1. Lakshay chooses two numbers, m and n, and draws two lines, $y = mx + 3$ and $y = nx + 23$. Given that the two lines intersect at $(20, 23)$, compute $m + n$.

   Answer: 1

   Solution: Since $(20, 23)$ lies on both lines, we can plug it in to our two equations: $23 = 20m+3$ and $23 = 20n + 23$. In particular, we get that $m = 1$ and $n = 0$, for a final answer of $\boxed{1}$.

2. For real numbers x and y, suppose that $|x| - |y| = 20$ and $|x| + |y| = 23$. Compute the sum of all possible distinct values of $|x - y|$.

   Answer: 43

   Solution: Adding the equations gives $2|x| = 43$, which means $(x, y) = (\pm\frac{43}{2}, \pm\frac{3}{2})$. Thus, the possible values of $x - y$ are $\pm 20, \pm 23$. However, the negative solutions are extraneous because we are interested in $|x - y|$. The sum of all unique possible values therefore is $20 + 23 = \boxed{43}$.

3. Consider two geometric sequences $16, a_1, a_2, \ldots$ and $56, b_1, b_2, \ldots$ with the same common nonzero ratio. Given that $a_{2023} = b_{2020}$, compute $b_6 - a_6$.

   Answer: 490

   Solution: Let r represent the common ratio of both sequences. Then $a_{2023} = b_{2020}$ implies $16 \cdot r^{2023} = 56 \cdot r^{2020}$. This means $\frac{r^{2023}}{r^{2020}} = r^3 = \frac{56}{16} = \frac{7}{2}$. Then we have $a_6 = 16 \cdot r^6 = 16 \cdot (r^3)^2 = 16 \cdot (\frac{7}{2})^2 = 196$. Similarly, $b_6 = 56 \cdot r^6 = 56 \cdot (r^3)^2 = 56 \cdot (\frac{7}{2})^2 = 686$. Then $b_6 - a_6 = 686 - 196 = \boxed{490}$.

4. Let $f(x)$ be a continuous function over the real numbers such that for every integer n, $f(n) = n^2$ and $f(x)$ is linear over the interval $[n, n + 1]$. There exists a unique two-variable polynomial g such that $g(x, \lfloor x \rfloor) = f(x)$ for all x. Compute $g(20, 23)$. (Here, $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x. For example, $\lfloor 2 \rfloor = 2$ and $\lfloor -3.5 \rfloor = -4$.)

   Answer: 388

   Solution: We wish to construct the function f so that over every interval $[n, n+1]$, $f(n) = n^2$, $f(n+1) = (n+1)^2$, and $f(x)$ is linear with these endpoints. Utilizing the equation for a line gives the general formula $f(x) = n^2 + (x-n)(2n+1)$ over $[n, n+1]$. Since $n = \lfloor x \rfloor$, we can simply plug this in to our equation for the general formula to get $f(x) = \lfloor x \rfloor^2 + (x - \lfloor x \rfloor)(2\lfloor x \rfloor + 1) = -\lfloor x \rfloor^2 + 2x\lfloor x \rfloor - \lfloor x \rfloor + x$. Thus, if we let $a = x$ and $b = \lfloor x \rfloor$, the polynomial g is $g(a, b) = -b^2 + 2ab - b + a$. Hence, our answer is $g(20, 23) = -23^2 + 2 \cdot 20 \cdot 23 - 23 + 20 = \boxed{388}$.

5. Let p, q, and r be the three roots of the polynomial $x^3 - 2x^2 + 3x - 2023$. Suppose that the polynomial $x^3 + Bx^2 + Mx + T$ has roots $p + q$, $p + r$, and $q + r$ for real numbers B, M, and T. Compute $B - M + T$.

   Answer: 2006

   Solution 1: We have $x^3 - 2x^2 + 3x - 2023 = (x - p)(x - q)(x - r) = 0$. By Vieta's, the following are true: $p + q + r = 2$, $pq + qr + pr = 3$, and $pqr = 2023$. Applying Vieta's on $x^3 + Bx^2 + Mx + T$, we have that

   $$B = -((p + q) + (p + r) + (q + r)) = -2p - 2q - 2r = -2(p + q + r) = -2(2) = -4.$$

   Also,

   $$M = (p+q)(q+r) + (p+q)(p+r) + (p+r)(q+r) = (2-r)(2-p) + (2-r)(2-q) + (2-q)(2-p) =$$

$$pq + qr + pr - 4(p + q + r) + 12 = 3 - 4(2) + 12 = 7.$$

Finally,

$$T = -(p + q)(q + r)(p + r) = -(2 - r)(2 - p)(2 - q) = (r - 2)(p - 2)(q - 2) =$$

$$pqr - 2(pq + qr + pr) + 4(p + q + r) - 8 = 2023 - 2(3) + 4(2) - 8 = 2017.$$

Thus, $B - M + T = -4 - 7 + 2017 = \boxed{2006}$.

