FEM for ODEs

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1 Introduction

We consider the smooth n-dimesniosnal ODE

$$u'(t) = f(u(t)) + l(t), \quad t \in I =]0, T[, \qquad u(0) = u_0. \tag{1}$$

With $U := H^1(I, \mathbb{R}^n)$ and $V := L^2(I, \mathbb{R}^n)$ a weak formulation is

$$u \in U : \int_0^T \langle u' - f(u), v\rangle + \langle u(0), v_0\rangle = \int_0^T \langle l, v\rangle + \langle u_0, v_0\rangle \quad \forall (v, v_0) \in V \times \mathbb{R}^n. \tag{2}$$

Using the relation $\int_0^T u'v = -\int_0^T uv' + uv|_0^T$ An alternative formulation is given by: Find $u, u_T \in V \times \mathbb{R}^n$, such that

$$-\int_0^T (\langle u, v'\rangle + \langle f(u), v\rangle) + \langle u_T, v(T)\rangle = \int_0^T \langle l, v\rangle + \langle u_0, v(0)\rangle \quad \forall v \in U. \tag{3}$$

2 FEM discretization

We let $\delta = (0 = t_0 < t_1 < \cdots < t_N = T)$ be a partition, $I_\ell :=]t_{\ell-1}, t_\ell[$, $1 \leqslant \ell \leqslant N$, $\delta_\ell := |I_\ell|$.

We let $U_\delta \subset U$ and $V_\delta \subset V$ be two conforming piecewise polynomial spaces and consider the semi-implicit discretization: Find $u_\delta \in U_\delta$ such that for all $(v, v_0) \in V_\delta \times \mathbb{R}^n$

$$\int_0^T \langle u_\delta' - (f(\tilde{u}) + f_u'(\tilde{u})(u_\delta - \tilde{u})), v\rangle + \langle u_\delta(0), v_0\rangle = \int_0^T \langle l, v\rangle + \langle u_0, v_0\rangle. \tag{4}$$

where

$$\tilde{u}_{|I_\ell} := u_{\ell-1}, \qquad u_{\ell-1} := u(t_{\ell-1}). \tag{5}$$

This gives on each time interval the linear system of equations

$$\int_{I_\ell} \langle u_\delta' - Au_\delta), v\rangle = \int_{I_\ell} \langle l + f(u_{\ell-1}) - Au_{\ell-1}, v\rangle \quad \forall v \in P^{k-1}(I_\ell).$$

with $A := f_u'(u_{\ell-1})$. Now we suppose that

$$u_{|I_\ell} = u_{\ell-1} + \sum_{j=1}^{k} c_j \phi_j(t), \quad \phi_j(0) = 0, \quad \deg \phi_j = j,\ 1 \leqslant j \leqslant k-1. \tag{6}$$

Then on each time interval we have to solve

$$\int_{I_\ell} \langle \phi_j' - A\phi_j, v\rangle c_j = \int_{I_\ell} \langle l + f(u_0), v\rangle \quad \forall v \in P^{k-1}(I_\ell).$$

Let l_i be the Legendre functions normalized by $\int_0^1 l_i l_j = \delta_{ij}, 0 \leqslant i, j$.

Taking as basis for $P^{k-1}(I)$ $\psi_i = l_{i-1}, 1 \leqslant i \leqslant k$ and $\phi_j(t) = \int_0^t l_{j-1}(s)\, ds$ for $1 \leqslant j \leqslant k-1$ we have for $j \leqslant k-1$

$$M_{ij} = \int_0^1 \phi_j' l_i = \int_0^1 l_j l_i$$

2.1 Dual scheme

In the same way, the weak formulation (**??**) leads to the dual scheme: Find $u_\delta, u_T \in V_\delta \times \mathbb{R}^n$, such that

$$-\int_0^T (\langle u_\delta, v' \rangle + \langle f(u_\delta), v \rangle) + \langle u_T, v(T) \rangle = \int_0^T \langle l, v \rangle + \langle u_0, v(0) \rangle \quad \forall v \in U_\delta. \quad (7)$$

