1. Find $f'\left(\frac{\pi}{4}\right)$, where $f(x) = 20\cos(x) + 23\sin(x)$.

   Answer: $\frac{3\sqrt{2}}{2}$

   Solution: $f'(x) = -20\sin x + 23\cos x$. As $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, this becomes $3\cos\left(\frac{\pi}{4}\right) = \boxed{\frac{3\sqrt{2}}{2}}$.

2. Compute
$$\int_0^4 (x-2)^5 + (x-2)^6 + (x-2)^7 \, \mathrm{d}x\,.$$

   Answer: $\frac{256}{7}$

   Solution: We can rewrite the integral by substituting $u = x - 2$, so $\mathrm{d}u = \mathrm{d}x$ and changing limits of integration yields $\int_{-2}^{2} u^5 + u^6 + u^7 \, \mathrm{d}u$. Since u^5 and u^7 are odd functions, the integral from -2 to 2 of both of them are equal to 0, so we are left with
$$\int_{-2}^{2} u^6 \, \mathrm{d}u = \frac{u^7}{7}\bigg|_{-2}^{2} = \frac{128}{7} - \frac{-128}{7} = \boxed{\frac{256}{7}}.$$

3. Let A be the area of the region bounded by $x = 0$, $y = 0$, $x = 6$, and $y = \sqrt{kx}$, for some real number $k > 0$. If $A = 36$, compute the value of k.

   Answer: $\frac{27}{2}$

   Solution: We see that the area under the curve $y = \sqrt{kx}$ is equal to
$$\int_0^6 \sqrt{kx} = \left(\frac{2}{3}x\sqrt{kx}\right)_0^6 = 4\sqrt{6k} = 36.$$

   Solving this for k gives us that $k = \frac{81}{6}$, so $k = \boxed{\frac{27}{2}}$.

4. An ice cube melts such that it always remains a cube, and its volume decreases at a constant rate. The initial side length of the cube is 10 inches, and it takes 50 minutes for the ice cube to completely melt. When the side length of the ice cube is 4 inches, what is the rate, in inches per minute, at which the side length of the ice cube is decreasing?

   Answer: $\frac{5}{12}$

   Solution: Since the rate at which the volume decreases is constant, it is $\frac{10^3}{50} = 20$.

   Let V be the volume of the cube, and let s be the side length of the cube. Since $V = s^3$, $20 = \frac{\mathrm{d}V}{\mathrm{d}t} = 3s^2\frac{\mathrm{d}s}{\mathrm{d}t}$. Thus, $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{20}{3s^2}$, and when $s = 4$, this gives $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{20}{3\cdot 4^2} = \boxed{\frac{5}{12}}$.

5. Let $f(a,b) = b^3 - a^3 + a^2b - ab^2$. There exists a real number C such that regardless of the choice of nonnegative real numbers $0 = x_0 < x_1 < x_2 < x_3 < \cdots < x_n = 4$, we have $C \le \sum_{i=1}^{n} f(x_{i-1}, x_i)$. Compute the maximum value of C.

   Answer: $\frac{128}{3}$

   Solution: We have $f(a,b) = (a^2+b^2)(b-a) = 2\cdot\frac{(a^2+b^2)(b-a)}{2}$.

   Consider the sum $\frac{1}{2}\sum_{i=1}^{n} f(x_{i-1}, x_i)$. We claim we are taking a trapezoidal integral of $y = x^2$ from $x = 0$ to $x = 4$. Indeed, we first partition $[0,4]$ into $0 < x_1 < \cdots < x_n$. Then, for each

i, we consider the trapezoid of coordinates $(x_i, 0), (x_{i+1}, 0), (x_{i+1}, x_{i+1}^2), (x_i, x_i^2)$. We verify we have trapezoidal sums, which overestimate the actual area, since the function x^2 is convex (or concave up).

Thus, the lower bound for the trapezoidal sums is the actual area which is $\int_0^4 x^2 dx = \frac{x^3}{3}\Big|_0^4 = \frac{64}{3}$.

