Optimization algorithms with approximation

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Contents

1 Introduction

We consider a Hilbert space $(X, \langle \cdot, \cdot \rangle)$ with induced norm $\|\cdot\|$ and the minimization of a smooth strictly convex function $f : X \to \mathbb{R}$:

$$\inf_{x \in X} f(x) = \inf\{f(x) \mid x \in X\}.$$

We suppose that a unique minimizer x^* exists. As a motivation we consider the solution of a scalar elliptic semi-linear PDE, where X is a Sobolev space and f corresponds to the underlying energy functional of the PDE.

Our purpose is to analyse gradient algorithms on a sequence of subspaces (finite element spaces for the PDE)

$$X_0 \subset \cdots \subset X_k \subset X_{k+1} \subset \cdots \subset X, \quad P_k : X \to X_k,$$

such that a typical iteration reads:

$$x_{k+1} = x_k - t_k P_k \nabla f(x_k), \tag{1}$$

where P_k is the orthogonal projector on X_k and $\nabla f(x) \in X$ is defined by the Riesz map. In order to generate the subspaces X_k, we suppose to have an error estimators $\eta_k : X_k \to \mathbb{R}$ and a refinement algorithm satisfying typical hypothesis from the theory of AFEM. In the case $f \in \mathcal{S}^{1,1}_{\mu,L}(X)$, X finite-dimensional, and $t_k = 2/(\mu + L)$ for all k we have the following convergence estimate (Theorem 2.1.15 in [1]) for the gradient method (**GM**):

$$\|x_n - x^*\| \leqslant \rho^n \|x_0 - x^*\|, \quad \rho = 1 - 1/\kappa, \tag{2}$$

such that $\varepsilon > 0$ is achieved in $n(\varepsilon) = O(\kappa)\ln(1/\varepsilon)$ iterations. It is well-known that **GM** is not optimal for this class of functions. The accelerated gradient method (**AGM**) [1] yields an improved estimate $n(\varepsilon) = O(\sqrt{\kappa})\ln(1/\varepsilon)$.

Our aim is to establish a similar iteration count for the method on a sequence of subspaces. There is important progress of adaptive finite element methods (AFEM) for nonlinear elliptic equations, see [2, 3, 4, 5, 6, 7, 8, 9], and our development is based on these works. However, here, we wish to work out the optimization point of view.

2 Notation

We throughout suppose that $f : X \to \mathbb{R}$ is convex and C^1 and we use the the Fréchet-Riesz theorem to define

$$\langle \nabla f(y), x\rangle = f'(y)(x) \quad \forall x, y \in X.$$

It is then easy to see, that $P_Y \nabla f(y) = \nabla f_{|_Y}$ for all y in a closed subspace $Y \subset X$ and $P_Y : X \to Y$ its orthogonal projector.

Let $X_0 \subset X$ be a subspace. We suppose to have a lattice of admissible closed subspaces

$$\mathcal{X}(X_0) = \{X_0 \subset Y \subset X\}. \tag{3}$$

The partial order on $\mathcal{X}(X_0)$ is given by $Y \geqslant Z$ if and only if Y is a superspace of Z. We then have the finest common coarsening $Y \wedge Z$ and the coarsest common refinement $Y \vee Z$, respectively. We let

$$\mathcal{X}(Y) = \{Z \in \mathcal{X}(X_0) \mid Y \wedge Z = Y\}, \quad Y \in \mathcal{X}(x_0).$$

We make the following assumptions There exist constants $C_{\mathrm{stab}}, C_{\mathrm{eff}}, C_{\mathrm{rel}}$ and $0 \leqslant q_{\mathrm{red}} < 1$ such that for all $Y \in \mathcal{X}(x_0)$:

$$\begin{aligned}
C_{\mathrm{eff}}^{-1}\eta(y, Y) &\leqslant \|(I - P_Y)\nabla f(y)\| \leqslant C_{\mathrm{rel}}\eta(y, Y) && \forall y \in Y, x \in X && \text{(H1)}\\
\eta^2(y^+, Y^+) &\leqslant q_{\mathrm{red}}\eta^2(y, Y) + C_{\mathrm{stab}}^2 \|y^+ - y\|^2 && \forall y \in Y, y \in Y^+ = \mathbf{REF}(Y, \eta(y, Y)), && \text{(H2)}\\
|\eta(y, Y) - \eta(z, Y)| &\leqslant C_{\mathrm{stab}} \|y - z\| && \forall y, z \in Y && \text{(E3)}
\end{aligned}$$

For the complexity estimate, we introduce notion from nonlinear approximation. Let for $Y \in \mathcal{X}(X_0)$ and $N \in \mathbb{N}$

$$\varepsilon(Y) := \inf_{y \in Y} (f(y) - f^*), \quad \varepsilon(N) := \inf\{\varepsilon(Y) \mid Y \in \mathcal{X}(X_0),\ \dim Y \leqslant N\}.$$

For $s > 0$, we suppose that

$$\alpha_f(s) := \sup\{\varepsilon(N)N^s \mid N \in \mathbb{N}\} < +\infty. \tag{4}$$

Newt we suppose that $(X_k)_{k\in\mathbb{N}} \subset \mathcal{X}(X_0)$ and $x_k \in X_k$ are sequences such that with $\rho < 1$ we have quasi-geometrical convergence, for all $m, k \in \mathbb{N}$

$$e_{k+m} \leqslant C\rho^m e_k, \quad e_k := (f(x_k) - f^*) + \eta_k^2(x_k, X_k). \tag{5}$$

We wish to avoid the technical details of AFEM, and instead make the following hypothesis.