If we use linearization, we have

$$-\int_0^T (\langle u_\delta, v' \rangle + \langle A u_\delta, v \rangle) + \langle u_T, v(T) \rangle = \int_0^T \langle l + f(\tilde{u}) - A\tilde{u}, v \rangle + \langle u_0, v(0) \rangle \quad \forall v \in U_\delta. \quad (8)$$

3 Abstract setting

In order to put (**??**) and (23) in conforming the Babuska-framework [**?**]

$$\begin{cases} X_\delta \times Y_\delta \subset X \times Y \\ x \in X: \; a(x)(y) = b(y) \quad \forall y \in Y, \\ x_\delta \in X_\delta: \; a_\delta(x_\delta)(y) = b(y) \quad \forall y \in Y_\delta. \end{cases} \quad (9)$$

We let

$$\begin{cases} X := H^1(I, \mathbb{R}^n), \quad \|x\|_X := \left(\|x'\|^2_{L^2(I,\mathbb{R}^n)} + \|x(0)\|^2 \right)^{\frac{1}{2}} \\ Y := L^2(I, \mathbb{R}^n) \times \mathbb{R}^n, \quad \|(y_1, y_0)\|_Y := \left(\|y_1\|^2_{L^2(I,\mathbb{R}^n)} + \|y_0\|^2 \right)^{\frac{1}{2}} \end{cases} \quad (10)$$

Let us suppose the continuous inf-sup uniform condition

$$\gamma := \inf_{x \in X \setminus \{0\}} \sup_{y \in Y \setminus \{0\}} \frac{a'(x_0)(x, y)}{\|x\|_X \, \|y\|_Y} > 0, \quad \forall x_0 \in X. \quad (11)$$

and its discrete version

$$\gamma_\delta := \inf_{x \in X_\delta \setminus \{0\}} \sup_{y \in Y_\delta \setminus \{0\}} \frac{a_\delta'(x_0)(x, y)}{\|x\|_X \, \|y\|_Y} > 0, \quad \forall x_0 \in X_\delta. \quad (12)$$

3.0.1 A priori

Let $\widetilde{x}_\delta \in X_\delta$

3.0.2 A posteriori

Theorem 1. *We have*

$$\begin{cases} \gamma \|x - x_\delta\|_X \leqslant R_1 + R_2, \\ \qquad R_1 := \sup_{y \in Y\setminus\{0\}} \inf_{y_\delta \in Y_\delta \setminus\{0\}} \dfrac{b(y-y_\delta) - a_\delta(x_\delta)(y-y_\delta)}{\|y\|_Y}, \\ \qquad R_2 := \sup_{y \in Y\setminus\{0\}} \dfrac{a(x_\delta)(y) - a_\delta(x_\delta)(y)}{\|y\|_Y}. \end{cases} \tag{13}$$

Proof. We have for any $y \in Y$ and $y_\delta \in Y_\delta$

$$\begin{aligned} &\int_0^1 a'(x_\delta + t(x-x_\delta))(x-x_\delta, y)\,dt = a(x)(y) - a(x_\delta)(y) = b(y) - a(x_\delta)(y) \\ &= b(y) - a_\delta(x_\delta)(y) + a(x_\delta)(y) - a_\delta(x_\delta)(y) \\ &= b(y-y_\delta) - a_\delta(x_\delta)(y-y_\delta) + a(x_\delta)(y) - a_\delta(x_\delta)(y) \leqslant (R_1 + R_2)\|y\|_Y \end{aligned}$$

Then

$$\begin{aligned} \gamma \|x - x_\delta\| &\leqslant \int_0^1 \sup_{y \in Y\setminus\{0\}} \frac{a'(x_\delta + t(x-x_\delta))(x-x_\delta, y)}{\|y\|_Y}\,dt \\ &= \sup_{y \in Y\setminus\{0\}} \frac{a(x)(y) - a(x_\delta)(y)}{\|y\|_Y} \leqslant R_1 + R_2. \end{aligned}$$

