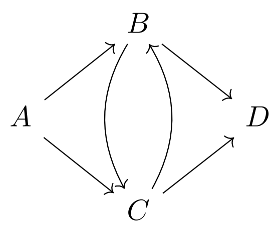
1. Aditya is walking in a park. There are four statues in the park and one-way roads between some pairs of statues, as shown by the letters and arrows in the diagram below. Aditya may visit the same statue more than once, but he refuses to travel along the same road more than once. In how many ways can he travel from statue A to statue D?



Answer: 6

Solution: Note that we initially have to go from A to either B or C. If we first travel from A to B, then we can travel to C and then back to B, but we cannot travel to C a second time because we've already taken the road from B to C. Therefore, this gives us three paths: $\{A, B, D\}, \{A, B, C, D\}$, and $\{A, B, C, B, D\}$. The same logic holds if we first go to C: we can go to B and back to C but cannot go to B a second time. So, we have $3 + 3 = 6$ total paths: $\{A, B, D\}, \{A, B, C, D\}, \{A, B, C, B, D\}, \{A, C, D\}, \{A, C, B, D\}, \{A, C, B, C, D\}$.

2. Let N_1 be the answer to Problem 1.

What is the sum of all positive integers x such that

$$x + N_1 = x^{N_1/x}?$$

Answer: 5

Solution: From question 1, we have that $N_1 = 6$. First, let's rearrange the equation as $6 = x^{6/x} - x$. For any x greater than 5, the right-hand side is not a positive number. Thus, $x \leq 5$. We can try all of the values: 1 fails because $1^6 - 1 = 1 - 1 = 0 \neq 6$. Additionally, $4^{3/2} - 4 = 8 - 4 = 4 \neq 6$ and $5^{6/5} \neq 11$ so $5^{6/5} - 5 \neq 6$. On ther other hand, when we try 2 and 3, they both work, which means the answer is $2 + 3 = \boxed{5}$.

3. Suppose a, b, c, and m are numbers satisfying the system of equations:

$$\frac{a+b+c}{3} = m,$$
$$a - m = -20,$$
$$b - m = -24.$$

What is the value of $c - m$?

Answer: 44

Solution: First, if we multiply the first equation by 3, we get that $a+b+c = 3m$. We can rewrite this as $(a-m)+(b-m)+(c-m) = a+b+c-3m = 0$, so we must have $-20-24+(c-m) = 0$. Thus, $(c - m) = 0 - (-20) - (-24) = \boxed{44}$.

4. A bag contains 9 cinnamon candies and 1 cherry candy. Nikki takes five candies from the bag uniformly at random without replacement. What is the probability that the second candy she takes is a cherry candy?

Answer: $\mathbf{\frac{1}{10}}$

Solution 1: Given no information about the other candies that were drawn, the probability that the second candy is cherry is the probability that any randomly drawn candy is a cherry candy, which is $\frac{1}{1+9} = \boxed{\frac{1}{10}}$.

Solution 2: We could also approach this by first computing the probability that the first candy is cinnamon, i.e. $\frac{9}{10}$, and multiplying it by the probability that the second candy is cherry given that the first one is cinnamon, i.e. $\frac{1}{9}$ since one cinnamon candy has been removed from the bag. Therefore, the answer is: $\frac{9}{10} \cdot \frac{1}{9} = \boxed{\frac{1}{10}}$.

5. Let N_3 be the **units digit** of the answer to Problem 3 and N_4 be the answer to Problem 4.

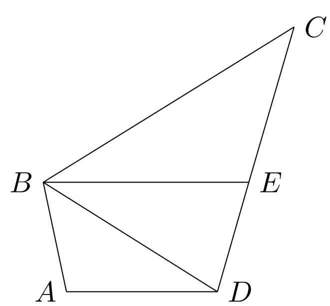
Luke the frog and Barti the dolphin are racing across the length of a pond, starting at the same time. Luke hops at the speed of N_3 meters per second, and Barti swims at the speed of $\frac{1}{N_4}$ meters per second. Given Luke reaches the other side of the pond 30 seconds after Barti, what is the length of the pond, in meters?

Answer: 200

Solution: First, from the previous questions, $N_3 = 4$ and $N_4 = \frac{1}{10}$, so Luke hops 4 meters per second and Barti swims 10 meters per second.

