1. Timothe is reserving 100 classrooms for BmMT this year, and each room has exactly one purpose: testing, grading, or activities. Half of the rooms are for testing and four rooms are for grading. How many rooms are left for activities?

   Answer: 46

   Solution: Out of the 100 rooms, $\frac{1}{2} \cdot 100 = 50$ rooms are for testing, and 4 rooms are for grading. This leaves $100 - 50 - 4 = \boxed{46}$ rooms for activities.

2. Points A, B, and C lie on a straight line, not necessarily in that order. The distance between A and B is 15, and the distance between A and C is 7. What is the sum of all distinct possible distances between B and C?

   Answer: 30

   Solution: Similar to the solution for the local version, $15 + 7 = 22$, $15 - 7 = 8$. The sum is $22 + 8 = \boxed{30}$.

3. Benji writes down $1, 2, 3, 4, 5$ and Kiran writes down $6, 7, 8, 9, 10$. Later, Kiran erases some of his numbers so that the sum of his remaining numbers equals the sum of Benji's numbers. What is the sum of the numbers that Kiran erased?

   Answer: 25

   Solution: Similar to the solution for the local version, the sum of Benji's numbers is 15, and the sum of Kiran's numbers is 40. So what Kiran erased sums to $\boxed{25}$.

   For example, Kiran can erase $6, 9, 10$, or $7, 8, 10$.

4. A bag contains two red marbles, two blue marbles, two green marbles, and two yellow marbles. Richard removes marbles from the bag one at a time without replacement. What is the least number of marbles Richard must remove to guarantee that two of the marbles that he has taken out are the same color?

   Answer: 5

   Solution: Similar to the solution for the local version, Richard need to take at least one for each color + one extra to guarantee a repeat, by pigeonhole principle.

   There are 4 colors, so Richard needs to remove $\boxed{5}$.

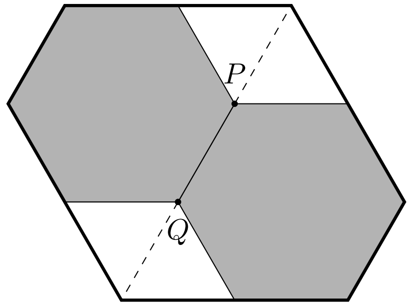
5. Four consecutive even integers sum to -20. What is the product of these four integers?

   Answer: 384

   Solution: Similar to the solution for the local version, the four even integers are $-2, -4, -6, -8$. Their product is

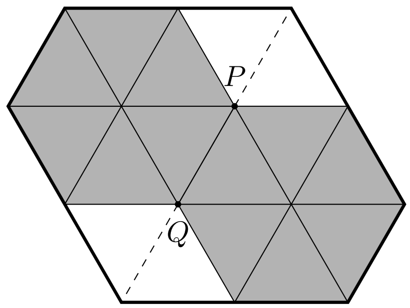
$$(-2)(-4)(-6)(-8) = \boxed{384}$$

6. Points P and Q are vertices shared by two congruent regular hexagons (shaded) and two equilateral triangles (unshaded) with the same side lengths, as shown below. Together, these shapes form a larger, non-regular hexagon (drawn with a thick border). What is the ratio of the **combined** area of both regular hexagons (the shaded area) to the area of the large hexagon?



Answer: $\mathbf{\frac{3}{4}}$

Solution:



Similar to the solution for the local version, as shown above, each of the regular hexagons can be cut into 6 equilateral triangles. Since the larger hexagon is made of 2 regular hexagons and 4 equilateral triangles, it has a total area equal to that of $2 \cdot 6 + 4 = 16$ equilateral triangles. Thus, the ratio of the combined area of the two regular hexagons to the larger hexagon is $\frac{2 \cdot 6}{16} = \boxed{\frac{3}{4}}$.

7. How many positive factors of 84 are divisible by 3? Note that 1 and 84 are positive factors of 84.

   Answer: 6

   Solution: Similar to the solution for the local version, a number $3n$ is a factor of 84 if and only if n is a factor of 28. The factors of 28 are

$$1, 2, 4, 7, 14, 28.$$

   So there are $\boxed{6}$ choices of $3n$ that are factors of 84.

8. Luke the frog lives on a pond with 4 lilypads, labeled 1 through 4. He starts at lilypad 1 and, at every step, hops to a lilypad with a larger number, chosen uniformly at random. Luke continues hopping until he reaches lilypad 4. What is the probability that it takes Luke exactly 2 steps to reach lilypad 4?