□

4 A posterior error estimator

4.1 Primal scheme

We have for $y = (v, v_0) \in L^2(I, \mathbb{R}^n) \times \mathbb{R}^n$ and $y_\delta = (v_\delta, v_0)$ with $v_\delta = \pi_\delta v$. Since $u'_\delta \in V_\delta$ and $f(\widetilde{u_\delta}) + f'(\widetilde{u_\delta})(\widetilde{u_\delta}) \in V_\delta$ we have

$$b(y - y_\delta) - a_\delta(u_\delta)(y - y_\delta) = \int_0^T \langle l + f'(\widetilde{u_\delta})(u_\delta), v - v_\delta \rangle$$

so

$$R_1 \leqslant \sum_{\ell=1}^{N} \eta_\ell^1(u_\delta)\,\|v\|_{L^2(I_\ell)} \leqslant \left(\sum_{\ell=1}^{N} \eta_\ell^1(u_\delta)^2\right)^{\frac{1}{2}} \|v\|_{L^2(I)}$$

With

$$\eta_\ell^1(u_\delta) := \|(I-\pi_\delta)\,(l+f'(u_{\ell-1})u_\delta)\|_{L^2(I_\ell)}$$

Lemma 1. *Let* u *have the development as in (6). Then*

$$\|(I-\pi_\delta)f'(u_{\ell-1})u_\delta\|_{I_\ell} = \|f'(u_{\ell-1})c_k\|^2\,\|\phi_k\|_{L^2(I_\ell)} \tag{14}$$

Proof. □

Similarly we have for R_2

$$a(x_\delta)(y) - a_\delta(x_\delta)(y) = \sum_{k=1}^{N} \int_{I_\ell} \langle f(u_\delta) - f(\widetilde{u_\delta}), v\rangle$$

$$\begin{aligned} f(u_\delta) - f(\widetilde{u_\delta}) =& f(u_\delta) - f(u_{\ell-1}) - f'(u_{\ell-1})(u_\delta - u_{\ell-1}) \\ =& \int_0^1 (f'((1-s)u_{\ell-1} + su_\delta) - f'(u_{\ell-1}))\ ds(u_\delta - u_{\ell-1}) \end{aligned}$$

If f' is quadratic, the simpson rule gives for the integral

$$\frac{2}{3}\left(f'(\frac{u_{\ell-1}+u_\ell}{2}) - f'(u_{\ell-1})\right) + \frac{1}{6}\,(f'(u_\ell) - f'(u_{\ell-1}))$$

For trapez we get

$$\frac{1}{2}\,(f'(u_\ell) - f'(u_{\ell-1}))$$

Lemma 2.

$$\eta_\ell^2(u_\delta) = \frac{1}{2}\,\|f'(u_\ell) - f'(u_{\ell-1})\|\,\|u_\delta - u_{\ell-1}\|_{L^2(I_\ell)} \tag{15}$$

4.2 Dual scheme

We have to consider with $v \in H^1(I, X)$ and $w = v - v_\delta$

$$-\int_0^T (\langle u_\delta, w'\rangle + \langle Au_\delta, w\rangle) + \langle u_T, w(T)\rangle = \int_0^T \langle l + f(\tilde{u}) - A\tilde{u}, w\rangle + \langle u_0, w(0)\rangle \quad \forall v \in U_\delta.$$

Integration by parts gives

$$\begin{aligned} -\int_0^T \langle u_\delta, w'\rangle + \langle u_T, w(T)\rangle - \langle u_0, w(0)\rangle &= \int_0^T \langle u_\delta', w\rangle \\ -\sum_{\ell=1}^{N-1} \langle [u_\delta(t_\ell)], w(t_\ell)\rangle + \langle u_\delta(0) - u0, w(0)\rangle - \langle u_\delta(T) - u_T, w(T)\rangle \end{aligned}$$