We formulate the following property: there exist $\gamma > 0$ and $C > 0$ such that for all refinement steps k and any $X_k^+ \in \mathcal{X}(X_K)$ there holds

$$\min_{x+\in X_k^+} f(x^+) - f^* \leqslant \gamma e_k \ \Rightarrow\ \|(I - P_{X_k})\nabla f(x_k)\| \leqslant C \left\|(P_{X_k^+} - P_{X_k})\nabla f(x_k)\right\|. \tag{6}$$

Then we make the hypothesis

$$(6)\ \&\ (4)\ \&\ (5) \quad \Rightarrow \quad \sum_{k=0}^{n} \dim X_k \leqslant C\varepsilon_n^{-1/s} \quad \forall n \in \mathbb{N}. \tag{H3}$$

Remark 1. *(6) mimics the argument in AFEM for optimality of the Dörfler marking [10] and [6]. At each step, the assumption on approximation speed implies to existence of a refinement leading to better error with controlled complexity. Then the implication of (6) shows that the overall estimator is dominated by the refined part only.*

For $t > 0$ and $Z \in \mathcal{X}(X_0)$ let

$$\begin{cases} Q(x;y,t) := f(y) + \langle \nabla f(y), x - y\rangle + \frac{1}{2t}\|x-y\|^2 & x, y \in X,\\ Q^*(y,t,Z) := \min_{x\in Z} Q(x;y,t) = f(y) - \frac{t}{2}\|P_Z\nabla f(y)\|^2, & y \in Z,\\ \widetilde{x}(y,t,Z) := \operatorname{argmin}_{x\in Z} Q(x;y,t) = y - tP_Z\nabla f(y), & y\in Z.\end{cases} \tag{7}$$

3 Gradient method

Algorithm 1: Adaptive **GM**

Inputs: X_0, $x_0 \in X_0$, $t_0 > 0$, $\lambda > 0$, $1 > \omega > 0$. Set $k = 0$.

(1) While $Q^*(x_k, t_k, X_k) < f(\widetilde{x}(x_k, t_k, X_k))$: $\quad t_k = \omega * t_k$.

(2) $x_{k+1} = \widetilde{x}(x_k, t_k, X_k)$.

(3) If $\eta^2(x_{k+1}, X_k) > q_{\text{red}}\eta^2(x_k, X_k) + \lambda t_k(f(x_k) - f(x_{k+1}))$: $\quad X_{k+1} = \mathbf{REF}(X_k, \eta(x_k, X_k))$

(4) $t_{k+1} = t_k/\omega$.

(5) Increment k and go to (1).

Lemma 1. *If the level set $\mathcal{L}_f(x_0) := \{x \in X \mid f(x) \leqslant f(x_0)\}$ is bounded and ∇f is L-Lipschitz on this set, the line-search step (1) terminates and its number of iterations is uniformly bounded with step sizes $t_k \geqslant 1/(2L)$. If in addition f is μ-strictly convex we have $t_k \leqslant 1/\mu$.*

Proof. The lower bound of the step-size follows from the following standard inequality for a function with L-Lipschitz gradient

$$f(x) \leqslant f(y) + \langle \nabla f(y), x-y\rangle + \frac{L}{2}\|x-y\|^2,$$

which implies for $t_k \leqslant 1/L$ with $\widetilde{x} := \widetilde{x}(x_k, t_k, X_k)$

$$f(\widetilde{x}) \leqslant Q(x_k, \widetilde{x}, \frac{1}{L}) = Q(x_k, \widetilde{x}, t_k) + \frac{1}{2}(L - \frac{1}{t_k})\|\widetilde{x} - x_k\|^2 \leqslant Q^*(x_k, t_k, X_k).$$

The upper bound of the step-size follows from the definition of μ-convexity:

$$f(x) \geqslant f(y) + \langle \nabla f(y), x-y\rangle + \frac{\mu}{2}\|x-y\|^2,$$

giving

$$f(\widetilde{x}) \geqslant Q_{1/\mu}(x_k, \widetilde{x}), X_k) = Q^*(x_k, t_k, X_k) + \frac{1}{2}(\mu - \frac{1}{t_k})\|\widetilde{x} - x_k\|^2.$$

The step-size-loop stops if $Q^*(x_k, t_k, X_k) \geqslant f(\widetilde{x})$, which implies $t_k \leqslant 1/\mu$.

$\square$

Lemma 2. *The iterates of GM satisfy*

$$f(x_{k+1}) - f(x_k) \leqslant -\frac{1}{2t_k} \|x_{k+1} - x_k\|^2 \tag{8}$$

and

$$f(x_{k+1}) - f(x^*) \leqslant \frac{1}{2t_k} \left(\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 \right) + \frac{C_{\mathrm{rel}}^2}{2\mu}\eta^2(x_k, X_k). \tag{9}$$

