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## 1 Lunar Ascent: From Earth to Moon's Orbit

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## 2 Abstract

We present a simulation of a rocket's path from Earth's surface to the orbit of the Moon. We model the complex dynamics involved in rocket propulsion, gravitational interactions, and orbital mechanics by leveraging differential equations and numerical methods. Our simulation accurately captures the stages of ascent, including liftoff, trajectory optimization, and orbital insertion. Through detailed analysis and visualization, we explore the critical factors influencing orbital maneuvers required for this orbital trajectory. The results offer valuable insights into the challenges and considerations of space exploration, paving the way for future missions to celestial bodies beyond Earth's orbit.

## 3 Background

Space travel presents a highly relevant and classic optimal control problem, influenced by numerous factors such as drag, fuel consumption, gravitational forces from multiple celestial bodies, and varying mass considerations. Navigating a rocket from Earth's surface to the Moon's orbit requires modeling complex dynamics encompassing rocket propulsion, gravitational interactions, and orbital mechanics. We rely on a set of differential equations and numerical methods to simulate and analyze these dynamics in order to accurately optimize trajectory and orbital insertion.

Understanding and optimizing orbital maneuvers for interplanetary travel have been longstanding challenges in aerospace engineering and space exploration. Previous research has explored various aspects of trajectory optimization, spacecraft propulsion, and orbital transfers to achieve efficient and successful missions beyond Earth's immediate vicinity. Notably, the formulation and solution of optimal control problems have been instrumental in advancing our understanding of spaceflight dynamics and guiding mission planning efforts in order to achieve recent feats such as landing on the moon and Mars.

This study builds upon existing research by focusing on the specific trajectory from Earth's surface to the moon's orbit. By conducting detailed analysis and visualizing critical factors affecting orbital maneuvers, our simulation provides valuable insights into the challenges and considerations of space exploration beyond Earth's orbit. The results obtained pave the way for enhanced mission planning and spacecraft design, contributing to the realization of future missions to celestial bodies throughout the solar system and beyond.

# 4 Mathematical Representation

To solve this problem, we defined a cost functional, a state-space evolution equation, and initial and endpoint conditions. We then utilized Pontryagin's maximum principle to formulate the co-state evolution equation and solve for the optimal control in terms of the co-state. This allowed us to set up a boundary-value problem to solve for the state and costate and then the control. The code and steps for how we accomplished this is defined below. We made several simplifying assumptions in this project defined below.

Assumptions 1. Rocket, earth, and moon have point mass for the sake of calculating gravity 2. Rocket, earth, and moon are the only objects for the sake of calculating gravity (no influence of the sun, stars, etc) 3. Rocket, earth, and moon lie on a 2D plane 4. Moon's orbit is perfectly circular 5. Rocket's mass does not change 6. Rocket is single stage 7. The earth's position remains at the origin of the 2D plane 8. The moon's position at time zero is on the $x$-axis

Unless otherwise specified, we the units for this project are defined as: - Mass: kilograms (kg) - Distance: meters (m) - Force: newtons (N) - Angle: degrees (ř) - Speed: meters per second (m/s) - Time: seconds (s)

## 4.1 Cost Functional

As mentioned above, we seek to find the optimal thrust for a rocket to leave Earth and begin orbiting the moon. To do so, we set up our cost functional as

$$J[\mathbf{u}] = \int_0^{t_f} c\|\mathbf{u}(t)\|^2 dt,$$

where we can interpret $\mathbf{u} = \begin{bmatrix} u_x & u_y \end{bmatrix}$ as the thrust in the $x$- and $y$-directions, respectively. We also define our initial conditions as

$$\begin{aligned} x(0) &= R_e, & x(t_f) &= L_m \cos(\theta) + (R_m + h_f)\cos(90 + \varphi) \\ y(0) &= 0, & y(t_f) &= L_m \sin(\theta) + (R_m + h_f)\sin(90 + \varphi) \\ x'(0) &= 0, & x'(t_f) &= v_f \cos(\theta) \\ y'(0) &= \omega_e, & y'(t_f) &= v_f \sin(\theta) \end{aligned}$$

where

- $c$ is the scaling constant for the cost of thrust
- $R_e$ is the radius of the Earth (in kilometers),
- $L_m$ is the distance from the Earth to the moon (in kilometers),
- $R_m$ is the radius of the moon (in kilometers),
- $h_f$ is the desired height above the moon to enter orbit (in kilometers),
- $v_f$ is the final velocity of the rocket (in kilometers per second),
- $\omega_e$ is the angular velocity of the Earth at the equator (in radians per second),
- $\theta$ is the angle between the $x$-axis and the line between the Earth and the moon, assuming the Earth is at the origin (in degrees),
- $\varphi$ is the angle between the $y$-axis and the line between the Earth and the moon.

### 4.2 State Space

We define our state space equation as

$$\bar{x} = \begin{bmatrix} x \\ y \\ x' \\ y' \end{bmatrix}$$

where $x''$ and $y''$ are the acceleration in the $x$ and $y$ directions, respectively. Using Newton's second law, this gives:

$$\begin{aligned} x'' &= -\frac{GM_e x}{(x^2+y^2)^{3/2}} + \frac{GM_m(L_x - x)}{((L_x-x)^2+(L_y-y)^2)^{3/2}} + \frac{u_x}{m_r} - \frac{\frac{1}{2}cA\rho(x,y)x'^2}{m_r} \\ y'' &= -\frac{GM_e y}{(x^2+y^2)^{3/2}} - \frac{GM_m(L_y - y)}{((L_x-x)^2+(L_y-y)^2)^{3/2}} + \frac{u_y}{m_r} - \frac{\frac{1}{2}cA\rho(x,y)y'^2}{m_r}. \end{aligned}$$

where

- $G$ is the universal gravitational constant
- $M_e$ is the mass of the Earth
- $M_m$ is the mass of the moon
- $m_r$ is the mass of the rocket
- $u_x$ is the rocket's thrust in the $x$ direction
- $u_y$ is the rocket's thrust in the $y$ direction
- $L_x$ is the $x$ position of the moon
- $L_y$ is the $y$ position of the moon
- $c$ is the drag constant $\frac{1}{4}$
- $a$ is the surface area of the top of the rocket
- $\rho(x,y)$ is the air density at position $(x,y)$

### 4.3 Hamiltonian

We now derive the Hamiltonian equation. We have

$$\begin{aligned} H = \mathbf{p}\cdot\mathbf{x}' - L = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ x'' \\ y'' \end{bmatrix} - c\|\mathbf{u}\|^2 \\ = p_1 x' + p_2 y' + p_3 x'' + p_4 y'' - cu_x^2 - cu_y^2. \end{aligned}$$

Plugging these values into the Hamiltonian equation, we get

$$\begin{aligned} H = p_1x' + p_2y' - p_3\left(-\frac{GM_e x}{(x^2+y^2)^{3/2}} + \frac{GM_m(L_x - x)}{((L_x-x)^2+(L_y-y)^2)^{3/2}} + \frac{u_x}{m_r} - \frac{\frac{1}{2}ca\rho(x,y)x'^2}{m_r}\right) \\ - p_4\left(-\frac{GM_e y}{(x^2+y^2)^{3/2}} - \frac{GM_m(L_y - y)}{((L_x-x)^2+(L_y-y)^2)^{3/2}} + \frac{u_y}{m_r} - \frac{\frac{1}{2}ca\rho(x,y)y'^2}{m_r}\right) - cu_x^2 - cu_y^2. \end{aligned}$$