   Answer: $\mathbf{\frac{1}{2}}$

   Solution: Similar to the solution for the local version, the cases are

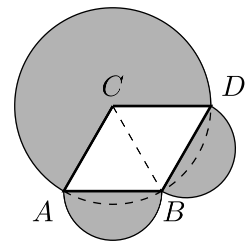
$$1 \to 2 \to 4$$

$$1 \to 3 \to 4$$

So the probability is

$$\frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1 = \boxed{\frac{1}{2}}.$$

9. Around equilateral triangles $\triangle ABC$ and $\triangle BCD$ shown below, a circle centered at C with radius $\overline{CA}$ and semicircles with diameters $\overline{AB}$ and $\overline{BD}$ are drawn. If the total area of the shaded region (the whole shape excluding $\triangle ABC$ and $\triangle BCD$) is 3300π, what is the perimeter of parallelogram $ABCD$?



Answer: 240

Solution: Similar to the solution for the local version, let $AC = AB = BD = 2r$. The sum of the shaded areas are

$$\frac{2}{3}\pi(2r)^2 + \frac{1}{2}\pi r^2 + \frac{1}{2}\pi r^2 = \left(\frac{8}{3} + \frac{1}{2} + \frac{1}{2}\right)\pi r^2 = \frac{11r^2}{3}\pi.$$

Since $\frac{11r^2}{3} = 3300$, $r = 30$, so the perimeter is $4 \cdot (2r) = \boxed{240}$.

10. Aaron has four coins. These coins have values of 1, 10, 100, and 1000 cents. Aaron makes a list of all the different numbers of cents he can make with some or all of his coins. If Aaron adds up all the numbers in his list, what is the result **in cents**?

Answer: 8888

Solution: Similar to the solution for the local version, in the list, on each digit, the number 1 appears eight times, so the sum is $1111 \times 8 = \boxed{8888}$.

The list is

$$1, 10, 100, 1000,$$

$$11, 101, 110, 1001, 1010, 1100,$$

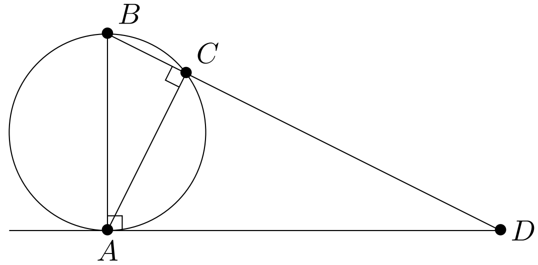
$$111, 1011, 1101, 1110, 1111.$$

They add up to 8888.

11. Points A, B, and C lie on a circle. Let segment $\overline{BC}$ extend through C to point D such that $\overline{AD}$ is tangent to the circle. If $AC = 11$, $BC = 6$, and $\angle ACD = 90^\circ$, what is CD?

Answer: $\mathbf{\frac{121}{6}}$

Solution:



Since $\angle ACD = 90°$, $\angle ACB = 90°$. Then we conclude that $\overline{AB}$ is a diameter of the circle. Then for $\overline{AD}$ to be tangent to the circle, we must have $\angle BAD = 90°$. Thus $\triangle BAD$ is a right triangle.

Now, note that $\angle CAB = 90° - \angle ABC = \angle ADC$, so $\triangle ADC$ is similar to $\triangle BAC$. Thus,

$$\frac{CD}{CA} = \frac{AC}{BC},$$

and thus $CD = \frac{AC^2}{BC} = \boxed{\frac{121}{6}}$.

12. Find the sum of all numbers n such that the equation $x^2 - nx + 72 = 0$ has two positive integer solutions, and one solution is an integer multiple of the other.

Answer: 156

Solution: Similar to the solution for the local version, let $f(x) = x^2 - nx + 72$.

We can rewrite $f(x)$ as $f(x) = (x - a)(x - ka)$, where k is some positive integer. Expanding this out gives us

$$f(x) = x^2 - (k+1)ax + ka^2$$

This yields the following two equations:

$$(k+1)a = n,$$

$$ka^2 = 72.$$

Looking at $ka^2 = 72$, since both k and a must be integers, and a^2 divides 72, $a = 1, 2, 3, 6$. The pairs of possible ka and a are

$$(ka, a) = \{(72, 1), (36, 2), (24, 3), (12, 6)\}.$$

Therefore, the sum of all possible values of n is $(72+1)+(36+2)+(24+3)+(12+6) = \boxed{156}$.

