Calculating the pseudoinverse

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This work is only a clarification and correction of [1].

1 Method

With a symmetric matrix S, we can calculate S^+ the following way:

$$\begin{Bmatrix} S \\ I_n \end{Bmatrix} \xrightarrow{\text{Gauss}} \begin{Bmatrix} \begin{pmatrix} & 0 \\ \Delta & I_r \end{pmatrix} \\ \begin{pmatrix} U \\ C \end{pmatrix} \end{Bmatrix}$$

And then, we do:

$$\begin{Bmatrix} \begin{pmatrix} & U \\ \Delta & I_r \end{pmatrix} \\ \begin{pmatrix} 0 \\ C \end{pmatrix} \end{Bmatrix} \xrightarrow{\text{Gauss}} \begin{Bmatrix} I_n \\ S^+ \end{Bmatrix}$$

For a usual matrix A, we use $A^+ = A^\tau \left(AA^\tau\right)^+$.

2 Demonstrations

The first transformation can be done as $M \mapsto P_1 M Q_1$ where P_1 operates on the lines, and Q_1 is only a permutation of the columns.

The second transformation should be doable only with a matrix P_2 for operations on the lines, provided that the top matrix is invertible.

2.1 Invertible matrix

Let's demonstrate this last fact. The first transformation gives us:

$$\begin{Bmatrix} P_1 S Q_1 \\ P_1 Q_1 \end{Bmatrix} = \begin{Bmatrix} \begin{pmatrix} & 0 & \\ \Delta & & I_r \end{pmatrix} \\ \begin{pmatrix} U \\ C \end{pmatrix} \end{Bmatrix}$$

We denote $P_1^{top} = (I_{n-r} \quad 0)\, P_1$ such that $P_1^{top} Q_1 = U$. Then:

$$\begin{pmatrix} P_1 & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} S \\ P_1^{top} \end{pmatrix} Q_1 = \begin{pmatrix} P_1 S \\ P_1^{top} \end{pmatrix} Q_1 = \begin{pmatrix} P_1 S Q_1 \\ P_1^{top} Q_1 \end{pmatrix} = \begin{pmatrix} & 0 & \\ \Delta & & I_r \\ & U & \end{pmatrix}$$

But the very first matrix above is invertible (because both P_1 and I_{n-r} are), and Q_1 is invertible too. If we alter a bit the disposition of the rows of the last matrix and remove the null ones, this gives us:

$$\mathrm{rk} \begin{pmatrix} S \\ P_1^{top} \end{pmatrix} = \mathrm{rk} \begin{pmatrix} & U & \\ \Delta & & I_r \end{pmatrix}$$

Thanks to this rank equality, we only need to prove that the rank of the matrix on the left is n.

We know that $\mathrm{rk}(S) = r$ because of the first transformation of the algorithm itself. P_1 being invertible, we also have that $\mathrm{rk}(P_1^{top}) = \mathrm{rk}\,(I_{n-r} \quad 0) = n - r$.

We know that $P_1^{top} S Q_1 = 0$, hence $P_1^{top} S = 0$ because Q_1 is invertible. But S is symmetric, therefore $P_1^{top} S^\tau = 0$.

Now, we have that $\mathrm{rk}(S) = r$, $\mathrm{rk}(P_1^{top}) = n - r$ and the rows of S and P_1^{top} are totally independent. This implies that:

$$\mathrm{rk} \begin{pmatrix} S \\ P_1^{top} \end{pmatrix} = n$$

which concludes our demonstration.

2.2 Correct result

For this part, we will assume the theorem from [1] stating that if $VS = 0$, $\mathrm{rk}(V) = n-r$, we have:

$$\begin{pmatrix} S & V^\tau \\ V & 0 \end{pmatrix}^{-1} = \begin{pmatrix} S^+ & V^+ \\ (V^\tau)^+ & 0 \end{pmatrix}$$

From this assumption, we will prove that our scheme works with $V = P_1^{top}$. We already know that $\mathrm{rk}(V) = n - r$ and that $VS = 0$.

The first step is to notice that, when doing the Gaussian elimination corresponding to the above inversion, we can restrict ourselves to the left part of the matrices:

$$\begin{Bmatrix} \begin{pmatrix} S \\ V \end{pmatrix} \\ \\ \begin{pmatrix} I_n \\ 0 \end{pmatrix} \end{Bmatrix} \xrightarrow{\text{Gauss}} \begin{Bmatrix} \begin{pmatrix} I_n \\ 0 \end{pmatrix} \\ \\ \begin{pmatrix} S^+ \\ P_3^{-1} \left(V^\tau\right)^+ \end{pmatrix} \end{Bmatrix}$$

This is due to the fact that the remaining operations for the left part of the matrix can be done with operations on the columns on the right side only and operations on the lines in the bottom only. P_3^{-1} is here to take these operations into account.

The next step is to break this transformation into two parts, thus revealing our intermediary steps:

$$\begin{Bmatrix} \begin{pmatrix} S \\ V \end{pmatrix} \\ \\ \begin{pmatrix} I_n \\ 0 \end{pmatrix} \end{Bmatrix} = \begin{Bmatrix} \begin{pmatrix} S \\ P_1^{top} \end{pmatrix} \\ \\ \begin{pmatrix} I_n \\ 0 \end{pmatrix} \end{Bmatrix} \xrightarrow{\text{Gauss}} \begin{Bmatrix} \begin{pmatrix} P_1 S Q_1 \\ P_1^{top} Q_1 \end{pmatrix} \\ \begin{pmatrix} P_1 Q_1 \\ 0 \cdot Q_1 \end{pmatrix} \end{Bmatrix} = \begin{Bmatrix} \begin{pmatrix} & 0 & \\ \Delta & & I_r \\ & U & \end{pmatrix} \\ \begin{pmatrix} U \\ C \\ 0 \end{pmatrix} \end{Bmatrix}$$

This is the first part. The operations on the columns correspond to Q_1 whereas the operations on the rows correspond to a left-side multiplication with P_1', where:

$$P_1' = \begin{pmatrix} P_1 & 0 \\ 0 & I_{n-r} \end{pmatrix} \qquad\qquad P_2' = \begin{pmatrix} P_2 & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

With a few permutation of rows, we continue with:

$$\begin{Bmatrix} \begin{pmatrix} & 0 & \\ \Delta & & I_r \\ & U & \end{pmatrix} \\ \begin{pmatrix} U \\ C \\ 0 \end{pmatrix} \end{Bmatrix} \xrightarrow{\text{Gauss}} \begin{Bmatrix} \begin{pmatrix} & U & \\ \Delta & & I_r \\ & 0 & \end{pmatrix} \\ \begin{pmatrix} 0 \\ C \\ U \end{pmatrix} \end{Bmatrix} \xrightarrow{\text{Gauss}} \begin{Bmatrix} P_2' \begin{pmatrix} & U & \\ \Delta & & I_r \\ & 0 & \end{pmatrix} \\ P_2' \begin{pmatrix} 0 \\ C \\ U \end{pmatrix} \end{Bmatrix} = \begin{Bmatrix} \begin{pmatrix} I_n \\ 0 \end{pmatrix} \\ \\ \begin{pmatrix} S^+ \\ U \end{pmatrix} \end{Bmatrix}$$

Interestingly, we learn that $U = P_3^{-1} \left(V^\tau\right)^+$, while concluding the demonstration.

References

[1] B. Germain-Bonne. *Méthodes de calcul de pseudo-inverses de matrices.* Departement de mathematique, 1967.