Solution 2: Let $f(x) = x^3 - 2x^2 + 3x - 2023$ and $g(x) = x^3 + Bx^2 + Mx + T$. We know by Vieta's that $p+q+r = 2$. Thus, the roots of $g(x)$ are $2-p$, $2-q$, and $2-r$. Then $B - M + T =$

$$g(-1) + 1 = (-1 - (2 - p))(-1 - (2 - q))(-1 - (2 - r)) + 1 = (-3 + p)(-3 + q)(-3 + r) + 1 =$$

$$-(3 - p)(3 - q)(3 - r) + 1 = -f(3) + 1 = -((3)^3 - 2(3)^2 + 3(3) - 2023) + 1 = \boxed{2006}.$$

6. Define a sequence $a_0, a_1, a_2, \ldots$ recursively by $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + xa_n$ for each $n \geq 0$ and some real number x. The infinite series

$$\sum_{n=0}^{\infty} \frac{a_n}{10^n} = 1.$$

Compute x.

Answer: 80

Solution: For brevity, let S denote the summation. Notice that

$$\begin{aligned} 100S &= 100a_0 + 10a_1 + \sum_{n=0}^{\infty} \frac{a_{n+2}}{10^n} \\ &= 100a_0 + 10a_1 + \sum_{n=0}^{\infty} \frac{a_{n+1}}{10^n} + \sum_{n=0}^{\infty} \frac{xa_n}{10^n} \\ &= 100a_0 + 10a_1 + 10(S - a_0) + xS. \end{aligned}$$

Rearranging, we find

$$x = \frac{90S - 90a_0 - 10a_1}{S} = \frac{90(1) - 90(0) - 10(1)}{1} = \boxed{80}.$$

7. Nikhil constructs a list of all polynomial pairs $(a(x), b(x))$ with real coefficients such that $a(x)$ has higher degree than $b(x)$ and $a(x)^2 + b(x)^2 = x^{10} + 1$. Danielle takes Nikhil's list and adds all polynomial pairs that satisfy the same conditions but have complex coefficients. If Nikhil's original list had N pairs and Danielle added D pairs, compute $D - N$.

Answer: 376

Solution: Based on the degree conditions and the form of $x^{10} + 1$, $a(x)$ should have degree 5 and leading coefficient ± 1. We may start with leading coefficient 1. Let $i = \sqrt{-1}$, and factor the LHS, RHS. We have the following: $(a(x) + b(x)i)(a(x) - b(x)i) = (x - e^{\frac{\pi i}{10}})(x - e^{\frac{3\pi i}{10}})(x - e^{\frac{5\pi i}{10}})(x - e^{\frac{7\pi i}{10}})(x - e^{\frac{9\pi i}{10}})(x + e^{\frac{\pi i}{10}})(x + e^{\frac{3\pi i}{10}})(x + e^{\frac{5\pi i}{10}})(x + e^{\frac{7\pi i}{10}})(x + e^{\frac{9\pi i}{10}})$. Each choice of $a(x), b(x)$ represents that we are writing $a(x) + b(x)i$ as the product of 5 factors on the right, and the choice of (1, 3, 5, 7, 9) in the exponents comes from computing $2k + 1$ for $k = 0, 1, 2, 3, 4$. Note that the reason we select 5 factors is because $a(x)+b(x)i$ must have degree 5. We know that $a(x)+b(x)i$ and $a(x)-b(x)i$ are complex conjugates, which guarantees that $a(x)$ and $b(x)$ both have real coefficients. We can also choose either $\pm$ for each exponent. Thus, Nikhil has $2^5 = 32$ pairs on his list with leading coefficient 1. In order to account for the fact that they can have leading coefficient -1, we multiply this answer by 2 to get a total of $N = 64$ pairs.

In the case of having complex coefficients, we can simply choose any 5 factors of $x^{10}+1$ and assign them the form $a(x)+b(x)i$. This means that after Danielle adds to the list, they have a total of $\binom{10}{5} = 252$ pairs with leading coefficient 1. For the same reason given at the end of the previous paragraph, we must multiply this answer by 2, for a total of 504 possible pairs. Thus, Danielle added $D = 504-64 = 440$ pairs to Nikhil's list. Our answer is $D-N = 440-64 = \boxed{376}$.

8. Compute the smallest real t such that there exist constants a, b for which the roots of $x^3 - ax^2 + bx - \frac{ab}{t}$ are the side lengths of a right triangle.

