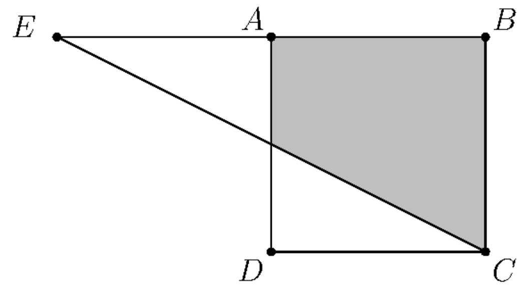
1. Given a square $ABCD$ of side length 6, the point E is drawn on the line AB such that the distance EA is less than EB and the triangle $\triangle BCE$ has the same area as $ABCD$. Compute the shaded area.



Answer: 27

Solution: Let the intersection of $\overline{CE}$ and $\overline{AD}$ be F. Then we see that the area of triangle $\triangle BCE$ is the shaded area plus the area of triangle $\triangle AFE$, which is similar to $\triangle BCE$ with half the side length. Therefore the area of $\triangle AFE$ is $\frac{1}{4}$ the area of $\triangle BCE$ which by givens is the area of $ABCD$. This is $6^2 = 36$, so the shaded area is $(1 - \frac{1}{4}) \cdot 36 = \boxed{27}$.

2. Jerry has red blocks, yellow blocks, and blue blocks. He builds a tower 5 blocks high, without any 2 blocks of the same color touching each other. Also, if the tower is flipped upside-down, it still looks the same. Compute the number of ways Jerry could have built this tower.

Answer: 12

Solution: We go step-by-step placing blocks until the whole tower is decided. The bottom block can be any of the 3 colors. Then, the next 2 blocks have 2 options each for what color they can be: the only option ruled out is the color directly below it in our stack. This gives $3 \cdot 2^2 = 12$ for the first 3 blocks in the stack. Then, the remaining 2 blocks are fixed as flipping the tower over tells us they are the same as the bottom 2 blocks. Therefore there are $\boxed{12}$ possible towers.

3. Compute the second smallest positive whole number that has exactly 6 positive whole number divisors (including itself).

Answer: 18

Solution: It is possible to just check all numbers up to 18 until you see that 12 is the first number with 6 factors and 18 is the second, but there is a faster approach. The number of divisors of a number is equal to the product of the number of choices you have for each prime factor; for example, take $60 = 2^2 \cdot 3 \cdot 5$. For a given divisor, you can take $2^0, 2^1$, or 2^2 as the power of 2, $3^0, 3^1$ as the power of 3, and $5^0, 5^1$ as the power of 5. So you have $3 \cdot 2 \cdot 2 = 12$ choices of divisor for 60.

Since we care about numbers with 6 divisors, we want to check for numbers that are written like p^2q or p^5 for primes p, q, so that we either have $3 \cdot 2$ choices or just 6 choices outright. We use the smallest primes we can, 2 and 3, to get $2^2 \cdot 3 = 12$ as the smallest number with 6 factors. Then we try $3^2 \cdot 2 = 18$, which is in fact the next smallest. Also worth checking are $2^2 \cdot 5 = 20 > 18$, and $2^5 = 32 > 18$, after which everything is definitely bigger than 18. This gives us our final answer of $\boxed{18}$.

4. A grasshopper is traveling on the coordinate plane, starting at the origin $(0, 0)$. Each hop, the grasshopper chooses to move 1 unit up, down, left, or right with equal probability. The

grasshopper hops 4 times and stops at point P. Compute the probability that it is possible to return to the origin from P in at most 3 hops.

Answer: $\frac{\mathbf{49}}{\mathbf{64}}$

Solution: We will compute the complement (that it will take exactly 4 hops to return to the origin) and then subtract from 1. Notice that if we ever hop in more than 2 distinct directions, one will be the opposite of the other direction and will "cancel" out, allowing us to return in less than 4 hops. Therefore, we must select at most 2 directions that we may hop in. However, these directions can't be up and down or left and right, as these are opposites. So, there are 4 ways of choosing the pairs of 2 directions the grasshopper can hop in.

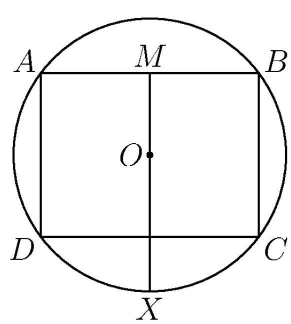
Then, each hop is selected from one of those 2 directions for each of the pairs, which gives 2^4 possible hops for each of the 4 pairs and a total count of $4(2^4)$. However, we count each of the cases where we hop only in one direction exactly twice, and there are 4 of those cases. So we should subtract 4 from our total to get $4(2^4-1)$ possible ways. We divide by the total number of ways of hopping which is just 4^4 giving the result $\frac{4(15)}{4^4} = \frac{15}{64}$. Then we compute the complement to get our answer $1 - \frac{15}{64} = \boxed{\frac{49}{64}}$

5. Two parabolas, $y = ax^2 + bx + c$ and $y = -ax^2 - bx - c$, intersect at $x = 2$ and $x = -2$. If the y-intercepts of the two parabolas are exactly 2 units apart from each other, compute $|a + b + c|$.

Answer: $\frac{\mathbf{3}}{\mathbf{4}}$

Solution: First, notice that either way we choose a, b, c, the absolute value of their sum is the same. For the parabola $y = ax^2 + bx + c$, we know that two of its points are $(-2, 0)$, and $(2, 0)$. Then, from symmetry across $y = 0$, we get that the y-intercepts of the two parabolas are $(0, \pm 1)$, and we choose $(0, -1)$ to solve for one of the parabolas. We can then substitute in the values of the points for (x, y) and solve for a, b, c. From $(0, -1)$, we know that $c = -1$. From $(-2, 0)$ and $(2, 0)$, we know that $a = \frac{1}{4}$ and $b = 0$. Therefore, our answer is $a + b + c = \boxed{\frac{3}{4}}$.

6. Let rectangle $ABCD$ have side lengths $AB = 8, BC = 6$. Let $ABCD$ be inscribed in a circle with center O, as shown in the diagram. Let M be the midpoint of side $\overline{AB}$, and let X be the intersection of ray $\overrightarrow{MO}$ with the circle. Compute the length AX.



Answer: $\mathbf{4\sqrt{5}}$

Solution: We have the right triangle $\triangle AMX$. The length AM is half of AB which is 4, and then length MX is $MO + OX$ where MO is half of BC and OX is a radius of the circle. The radius of the circle is $\frac{1}{2}\sqrt{AB^2 + BC^2} = 5$, which means $MX = 3 + 5 = 8$. Then $AX^2 = AM^2 + MX^2 = 4^2 + 8^2 = 80$, so $AX = \sqrt{80} = \boxed{4\sqrt{5}}$.

7. For an integer $n > 0$, let $p(n)$ be the product of the digits of n. Compute the sum of all integers n such that $n - p(n) = 52$.

Answer: 157

Solution: Let n be a natural number with d digits. Now we represent n as a number in base 10 like so: $n = a_{d-1}10^{d-1} + a_{d-2}10^{d-2} + \cdots + a_1 10^1 + a_0$, where a_{d-1} is an integer 1 to 9 and a_j for $0 \le j < d-1$ is an integer 0 to 9.

We know that $p(n)$ would be maximized if all d digits of n were 9, so we can set an upper bound for $p(n)$ as $9^{d-1}a_{d-1}$. We can get a lower bound for n by simply taking the first term of the base 10 expansion: $a_{d-1}10^{d-1}$. Therefore, we can only have solutions when $10^{d-1}a_{d-1} - 9^{d-1}a_{d-1} \le 52$, as otherwise we would have $n - p(n) > 52$. We can rewrite the inequality as $a_{d-1}(10^{d-1} - 9^{d-1}) < 52$, which holds for $d \le 2$, and for $d = 3$ if $a_2 < 3$. It's fairly simple to see that this inequality doesn't hold for $d > 3$.

