Definition 1. *A subset $\mathcal{U} \subseteq \mathcal{F}$ is **minimal unsatisfiable subset** (MUS) if $\mathcal{U}$* unsatisfiable *and $\forall \mathcal{U}' \subset \mathcal{U}$, $\mathcal{U}'$ is* satisfiable. *An MUS of $\mathcal{F}$ with an optimal value w.r.t an* objective function *f is called an **optimal MUS** (OMUS).*

Definition 2. *A subset $\mathcal{C}$ of $\mathcal{F}$ is an **minimal correction subset** (MCS) if $\mathcal{F} \setminus \mathcal{C}$ is* satisfiable *and $\forall \mathcal{C}' \subseteq \mathcal{C} \wedge \mathcal{C}' \neq \emptyset$, $(\mathcal{F} \setminus \mathcal{C}) \cup \mathcal{C}'$ is* unsatisfiable.

Definition 3. *A* satisfiable subset *$\mathcal{S} \subseteq \mathcal{F}$ is a **Maximal Satisfiable Subset** (MSS) if $\forall$ $\mathcal{S}' \subseteq \mathcal{F}'$ s.t $\mathcal{S} \subseteq \mathcal{S}'$, $\mathcal{S}'$ is* unsatisfiable.

An MSS can also be defined as the complement of an MCS (and vice versa). If $\mathcal{C}$ is a MCS then $\mathcal{S} = \mathcal{F} \setminus \mathcal{C}$ is a MSS. On the other hand, MUSes and MCSes are related by the concept of minimal hitting set.

Definition 4. *A* minimal correction subset *$\mathcal{C}$ of $\mathcal{F}$ is an **Optimal Correction Subset** of $\mathcal{F}$, OCS($\mathcal{F}$) if $\forall$ $\mathcal{C}' \in MCSes(\mathcal{F}') : f(\mathcal{C}) \leq f(\mathcal{C}')$.*

Definition 5. *A* maximal satisfiable subset *$\mathcal{S} \subseteq \mathcal{F}$ is an **Optimal Satisfiable Subset** (OSS) if $\forall$ $\mathcal{S}' \in MSSes(\mathcal{F}') : f(\mathcal{S}) \leq f(\mathcal{S}')$.*

Proof. Let $\mathcal{C}$ be the collection of MCSes($\mathcal{F}$), $\mathcal{S}$ be the collection of MSSes($\mathcal{F}$), we know that

- $|\mathcal{C}| = |\mathcal{S}|$ and,
- $\forall$ $i \in \{1..|\mathcal{C}|\} : \mathcal{S}_i = \mathcal{F} \setminus \mathcal{C}_i$

Let $\mathcal{S}*$ be the OSS,

Let $\mathcal{C}*$ be the OCS, □

Lemma 1. *Given a CNF formula $\mathcal{F}$, let OMUSes($\mathcal{F}$) and OCSes($\mathcal{F}$) be the set of all OMUSes and OCSes of $\mathcal{F}$ respectively. Then the following holds:*

1. *A subset $\mathcal{U}$ of $\mathcal{F}$ is an OMUS iff $\mathcal{U}$ is an optimal hitting set of OCSes($\mathcal{F}$)*
2. *A subset $\mathcal{C}$ of $\mathcal{F}$ is an OCS iff $\mathcal{U}$ is an optimal hitting set of OMUSes($\mathcal{F}$)*

Definition 6. *Let Γ be a collection of sets and MHS(Γ) the set of all minimal hitting sets on Γ and let f be an objective function with input a set of constraints. Then a hitting set $h \in \Gamma$ is said to be an **optimal** hitting set if $\forall$ $h' \in OHS(\Gamma)$ we have that $f(h) \leq f(h')$ [Davies and Bacchus, 2011].*

Property 1. *The **optimal** hitting set of a collection of sets Γ is denoted by $OHS(\Gamma)$.*

