

# 1 Introduction

## 1.1 Ideas

- explanations in constraint satisfaction problems **TODO:cite: xai special issue** easy, understandable, human-interpretable
- Explanations can be small, but also difficult (combinations of constraints/clues)
- **TODO:which complexity is extracting an OMUS ?**
- use a/different proxy(s) to qualify interpretability of an explanation
- extension SMUS to OMUS based on enumeration of optimal hitting set
- approach tested on Boolean Satisfiability instances and on a high level problem

## 1.2 Todo

- **TODO:which complexity is extracting an OMUS ?**

# 2 Background

The algorithm presented in this section is based on the key ideas and observations of Ignatiev et al. presented in [Ignatiev et al., 2015]. The algorithm is adapted to incorporate an optimality criterion in order to guide the search not in the direction of the SMUS, but towards the OMUS. To do so, we first define the objective function  $f$ :

**Definition 1.** Given a CNF Formula  $\mathcal{F}$ , let  $f : 2^{\mathcal{F}} \rightarrow \mathbb{R}$  be a mapping of a set of clauses to a real number.  $f$  is said to be a consistent objective function if for any subsets  $\mathcal{A}, \mathcal{B}$  of  $\mathcal{F}$  if  $\mathcal{A} \subseteq \mathcal{B}$  then  $f(\mathcal{A}) \leq f(\mathcal{B})$ . *function  $f$  is said to be set increasing if ...*

For an unsatisfiable CNF formula  $\mathcal{F}$ , a subset of clauses that are still unsatisfiable, but if any of the clauses are removed then the reduced formula becomes satisfiable. Formally, we define this set as a Minimum unsatisfiable Subset (MUS):

**Definition 2.** A subset  $\mathcal{U} \subseteq \mathcal{F}$  is a **minimal unsatisfiable subset (MUS)** if  $\mathcal{U}$  is unsatisfiable and  $\forall \mathcal{U}' \subset \mathcal{U}$ ,  $\mathcal{U}'$  is satisfiable. An MUS of  $\mathcal{F}$  with an optimal value w.r.t an objective function  $f$  is called an **optimal MUS (OMUS)**.

**Definition 3.** A subset  $\mathcal{C}$  of  $\mathcal{F}$  is an **minimal correction subset (MCS)** if  $\mathcal{F} \setminus \mathcal{C}$  is satisfiable and  $\forall \mathcal{C}' \subseteq \mathcal{C} \wedge \mathcal{C}' \neq \emptyset$ ,  $(\mathcal{F} \setminus \mathcal{C}) \cup \mathcal{C}'$  is unsatisfiable.

**Definition 4.** A satisfiable subset  $\mathcal{S} \subseteq \mathcal{F}$  is a **Maximal Satisfiable Subset (MSS)** if  $\forall \mathcal{S}' \subseteq \mathcal{F}$  s.t  $\mathcal{S} \subseteq \mathcal{S}'$ ,  $\mathcal{S}'$  is unsatisfiable.

An MSS can also be defined as the complement of an MCS (and vice versa). If  $\mathcal{C}$  is a MCS then  $\mathcal{S} = \mathcal{F} \setminus \mathcal{C}$  is a MSS. On the other hand, MUSes and MCSes are related by the concept of minimal hitting set.

**Definition 5.** Given a collection of sets  $\Gamma$  from a universe  $\mathbb{U}$ , a hitting set on  $\Gamma$  is a set such that  $\forall S \in \Gamma, h \cap S \neq \emptyset$ .

**Proposition 1.** Given a CNF formula  $\mathcal{F}$ , let  $MUSes(\mathcal{F})$  and  $MCSes(\mathcal{F})$  be the set of all MUSes and MCSes of  $\mathcal{F}$  respectively. Then the following holds:

1. A subset  $\mathcal{U}$  of  $\mathcal{F}$  is an MUS iff  $\mathcal{U}$  is a minimal hitting set of  $MCSes(\mathcal{F})$
2. A subset  $\mathcal{C}$  of  $\mathcal{F}$  is an MCS iff  $\mathcal{C}$  is a minimal hitting set of  $MUSes(\mathcal{F})$

### 3 Explanation Generation

For each node  $(E, \mathcal{S}, n)$  of the graph, we have a structure that contains four components:

- $n.state$  a partial interpretation
- $n.Parent$  parent explanation
- $n.Action$   $E$  (*facts used*),  $S$  (*Constraint used*)
- $n.Path-Cost$  the cost denoted by  $f(E, S, n) + \# \text{ elements until goal}$

### 4 OMUS Algorithm

Note that if we assign a unit weight to every element in the subset, we reduce the problem of finding an OMUS to finding a SMUS.

**Definition 6.** Let  $\Gamma$  be a collection of sets and  $HS(\Gamma)$  the set of all hitting sets on  $\Gamma$  and let  $f$  be an valid objective function. Then a hitting set  $h \in \Gamma$  is said to be an **optimal** hitting set if  $\forall h' \in HS(\Gamma)$  we have that  $f(h) \leq f(h')$ .