If x is the amount of time in seconds Barti spends getting to the other side, Luke takes $x + 30$ seconds to get to the other side. They both travel the distance of the whole pond, so using the relationship $d = rt$, we have that the length of the pond is equal to both $4(x + 30)$ and $10x$. Solving the equation $4(x + 30) = 10x$ gives us $x = 20$, so the length is $10 \cdot 20 = \boxed{200}$ meters.

6. In quadrilateral $ABCD$, point E lies on $\overline{CD}$ such that angle $\angle EBC = 32°$, angle $\angle BCE = 40°$, $BD = BE$, and $\overline{AD}$ and $\overline{BE}$ are parallel. Compute angle $\angle ADB$ in degrees. Note that the diagram below is not drawn to scale.



Answer: 36

Solution: First, the sum of the angles of triangle $\triangle BCE$ is $180°$, so $\angle CEB = 180° - 32° - 40° = 108°$. Next, E lies on line $\overline{CD}$ so $\angle BED = 180° - \angle CEB = 72°$. Since triangle $\triangle BED$ is isosceles with $BD = BE$, we have that $\angle BED = \angle\angle BDE$ so $\angle DBE = 180 - 2 \cdot \angle BED = 36°$. Finally, since $\overline{BE}$ and $\overline{AD}$ are parallel, $\angle ADB = \angle DBE = \boxed{36}$ degrees.

7. Let N_6 be the answer to Problem 6.

Define a function $f(x) = \frac{x^2}{1+x^2}$. Let

$$A = f(1) + f(2) + f(3) + \cdots + f(N_6)$$

and

$$B = f(1) + f\left(\frac{1}{2}\right) + f\left(\frac{1}{3}\right) + \cdots + f\left(\frac{1}{N_6}\right).$$

Find $A + B$.

Answer: 36

Solution: First, from the prior problem, $N_6 = 36$. Consider pairing the terms $f(a) + f\left(\frac{1}{a}\right)$ for each integer a. This is equal to $\frac{a^2}{1+a^2} + \frac{\frac{1}{a^2}}{1+\frac{1}{a^2}} = \frac{a^2}{1+a^2} + \frac{1}{a^2+1} = 1$. Since we discovered $f(a)+f\left(\frac{1}{a}\right) = 1$, we know that the sum of $f(a)+f\left(\frac{1}{a}\right)$ for a ranging from 1 to 36 is $A+B = \boxed{36}$.

8. Let N_7 be the answer to Problem 7.

   Theo draws rectangle $ABCD$ in the plane. He makes a copy of this rectangle in the same plane, rescales the copy by some factor, and shifts the copy in the plane to produce a similar rectangle, $A_1B_1C_1D_1$, such that $\overline{AB}$ is parallel to $\overline{A_1B_1}$. Given $AA_1 = \sqrt{65}$, $BB_1 = \sqrt{20}$, and $CC_1 = \sqrt{N_7}$, compute DD_1.

   Answer: 9

   Solution: From the prior problem, $N_7 = 36$. Without loss of generality, let vertex A be in the top left corner and B, C, and D be the remaining vertices labeled counterclockwise. Note that the vertical distance from A to A_1 is the same as the vertical distance from D to D_1; we'll call this v_1. The same is true of the vertical distances from B to B_1 and from C to C_1; call this distance v_2. A similar relationship exists between horizontal distances: the horizontal component of AA_1 is equal to that of BB_1, say h_1, and define h_2 similarly for CC_1 and DD_1.

   Let's express the distances AA_1, BB_1, CC_1, and DD_1 in terms of v_1, v_2, h_1, and h_2 using the Pythagorean Theorem:

$$AA_1^2 = v_1^2 + h_1^2,$$
$$BB_1^2 = v_2^2 + h_1^2,$$
$$CC_1^2 = v_2^2 + h_2^2,$$
$$DD_1^2 = v_1^2 + h_2^2.$$

   Observe that $65 + 36 = AA_1^2 + CC_1^2 = v_1^2 + v_2^2 + h_1^2 + h_2^2 = BB_1^2 + DD_1^2 = 20 + DD_1^2$, and so $DD_1 = \boxed{9}$.

9. Nine cupcakes are arranged in a line in increasing size from left to right. In the line, the cupcake flavors alternate between chocolate and vanilla with the leftmost cupcake being chocolate. Andrew eats one chocolate cupcake, one vanilla cupcake, and then another chocolate cupcake, so that each cupcake eaten is larger than all cupcakes eaten before it. In how many ways can Andrew choose his three cupcakes?