Answer: $\mathbf{5+3\sqrt{2}}$

Solution: Let the roots be $r\cos\theta, r\sin\theta, r$ for some $r > 0$ or $0 < \theta < \pi/2$, so by Vieta's formulas

$$x^3 - ax^2 + bx - \frac{ab}{t} = x^3 - r(1+\cos\theta+\sin\theta)x^2 + r^2(\cos\theta+\sin\theta+\cos\theta\sin\theta)x - r^3(\cos\theta\sin\theta).$$

Thus:

$$\begin{aligned} r^3(\cos\theta\sin\theta) &= \frac{r^3}{t}(1+\cos\theta+\sin\theta)(\cos\theta+\sin\theta+\cos\theta\sin\theta) \\ (t-3)\cos\theta\sin\theta &= 1+\cos\theta+\sin\theta+\cos^2\theta\sin\theta+\cos\theta\sin^2\theta \\ t &= 3+\frac{1}{\cos\theta\sin\theta}+\frac{1}{\sin\theta}+\frac{1}{\cos\theta}+\cos\theta+\sin\theta. \end{aligned}$$

To now find the minimum value of t, we first compute the location of the minimum of $\frac{1}{\cos\theta\sin\theta} + \frac{1}{\cos\theta} + \frac{1}{\sin\theta} + 1$. We claim these two minima arise from the same value of θ: since the minima of $\sin\theta + \frac{1}{\sin\theta}$ and $\cos\theta + \frac{1}{\cos\theta}$ occur when $\sin\theta$ and $\cos\theta$ are maximized respectively, $\sin\theta+\cos\theta$ has essentially no effect on the minimum of $\frac{1}{\sin\theta}+\frac{1}{\cos\theta}$, and by extension, the minimum that we are looking for. Now, using the QM-AM-GM inequalities (can also use AM-GM with Cauchy-Schwartz to derive this) with $\sin\theta$ and $\cos\theta$, we get:

$$\sin\theta\cos\theta \le \frac{(\sin\theta+\cos\theta)^2}{4} \le \frac{\sin^2\theta+\cos^2\theta}{2} = \frac{1}{2}$$

with equality when $\sin\theta = \cos\theta$, where $\theta = \frac{\pi}{4}$. This means $\frac{1}{\sin\theta\cos\theta}$ and $\frac{1}{\sin\theta}+\frac{1}{\cos\theta}$ are both minimized at $\frac{\pi}{4}$ with values $\frac{1}{2}$ and $\sqrt{2}$. This gives us that their sum is also minimized at $\frac{\pi}{4}$, and so we have that t is minimized at $\frac{\pi}{4}$ by our prior argument.

Since the minimum is achieved when $\theta = \frac{\pi}{4}$, we therefore have

$$t = 3+\frac{1}{\cos\frac{\pi}{4}\sin\frac{\pi}{4}}+\frac{1}{\sin\frac{\pi}{4}}+\frac{1}{\cos\frac{\pi}{4}}+\cos\frac{\pi}{4}+\sin\frac{\pi}{4} = \boxed{5+3\sqrt{2}}.$$

9. A sequence of real numbers $\{x_n\}$ satisfies the recursion $x_{n+1} = 4x_n - 4x_n^2$, where $n \ge 1$. If $x_{2023} = 0$, compute the number of distinct possible values for x_1.

Answer: $\mathbf{2^{2021}+1}$

Solution: We can factor $x_{n+1} = 4x_n - 4x_n^2 = 4x_n(1-x_n)$. Thus, we can define $f(x) = 4x(1-x)$. If we evaluate the function, notice we have $x_1 = y$, $x_2 = f(y)$, $x_3 = f(f(y) = f^2(y)$, etc. By a simple induction argument, we find $x_n = f^{n-1}(y)$. This means we want to find where $x_{2023} = f^{2022}(y) = 0$. The roots of $f(x)$ occur at $x = 0$ and $x = 1$. Therefore, we have $x \in [0, 1]$, which means $f(x)$ will also similarly have its image in the interval $[0, 1]$. This is clear because if we plug in values where $x < 0$ or $x > 1$, then $f(x) < 0$. In order for roots to exist, y must lie in the interval $[0, 1]$. We can see this by computing the inverse: let $x = 4y(1-y)$ and solve for y. We have $-4y^2 + 4y - x = 0$, which means $y = \frac{-4\pm\sqrt{16-16x}}{8} = \frac{-4\pm4\sqrt{1-x}}{8} = \frac{-1\pm\sqrt{1-x}}{2}$. Since $x \in [0, 1]$, $y \in [0, 1]$.

