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Tie-breaking the Highest Median: Alternatives to the Majority Judgment

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Abstract The paper deals with voting rules that require voters to rate the candidates on a finite evaluation scale and then elect a candidate whose median grade is maximum. These rules differ by the way they choose among candidates with the same median grade. Call proponents (resp. opponents) of a candidate the voters who rate this candidate strictly above (resp. strictly below) her median grade. A simple rule, coined the typical judgment, orders tied candidates by the difference between their share of proponents and opponents. A natural rule, coined the usual judgment, divides this difference by the share of median votes. An alternative rule, coined the central judgment, compares the relative shares of proponents and opponents. The usual judgment is continuous with respect to these shares. The majority judgment of Balinski & Laraki (2007) considers the largest of these shares and loses continuity. A result in Balinski & Laraki (2014) aims to characterize the majority judgment and states that only a certain class of functions share some valuable characteristics, like monotonicity. We relativize this result, by emphasizing that it only holds for continuous scales of grades. Properties remaining specific to the majority judgment in the discrete case are idiosyncratic features more than universally sought criteria, and other median-based rules exist that are both monotonic and continuous.

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1 Introduction

In a series of papers and a book, Balinski & Laraki (2007; 2011; 2014; 2016) have developed a "theory of measuring, electing and ranking". This theory demonstrates the valuable properties of a voting system based on evaluations rather than on rankings or single marks, where the choice (or candidate) obtaining the highest median evaluation is elected. Balinski & Laraki (BL) have proposed an elegant set of rules electing a choice with the highest median, and coined this voting system the "majority judgment". Ties between choices with the same median grades are resolved by comparing the shares of grades above and below the median, respectively: the choice with the largest share wins if the corresponding grades are above the median, and loses if they are below. The majority judgment (MJ) holds remarkable properties, making it the ideal voting system according to its proponents.[1] It is independent of irrelevant alternatives, it reduces manipulability, and it does not fall into the scope of Arrow's impossibility theorem (Balinski & Laraki, 2007). These properties emerge from the expressiveness of the information contained in each vote, as well as from the election of the highest median. They do not rely on the particular tie-breaking rules of the majority judgment. Contrarily to what Balinski & Laraki (2014) convey, neither does the property of monotonicity, which ensures that a choice can only benefit from an increase of its grades. Indeed, the uniqueness property of one of the various theorems that BL have established in their theory turns out to be inapplicable. Actually, Balinski & Laraki (2014) do not make clear that their theorem only applies to a continuous set of grades, whereas in all practical application, the set of grades is finite. As a consequence, other voting systems electing the highest median are as credible as the majority judgment. Rather than discrediting the outstanding theory elaborated by BL, this observation enriches it by a variety of simpler voting rules. We will present two such rules in section 3, after describing the setting and notations of the paper in section 2. Then, the main results are exposed in section 4: we clear up ambiguity of a proposition of BL on the uniqueness of monotonicity by showing that our three voting rules are counter-examples. In section 5, we compare the properties of the different tie-breaking rules. Finally, section 6 provides a practical understanding of the difference between these voting systems, by exhibiting how they behave on four real-world examples.

Before the formal analysis, let us grasp an intuition of the tie-breaking rules with a graphical example. Figure 1 and Table 1 show the voting profiles of four choices —*A*,

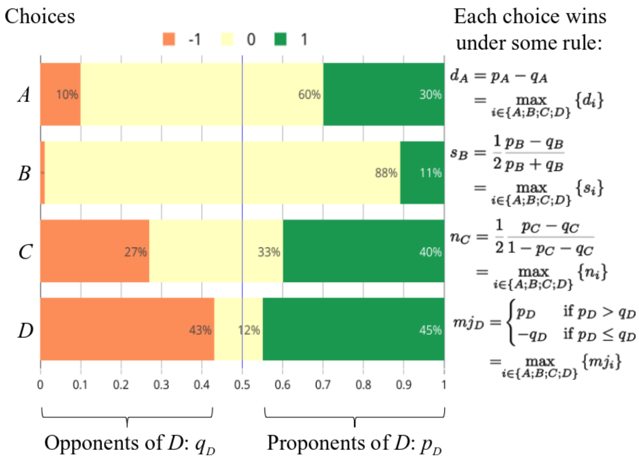

Fig. 1 Example of vote outcome where each choice *A*, *B*, *C* or *D* wins according to one of the four tie-breaking rule studied: respectively *difference* (d), *relative share* (s), *normalized difference* (n), and *majority judgment* (mj).

c	α_c	p_c	q_c	d_c	s_c	n_c	mj_c
A	0	$\frac{30}{100}$	$\frac{10}{100}$	$\mathbf{\frac{20}{100}}$	$\frac{20}{80}$	$\frac{20}{120}$	$\frac{30}{100}$
B	0	$\frac{11}{100}$	$\frac{1}{100}$	$\frac{10}{100}$	$\mathbf{\frac{10}{24}}$	$\frac{10}{176}$	$\frac{11}{100}$
C	0	$\frac{40}{100}$	$\frac{27}{100}$	$\frac{13}{100}$	$\frac{13}{134}$	$\mathbf{\frac{13}{66}}$	$\frac{40}{100}$
D	0	$\frac{45}{100}$	$\frac{43}{100}$	$\frac{2}{100}$	$\frac{2}{176}$	$\frac{2}{24}$	$\mathbf{\frac{45}{100}}$

Table 1 Comparison of different tie-breaking rules on an example: the score of the winner is in bold and blue for each rule.

B, *C* and *D*— evaluated by 100 voters. All choices share the same median grade $\alpha = 0$. Ties are resolved by looking at what we abusively call the *proponents* and the *opponents* to each choice c: the voters attributing to c a higher or a lower grade than c's median grade. The shares of proponents and opponents are respectively noted p and q—in this simple example, they correspond to the votes 1 and −1. For example, *A* has 30 proponents and 10 opponents, so that $p_A = 0.3$ and $q_A = 0.1$. Choice D wins the majority judgment, because its share of proponents exceeds all other shares of proponents or opponents. *A* wins with the *difference* tie-breaking rule relying on the score d, because its difference between proponents' and opponents' shares is the highest. *B* wins the tie-breaking rule relying on s, because its ratio of proponents (or equivalently, of $p_B - q_B$) over non-median graders ($p_B + q_B$) is the highest. *C* wins with tie-breaking rule of n, because its difference between proponents' and opponents' shares normalized by its share of median grades ($1 - p_C - q_C$) is the highest. This example shows that the tie-breaking rules described can lead to different outcomes. It is worth noticing that, as MJ only takes into account the largest of the two groups of non-median grades, it is more sensitive to a small variation in their sizes when these are close: in our example, if 3% of votes shifts from 0 to −1 for *D*, the highest group of

[1] Nuanced reviews of MJ have also been published (Felsenthal & Machover, 2008; Brams, 2011; Laslier, 2018).

D becomes its opponents and D is ranked last instead of first (in which case C wins MJ). However, a similar variation would not impact the other tie-breaking rules as heavily.

2 Setting

Let $\mathscr{C}$ be a finite set of choices, $\mathscr{V}$ a finite set of V voters, and $\mathscr{G}$ a finite[2] ordered set of G grades. We assume that $G \geq 3$, as all relevant rules boil down to approval voting when $G = 2$.

The grade (or evaluation) of choice c by voter v is noted $g_{c,v} \in \mathscr{G}$. The family $\Phi = \left(g_{c,v}\right)_{(c,v)\in\mathscr{C}\times\mathscr{V}}$ is called the *voting profile*. In Appendix B, we allow for partial abstention, i.e. $g_{c,v} \in \mathscr{G} \cup \{\emptyset\}$, where $g_{c,v} = \emptyset$ indicates that voter v does not attribute any grade to c. All results are preserved when allowing for partial abstention, but to simplify the presentation, we follow BL and consider that each choice receives the same number of expressed grades V.

$\mathscr{G}$ is isomorphic to the ordered set of integers $[\![1; G]\!]$. Hence, each grade $g \in \mathscr{G}$ is identified with an integer, so that its successor (when it exists) (resp. its predecessor) is simply noted $g+1$ (resp. $g-1$), and the order relation on $\mathscr{G}$ is denoted by "$\geq$". Note that values or distances between values do not play any role, as all reasonings rely on quantiles of grades, and not on averages.