13. A positive integer n is *two-cool* if the decimal expansion of $n/250$ has exactly one or three digits past the decimal point, excluding trailing zeros. For example, $10/250 = 0.04$ has exactly two digits past the decimal point, and $100/250 = 0.4$ has exactly one digit past the decimal point, so 100 is *two-cool* but 10 is not. How many positive integers less than 35 are *two-cool*?

Answer: 29

Solution: Similar to the solution for the local version, notice that $\frac{1}{250} = 0.004$. So $1, 2, 3, 4$ are two-cool, but $\frac{5}{250} = 0.02$ is not. $6, 7, 8, 9$ are two-cool, but 10 is not.

Out of multiples of 5 under 35, only 25 is two-cool. So we have

$$4 \cdot 7 + 1 = \boxed{29}$$

two-cool numbers under 35.

Out of the numbers $1 - 34$, $5, 10, 15, 20, 30$ are not two-cool.

14. Define a sequence of positive integers $a_1, a_2, a_3, \dots$ such that $a_1 = 1$ and

$$a_i = 61a_{i-1} + i$$

for all integers $i \geq 2$. What is the smallest positive integer n such that both a_n and a_{n-1} are divisible by 12?

Answer: 24

Solution: $61 \equiv 1 \pmod{12}$, so $a_i \equiv a_{i-1} + i \pmod{12}$.

The general formula for a_n is

$$a_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \pmod{12}.$$

We need $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$ to both be divisible by 12.

Out of $(n-1)$, n, $(n+1)$: if $n-1$ or $n+1$ is divisible by 3, the other two cannot be, so we can't have both $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$ divisible by 12. So n is divisible by 3.

Similarly, $(n+1)$ and $(n-1)$ cannot both be divisible by 4 if they are even. Therefore n is even. In fact n needs to be divisible by 8. The smallest such n is $\boxed{24}$.

15. Let (a, b, c, d, e) be a permutation of $(1, 3, 5, 7, 9)$. For example, a possible permutation is $(a, b, c, d, e) = (7, 3, 9, 1, 5)$. What is the maximum possible value of $a - ab + abc - abcd + abcde$, where ab, abc, $abcd$, and $abcde$ all represent multiplication (for example, $ab = a \times b$)?

Answer: 935

Solution: Fixing c, d, e, the only two terms where the order of a and b matters is a. The other term either has both a and b, or has neither of them. The value of a is maximized when $a > b$.

Similarly, fixing a, d, e, we find that $b < c$.

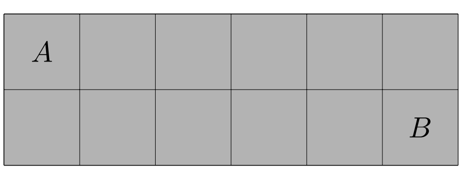
Similarly, $c > d$, $d < e$.

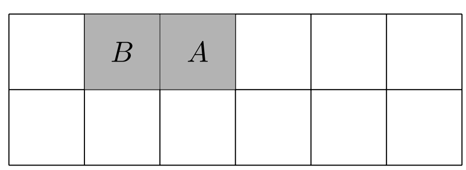
Fixing b, d, e, we consider a and c. The only terms where the order of a and c matters is $a - ab = a(1-b)$. Since $(1-b) < 0$, we want $a < c$.

Similarly, we found that $b > d$, $c < e$. Altogether, $d < b < a < c < e$. So the answer is

$$\begin{aligned} &5 - 5 \cdot 3 + 5 \cdot 3 \cdot 7 - 5 \cdot 3 \cdot 7 \cdot 1 + 5 \cdot 3 \cdot 7 \cdot 1 \cdot 9 \\ &= 5 - 15 + 105 - 105 + 945 = \boxed{935}. \end{aligned}$$

16. Nikki chooses three distinct square cells, A, B, and C, from a 2×6 square grid uniformly at random. What is the probability that square C is contained within the rectangle whose opposite corners are squares A and B? Examples of rectangles with opposite corners at A and B are shown as shaded regions below.



