BMT to AIME - 2020 Edition

Instructions:

- This is a 3-hour timed exam of 15 problems. Do not begin to work on the problems before starting a 3-hour timer, and stop working when 3 hours is up.
- All answers are integers from 000 to 999 (you may choose to include or exclude leading zeros for this test). Each correct answer is worth 1 point; an incorrect or blank answer is worth 0 points.
- No calculators or computational aids are allowed.
- Answers and problem sources will be provided; find the solutions by checking out the original problem from the BMT archives. Every problem is sourced from BMT (mostly the original with modified answer extraction, sometimes a slight variant).

1. Let a, b, and c be integers that satisfy $2a + 3b = 52, 3b + c = 41$, and $bc = 60$. Find $a + b + c$.

2. How many integers $100 \le x \le 999$ have the property that, among the six digits in $\left\lfloor 280 + \frac{x}{100} \right\rfloor$ and x, there is exactly one pair of identical digits? ($\lfloor r \rfloor$ denotes the greatest integer less than r.)

3. Triangle $\triangle ABC$ has side lengths $AB = 5, BC = 12$, and $AC = 13$. Points $M \neq A$ and $N \neq B$ are located on $\overline{AC}$ and $\overline{BC}$ respectively such that $\overline{MN}$ is tangent to the incircle of $\triangle ABC$. The area of $\triangle MNC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

4. Let N be the number of ways to choose 4 points from a 6×6 grid of lattice points such that they are vertices of a non-degenerate quadrilateral with at least one pair of opposite sides parallel to the sides of the grid. Find the remainder when N is divided by 1000.

5. A hollow box (with negligible thickness) shaped like a rectangular prism has a volume of 108 cubic units. The top of the box is removed, exposing the faces on the inside of the box. What is the least possible surface area of the exterior and interior of the box, in square units?

6. A random number C from 1 to 999, inclusive, is generated. Alice writes down the number and then appends the digit 9 to the beginning of the number, creating the number A. Bob writes down the number and then appends the digit 9 to the end of the number, creating the number B. (For example, if $C = 234$, then $A = 9234$ and $B = 2349$.) The expected value of $\gcd(A, B, C)$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

7. A tetrahedron has four congruent faces, each of which is a triangle with side lengths 6, 5, and 5. If the volume of the tetrahedron is V, compute V^2.

8. Let f be a function over the positive real numbers such that for all positive real x and y, $f(x)f(y) = f(xy) + f\left(\frac{x}{y}\right)$. Given that $f(2) = 3$, find the remainder when $f\left(2^{2^{2020}}\right)$ is divided by 100.

9. Non-degenerate quadrilateral $ABCD$ has side lengths $AB = AD = 3$ and $BC = CD = n$ for some integer n, and $\angle ABC = \angle BCD = \angle CDA$. Find the sum of all possible values of n.

10. John writes down all of the ordered triples of positive integers (a, b, c) such that $a + b + c + ab + bc + ac = abc + 1$. Then, he sums together all of the numbers in the ordered triples. What is the resulting sum?

11. Let a, b, and c be real numbers such that $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $abc=5$. The least possible value of the expression

$$\left(a-\frac{1}{b}\right)^3+\left(b-\frac{1}{c}\right)^3+\left(c-\frac{1}{a}\right)^3$$

is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

12. Circle Γ has radius 10, center O, and diameter AB. Point C lies on Γ such that $AC=12$. Let P be the circumcenter of $\triangle AOC$. Line $\overleftrightarrow{AP}$ intersects Γ again at $Q \neq A$. Then $\frac{AP}{AQ}=\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

13. Let $\psi(n)$ be the number of integers $0 \leq r < n$ such that there exists an integer x that satisfies $x^2+x \equiv r \pmod{n}$. Let S be the set of positive divisors of 225. Compute

$$\sum_{d \in S} \psi(d).$$

14. Let S be the set of points (x, y) in the coordinate plane with $0 \leq x < 1$ and $0 \leq y < 1$. For any point (x, y) in S, Jenny can perform a *shuffle* on that point, which takes the point to $(\{3x+y\}, \{x+2y\})$ where $\{\alpha\}$ denotes the fractional part of α (so for example, $\{\pi\}=\pi-3=0.1415\ldots$). How many points p in S satisfy the property that after 3 shuffles on p, p ends up in its original position?

15. For $k \geq 0$, let $a_k=2^k$. Let

$$\theta=\sum_{k=1}^{100} \cos^{-1}\left(\frac{2a_k^2-6a_k+5}{\sqrt{(a_k^2-4a_k+5)(4a_k^2-8a_k+5)}}\right).$$

Then $\cos^2(\theta)=\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m+n$ is divided by 1000.