5 Analysis in the linear case $f(u) = -Au$ with SPD A

The equation reads

$$u \in U : \int_0^T \langle u' + Au, v\rangle + \langle u(0), v_0\rangle = \int_0^T \langle l, v\rangle + \langle u_0, v_0\rangle \quad \forall (v, v_0) \in V \times \mathbb{R}^n. \tag{16}$$

We suppose A to symmetric and positive definite and denote $\|u\|_{A^k} = \left\|A^{k/2}u\right\|$, $k \in \mathbb{Z}$. We equip U and V with the norms

$$\|u\|_U^2 := \left\|A^{-\frac{1}{2}}u'\right\|_{L^2(I,\mathbb{R}^n)}^2 + \left\|A^{\frac{1}{2}}u\right\|_{L^2(I,\mathbb{R}^n)}^2 + \|u(0)\|_{\mathbb{R}^n}^2 + \|u(T)\|_{\mathbb{R}^n}^2, \quad \|v\|_V^2 := \left\|A^{\frac{1}{2}}v\right\|_{L^2(I,\mathbb{R}^n)}^2$$

Denoting the bilinear form on the left of (16) by a, we wish to show that

$$\inf_{u \in U\setminus\{0\}} \inf_{(v,v_0)\in V\times\mathbb{R}^n\setminus\{0\}} \frac{a(u,v)}{\|u\|_U \|v\|_V} = \gamma > 0. \tag{17}$$

First, testing with $(v, v_0) = (A^{-1}(u' + Au), 2u(0))$ we have

$$\|(v, v_0)\|_{V\times\mathbb{R}^n}^2 \leqslant 4\|u(0)\|_{\mathbb{R}^n}^2 + \left\|A^{-\frac{1}{2}}(u' + Au)\right\|_{L^2(I,\mathbb{R}^n)}^2 \leqslant 4\|u\|_U^2.$$

and

$$\begin{aligned} a(u, (v, v_0)) &= \left\|A^{-\frac{1}{2}}u'\right\|_{L^2(I,\mathbb{R}^n)}^2 + 2\|u(0)\|^2 + 2\int_0^T \langle u, u'\rangle + \left\|A^{\frac{1}{2}}u\right\|_{L^2(I,\mathbb{R}^n)}^2 \\ &= \left\|A^{-\frac{1}{2}}u'\right\|_{L^2(I,\mathbb{R}^n)}^2 + \|u(1)\|^2 + \|u(0)\|^2 + \left\|A^{\frac{1}{2}}u\right\|_{L^2(I,\mathbb{R}^n)}^2 \end{aligned}$$

since

$$2\int_0^T \langle u, u' \rangle = \|u(1)\|^2 - \|u(0)\|^2 .$$

This yields $\gamma \geqslant \frac{1}{2}$ in (17).

For the discrete scheme

$$u_\delta \in U_\delta : \int_0^T \langle u_\delta' + A u_\delta, v \rangle + \langle u(0), v_0 \rangle = \int_0^T \langle l, v \rangle + \langle u_0, v_0 \rangle \quad \forall (v, v_0) \in V_\delta \times \mathbb{R}^n. \tag{18}$$

we let $\pi_\delta : L^2(I, \mathbb{R}^n) \to V_\delta$ be the $L^2(I, \mathbb{R}^n)$ projection. Then, testing with $(v, v_0) = (A^{-1}\pi_\delta(u_\delta' + A u_\delta), 2u_\delta(0))$ yields

$$a(u_\delta, (v, v_0)) = \left\| A^{-\frac{1}{2}} \pi_\delta u_\delta' \right\|^2_{L^2(I,\mathbb{R}^n)} + 2\|u_\delta(0)\|^2 + 2\int_0^T \langle \pi_\delta u_\delta, u_\delta' \rangle + \left\| A^{\frac{1}{2}} \pi_\delta u_\delta \right\|^2_{L^2(I,\mathbb{R}^n)}$$