   Answer: 20

   Solution: If Andrew chooses the first vanilla cupcake, then one of the chocolate cupcakes must lie to the left (1 way to choose) and the other must lie to the right (4 ways to choose). Similarly, when he chooses the last vanilla cupcake, there are $4 \cdot 1$ ways to choose two chocolate cupcakes according to the rules of the problem. Finally, if Andrew chooses one of the vanilla cupcakes in the middle, he could choose one of two chocolate cupcakes on one side and one of three on the other side, giving $2 \cdot 3$ additional solutions. Thus, the answer is $1 \cdot 4 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1 = 4 + 6 + 6 + 4 = \boxed{20}$.

10. Let N_9 be the answer to Problem 9.

Clara plays a game with a list of the integers from 1 to N_9, inclusive. First, Clara randomizes the order of the list. Then, in each round of the game, she removes the first and last elements from the list and adds the bigger of these numbers to her score. If she began the game with a score of 0, what is the smallest possible score that Clara could have achieved at the end of the game?

Answer: 110

Solution: First, from the prior question, we have $N_9 = 20$.

To minimize Clara's score, we should consider the situation where she chooses the lowest numbers possible on each turn, thus adding the minimum possible numbers to her score. Clara can never add 1 to her score, as it will be smaller than any other option, so we can let Clara remove 1 and 2 on the first turn because this is the only way she can add 2 to her score. From here, the lowest number Clara could add to her score is 4 (1 and 2 are not available, and 3 is still in play). This process continues, and Clara adds $2n$ to her score on the n^{th} turn. Thus, the lowest possible score Clara could have at the end of the game is $2 + 4 + 6 + \cdots + 20 = \boxed{110}$.

11. Let N_9 be the answer to Problem 9.

A sequence of real numbers $a_0, a_1, a_2, \ldots$ satisfies $a_0 = N_9$ and $a_{n+1} = a_n - a_{n-1}$ for $n \geq 1$. Given that

$$a_0 + a_1 + a_2 + \cdots + a_{2024} = 2024,$$

compute a_2.

Answer: 992

Solution: From the prior problems, we have that $N_9 = 20$.

An important observation is that the sequence $a_0, a_1, a_2, \ldots$ has a period of 6. Indeed, solving the for first few terms in the recurrence in terms of a_0 and a_1, we get:

n	0	1	2	3	4	5	6	7
a_n	a_0	a_1	$a_1 - a_0$	$-a_0$	$-a_1$	$a_0 - a_1$	a_0	a_1

Because we have $a_6 = a_0$ and $a_7 = a_1$, and given the fact that the recurrence relation begins with the two terms a_0 and a_1, we may conclude that $a_8 = a_2$ and $a_9 = a_3$ and so on.

We now begin the computation. Writing $2024 = 6 \cdot 337 + 2$, we see that

$$2024 = a_0 + a_1 + \cdots + a_{2024} = 337(a_0 + a_1 + a_2 + a_3 + a_4 + a_5) + a_0 + a_1 + a_2.$$

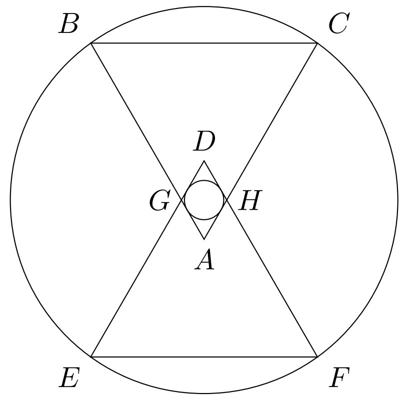
As such, we now compute

$$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = a_0 + a_1 + (a_1 - a_0) - a_0 - a_1 + (a_0 - a_1) + a_0 + a_1 = 0,$$

so we conclude that $2024 = 337 \cdot 0 + a_0 + a_1 + a_2 = a_0 + a_1 + a_2 = a_0 + a_1 + (a_1 - a_0) = 2 \cdot a_1$. Finally, we have $a_1 = 1012$, so $a_2 = a_1 - a_0 = 1012 - N_9 = 1012 - 20 = \boxed{992}$.

