Calculating the pseudoinverse

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This work is only a clarification and correction of [1].

1 Method

1.1 Regular method

If $A \in \mathcal{M}_{m,n}(\mathbb{R})$, then we calculate A^+ the following way:

$$\begin{pmatrix} A & I_m \\ I_n & * \end{pmatrix} \underset{M \mapsto PMQ}{\overset{\text{Gauss}}{\longrightarrow}} \begin{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} * \\ V \end{pmatrix} \\ \begin{pmatrix} * & U \end{pmatrix} & * \end{pmatrix}$$

$$\begin{pmatrix} A & V^\tau \\ U^\tau & 0 \end{pmatrix}^{-1} = \begin{pmatrix} A^+ & * \\ * & * \end{pmatrix}$$

Note that $*$ denotes some irrelevant data, and that P and Q operate only on rows and columns that cross A.

1.2 Symmetric matrices

With a symmetric matrix S:

$$\begin{pmatrix} S & I_n \end{pmatrix} \underset{M \mapsto PMQ}{\overset{\text{Gauss}}{\longrightarrow}} \begin{pmatrix} \begin{pmatrix} I_r & * \\ & 0 \end{pmatrix} & \begin{pmatrix} * \\ V \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} S & V^\tau \\ V & 0 \end{pmatrix}^{-1} = \begin{pmatrix} S^+ & * \\ * & * \end{pmatrix}$$

Note that the calculation of the inverse in the second step can be done with only half the matrix, given that it still is symmetric.

For a usual matrix A, we can use $A^+ = A^\tau (AA^\tau)^+$ as a way to speed up the computation compared to the regular calculations when A is very flat ($n \gg m$ or $n \ll m$). However, we won't use it because of the speed-up we can get in the regular method by discarding the left side of Q' or the top side of P' (see 2.2).

2 Demonstrations

The symmetric case comes directly from the regular case.

The first transformation can be done as $M \mapsto PMQ$ where P operates on the lines, and Q is only a permutation of the columns.

The second transformation implies that the matrix on the left is invertible.

2.1 Invertible matrix

Let's demonstrate this last fact. We know that:

$$\operatorname{rk}\begin{pmatrix} A & I_m \\ I_n & * \end{pmatrix} = n+m \quad \text{and} \quad \operatorname{rk}\begin{pmatrix} A & I_m \\ I_n & * \end{pmatrix} = \operatorname{rk}\begin{pmatrix} I_r & 0 & * \\ 0 & 0 & V \\ * & U & * \end{pmatrix}$$

Which means that:

$$\operatorname{rk}\begin{pmatrix} I_r & 0 & * \\ 0 & 0 & V \\ * & U & * \end{pmatrix} = n+m \quad \text{thus} \quad \operatorname{rk}\begin{pmatrix} 0 & V \\ U & * \end{pmatrix} = n+m-r$$

This last matrix being $(n+m-r)\times(n+m-r)$, this means that it is invertible. U and V are of maximal rank and the matrix

$$\begin{pmatrix} A & V^\tau \\ U^\tau & 0 \end{pmatrix}$$

is therefore invertible.

2.2 Correct result

For this part, we will assume the theorem from [1] stating that if $AU = 0$, $VA = 0$, and if we have the previously demonstrated invertibility, then:

$$\begin{pmatrix} A & V^\tau \\ U^\tau & 0 \end{pmatrix}^{-1} = \begin{pmatrix} A^+ & (U^\tau)^+ \\ (V^\tau)^+ & 0 \end{pmatrix}$$

We only have left to demonstrate that $AU = 0$ and $VA = 0$. Because P and Q only operate on A (apart from side-effects), we can write:

$$P = \begin{pmatrix} P' & 0 \\ 0 & I_n \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} Q' & 0 \\ 0 & I_m \end{pmatrix}$$

Then, we have:

$$P'AQ' = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \qquad Q' = \begin{pmatrix} * & U \end{pmatrix} \qquad P' = \begin{pmatrix} * \\ V \end{pmatrix}$$

$$U = Q' \begin{pmatrix} 0_{r \times (n-r)} \\ I_{n-r} \end{pmatrix}$$

$$P'AU = P'AQ' \begin{pmatrix} 0_r \\ I_{n-r} \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0_{r \times (n-r)} \\ I_{n-r} \end{pmatrix} = 0_{m \times (n-r)}$$

But P' being invertible, we have $AU = 0$.

Likewise,

$$V = \begin{pmatrix} 0_{(m-r) \times r} & I_{m-r} \end{pmatrix} P'$$

$$VAQ' = \begin{pmatrix} 0_{(m-r) \times r} & I_{m-r} \end{pmatrix} P'AQ' = \begin{pmatrix} 0_{(m-r) \times r} & I_{m-r} \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = 0_{(m-r) \times n}$$

And Q' being invertible, $VA = 0$.

References

[1] B. Germain-Bonne. *Méthodes de calcul de pseudo-inverses de matrices*. Departement de mathematique, 1967.