# BMT to AIME - 2024 Edition

**Instructions:**

- This is a 3-hour timed exam of 15 problems. Do not begin to work on the problems before starting a 3-hour timer, and stop working when 3 hours is up.
- All answers are integers from 000 to 999 (you may choose to include or exclude leading zeros for this test). Each correct answer is worth 1 point; an incorrect or blank answer is worth 0 points.
- No calculators or computational aids are allowed.
- Answers and problem sources will be provided; find the solutions by checking out the original problem from the BMT archives. Every problem is sourced from BMT (mostly the original with modified answer extraction, sometimes a slight variant).

1. Find the least three-digit positive integer with distinct nonzero digits satisfying the property that it is not divisible by any of its digits.

2. Suppose $a_1, a_2, \ldots$ is an arithmetic sequence, and suppose $g_1, g_2, \ldots$ is a geometric sequence with common ratio 2. Suppose $a_1 + g_1 = 1$ and $a_2 + g_2 = 1$. If $a_{24} = g_7$, find $|a_{2024}|$.

3. Eight players are seated around a circular table. Each player is assigned to either Team Green or Team Yellow so that each team has at least one player. In how many ways can the players be assigned to the teams such that each player is on the same team as at least one player adjacent to them?

4. Right triangle $\triangle ABC$ has $\angle B = 90^\circ$. Points $D$ and $E$ are on $\overline{AC}$ such that $AB = AE$ and $BC = DC$. If $AD = 2$ and $EC = 9$, then $BD \cdot BE = \frac{a\sqrt{b}}{c}$, where $a, b$, and $c$ are positive integers such that $a$ and $c$ are relatively prime and $b$ is squarefree. Find $a + b + c$.

5. Gigi randomly rearranges three $G$'s and seven $I$'s to form an ten-letter string. The probability that there is a group of four consecutive letters that form "$GIGI$" is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

6. Let $N$ be the greatest multiple of 43 whose base 6 representation is a permutation of the digits 1, 2, 3, 4, and 5. Find $N/43$.

7. Let $B, M$, and $T$ be real numbers that satisfy the equations:

$$\begin{aligned}
2B + M + T - 2B^2 - 2BM - 2MT - 2BT &= 0, \\
B + 2M + T - 3M^2 - 3BM - 3MT - 3BT &= 0, \\
B + M + 2T - 4T^2 - 4BM - 4MT - 4BT &= 0.
\end{aligned}$$

   Then $B + M + T = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

8. Let $U$ and $C$ be two circles, and kite $BERK$ have vertices that lie on $U$ and sides that are tangent to $C$. If the lengths of diagonals of the kite are 5 and 6, then the ratio of the area of $U$ to the area of $C$ is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

9. For an arbitrary positive integer $n$, we define $f(n)$ to be the number of ordered 5-tuples of positive integers, $(a_1, a_2, a_3, a_4, a_5)$, such that $a_1a_2a_3a_4a_5 | n$. Let $N$ the sum of all $n$ for which $f(n)/n$ is maximized. Find the remainder when $N$ is divided by 1000.

10. Let $\alpha$ be a positive real number. Over all choices of positive real numbers $w, x, y, z$ satisfying

$$\begin{aligned} wx + yz &= \alpha, \\ wy + xz &= \alpha, \\ wz + xy &= \alpha, \end{aligned}$$

the minimum value of $w + 2x + 3y + 4z$ is $\alpha/2$. Compute $\alpha$.

11. Points $A, B, C, D, E$, and $F$ lie on a sphere with radius $\sqrt{10}$ such that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent at point $P$ inside the sphere and are pairwise perpendicular. If $PA = \sqrt{6}$, $PB = \sqrt{10}$, and $PC = \sqrt{15}$, then the volume of tetrahedron $DEFP$ is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

12. Let $d_n(x)$ be the nth decimal digit (after the decimal point) of $x$. For example, $d_3(\pi) = 1$ because $\pi = 3.14\underline{1}5\ldots$ For a positive integer $k$, let $f(k) = p_k^4$, where $p_k$ is the $k$th prime number. Find the remainder when

$$\sum_{i=1}^{2023} d_{f(i)}\left(\frac{1}{1275}\right)$$

is divided by 1000.

13. Define two sequences of real numbers, $a_n$ and $b_n$, such that $a_0 = b_0 = \sqrt{3}$ and for $n \geq 0$:

$$\begin{aligned} a_{n+1} &= a_n + \sqrt{1 + a_n^2} \\ b_{n+1} &= \frac{b_n}{1 + \sqrt{1 + b_n^2}}. \end{aligned}$$

Let $M$ be the least real number such that $M > |a_i b_i - a_j b_j|$ for any integers $i, j > 0$. Then $M = \frac{p\sqrt{q} - r}{s}$, where $p, q, r$, and $s$ are positive integers such that $\gcd(p, s) = 1$ and $q$ is not divisible by the square of any prime. Find $p + q + r + s$.

14. Let $\triangle ABC$ have incenter $I$, and let $M$ be the midpoint of $\overline{BC}$. Line $\overleftrightarrow{AM}$ intersects the circumcircle of $\triangle IBC$ at points $P$ and $Q$. Suppose that $AP = 13$, $AQ = 83$, and $BC = 56$. Find the perimeter of $\triangle ABC$.

15. The positive integers 1 through 9 are placed in the 9 cells of a $3 \times 3$ grid. Then, for every pair of cells sharing a side, the sum of the numbers in that pair is recorded in a list. The most number of times any number occurs in the list is 4. Let $N$ be the number of ways that the numbers could have been placed in the grid. Find the remainder when $N$ is divided by 1000.

Answer key and sources next page. Go to the BMT website archives tab to find solutions for these problems.

Answer Key and Sources:

1. **239** (Guts Round Problem 5)
2. **044** (Algebra Round Problem 3)
3. **022** (Discrete Round Problem 4)
4. **739** (Guts Round Problem 11)
5. **157** (Discrete Round Problem 7, easier variation)
6. **140** (Discrete Round Problem 6)
7. **038** (Algebra Round Problem 6)
8. **091** (Geometry Round Problem 5)
9. **160** (Discrete Round Problem 8)
10. **128** (Algebra Round Problem 8)
11. **101** (Geometry Round Problem 8)
12. **199** (Guts Round Problem 22)
13. **011** (Algebra Round Problem 9)
14. **128** (Geometry Round Problem 9)
15. **616** (Discrete Round Problem 10)