Maximum score: 200 points.

Instructions: For this test, you work in teams to solve a multi-part, proof-oriented question.

Problems that use the words "compute," "list," or "draw" require only an answer; no explanation or proof is needed. Unless otherwise stated, all other questions require explanation or proof.

The problems are ordered by content, *not difficulty*. The difficulties of the problems are generally indicated by the point values assigned to them; it is to your advantage to attempt problems throughout the test.

While completing the round, you should not consult the internet or any materials outside of the content of this test as it relates to induction, the pigeonhole principle, probability, or Ramsey theory (*including results not covered in this power round*). You may not use calculators.

Please clearly label every page of your solutions with your team ID number and problem number. Remember: your submissions will be read by a human. Please write legibly, and only upload work that you wish to be graded.

Good luck!

Combinatorial species

This power round focuses on *combinatorial species*, a powerful tool for counting certain types of objects, such as trees, graphs, permutations, and subsets.

1. asdf

1 Generating functions

Often in combinatorics and other fields, we find ourselves working with sequences $a_0, a_1, a_2, \ldots$. *Generating functions* furnish us with a powerful tool for working with sequences and, in many cases, discovering new properties of them. There are two types of generating functions that one often encounters in combinatorics: *ordinary* generating functions and *exponential* generating functions, which we define presently.

Definition. Let $(a_n)_{n\geq 0}$ be a sequence of real numbers. The *ordinary generating function (OGF)* of (a_n) is given by

$$A^o(x) = \sum_{n=0}^{\infty} a_n x^n$$

The *exponential generating function (EGF)* of (a_n) is given by

$$A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

Remark. It is important to note that generating functions are *formal* power series. This means that we pay no mind to whether or not these sums converge. In return for this nonchalance, we pay the price of not being able to evaluate a generating function for a specific value of x without first checking certain conditions. During this power round, this won't be an issue.

Example. Perhaps the simplest sequences we can consider are the sequences $0, 0, \ldots$ and $1, 1, \ldots$ given by $z_n = 0$ for all n and $e_n = 1$ for all n. We have

$$Z^o(x) = \sum_{n=0}^{\infty} 0x^n = 0 \qquad\qquad Z(x) = \sum_{n=0}^{\infty} 0 \cdot \frac{x^n}{n!} = 0$$

For (e_n) the situation is slightly less trivial. We have

$$E^o(x) = \sum_{n=0}^{\infty} 1 \cdot x^n = 1 + x + x^2 + \cdots$$

This is a geometric series, so we can evaluate its sum to be

$$E^o(x) = \frac{1}{1-x}$$

For the EGF, we have

$$E(x) = \sum_{n=0}^{\infty} 1 \cdot \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

This is an important function called the *exponential function*, and we write it as $\exp(x)$.

2. Calculate the following generating functions. None of your final answers should be in the form of a sum.

   (a) The EGF of $f_n =$ the number of permutations of $\{1, \dots, n\}$. *Hint:* this sequence begins $1, 1, 2, 6, 24, \dots$.

   (b) The EGF *and* the OGF of $p_n =$ the number of subsets of $\{1, \dots, n\}$. *Hint:* this sequence begins $1, 2, 4, 8, 16, \dots$

   (c) The EGF of (a_n), where for a given positive integer a, we define

$$a_n = \begin{cases} \frac{1}{(a-n)!} & \text{if } 0 \le n \le a \\ 0 & \text{else} \end{cases}$$