Proof. We have by the line-search step (1)

$$f(x_{k+1}) \leqslant Q^*(x_k, t_k, X_k) = f(x_k) - \frac{t_k}{2} \|P_k \nabla f(x_k)\|^2 \tag{10}$$

which immediately gives (8), and by μ-convexity

$$f(x^*) \geqslant f(x_k) + \langle \nabla f(x_k), x^* - x_k \rangle + \frac{\mu}{2} \|x^* - x_k\|^2,$$

such that with (H1)

$$\begin{aligned}
f(x_k) - f(x^*) \leqslant & \langle \nabla f(x_k), x_k - x^* \rangle - \frac{\mu}{2} \|x^* - x_k\|^2 \\
= & \langle P_k \nabla f(x_k), x_k - x^* \rangle + \langle (I - P_k) \nabla f(x_k), x_k - x^* \rangle - \frac{\mu}{2} \|x^* - x_k\|^2 \\
\leqslant & \langle P_k \nabla f(x_k), x_k - x^* \rangle + \frac{1}{2\mu} \|(I - P_k) \nabla f(x_k)\|^2 \\
\leqslant & \langle P_k \nabla f(x_k), x_k - x^* \rangle + \frac{C_{\mathrm{rel}}^2}{2\mu}\eta^2(x_k, X_k)
\end{aligned}$$

Adding (10), it then follows with the binomial identity that

$$\begin{aligned}
f(x_{k+1}) - f(x^*) \leqslant & \langle P_k \nabla f(x_k), x_k - x^* \rangle - \frac{t_k}{2} \|P_k \nabla f(x_k)\|^2 + \frac{C_{\mathrm{rel}}^2}{2\mu}\eta^2(x_k, X_k) \\
= & \frac{1}{2t_k} \left(2\langle t_k P_k \nabla f(x_k), x_k - x^* \rangle - \|t_k P_k \nabla f(x_k)\|^2 \right) + \frac{C_{\mathrm{rel}}^2}{2\mu}\eta^2(x_k, X_k) \\
= & \frac{1}{2t_k} \left(\|x_k - x^*\|^2 - \|x_k - x^* - t_k P_k \nabla f(x_k)\|^2 \right) + \frac{C_{\mathrm{rel}}^2}{2\mu}\eta^2(x_k, X_k) \\
= & \frac{1}{2t_k} \left(\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 \right) + \frac{C_{\mathrm{rel}}^2}{2\mu}\eta^2(x_k, X_k).
\end{aligned}$$

□

Theorem 1. *We suppose that* f *is continuously differentiable,* μ*-strongly convex, the level set* $\mathcal{L}_f(x_0)$ *is bounded and* ∇f *is* L*-Lipschitz on* $\mathcal{L}_f(x_0)$.

Suppose that

$$\bar{t} \geqslant t_k \geqslant \underline{t} > 0 \quad \forall k \in \mathbb{N}. \tag{11}$$

Let

$$e_k := f(x_k) - f(x^*) + C_1\eta^2(x_k, X_k), \qquad C_1 := \frac{C_{\mathrm{rel}}^2}{\mu}. \tag{12}$$

Then we have for all $m, n \in \mathbb{N}$ *and arbitrary* $\lambda > 0$

$$e_{m+n} \leqslant (C{+}1)\rho^n e_m, \quad C = \max\left\{ \frac{1}{4\mu\underline{t}} + \frac{2\max\left\{\lambda, 2C_{\mathrm{stab}}^2\right\} C_{\mathrm{rel}}^2}{\mu(1 - q_{\mathrm{red}})}\bar{t},\ \frac{1 + q_{\mathrm{red}}}{1 - q_{\mathrm{red}}} \right\}, \quad \rho = 1{-}1/(C{+}1). \tag{13}$$

Remark 2. *Supposing that $\underline{t}$ and $\bar{t}$ are proportional to $1/L$, we find that C is proportional to $\kappa_f = L/\mu$ as in the standard gradient method.*

Proof. We first claim that

$$\eta^2(x_{k+1}, X_{k+1}) \leqslant q_{\mathrm{red}}\eta^2(x_k, X_k) + \widetilde{\lambda}\bar{t}\left(f(x_k) - f(x_{k+1})\right), \quad \widetilde{\lambda} := \max\left\{\lambda, 2C_{\mathrm{stab}}^2\right\}. \tag{14}$$

If no refinement happens, this follows by rule (3) of the algorithm and the assumption (21). If a refinement step happens from k to $k+1$, we have by (H2) and (8)

$$\begin{aligned}\eta^2(x_{k+1}, X_{k+1}) &\leqslant q_{\mathrm{red}}\eta^2(x_k, X_k) + C_{\mathrm{stab}}^2 \left\|x_{k+1} - x_k\right\|^2 \\ &\leqslant q_{\mathrm{red}}\eta^2(x_k, X_k) + 2C_{\mathrm{stab}}^2 t_k \left(f(x_k) - f(x_{k+1})\right).\end{aligned}$$

Now let

$$\Delta_k := f(x_k) - f(x^*), \qquad \eta_k^2 = \eta^2(x_k, X_k), \qquad \zeta_k := \|x_k - x^*\|^2.$$

From (9), (24) and the assumption on the step-length (21), we have for $\beta := \frac{2C_{\mathrm{rel}}^2}{\mu(1-q_{\mathrm{red}})}$

$$\Delta_{k+1} + \beta\eta_{k+1}^2 \leqslant \left(q_{\mathrm{red}} + \frac{C_{\mathrm{rel}}^2}{\mu\beta}\right)\beta\eta_k^2 + \widetilde{\lambda}\beta\bar{t}\left(f(x_k) - f(x_{k+1})\right) + \frac{1}{2\underline{t}}\left(\zeta_k - \zeta_{k+1}\right).$$

such that with $\widetilde{q_{\mathrm{red}}} := q_{\mathrm{red}} + \frac{C_{\mathrm{rel}}^2}{\mu\beta} = \frac{1}{2}(1 + q_{\mathrm{red}}) < 1$

$$\Delta_{k+1} + \beta\eta_{k+1}^2 \leqslant \widetilde{q_{\mathrm{red}}}\beta\eta_k^2 + \widetilde{\lambda}\beta\bar{t}\left(f(x_k) - f(x_{k+1})\right) + \frac{1}{2\underline{t}}\left(\zeta_k - \zeta_{k+1}\right)$$