12. Let N_{11} be the **smallest prime that does not divide** the answer to Problem 11.

In the diagram below, let G be the intersection of $\overline{AB}$ and $\overline{DE}$, and let H be the intersection of $\overline{AC}$ and $\overline{DF}$. Suppose that equilateral triangle $\triangle ABC$ is the reflection of triangle $\triangle DEF$ about the line $\overleftrightarrow{GH}$ and that $AB = N_{11} \cdot AG$. Let circle O_1 be tangent to all four sides of quadrilateral $AGDH$, and let points B, C, F, and E all lie on circle O_2. Find the ratio of the area of O_1 to the area of O_2. Note that the diagram below is not drawn to scale.



Answer: $\mathbf{\frac{1}{28}}$

Solution: The answer to problem 11 is 992, which is divisible by 2 but not 3, so $N_{11} = 3$. Let $AG = x$, meaning that $AB = 3x$. Since $\triangle ABC$ and $\triangle DEF$ are equilateral triangles with $\overline{BC}$ and $\overline{EF}$ parallel to $\overline{GH}$, triangle $\triangle AGH$ and triangle $\triangle DGH$ are also equilateral triangles. Now, let O be the center of O_1 and O_2. In order to find the radius of O_1, we drop a perpendicular from O to $\overline{DG}$, call this point P. We have that triangles $\triangle OGP$ and $\triangle DOP$ are 30-60-90 triangles, which implies that the radius of O_1 is $OP = \frac{\sqrt{3}}{4}x$.

Let the intersection of $\overleftrightarrow{GH}$ and $\overline{BE}$ be J. Since $AB - AG = DE - DG$, $BG = EG$. So triangle $\triangle BEG$ is isosceles and $\angle BJG = 90°$, $\angle BGJ = 60°$. Since $BG = (3-1)x$ and $\triangle BGJ$ is a $30-60-90$ triangle with $BJ = \frac{2\sqrt{3}x}{2} = \sqrt{3}x$ and $GJ = x$. We know the radius of O_2 is BO, by Pythagorean Theorem, $BO^2 = BJ^2 + OJ^2 = BJ^2 + \left(\frac{BC}{2}\right)^2 = \left(\frac{2\sqrt{3}x}{2}\right)^2 + \left(\frac{3x}{2}\right)^2$. Since the ratio of the area of the circles is equal to the ratio of the square of the radii, the answer would be $\frac{\left(\frac{\sqrt{3}}{4}x\right)^2}{\left(\frac{2\sqrt{3}x}{2}\right)^2+\left(\frac{3x}{2}\right)^2} = \boxed{\frac{1}{28}}$.

13. Let N_{14} be the answer to Problem 14.

The lines $y = \frac{1}{20}x$ and $y = \frac{1}{20-N_{14}}x$ intersect the line $y = \frac{\sqrt{2}}{2}$ at distinct points A and B, respectively. Let point O be the origin of the coordinate plane. Find the area of triangle $\triangle ABO$, given that it is a positive integer.

Answer: 4

Solution: Let point $A = (x_1, \frac{\sqrt{2}}{2}), B = (x_2, \frac{\sqrt{2}}{2})$. Solving the following equations:

$$\frac{1}{20}x_1 = \frac{\sqrt{2}}{2},$$
$$\frac{1}{20 - N_{14}}x_2 = \frac{\sqrt{2}}{2}.$$

After multiplying out the coefficients of x_1 and x_2 and subtracting, we have $x_1 - x_2 = \frac{\sqrt{2}}{2}N_{14}$ and $x_1 > x_2$. Thus, the area of $\triangle ABO$ is $\frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot N_{14} \cdot \frac{\sqrt{2}}{2} = \frac{1}{4}N_{14}$.

Setting up the sets of equations by what we find in Problem 14 and the formula of the area we found in Problem 13, we have

$$N_{14} = N_{13}^2,$$
$$N_{13} = \frac{1}{4}N_{14}.$$

Thus, $N_{13} = \frac{1}{4}N_{13}^2$. Since the area is not equal to 0, $N_{13} = \boxed{4}$.

14. Let N_{13} be the answer to Problem 13.

Find the number of ways to choose two positive integers a and b such that $a \le b < a+b \le 2N_{13}$.

Answer: 16

Solution: If both of a and b are less than or equal to N_{13}, there are 2 cases. In the case that the a and b are different, there are $\binom{N_{13}}{2}$ ways to pick the points. If a and b are the same, there are N_{13} ways to pick the points.

