Random Variables : A Real Variable X whose Value is determined by the outcome of a random experiment is Called a random Variable. A random Variable X Can also be regarded as a real-Value function defined on the Sample Space S of a random experiment Such that for each point x of the sample space, f(x) is the probability of occurrence of the event represented by x.

Ex: Consider a random experiment Consisting of tossing a coin twice. The Sample Space $S = \{s_1, s_2, s_3, s_4\}$ where $s_1 = HH$, $s_2 = HT$, $s_3 = TH$ and $s_4 = TT$ Consists of four elements (Sample points). Define a function $X : S \to R$ by $X(s) =$ " number of heads". Then

$$X(s_1) = 2, \quad X(s_2) = 1, \quad X(s_3) = 1 \text{ and } X(s_4) = 0$$

$$\text{Range of } X = \{X(s) : s \in S\}$$

$$= \{X(s_1), X(s_2), X(s_3), X(s_4)\}$$

$$= \{0, 1, 2\}$$

Types of Random Variables : Random Variable is of two types.

(i) Discrete Random Variable (ii) Continuous Random Variable

(i) Discrete Random Variable : A Random Variable X which Can take only a finite number of discrete Values in an interval of domain is Called a discrete Random Variable. In other words, if the random Variable takes the Values only on the set $\{0, 1, 2, ----n\}$ is Called a discrete random Variable.

Ex: $X(s) = \{s : s = 0, 1, 2\}$ (or) Range of $X = \{0, 1, 2\}$

The Random Variable X is a discrete random Variable

...ction :

② Continuous Random Variable : A Random Variable X which Can take Values Continuously i.e, which takes all possible values in a given interval is Called a Continuous random Variable.

Ex: The height, age, weight of individuals are examples of Continuou random Variable. Also temperature & time are Continuous random Variables

Probability Function of a Discrete Random Variable :
If for a discrete random Variable X, the real valued function $p(x)$ is such that $p(X=x) = p(x)$ then $p(x)$ is Called probability function (or) probability density function of a discrete random Variable X. probability function $p(x)$ gives the measure of probability for different values of X.

properties of a probability Function : If $p(x)$ is a probability functio of a random Variable X, then it possesses the following properties.

1. $p(x) \geq 0$ for all x
2. $\sum p(x) = 1$, Summation is taken over for all values of x.
3. $p(x)$ Can not be negative for any value of x.

Probability Distribution Function : Let X be a random Variable. Then the probability distribution function associatated with X is defined as the probability that the outcome of an experiment will be one of the outcomes for which $X(s) \leq x$, $x \in R$. That is, the function $F(x)$ or $F_X(x)$ defined by

$$F_X(x) = P(X \leq x)$$
$$= P\{s : X(s) \leq x\}, \quad -\infty < x < \infty$$

is Called the distribution function of X.

Properties of Distribution Function :

1. If F is the distribution function of a random Variable X and if $a<b$, then

(i) $P(a < X \leq b) = F(b) - F(a)$

(ii) $P(a \leq X \leq b) = P(X=a) + [F(b) - F(a)]$

(iii) $P(a < X < b) = [F(b) - F(a)] - P(X=b)$

(iv) $P(a \leq X < b) = [F(b) - F(a)] - P(X=b) + P(X=a)$

Note : If $P(X=a) = P(X=b) = 0$ then

$$P(a < X \leq b) = P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = F(b) - F(a)$$

2. All distribution functions are monotonically increasing & lie between 0 and 1. That is, if F is the distribution function of the random Variable X, then

(i) $0 \leq F(x) \leq 1$

(ii) $F(x) < F(y)$ when $x < y$

3. (i) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

(ii) $F(\infty) = \lim_{x \to \infty} F(x) = 1$

Discrete probability Distribution (probability mass function) : probability distribution of a random Variable is the set of its possible values together with their respective probabilities. Suppose X is a discrete random Variable with possible outcomes (values) $x_1, x_2, x_3, \ldots$. The probability of each possible outcome x_i is

$$P_i = P(X = x_i) = P(x_i) \quad \text{for } i = 1, 2, 3, \ldots$$

If the numbers $P(x_i)$, $i = 1, 2, 3, \ldots$ Satisfy two Conditions

(i) $P(x_i) > 0$ for all values of i; $0 < P_i \leq 1$

(ii) $\sum P(x_i) = 1$, $i = 1, 2, 3, \ldots$

(4)

Then, the function $p(x)$ is called the probability mass function of the random variable x and the set $\{p(x_i)\}$, $i = 1, 2, ----$ is called the discrete probability distribution of the discrete random variable x.

Probability Density Function: The probability density function $f_X(x)$ is defined as the derivative of the probability distribution function, $F_X(x)$ of the random variable X.