Summing up yields

$$\sum_{k=n+1}^{N+1}\left(\Delta_k + \beta\eta_k^2\right) \leqslant \widetilde{q_{\mathrm{red}}}\beta\sum_{k=n}^{N}\eta_k^2 + \widetilde{\lambda}\beta\bar{t}\left(f(x_n) - f(x_{N+1})\right) + \frac{1}{2\underline{t}}\left(\zeta_n - \zeta_{N+1}\right)$$

such that

$$\sum_{k=n+1}^{N+1}\Delta_k + (1 - \widetilde{q_{\mathrm{red}}})\beta\sum_{k=n+1}^{N+1}\eta_k^2 \leqslant \widetilde{q_{\mathrm{red}}}\beta\eta_n^2 + \widetilde{\lambda}\beta\bar{t}\left(f(x_n) - f(x_{N+1})\right) + \frac{1}{2\underline{t}}\left(\zeta_n - \zeta_{N+1}\right)$$

This proves $\lim\limits_{N\to\infty} x_N \to x^*$ and then, with $\zeta_n = \|x_n - x^*\|^2 \leqslant \frac{2}{\mu}\Delta_n$,

$$\begin{aligned}\sum_{k=n+1}^{\infty}\Delta_k + (1 - \widetilde{q_{\mathrm{red}}})\beta\sum_{k=n+1}^{\infty}\eta_k^2 &\leqslant \widetilde{q_{\mathrm{red}}}\beta\eta_n^2 + \widetilde{\lambda}\beta\bar{t}\left(f(x_n) - f(x^*)\right) + \frac{1}{2\underline{t}}\zeta_n \\ &\leqslant \left(\widetilde{\lambda}\beta\bar{t} + \frac{1}{4\mu\underline{t}}\right)\Delta_n + \widetilde{q_{\mathrm{red}}}\beta\eta_n^2\end{aligned}$$

With

$$C_1 = (1 - \widetilde{q_{\mathrm{red}}})\beta = \frac{C_{\mathrm{rel}}^2}{\mu}$$

we have

$$\begin{aligned}\sum_{k=n+1}^{\infty}\left(\Delta_k + C_1\eta_k^2\right) &\leqslant \left(\frac{1}{4\mu\underline{t}} + \widetilde{\lambda}\beta\bar{t}\right)\Delta_n + \widetilde{q_{\mathrm{red}}}\beta\eta_n^2 \\ &\leqslant \left(\frac{1}{4\mu\underline{t}} + \frac{\widetilde{\lambda}C_1}{1 - \widetilde{q_{\mathrm{red}}}}\bar{t}\right)\Delta_n + \frac{\widetilde{q_{\mathrm{red}}}}{1 - \widetilde{q_{\mathrm{red}}}}C_1\eta_n^2 \\ &= \left(\frac{1}{4\mu\underline{t}} + \frac{2\widetilde{\lambda}C_1}{1 - q_{\mathrm{red}}}\bar{t}\right)\Delta_n + \frac{1 + q_{\mathrm{red}}}{1 - q_{\mathrm{red}}}C_1\eta_n^2\end{aligned}$$

□

Theorem 2. *If λ satisfies*

$$\lambda \geqslant 2C_{\text{stab}} + 8\kappa_f^2 \frac{1-q_{\text{red}}}{\underline{t}^2} \tag{15}$$

we have

$$\sum_{k=0}^{n} \dim X_k \leqslant C\varepsilon_n^{-1/s} \quad \forall n \in \mathbb{N}.$$

Proof. By the Lipschitz-continuity we have

$$\|(I - P_{X_k})\nabla f(x_k)\| = \|(I - P_{X_k})(\nabla f(x_k) - \nabla f(x^*)\| \leqslant L\,\|x^* - x_k\|\,,$$

such that

$$\|(I - P_{X_k})\nabla f(x_k)\|^2 \leqslant \frac{L^2}{\mu}(f(x_k) - f(x^*)).$$

Let $\widetilde{X}_k \in \mathcal{X}(X_0)$ and $\widetilde{x}_k := \operatorname{argmin}_{x\in\widetilde{X}_k} f(x)$. If $f(\widetilde{x}_k) - f(x^*) \leqslant \gamma e_k$ we have

$$\begin{aligned} f(x_k) - f(x^*) =& f(x_k) - f(\widetilde{x}_k) + f(\widetilde{x}_k) - f(x^*) \\ \leqslant& f(x_k) - f(\widetilde{x}_k) + \gamma(f(x_k) - f(x^*) + C_1\eta^2(x_k, X_k)) \end{aligned}$$

and then for $\gamma < 1$

$$(1-\gamma)f(x_k) - f(x^*) \leqslant (f(x_k) - f(\widetilde{x}_k)) + \gamma C_1\eta^2(x_k, X_k)$$