We define the jth order function $\ell^j : \mathscr{C} \to \mathscr{G}$ as the function that returns the jth lowest grade of a candidate. We define the *lower middlemost* grade α_c of choice c, as the median grade of c when V is odd, and as the $\frac{V}{2}$th lowest grade when V is even. Formally, $\alpha_c = \ell^{\lceil \frac{V}{2} \rceil}(c)$, where $\lceil \cdot \rceil$ is the ceiling function. We often abusively refer to α_c as the *median* grade of c. We call *the middlemost grades* the two central grades of c when V is even and the three central grades when V is odd. Formally, the middlemost grades of c are $\left(\ell^i(c)\right)_{i\in[\![\lceil \frac{V+1}{2}\rceil -1; \lceil \frac{V}{2}\rceil +1]\!]}$. We also define the *first middlemost interval*, which is the set of middlemost grades: $\left[\!\left[\ell^{\lceil \frac{V+1}{2}\rceil -1}(c); \ell^{\lceil \frac{V}{2}\rceil +1}(c)\right]\!\right]$, and the k*th middlemost interval* (for any integer $k < \frac{V}{2}$) as $\left[\!\left[\ell^{\lceil \frac{V+1}{2}\rceil -k}(c); \ell^{\lceil \frac{V}{2}\rceil +k}(c)\right]\!\right]$. For example, with the tuple of grades $(1, 1, 2, 4, 6, 7, 8)$, the lowest middlemost grade —also called the *zeroth* middlemost interval— is the median: 4, the middlemost grades are $(2, 4, 6)$, the first middlemost interval is $[\![2; 6]\!]$, the second middlemost interval is $[\![1; 7]\!]$, and the third middlemost interval is $[\![1; 8]\!]$.

For $n \in (0; G)$, we denote by p_c^n (resp. q_c^n) the proportion of c's grades at or above $\alpha_c + n$ (resp. at or below $\alpha_c - n$). Formally, as grades outside $\mathscr{G}$ are not defined, we use the function min and set $p_c^n := \frac{1}{V}\left|\left\{v \in \mathscr{V} \mid g_{c,v} > \min\{\alpha_c + n - 1; G\}\right\}\right|$ (resp. $q_c^n := \frac{1}{V}\left|\left\{v \in \mathscr{V} \mid g_{c,v} < \max\{\alpha_c - n + 1; 1\}\right\}\right|$). For simplicity, we abusively qualify $p_c := p_c^1$ as the share of proponents and $q_c := q_c^1$ as the share of opponents to c.

Finally, a *rule* is a total preorder on $\mathscr{C}$, function of the profile Φ; while a *tie-breaking rule* is an order on $\mathscr{C}$ restricted to profiles with the same median grade. Rules often rely on *scores*, which are real-valued functions of the grades of a choice. We call *tie-breaking score* a function of p and q; *primary score* a function of α, p and q (more precisely the sum of α and of a tie-breaking score); and a *complementary score* any other function of the grades which is used to rank choices sharing the same primary score. Without further precision, *score* refers to *primary score*.

3 Different tie-breaking rules

Hereafter we describe different tie-breaking rules, which share the common characteristic that choices c of $\mathscr{C}$ are ordered lexicographically starting with a (primary) score which depends exclusively on the tuple $\left(\alpha_c, p_c, q_c\right)$, while complementary scores decide in the rare cases when ties remain. One can grasp the intuition behind these rules with the graphical example of section 1 as well as with Figure 2.

Other types of tie-breaking rules have been proposed, by Falcó & García-Lapresta (2011) and García-Lapresta & Pérez-Román (2018). We do not further detail these approaches, because they contain theoretical inputs that are beyond the scope of the present paper.

3.1 Majority Judgment

Balinski & Laraki (2016) propose the following *majority-gauge rule* between two choices A and B sharing the same median grade. They write:

$$A \succ_{mg} B \iff p_A > \max\left\{p_B; q_A; q_B\right\} \text{ or } q_B > \max\left\{p_A; p_B; q_A\right\} \quad (1)$$

In other words, the primary score mj_c of a choice c can be defined as:[3]

$$mj_c := mj\left(\alpha_c, p_c, q_c\right) := \alpha_c + \mathbb{1}_{p_c > q_c} p_c - \mathbb{1}_{p_c \leq q_c} q_c. \quad (2)$$

Intuitively in MJ, the largest share of proponents (p_c) or opponents (q_c) determines the fate of the vote: if the

[2] We restrict our analysis to finite sets of grades as they cover all practical applications.

[3] $\mathbb{1}_{\mathscr{P}(\boldsymbol{x})}$ denotes the indicator function of the property $\mathscr{P}$ evaluated in $\boldsymbol{x}$. For example, $\mathbb{1}_{p_c > q_c} = 1$ if $p_c > q_c$ and 0 otherwise.

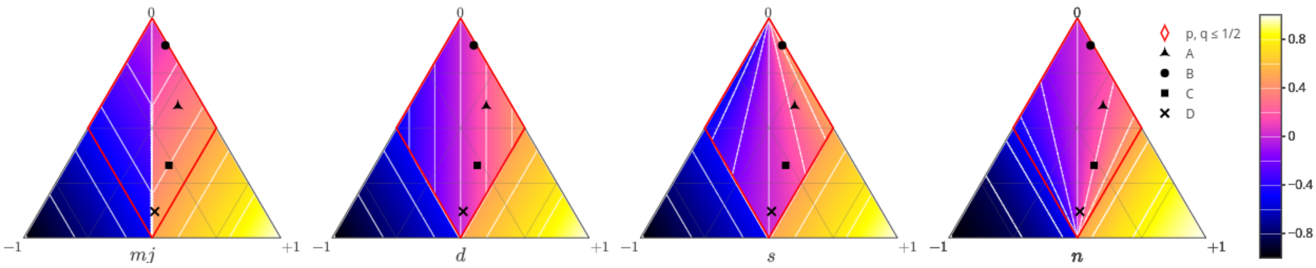

Fig. 2 Ternary plot showing the scores of different tie-breaking rules. Tuples $(p, q, 1-p-q)$ for $p, q \in (0;1)$ are represented in barycentric coordinates on the frame $(+1, -1, 0)$. The interior of the red rhombus corresponds to $p, q \leq \frac{1}{2}$. In such cases, the median grade is 0, and the grades above (resp. below) 0 are treated as +1 (resp. −1). *A, B, C* and *D* are the example candidates described in Table 1.

largest share is the proponents of *c*, *c* is elected; if instead it is the opponents of *c*, *c* is dismissed (and the operation is possibly iterated on the —now smaller— set of tied winners).

By construction, it is possible that several choices share the same score *mj* even though their tuples (α_c, p_c, q_c) are distinct. This is so when tied tuples[4] T are of the forms (i) (α, p_c, q), with $q \geq \max_{c \in T}\{p_c\}$, or (ii) (α, p, q_c), with $p > \max_{c \in T}\{q_c\}$. In such cases, choices are ordered using a *complementary* score mj_c^2 equal to $mj(\alpha_c, p_c, q_c^2)$ in case (i) and equal to $mj(\alpha_c, p_c^2, q_c)$ in case (ii). If a tie remains, a new complementary score is constructed along the same procedure until a unique winner is found, by replacing the shared group (p^n or q^n) by its *successors* (p_c^{n+1} or q_c^{n+1}). This procedure specifies a total order on choices called the *majority ranking*, and it characterizes the majority judgment.

For example, let $\mathscr{C} = \{E; F\}$ and assume the following voting profile: $\Phi = \begin{pmatrix} g_{E,v} \\ g_{F,v} \end{pmatrix}_{v \in \mathscr{V}} = \begin{pmatrix} -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. *E* and *F* share the same median grade 0, and the same score $mj_E = mj_F = \frac{2}{5}$. As the share of proponents is higher than the share of opponents for both *E* and *F*, and as these share of proponents are equal, our example belongs to case (ii). We resolve the tie by comparing $mj(\alpha_c, p_c^2, q_c)$ for $c \in \{E; F\}$. As $p_E^2 = p_F^2 = 0$ (because no grade higher than 1 has been attributed), this amounts to comparing the shares of opponents of *E* and *F*. As $q_E = \frac{1}{5} > q_F = 0$, $mj_E^2 = -\frac{1}{5} < 0 = mj_F^2$, and *F* wins.