We cannot have $a > N_{13}$ because if so, $a + b > 2N_{13}$. Therefore, the other case is when $b > N_{13}$ and $a \le N_{13}$. Specifically, if b has the label $2N_{13} - k$ for some non-negative integer k, then a must have a label that is k or less. Now, k can range from 1 to $N_{13} - 1$, meaning that the number of choices for this case is $1 + 2 + \cdots + (N_{13} - 1) = \frac{(N_{13}-1)(N_{13})}{2}$.

Therefore, the number of ways to choose 2 points is $\binom{N_{13}}{2} + N_{13} + \frac{(N_{13}-1)(N_{13})}{2} = N_{13}^2$.

Using $N_{13} = 4$ found from Problem 13, $N_{14} = N_{13}^2 = 4^2 = \boxed{16}$.

15. Justin is playing a game. He initially writes the following list of 7 numbers:

$$1, 1, 1, 2, 3, 5, 8.$$

Then, on every turn of the game, he chooses either the median (the 4^{th} greatest element) or arithmetic mean (the sum of all elements divided by 7) of the list. He adds that number to every element of the list, and this becomes his new list. If the arithmetic mean of Justin's list is greater than 500, the game ends. What is the minimum number of turns it could take for the game to end?

Answer: 8

Solution: The sum of the list is $1 + 1 + 1 + 2 + 3 + 5 + 8 = 21$. This means that the mean is $\frac{21}{7} = 3$ while the median is 2.

If any number x is added to every element of the list, the median will always be 1 less than the mean of this list as both values will increase by x. Notably, the median will always be smaller. Therefore, if we want to increase the arithmetic mean of the list as quickly as possible, we want to always choose the list's current arithmetic mean to add to each element. Next, we note that adding the mean of the list to each element will double the mean of the list. If the mean of the list is m, the sum of the elements of the list is $7m$. Adding m to each of the 7 elements will increase the sum by $7m$ to $14m$, meaning that the arithmetic mean is now $2m$, which is twice m.

This means, if we use the most optimal strategy of adding the mean to the list every turn, after n turns, the mean will be $3 \cdot 2^n$. After 7 turns, the mean becomes $3 \cdot 2^7 = 3 \cdot 128 = 384 < 500$.

After one more turn, we have $3 \cdot 2^8 = 3 \cdot 256 = 768$, which is larger than the target of 500. Therefore, the answer is $\boxed{8}$.

16. For a non-negative integer a and a positive integer $b \geq 2$, let $a \star b$ denote the remainder when $a+1$ is divided by b. For example, $2 \star 3 = 0$ and $1 \star 10 = 2$. Compute

$$(((((1 \star 2024) \star 2023) \star 2022) \star \cdots) \star 61).$$

Answer: 3

Solution: Let the value of the above expression equal S. First, we note that when $a < b+1$, we have that $a \star b = a+1$. So, for small values of i, we have that $((((1 \star 2024) \star 2023) \star \cdots) \star (2025-i)) = i+1$. For example, $(1 \star 2024) = 2$ and $((1 \star 2024) \star 2023) = 3$.

Now, let's determine the first time that calling the $\star$ operator doesn't simply increase the first value by 1. For any i before the first such "overflow" instance, the ith time that we call $\star$, we consider $i \star (2025 - i)$. So, we want the smallest i such that $i \geq (2025 - i) - 1$, namely $i = \frac{2024}{2} = 1012$. We have that

$$S = (((((1012 \star 1013) \star 1012) \star 1011) \star \cdots) \star 61) = ((((0 \star 1012) \star 1011) \star \cdots) \star 61).$$

Now, we're in the same position: $(0 \star 1012) = 1, ((0 \star 1012) \star 1011) = 2$, etc. To find the next instance where $\star$ doesn't increment by 1, we solve $j - 1 \geq (1013 - j) - 1$, or namely $j = 506$. With this logic repeatedly, we get that

$$\begin{aligned}
S &= (((((506 \star 506) \star 505) \star 504) \star \cdots) \star 61) \\
&= ((((1 \star 505) \star 504) \star \cdots) \star 61) \\
&= (((((253 \star 253) \star 252) \star 251) \star \cdots) \star 61) \\
&= ((((1 \star 252) \star 251) \star \cdots) \star 61) \\
&= (((((126 \star 127) \star 126) \star 125) \star \cdots) \star 61) \\
&= ((((0 \star 126) \star 125) \star \cdots) \star 61) \\
&= (((63 \star 63) \star 62) \star 61) \\
&= ((1 \star 62) \star 61) \\
&= (2 \star 61) \\
&= \boxed{3}.
\end{aligned}$$

17. Zain is riding a bike in the coordinate plane. They start at $(0,0)$ and bike to all the points with integer coordinates in a spiral pattern: they first bike to $(0,1)$, then to $(-1,1)$, $(-1,0)$, $(-1,-1)$, $(0,-1)$, and so on, turning left if they haven't visited the point to their left and going straight otherwise. Every time Zain reaches a point (x,y) whose coordinates satisfy $|x+y| = 2024$, they briefly celebrate on that point. Suppose the 4144^{th} point Zain celebrates on is (a,b). Compute b.