By strong convexity we have

$$\begin{aligned} f(x_k) - f(\widetilde{x}_k) \leqslant& \langle \nabla f(x_k), x_k - \widetilde{x}_k\rangle - \frac{\mu}{2}\|x_k - \widetilde{x}_k\|^2 \\ =& \langle P_{X_k}\nabla f(x_k), x_k - \widetilde{x}_k\rangle + \langle (P_{\widetilde{X}_k} - P_{X_k})\nabla f(x_k), x_k - \widetilde{x}_k\rangle - \frac{\mu}{2}\|x_k - \widetilde{x}_k\|^2 \\ \leqslant& \frac{1}{t_k}\|x_{k+1} - x_k\|\,\|x_k - \widetilde{x}_k\| + \left\|(P_{\widetilde{X}_k} - P_{X_k})\nabla f(x_k)\right\|\,\|x_k - \widetilde{x}_k\| - \frac{\mu}{2}\|x_k - \widetilde{x}_k\|^2 \\ \leqslant& \frac{1}{\underline{t}^2\mu}\|x_{k+1} - x_k\|^2 + \frac{1}{\mu}\left\|(P_{\widetilde{X}_k} - P_{X_k})\nabla f(x_k)\right\|^2 \end{aligned}$$

From the refinement criterion we have

$$\eta^2(x_{k+1}, X_k) > q_{\text{red}}\eta^2(x_k, X_k) + \lambda t_k(f(x_k) - f(x_{k+1})) \geqslant q_{\text{red}}\eta^2(x_k, X_k) + \lambda\,\|x_{k+1} - x_k\|^2$$

With (E3) and (8) we have

$$\lambda\,\|x_{k+1} - x_k\|^2 \leqslant (1 - q_{\text{red}})\eta^2(x_k, X_k) + C_{\text{stab}}^2\,\|x_{k+1} - x_k\|^2$$

such that with $\xi := \lambda - 2C_{\text{stab}}^2 > 0$

$$\|x_{k+1} - x_k\|^2 \leqslant \frac{1 - q_{\text{red}}}{\xi}\eta^2(x_k, X_k).$$

Combining these inequalities we get with (H1)

$$\begin{aligned} \|(I - P_{X_k})\nabla f(x_k)\|^2 \leqslant& \frac{L^2}{\mu}(f(x_k) - f(x^*)) \leqslant \frac{L^2}{\mu(1-\gamma)}\left((f(x_k) - f(\widetilde{x}_k)) + \gamma C_1\eta^2(x_k, X_k)\right) \\ \leqslant& \frac{L^2}{\mu(1-\gamma)}\left(\left(\frac{1 - q_{\text{red}}}{\underline{t}^2\mu\xi} + \gamma C_1\right)\eta^2(x_k, X_k) + \frac{1}{\mu}\left\|(P_{\widetilde{X}_k} - P_{X_k})\nabla f(x_k)\right\|^2\right) \\ \leqslant& \frac{L^2}{\mu(1-\gamma)}\left(\left(\frac{1 - q_{\text{red}}}{\underline{t}^2\mu\xi} + \gamma C_1\right)C_{\text{eff}}^2\,\|(I - P_{X_k})\nabla f(x_k)\|^2 + \frac{1}{\mu}\left\|(P_{\widetilde{X}_k} - P_{X_k})\nabla f(x_k)\right\|^2\right) \end{aligned}$$

Then for

$$\gamma \leqslant \min\left\{\frac{1}{2}, \frac{\mu}{4L^2C_1C_{\text{eff}}^2}\right\}, \quad \xi \geqslant \frac{8L^2}{\mu}\frac{1-q_{\text{red}}}{\underline{t}^2\mu}$$

we finally have

$$\|(I-P_{X_k})\nabla f(x_k)\|^2 \leqslant 4\kappa_f^2 \left\|(P_{\widetilde{X}_k} - P_{X_k})\nabla f(x_k)\right\|^2$$

□

4 Accelerated gradient method

Algorithm 2: Adaptive **AGM**

Inputs: X_0, $x_0 \in X_0$, $t_0 > 0$, $\lambda > 0$, $\beta > 0$. Set $y_0 = x_0$ and $k = 0$.

(1) While $Q_{t_k}^*(y_k, X_k) < f(\widetilde{x}(y_k, t_k, X_k))$: $\quad t_k = t_k/2$.

(2) $x_{k+1} = \widetilde{x}(y_k, t_k, X_k)$.

(3) $y_{k+1} = x_{k+1} + \beta(x_{k+1} - x_k)$.

(4) If $\eta^2(x_{k+1}, X_k) > q_{\text{red}}\eta^2(x_k, X_k) + \lambda t_k(f(x_k) - f(x_{k+1}))$: $\quad X_k = \mathbf{REF}(X_k, \eta(x_k, X_k))$

(5) $t_k = 2t_k$.

(6) Increment k and go to (1).