Therefore we only have to consider the cases: $\overline{a}, \overline{ab}, \overline{1ab}, \overline{2ab}$. These can be done individually fairly quickly.

The single-digit case has $p(n) = n$, so this doesn't work.

The two-digit case looks like this: $10a + b - ab = 52$, which we can factor like this: $ab - 10a - b + 10 = -42 \implies (a-1)(10-b) = 42$. Both of these numbers are between 0 and 9 inclusive, so we have the single factor pair $6 \cdot 7$ as a possibility. We then have $(a-1) = 7, (10-b) = 6$ giving the solution 84, or $(a-1) = 6, (10-b) = 7$ giving 73 as a second solution.

The first three-digit restricted case looks like this: $100 + 10a + b = ab + 52$. We may factor: $100 + (10-b)a + b = 52$, which is impossible as $100 + (10-b)a + b \ge 100$. Therefore there are no solutions of the form $\overline{1ab}$.

Similarly, we have $200 + 10a + b = 2ab + 52$, factoring as $200 + (10-2b)a + b = 52$. We have $(10-2b)a > (-100)$ by taking absolute values, so we get $200 + (10-2b)a + b > 100 + b > 52$, so there are no solutions of the form $\overline{2ab}$.

Therefore the only solutions are in the two digit case, which sum to give $\boxed{157}$.

8. Circle ω_1 is centered at O_1 with radius 3, and circle ω_2 is centered at O_2 with radius 2. Line ℓ is tangent to ω_1 and ω_2 at X, Z, respectively, and intersects segment $\overline{O_1O_2}$ at Y. The circle through O_1, X, Y has center O_3, and the circle through O_2, Y, Z has center O_4. Given that $O_1O_2 = 13$, find O_3O_4.

Answer: $\mathbf{\frac{13}{2}}$

Solution: By tangency, O_1X and O_2Z are both perpendicular to ℓ. That is, $\triangle O_1XY$ and $\triangle O_2YZ$ are both right, so O_3 and O_4 are the midpoints of $\overline{O_1Y}$ and $\overline{O_2Y}$, respectively. Thus, $O_3O_4 = \frac{O_1O_2}{2} = \boxed{\frac{13}{2}}$.

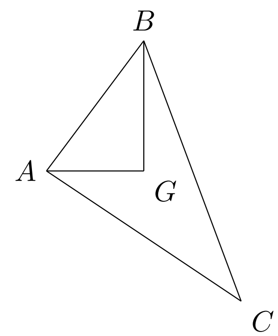
9. For positive integers a and b, consider the curve $x^a + y^b = 1$ over real numbers x, y and let $S(a, b)$ be the sum of the number of x-intercepts and y-intercepts of this curve. Compute $\sum_{a=1}^{10} \sum_{b=1}^{5} S(a, b)$.

Answer: 145

Solution: Notice that $S(a, b)$ is just dependent on a and b modulo 2. In particular, the number of x intercepts is the number of solutions to $x^a = 1$ which is 1 if a is odd and 2 if a is even; let's call this $X(a)$. Symmetric logic holds for b, whose number y intercepts we will call $Y(b) = X(b)$. Therefore, we can rewrite the sum as

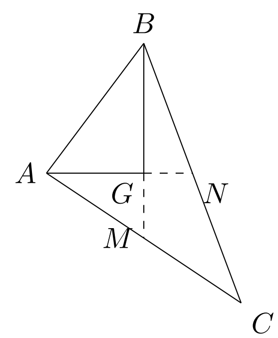
$$\sum_{a=1}^{10} 3(X(a)+1) + 2(X(a)+2) = \sum_{a=1}^{10} 5X(a) + 7 = 70 + 5(5 \cdot 1 + 5 \cdot 2) = \boxed{145}$$

10. Let $\triangle ABC$ be a triangle with G as its centroid, which is the intersection of the three medians of the triangle, as shown in the diagram. If $\overline{GA} \perp \overline{GB}$ and $AB = 7$, compute $AC^2 + BC^2$.



Answer: 245

Solution:



Extend $\overline{BG}$ to M and $\overline{AG}$ to N. Since G is the centroid of $\triangle ABC$, M, N will be the midpoints of $\overline{AC}$ and $\overline{BC}$, $GM = \frac{1}{2}BG$, $GN = \frac{1}{2}AG$. Then we can set up the following equations:

$$\left(\frac{AC}{2}\right)^2 = AM^2 = AG^2 + GM^2$$
$$\left(\frac{BC}{2}\right)^2 = BN^2 = BG^2 + GN^2$$

Since $BG^2 + AG^2 = AB^2 = 7^2 = 49$, $GM^2 + GN^2 = MN^2 = \left(\frac{AB}{2}\right)^2 = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$. Then we can get $AC^2 + BC^2 = 4\left(AG^2 + BG^2 + GM^2 + GN^2\right) = 4\left(49 + \frac{49}{4}\right) = \boxed{245}$.

11. Compute the sum of all positive integers n for which there exists a real number x satisfying

$$\left(x + \frac{n}{x}\right)^n = 2^{20}.$$

Answer: 36

Solution: Observe that $x + \frac{n}{x} = y \implies x^2 - yx + n = 0$ has a real solution x if and only if $y^2 - 4n \geq 0$ by the quadratic formula. In particular, $x + \frac{n}{x} = 2^{20/n}$ has a real solution if and only if $2^{40/n-2} \geq n$. Note that $n \leq 8$ implies $2^{40/n-2} \geq 2^3 \geq n$, and $n > 8$ implies $2^{40/n-2} < 2^3 < n$. The answer is $1 + 2 + \cdots + 8 = \boxed{36}$.

12. Call an n-digit integer with distinct digits *mountainous* if, for some integer $1 \leq k \leq n$, the first k digits are in strictly ascending order and the following $n - k$ digits are in strictly descending order. How many 5-digit mountainous integers with distinct digits are there?

Answer: 3024

Solution: This problem can be solved via casework on k, but we present a faster solution. Suppose we have 5 distinct digits $0 \leq d_1 \leq d_2 \leq \cdots \leq d_5 \leq 9$. We have d_5 as the largest digit, which must be the "peak" of the mountain. The rest of the digits must either go before or after d_5, and there is only one way of ordering them so that they are increasing or decreasing once they have been sorted to left and right. Then we have two cases: $d_1 > 0$, or $d_1 = 0$.

If $d_1 > 0$, then there are $\binom{9}{5}$ ways of picking the 5 digits, combined with 2^4 ways of choosing which side of the peak each of the 4 digits $d_1, \cdots, d_4$ can go. If $d_1 = 0$, then there are $\binom{9}{4}$ ways of picking the remaining 4 digits, and there are 2^3 ways of choosing which side of the peak each of the 3 digits d_2, d_3, d_4 can go (d_1 is forced to the right since the number can't start with 0). So the final answer is $2^4 \cdot \binom{9}{5} + 2^3 \cdot \binom{9}{4} = \boxed{3024}$.

13. Consider the set of triangles with side lengths $1 \leq x \leq y \leq z$ such that x, y, and z are the solutions to the equation $t^3 - at^2 + bt = 12$ for some real numbers a and b. Compute the smallest real number N such that $N > ab$ for any choice of x, y, and z.

