has maximum at $x = -4$.

(GATE-94): The function $y = x^2 + \frac{250}{x}$ at $x = 5$ attains

Sol: $y' = 2x - \frac{250}{x^2} = 0 \Rightarrow x = 5$

$y'' = 2 + \frac{500}{x^3} \Rightarrow y''(5) = 2 + 4 = 6 > 0$

$\therefore$ y has minimum at $x = 6$

(GATE-97 CS): Maximum value of function $f(x) = 2x^2 - 2x + 6$ in the interval $[0, 2]$? a) 6 ✓b) 10 c) 12 d) 5.5 → $f(1/2)$

Sol: $f'(x) = 4x - 2 = 0 \Rightarrow x = 1/2$

$f''(x) = 4$

$f(0) = 0 + 6 = 6$ $\quad f(2) = 8 - 4 + 6 = 10$

$\therefore$ Maximum value $= 10$ in $[0, 2]$

(GATE-99 CE): Number of inflection points for the curve $y = 2x^4 + x$

Sol: $\frac{d^2y}{dx} = 0 \Rightarrow 8 \times 3x^2 = 0$

$\Rightarrow x = 0 \Rightarrow y = 0$

$\therefore$ (0,0) is the only inflection point.

(GATE-05 EE): For the function $f(x) = x^2 e^{-x}$, the max. occurs when $x = \underline{?}$

Sol: $f'(x) = e^{-x}[-x^2 + 2x] = 0 \Rightarrow x = 0, 2$ $\quad f''(0) = 2 > 0$

$f''(x) = -e^{-x}[-x^2 + 2x] + e^{-x}[-2x + 2]$ $\quad f''(2) = -2e^{-2} < 0$

$\therefore f(x)$ has max. at $x = 2$

(GATE-07 EE): The function $f(x) = (x^2-4)^2$ has (x is real number)

(a) only one minimum ✓(b) only two minima (c) Three minima (d) Three maxima.

Sol: $f'(x) = 2(x^2-4)\cdot 2x = 0 \Rightarrow x = \pm 2, 0$

$f''(x) = 4(3x^2-4)$

$f''(0) = -16 < 0$ $\quad f''(2) = 32 > 0$ $\quad f''(-2) = 32 > 0$

$\therefore f(x)$ has minimum at $x = \pm 2$

(GATE-08 CS): The number of distinct extrema for the curve $3x^4 - 16x^3 + 24x^2 + 37$ is

Sol: $f'(x) = 12x^3 - 48x^2 + 48x = 0 \Rightarrow x = 0, 2, 2$

$f''(x) = 36x^2 - 96x + 48$

$f''(0) = 48 > 0$ $\quad f''(2) = 0$ No extrema

$\therefore f(x)$ has only one extrema (minimum) at $x = 0$

(GATE-08 EC): For real values of 'x', the minimum value of function $f(x) = e^x + e^{-x}$ is

Sol: $f(x) = e^x + e^{-x}$ where $x \in R$

$f'(x) = e^x - e^{-x} = 0 \Rightarrow e^x = e^{-x} \Rightarrow x = 0$

$f''(x) = e^x + e^{-x} \Rightarrow f''(0) = 1 + 1 = 2 > 0$

$\therefore f(x)$ has Minimum at $x = 0$, Minimum value $f(0) = 2$

(GATE-10 EC): If $e^y = x^{1/x}$ then y has a

Sol: $e^y = x^{1/x} \Rightarrow y = \log x^{1/x} = \dfrac{\log x}{x}$

$y' = \dfrac{1 - \log x}{x^2} = 0 \Rightarrow x = e$

$y'' = \dfrac{x^2(-1/x) - (1-\log x)x}{x^4}$

$y''(e) = -e/e^4 < 0 \Rightarrow y$ has maximum at $x = e$

(GATE-12 EC, EE, IN): The maximum value of $f(x) = x^3 - 9x^2 + 24x + 5$ in the interval $[1, 6]$

Similar Ques. in GATE-14 EC

Sol: $f'(x) = 3x^2 - 18x + 24 = 0 \Rightarrow x = 2, 4$

$f''(x) = 6x - 18$ $\quad f''(2) = -6 < 0$ $\quad f'(4) = 6 > 0$

↳ local maximum at $x = 2$

Local maximum = f(2) = 25

In the interval [1, 6] $f(1) = 21$ $f(6) = 41$

$\therefore$ Absolute maximum = max { f(1), f(6), local max. }

= max { 21, 41, 25 } = 41.