Thus, let $y = \sin^2(\theta)$, where θ takes on values in the interval $[0, \frac{\pi}{2}]$. We have

$$f(y) = f(\sin^2(\theta)) = 4\sin^2(\theta)(1-\sin^2(\theta)) = 4\sin^2(\theta)(\cos^2(\theta)) = (2\sin(\theta)\cos(\theta))^2 = \sin^2(2\theta).$$

We can similarly compute

$$f^2(y) = f(f(y)) = f(\sin^2(2\theta)) = 4\sin^2(2\theta)(1-\sin^2(2\theta)) = 4\sin^2(2\theta)\cos^2(2\theta) =$$

$$4(\frac{1}{2}(1-\cos(4\theta)))(\frac{1}{2}(1-\cos(4\theta))) = (1-\cos(4\theta))(1+\cos(4\theta)) = 1-\cos^2(4\theta) = \sin^2(4\theta).$$

By induction, it follows that

$$f^n(t) = 2^n\sin^2(n\theta)(1-\sin^2(n\theta)) = 2^n\sin^2(n\theta)(\cos^2(n\theta)) = \sin^2(2^n\theta).$$

Therefore, the roots of $f^{2022}(t) = 0$ must be solutions to $2^{2022}(\theta) = k\pi$, where k is an integer. This yields $\theta = \frac{k\pi}{2^{2022}}$, where $k = 0, 1, 2, \ldots, 2^{2021}$, as we claimed $\theta \in [0, \frac{\pi}{2}]$. Thus, our answer is $\boxed{2^{2021}+1}$ possible values for x_1.

10. There exists a unique triple of integers (B, M, T) such that $B > T > M$ and

$$3B^2(3T-M) + 8M^2(B-T) + 3T^2(5M-B) - (2B^3 + 3M^3 + 4T^3) + 15BMT = 2023.$$

Compute $B + M + T$.

Answer: 55

Solution: First, we expand and group terms. This gives

$$-2B^3 - 3B^2M + 8BM^2 - 3M^3 + T(-8M^2 + 15BM + 9B^2) + T^2(15M - 3B) - 4T^3 = 2023.$$

Notice the first few terms can be factored as $-2B^3 - 3B^2M + 8BM^2 - 3M^3 = -2B^3 - M(3B^2 + 8BM - 3M^2) = -2B^3 - M(3B-M)(B+3M) = -(B-M)(2B-M)(B+3M)$.

This means

$$-(B-M)(2B-M)(B+3M) + T(-8M^2 + 15BM + 9B^2) + T^2(15M - 3B) - 4T^3 =$$

$$-(B-M)(2B-M)(B+3M) + T(9B^2 + 15BM - 8M^2) + T^2(-3B + 15M - 4T) = 2023.$$

Thus, we suspect the original equation can be factored as $-(B - M + aT)(2B - M + bT)(B + 3M + cT)$ for integers a, b, c. We know that $abc = 4$ given that the constant in front of T^3 is 4. This means (a, b, c) is some iteration of $(\pm1, \pm1, \pm4)$ or $(\pm1, \pm2, \pm2)$ seeing as our solutions must be integers.

Since there is a $15BMT$ term in our factorization, it is likely that $a = -4$ seeing as expanding will give $24BMT$, from which we then subtract $9BMT$. We can also see that $T = B + 3M$ is a solution to the original equation:

$$-(B-M)(2B-M)(B+3M)+(B+3M)(9B^2+15BM-8M^2)+(B+3M)^2(-3B+15M-4T) =$$

$$(B+3M)(-(B-M)(2B-M)+9B^2+15BM-8M^2+(B+3M)(-3B+15M-4T)) = 0.$$

Thus, we conclude $(a, b, c) = (-4, 1, -1)$. We have $-(B-M-4T)(B+3M-T)(2B-M+T) = (4T-B+M)(B+3M-T)(2B-M+T) = 2023$.

We can let $4T - B + M = x, B + 3M - T = y, 2B - M + T = z$ where $xyz = 2023$. After some manipulation, we will find $2M + 3B = y + z$ and $13M + 3B = x + 4y$. Subtraction yields $11M = x + 4y - (y + z) = x + 3y - z$. Given $B > T > M$, we see that $z > x > y$ and that $x, y, z > 0$. Now, we are interested in the prime factorization of $2023 = 7 \cdot 17^2$. Thus, we have $(x, y, z) = (7, 1, 289)$ or $(17, 1, 119)$. Testing both, we see that $(x, y, z) = (17, 1, 119)$ is in fact the only solution that is valid. Thus, $11M = -99$ and so $M = -9$. It follows directly that $B = 46$ and $T = 18$. Thus, our answer is $B + M + T = 46 - 9 + 18 = \boxed{55}$.