Answer: 152

Solution: First, we apply Vieta's formulas to get that $ab = (x + y + z)(xy + yz + zx)$. We multiply by $1 = \frac{12}{xyz}$ and expand out to get

$$ab = 12\left(3 + \frac{x}{y} + \frac{y}{x} + \frac{x}{z} + \frac{z}{x} + \frac{z}{y} + \frac{y}{z}\right)$$

We observe that this is maximized when x, y, z are as far apart as possible (can use AM-GM to verify), so we minimize x by taking $x = 1$, then because of the triangle inequality $x + y > z$, we want to approach $z = x + y$ to get an upper bound for the possible value of ab. So we get the equation $1(y)(y + 1) = xyz = 12$, meaning $y = 3$ and $z = 4$. From here, we can calculate ab explicitly as

$$ab = 12\left(3 + \frac{1}{3} + \frac{3}{1} + \frac{1}{4} + \frac{4}{1} + \frac{3}{4} + \frac{4}{3}\right) = 12(\frac{38}{3}) = \boxed{152}$$

14. Right triangle $\triangle ABC$ with $\angle A = 30^\circ$ and $\angle B = 90^\circ$ is inscribed in a circle ω_1 with radius 4. Circle ω_2 is drawn to be the largest circle outside of $\triangle ABC$ that is tangent to both $\overline{BC}$ and ω_1, and circles ω_3 and ω_4 are drawn this same way for sides $\overline{AC}$ and $\overline{AB}$, respectively. Suppose that the intersection points of these smaller circles with the bigger circle are noted as points D, E, and F. Compute the area of triangle $\triangle DEF$.

Answer: $12 + 4\sqrt{3}$

Solution: If we say that O is the center of the circumcircle ω_1, our problem reduces down to finding $[ODE] + [OEF] + [OFD]$. Furthermore, hypotenuse AC is the diameter of the circle that triangle $\triangle ABC$ is inscribed in, so OD is the radius of the circle ω_1, and $OD = OE = OF$. Through angle chasing, $\angle DOE = 120^\circ$ and $\angle FOE = 150^\circ$. Therefore, $[OED] = \frac{1}{2} \cdot 4 \cdot 4 \cdot \sin(120^\circ) = 4\sqrt{3}$, and $[OEF] = \frac{1}{2} \cdot 4 \cdot 4 \cdot \sin(150^\circ) = 4$. As for $[ODF]$, $OD = OF = 4$, so the area is equal to 8. Our final answer is therefore $\boxed{12 + 4\sqrt{3}}$.

15. Given a positive integer k, let $s(k)$ denote the sum of the digits of k. Let $a_1, a_2, a_3, \dots$ denote the strictly increasing sequence of all positive integers n such that $s(7n + 1) = 7s(n) + 1$. Compute a_{2023}.

Answer: 11111100111

Solution: Let $S = \{a_1, a_2, \dots\}$. We claim that $n \in S$ if and only if n is a positive integer whose digits are only 1s and 0s. Certainly this is sufficient because $7n + 1$ will have no carries in the multiplication and addition, so $s(7n+1) = 7s(n)+1$ follows. To see that this is necessary, write

$$n = \sum_{i=0}^{m} d_i 10^i$$

for digits $d_0, \dots, d_m$. Then $s(a + b) \le s(a) + s(b)$ implies

$$s(7n + 1) = s\left((7d_0 + 1) + \sum_{i=1}^{m} 7d_i 10^i\right) \le 7d_0 + 1 + \sum_{i=1}^{m} 7d_i = 7s(n) + 1$$

with equality if and only if there were no carries in any of the additions. In particular, we require no carries in $s(7d_i) = 7s(d_i)$, which requires $d_i \in \{0, 1\}$.

Thus, the sequence $a_1, a_2, \dots$ is just counting in binary. So the answer will be 2023 written in binary. So we write

$$\begin{aligned} 2023 &= 2048 - 25 \\ &= 2047 - 16 - 8 \\ &= 1,11111,11111_2 - 10000_2 - 1000_2 \\ &= 1,11111,00111_2, \end{aligned}$$

so our answer is $\boxed{11111100111}$.

16. Sabine rolls a fair 14-sided die numbered 1 to 14 and gets a value of x. She then draws x cards uniformly at random (without replacement) from a deck of 14 cards, each of which labeled a different integer from 1 to 14. She finally sums up the value of her die roll and the value on each card she drew to get a score of S. Let A be the set of all obtainable scores. Compute the probability that S is greater than or equal to the median of A.

Answer: $\mathbf{\frac{15}{28}}$

Solution: The key insight is to develop the following bijection: For every event where Sabine scores $S < 119$ points, there is a corresponding event where she scores $119 - S$ points. Consider a *turn* $t = (d, p)$ of the game where d is the value of the dice roll and p is a permutation of the integers from 1 to 14. The score of this turn $s(t)$ is given by the sum of d and the sum of the first d values of the permutation p (denoted by p_d). In short, $s(t) = d + p_d$. Then, for a given turn t scoring $s(t)$ points where $d < 14$, there is a corresponding turn $t' = (14 - d, p')$ where p'

is the reverse permutation of p. We have $s(t')$ as $(14-d)+(1+2+\cdots+14)-p_d$, as we sum $14-d$ and the remaining $14-d$ integers after the first d totaled in p_d. This totals to

$$s(t') = 14 + (1+2+\cdots+14) - (d+p_d) = 14 + \frac{14(15)}{2} - s(t) = 119 - s(t)$$

which was what we wanted.

Still ignoring the case where $d = 14$, this means the median score is $\frac{119}{2} = 59.5$ as every score larger than 59.5 is complemented by a score lower than 59.5. When $d = 14$, we get a guaranteed score of $14 + \frac{14(15)}{2} = 119$, as we are guaranteed to draw every card. This "pulls the median" to the right to the smallest achievable score greater than 59.5, which means we just need to find the probability that our score is greater than 59.5. Now, we observe that the bijection also carries over to probabilities: given that we do not roll a 14, we have an equal likelihood of scoring above 59.5 as we do of scoring below it. If we do roll 14, we are of course guaranteed to score higher than 59.5. Thus, the final probability is $\frac{13}{14} \cdot \frac{1}{2} + \frac{1}{14} = \boxed{\frac{15}{28}}$

17. Let N be the smallest positive integer divisble by $10^{2023} - 1$ that only has the digits 4 and 8 in decimal form (these digits may be repeated). Compute the sum of the digits of $\frac{N}{10^{2023}-1}$.

Answer: 20234

Solution: Let $f_n(m)$ be the number created by repeating each digit of m n times. (For example, $f_3(253) = 222555333$). Let $d = 10^{2023} - 1$, which is also equal to $f_{2023}(9)$. Observe that we may write the remainder of $N \pmod{d}$ as the sum of 2023-digit strings of N; because $10^{2023} \equiv 1 \pmod{d}$, we can split N into strings of 2023 digits and sum them to get the same remainder $\pmod{d}$. Then, it becomes easy to construct a possible value of N by simply getting the sum of the ones digits of each string to be a multiple of 9, and then copying that sum to each place in the sum. One such sum is $4+8+8+8+8 = 36$, and in fact this sum is minimal. We know this because the fact that the digits are 4 and 8 means the sum will be a multiple of 4, and we are aiming for a multiple of 9, so the least possible sum is 36, the least common multiple of these two digits. Therefore we have the minimal value of N as $N = f_{2023}(48888)$.

Now, we have to divide N by $10^{2023} - 1$. Let $n = 2023$ for convenience. We write that

$$N = f_n(48888) = \frac{4}{9}(10^n - 1)(10^{4n}) + \frac{8}{9}(10^n - 1)(10^{3n} + 10^{2n} + 10^n + 1)$$

$$N = (10^n - 1)(\frac{4}{9}10^{4n} + (10^{3n} + 10^{2n} + 10^n + 1) - \frac{1}{9}(10^{3n} + 10^{2n} + 10^n + 1))$$

$$\frac{N}{10^n - 1} = \frac{4}{9}10^{4n} + (10^{3n} + 10^{2n} + 10^n + 1) - \frac{1}{9}(10^{3n} + 10^{2n} + 10^n + 1)$$

We will use this expression to compute the sum of digits, proceeding carefully so that no carries occur in the addition (which would be troublesome to deal with.) Now, we see that $\frac{4}{9}10^{4n}$ contributes $4 \cdot 4n$ to the total sum of digits of $\frac{N}{10^n-1}$ (ignoring the digits after the decimal point, which we know will vanish since the number is an integer). Since all of the digits so far are ≥ 4, we can now subtract $\frac{1}{9}(10^{3n} + 10^{2n} + 10^n + 1)$ to remove at most 4 from each digit safely without needing a carry. This subtracts $3n + 2n + n = 6n$ from the total sum of digits, leaving us at $10n$. Then, finally, the terms $(10^{3n} + 10^{2n} + 10^n + 1)$ add 1 to the sum of digits each giving us $10n + 4$. When we plug in $n = 2023$, we get our answer $10(2023) + 4 = \boxed{20234}$

18. Consider the sequence $b_1, b_2, b_3, \ldots$ of real numbers defined by $b_1 = \frac{3+\sqrt{3}}{6}$, $b_2 = 1$, and for $n \geq 3$,

$$b_n = \frac{1 - b_{n-1} - b_{n-2}}{2b_{n-1}b_{n-2} - b_{n-1} - b_{n-2}}.$$

Compute b_{2023}.