Balinski & Laraki (2014) propose an equivalent description MJ. The choice with the highest median grade is elected. In case of a tie, the order must depend on remaining grades, so the median grade is dropped. The operation is then repeated on remaining grades, until one choice has a median grade higher than the others. In our example above, this amounts to compare, in that order and for $c \in \mathscr{C}$, the median grades: $\ell^3(c) = 0$, then the second lowest grades: $\ell^2(c) = 0$, then $\ell^4(c) = 1$. Finally, after dropping the middlemost grades, the median grades of *E* and *F* are their lowest grades, and they differ; as $\ell^1(E) = -1 < 0 = \ell^1(F)$, *F* wins. Notice that we compared grades of same rank, starting with the median and progressively moving away from it, alternating between the "left-" (lower) and the "right-" (higher) hand side of the median. Example 2 formalizes this idea.

3.2 Alternatives to the Majority Judgment

Here, we define three alternative tie-breaking rules for the election of the highest median. They all rank choices using a primary score and, in case of ties, rely on complementary scores.

3.2.1 Primary scores

The primary score of a candidate *c* is the sum of the median grade α_c and of a tie-breaking score. The latter is comprised in $\left(-\frac{1}{2}; \frac{1}{2}\right)$ so that primary scores do not overlap when median grades are distinct. We denote by Δ, σ and ν the *difference*, *relative share* and *normalized difference* tie-breaking scores defined below, and by d, s and n their respective primary score (e.g. $d_c = \alpha_c + \Delta_c$).

Difference between non-median groups This tie-breaking score is the difference in size between the shares of proponents and opponents to *c*:[5]

$$\Delta_c := p_c - q_c. \tag{3}$$

Intuitively, the winner is the one with the highest balance between proponents and opponents.

[4] We define tied tuples T as tuples of choices sharing the highest mj score: $T := \left\{(\alpha_c, p_c, q_c) \mid c \in \mathscr{C} \text{ and } mj_c = \max_k \{mj_k\}\right\}$.

[5] According to (Balinski & Laraki, 2011), it was very first proposed by David Gale.

Relative share of proponents Here, the tie-breaking score is given by the following formula:[6]

$$\sigma_c := \frac{1}{2}\frac{p_c - q_c}{p_c + q_c} \tag{4}$$

Intuitively, the winner is the one with the highest share of proponents within its non-median voters, i.e. the highest $\frac{p_c}{p_c+q_c}$. Indeed, $\sigma_c = \frac{1}{2}\left(\frac{p_c}{p_c+q_c} - \left(1 - \frac{p_c}{p_c+q_c}\right)\right) = \frac{p_c}{p_c+q_c} - \frac{1}{2}$.

Normalized difference between non-median groups Notice that, for a given size of *median group* $r_c := 1 - p_c - q_c$, the difference Δ_c is bounded above by the limit case $p_c = \frac{1}{2}$, which entails $q_c = \frac{1}{2} - r_c$ and $\Delta_c = r_c$ (or an even lower Δ_c if $r_c > \frac{1}{2}$, as $q_c \geq 0$). By symmetry, the difference Δ_c is thus bounded by $(-r_c\,; r_c)$, and it is tempting to normalize it by $2 \cdot r_c$ to define another tie-breaking score:[7]

$$\nu_c := \frac{1}{2}\frac{p_c - q_c}{1 - p_c - q_c} \tag{5}$$

Intuitively, the winner is the one with the highest balance between proponents and opponents relative to the share of median voters. Hence, for a given difference Δ, the candidate with the lowest share of median voters has the largest score ν in absolute value. When the difference is positive, the normalized difference rewards candidates with relatively less median voters, but it is the contrary when the difference is negative.

Visualizing the rules Figure 2 presents each score in a ternary plot, where the voting profile of any candidate can be drawn using barycentric coordinates. Take candidate A defined in Table 1 for example. Assuming that all altitudes of the triangles are unitary, the distance from point A to the (bottom) base of the triangle is the share of median voters r_A, while its distance to the upper left (resp. right) edge is the share p_A of proponents (resp. the share q_A of opponents). This plot makes clear that different rules yield different rankings, and reveals the discontinuities present in all rules but the normalized difference.

Extensions These rules extend naturally to similar voting schemes where the winner is determined using the (lowest) kth quantile $\ell^{\lceil kV\rceil}$ (instead of the median). Such extensions can be preferred e.g. to allow least satisfied people to get more influence on the final decision: the grade with the highest first quartile could then be elected instead of that with the highest median. In such cases, the score mj turns into $mj^k_c = \ell^{\lceil kV\rceil}(c) + \mathbb{1}_{\frac{k}{1-k}p_c > q_c}\, p_c - \mathbb{1}_{\frac{k}{1-k}p_c \leq q_c}\, q_c$; the *difference* Δ and the *relative share* σ can write the same; while the *normalized difference* becomes $\nu^k_c = \nu_c \cdot \frac{k\cdot(1-k)}{k^2+(1-k)^2}$. These extensions conserve most of the properties of their original voting systems (even if properties such as the majority rule need to be reformulated). However, Balinski & Laraki (2014) show that voting systems relying on the highest median minimize the probability of cheating.

3.2.2 *Complementary scores and ultimate tie-breaking*

When a primary score is shared by several choices, a secondary score determines the ranking between these choices. It is obtained by applying the tie-breaking score to $\left(p^2_c, q^2_c\right)$. If a tie remains after n steps, a complementary score of degree $n+1$ is computed using $\left(p^{n+1}_c, q^{n+1}_c\right)$; it is noted Δ^{n+1}_c, σ^{n+1}_c or ν^{n+1}_c. We qualify these voting procedures as *highest median with nested tie-breaking scores*. For example, let $\mathscr{C} = \{E; F\}$, assume the following voting profile: $\Phi = \begin{pmatrix} g_E \\ g_F \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, and consider the *difference* rule. E and F share the same median 0, and we have $\Delta_E = \frac{1}{3} - \frac{1}{3} = 0 = 0 - 0 = \Delta_F$. Thus, we compare E and F using $\Delta^2_E = \frac{1}{3} - 0 = \frac{1}{3} > 0 = \Delta^2_F$, and E wins.

When a tie remains after all these steps although the grades are distinct, the tied choices T are ranked according to the lexicographic order of vectors $\left(-q^1_c, p^1_c, -q^2_c, p^2_c, ..., -q^{G-1}_c, p^{G-1}_c\right)_{c\in T}$,[8] i.e. the winner is the one with the lowest proportion of opponents. For example, if we have $\mathscr{C} = \{E'; F\}$ and the following voting profile: $\Phi = \begin{pmatrix} g_{E'} \\ g_F \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, the tie remains after comparing median grades, the primary score and the secondary score, all equal to zero. Thus, we resort to the ultimate tie-breaking rule and declare F winner because $-q^1_F = 0 > -\frac{1}{3} = -q^1_{E'}$.

We say that this ultimate tie-breaking rule is *consensual* because a choice with a shared opinion is preferred to a polarizing one (the opposite option would be to *follow dissent*, by favoring the highest q^i_c instead of the lowest). We say that it is *innermost* because the middlemost grades are preeminent (we would say it is *outermost* if we were to compare $\left(-p^{G-1}_c\right)_c$ first and $\left(-q^1_c\right)_c$ last).

[6] In the case $p_c + q_c = 0$, the formula is not valid so σ_c is set to 0.

[7] In the case $p_c = q_c = \frac{1}{2}$, the formula is not valid so ν_c is set to 0.

[8] Admittedly, the $\left(p^i\right)_i$ are redundant with the $\left(-q^i\right)_i$ for d and n as, for any $i \in [\![1; G-1]\!]$, the common complementary score conveys a bijection from q^i to p^i. However, this is not the case for s^i when $p^i \cdot q^i = 0$. In all cases, it is equivalent to order the $\left(-q^i\right)_i$ before or after the $\left(p^i\right)_i$, and we give priority to the $\left(-q^i\right)_i$ only by analogy with MJ.

Table 2 Summary of four ultimate tie-breaking rules when A and B share the same primary and complementary scores. Here $\bar{n} := \underset{n\in(0;G),\, c\in\{A;B\}}{\arg\max} \left\{q_c^n > 0\right\}$. The variant in bold is the one defended in (Balinski & Laraki, 2014) and followed in this paper.