Proposition 1. *A set $\mathcal{U} \subseteq \mathcal{F}$ is an OMUS of $\mathcal{F}$ if and only if $\mathcal{U}$ is an optimal hitting set of MCSes($\mathcal{F}$)*

Lemma 2. *Let $\mathcal{K} \subseteq$ MCSes($\mathcal{F}$). Then a subset $\mathcal{U}$ of $\mathcal{F}$ is an OMUS if $\mathcal{U}$ is a optimal hitting set on $\mathcal{K}$ and $\mathcal{U}$ is unsatisfiable*

Proof Lemma 2

- Is an OCS obligatory an MCS ?
- Does an OCS have to be optimal and minimal ?
- How is MSS afected ?

Algorithm 1: OMUS($\mathcal{F}, f_o, \mathcal{H}_0$) [Ignatiev et al., 2015]

```
input          : F a CNF formula
input          : f a cost function
optional input: H_0 initial set of disjoint Minimum Correction Sets
begin
    H ← H_0 ;
    while true do
        // cost minimal hitting set w.r.t cost function f
        h ← OHS(H, f);
        // set with all clauses from hitting set
        F' ← {c_i | e_i ∈ h} ;
        if not SAT(F') then
            return OMUS ← F' ;
        end
        // written as grow(F') which is Minimum Correction Set of F'
        // find the biggest one with the biggest cost
        C ← F \ grow(F', F, f) ;
        H ← H ∪ {C} ;
    end
end
```

Proof 1.

- use the cost of the constraints
- if cost $\leq$ ... and cost $\geq$ ... then it means cost ....
- Use the min cost of OCS and OMUS

Proof 1.1 $\Rightarrow$ *A subset $\mathcal{U}$ of $\mathcal{F}$ is an OMUS $\Rightarrow$ $\mathcal{U}$ is an optimal hitting set of OCSes($\mathcal{F}$)*

Proof 1.2 $\Leftarrow$ *A subset $\mathcal{U}$ of $\mathcal{F}$ is an OMUS $\Leftarrow$ $\mathcal{U}$ is an optimal hitting set of OCSes($\mathcal{F}$)*

Proof 2.

Proof 2.1 $\Rightarrow$ *A subset $\mathcal{C}$ of $\mathcal{F}$ is an OCS $\Rightarrow$ $\mathcal{U}$ is an optimal hitting set of OMUSes($\mathcal{F}$)*

Proof 2.2 $\Leftarrow$ *A subset $\mathcal{C}$ of $\mathcal{F}$ is an OCS $\Leftarrow$ $\mathcal{U}$ is an optimal hitting set of OMUSes($\mathcal{F}$)*

For the set of clauses $\{c_1, ...c_{|C|}\}$ in the collection of sets $\mathcal{H}$. For Example:

$$\{c_3\},\ \{c_2, c_4\},\ \{c_1, c_4\},\ \{c_1, c_5, c_7\}$$

The optimal hitting set can be formulated as an integer linear program.

$$min_x \sum_{i \in \{1..|C|\}} c_i \cdot x_i \tag{1}$$

$$\sum_{i \in \{1..|C|\}} x_i \cdot w_{ij} \geq 1,\ \forall j \in \{1..|hs|\} \tag{2}$$

$$x_i = \{0, 1\} \tag{3}$$

$$w_{ij} = \{0, 1\} \tag{4}$$

- c_i is the cost associated with clause i in
- x_i is a boolean decision variable if constraint/clause c_i is chosen or not.
- Equation 2 is a boolean decision variable if constraint/clause i is in hitting set j.

References

[Davies and Bacchus, 2011] Davies, J. and Bacchus, F. (2011). Solving maxsat by solving a sequence of simpler sat instances. In *International conference on principles and practice of constraint programming*, pages 225–239. Springer.

[Ignatiev et al., 2015] Ignatiev, A., Previti, A., Liffiton, M., and Marques-Silva, J. (2015). Smallest mus extraction with minimal hitting set dualization. In *International Conference on Principles and Practice of Constraint Programming*, pages 173–182. Springer.