Answer: $\mathbf{\frac{3-\sqrt{3}}{6}}$

Solution: We apply a series of substitutions. The denominator can be rewritten via Simon's favorite factoring trick,

$$b_n = \frac{2 - 2b_{n-1} - 2b_{n-2}}{(2b_{n-1} - 1)(2b_{n-2} - 1) - 1} = \frac{(2b_{n-1} - 1) + (2b_{n-2} - 1)}{1 - (2b_{n-1} - 1)(2b_{n-2} - 1)}.$$

Take $a_n = 2b_n - 1$, so that $a_1 = \frac{1}{\sqrt{3}}$, $a_2 = 1$, and

$$a_n = \frac{a_{n-1} + a_{n-2}}{1 - a_{n-1}a_{n-2}}$$

for $n \geq 2$. We recognize this as the tangent addition formula: taking $\theta_n := \arctan a_n$ yields $\theta_1 = \frac{2\pi}{12}$, $\theta_2 = \frac{3\pi}{12}$, and

$$\tan\theta_n = \frac{\tan\theta_{n-1} + \tan\theta_{n-2}}{1 - \tan\theta_{n-1}\tan\theta_{n-2}} = \tan(\theta_{n-1} + \theta_{n-2}) \implies \theta_n = \theta_{n-1} + \theta_{n-2}$$

for $n \geq 2$. Equivalently, since the period of $\tan\theta$ is π, it suffices to $c_n := \frac{12}{\pi}\theta_n$ modulo 12, where $c_1 = 2$, $c_2 = 3$, and $c_n = c_{n-1} + c_{n-2}$ for $n \geq 2$.

We compute $c_3 = 5$, $c_4 = 8$, $c_5 = 1$, $c_6 = 9$, $c_7 = 10$, $c_8 = 7$, $c_9 = 5$, $c_{10} = 0$, $c_{11} = 5$, $c_{12} = 5$, $c_{13} = 10$, $c_{14} = 3$. Thus, $c_{k+12} = 5c_k \pmod{12}$, so $c_{k+24} = 25c_k \equiv c_k \pmod{12}$. Since the period of c is 24, $c_{2023} \equiv c_7 \equiv 10 \pmod{12}$, and

$$b_{2023} = \frac{a_{2023} + 1}{2} = \frac{\tan\theta_{2023} + 1}{2} = \frac{\tan\frac{10\pi}{12} + 1}{2} = \boxed{\frac{3 - \sqrt{3}}{6}}.$$

19. Let N_{21} be the answer to question 21. Suppose a jar has $3N_{21}$ colored balls in it: N_{21} red, N_{21} green, and N_{21} blue balls. Jonathan takes one ball at a time out of the jar uniformly at random without replacement until all the balls left in the jar are the same color. Compute the expected number of balls left in the jar after all balls are the same color.

Answer: $\mathbf{\frac{108}{73}}$

Solution 1: Let $N = N_{21}$ for convenience. Equivalently, suppose we continue drawing balls until the jar is empty, and we are asked to compute the expected value of X, the length of the streak of same-colored balls at the end. By symmetry, it is equivalent to compute

$$\begin{aligned}
\mathbb{E}[X \mid \text{last ball is red}] &= \sum_{x=1}^{N} \mathbb{P}(X \geq x \mid \text{last ball is red}) && (X \in \mathbb{Z}^+) \\
&= \sum_{x=1}^{N} \frac{\binom{3N-x}{N,N,N-x}}{\binom{3N-1}{N,N,N-1}} \\
&= \frac{1}{\frac{(3N-1)!}{N!N!(N-1)!}} \sum_{x=1}^{N} \frac{(3N-x)!}{N!N!(N-x)!} \\
&= \frac{(2N)!(N-1)!}{(3N-1)!} \sum_{y=2N}^{3N-1} \binom{y}{2N} && (y := 3N-x) \\
&= \frac{(2N)!(N-1)!}{(3N-1)!} \frac{(3N)!}{(2N+1)!(N-1)!} && \text{(Hockey stick)} \\
&= \frac{3N}{2N+1}.
\end{aligned}$$

Since $N_{21} = 36$, our answer is $\boxed{\frac{108}{73}}$.

Solution 2: We make use of indicator variables. We want to compute the probability that a given ball of some color c appears in the final streak of same-colored balls at the end of the drawing. This means, out of the $2N_{21}+1$ balls consisting of the single ball of color c and the $2N_{21}$ balls of different colors than c, the ball of color c must occur last. This has probability $\frac{1}{2N_{21}+1}$. Therefore, by linearity of expectation, we have that our desired answer is $3N_{21} \cdot \frac{1}{2N_{21}+1} = \frac{3N_{21}}{2N_{21}+1}$. Plugging in $N_{21} = 36$ gives the answer $\boxed{\frac{108}{73}}$.

20. Let N_{19} be the answer to question 19. For every non-negative integer k, define

$$f_k(x) = x(x-1) + (x+1)(x-2) + \cdots + (x+k)(x-k-1),$$

and let r_k and s_k be the two roots of $f_k(x)$. Compute the smallest positive integer m such that $|r_m - s_m| > 10N_{19}$.

Answer: 12

Solution: First, we simplify

$$\begin{aligned}
f_k(x) &= \sum_{i=0}^{k} (x+i)(x-i-1) \\
&= \sum_{i=0}^{k} (x^2 - x - i(i+1)) \\
&= (k+1)x^2 - (k+1)x - \left(\frac{k(k+1)(2k+1)}{6} + \frac{k(k+1)}{2}\right) \\
&= (k+1)\left(x^2 - x - \frac{k^2+2k}{3}\right).
\end{aligned}$$

By the quadratic formula, $|r_k - s_k| = \sqrt{1 + 4 \cdot \frac{k^2+2k}{3}}$. Algebra yields

$$|r_{m-1} - s_{m-1}| \leq 10N_{19} < |r_m - s_m| \implies m^2 \leq 75N_{19}^2 + \frac{1}{4} < (m+1)^2.$$

Next, we invoke our solutions to problems 19 and 21; let N_i denote the answer to problem i. Then $N_{19} = \frac{3N_{21}}{2N_{21}+1} = \frac{3N_{20}^2}{2N_{20}^2+4} \in [1, \frac{3}{2})$.