Lemma 3. *The iterates of AGM satisfy*

$$f(x_{k+1}) \leqslant f(y_k) - \frac{t_k}{2}\|P_k\nabla f(y_k)\|^2 \tag{16}$$

$$f(x_{k+1}) - f(x_k) \leqslant \langle P_k\nabla f(y_k), y_k - x_k\rangle - \frac{t_k}{2}\|P_k\nabla f(y_k)\|^2 - \frac{1}{2L}\|P_k\nabla f(y_k) - P_k\nabla f(x_k)\|^2 \tag{17}$$

and

$$f(x_{k+1}) - f(x^*) \leqslant \langle P_k\nabla f(y_k), y_k - x^*\rangle - \frac{t_k}{2}\|P_k\nabla f(y_k)\|^2 + \frac{C_{\text{rel}}^2}{2\mu}\eta^2(y_k, X_k)). \tag{18}$$

Proof. (16) follows from the line search:

$$f(x_{k+1}) \leqslant Q_{t_k}(y_k, x_{k+1}) = Q_{t_k}^*(y_k) = f(y_k) - \frac{t_k}{2}\|P_k\nabla f(y_k)\|^2.$$

Next we have by convexity and Lipschitz continuity

$$f(x_k) \geqslant f(y_k) + \langle P_k\nabla f(y_k), x_k - y_k\rangle + \frac{1}{2L}\|P_k\nabla f(y_k) - P_k\nabla f(x_k)\|^2,$$

which, subtracted from (16) gives (17).

Similarly, we have

$$f(x^*) \geqslant f(y_k) + \langle \nabla f(y_k), x^* - y_k\rangle + \frac{\mu}{2}\|x^* - y_k\|^2,$$

which gives with (16)

$$
\begin{aligned}
f(x_{k+1}) - f(x^*) \leqslant & \langle P_k \nabla f(y_k), y_k - x^* \rangle - \frac{t_k}{2} \|P_k \nabla f(y_k)\|^2 + \langle (I - P_k)\nabla f(y_k), y_k - x^* \rangle - \frac{\mu}{2} \|x^* - y_k\|^2 \\
\leqslant & \langle P_k \nabla f(y_k), y_k - x^* \rangle - \frac{t_k}{2} \|P_k \nabla f(y_k)\|^2 + \frac{1}{2\mu} \|(I - P_k)\nabla f(y_k)\|^2 \\
\leqslant & \langle P_k \nabla f(y_k), y_k - x^* \rangle - \frac{t_k}{2} \|P_k \nabla f(y_k)\|^2 + \frac{C_{\text{rel}}^2}{2\mu} \eta^2(y_k, X_k)
\end{aligned}
$$

□

Lemma 4. *Let*

$$
\Delta f_k := f(x_k) - f(x^*) \quad \textit{and} \quad \overline{x}_{k+1} = x_{k+1} + \frac{\beta}{1-\beta}(x_{k+1} - x_k). \tag{19}
$$

We have

$$
\Delta f_{k+1} - \beta \Delta f_k \leqslant \frac{1-\beta}{2t_k} \left(\|\overline{x}_k - x^*\|^2 - \|\overline{x}_{k+1} - x^*\|^2 \right) + \frac{(1-\beta) C_{\text{rel}}^2}{2\mu} \eta^2(y_k, X_k) \tag{20}
$$

Proof. Multiplying (17) by β and (18) by $1-\beta$ we have

$$
\Delta f_{k+1} - \beta \Delta f_k \leqslant \langle P_k \nabla f(y_k), y_k - \beta x_k - (1-\beta) x^* \rangle - \frac{t_k}{2} \|P_k \nabla f(y_k)\|^2 + (1-\beta) \frac{C_{\text{rel}}^2}{2\mu} \eta^2(y_k, X_k)
$$

By the binomial formula $2ab - a^2 = b^2 - (b-a)^2$ and the update rule for y_k we have

$$
\begin{aligned}
& 2\langle t_k P_k \nabla f(y_k), y_k - \beta x_k - (1-\beta) x^* \rangle - \|t_k P_k \nabla f(y_k)\|^2 \\
= & \|y_k - \beta x_k - (1-\beta) x^*\|^2 - \|y_k - \beta x_k - (1-\beta) x^* - t_k P_k \nabla f(y_k)\|^2 \\
= & \|x_k - \beta x_{k-1} - (1-\beta) x^*\|^2 - \|x_{k+1} - \beta x_k - (1-\beta) x^*\|^2 \\
= & (1-\beta) \left(\|\overline{x}_k - x^*\|^2 - \|\overline{x}_{k+1} - x^*\|^2 \right)
\end{aligned}
$$

□

Theorem 3. *We suppose that f is continuously differentiable, μ-strongly convex, the level set $\mathcal{L}_f(x_0)$ is bounded and ∇f is L-Lipschitz on $\mathcal{L}_f(x_0)$.*

Suppose that

$$
\overline{t} \geqslant t_k \geqslant \underline{t} > 0 \quad \forall k \in \mathbb{N}. \tag{21}
$$

Let

$$
e_k := f(x_k) - f(x^*) + C_1 \eta^2(x_k, X_k), \qquad C_1 := \frac{C_{\text{rel}}^2}{\mu}. \tag{22}
$$

Then we have for all $m, n \in \mathbb{N}$ and arbitrary $\lambda > 0$

$$
e_{m+n} \leqslant (C{+}1)\rho^n e_m, \quad C = \max\left\{ \frac{1}{4\mu \underline{t}} + \frac{2\max\left\{\lambda, 2C_{\text{stab}}^2\right\} C_{\text{rel}}^2}{\mu(1 - q_{\text{red}})} \overline{t},\ \frac{1 + q_{\text{red}}}{1 - q_{\text{red}}} \right\}, \quad \rho = 1{-}1/(C{+}1). \tag{23}
$$

Remark 3. *Supposing that $\underline{t}$ and $\overline{t}$ are proportional to L, we find that C is proportional to $\kappa_f = L/\mu$ as in the standard gradient method.*