Answer: 24

Solution: Consider the simpler case where Carlson celebrates on (x,y) if $|x+y| = 2$. If we draw the coordinate plane with the lines $x+y=2$ and $x+y=-2$ and trace Carlson's path, it is clear that the first time Carlson celebrates is on $(-1,-1)$, and every time he completes a full "lap" around the origin after reaching $(-1,-1)$, he celebrates four more times. In fact, a "lap" consists of Carlson walking over all points where $\max(|x|,|y|) = k$ for some positive integer k. The fourth time Carlson celebrates, he is on $(-2,0)$, the eighth time is on $(-3,1)$, and so on. We can therefore conclude that if the first time Carlson celebrates is at $(-1,-1)$, then the $4k^{\text{th}}$ celebration occurs at $(-1-k,-1+k)$.

If we shift back to thinking about $|x+y| = 2024$, we'll see a very similar situation except that Carlson's first celebration is at the point $(-1012,-1012)$, so his $4k^{\text{th}}$ celebration is at $(-1012-k,-1012+k)$. Since $4144 = 4\times 1036$, we are interested in the $4k^{\text{th}}$ celebration when $k = 1036$. This is the intersection point in the lap with the smallest x-coordinate, i.e. when $a = -1012 - 1036 = -2048$. Thus, the point is $(-2048, 24)$ and $b = \boxed{24}$.

18. Let N_{15} be the answer to Problem 15, N_{16} be the answer to Problem 16, and N_{17} be the answer to Problem 17.

    Triangle $\triangle XYZ$ has a right angle at Y. Points A and D lie on $\overline{XY}$ and $\overline{YZ}$, respectively, such that $\overline{AD}$ is parallel to $\overline{XZ}$. Let $\overline{AD}$ intersect the inscribed circle of $\triangle XYZ$ at points B and C, with A closer to B than C. Suppose $AB = N_{16}$, $BC = N_{17}$, and $CD = N_{15}$. Compute the smallest possible value of XZ.

    Answer: 60

    Solution: First, we have $N_{15} = 8, N_{16} = 3, N_{17} = 24$.

    Let E be the point on $\overline{XY}$ that is tangent to the incircle of $\triangle XYZ$, and similarly define F on $\overline{YZ}$. By power of a point on A with respect to the incircle,

    $$\begin{aligned} AB\cdot AC &= AE^2 \\ 3(27) &= AE^2 \\ 9 &= AE, \end{aligned}$$

    and similarly, $DF = 16$. Furthermore, because they are both tangents, $EY = FY$. We can use the Pythagorean theorem on right triangle $\triangle AYD$ to get

    $$\begin{aligned} AY^2 + YD^2 &= AD^2 \\ (9+EY)^2 + (16+FY)^2 &= 35^2 \\ (9+EY)^2 + (16+EY)^2 &= 35^2 \\ EY &= 12. \end{aligned}$$

    Let the center of the incircle of $\triangle XYZ$ be O. Quadrilateral $OEYF$ is a square, so the radius of the incircle is 12. Also, $\overline{AD} \parallel \overline{XZ}$ so $\triangle AYD \sim \triangle XYZ$. So, if we can determine the inradius of $\triangle AYD$, then we can determine the ratio of similarity between $\triangle AYD$ and $\triangle XYZ$ and thus find XZ using $AD = 35$. The inradius is given by $\frac{2\times\text{area}}{\text{perimeter}}$. The area of $\triangle AYD$ is $\frac{1}{2}\cdot AY\cdot YD = \frac{21\cdot 28}{2}$ and the perimeter is $21+28+35 = 84$, so the inradius is $\frac{21\cdot 28}{84} = 7$. Thus, the ratio $\triangle AYD : \triangle XYZ = 7 : 12$, $\frac{12}{7} = \frac{XZ}{AD} = \frac{XZ}{35}$, and $XZ = \boxed{60}$.