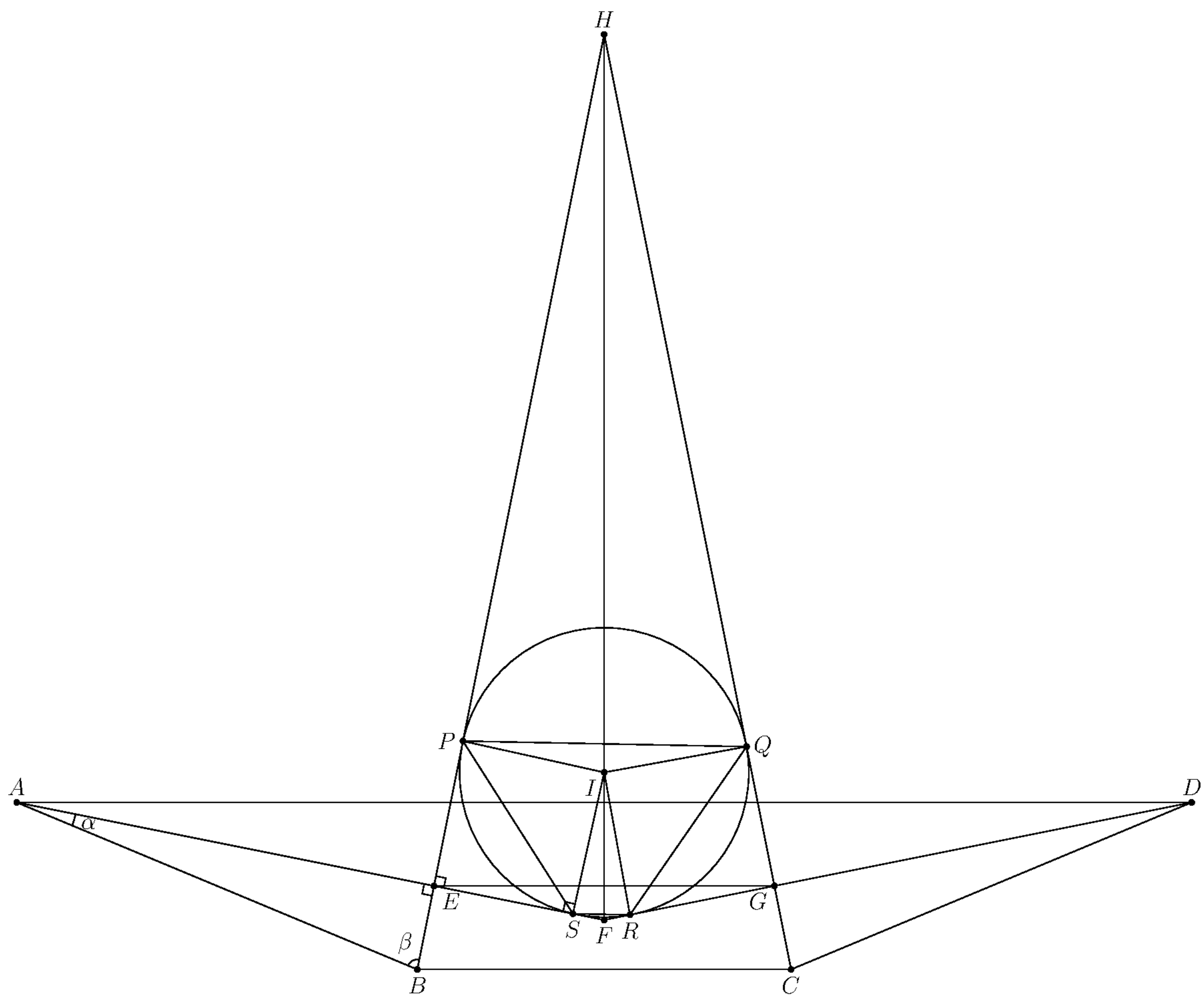
- $N_{19} \geq 1$ implies $(N_{20}+1)^2 > 75 \cdot 1^2 + \frac{1}{4}$, or $N_{20} \geq 8$. Since N_{19} increases with N_{20}, it follows that $N_{19} \geq \frac{3 \cdot 8^2}{2 \cdot 8^2 + 4} = \frac{16}{11} > \sqrt{2}$. Thus, $(N_{20}+1)^2 > 75 \cdot \sqrt{2}^2 + \frac{1}{4} > 144$ implies $N_{20} \geq 12$.
- $N_{19} < \frac{3}{2}$ implies $N_{20}^2 = m^2 < 75 \cdot \left(\frac{3}{2}\right)^2 + \frac{1}{4} = 13^2$, so $N_{20} \leq 12$.

Our answer is $N_{20} = \boxed{12}$.

21. Let N_{20} be the answer to question 20. In isosceles trapezoid $ABCD$ (where $\overline{BC}$ and $\overline{AD}$ are parallel to each other), the angle bisectors of A and D intersect at F, and the angle bisectors of points B and C intersect at H. Let $\overline{BH}$ and $\overline{AF}$ intersect at E, and let $\overline{CH}$ and $\overline{DF}$ intersect at G. If $CG = 3$, $AE = 15$, and $EG = N_{20}$, compute the area of the quadrilateral formed by the four tangency points of the largest circle that can fit inside quadrilateral $EFGH$.

Answer: 36

Solution:



Let $\alpha = \angle BAE$ and $\beta = \angle ABE$. Then $\alpha + \beta = \frac{1}{2}\angle DAB + \angle ABC = 90^\circ$. By symmetry, $\angle AEB = \angle DGC = 90^\circ$, and $EFGH$ is a kite.

The kite's area is $K = \frac{1}{2} \cdot N_{20} \cdot FH$. Since $\angle BHF = \frac{1}{2}(180^\circ - 2\beta) = \alpha$, it follows that $\triangle ABE \sim \triangle HFE$, and the kite's semiperimeter is

$$s = FE + EH = \frac{FH}{AB}(BE + EA) = \frac{3+15}{\sqrt{3^2+15^2}}FH.$$

Thus, the kite's inradius is $r = K/s = \frac{\sqrt{26}}{12}N_{20}$.

Let the intouch points of $EFGH$ be P, Q, R, S, as labeled above. Then $PISE$ and $QIRG$ are squares with side length r. Furthermore, by similar triangles, $[IPQ] = [IRS] = \frac{3}{\sqrt{3^2+15^2}}r \cdot \frac{15}{\sqrt{3^2+15^2}}r = \frac{5}{26}r^2$, and

$$[PQRS] = 2 \cdot \frac{1}{2}r^2 + 2 \cdot \frac{5}{26}r^2 = \frac{18}{13}r^2 = \frac{N_{20}^2}{4}.$$

Since $N_{20} = 12$, our answer is $\boxed{36}$.

22. Let $d_n(x)$ be the nth decimal digit (after the decimal point) of x. For example, $d_3(\pi) = 1$ because $\pi = 3.14\underline{1}5\ldots$. For a positive integer k, let $f(k) = p_k^4$, where p_k is the kth prime number. Compute the value of $\sum_{i=1}^{2023} d_{f(i)}\left(\frac{1}{1275}\right)$.

Answer: 18199

Solution: First, we want to show that

$$d_n\left(\frac{h}{k}\right) = \left\lfloor \frac{10^n h \pmod{10k}}{k} \right\rfloor$$

for a fraction $\frac{h}{k}$ with a non-terminating decimal expansion. We will analyze the long-division algorithm for computing decimal digits to show this. Let $x_n = 0.d_1 d_2 \cdots d_n$ be the first n decimal digits of $\frac{h}{k}$ so that $kx_n \le h$, which allows us to define the remainder $r_n = \frac{h}{k} - x_n$. We see that in particular $0 < 10^n(h - kx_n) < k$, as otherwise we could add 10^{-n} to x_n, incrementing d_n by 1 and decreasing $10^n(h - kx_n)$ by k. (We cannot have equality as otherwise the decimal would have terminated.) Since the non-terminating decimal representation of a number is unique, this changing of d_n is not allowed. Therefore, we take $r_n = 10^n h \pmod{k}$, as $10^n kx_n$ is a multiple of k such that $0 < 10^n h - 10^n kx_n < k$. To compute the next digit, we would multiply the remainder by 10 and subtract the largest possible multiple of k from it. This looks like $d_{n+1} = \left\lfloor \frac{10 \cdot r_n}{k} \right\rfloor$, which gives the formula we wanted.