Thus $$f_X(x) = \frac{d}{dx}\left[F_X(x)\right]$$

Expectation, Mean, Variance & Standard Deviation of a probability Distribution

① Expectation of a Discrete Variable: As defined earlier, a discrete variable takes only some finite values like number on dice, number of children in a family etc. Suppose a random variable X assumes the values $x_1, x_2, ----x_n$ with respective probabilities $P_1, P_2, ----P_n$. Then the mathematical expectation or mean or expected value of X, denoted by $E(x)$, is defined as the sum of products of different values of X and the corresponding probabilities.

$$E(x) = x_1 P_1 + x_2 P_2 + x_3 P_3 + -- + x_n P_n$$

$$E(x) = \sum_{i=1}^{n} P_i x_i$$

Similarly $$E(x^r) = \sum_{i=1}^{n} P_i \cdot x_i^r$$

In General, the expected value of any function $g(x)$ of a random variable X is defined as

$$E|g(x)| = \sum_{i=1}^{n} P_i \, g(x_i)$$

Note: Expected value of X is a population mean. If population mean is μ then $E(x) = \mu$

(2) Mean: The mean value μ of the discrete distribution function is given by

$$\mu = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i = E(x)$$

Note: If $E(x) = \mu$, Then $E(x - \mu) = 0$

(3) Variance: Variance characterizes the variability in the distributions since two distributions with same mean can still have different dispersion of data about their means. Variance of the probability distribution of a random variable x is the mathematical expectation of $[X - E(x)]^2$. Then

$$Var(x) = E[x - E(x)]^2$$

$$Var(x) = \sum_{i=1}^{n} \{(x_i - E(x))^2 \times P(x_i)\}$$

(or)

$$Var(x) = E(x)^2 - [E(x)]^2$$

Note: The variance of a random variable X is also denoted by $V(x)$

(4) Standard Deviation: It is the positive square root of the variance.

$$S.D = \sigma = \sqrt{\sum_{i=1}^{n} p_i x_i^2 - \mu^2} = \sqrt{E(x)^2 - \mu^2} = \sqrt{E[x - E(x)]^2}$$

Continuous Probability Distribution: When a random variable X takes every value in an interval, it gives rise to continuous distribution of x. The distributions defined by the variates like temperature, heights and weights are continuous distributions.

Probability Density Function: For Continuous Variable, the probability distribution is called probability density function because it is defined for every point in the range & not only for certain values.

Consider the small interval $\left[x-\frac{dx}{2}, x+\frac{dx}{2}\right]$ of length dx round the point x. Let $f(x)$ be any continuous function of x so that $f(x)\,dx$ represents the probability that the variable X falls in the infinitesimal interval $\left[x-\frac{dx}{2}, x+\frac{dx}{2}\right]$. Symbolically it can be expressed as $P\left[x-\frac{dx}{2} \leq X \leq x+\frac{dx}{2}\right] = f(x)\,dx$. Then $f(x)$ is called the probability density function (or) simply density function of the variate X and the continuous curve $y=f(x)$ is known as the probability density curve (or) simply probability curve.

Properties of probability density function $f(x)$:

(i) $f(x) \geq 0, \ \forall x \in R$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii) The probability $P(E)$ is given by

$P(E) = \int_E f(x)\,dx$ is well defined for any event E

Note: In the case of continuous random variable, we associate the probabilities with intervals. In this case, the probability of the variable at a particular point is always zero.

Thus $P(a < X \leq b) = P(a \leq X < b) = P(a < X < b) = P(a \leq X \leq b) =$

$[F(b) - F(a)]$ $\qquad [\because P(X=a) = P(X=b) = 0]$

That is inclusion or non inclusion of end points, does not change the probability, which is not the case in the discrete distributions.

Cumulative Distribution Function of A Continuous Random Variable :

The Cumulative distribution function or simply the distribution function of a continuous random variable X is denoted by F(x) and is defined as

$$F(x) = P(X \le x) = \int_{-\infty}^{\infty} f(x)\,dx$$

Thus F(x) gives the probability that the value of variable X will be $\le x$.

properties of F(x) :

(i) $0 \le F(x) \le 1$, $\quad -\infty < x < \infty$

(ii) $F'(x) = f(x) \geq 0$, So that F(x) is non-decreasing function

(iii) $F(-\infty) = 0$

(iv) $F(\infty) = 1$

(v) F(x) is a Continuous function of x on the right

(vi) The discontinuties of F(x) are Countable

(vii) $P(a \le X \le b) = \int_a^b f(x)\,dx = F(b) - F(a)$

(viii) Since $F'(x) = f(x)$, we have $\frac{d}{dx}[F(x)] = f(x)$

$$dF = f(x).dx$$

This is known as probability differential of X

Measures of Central Tendency for Continuous probability Distribution

(i) Mean : Mean of a distribution is given by

$$\mu = E(x) = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

If X is defined from a to b, Then $\mu = E(X) = \int_a^b x f(x)\,dx$

In General, mean or expectation of any function $\phi(x)$ is given by $$E(\phi(x)) = \int_{-\infty}^{\infty} \phi(x) \cdot f(x)\,dx$$

(ii) Median : Median is the point which divides the entire distribution into two parts. In case of continuous distribution, median is the point which divides the total area into two equal parts. Thus if X is defined from a to b and M is the median, then

$$\int_{a}^{M} f(x)\,dx = \int_{M}^{b} f(x)\,dx = \frac{1}{2}$$

Solving for M, we get the median

(iii) Mode : Mode is the value of x for which f(x) is maximum. Mode is thus given by $f'(x) = 0$ and $f''(x) < 0$ for $a < x < b$

(iv) Variance : Variance of a distribution is given by

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \quad \text{(or)} \quad \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

Suppose that the Variable/variate X is defined from a to b. then

$$\sigma^2 = \int_{a}^{b} (x-\mu)^2 f(x)\,dx \quad \text{(or)} \quad \sigma^2 = \int_{a}^{b} x^2 f(x)\,dx - \mu^2$$

(v) Mean deviation : Mean deviation about the mean (μ) is given by

$$\int_{-\infty}^{\infty} |x-\mu|\, f(x)\,dx$$