# BMT to AIME - 2022 Edition

**Instructions:**

- This is a 3-hour timed exam of 15 problems. Do not begin to work on the problems before starting a 3-hour timer, and stop working when 3 hours is up.
- All answers are integers from 000 to 999 (you may choose to include or exclude leading zeros for this test). Each correct answer is worth 1 point; an incorrect or blank answer is worth 0 points.
- No calculators or computational aids are allowed.
- Answers and problem sources will be provided; find the solutions by checking out the original problem from the BMT archives. Every problem is sourced from BMT (mostly the original with modified answer extraction, sometimes a slight variant).

1. Each box in the expression

$$\square \times \square \times \square - \square \times \square \times \square$$

is filled in with a different number in the list $2, 3, 4, 5, 6, 7, 8$ so that the value of the expression is 9. What would the value of the expression be if the $-$ sign was replaced by a $+$ sign?

2. Suppose we have four real numbers $a, b, c, d$ such that $a$ is nonzero, $a, b, c$ form a geometric sequence, in that order, and $b, c, d$ form an arithmetic sequence, in that order. The least possible value of $\frac{d}{a}$ is $-\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

3. Richard and Shreyas are arm wrestling against each other. They will play 10 rounds, and in each round, there is exactly one winner. If the same person wins in consecutive rounds, these rounds are considered part of the same "streak". How many possible outcomes are there in which there are strictly more than 3 streaks? For example, if we denote Richard winning by $R$ and Shreyas winning by $S$, $SSRSSRRRRR$ is one such outcome, with 4 streaks.

4. Let circles $C_1$ and $C_2$ be internally tangent at point $P$, with $C_1$ being the smaller circle. Consider a line passing through $P$ which intersects $C_1$ at $Q$ and $C_2$ at $R$. Let the line tangent to $C_2$ at $R$ and the line perpendicular to $\overline{PR}$ passing through $Q$ intersect at a point $S$ outside both circles. Given that $SR = 5, RQ = 3$, and $QP = 2$, the radius of $C_2$ is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

5. Given a positive integer $n$, let $s(n)$ denote the sum of the digits of $n$. Compute the largest positive integer $n$ such that $n = s(n)^2 + 2s(n) - 2$.

6. For real numbers $B, M$, and $T$, we have $B^2 + M^2 + T^2 = 2022$ and $B + M + T = 72$. Compute the sum of the minimum and maximum possible values of $T$.

7. Triangle $\triangle BMT$ has $BM = 4, BT = 6$, and $MT = 8$. Point $A$ lies on line $\overleftrightarrow{BM}$ and point $Y$ lies on line $\overleftrightarrow{BT}$ such that $\overline{AY}$ is parallel to $\overline{MT}$ and the center of the circle inscribed in triangle $\triangle BAY$ lies on $\overline{MT}$. Then $AY = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

8. Bayus has eight slips of paper, which are labeled $1, 2, 4, 8, 16, 32, 64$, and $128$. Uniformly at random, he draws three slips with replacement; suppose the three slips he draws are labeled $a, b$, and $c$. The probability that Bayus can form a quadratic polynomial with coefficients $a, b$, and $c$, in some order, with 2 distinct real roots, is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

9. Compute the number of ordered triples $(a, b, c)$ with $-100 \leq a, b, c \leq 100$ satisfying the simultaneous equations:

$$\begin{aligned} a^3 - 2a &= abc - b - c \\ b^3 - 2b &= abc - c - a \\ c^3 - 2c &= abc - a - b. \end{aligned}$$

10. In triangle $\triangle ABC$ with orthocenter $H$, the internal angle bisector of $\angle BAC$ intersects $\overline{BC}$ at $Y$. Given that $AH = 4, AY = 6$, and the distance from $Y$ to $\overline{AC}$ is $\sqrt{15}$, compute $BC^2$.

11. Luke the frog has a standard deck of 52 cards shuffled uniformly at random placed face down on a table. The deck contains four aces and four kings (no card is both an ace and a king). He now begins to flip over the cards one by one, leaving a card face up once he has flipped it over. He continues until the set of cards he has flipped over contains at least one ace and at least one king, at which point he stops. The expected value of the number of cards he flips over is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

12. Lysithea and Felix each have a take-out box, and they want to select among 42 different types of sweets to put in their boxes. They each select an even number of sweets (possibly 0) to put in their box. In each box, there is at most one sweet of any type, although the boxes may have sweets of the same type in common. The total number of sweets they take out is 42. Let $N$ be the number of ways can they select sweets to take out. Find the remainder when $N$ is divided by 1001.

13. Define the two sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ by $a_0 = 3$ and $b_0 = 1$ with the recurrence relations $a_{n+1} = 3a_n + b_n$ and $b_{n+1} = 3b_n - a_n$ for all nonnegative integers $n$. Let $r$ and $s$ be the remainders when $a_{128}$ and $b_{128}$ are divided by 127, respectively. Compute $10r + s$.

14. Let $p, q$, and $r$ be the roots of the polynomial $f(t) = t^3 - 2022t^2 + 2022t - 337$. Define:

$$\begin{aligned} x &= (q-1)\left(\frac{2022-q}{r-1} + \frac{2022-r}{p-1}\right) \\ y &= (r-1)\left(\frac{2022-r}{p-1} + \frac{2022-p}{q-1}\right) \\ z &= (p-1)\left(\frac{2022-p}{q-1} + \frac{2022-q}{r-1}\right) \end{aligned}$$

Compute $qrx + rpy + pqz - xyz$.

15. In triangle $\triangle ABC$, $E$ and $F$ are the feet of the altitudes from $B$ to $\overline{AC}$ and $C$ to $\overline{AB}$, respectively. Line $\overleftrightarrow{BC}$ and the line through $A$ tangent to the circumcircle of $\triangle ABC$ intersect at $X$. Let $Y$ be the intersection of line $\overleftrightarrow{EF}$ and the line through $A$ parallel to $\overleftrightarrow{BC}$. If $XB = 4, BC = 8$, and $EF = 4\sqrt{3}$, compute $XY^2$.

Answer key and sources next page. Go to the BMT website archives tab to find solutions for these problems.

Answer Key and Sources:

1. **201** (General Round Problem 10)
2. **009** (Algebra Round Problem 3)
3. **932** (Discrete Round Problem 4)
4. **033** (Guts Round Problem 12)
5. **397** (Discrete Round Problem 5)
6. **048** (Algebra Round Problem 5)
7. **077** (Geometry Round Problem 6)
8. **139** (Discrete Round Problem 6)
9. **207** (Guts Round Problem 17)
10. **560** (Geometry Round Problem 7)
11. **734** (Discrete Round Problem 7)
12. **473** (Discrete Round Problem 9)
13. **310** (Discrete Round Problem 8, less bashable variation)
14. **674** (Algebra Round Problem 10)
15. **084** (Geometry Round Problem 10)