Now, the important part is determining what the period of this sequence is. This requires computing the order of 10 modulo 12750. We factor: $12750 = 10(1275) = 10(25)(51) = 2 \cdot 3 \cdot 5^3 \cdot 17$. The order of 10 modulo 12750 is then just the LCM of the orders mod each prime power. We can ignore $2, 5^3$ since they both divide 10^3 and 3 is less than any prime to the fourth power. So we just need the order of 10 modulo 3 and 17; since $10 \equiv 1 \pmod{3}$ that order is 1, and we compute $10^2 = 100 \equiv -2 \pmod{17}$, and $(-2)^4 \equiv -1 \pmod{17}$ meaning that the order of 10 is exactly $17 - 1 = 16$. So the overall period of the sequence is 16 (after the first 3 terms where we are eliminating the powers of 2 and 5).

It remains to compute $p_n^4 \pmod{16}$. We see that $2^4 \equiv 0 \pmod{16}$, and then all other primes are odd so their square is $1 \pmod{8}$ (by computation). We obtain that $(8k+1)^2 = 64k^2 + 16k + 1$ is equivalent to $1 \pmod{16}$ always. Therefore the sum we want to compute is equal to $d_{16}(\frac{1}{1275}) + 2022 d_{17}(\frac{1}{1275})$ (we can't use d_0 and d_1 as we need to wait for the sequence to stabilize by becoming equivalent to 0 modulo the powers of 2 and 5.)

So, we have to compute what r_{15}, r_{16} are, which we can do via CRT and repeated squaring. We know what r_n will be modulo 2, 3, and 5^3 so we only need to compute what it will be modulo 17 to get the value. We'll need to compute $10^{15}, 10^{16} \pmod{17}$. We've already shown that $10^{16} \equiv 1 \pmod{17}$, so $10^{15} \equiv 10^{-1} = 12 \pmod{17}$. Therefore $r_{15}, r_{16} \equiv 0 \pmod{250}$, $r_{15} \equiv 2(17) + 12 = 46 \pmod{51}$ and $r_{16} \equiv 1 \pmod{51}$. We see that $250 \equiv -5 \pmod{51}$, so we can quickly obtain that $r_15 = 250$ and $r_{16} = 10(250) = 2500 \equiv 1225 \pmod{1275}$. Finally, we compute

$$d_{16} = \left\lfloor \frac{10 \cdot r_{15}}{1275} \right\rfloor = \left\lfloor \frac{2500}{1275} \right\rfloor = 1$$

and

$$d_{17} = \left\lfloor \frac{10 \cdot r_{16}}{1275} \right\rfloor = \left\lfloor \frac{12250}{1275} \right\rfloor = 9$$

So our final answer is $1 + 2022(9) = \boxed{18199}$

23. A robot initially at position 0 along a number line has a *movement function* $f(u, v)$. It rolls a fair 26-sided die repeatedly, with the kth roll having value r_k. For $k \ge 2$, if $r_k > r_{k-1}$, it moves $f(r_k, r_{k-1})$ units in the positive direction. If $r_k < r_{k-1}$, it moves $f(r_k, r_{k-1})$ units in the negative

direction. If $r_k = r_{k-1}$, all movement and die-rolling stops and the robot remains at its final position x. If $f(u,v) = (u^2 - v^2)^2 + (u-1)(v+1)$, compute the expected value of x.

Answer: 225

Solution 1: Let $g(u,v) = (u^2 - v^2)^2 + uv - 1$ and $h(u,v) = u - v$ be the symmetric and asymmetric parts of f. We will compute the robot's expected movement C on the kth step, which is a constant independent of k.

$$\begin{aligned} C &= \mathbb{E}_{R_{k-1},R_k}\left[\begin{cases} f(R_k,R_{k-1}) & \text{if } R_k > R_{k-1} \\ -f(R_k,R_{k-1}) & \text{if } R_k < R_{k-1} \\ 0 & \text{if } R_k = R_{k-1}\end{cases}\right] \\ &= \mathbb{E}_{R_{k-1}}\left[\mathbb{E}_{R_k}\left[\begin{cases} g(R_k,R_{k-1}) + h(R_k,R_{k-1}) & \text{if } R_k > R_{k-1} \\ -g(R_k,R_{k-1}) - h(R_k,R_{k-1}) & \text{if } R_k < R_{k-1} \\ 0 & \text{if } R_k = R_{k-1}\end{cases} \middle| \, R_{k-1}\right]\right]. \end{aligned}$$

We split this calculation into two parts. For the g terms, since R_k and R_{k-1} are iid, it follows that we can swap them in the symmetric case. We can simplify as

$$\begin{aligned} &\mathbb{E}_{R_{k-1},R_k}\left[g(R_k,R_{k-1}) \,|\, R_k > R_{k-1}\right]\mathbb{P}(R_k > R_{k-1}) \\ &\quad - \mathbb{E}_{R_{k-1},R_k}\left[g(R_k,R_{k-1}) \,|\, R_k < R_{k-1}\right]\mathbb{P}(R_k < R_{k-1}) \\ &= \mathbb{E}_{R_{k-1},R_k}\left[g(R_k,R_{k-1}) \,|\, R_k < R_{k-1}\right]\mathbb{P}(R_k < R_{k-1}) \\ &\quad - \mathbb{E}_{R_{k-1},R_k}\left[g(R_k,R_{k-1}) \,|\, R_k > R_{k-1}\right]\mathbb{P}(R_k > R_{k-1}). \end{aligned}$$

The two sides of the equation are also negations, so both sides must be 0.

For the h terms, we compute

$$\begin{aligned} &\mathbb{E}_{R_{k-1},R_k}\left[R_k - R_{k-1} \,|\, R_k > R_{k-1}\right]\mathbb{P}(R_k > R_{k-1}) \\ &\quad - \mathbb{E}_{R_{k-1},R_k}\left[R_k - R_{k-1} \,|\, R_k < R_{k-1}\right]\mathbb{P}(R_k < R_{k-1}) \\ &= \frac{1}{N^2}\sum_{r_{k-1}=1}^{N}\left(\sum_{r_k=1}^{r_{k-1}}(r_{k-1} - r_k) + \sum_{r_k=r_{k-1}+1}^{N}(r_k - r_{k-1})\right) \\ &= \frac{1}{N^2}\sum_{r_{k-1}=1}^{N}\left(\binom{r_{k-1}}{2} + \binom{N+1-r_{k-1}}{2}\right) \\ &= \frac{1}{N^2}\cdot 2\binom{N+1}{3}, \end{aligned}$$

where $N = 26$ and the last line follows from Hockey stick. The number of times the robot rolls the die is distributed as $T \sim \text{Geometric}(\frac{1}{N})$, so by linearity of expectation

$$\mathbb{E}[X] = C\mathbb{E}[T] = \frac{(N-1)(N+1)}{3N}\cdot N = \boxed{225}.$$

Solution 2: By linearity of expectation and some symmetry, we can remove most of the complicated parts of $f(x,y)$, since in expectation symmetric functions in x,y will sum to 0 expected movement.

Subtracting off all symmetric parts gives $g(r_k, r_{k-1}) = r_k - r_{k-1}$, and then by antisymmetry the expected value of the sums will be equal to $2r_k$.

So we can replace $f(x, y)$ with $2x$. Let $n = 26$ be the number of sides of the die for convenience. Then given that $r_2 = k$, we expect to add

$$E_k = 2 \cdot \frac{1}{n}\left((k+1+\cdots+n) - (1+\cdots+k-1)\right) = \frac{1}{n} \cdot \left(\binom{n+1}{2} - \binom{k+1}{2} - \binom{k}{2}\right)$$

to our x coordinate. Therefore our expected total movement is

$$\frac{1}{n}\sum_{i=1}^{n} E_i = 2 \cdot \frac{1}{n^2}\left(n\binom{n+1}{2} - \binom{n+2}{3} - \binom{n+1}{3}\right) = 2 \cdot \frac{1}{n^2}\left(\binom{n+1}{3}\right)$$

by use of the hockeystick identity. Then since we have probability $\frac{1}{n}$ of stoppping on any given turn, we expect to move n times so the expected value of the x coordinate is therefore $\frac{2}{n}\binom{n+1}{3}$. Plugging in $n = 26$ gives $\frac{2(27)(26)(25)}{6(26)} = (9)(25) = \boxed{225}$.