$A \succ B$ if...	**consensus**	dissent
innermost	$\mathbf{q_A^1 < q_B^1}$	$q_A^1 > q_B^1$
outermost	$q_A^{\bar{n}} < q_B^{\bar{n}}$	$q_A^{\bar{n}} > q_B^{\bar{n}}$

Of course, we could have chosen an ultimate tie-breaking rule following dissent instead of consensus, and/or an outermost one (see Table 2). However, as Balinski & Laraki (2014) argue, following consensus is consistent with deciding upon the lower middlemost grade (instead of the upper one, $\ell^{\lceil \frac{V+1}{2} \rceil}$), in that a majority always grades the winner at least as much as its lower middlemost grade; while being innermost is consistent with deciding upon the middlemost grades. That being said, choosing another ultimate tie-breaking rule would have virtually no consequence on votes with a large electorate, as an ultimate tie is highly improbable in such cases.

The voting systems resulting from these rules each convey an order on choices. We denote by D, S, and N respectively, the highest median ranking function with nested tie-breaking score Δ, σ, and ν following the innermost consensus. We coin D as the *typical judgment*; S as the *central judgment*; and N as the *usual judgment*.

4 Common properties of these rules

Most of the results on the majority judgment proven by Balinski & Laraki apply to all voting rules electing the choice with the highest median grade. The main properties that rely on their tie-breaking rule also apply to the alternative tie-breaking rules. Indeed, although they convey that only a certain class of tie-breaking rules (which does not encompass our alternative rules) satisfies some valuable properties like monotonicity, this proposition is false in general. Indeed, BL apply a result from social choice theory that is only valid for a continuous set of grades. Yet in our setting (as in most –if not all– practical applications), the set of grades is finite. In any case, it could not be otherwise, as the definitions of the alternative tie-breaking rules require a finite set of grades.

We first need to recall some definitions adapted from Balinski & Laraki (2014) in order to expose the limit of their proposition.

Definition 1 *Independence of irrelevant alternatives in ranking* (IIAR). A preorder $\succeq_\Phi$ on $\mathscr{C}$ (function of the profile Φ) is IIAR when, for any profile Φ' obtained by eliminating or adjoining other choices (and corresponding votes) from a profile Φ, and for any choices c and c' present in both profiles, $c \succeq_\Phi c' \iff c \succeq_{\Phi'} c'$.

Definition 2 A preorder $\succeq_\Phi$ on $\mathscr{C}$ (function of the profile Φ) is *impartial* when it is independent of a permutation of choices (or rows) and voters (or columns).

Definition 3 A *social-ranking function (SRF)* $\succeq$ is a total preorder on $\mathscr{C}$, function of the profile Φ, that is impartial and IIAR.

Example 1 Along with majority judgment, its alternatives D, S and N define social-ranking functions. In addition, they specify an order (not only a preorder), as choices with distinct grades are never tied.

Definition 4 A social-ranking function $\succeq$ is *choice monotone* if $A \succeq B$ and a voter increases the grade of A implies $A \succ B$.[9]

Definition 5 A social-ranking function $\succeq$ is *order consistent* if the order between any two choices for some profile Φ implies the same order for any profile Φ' obtained from Φ by any strictly increasing transformation ϕ of all grades.

Remark 1 Order consistency requires that the social-ranking function be insensitive to a relabeling of grades, provided that the relabeling preserves the order between grades. In our setting, any social-ranking is trivially order consistent, as the only strictly increasing transformation of a finite set (of grades) is the identity function.[10]

Lemma 1 *The social-ranking functions defined by the highest median with the nested tie-breaking scores Δ, σ and ν are order consistent and choice monotone.*

Proof As explained in the previous remark, the order consistency is trivial. To prove the choice monotonicity, let A and B be choices such that $A \succeq B$. Either (i) $A = B$ or (ii) $A \succ B$. In case (i), if a voter increases her grade of A, either α_A will increase, or there is a minimum step n_0 for which $p_A^{n_0}$ will increase or $q_A^{n_0}$ will decrease. Thus, the score of A of degree n_0 will (strictly) increase while there will be no change on the scores of B, leading to $A \succ B$. The same reasoning applies on case (ii).

Definition 6 Let $\mathscr{S} = \{r_1; \ldots; r_V\}$ be an ordered set, such that $r_1 < \ldots < r_V$. A *lexi-order social-ranking function* uses a bijection $\pi : \mathscr{S} \to [\![1; V]\!]$ to rank the choices by $A \succ B \iff \left(\ell^{\pi(r_1)}(A), ..., \ell^{\pi(r_V)}(A)\right) \succ_{\text{lex}} \left(\ell^{\pi(r_1)}(B), ..., \ell^{\pi(r_V)}(B)\right)$, where $\succ_{\text{lex}}$ is the lexicographic order.

[9] Balinski & Laraki (2011) define a related notion (p. 204), *monotonicity*, equivalent to *choice monotonicity* as long as the social-ranking function is antisymmetric (i.e. $A \succeq B$ and $B \succeq A \Rightarrow A = B$). All tie-breaking rules dealt with in this paper are antisymmetric and thus monotonic, as each specifies an order. Hence, we sometimes use *monotonicity* as a shortcut for *choice monotonicity*.

[10] Let us precise that in absence of any explicit definition from BL, we understand *transformation* in its usual sense of a function from a set to itself. Indeed, it would make little sense to take a codomain of ϕ larger than its domain, because then some new grades could not be used.

Example 2 The majority judgment is a lexi-order social-ranking function, whose permutation π_{MJ} is defined by $\pi_{MJ}(2k+1) = \left\lceil \frac{V+1}{2} \right\rceil - k$ and $\pi_{MJ}(2k) = \left\lceil \frac{V}{2} \right\rceil + k$ for $k \in \left[\!\left[0; \left\lceil \frac{V-1}{2} \right\rceil \right]\!\right]$.[11] This was observed in Section 3.1 and is detailed in Balinski & Laraki (2011).

Lemma 2 *Consider a social-ranking function R that relies on a tie-breaking score whose sign is the sign of $p-q$. Then R is not a lexi-order function. In particular, the typical judgment (D), the central judgment (S) and the usual judgment (N) are not lexi-order functions.*

Proof Let $\mathscr{C} = \{A; B\}$ be the set of choices, let $N = 3$ and let $\mathscr{G} = [\![-1; 2]\!]$.[12] Let us exhibit two profiles Φ and Φ' for which $A \succ B$ but for which R cannot be expressed as a lexi-order function. We further denote with (resp. without) a "′" the variables related to profile Φ' (resp. Φ). Let $\Phi = \begin{pmatrix} g_{A,v} \\ g_{B,v} \end{pmatrix}_{v \in \mathscr{V}} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \end{pmatrix}$ and $\Phi' = \left(g'_{c,v}\right) = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$. The medians of A and B are the same in each case, so the tie-breaking rule applies. We have $p_A = \frac{1}{3} > 0 = q_A$ while $p_B = q_B = \frac{1}{3}$ on the one hand, and $p'_A = q'_A = \frac{1}{3}$ while $p'_B = 0 < \frac{1}{3} = q'_B$ on the other hand. As the sign of the tie-breaking score of R is the sign of $p-q$, we have $A \succ B$ in each profile. To rationalize this ranking in Φ with a lexi-order function, the lowest grades should be compared before the highest ones as, in Φ, the lowest grade of A is the only one above that of B. However by the same reasoning on Φ', the highest grades should be compared before the lowest ones, if R is a lexi-order function. There is a contradiction, proving that R is not a lexi-order function.

Claim (Theorem 11 in (Balinski & Laraki, 2014)) The unique choice-monotone and order consistent social-ranking functions are the lexi-order functions.

Remark 2 This claim is equivalent to the remark following theorem 11.5b in Chapter 11 of (Balinski & Laraki, 2011) which is supposed to prove it: "Repeated application of the theorem 11.5b shows that an SRF is order consistent and monotonic if and only if there is a sequence of order functions that decide: if the first does not strictly rank the candidates, the second does; if the second doesn't either, then the third does; and so on." It is true that for any profile, a monotonic and order consistent SRF can be characterized by a lexi-order function, but in general the sequence of order functions depends on the profile. Admittedly, the remark of BL can be justified using Theorem 5 from (Gevers, 1979) when the set of grades is continuous, an assumption which is made in (Balinski & Laraki, 2011). However, (Balinski & Laraki, 2014) do not state clearly the assumption of continuity, suggesting a more general result. The following proposition removes ambiguity.