19. Let N_{19} be the answer to this problem and N_{20} be the answer to Problem 20.

    Parallel lines ℓ_1, ℓ_2, and ℓ_3 are arranged with ℓ_2 between ℓ_1 and ℓ_3 such that adjacent lines are distance N_{20} apart. Point A lies on ℓ_1, point B lies on ℓ_2, and point C lies on ℓ_3 such that $AB = N_{19}$ and $BC = N_{20}\sqrt{10}$. Compute AC.

    Answer: $\mathbf{1+\sqrt{5}}$

Solution: First, we show that triangle $\triangle ABC$ is a 45-45-90 right triangle by analyzing $\angle BAC$. Suppose that $\angle BAC$ were acute. We drop perpendiculars from A and B to the lines: let the intersection of the perpendicular line from B to ℓ_3 be D, B to ℓ_1 be E, and A to ℓ_3 be F. By definition, we have that $AB = AC = N_{19}$. For the ease of notation, let $BD = BE = x$ and $CF = y$. Note that $BD = N_{20}$ because lines ℓ_2 and ℓ_3 are N_{20} apart. This means that triangle $\triangle BDC$ is a right triangle with $BD = N_{20}$ and $BC = \sqrt{10}N_{20}$, and so $DC = 3N_{20}$, meaning that we have $AE = 3x + y$, $AF = 2x$. By Pythagorean Theorem:

$$AB^2 = x^2 + (3x + y)^2 = AC^2 = y^2 + (2x)^2.$$

Simplifying the equation we have:

$$6x^2 + 6xy = 0$$

However, since $x, y > 0$, we must have $6x^2 + 6xy \neq 0$ which contradicts the above equation we got. Thus, $\angle A$ cannot be acute. Thus, we know that $\angle BAC \geq 90°$.

Now, we show that $\angle BAC = 90°$ exactly. We again drop perpendicular lines: let the intersection of the perpendicular line from B to ℓ_3 be D, B to ℓ_1 be E, and C to ℓ_1 be G. Additionally, let $BD = BE = x$, $AE = y$. Using similar logic to before, we have that $AG = 3x - y$, $CG = 2x$. So, by the Pythagorean Theorem:

$$AB^2 = x^2 + y^2 = AC^2 = (3x - y)^2 + (2x)^2.$$

Simplifying the equation we have:

$$12x^2 - 6xy = 0.$$

Since $x \neq 0$, we have $y = 2x$. Thus, $AE = CG = 2x$, $BE = AG = x$, and $\angle BEA = \angle AGC = 90°$. Thus, $\triangle BEA \cong \triangle AGC$, $\angle BAC = 180° - \angle EAB - \angle GAC = 180° - \angle EAB - \angle EBA = 90°$. As a result, $BD = N_{20}$, $BC = \sqrt{10} \cdot N_{20}$, and $AC = \sqrt{5} \cdot N_{20}$.

Plugging in $N_{20} = \frac{1+\sqrt{5}}{\sqrt{5}}$, we get $AC = \sqrt{5} \cdot \frac{1+\sqrt{5}}{\sqrt{5}} = \boxed{1 + \sqrt{5}}$.

20. Let N_{19} be the answer to Problem 19 and N_{20} be the answer to this problem.

Suppose that exactly 2 of the following 3 expressions are equal:

$$N_{20} + N_{19},\ 2N_{20} - N_{19}, \text{ and } N_{19}N_{20}.$$

Compute N_{20}.

Answer: $\boldsymbol{\frac{\sqrt{5}}{5} + 1}$

Solution: From Problem 19, we learned that $N_{19} = \sqrt{5} \cdot N_{20}$. Since it is the length of a side of a triangle, we have $N_{19}, N_{20} > 0$. Thus, we can get bounds for our three expressions, namely:

$$\begin{aligned} N_{20} + N_{19} &> 0, \\ 2N_{20} - N_{19} &< 0, \\ N_{19}N_{20} &> 0. \end{aligned}$$

Thus, it is only possible to have $N_{20} + N_{19} = N_{19} \cdot N_{20}$.

Plugging in $N_{19} = \sqrt{5} \cdot N_{20}$, we have $(1 + \sqrt{5}) \cdot N_{20} = \sqrt{5} \cdot N_{20}^2$. Thus, $N_{20} = \frac{1+\sqrt{5}}{\sqrt{5}} = \boxed{\frac{\sqrt{5}}{5} + 1}$.