24. Define the sequence $s_0, s_1, s_2, \dots$ by $s_0 = 0$ and $s_n = 3s_{n-1}+2$ for $n \geq 1$. The monic polynomial $f(x)$ defined as

$$f(x) = \frac{1}{s_{2023}}\sum_{k=0}^{32} s_{2023+k}x^{32-k}$$

can be factored uniquely (up to permutation) as the product of 16 monic quadratic polynomials $p_1, p_2, \cdots, p_{16}$ with real coefficients, where $p_i(x) = x^2 + a_i x + b_i$ for $1 \leq i \leq 16$. Compute the integer N that minimizes $\left|N - \sum_{k=1}^{16}(a_k + b_k)\right|$.

Answer: 141

Solution: Let $d = 32$ (an even number) be the degree of $f(x)$, and suppose we are decomposing f into the product of $\frac{d}{2}$ monic irreducible quadratic polynomials.

This problem can be broken down into 3 phases. Phase 1 is approximating the polynomial with something much easier to work with. Observe that the approximate ratio of $\frac{a_{n+1}}{a_n} \approx 3$. Let this common ratio be $r = 3$. Since $\frac{a_{n+1}}{a_n}$ converges very quickly to r, we assume that $\frac{a_{n+k}}{a_n} \approx r^k$. In this case, we have $n = 2023$ which is sufficiently large for this approximation to be very accurate. This gives us the approximation

$$f(x) \approx g(x) = \sum_{i=0}^{d} r^i x^{d-i}$$

Some notes on why this approximation is good enough to get our answer: we notice that $a_n = 3^n - 1$, which means the error in the coefficient $\frac{a_{n+k}}{a_n}$ is about $\frac{3^k}{3^{2023}}$. This means the error never goes above about 3^{-2000}, which is vanishingly small (which is the guiding principle of our argument). The difference between $f(x)$ and $g(x)$ is a polynomial with coefficients at most 3^{-2000}. Since the roots of $g(x)$ have magnitude 3, the size of the error from the root is bounded by at most $33 \cdot 3^{32-2000}$ which is still vanishingly small. These roots similarly determine what the factor polynomials' coefficients a_k, b_k will be, and we can similarly bound the error of the coefficients below $3^{(-1900)}$. Therefore the sum of a_k, b_k for $g(x)$ is definitely less than $\frac{1}{2}$ away from the sum of a_k, b_k for $f(x)$. We will determine later that this sum is an integer for $g(x)$, so this is enough to guarantee the closest integer to this sum for $f(x)$ will be the same.

This is the starting point of phase 2, where we find the roots of this polynomial and factor into monic quadratic polynomials. The next observation is to recognize that

$$g(x) = \frac{x^{d+1} - r^{d+1}}{x - r}$$

from the finite geometric series sum formula, which has roots equal to $r\omega^k$ for $1 \le k \le d$, where $\omega^k = e^{\frac{2k\pi i}{d+1}}$ is a $(d+1)$th root of unity (excluding 1). Then, the unique factorization into $\frac{d}{2}$ monic quadratic polynomials comes from pairing off complex conjugate root pairs like $r\omega^k$ and $r\omega^{-k}$. When we take the product of $(x - r\omega^k)(x - r\omega^{-k})$, we get a monic quadratic polynomial $x^2 - r(\omega^k + \omega^{-k})x + r^2$. These coefficients are real because the sum of conjugate complex numbers is real, and so is their product.

Now, we enter phase 3, which is to compute the sum of these coefficients. We see that the b_k sum to $\frac{dr^2}{2}$ as there are $\frac{d}{2}$ polynomials and their ending terms are all r^2. Then, all that remains is to sum the a_k. We can observe that this is equal to the coefficient of x_31 in $g(x)$ (by just multiplying the p_i together). Thus the sum of the a_k is $-r$. Our final answer would then be $\frac{d}{2}r^2 - r$. Plugging $d = 32, r = 3$ gives $16 \cdot 9 - 3 = \boxed{141}$.

25. Let triangle $\triangle ABC$ have side lengths $AB = 6$, $BC = 8$, and $CA = 10$. Let S_1 be the largest square fitting inside of $\triangle ABC$ (sharing points on edges is allowed). Then, for $i \ge 2$, let S_i be the largest square that fits inside of $\triangle ABC$ while remaining outside of all other squares $S_1, \cdots, S_{i-1}$ (with ties broken arbitrarily). For all $i \ge 1$, let m_i be the side length of S_i and let S be the set of all m_i. Let x be the 2023rd largest value in S. Compute $\log_2\left(\frac{1}{x}\right)$.

Submit your answer as a decimal E to at most 3 decimal places. If the correct answer is A, your score for this question will be $\max(0, 25 - 2|A - E|)$, rounded to the nearest integer.

Answer: 60.41144149627205

Solution: We want to compute the largest square fitting in a 3-4-5 right triangle $\triangle XYZ$ with $XY = 3, YZ = 4, XZ = 5$ (and then scale appropriately). We test two possible configurations of the square: with sides on the legs of the triangle, or with a side on the hypotenuse of the triangle. Let s be the side length of this maximal square. With sides on the base, we get an equation in terms of s: $s = 4 - \frac{4s}{3}$, giving $s = \frac{12}{7}$. On the other hand, we can check if the largest square has base on the hypotenuse. Dropping an altitude from Y to XZ at point H gives the maximal height of such a square as $\frac{12}{5}$. Then we have two right triangles $\triangle ZHY \sim \triangle YHX \sim \triangle XYZ$. We want to draw a segment PQ on XZ containing H such that the height from P and Q to the sides YZ and XY are both equal to PQ. If we set the height above the hypotenuse of the square to be $\frac{12}{5} - k$, then $PH = \frac{3k}{4}$ and $HQ = \frac{4k}{3}$ by similar triangles. We then have that $\frac{25k}{12} = \frac{12}{5} - k$, which tells us that the overall side length $\frac{25k}{12} = \frac{25}{37} \cdot \frac{12}{5} = \frac{60}{37}$. However, $\frac{60}{37} < \frac{12}{7}$ so we see that the largest square fitting inside has side length $\frac{12}{7}$.

This gives that $s_1 = 2\frac{12}{7} = \frac{24}{7}$. Placing this largest square inside the right triangle gives another two right triangles scaled by $\frac{3}{7}$ and $\frac{4}{7}$ respectively (again by similar triangles). Now, we have to balance out all of the multiplicative scaling and estimate which answer we will get in the end. From the 3-4-5 right triangle, placing the maximal square of side length s gives two right triangles, one with maximal square side length $\frac{3s}{7}$ and one with maximal square side length $\frac{4s}{7}$. Therefore, the values of side lengths look like $s_1 \cdot \left(\frac{3}{7}\right)^m \left(\frac{4}{7}\right)^n$ for some nonnegative integers m, n. Then, we want to find the 2023rd largest element in this set.

We can ignore s_1 for now and multiply by it later. We essentially try to find some constant K such that $K \le \left(\frac{3}{7}\right)^m \left(\frac{4}{7}\right)^n$ for exactly 2023 pairs of nonnegative integers m, n. Taking logs (base

2) gives the equation

$$K_1 \le m(\log 3 - \log 7) + n(\log 4 - \log 7)$$

for some constant $K_1 < 0$. We multiply by -1 to obtain

$$K_2 \ge m(\log 7 - \log 3) + n(\log 7 - \log 4)$$

Our goal is to solve so that $K_2 = \log \frac{1}{xs_1}$ and then add $\log s_1$ to get our final answer. Note that our current inequality describes a line in the mn plane where we want to find the lattice points that are under or on the line. In order to estimate this, we need to compute estimations of $\log 7, \log 3, \log 4$.