Proposition 1 *Theorem 11 of (Balinski & Laraki, 2014) is false when the set of grades is finite.*

Proof Lemma 2 provides counter-examples of social-ranking functions that are not lexi-order functions although, from Lemma 1, they are choice monotone and order consistent.

5 Properties specific to each rule

The previous proposition shows that the most valuable property of MJ is not as specific as was thought. Now that we have dismissed the main rationale to overlook alternative tie-breaking rules, we explore the properties specific to each tie-breaking rule, to understand which one should be preferred. We first describe how each score reacts to a marginal change in the votes, then study the sensitivity of each rule to a small fluctuation in the votes, and finally we analyze the properties specific to the majority judgment.

5.1 Influence of each group

Table 3 give the relative influence of an infinitesimal shift of voters from opponents to proponents. More precisely, the first row provides the ratio of the derivative of each score with respect to p over its derivative with respect to $-q$. This ratio is equal to 1 for Δ, because one more proponent yields the same influence on Δ as one less opponent. The second row displays the ratio of elasticities of each score instead of the ratio of derivatives. This elasticity ratio is equal to 1 for σ, as σ is an increasing transformation[13] of $\frac{p}{q}$ and thus the multiplicative analog

Table 3 Marginal effects on each primary score of infinitesimal increase or growth in p relative to an equivalent reduction in q.

	d	s	n	mj
$-\frac{d\cdot/dp}{d\cdot/dq}$	1	$\frac{q}{p}$	$\frac{1-2q}{1-2p}$	$\begin{cases} +\infty & \text{if } p \geq q \\ 0 & \text{if } p < q \end{cases}$
$-\frac{p}{q}\frac{d\cdot/dp}{d\cdot/dq}$	$\frac{p}{q}$	1	$\frac{p}{q}\frac{1-2q}{1-2p}$	

[11] When V is odd, the domain $\mathscr{S}$ of π_{MJ} is $[\![0; V-1]\!]$, and when V is even, π_{MJ} is instead defined on $[\![1; V]\!]$.

[12] The assumption on $\mathscr{G}$ (which amounts to take $G \geq 4$) is made to simplify the argument, but a similar proof exists for $G = 3$. Take $\Phi_1 = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$, $\Phi_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 \end{pmatrix}$, $\Phi_3 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}$ and $\Phi_4 = \begin{pmatrix} -1 & -1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$. In each case, $A \succ B$. Using Φ_1, this ranking implies that quintile 2 should be compared before quintile 1, which we denote $2 \lhd 1$. Combining this with the condition given by Φ_2: $2 \lhd 4$ or $1 \lhd 4$, we obtain that $2 \lhd 4$. Similarly, $4 \lhd 5$ (Φ_3), and $4 \lhd 2$ or $5 \lhd 2$ (Φ_4) imply that $4 \lhd 2$, which yields a contradiction.

[13] Indeed, $\sigma = \frac{1}{2}\frac{p-q}{p+q} = \frac{1}{2}\frac{1-\frac{q}{p}}{1+\frac{q}{p}} = f\left(\frac{p}{q}\right)$ with $f(x) := \frac{1}{2}\frac{1-\frac{1}{x}}{1+\frac{1}{x}}$.

to $\Delta = p - q$. Except for these two cases, the alternative tie-breaking rules do not provide equal influence (in an additive or multiplicative sense) to each group, as other ratios are different from 1. However, their ratios always remain finite, which is not the case for the majority judgment, where additional voters have no influence at all when they join the smallest non-median group. Indeed, the majority judgment does not fully exploit the information available in (α, p, q), as the value of the smallest non-median group has no influence on the value of the score mj. Hence, to the extent that the influence of all marginal voters is desired, MJ should be avoided.

5.2 Sensitivity to small fluctuations

It is arguably appealing that a rule be insensitive to small fluctuations in the profiles. Indeed, it may seem unfair (or too random) if a score can vary substantially with a small change in the profile. Furthermore, a high sensitivity to small fluctuations could lead to more frequent allegations of irregularities in ballots among large electorates, as a losing candidate could rationally hope that the results would change after a new vote (or a recount).

Sensitivity to small fluctuations appear when the score of a choice is discontinuous with respect to the shares of each grade. Figures 2 and 3 help understand where the discontinuities occur. Figure 2 displays all possible score in a ternary plot when $G = \{-1; 0; +1\}$. Figure 3 uses $G = [\![-1; 2]\!]$ and shows how the scores vary with the share x of grades $+1$, keeping the shares of grades -1 and $+2$ constant at 0.1. One can notice that a discontinuity occurs for the majority judgment where $p = q$, as the largest group flips when the share of proponents exceeds the share of opponents. However, when the share of proponents increases so much that the median grade changes (i.e. when x go beyond $\frac{1}{2}$), the score mj adapts smoothly, as the largest non-median group flips from one half of proponents with a median of 0 to one half of opponents with a median of 1. Conversely, for D and S the discontinuity occurs when the median grade changes (i.e. at $p = \frac{1}{2}$ or $q = \frac{1}{2}$). Indeed, when x exceeds $\frac{1}{2}$, the share of grades -1 leaves the formula of the score while a new share of proponents (0.1) enters. Contrarily to D and S, the normalized difference succeeds in keeping the continuity when the median grade changes. Indeed, the difference is then equal to the share of median grades in absolute terms, making their ratio unitary. Thus, the location of discontinuities is restricted to $p = q = \frac{1}{2}$ for the normalized difference, which should be more robust to small fluctuations.

Table 4 Sensitivity to small fluctuations for each rule: locations of discontinuities and probability that a random reallocation of 2% of grades shift the ranking by more than 20 or 50 percentiles.

	d	s	n	mj
Discontinuity at:	$p = \frac{1}{2}$ or $q = \frac{1}{2}$		$p = q = \frac{1}{2}$	$p = q$
Proba shift > 20% (in %)	0.80	1.64	0.39	1.64
Proba shift > 50% (in %)	0.02	0.09	0.05	0.90

We can also assess the sensitivity to small fluctuations numerically. Taking a profile with many choices sharing the same median grade, we can measure the shift in ranking triggered by a small fluctuation in the grades of one choice. To this end, we draw randomly 100,000 p and q independently from the uniform distribution over $(0; 0.5)$.[14] We rank all choices according to each rule. Then, we reallocate 2% of the grades of each choice independently. For each choice, we draw ε uniformly in $[0; 0.02]$ and draw with probability $\frac{1}{2}$ the sign of the variation in p and, independently, the sign for q. Then, we increase or decrease p by ε and q by $0.02 - \varepsilon$. Finally, we measure the quantile shift in the rankings following each reallocation. Figure 4 shows the probability that the quantile shift is lower than a given quantile r ($r > 0.1$), for each rule. Table 4

[14] We choose the uniform distribution since we have no good prior on the real-world distribution of grades of an ordinary choice.

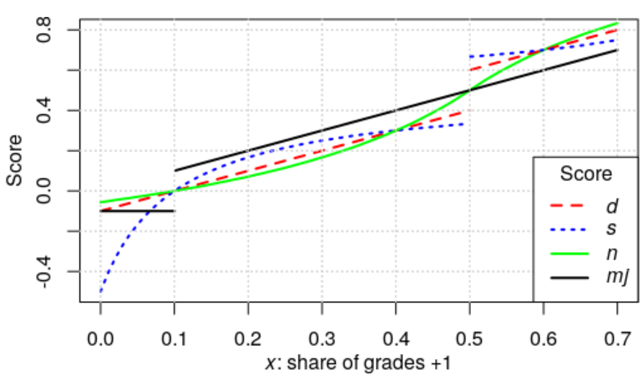

Fig. 3 Score of each rule given the following distribution of grades: 10% of -1, 10% of $+2$, the abscissa x of $+1$, and the remaining of 0.