**Property 1.** The **optimal** hitting set of a collection of sets  $\Gamma$  is denoted by  $OHS(\Gamma)$ .

The algorithm is based on the following observation:

**Proposition 2.** A set  $\mathcal{U} \subseteq \mathcal{F}$  is an OMUS of  $\mathcal{F}$  if and only if  $\mathcal{U}$  is an optimal hitting set of  $MCSes(\mathcal{F})$

**Lemma 1.** Let  $\mathcal{K} \subseteq MCSes(\mathcal{F})$ . Then a subset  $\mathcal{U}$  of  $\mathcal{F}$  is an OMUS if  $\mathcal{U}$  is a optimal hitting set on  $\mathcal{K}$  and  $\mathcal{U}$  is unsatisfiable

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**Algorithm 1:** CSP-Explain( $\mathcal{T}$ ,  $f$  [,  $\mathcal{I}_0$ ])

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input      :  $\mathcal{T}$  set of constraints
input      :  $f$  a consistent objective function
optional input:  $\mathcal{I}_0$  a partial interpretation
output     : Explanation sequence
1 begin
  // State with current interpretation
2   $\mathcal{I} \leftarrow \mathcal{I}_0$  ;
  // Goal state
3   $\mathcal{I}_{end} \leftarrow \text{propagate}(\mathcal{I}, \mathcal{T})$  ;
  // nodes to expand
4   $\text{frontier} \leftarrow \text{Priority queue ordered by Path-Cost + cost to goal, with } \mathcal{I} \text{ as only element}$  ;
  // explanation sequence
5   $\text{Seq} \leftarrow \text{empty set}$  ;
6  while true ? do
7     $(E_n, \mathcal{S}_n, n) \leftarrow \text{POP}(\text{frontier})$  ;
8     $\text{nested} \leftarrow \text{nested-explanation}(E_n, \mathcal{S}_n, n)$ ;
9    append  $((E_n, \mathcal{S}_n, n), \text{nested})$  to  $\text{Seq}$  ;
10    $\mathcal{I} \leftarrow \mathcal{I} \cup n$  ;
11   if  $\mathcal{I} == \mathcal{I}_{end}$  then
12     return  $\text{Seq}$ 
13   end
  // for every subset of constraints
14  for  $S \subseteq \mathcal{T}$  ordered ascending by  $g(S)$  do
15     $\mathcal{J} \leftarrow \text{propagate}(\mathcal{I}, S)$  ;
  // For every newly derivable fact
16    for  $n_j \in \mathcal{J} \setminus \mathcal{I}$  do
17       $\neg n_j$  has to be in initialisation of H, as well as S ;
18       $X \leftarrow \text{OMUS}(\neg n_j \wedge \mathcal{I} \wedge S)$  ;
19       $E \leftarrow \mathcal{I} \cap X$ ;
20       $N \leftarrow \text{propagate}(E \wedge S)$  ;
21      for  $n' \in \mathcal{N}$  do
22        if  $n' \notin \text{frontier}$  then
23          add  $(E, \mathcal{S}, n')$  to frontier ;
24        else if  $n' \in \text{frontier}$  with higher Path-Cost then
25          replace previous explanation with  $(E, \mathcal{S}, n')$  in Frontier ;
26        end
27      end
28    end
29  end
30 end
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**Algorithm 2:** OMUS( $\mathcal{F}$ ,  $f$  [,  $\mathcal{H}_0$ ]) [Ignatiev et al., 2015]

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input           :  $\mathcal{F}$  a CNF formula
input           :  $f$  a cost function
optional input:  $\mathcal{H}_0$  initial collection of disjoint Minimum Correction Sets
output          : OMUS( $\mathcal{F}$ )
1 begin
2    $\mathcal{H} \leftarrow \mathcal{H}_0$  ;
3   while true do
4     // cost minimal hitting set w.r.t cost function f
4      $h \leftarrow \text{OHS}(\mathcal{H}, f)$ ;
5     // set with all unique clauses from hitting set
5      $\mathcal{F}' \leftarrow \{c_i | e_i \in h\}$  ;
6     if not SAT( $\mathcal{F}'$ ) then
7       | return OMUS  $\leftarrow \mathcal{F}'$  ;
8     end
9     // written as grow( $\mathcal{F}'$ ) which is Minimum Correction Set of  $\mathcal{F}'$ 
9     // find the biggest one with the biggest cost
9      $\mathcal{C} \leftarrow \mathcal{F} \setminus \mathcal{F}'$  ;
10     $\mathcal{H} \leftarrow \mathcal{H} \cup \{\mathcal{C}\}$  ;
11  end
12 end

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For the set of clauses  $\mathcal{C} = \{c_1, \dots, c_{|C|}\}$  with weights  $\mathcal{W} = \{w_1, \dots, w_{|C|}\}$  in the collection of sets  $\mathcal{H}$ . For Example:

$$\begin{aligned}
 \mathcal{C} &= \{c_1, \dots, c_6\} \\
 \mathcal{W} &= \{w_1 = 20, w_2 = 20, w_3 = 10, w_4 = 10, w_5 = 10, w_6 = 20\} \\
 \mathcal{H} &= \{c_3\}, \{c_2, c_4\}, \{c_1, c_4\}, \{c_1, c_5, c_6\}
 \end{aligned} \tag{1}$$

The optimal hitting set can be formulated as an integer linear program.

$$\min \sum_{i \in \{1..|C|\}} w_i \cdot x_i \tag{2}$$

$$\sum_{i \in \{1..|C|\}} x_i \cdot h_{ij} \geq 1, \forall j \in \{1..|\mathcal{H}|\} \tag{3}$$

$$x_i = \{0, 1\} \tag{4}$$

- $w_i$  is the input cost/weight associated with clause  $i$  in
- $x_i$  is a boolean decision variable if constraint/clause  $c_i$  is chosen or not.
- Equation 3,  $h_{ij}$  is a boolean input variable corresponding to if constraint/clause  $i$  is in set to hit  $j$ .

## 5 Future Work

## References

- [Ignatiev et al., 2015] Ignatiev, A., Previti, A., Liffiton, M., and Marques-Silva, J. (2015). Smallest mus extraction with minimal hitting set dualization. In *International Conference on Principles and Practice of Constraint Programming*, pages 173–182. Springer.