In case

$$u_\delta' \in V_\delta \tag{19}$$

we get

$$a(u_\delta, (v, v_0)) = \left\| A^{-\frac{1}{2}} u_\delta' \right\|^2_{L^2(I,\mathbb{R}^n)} + \|u_\delta(1)\|^2 + \|u_\delta(0)\|^2 + \left\| A^{\frac{1}{2}} \pi_\delta u_\delta \right\|^2_{L^2(I,\mathbb{R}^n)}$$

which is weaker, since it only controls the projection of the solution in the A-norm. This explains the oscillations of the Crank-Nicolson scheme for the heat equation.

Without (19) we have

$$\begin{aligned} a(u_\delta, (v, v_0)) = &\left\| A^{-\frac{1}{2}} \pi_\delta u_\delta' \right\|^2_{L^2(I,\mathbb{R}^n)} + \|u_\delta(1)\|^2 + \|u_\delta(0)\|^2 \\ &+ 2\int_0^T \langle \pi_\delta u_\delta - u_\delta, u_\delta' \rangle + \left\| A^{\frac{1}{2}} \pi_\delta u_\delta \right\|^2_{L^2(I,\mathbb{R}^n)} \end{aligned}$$

We get the correct norms under the assumptions

$$\left\| A^{-\frac{1}{2}} u_\delta' \right\|^2_{L^2(I,\mathbb{R}^n)} \lesssim \left\| A^{-\frac{1}{2}} \pi_\delta u_\delta' \right\|^2_{L^2(I,\mathbb{R}^n)} \tag{20}$$

and

$$\left\| A^{\frac{1}{2}} (\pi_\delta u_\delta - u_\delta) \right\|^2_{L^2(I,\mathbb{R}^n)} \lesssim \int_0^T \langle \pi_\delta u_\delta - u_\delta, u_\delta' \rangle + \left\| A^{\frac{1}{2}} \pi_\delta u_\delta \right\|^2_{L^2(I,\mathbb{R}^n)} \tag{21}$$

6 Linearization (semi-implicit scheme)

by D_δ^k and P_δ^k the spaces of general and continuous piecewise k-th order polynomials, respectively. We note that $\dim D_\delta^k = (N-1)(k+1)$ and $\dim P_\delta^k = N + (N-1)(k-1) = \dim D_\delta^{k-1} + 1$. Let f_δ be a piecewise polynomial approximation of f. We define

$$a_\delta(u)(v, v_0) = \sum_{k=0}^{N-1} \int_{I_\ell} \langle u'(t) - f_k^\delta(u), v(t) \rangle \, dt + \langle u(0), v_0 \rangle \tag{22}$$

and the discrete problem for $k \in \mathbb{N}$

$$u_\delta \in P_\delta^k : \; a(u_\delta)(v, v_0) = b(v, v_0) \quad \forall (v, v_0) \in D_\delta^{k-1} \times \mathbb{R}^n. \tag{23}$$

The choice, with $u_k := u(t_\ell)$,

$$f_k^\delta(u) = f(u_k) + f'(u_k)(u - u_k) \tag{24}$$

leads to a semi-implicit scheme.

