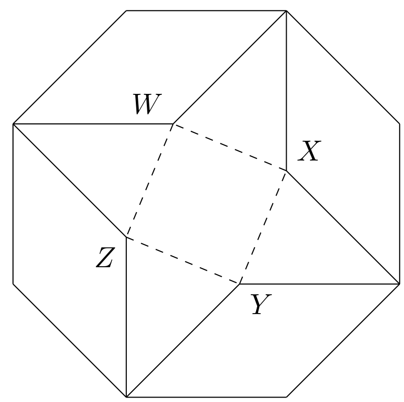
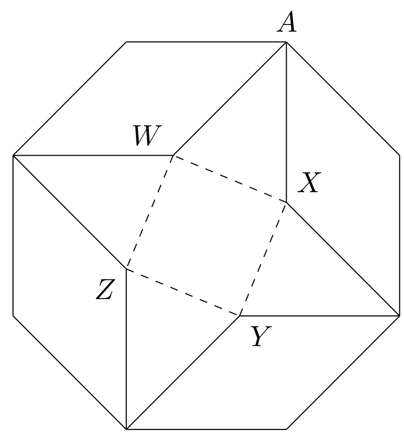
1. Points W, X, Y, and Z are chosen inside a regular octagon so that four congruent rhombuses are formed, as shown in the diagram below. If the side length of the octagon is 1, compute the area of quadrilateral $WXYZ$.



Answer: $\mathbf{2 - \sqrt{2}}$

Solution: Let the closest vertex of the octagon between W and X be A.



Since rhombuses are formed, $WA = XA = 1$. Using formula

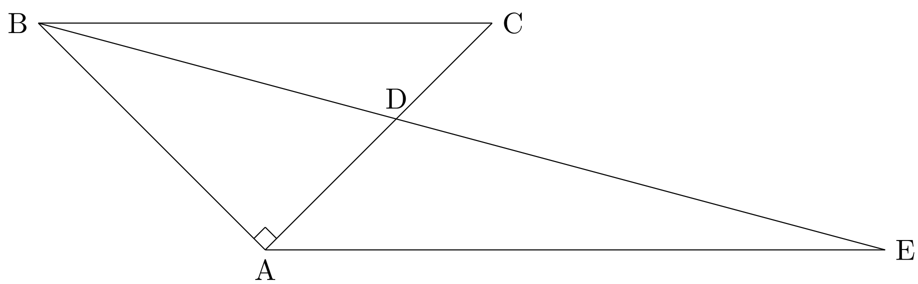
$$\text{interior angle} = \frac{180^\circ \cdot (n-2)}{n}$$

where n is the number of the sides of the octagon, we know that the interior angle of the octagon is 135°. Thus, $\angle WAX = 135^o - (180^o - 135^o) \cdot 2 = 45^o$. Using law of cosines on $\triangle WAX$, we can see $WX^2 = WA^2 + XA^2 - 2 \cdot WA \cdot XA \cdot \cos(\angle WAX) = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \frac{\sqrt{2}}{2} = 2 - \sqrt{2}$. By symmetry, $WXYZ$ is a square whose area is $WX^2 = \boxed{2 - \sqrt{2}}$.

2. Triangle $\triangle ABC$ has $\angle ABC = \angle BCA = 45^\circ$ and $AB = 1$. Let D be on $\overline{AC}$ such that $\angle ABD = 30^\circ$. Let $\overleftrightarrow{BD}$ and the line through A parallel to $\overleftrightarrow{BC}$ intersect at E. Compute the area of $\triangle ADE$.

Answer: $\mathbf{\frac{3+\sqrt{3}}{12}}$

Solution:

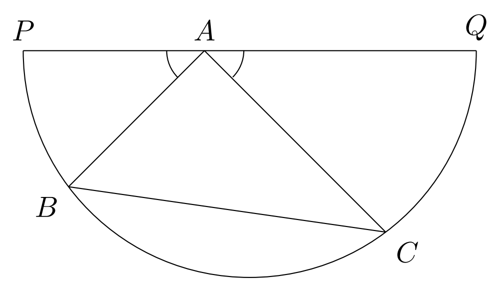


Triangles $\triangle ADE$ and $\triangle CDB$ are similar. Thus, the area of $\triangle ADE$ is $(\frac{AD}{DC})^2$ times the area of $\triangle CDB$. Since $\angle ABD = 30^\circ$ and $AB = 1$, we have $AD = \frac{1}{\sqrt{3}}$ and $DC = 1 - \frac{1}{\sqrt{3}}$. Thus, the area of $\triangle ADE$ is $\frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\left(\frac{1/\sqrt{3}}{1-(1/\sqrt{3})}\right)^2 = \boxed{\dfrac{3+\sqrt{3}}{12}}$.

3. Points A, B, and C lie on a semicircle with diameter $\overline{PQ}$ such that $AB = 3$, $AC = 4$, $BC = 5$, and A is on $\overline{PQ}$. Given $\angle PAB = \angle QAC$, compute the area of the semicircle.

Answer: $\mathbf{\frac{25\pi}{4}}$

Solution: Since $\angle PAB = \angle QAC$, B and C cannot lie on diameter PQ. Otherwise, one of $\angle PAB$ and $\angle QAC$ would be 0° or 180° and the other one would be 90°. So we can construct the diagram below.



Consider reflecting the whole picture across $\overline{PQ}$, creating points A', B', and C' (A' is the same point as A). Since $\angle PAB = \angle QAC = \frac{\pi}{4}$, we must have $\angle B'AB = \angle C'AC = \frac{\pi}{2}$. Thus, two isosceles right triangles are formed, and $BB' = 3\sqrt{2}$ and $CC' = 4\sqrt{2}$. If the center of the semicircle is O, then the central angle $\angle B'OB$ equals $2\angle B'CB = 2\sin^{-1}(\frac{3}{5})$. Then, $\angle B'OP = \sin^{-1}(\frac{3}{5}) \rightarrow \frac{3}{5} = \frac{0.5 \cdot B'B}{r}$, so $r = \frac{5\sqrt{2}}{2}$ and the area of the semicircle is therefore $\boxed{\dfrac{25\pi}{4}}$.