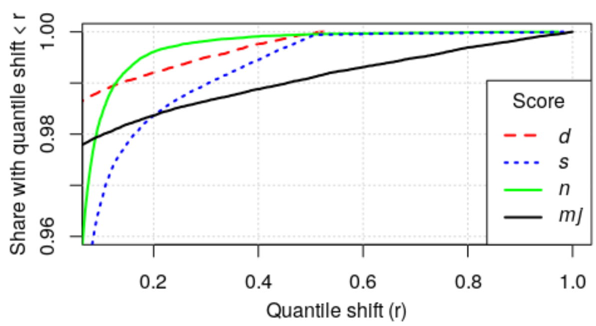

Fig. 4 CDF of quantile shift following a random reallocation of 2% of grades of a random choice. *E.g. the probability that the rank of a choice shifts by less than 45 percentiles after a random reallocation of 2% of grades is 99% for MJ.*

reports the probabilities that a quantile shift is large (i.e. higher than 20 percentiles) or very large (i.e. higher than 50 percentiles). It confirms that the majority judgment is the rule most prone to large or very large shifts following small fluctuations, as these probabilities are 4 times and 20 times higher for MJ than for N, respectively. Compared to N, D and S have similar probabilities of very large shifts (below 1‰) but they have higher probabilities of large shifts (0.8 and 1.6 respectively, compared to 0.4 for N).

Overall, the normalized difference N is the most appealing rule regarding the criteria of continuity and of low sensitivity to small fluctuations. Concerning this latter criterion, the majority judgment is hardly satisfactory, as it is 20 times more prone to very large shifts following small fluctuations than the normalized difference.

5.3 Properties specific to the majority judgment

Since the alternative tie-breaking scores appear to take better account of all non-median groups than *mj*, which varies only with the larger one, and since *mj* compares poorly to the normalized difference regarding the sensitivity to small fluctuations, one wonders what properties can make MJ attractive.

Advantages with polarized pairs MJ holds interesting and specific properties when the choices consist in a polarized pair, i.e. when the more a voter appreciates one candidate in a pair, the less the voter appreciates the other one. Balinski & Laraki (2016) formalize these properties.

Definition 7 (Balinski & Laraki, 2016) Two choices A and B are *polarized* if, for any two voters, i evaluates A higher (respectively, lower) than j then i evaluates B no higher (respectively, no lower) than j.

Definition 8 (Balinski & Laraki, 2016) A method is *strategy-proof* when it is an optimal strategy for every voter to express their opinion honestly. It is assumed that a voter's utility is the grade they attach to the elected choice.

Proposition 2 (Theorem 6 in (Balinski & Laraki, 2016)) *A social-ranking function $\succeq$ that is strategy-proof on the limited domain of polarized pairs of choices must coincide with the majority-gauge rule when the language of grades is sufficiently rich (i.e. when there are no less grades than choices*[15]*).*

[15] BL prefer to say that a language is sufficiently rich if "a voter who gives the same grade to two candidates has no preference between them".

Corollary 1 *D, S and N are not strategy-proof on the limited domain of polarized pairs of candidates.*

Definition 9 *(adapted from (Balinski & Laraki, 2016))* A social-ranking function $\succeq$ is *consistent with the majority rule on polarized pairs* of choices A and B if $A \succ B$ whenever $\left|\left\{v \,|\, g_{A,v} > g_{B,v}\right\}\right| > \left|\left\{v \,|\, g_{A,v} < g_{B,v}\right\}\right|$.

Remark 3 This definition relies on the concept of *relative* majority. If instead one was interested in strict majority, one should require that $A \succ B$ whenever $\left|\left\{v \,|\, g_{A,v} > g_{B,v}\right\}\right| > \frac{V}{2}$. All tie-breaking rules studied in this paper are consistent with this strict majority rule on polarized pairs.[16] However, the alternatives to majority judgment are not consistent with the broader, relative majority rule on polarized pairs of the previous definition, as can be deduced from the following theorem.

Proposition 3 (adapted from Theorem 4 of (Balinski & Laraki, 2016)) *A choice monotone social-ranking function that is consistent with the majority rule on polarized pairs of choices must coincide with the majority-gauge rule when the language of grades is sufficiently rich.*

Balinski & Laraki (2016) summarize the specific advantages of the majority judgment in their footnote 20: "only the majority-gauge rule coincides with the [relative] majority rule and combats strategic manipulation on polarized pairs". Understandably, they do not recall other properties of MJ, detailed in Appendix C, because these properties are not universally sought. That being said, MJ is arguably the best-behaved tie-breaking rule when the choice involves a polarized pair. However, the respect of the majority rule on polarized pairs should not be exaggerated, as alternative tie-breaking rules also coincide with the strict majority rule. More importantly, the universally sought properties of MJ only apply to polarized pairs of choices, which generally do not occur in practice (see section 6).[17]

6 The different rules in practice

In this section, we examine how the different tie-breaking rules compare in practice amongst themselves, and with respect to another voting system relying on grades: evaluative voting. Evaluative voting simply

[16] To see this, let us define $M = \left\{v \,|\, g_{A,v} > g_{B,v}\right\}$, $\hat{g} = \min_{v \in M}\left\{g_{A,v}\right\}$ and $\hat{M} = \left\{v \,|\, g_{A,v} = \hat{g}\right\}$. Take $k \in \hat{M}$. $\forall j \in M \setminus \hat{M}$, $g_{A,j} > \hat{g} = g_{A,k}$. Thus, as A and B are polarized, $g_{B,j} \leq g_{B,k} < g_{A,k} = \hat{g}$. In addition, $\forall i \in \hat{M}$, $g_{B,i} < \hat{g}$. Hence, $\forall v \in M$, $g_{B,v} < \hat{g} \leq g_{A,v}$. Finally, we deduce that $\alpha_B < \hat{g} \leq \alpha_A$ from $|M| > \frac{V}{2}$.

[17] One could also argue that in practice, there are often more choices than grades, so that the universally sought properties do not apply. However, even with a large number of choices, one must acknowledge that it is unlikely that the number of choices that are tied together exceeds the number of grades.

amounts to averaging the grades of each choice (see e.g. Hillinger, 2004). Four real datasets are used, where people had to express their preferences over several choices using grades. Two of them are polls on 2012 and 2017 French presidential elections, and are reported in (Balinski & Laraki, 2016). Their samples have not been weighted to account for under or over-representation of some socio-demographic groups in the samples relative to the French population, so their results are not representative of French preferences. The third dataset is the results to a citizens' primary for the 2017 French presidential election –la Primaire–, which constitutes the widest use of the majority judgment to date, as 11,304 persons have graded 5 candidates drawn randomly within a list of 12. The last one comes from a survey on French preferences for income distribution, where respondents were asked to grade different distributions (Fabre, 2017). Overall, 40 choices have been evaluated, ranging from 7 choices in the last dataset to 12 in the citizens' primary. In Appendix A, one can see the grades of each choice in the Figures, and read the rankings they imply in the corresponding Tables.

The winner to each of these real examples is the same irrespective of the voting system. For each dataset, we obtain five different rankings (one per rule); we then measure the distance between these rankings using the Kendall distance and aggregate these distances.[18] The rankings somewhat vary amongst one another. For example, for 4.8% pairs of choices (i.e. 9 pairs of among $\sum_{k\in\{7;10;11;12\}} \frac{k\cdot(k-1)}{2} = 187$), the order within the pair is reversed between majority judgment and evaluative voting (see Table 5). Empirically, 4.8% is the (normalized) Kendall distance between MJ and evaluative voting, and this is the highest distance between the 5 rules studied. The closest ones are the three alternative tie-breaking rules for the highest median, all at a distance lower than 2% from one another, while each of them has a Kendall distance of $2.9 \pm 0.8\%$ to the majority judgment and to evaluative voting. Similar figures are obtained if we compute the Kendall distances on the rankings over the four datasets combined. This analysis shows that in practice, all systems are usually equivalent; but in some occasions, the tie-breaking rule will decide the election.

Finally, the discontinuity of the majority judgment at $p = q$ discussed in section 5.2 is not far from occurring in the 2012 French presidential election, as the winner, Hollande, obtains $p = 0.451$ and $q = 0.433$ (see Table 6 and Figure 5 in Appendix A). Hence, the tie-breaking score of Hollande would reverse from +0.451 to −0.453 if 2% of median voters decreased their grade: in such a case, Bayrou would win the majority judgment. Yet, such a small variation would not have put the victory of Hollande at risk with respect to the other rules, because they rely on both the proponents and the opponents, and not only on the largest between the two. In a way, they use more information than the majority judgment, and this helps them be more robust to an fluctuations in the results.