Proof. We first claim that

$$\eta^2(x_{k+1}, X_{k+1}) \leqslant q_{\text{red}}\eta^2(x_k, X_k) + \widetilde{\lambda}\bar{t}\,(f(x_k) - f(x_{k+1}))\,, \quad \widetilde{\lambda} := \max\left\{\lambda, 2C_{\text{stab}}^2\right\}. \tag{24}$$

If no refinement happens, this follows by rule (4) of the algorithm and the assumption (21). If a refinement step happens from k to $k+1$, we have by (H2) and (8)

$$\begin{aligned}\eta^2(x_{k+1}, X_{k+1}) \leqslant& q_{\text{red}}\eta^2(x_k, X_k) + C_{\text{stab}}^2 \left\|x_{k+1} - x_k\right\|^2 \\ \leqslant& q_{\text{red}}\eta^2(x_k, X_k) + 2C_{\text{stab}}^2 t_k\,(f(x_k) - f(x_{k+1}))\,.\end{aligned}$$

Now let

$$\Delta_k := f(x_k) - f(x^*), \qquad \eta_k^2 = \eta^2(x_k, X_k), \qquad \zeta_k := \left\|x_k - x^*\right\|^2.$$

From (9), (24) and the assumption on the step-length (21), we have for $\beta := \frac{2C_{\text{rel}}^2}{\mu(1-q_{\text{red}})}$

$$\Delta_{k+1} + \beta\eta_{k+1}^2 \leqslant \left(q_{\text{red}} + \frac{C_{\text{rel}}^2}{\mu\beta}\right)\beta\eta_k^2 + \widetilde{\lambda}\beta\bar{t}\,(f(x_k) - f(x_{k+1})) + \frac{1}{2\underline{t}}\,(\zeta_k - \zeta_{k+1})\,.$$

such that with $\widetilde{q_{\text{red}}} := q_{\text{red}} + \frac{C_{\text{rel}}^2}{\mu\beta} = \frac{1}{2}(1 + q_{\text{red}}) < 1$

$$\Delta_{k+1} + \beta\eta_{k+1}^2 \leqslant \widetilde{q_{\text{red}}}\beta\eta_k^2 + \widetilde{\lambda}\beta\bar{t}\,(f(x_k) - f(x_{k+1})) + \frac{1}{2\underline{t}}\,(\zeta_k - \zeta_{k+1})$$

Summing up yields

$$\sum_{k=n+1}^{N+1}\left(\Delta_k + \beta\eta_k^2\right) \leqslant \widetilde{q_{\text{red}}}\beta\sum_{k=n}^{N}\eta_k^2 + \widetilde{\lambda}\beta\bar{t}\,(f(x_n) - f(x_{N+1})) + \frac{1}{2\underline{t}}\,(\zeta_n - \zeta_{N+1})$$

such that

$$\sum_{k=n+1}^{N+1}\Delta_k + (1 - \widetilde{q_{\text{red}}})\beta\sum_{k=n+1}^{N+1}\eta_k^2 \leqslant \widetilde{q_{\text{red}}}\beta\eta_n^2 + \widetilde{\lambda}\beta\bar{t}\,(f(x_n) - f(x_{N+1})) + \frac{1}{2\underline{t}}\,(\zeta_n - \zeta_{N+1})$$

This proves $\lim_{N\to\infty} x_N \to x^*$ and then, with $\zeta_n = \left\|x_n - x^*\right\|^2 \leqslant \frac{2}{\mu}\Delta_n$,

$$\begin{aligned}\sum_{k=n+1}^{\infty}\Delta_k + (1 - \widetilde{q_{\text{red}}})\beta\sum_{k=n+1}^{\infty}\eta_k^2 \leqslant& \widetilde{q_{\text{red}}}\beta\eta_n^2 + \widetilde{\lambda}\beta\bar{t}\,(f(x_n) - f(x^*)) + \frac{1}{2\underline{t}}\zeta_n \\ \leqslant& \left(\widetilde{\lambda}\beta\bar{t} + \frac{1}{4\mu\underline{t}}\right)\Delta_n + \widetilde{q_{\text{red}}}\beta\eta_n^2\end{aligned}$$

With

$$C_1 = (1 - \widetilde{q_{\text{red}}})\beta = \frac{C_{\text{rel}}^2}{\mu}$$

we have

$$\begin{aligned}\sum_{k=n+1}^{\infty}\left(\Delta_k + C_1\eta_k^2\right) \leqslant& \left(\frac{1}{4\mu\underline{t}} + \widetilde{\lambda}\beta\bar{t}\right)\Delta_n + \widetilde{q_{\text{red}}}\beta\eta_n^2 \\ \leqslant& \left(\frac{1}{4\mu\underline{t}} + \frac{\widetilde{\lambda}C_1}{1 - \widetilde{q_{\text{red}}}}\bar{t}\right)\Delta_n + \frac{\widetilde{q_{\text{red}}}}{1 - \widetilde{q_{\text{red}}}}C_1\eta_n^2 \\ =& \left(\frac{1}{4\mu\underline{t}} + \frac{2\widetilde{\lambda}C_1}{1 - q_{\text{red}}}\bar{t}\right)\Delta_n + \frac{1 + q_{\text{red}}}{1 - q_{\text{red}}}C_1\eta_n^2\end{aligned}$$