The rough estimations come from $2^3 \approx 7$, $2^3 \approx 3^2$, and $2^2 = 4$, giving $\log 7 \approx 3, \log 3 \approx \frac{3}{2}, \log 4 = 2$, which can potentially be improved by nudging $\log 7$ down and $\log 3$ up by a small amount (say, 0.1). Better estimations can be obtained by using these estimations; simply raise both sides of the equation to the same power and then make small adjustments when they make improvements. For example, $2^{12} = 4096 \approx 2401 = 7^4$ by raising both sides to the 4th power, but then it's clear that a division by 2 on the left improves the approximation greatly. This gives $\log 7 \approx \frac{11}{4}$. Similarly, we can improve $\log 3$: $2^9 = 512 \approx 3^6 = 729$, dividing by 2 on the left and 3 on the right gives $2^8 = 256 \approx 3^5 = 243$ which gives $\log 3 \approx 1.6$. If you want to make tweaks now, note that we now have an underestimation of $\log 7$ and overestimation of $\log 3$, so the tweaks should be in the opposite direction as before. We will adjust to $\log 7 \approx 2.8$ (noticing that $2^{14} = 16384 \approx 7^5 = 16807$ using the same strategy described above) and leave $\log 3$ untouched.

We then get the two approximations of coefficients $\log 7 - \log 3 \approx 1.2$ and $\log 7 - \log 4 \approx 0.8$. This lets us get the equation $K_2 \ge 1.2m + 0.8n$, and we want to pick K_2 so there are 2023 solutions to this equation. We can write the equation of the line as $n = -1.5m + K_3$ (with $K_3 = 1.25K_2$), which has intercepts $(0, K_3)$ and $(\frac{2}{3}K_3, 0)$. We want to calculate the number of lattice points inside, we can either estimate with the area of the triangle or count them more carefully. We can count the points on the boundary and use Pick's theorem to compute the exact number of lattice points (assuming integer K_3). We can count roughly $K_3 + \frac{2}{3}K_3 + \frac{1}{3}K_3 = 2K_3$ lattice points on the boundary, and then since the area $\frac{1}{3}K_2^2 = I + \frac{B}{2} + 1$ we get that there are $\frac{1}{3}K_3^2 + K_3 - 1$ lattice points inside the triangle. We want this to equal 2023, and we can estimate by the following manipulation:

$$\frac{1}{3}K_3^2 + K_3 - 1 = \frac{1}{3}(K_3 + \frac{3}{2})^2 - \frac{7}{4} = 2023$$

which gives $K_3 \approx \sqrt{3 \cdot 45^2} - \frac{3}{2} \approx 77.25$.

We now solve backwards to get our original constant: $K_2 = 0.8K_3 \approx 61.8$ and then we just need to subtract $\log s_1 = \log 3 + \log 8 - \log 7 = 1.8$ which gives our final answer as $\boxed{60}$ which is within 0.4 of the correct answer and earns 24 points.

Alternate solutions are available with different approximations of logarithms, estimates of number of lattice points, or a completely different method of counting the squares all exist as well.

26. For positive integers i and N, let $k_{N,i}$ be the ith smallest positive integer such that the polynomial $\frac{x^2}{2023} + \frac{Nx}{7} - k_{N,i}$ has integer roots. Compute the minimum positive integer N satisfying the condition $\frac{k_{N,2023}}{k_{N,1000}} < 3$. Submit your answer as a positive integer E. If the correct answer is A, your score for this question will be $\max\left(0, 25\min\left(\frac{A}{E}, \frac{E}{A}\right)^{\frac{3}{2}}\right)$, rounded to the nearest integer.

Answer: 232

Solution:

We first claim that Will write up soon, idea is that, with a little CRT,

$$k_{N,i} \approx \begin{cases} \frac{1}{2023} * (i * 119)(i * 119 + 289 * N) & \text{if } 7 \mid N \\ \frac{1}{2023} * (\frac{i}{2} * 119)(\frac{i}{2} * 119 + 289 * N) & \text{if } 7 \nmid N \end{cases}.$$

Now, $\frac{(2023*119)}{(1000*119)} = \frac{2023}{1000}$, so we want $\frac{\frac{2023}{2}*119+289*N}{\frac{1000}{2}*119+289*N} \approx \frac{3000}{2023}$.

In general, we we can see that $\frac{\frac{2023}{2}*119+289*N}{\frac{1000}{2}*119+289*N}$ will be closer to 1 than $\frac{\frac{2023}{2}*119+289*N}{\frac{1000}{2}*119+289*N}$, so we choose that to estimate N. (Part of the problem is determining whether N being divisible by 7 or not divisible by 7 is better)

27. Let ω be a circle with positive integer radius r. Suppose that it is possible to draw isosceles triangle with integer side lengths inscribed in ω. Compute the number of possible values of r where $1 \le r \le 2023^2$. Submit your answer as a positive integer E. If the correct answer is A, your score for this question will be $\max\left(0, 25\left(3 - 2\max\left(\frac{A}{E}, \frac{E}{A}\right)\right)\right)$, rounded to the nearest integer.

Answer: 217602

Solution: sketch: will add diagrams later and detail later.

Let $N = 2023^2$ for convenience. We essentially need that for some integers r, s we have that $(r+s)^2 + r^2 - s^2 = k^2$ or $(r-s)^2 + r^2 - s^2 = k^2$ for some integer k, and we also need $r^2 - s^2 = l^2$ for an integer l in both cases. We begin with the first case: Then we have the Pythagorean triple $r+s, l, k$. However, $r^2 - s^2 = l^2$ so that s, l, r is another Pythagorean triple. Then, to generate primitive Pythagorean triples for s, l, r, we set $s = a^2 - b^2, l = 2ab, r = a^2 + b^2$ for integers a, b where we need a, b coprime and exactly one of a, b to be even. However, we need to make sure $r + s, l, k$ will also be a Pythagorean triple; we have that $k^2 = 2r(r+s)$ by simplification, and we know that $r + s = 2a^2$ means that $k^2 = 4a^2 r$. Therefore, r must also be a perfect square $r = c^2$, which means that we have another Pythagorean triple $a^2 + b^2 = c^2$ and this means we must parameterize a, b, c like we did for s, l, r, getting $a = x^2 - y^2, b = 2xy, c = x^2 + y^2$. So, our characterization of all possible r is given by $\lambda\alpha^2$, where λ is some positive integer and α is some prime expressible as the sum of two squares (which just means $1 \pmod 4$). Let P be the set of all valid α.

We then need to compute the sum

$$\sum_{p \in P} \left\lfloor \frac{N}{p^2} \right\rfloor.$$

We can estimate this sum by estimating $\sum_{p \in P} \frac{1}{p^2}$ and multiplying by N. We estimate this constant coefficient by adding up a few terms:

$$\frac{1}{25} + \frac{1}{169} + \frac{1}{289} + \frac{1}{841} \approx 0.04 + 0.006 + 0.0035 + 0.0011 = 0.0506$$

We then multiply by N to get the approximation $0.0506(2023^2) = 0.0511(2000^2 + 2(23)(2000) + 23^2) \approx 202500 + 4600 + 26 = 207126$, which is within 10,000 of the real answer $\boxed{217602}$ and gives you a score of 22. Adding a few more terms more meticulously (or guessing a good value for the coefficient like $\frac{1}{19}$) can get you a perfect score of 25. Alternatively, you can guess that the constant coefficient is something like $\frac{1}{20}$ and that $2023^2 \approx 4 \cdot 10^6$ and get the answer 200000, which is also a fairly good estimate and gets a score of 20.

If you don't find the parameterization of r, but somehow stumble into the $r = 25$ solution and scale appropriately, you can multiply $\frac{1}{25}$ into N and get 163000 giving about 10 points.