Table 5 Kendall distance between different rules estimated on real data involving 187 pairs of choices (in %). A distance of 5% means that –on average– there is a 5% chance that the order within a pair of choices would be reversed between the two rankings.

	mean	mj	d	s	n
mean	0.0	4.8	2.1	3.2	2.7
mj	4.8	0.0	2.7	3.7	2.1
d	2.1	2.7	0.0	1.1	0.5
s	3.2	3.7	1.1	0.0	1.6
n	2.7	2.1	0.5	1.6	0.0

7 Conclusion

We enriched the theory of Balinski & Laraki on voting systems electing the choice with the highest median grade, by proposing tie-breaking rules alternative to the majority judgment. We coined the voting systems resulting from these rules the *typical judgment* (abbreviated *D*), the *central judgment* (*S*) and the *usual judgment* (*N*). We disputed the claim of BL that MJ "invokes no tie-breaking rules" and that the class it belongs –the lexi-order functions– has unique and valuable properties. In particular, we showed that our alternative tie-breaking rules also share this valuable property that the score of a choice necessarily increases if some voter increase their grades for this choice. Then, we detailed the different characteristics of the different tie-breaking rules: the majority judgment combats well manipulability on polarized pairs of choices, while rankings from alternative tie-breaking rules are more robust to small fluctuations in the grades. The usual judgment is the most robust to small fluctuations, as it is continuous where other rules are discontinuous. The lack of robustness of the majority judgment can have practical consequences: as shown in the last paragraph, the result of the vote could have been reversed in the example of the 2012 French election. This might be a decisive argument in favor of more robust rules, such as the usual judgment.

[18] The Kendall distance counts the number of pairwise disagreements between two rankings. In other words, it gives the minimal number of swaps between neighboring choices required to transform one ranking into the other.

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Appendix

A Applying the tie-breaking rules on real examples

Table 6 Different tie-breaking rules for the highest median applied on a poll on 2012 French presidential election (737 respondents). Data from Balinski & Laraki (2016).

choices	mean	MG	mj	d	s	n
Hollande	1.00	1+	1.450	1.018	1.010	1.076
Bayrou	0.79	1-	0.593	0.934	0.956	0.868
Sarkozy	0.48	0+	0.493	0.096	0.054	0.433
Mélenchon	0.25	0+	0.425	0.020	0.012	0.060
Dupont-Aignan	-0.60	-1+	-0.594	-0.934	-0.955	-0.870
Joly	-0.65	-1-	-1.385	-1.018	-1.012	-1.036
Poutou	-0.98	-1-	-1.457	-1.195	-1.136	-1.348
Le Pen	-0.27	-1-	-1.476	-1.015	-1.008	-1.119
Arthaud	-1.07	-1-	-1.499	-1.251	-1.168	-1.497
Cheminade	-1.16	-2	-1.520	-1.520	-1.510	-1.538
Hollande changed	0.98	1-	0.547	0.998	0.999	0.988

Table 7 Different tie-breaking rules for the highest median applied on a poll on 2017 French presidential election (1000 respondents). Data from Laraki's website.

choices	mean	MG	mj	d	s	n
Mélenchon	0.71	1-	0.644	0.999	0.999	0.998
Macron	0.49	1-	0.581	0.905	0.936	0.815
Hamon	0.12	0+	0.466	0.102	0.061	0.300
Dupont-Aignan	-0.15	0-	-0.448	-0.075	-0.046	-0.209
Le Pen	0.08	0-	-0.477	-0.021	-0.011	-0.157
Poutou	-0.26	0-	-0.485	-0.146	-0.089	-0.415
Fillon	-0.19	-1+	-0.514	-0.849	-0.908	-0.578
Lassale	-0.55	-1+	-0.564	-0.856	-0.901	-0.735
Arthaud	-0.52	-1+	-0.576	-0.868	-0.908	-0.768
Asselineau	-0.65	-1+	-0.610	-0.926	-0.948	-0.874
Cheminade	-0.73	-1+	-0.632	-0.955	-0.967	-0.927

Table 8 Different tie-breaking rules for the highest median applied on a 2017 citizens' primary for French presidential election (11304 voters). Data from laprimaire.org.

choices	mean	MG	mj	d	s	n
Marchandise	0.96	1+	1.477	1.193	1.127	1.405
Bernabeu	0.48	1-	0.558	0.826	0.878	0.701
Revon	0.47	1-	0.536	0.790	0.854	0.629
Bourgeois	0.08	0+	0.403	0.066	0.045	0.127
Pettini	0.04	0+	0.382	0.029	0.020	0.056
Mazuel	-0.06	0-	-0.388	-0.039	-0.027	-0.075
Fortané	-0.10	0-	-0.389	-0.058	-0.040	-0.103
Vitalis	-0.05	0-	-0.401	-0.019	-0.012	-0.043
Nonnez	-0.16	0-	-0.404	-0.073	-0.049	-0.137
Billaut	-0.19	0-	-0.431	-0.105	-0.069	-0.217
André	-0.31	0-	-0.479	-0.177	-0.113	-0.403
Bussard	-0.38	0-	-0.482	-0.233	-0.160	-0.432

Fig. 5 Grades for 2012 French presidential election (cf. Table 6).

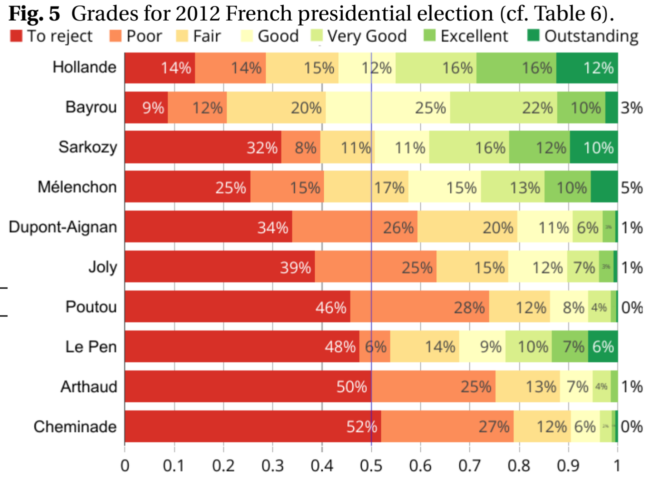


Fig. 6 Grades for 2017 French presidential election (cf. Table 7).

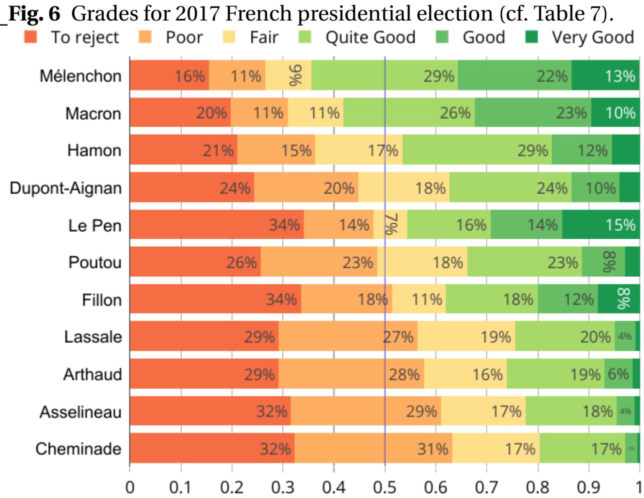


Fig. 7 Grades of a 2017 citizens' primary for French presidential election (cf. Table 8).

