Combinatorial species

This power round focuses on *combinatorial species*, a powerful tool for counting certain types of objects, such as trees, graphs, permutations, and subsets.

1. asdf

   Solution to Problem 1: asdf

1 Generating functions

Often in combinatorics and other fields, we find ourselves working with sequences $a_0, a_1, a_2, \ldots$. *Generating functions* furnish us with a powerful tool for working with sequences and, in many cases, discovering new properties of them. There are two types of generating functions that one often encounters in combinatorics: *ordinary* generating functions and *exponential* generating functions, which we define presently.

Definition. Let $(a_n)_{n\geq 0}$ be a sequence of real numbers. The *ordinary generating function (OGF)* of (a_n) is given by

$$A^o(x) = \sum_{n=0}^{\infty} a_n x^n$$

The *exponential generating function (EGF)* of (a_n) is given by

$$A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

Remark. It is important to note that generating functions are *formal* power series. This means that we pay no mind to whether or not these sums converge. In return for this nonchalance, we pay the price of not being able to evaluate a generating function for a specific value of x without first checking certain conditions. During this power round, this won't be an issue.

Example. Perhaps the simplest sequences we can consider are the sequences $0, 0, \ldots$ and $1, 1, \ldots$ given by $z_n = 0$ for all n and $e_n = 1$ for all n. We have

$$Z^o(x) = \sum_{n=0}^{\infty} 0x^n = 0 \qquad\qquad Z(x) = \sum_{n=0}^{\infty} 0 \cdot \frac{x^n}{n!} = 0$$

For (e_n) the situation is slightly less trivial. We have

$$E^o(x) = \sum_{n=0}^{\infty} 1 \cdot x^n = 1 + x + x^2 + \cdots$$

This is a geometric series, so we can evaluate its sum to be

$$E^o(x) = \frac{1}{1-x}$$

For the EGF, we have

$$E(x) = \sum_{n=0}^{\infty} 1 \cdot \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

This is an important function called the *exponential function*, and we write it as $\exp(x)$.

2. Calculate the following generating functions. None of your final answers should be in the form of a sum.

   (a) The EGF of $f_n =$ the number of permutations of $\{1, \ldots, n\}$. *Hint:* this sequence begins $1, 1, 2, 6, 24, \ldots$.

   (b) The EGF *and* the OGF of $p_n =$ the number of subsets of $\{1, \ldots, n\}$. *Hint:* this sequence begins $1, 2, 4, 8, 16, \ldots$

   (c) The EGF of (a_n), where for a given positive integer a, we define

$$a_n = \begin{cases} \frac{1}{(a-n)!} & \text{if } 0 \leq n \leq a \\ 0 & \text{else} \end{cases}$$

Solution to Problem 2:

(a) We have

$$F(x) = \sum_{n=0}^{\infty} n! \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n$$

which is the geometric series from the example. Evaluating as before, we obtain

$$F(x) = \frac{1}{1-x}$$

(b) We have

$$P^o(x) = \sum_{n=0}^{\infty} 2^n \cdot x^n = \sum_{n=0}^{\infty} (2x)^n$$

which is another geometric series which evaluates to

$$P^o(x) = \frac{1}{1-2x}$$

Similarly, we have

$$P(x) = \sum_{n=0}^{\infty} 2^n \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \exp(2x)$$

(c) We have

$$\begin{aligned} A(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot \begin{cases} \frac{1}{(a-n)!} & \text{if } 0 \leq n \leq a \\ 0 & \text{else} \end{cases} \\ &= \sum_{n=0}^{a} \frac{1}{(a-n)!} \cdot \frac{x^n}{n!} \\ &= \sum_{n=0}^{a} \frac{1}{a!} \cdot x^n \cdot \frac{a!}{n!(a-n)!} \\ &= \frac{1}{a!} \sum_{n=0}^{a} x^n \binom{a}{n} \\ &= \frac{(1+x)^a}{a!} \end{aligned}$$

applying the binomial theorem in the last step.