(GATE-14 EC): For $0 \le t < \infty$ the maximum value of the function $f(t) = e^{-t} - 2e^{-2t}$ occurs at

sol: $f'(t) = (-e^{-t} + 4e^{-2t}) = 0 \Rightarrow e^{-t} = 1/4$

$\Rightarrow t = \log_e 4$

$f''(t) = (e^{-t} - 8e^{-2t})$

$f''(\log_e 4) = e^{-\log_e 4} - 8e^{-2\log_e 4} = \frac{1}{4} - 8 \cdot \frac{1}{16} = -1/4 < 0$

$\therefore$ f(t) has maximum value at $t = \log_e 4$

(GATE-14 EC): The max value of function $f(x) = \ln(1+x) - x$ (where $x > -1$) occurs at $x =$

sol: $f'(x) = \frac{1}{1+x} - 1 = 0 \Rightarrow x = 0$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1 < 0$

$\therefore$ f(x) has maximum at $x = 0$

(GATE-14 EE): The max. value of the function $f(x) = xe^{-x}$ in the interval $(0, \infty)$ is

sol: $f'(x) = (-x \cdot e^{-x} + e^{-x}) = 0 \Rightarrow x = 1$

At $x = 1$, $f''(x) < 0$

$\therefore$ Maximum exists at $x = 1$ and $f(1) = 1 \cdot e^{-1} = 1/e$

(GATE-14 EE): Minimum of the real valued function $f(x) = (x-1)^{2/3}$ occurs at $x =$

sol: $f(x) = [(x-1)^{1/3}]^2$ $\therefore f(x) \ge 0$ always

The minimum value of f(x), $f(x) = 0$

$(x-1)^{2/3} = 0 \Rightarrow x = 1$

(GATE-14 EE): The minimum value of $f(x) = x^3 - 3x^2 - 24x + 100$ in interval $[-3, 3]$ is

sol: $f'(x) = 3x^2 - 6x - 24 = 0 \Rightarrow x = -2, 4$

but $x = 4 \notin [-3, 3]$

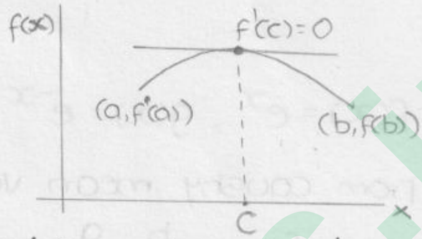
$f''(x) = 6x - 6 \Rightarrow f''(-2) = -18 < 0$ we get maximum

$\therefore$ Absolute minimum $= \min\{f(-3), f(3)\}$ $\quad f(-3) = 118 \quad f(3) = 28$

$= \min\{118, 28\} = 28$

MEAN VALUE THEOREMS:

Rolle's theorem: If $f(x)$ is a function defined in an interval $[a,b]$ such that

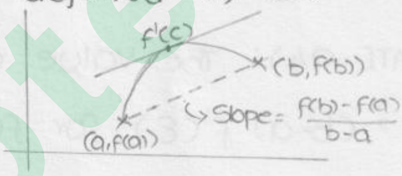


(1) $f(x)$ is continous in $[a,b]$

(2) $f'(x)$ exists in (a,b)

(3) $f(a) = f(b)$ then there exist $c \in (a,b)$ such that $f'(c) = 0$

Langranges theorem: If $f(x)$ is a function defined in an interval $[a,b]$ such that



(1) $f(x)$ is continous in $[a,b]$

(2) $f'(x)$ is exists in (a,b) then there exist $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a} \qquad \min_{a \le x \le b} f'(x) \le \frac{f(b)-f(a)}{b-a} \le \max_{a \le x \le b} f'(x)$$

cauchy's mean value theorem: If $f(x), g(x)$ are two functions defined in an interval such that

(1) $f(x), g(x)$ are continous in $[a,b]$

(2) $f'(x), g'(x)$ exists in (a,b)

(3) $g'(x) \neq 0 \ \forall \ x \in (a,b)$ then there exist $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

prob: The value c of mean value theorem for $f(x) = x^2 - 5x + 6$ in the interval $[2,3]$ is