We double the answer to get $\boxed{\frac{128}{3}}$.

6. For a positive number x, let $f_0(x) = \frac{1}{x}$ and $f_n(x) = \frac{d^n}{dx^n}\left(\frac{1}{x}\right)$ for all positive integers n. If

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{f_n(x)},$$

compute $g(1)$.

Answer: $\boldsymbol{\frac{1}{e}}$

Solution: Note that $f_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}$. To see this inductively, clearly this is true if $n = 0$, and if it's true for n, then $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(-1)^n n!}{x^{n+1}}\right) = \frac{(-1)^n n! \cdot (-1) \cdot (n+1)}{x^{n+2}} = \frac{(-1)^{n+1}(n+1)!}{x^{n+2}}$, showing it holds for $n + 1$. Then,

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(-1)^n n!} = x \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = xe^{-x}$$

Hence, $g(1) = \boxed{\frac{1}{e}}$.

7. Define a sequence $a_0, a_1, a_2, \ldots$ by $a_0 = 24$, $a_1 = 23$, and $a_{n+2} = -a_{n+1} + 6a_n$ for $n \geq 0$. Compute

$$\sum_{n=1}^{\infty} \frac{a_n}{n6^n}.$$

Answer: $\boldsymbol{14 \ln\left(\frac{3}{2}\right)}$

Solution: The recurrence relation solves to be $a_n = x2^n + y(-3)^n$. We split into two sums:

$$\sum_{n=1}^{\infty} \frac{x}{n3^n} + \sum_{n=1}^{\infty} \frac{y}{n(-2)^n}.$$

Using the Taylor expansion for $\ln(1 - x)$ we get the two sums equal to $-x \ln \frac{2}{3} - y \ln \frac{3}{2}$. This is equal to $(x - y) \ln \frac{3}{2}$. We see that $a_0 = x + y = 24$, and $a_1 = 2x - 3y = 23$, so $5x = 3(x + y) + (2x - 3y) = 95$, so $x = 19$. Hence, $y = 5$, and we have $S = \boxed{14 \ln \frac{3}{2}}$.

8. Let $f : [1, \infty) \to \mathbb{R}$ be a continuous function such that

$$I(f) = \int_1^{\infty} \left(\sqrt{2023} x e^{-x} f(x) - \frac{1}{4} x^2 f(x)^2\right) \mathrm{d}x$$

converges and is maximized over all continuous functions on $[1, \infty) \to \mathbb{R}$. Compute $f(1) + I(f)$.

Answer: $\frac{2\sqrt{2023}}{e} + \frac{2023}{2e^2}$

Solution: Consider the function $-(axf(x) - be^{-x})^2$ for some constants a, b. Expanding that we get the following:

$$2abxe^{-x}f(x) - a^2x^2f(x)^2 - b^2e^{-2x}$$

Then we get the following system of equations:

$$\begin{cases} 2ab = \sqrt{2023} \\ a^2 = \frac{1}{4} \end{cases} \tag{1}$$

which evaluates to $(a, b) = (\frac{1}{2}, \sqrt{2023}), (-\frac{1}{2}, -\sqrt{2023})$. It is apparent that either ordered-pair would yield the same result. Substituting $(a, b) = (\frac{1}{2}, \sqrt{2023})$ yields:

$$I(f) = \int_1^\infty \left(2023e^{-2x} - \left(\frac{1}{2}xf(x) - \sqrt{2023}e^{-x}\right)^2\right) \mathrm{d}x$$

.

Because $-\left(\frac{1}{2}xf(x) - \sqrt{2023}e^{-x}\right)^2 \leq 0$ for all $x \geq 1$, $I(f)$ would be maximized when $f(x) = \frac{2\sqrt{2023}}{xe^x}$. Setting f equal to that we get:

$$I(f) = \int_1^\infty 2023e^{-2x}dx = \frac{2023}{2e^2}$$

.