$\square$

Theorem 4. *If λ satisfies*

$$\lambda \geqslant 2C_{\text{stab}} + 8\kappa_f^2 \frac{1-q_{\text{red}}}{\underline{t}^2} \tag{25}$$

we have

$$\sum_{k=0}^{n} \dim X_k \leqslant C\varepsilon_n^{-1/s} \quad \forall n \in \mathbb{N}.$$

Proof. By the Lipschitz-continuity we have

$$\|(I - P_{X_k})\nabla f(x_k)\| = \|(I - P_{X_k})(\nabla f(x_k) - \nabla f(x^*)\| \leqslant L \|x^* - x_k\|,$$

such that

$$\|(I - P_{X_k})\nabla f(x_k)\|^2 \leqslant \frac{L^2}{\mu}(f(x_k) - f(x^*)).$$

Let $\widetilde{X}_k \in \mathcal{X}(X_0)$ and $\widetilde{x}_k := \operatorname{argmin}_{x \in \widetilde{X}_k} f(x)$. If $f(\widetilde{x}_k) - f(x^*) \leqslant \gamma e_k$ we have

$$\begin{aligned} f(x_k) - f(x^*) =& f(x_k) - f(\widetilde{x}_k) + f(\widetilde{x}_k) - f(x^*) \\ \leqslant& f(x_k) - f(\widetilde{x}_k) + \gamma(f(x_k) - f(x^*) + C_1\eta^2(x_k, X_k)) \end{aligned}$$

and then for $\gamma < 1$

$$(1-\gamma)f(x_k) - f(x^*) \leqslant (f(x_k) - f(\widetilde{x}_k)) + \gamma C_1 \eta^2(x_k, X_k)$$

By strong convexity we have

$$\begin{aligned} f(x_k) - f(\widetilde{x}_k) \leqslant& \langle \nabla f(x_k), x_k - \widetilde{x}_k \rangle - \frac{\mu}{2}\|x_k - \widetilde{x}_k\|^2 \\ =& \langle P_{X_k}\nabla f(x_k), x_k - \widetilde{x}_k\rangle + \langle (P_{\widetilde{X}_k} - P_{X_k})\nabla f(x_k), x_k - \widetilde{x}_k\rangle - \frac{\mu}{2}\|x_k - \widetilde{x}_k\|^2 \\ \leqslant& \frac{1}{t_k}\|x_{k+1} - x_k\|\,\|x_k - \widetilde{x}_k\| + \left\|(P_{\widetilde{X}_k} - P_{X_k})\nabla f(x_k)\right\| \|x_k - \widetilde{x}_k\| - \frac{\mu}{2}\|x_k - \widetilde{x}_k\|^2 \\ \leqslant& \frac{1}{\underline{t}^2\mu}\|x_{k+1} - x_k\|^2 + \frac{1}{\mu}\left\|(P_{\widetilde{X}_k} - P_{X_k})\nabla f(x_k)\right\|^2 \end{aligned}$$

From the refinement criterion we have

$$\eta^2(x_{k+1}, X_k) > q_{\text{red}}\eta^2(x_k, X_k) + \lambda t_k(f(x_k) - f(x_{k+1})) \geqslant q_{\text{red}}\eta^2(x_k, X_k) + \lambda\|x_{k+1} - x_k\|^2$$

With (E3) and (8) we have

$$\lambda\|x_{k+1} - x_k\|^2 \leqslant (1 - q_{\text{red}})\eta^2(x_k, X_k) + C_{\text{stab}}^2\|x_{k+1} - x_k\|^2$$

such that with $\xi := \lambda - 2C_{\text{stab}}^2 > 0$

$$\|x_{k+1} - x_k\|^2 \leqslant \frac{1 - q_{\text{red}}}{\xi}\eta^2(x_k, X_k).$$

Combining these inequalities we get with (H1)

$$\begin{aligned} \|(I - P_{X_k})\nabla f(x_k)\|^2 \leqslant& \frac{L^2}{\mu}(f(x_k) - f(x^*)) \leqslant \frac{L^2}{\mu(1-\gamma)}\left((f(x_k) - f(\widetilde{x}_k)) + \gamma C_1\eta^2(x_k, X_k)\right) \\ \leqslant& \frac{L^2}{\mu(1-\gamma)}\left(\left(\frac{1-q_{\text{red}}}{\underline{t}^2\mu\xi} + \gamma C_1\right)\eta^2(x_k, X_k) + \frac{1}{\mu}\left\|(P_{\widetilde{X}_k} - P_{X_k})\nabla f(x_k)\right\|^2\right) \\ \leqslant& \frac{L^2}{\mu(1-\gamma)}\left(\left(\frac{1-q_{\text{red}}}{\underline{t}^2\mu\xi} + \gamma C_1\right)C_{\text{eff}}^2\|(I - P_{X_k})\nabla f(x_k)\|^2 + \frac{1}{\mu}\left\|(P_{\widetilde{X}_k} - P_{X_k})\nabla f(x_k)\right\|^2\right) \end{aligned}$$

Then for

$$\gamma \leqslant \min\left\{\frac{1}{2}, \frac{\mu}{4L^2C_1C_{\mathrm{eff}}^2}\right\}, \quad \xi \geqslant \frac{8L^2}{\mu}\frac{1-q_{\mathrm{red}}}{\underline{t}^2\mu}$$

we finally have

$$\|(I - P_{X_k})\nabla f(x_k)\|^2 \leqslant 4\kappa_f^2 \left\|(P_{\tilde{X}_k} - P_{X_k})\nabla f(x_k)\right\|^2$$

□

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