Sol: $f(2) = 4 - 10 + 6 = 0$

$f(3) = 9 - 15 + 6 = 0$

From rolle's theorem $\exists \ c \in (2,3)$ such that $f'(c) = 0$

$\Rightarrow 2c - 5 = 0 \Rightarrow c = 5/2$

prob: $f(x) = x^3 - 4x^2 + 4x$ in the interval $[0,2]$ is

$f(0) = 0 \qquad f(2) = 8 - 16 + 8 = 0$

$\therefore$ From rolle's theorem $\exists \ c \in (0,2)$ such that $f'(c) = 0$

$3c^2-8c+4=0 \Rightarrow c=2,\ 2/3 \qquad c=2 \notin (0,2)$

$c=2/3 \in (0,2)$

prob: $f(x)=1+x^2$ in $[1,2]$ is

$f(1)=2 \qquad f(2)=5 \qquad f(a)\neq f(b)$

from langranges theorem $f'(c)=\frac{5-2}{1}=3$

$2c=3 \Rightarrow c=3/2 \in (1,2)$

prob: $f(x)=e^x,\ g(x)=e^{-x}$ in $[a,b]$

sol: From caushy mean value theorem

$$\frac{e^c}{-e^{-c}}=\frac{e^b-e^a}{e^{-b}-e^{-a}} \Rightarrow -e^{2c}=-e^{a+b}$$

$$\Rightarrow 2c=a+b \Rightarrow c=\frac{a+b}{2} \in (a,b)$$

(GATE-94): The value of ε in the mean value theorem of $f(b)-f(a) = (b-a)f'(\varepsilon)$ for $f(x)=Ax^2+Bx+C$ in (a,b) is

sol: $f'(\varepsilon)=\frac{f(b)-f(a)}{b-a}$

$$2A\varepsilon+B=\frac{Ab^2+Bb+C-Aa^2-Ba-C}{b-a}$$

$$2A\varepsilon+B=A(b+a)+B$$

By comparing

$$2\varepsilon=a+b \Rightarrow \varepsilon=\frac{a+b}{2}$$

(GATE-95): If $f(0)=2$ and $f'(x)=\frac{1}{5-x^2}$, then the lower and upper bounds of $f(1)$ estimated by the mean value theorem are

sol: Let $f(x)$ be defined in $[0,1]$ by lagrange's M.V.T

$\exists\, c\in(0,1)$ such that $f'(c)=\frac{f(1)-f(0)}{1-0}$

$$\frac{1}{5-c^2}=\frac{f(1)-2}{1}$$

We know that $\text{Min}\{f'(x)\} < f'(c) < \text{Max}\{f'(x)\} \qquad c\in(0,1)$

$$\frac{1}{5} < f(1)-2 < \frac{1}{4} \Rightarrow 2.2 < f(1) < 2.25$$

TAYLOR SERIES: The Taylor series of $f(x)$ about $x=a$ is given by

$$f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\cdots$$

Maclaurin's Series: put $a=0$

$$f(x)=f(0)+x\cdot f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots$$

Important expansions:

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
4. $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$
5. $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$
6. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

prob: In the taylor series of $f(x) = e^x$ about $x = 2$ the co.eff of $(x-2)^4$ is

sol: $$= \frac{1}{4!} f^{IV}(2) = \frac{1}{4!} e^x \Big|_{x=2} = \frac{e^2}{24}$$

prob: In the taylor series of $f(x) = e^x + \sin x$ about $x - \pi$ the co. eff of $(x-\pi)^2$ is

sol: $$= \frac{1}{2!} f''(\pi) = \frac{1}{2!} \left[e^x - \sin x \right]\Big|_{x=\pi} = \frac{e^\pi}{2}$$

prob: The linear approximation for e^{-x} around $x = 2$ is

sol: $$f(x) = f(2) + (x-2) f'(2)$$
$$= e^{-2} + (x-2)(-e^{-2}) = e^{-2}(3-x)$$

(GATE-00 CE): The Taylor series expansion of $\sin x$ about $x = \pi/6$ is

sol: $$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$f(x) = \sin x$ and $a = \pi/6 \Rightarrow \sin(\pi/6) = 1/2 = f(a)$

$f'(x) = \cos x \Rightarrow f'(a) = \cos \pi/6 = \sqrt{3}/2$

$f''(x) = -\sin x \Rightarrow f''(a) = -\sin \pi/6 = -1/2$

$$\sin x = \frac{1}{2} + \left(x - \pi/6\right) \sqrt{3}/2 + \frac{(x - \pi/6)^2}{2!}\left(-\frac{1}{2}\right) + \cdots$$
$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 + \cdots$$

(GATE-01 CE): Limit of the following series as x approaches $\pi/2$ is $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin x$

sol: $$\lim_{x \to \pi/2} \sin x = \sin \pi/2 = 1$$