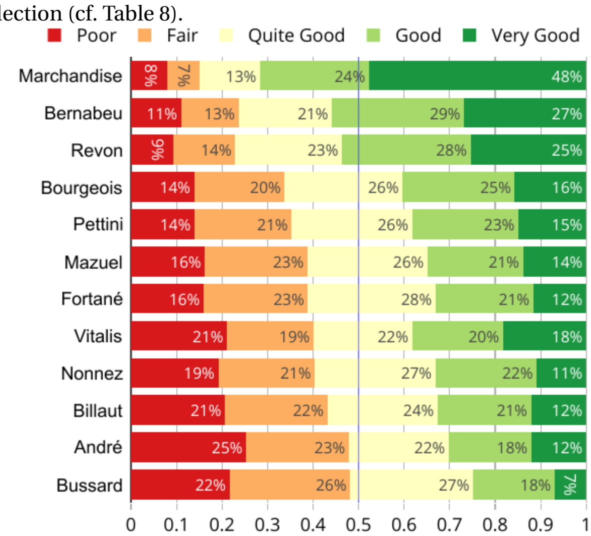
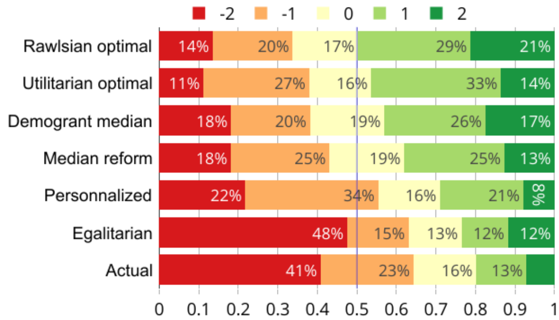


Table 9 Different tie-breaking rules for the highest median applied on a 2016 survey on French preferences over income distributions (1000 respondents). Data from Fabre (2017).

choices	mean	MG	mj	d	s	n
Rawlsian optimal	0.24	0+	0.497	0.160	0.096	0.484
Utilitarian optimal	0.11	0+	0.464	0.084	0.050	0.268
Demogrant median	0.04	0+	0.431	0.050	0.031	0.132
Median reform	-0.10	0-	-0.430	-0.050	-0.031	-0.132
Personnalized	-0.40	-1+	-0.554	-0.772	-0.828	-0.660
Actual	-0.78	-1-	-1.408	-1.051	-1.033	-1.109
Egalitarian	-0.75	-1-	-1.475	-1.106	-1.062	-1.341

Fig. 8 Grades of income distributions (cf. Table 9).



B Allowing for partial abstention

As some voters may not express an opinion over all choices, for example because there are plenty of choices, it is useful to allow for partial abstention. In this Appendix, we extend our setting in such a way, and show that all previous results hold. The formalization simply needs some adjustments.

The set of grades becomes $\mathscr{G} \cup \{\emptyset\}$, where $g_{c,v} = \emptyset$ indicates that voter v does not attribute any grade to c. The number of expressed grades for c is $E_c := \left|\{v \in \mathscr{V} \,|\, g_{c,v} \in \mathscr{G}\}\right|$. We then define α_c as the lower middlemost grade among expressed grades to c, and we define the shares of proponents and opponents to a choice relative to its number of expressed grades: $p_c^n := \frac{1}{E_c}\left|\{v \in \mathscr{V} \,|\, g_{c,v} > \min\{\alpha_c + n - 1; G\}\}\right|$ and $q_c^n := \frac{1}{E_c}\left|\{v \in \mathscr{V} \,|\, g_{c,v} < \max\{\alpha_c - n + 1; 1\}\}\right|$, for $n \in (0; G)$. We also redefine each order function $\ell^j(c)$, as the jth (lowest) quantile of expressed grades of c, and we adopt the convention that when this quantile falls between two grades of c, then $\ell^j(c)$ equals the lowest of the two. For example, assuming that there are two voters and that the grades of E are: $(g_{E,v}) = \begin{pmatrix} 0 & 1 \end{pmatrix}$, we have $\ell^j(E) = 0$ for $j \le \frac{1}{2}$ and $\ell^j(E) = 1$ for $j > \frac{1}{2}$; and in particular, $\alpha_E = \ell^{1/2}(E) = 0$. Then, the characterization of MJ as a lexi-order social-ranking function (see Section 3.1 and Example 2) requires that, when comparing grades of same rank, we move away from the median with a step small enough to capture any change in grade "at the right quantile".[19] Thus, we introduce Π, the least common multiple of $(E_c)_{c\in\mathscr{C}}$ (or more simply, $\Pi := \prod_{c\in\mathscr{C}} E_c$), and redefine the bijection π_{MJ} that characterizes MJ as a lexi-order social-ranking function as follows: $\pi_{MJ}(2k+1) = \frac{1}{2} - \frac{k}{2\Pi} - \frac{1}{4\Pi}$ and $\pi_{MJ}(2k) = \frac{1}{2} + \frac{k}{2\Pi} - \frac{1}{4\Pi}$ for $k \in [\![0; \Pi - 1]\!]$.[20]

With the appropriate formalization just described, all the results of the paper can be as easily derived when allowing for partial abstention as for the restrictive case used in the main text.

[19] For example, assume that among four voters, all attribute a grade to E: $(g_{E,v}) = \begin{pmatrix} -1 & 0 & 1 & 1 \end{pmatrix}$, but only two attribute a grade to F: $(g_{F,v})_{v|g_{F,v}\in\mathscr{G}} = \begin{pmatrix} 0 & 0 \end{pmatrix}$. As E and F share the same lower middlemost grade 0, the lexi-order characterization of MJ requires that we compare another rank, say a lower one (the counter-example should be adapted if we were to compare a higher one). One could naively think that comparing the first quartile would be natural, as $V = 4$. However, as $\ell^{1/4}(E) = -1 < 0 = \ell^{1/4}(F)$, this would lead to elect F, against the spirit of MJ and the ranking of mj scores: $mj_E = \frac{1}{2} > 0 = mj_F$.

[20] In our example, as $\Pi = 4$, the step $\frac{k}{2\Pi}$ between each (same side) rank used in the comparison is one eighth (and not one quarter), and one can check that this leads to electing E, as it should be.

C Other properties specific to the majority judgment

Resists manipulability In Theorem 13 of (Balinski & Laraki, 2014) and Theorem 13.5 of (Balinski & Laraki, 2011), BL show that electing the choice with the highest median grade minimizes manipulability. This result applies also to alternative tie-breaking rules. However, among social-ranking functions, MJ is the least manipulable because ties are resolved using middlemost grades. That being said, it is not clear if this theoretical nuance would have any behavioral implication in practice, as the different voting systems differ only by their tie-breaking rules, and elect the same choice in most of cases (see section 6).

*Other features*The following characteristics of MJ consists more in idiosyncratic features than in universally sought criteria for a rule.

Definition 10 (Balinski & Laraki, 2007) *Decisive for the center grades*: the ranking between A and B is the ranking determined by the middlemost grades unless that ranking is a tie; in that case, the ranking is determined by the residual grades.

Example 3 Majority judgment is decisive for the center grades (Balinski & Laraki, 2007), while D, S and N are not. Indeed, with the latter rules, choices with distinct middlemost grades can share the same primary score, in which case the tie is resolved using secondary score instead of comparing the middlemost grades. For example, take $\mathscr{C} = \{E; F\}$, $\Phi = \begin{pmatrix} g_{E,v} \\ g_{F,v} \end{pmatrix}_{v\in\mathscr{V}} = \begin{pmatrix} -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$, and consider the rule D. We have $\alpha_E = \alpha_F = 0$ and $\Delta_E = \Delta_F = \frac{1}{5}$, so that F is the winner for D as $\Delta_F^2 = \frac{1}{5} > 0 = \Delta_E^2$. Conversely, MJ decides with the middlemost grades and elects E.

Definition 11 (Balinski & Laraki, 2014) Suppose the first of the jth-middlemost intervals ($j \ge 0$) where A's and B's grades differ is the kth. A social-ranking function *rewards consensus* when all of A's grades strictly belong to the kth-middlemost interval of B's grades implies that A is ranked above B.

Example 4 Majority judgment rewards consensus (Balinski & Laraki, 2014), while D, S and N do not. For example, take $\mathscr{C} = \{E; F\}$ and $\Phi = \begin{pmatrix} g_{E,v} \\ g_{F,v} \end{pmatrix}_{v\in\mathscr{V}} = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$: MJ elects F (as lowest grades decide the ranking) while the alternatives elect E (because $\Delta_F = -\frac{1}{3} < 0 = \Delta_E$).

Proposition 4 *(theorem 17 in (Balinski & Laraki, 2007), theorem 15 in (Balinski & Laraki, 2014)) The majority-ranking is the unique monotone social-ranking function that is decisive for the center and rewards consensus.*