7 Definition of the method

We consider the smooth autonomous ODE

$$u'(t) = f(u(t)), \quad t \in I =]0, T[, \qquad u(0) = u_0. \tag{25}$$

We let $\delta = (0 = t_0 < t_1 < \cdots < t_N = T)$ be a partition, $I_\ell :=]t_{\ell-1}, t_k[$, $1 \leqslant k \leqslant N$. We denote by D_δ^k and P_δ^k the spaces of general and continuous piecewise k-th order polynomials. We note that $\dim D_\delta^k = (N-1)(k+1)$ and $\dim P_\delta^k = N + (N-1)(k-1) = \dim D_\delta^{k-1} + 1$. We define the function spaces $X = H^1(I, \mathbb{R}^n)$ and $Y = L^2(I, \mathbb{R}^n) \times \mathbb{R}^n$ and the form $a : X \times Y \to \mathbb{R}$

$$a(u)(v, w) := \int_I (u'(t) - f(u(t))) v(t) \, dt + \langle u(0), w \rangle. \tag{26}$$

Then with the linear form

$$b(v, w) := \langle u_0, w \rangle \tag{27}$$

a weak formulation of (25) reads

$$u \in X : \; a(u)(v, w) = b(v, w) \quad \forall (v, w) \in Y.$$

Let f_δ be a piecewise polynomial approximation of f. We define for $k \in \mathbb{N}$ $X_\delta := P_\delta^k$, $Y_\delta := D_\delta^{k-1} \times \mathbb{R}^n$,

$$a_\delta(u)(v) = \sum_{\ell=1}^{N} \int_{I_\ell} (u'(t) - f_\delta(u))v(t)\,dt + \langle u(0), w\rangle \tag{28}$$

and the discrete problem

$$u_\delta \in X_\delta: \; a(u_\delta)(v,w) = b(v,w) \quad \forall (v,w) \in Y_\delta.$$

8 CG2 variants with linearization

We use a quadratic approximation written be means of an hierarchical basis with piecewise linear test functions and linearization

$$f_\delta(u) = f(u_0) + f'(u_0)(u - u_0). \tag{29}$$

Transforming all intervals to $[0,1]$ we have the development

$$u(t) = (1-t)u_0 + tu_1 + t(1-t)u_2 \tag{30}$$

with u_0 known and u_1 and u_2 verifying

$$\int_0^1 (u'(t) - (f(u_0) + f'(u_0)(u - u_0)))\,\psi(t)\,dt = 0 \quad \psi$$

$$\int_0^1 ((u_1 - u_0) + (1-2t)u_2 - (f(u_0) + f'(u_0)(t(u_1 - u_0) + t(1-t)u_2))\,\psi(t)\,dt = 0 \quad \psi$$

$$\int_0^1 (u_1 + (1-2t)u_2 - f'(u_0)(tu_1 + t(1-t)u_2))\,\psi(t)\,dt = \int_0^1 (u_0 + f(u_0) - tf'(u_0)u_0))\,\psi(t)\,dt \quad \psi$$

$$\int_0^1 (u_1 + (1-2t)u_2 - f'(u_0)(tu_1 + t(1-t)u_2))\,\psi(t)\,dt = \int_0^1 (u_0 + f(u_0) - tf'(u_0)u_0))\,\psi(t)\,dt \quad \psi$$

Denoting $A := f'(u_0)$ and be $b := u_0 + f(u_0)$ we have

$$\alpha(\psi)u_1 + \beta(\psi)u_2 = \int_0^1 b\psi(t)\,dt + \alpha(\psi)u_0, \quad \psi \in \Psi$$

$$\alpha(\psi) := \int_0^1 (M - tA)\,\psi(t)\,dt, \quad \beta(\psi) := \int_0^1 ((1-2t)M - t(1-t)A)\,\psi(t)\,dt$$

8.1 CG2-DG1

With $\Psi = \{1, 1-t\}$ we have

$$\int_0^1 t(1-t) = \frac{1}{6}, \quad \int_0^1 (1-2t)(1-t) = \frac{1}{6}, \quad \int_0^1 t(1-t)^2 = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}$$

$$\begin{bmatrix} M - \frac{1}{2}A & \frac{1}{6}A \\ \frac{1}{2}M - \frac{1}{6}A & \frac{1}{6}M - \frac{1}{12}A \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b \\ \frac{1}{2}b \end{bmatrix} + \begin{bmatrix} (M - \frac{1}{2}A)u_0 \\ (\frac{1}{2}M - \frac{1}{6}A)u_0 \end{bmatrix}$$

