APL SIMD Boolean Array Algorithms

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Abstract

Computation on large Boolean arrays is becoming more prevalent, due to applications such as cryptography, data compression, and image analysis and synthesis. The advent of bit-oriented vector extensions for microprocessors and of GPUS presents opportunities for significant performance improvements in such Boolean-dominated applications. Since APL is one of the few computer languages that supports dense (one bit per element, eight bits per byte), multi-dimensional Boolean arrays as first-class objects, it has naturally attracted research into optimizations for improved performance of Boolean array operations. This paper presents some of the Single Instruction, Multiple Data (SIMD) Boolean-related optimizations that have appeared in APL implementations, and suggests ways in which those optimizations might be exploited using contemporary hardware.

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- Boolean arrays are grist to APL's data-parallel, expressive mill!

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- GPU and SIMD vector facilities can exploit them

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- rbemove will copy 12 adjacent array elements at once

Structural and Selection Verbs III

- 2 3 4⍴⍳24

```
 0  1  2  3
 4  5  6  7
 8  9 10 11

12 13 14 15
16 17 18 19
20 21 22 23
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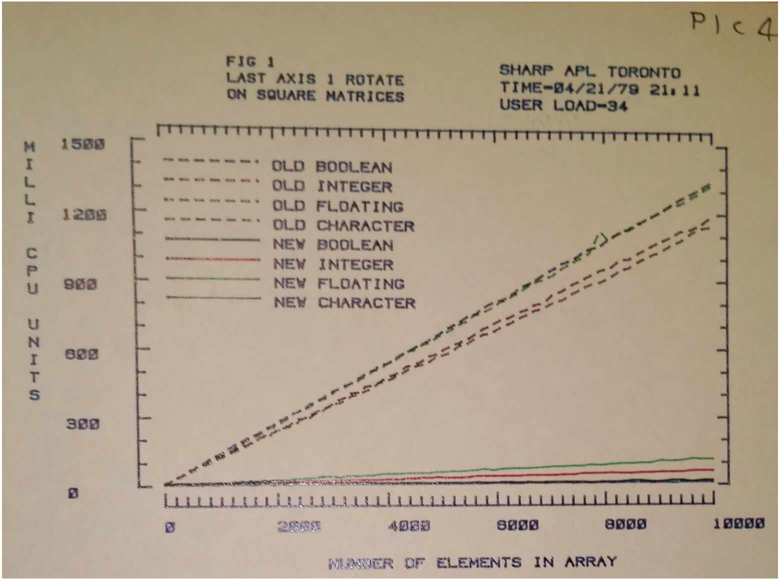
- All non-last-axis operations copied entire cells at once, using rbemove

Reverse and Rotate Performance on Booleans



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- Do an overlapped move, or “smear" of the result to its tail

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- Kernel generalizes to any power of two, *e.g.*, 16x16, 32x32

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- Algorithm used briefly for ∨/ω and ∧/ω

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- Result: linear-time, word-at-a-time, SIMD Boolean scan & reduce

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- Bernecky's simple C Heckman implementation is about 3X faster than Dyalog APL 15.0 (vector only)

Scan

- John Heckman, 1970 or 1971: user-defined APL scan verbs
- Now widely used in GPUs
- Recursive doubling:

```
r←nescanall y;s;biw
  ⍝ Not-equal scan
  r←y
  biw←⌈2⍟1⌈⍴y
  :For s :In 2*⍳biw ⍝ Heckman
    r←r≠(-⍴r)↑(-s)↓r
  :EndFor
```

- SIMD, word-at-a-time algorithm for Boolean ≠\⍵ and =\⍵ along last axis
- Bernecky's simple C Heckman implementation is about 3X faster than Dyalog APL 15.0 (vector only)
- So far, no X86 vectorization; perhaps we can do even better

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- redesigned Boolean inner product to use STAR algorithm

Classic Inner Product Algorithm

```
Z←X ipclassic Y;RX;CX;CY;I;J;K
 RX←(⍴X)[0]
 CX←(⍴X)[1]
 CY←(⍴Y)[1]
 Z←(RX,CY)⍴0.5
 :For I :In ⍳RX
  :For J :In ⍳CY
   Z[I;J]←0
   :For K :In ⍳CX
    Z[I;J]←Z[I;J]+X[I;K]×Y[K;J]
   :EndFor
  :EndFor
 :EndFor
```

STAR Inner Product Algorithm

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Z←X ipstar Y;RX;CX;CY;I;J;Xel
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 CY←(⍴Y)[1]
 Z←(RX,CY)⍴0
 :For I :In ⍳RX
  :For J :In ⍳CX
   Xel←X[I;J]
   Z[I;]←Z[I;]+Xel×Y[J;]
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- Unfortunately, the APL primitive is still 30X faster than the APL model

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- Similarly, if `Xel` is `1`, we can do the `g`-reduction using `RO`
- This gives us poor man's sparse arrays, which works on other data types, too
- Final result: Boolean inner products on SHARP APL/PC ran much faster than APL2 on huge mainframe

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- See also the Bernecky-Scholz PLDI2014 Arrays Workshop paper: *Abstract Expressionism*

Boolean Sort

- Sort ascending for Booleans:
  ```
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- Second phase can be performed in SIMD, *e.g.*, by a single SAC data-parallel with-loop.

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- SIMD Boolean upgrade:
  `ug←{((~⍵)/⍳⍴⍵),⍵/⍳⍴⍵}`

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  `ug←{((~⍵)/⍳⍴⍵),⍵/⍳⍴⍵}`
- Not stunningly SIMD, though.

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- A similar definition holds for word-oriented algorithms

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