Therefore, $f(1) + I(f) = \boxed{\frac{2\sqrt{2023}}{e} + \frac{2023}{2e^2}}$.

9. Compute

$$\int_0^{2\pi} (\sin(x) + \cos(x))^6 \,\mathrm{d}x\,.$$

Answer: $\boldsymbol{5\pi}$

Solution: We will use periodic cancelling technique. Since the bounds are from 0 to 2π, any leftover $\sin(nx)$ or $\cos(nx)$ will be 0 for any natural number n. After expanding, the only terms that survive are

$$= \int_0^{2\pi} \left[\sin^6(x) + 15\sin^4(x)\cos^2(x) + 15\sin^2(x)\cos^4(x) + \cos^6(x)\right] dx$$

This is because squared sines and squared cosines leave off constants due to the identity $\sin^2(x) = \frac{1-\cos(2x)}{2}$ and $\cos^2(x) = \frac{1+\cos(2x)}{2}$. Now, we abbreviate $\sin(x)$ with s and $cos(x)$ with c. Notice that $s^6 + c^6 = (s^2 + c^2)(s^4 - s^2c^s + c^4) = (s^4 - s^2c^2 + c^4)$. Also, $s^4c^2 + s^2c^4 = s^2c^2(s^2 + c^2) = s^2c^2$. So now we have

$$\int_0^{2\pi} \left[\sin^4(x) + \cos^4(x) + 14\sin^2(x)\cos^2(x)\right] dx$$

With more trig manipulation, we have $s^4 + c^4 + 2s^2c^2 = (s^2 + c^2) = 1 \implies s^4 + c^4 = 1 - 2s^2c^2$. This simplifies our integral to

$$\int_0^{2\pi} \left[1 + 12\sin^2(x)\cos^2(x)\right] dx = \int_0^{2\pi} \left[1 + 3\sin^2(2x)\right] \mathrm{d}x$$
$$= \int_0^{2\pi} \left[1 + \frac{3}{2}(1 - \cos(4x))\right] \mathrm{d}x = 2\pi\left(1 + \frac{3}{2}\right) = \boxed{5\pi}.$$

10. Compute

$$\int_0^\infty \frac{\sin(x)}{x^2} \sum_{n=1}^\infty \frac{\sin(nx)}{n!} \, \mathrm{d}x \, .$$

Answer: $\boldsymbol{\frac{\pi}{2}(e - 1)}$

Solution:

$$\int_0^\infty \frac{\sin(x)}{x^2} \sum_{n=1}^\infty \frac{\sin(nx)}{n!} \, \mathrm{d}x$$
$$= \sum_{n=1}^\infty \frac{1}{n!} \int_0^\infty \frac{\sin(x)\sin(nx)}{x^2}$$
$$= \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n!} \int_0^\infty \frac{\cos((n-1)x) - \cos((n+1)x)}{x^2} \, \mathrm{d}x$$
$$= \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n!} \int_0^\infty \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{x}\right)\right] (\cos((n-1)x) - \cos((n+1)x)) \, \mathrm{d}x$$
$$= \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n!} \int_0^\infty \frac{(n+1)\sin((n+1)x) - (n-1)\sin((n-1)x)}{x} \, \mathrm{d}x$$
$$= \frac{1}{2}\sum_{n=1}^\infty \frac{n+1}{n!} \int_0^\infty \frac{\sin((n+1)x)}{x} \, \mathrm{d}x$$
$$- \frac{1}{2}\sum_{n=1}^\infty \frac{n-1}{n!} \int_0^\infty \frac{\sin((n-1)x)}{x} \, \mathrm{d}x$$
$$= \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n!} \left[\frac{\pi}{2}(n+1) - \frac{\pi}{2}(n-1)\right]$$
$$= \frac{\pi}{2}\sum_{n=1}^\infty \frac{1}{n!}$$
$$= \boxed{\frac{\pi}{2}(\mathrm{e} - 1)},$$

as desired.