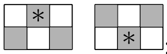
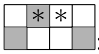
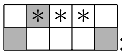
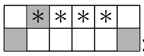


Answer: $\mathbf{\frac{3}{11}}$

Solution: We do complementary counting using by casework.

If three squares are in a configuration such that one is contained in the rectangle formed by the other two, there are always two choices out of the six ways to assign A, B to the diagonal squares and C to the remaining square. So we count the number of label-less configurations such that no square is within the other two, take the complement, and multiply by 2.

There are $\frac{12 \cdot 11 \cdot 10}{6} = 220$ ways to pick 3 squares out of 12, without labels. If no square is contained in the rectangle formed by the other two on the corners, it should look like one of:

- : 8 ways. 4 ways to place this in the grid, 1 slot for the $*$ square, and 2 choices of whether to flip it.
- : 12 ways. 3 locations in the grid, 2 slots for the $*$ square, and 2 flips.
- : 12 ways. 2 locations in the grid, 3 slots for the $*$ square, and 2 flips.
- : 8 ways. 1 location in the grid, 4 slots for the $*$ square, and 2 flips.

$$220 - 8 - 12 - 12 - 8 = 180.$$

There are 180 ways to pick 3 squares such that one square is in the rectangle defined by the other two. For each of the 180 ways, the "inside" square is always C, and there are two ways to put A and B in the other two squares. So out of $12 \cdot 11 \cdot 10$ ways to label 3 squares out of 12, the probability is

$$\frac{180 \cdot 2}{12 \cdot 11 \cdot 10} = \boxed{\frac{3}{11}}.$$

17. Let a, b, and c be positive integers such that $\{a, b, c, 2025 \cdot 6^7\}$ is a geometric sequence that is strictly increasing, meaning $a < b < c < 2025 \cdot 6^7$. Find the number of distinct possible values of a.

Answer: 107

Solution: The common ratio of the geometric sequence is a rational number since c is an integer. Denote the common ratio as $r = \frac{x}{y}$, where x, y are coprime positive integers.

Then, we can express the geometric sequence as a, ar, ar^2, ar^3. Therefore, $2025 \cdot 6^7 = ar^3$, and

$$a = \frac{ar^3}{r^3} = 2025 \cdot 6^7 \frac{y^3}{x^3}$$

is a positive integer. The prime factorization of $2025 \cdot 6^7$ is

$$2025 \cdot 6^7 = 2^7 \cdot 3^{11} \cdot 5^2.$$

Notice that since x^3 must be a perfect cube, the exponents in its prime factorization must all be multiples of 3. The maximum value of x^3 is therefore $2^6 \cdot 3^9$, and the maximum value of x is $2^2 \cdot 3^3 = 108$. Furthermore, since $r > 1$, y can only range from 1 to 107, and each such y gives a valid geometric sequence. Therefore, a can take on $\boxed{107}$ values.

18. Let $ABCDEF$ be a regular hexagon. Let P be a point on segment $\overline{BF}$, and let line $\overleftrightarrow{CP}$ intersect segment $\overline{AF}$ at Q. If $AB = 20$ and $\frac{BP}{PF} = \frac{5}{2}$, what is PQ?

Answer: $\frac{20}{21}\sqrt{31}$

Solution: When $\frac{BP}{PF} = \frac{5}{2}$:

Extend $\overline{FB}$ and $\overline{DC}$ to meet at R. The key observation is that triangles $\triangle QPF$ and $\triangle CPR$ are similar.

Next, note that triangles $\triangle BFC$ and $\triangle BRC$ are congruent, so $BF = BR$. Thus

$$\frac{CR}{QF} = \frac{CP}{PQ} = \frac{RP}{PF} = \frac{BR + BP}{PF} = \frac{BF + BP}{PF} = \frac{7+5}{2} = \frac{6}{1}.$$

Since $CR = CF = 2AB = 40$, $FQ = \frac{40}{6} = \frac{20}{3}$, $AQ = 20 - \frac{20}{3} = \frac{40}{3}$, and by Pythagoras's theorem

$$CQ = \sqrt{1200 + \frac{1600}{9}} = \frac{20}{3}\sqrt{31}$$

$$PQ = \frac{1}{1+6}CQ = \frac{1}{7}CQ = \boxed{\frac{20}{21}\sqrt{31}}.$$

19. Danielle calculates the ones digit of each of the 2048 integers $1^{1!}, 2^{2!}, 3^{3!}, \cdots, 2048^{2048!}$. If Danielle adds up all these ones digits, what is the resulting value?