8.2 CG2-2DG0

With $\Psi = \left\{\chi_{[0,1]}, \chi_{[0,\frac{1}{2}]}\right\}$ we have

$$\int_0^{\frac{1}{2}} t \, dt = \int_0^{\frac{1}{2}} (1-2t) \, dt = \frac{1}{4}, \quad \int_0^{\frac{1}{2}} t(1-t) \, dt = \frac{1}{2}(\frac{2}{3} \times \frac{1}{4} \times \frac{3}{4} + \frac{1}{6} \times \frac{1}{4}) = \frac{1}{12}, \quad \int_0^{\frac{1}{2}} (1-t) \, dt = \frac{1}{4}$$

$$\begin{bmatrix} M - \frac{1}{2}A & \frac{1}{6}A \\ \frac{1}{2}M - \frac{1}{4}A & \frac{1}{4}M - \frac{1}{12}A \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b \\ \frac{1}{2}b \end{bmatrix} + \begin{bmatrix} (M - \frac{1}{2}A)u_0 \\ (\frac{1}{2}M - \frac{1}{4}A)u_0 \end{bmatrix}$$

8.3 2CG1-DG1

We replace the quadratic in (32) by a piecewise linear

$$u(t) = (1-t)u_0 + tu_1 + \phi(t)u_2, \quad \phi(t) = \frac{1}{2}\min\{t, 1-t\} = \frac{1 - |2t-1|}{4} \quad (31)$$

Then with

$$\int_0^1 \phi(t) = \frac{1}{8}, \quad \int_0^1 \phi(t)(1-t) = \frac{1}{16}, \quad \int_0^1 \phi'(t) = 0, \quad \int_0^1 \phi'(t)(1-t) = \frac{1}{2}\frac{3}{4}\frac{1}{2} - \frac{1}{2}\frac{1}{2}\frac{1}{4} = \frac{1}{8}$$

With $\Psi = \{1, 1-t\}$ we have

$$\begin{bmatrix} M - \frac{1}{2}A & \frac{1}{8}A \\ \frac{1}{2}M - \frac{1}{6}A & \frac{1}{8}M - \frac{1}{16}A \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b \\ \frac{1}{2}b \end{bmatrix} + \begin{bmatrix} (M - \frac{1}{2}A)u_0 \\ (\frac{1}{2}M - \frac{1}{4}A)u_0 \end{bmatrix}$$

8.4 2CG1-2DG0

We replace the quadratic in (32) by a piecewise linear

$$u(t) = (1-t)u_0 + tu_1 + \phi(t)u_2, \quad \phi(t) = \frac{1}{2}\min\{t, 1-t\} = \frac{1-|2t-1|}{4} \quad (32)$$

Then with $\Psi = \left\{\chi_{[0,1]}, \chi_{[0,\frac{1}{2}]}\right\}$ and

$$\int_0^1 \phi(t) = \frac{1}{8}, \quad \int_0^{\frac{1}{2}} \phi(t) = \frac{1}{16}, \quad \int_0^1 \phi'(t) = 0, \quad \int_0^{\frac{1}{2}} \phi'(t) = \frac{1}{4}$$

$$\begin{bmatrix} M - \frac{1}{2}A & \frac{1}{8}A \\ \frac{1}{2}M - \frac{1}{8}A & \frac{1}{4}M - \frac{1}{16}A \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b \\ \frac{1}{2}b \end{bmatrix} + \begin{bmatrix} (M - \frac{1}{2}A)u_0 \\ (\frac{1}{2}M - \frac{1}{4}A)u_0 \end{bmatrix}$$