Answer: 6770

Solution: Notice that, for every term, we do only care about the units digit of the base. Now, let's consider unit digits.

- For any odd multiple of 5, x, and any positive integer n, x^{4n} ends in 5.
- For any even multiple of 5, x, and any positive integer n, x^{4n} ends in 0.
- For any odd number, x, that does not end in 5 and any positive integer n, x^{4n} ends in 1.
- For any even number, x, that does not end in 0 and any positive integer n, x^{4n} ends in 6.

Now, notice that $n!$ is divisible by 4 for all $n \geq 4$. Therefore, we count the number of values between 4 and 2048 inclusive in each of the above categories:

- All even multiples of 5 are multiples of 10, so there are $\lfloor \frac{2048}{10} \rfloor = 204$ of them (because neither $1, 2$, or 3 are divisible by 5).
- All other multiples of 5 are odd multiple of 5, so there are $\lfloor \frac{2048}{5} \rfloor - 204 = 205$ of them.
- There are $(2048 - 4 + 1) - 409 = 1636$ numbers between 4 and 2048 that are not multiples of 5. Since there are 2 more odd multiples of 5 than even within these 1636 numbers, we then have 819 even non-multiples of 5 and 817 odd non-multiples of 5.

Thus, the answer is $1 + 4 + 9 + (204 \cdot 0) + (205 \cdot 5) + (819 \cdot 6) + (817 \cdot 1) = \boxed{6770}$.

20. There exist strictly increasing arithmetic sequences of real numbers, $\{a, b, c\}$ and $\{p, q, r\}$, having the properties that q is a positive integer greater than 1 and that the equation $x^3 - ax^2 + bx - c = 0$ has solutions p, q, and r. Over all such pairs of increasing arithmetic sequences, what is the least possible value of the product pqr?

Answer: 189

Solution: Let $p = q - n$ and $r = q + n$ for some positive number n. Using Vieta's relations, we can generate the following equations:

$$a = (q - n) + q + (q + n) = 3q$$

$$b = (q - n)(q) + (q - n)(q + n) + q(q + n) = 3q^2 - n^2$$

$$c = (q - n)(q)(q + n) = q^3 - qn^2$$

Furthermore, $a + c = 2b$ since $\{a, b, c\}$ is an arithmetic sequence. Substituting our above expressions for a, b, c, we generate

$$q^3 - qn^2 + 3q = 6q^2 - 2n^2 \to q(q^2 - 6q + 3) = n^2(q - 2)$$

Since $n^2 > 0$, it must also be true that $\frac{q(q^2-6q+3)}{q-2} > 0$. Note that we don't have to worry about $q = 2$ causing a division by 0, as $q(q^2 - 6q + 3) = n^2(q - 2)$ fails for $q = 2$, so we can multiply both sides by $(q - 2)$ and divide both sides by q to yield $q^2 - 6q + 3 = (q - 3)^2 - 6 > 0$. Thus, $(q - 3)^2 > 6$ and $q \geq 6$.

Minimizing pqr is minimizing $q^3 - qn^2$. We already have $n^2 = \frac{q(q^2-6q+3)}{q-2}$, so

$$\begin{aligned}
pqr = c &= q^3 - q\frac{q(q^2 - 6q + 3)}{q - 2} \\
&= q^3 - \frac{q^2q(q - 2) + q^2(-4q + 3)}{q - 2} = q^3 - q^3 + \frac{q^2(4q - 3)}{q - 2} \\
&= 4q^2 + 5q + 10 + \frac{20}{q - 2} = 4q^2 + 20 + 5\left((q - 2) + \frac{4}{q - 2}\right)
\end{aligned}$$

Since $(q - 2) + \frac{4}{q-2}$ monotonically increases when $q - 2 > 2$, or $q > 4$. We already got $q \geq 6$, so pqr is minimized when $q = 6$.

Plugging in $q = 6$, these three terms form an arithmetic sequence:

$$18, 108 - n^2, 216 - 6n^2.$$

The common difference is $90 - n^2 = 108 - 5n^2$, so $n^2 = \frac{9}{2}$.

The minimum possible value for the product pqr is $216 - 6 \cdot \frac{9}{2} = \boxed{189}$.