8.5 CG2$^+$-DG0

Another variant is to force the quadratic approximation to be C_1. One needs careful scaling on variable intervals and we have to decide what to do on the first interval. On $[t_n, t_{n+1}]$, $t_{n+1} = t_n + \delta_n$ we have

$$u_{|I_n}(t) = \frac{t_{n+1} - t}{\delta_n}u_0 + \frac{t - t_n}{\delta_n}u_1 + \frac{(t_{n+1} - t)(t - t_n)}{\delta_n^2}u_2$$
$$\Rightarrow \quad {u_{|I_n}}'(t_n) = \frac{u_1 - u_0 + u_2}{\delta_n}$$

Denoting the previous values by u_{-1} and u_{-2}, i.e on $[t_{n-1}, t_n]$ we have

$$u_{|I_{n-1}}(t) = \frac{t_n - t}{\delta_{n-1}}u_{-1} + \frac{t - t_{n-1}}{\delta_{n-1}}u_0 + \frac{(t_n - t)(t - t_{n-1})}{\delta_{n-1}^2}u_2$$
$$\Rightarrow \quad {u_{|I_{n-1}}}'(t_n) = \frac{u_0 - u_{-1} - u_{-2}}{\delta_{n-1}}$$

So the C^1-condition reads

$$u_1 + u_2 = \left(1 + \frac{\delta_{n-1}}{\delta_n}\right)u_0 - \frac{\delta_{n-1}}{\delta_n}\left(u_{-1} + u_{-2}\right).$$

$$\begin{bmatrix} M - \frac{1}{2}A & \frac{1}{6}A \\ M & M \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} (M + \frac{1}{2}A)u_0 + b \\ 2u_0 - u_{-1} - u_{-2} \end{bmatrix}$$

9 General analysis

We consider

$$\begin{cases} \quad x \in X: & a(x)(y) = l(y) \quad \forall y \in Y \\ x_\delta \in X_\delta: & a(x_\delta)(y) = l(y) \quad \forall y \in Y_\delta \end{cases} \tag{33}$$

Let us suppose the continuous inf-sup uniform condition

$$\gamma := \inf_{x \in X \setminus \{0\}} \sup_{y \in Y \setminus \{0\}} \frac{a'(x_0)(x, y)}{\|x\|_X \|y\|_Y} > 0, \quad \forall x_0 \in X. \tag{34}$$

and its discrete version

$$\gamma_\delta := \inf_{x \in X_\delta \setminus \{0\}} \sup_{y \in Y_\delta \setminus \{0\}} \frac{a'_\delta(x_0)(x, y)}{\|x\|_X \|y\|_Y} > 0, \quad \forall x_0 \in X_\delta. \tag{35}$$

9.1 A priori

Let $\widetilde{x}_\delta \in X_\delta$

9.2 A posteriori

We have for any $y \in Y$

$$\begin{aligned} \int_0^1 a'(x_\delta + t(x - x_\delta))(x - x_\delta, y)\, dt &= a(x)(y) - a(x_\delta)(y) = l(y) - a(x_\delta)(y) \\ &\leqslant (R_1 + R_2) \|y\|_Y, \end{aligned}$$

with

$$R_1 := \sup_{y \in Y \setminus \{0\}} \inf_{y_\delta \in Y_\delta \setminus \{0\}} \frac{l(y - y_\delta) - a_\delta(x_\delta)(y - y_\delta)}{\|y\|_Y}, \quad R_2 := \sup_{y \in Y \setminus \{0\}} \frac{a(x_\delta)(y) - a_\delta(x_\delta)(y)}{\|y\|_Y}.$$

Then

$$\begin{aligned} \gamma \|x - x_\delta\| &\leqslant \int_0^1 \sup_{y \in Y \setminus \{0\}} \frac{a'(x_\delta + t(x - x_\delta))(x - x_\delta, y)}{\|y\|_Y}\, dt \\ &= \sup_{y \in Y \setminus \{0\}} \frac{a(x)(y) - a(x_\delta)(y)}{\|y\|_Y} \leqslant R_1 + R_2. \end{aligned}$$