SABR Model and Optimal Delta Hedging

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Outline

1. Introduction
2. Data
3. SABR Model Calibration
4. Delta Calibration
5. Hedging Performance

Project Outline

1. SABR model calibration on SPX options data for one year time series
2. Delta calculations (BS delta, SABR delta, Bartlett's delta)
3. Hedging performance comparison for various deltas

Data Source

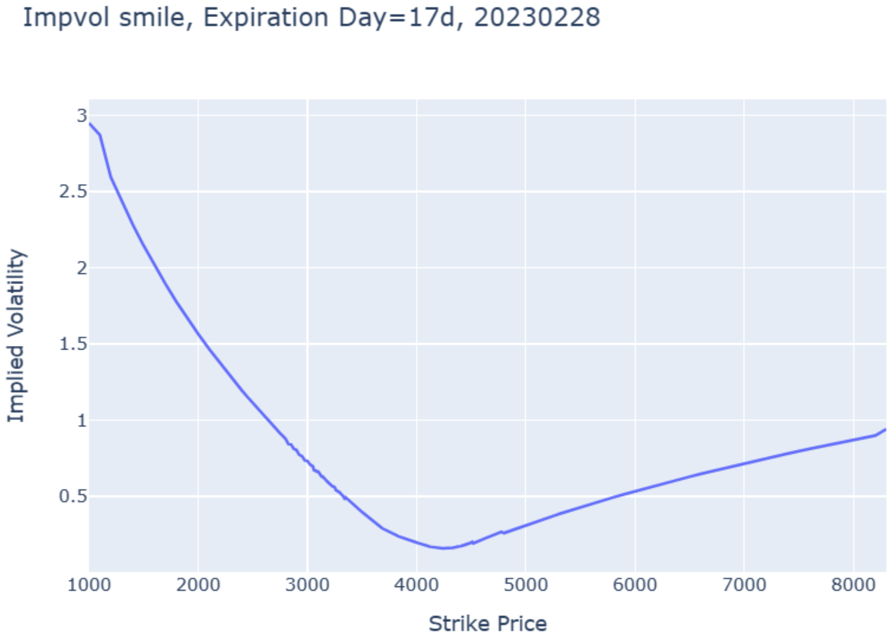
- Data source: SPX option data from WRDS and OptionMetrics. The data includes option strike prices, maturities, and hedge parameters based on the BS model. The period covered by the data we used is Mar 1, 2022 to Feb 28, 2023. The number of observation is 2093178.

date	exdate	Days_to_expiration	strike_price	impl_volatility
2022-03-01	2022-03-18	17 days	3100.0	0.724097
2022-03-01	2022-03-18	17 days	3125.0	0.716491
2022-03-01	2022-03-18	17 days	3150.0	0.706807
2022-03-01	2022-03-18	17 days	3175.0	0.696787
2022-03-01	2022-03-18	17 days	3200.0	0.686463
...	...	...	...	...
2023-02-28	2023-12-29	304 days	5025.0	0.136739
2023-02-28	2023-12-29	304 days	5100.0	0.137016
2023-02-28	2023-12-29	304 days	5200.0	0.138055
2023-02-28	2023-12-29	304 days	5300.0	0.140171
2023-02-28	2023-12-29	304 days	5400.0	0.142722

Figure 1: One year SPX option data

An example of the observed Implied volatility smile

This pattern aligns with the market experience, which shows the necessity to consider the implied volatility as a stochastic process.



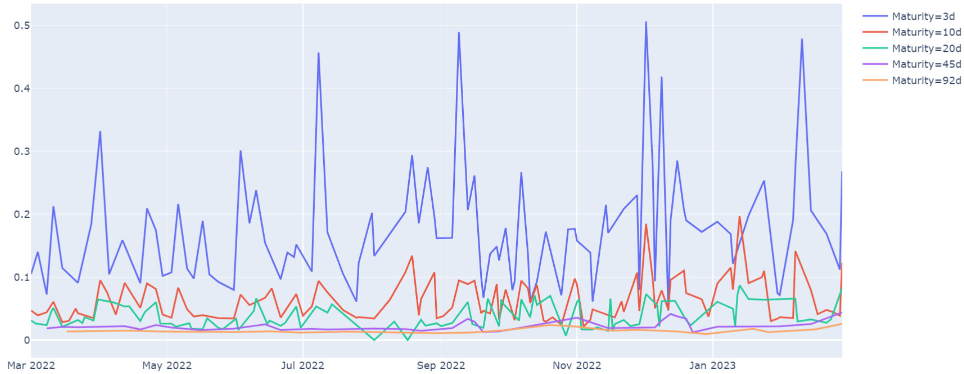
Single date: Different maturity results on 2023.02.28

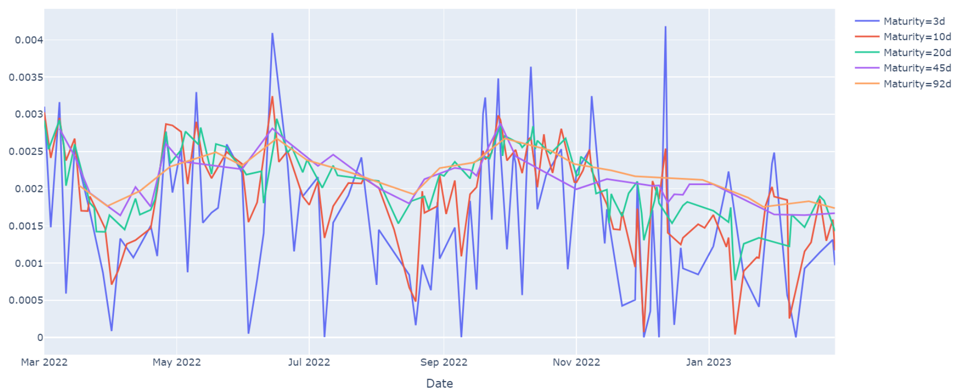
We use the lognormal SABR Model and calibrate model parameters non-linear least square algorithm.

Here we show the fitting performance in the strike price range of [3800,4100]. The model fits well for options with different maturities, especially better for near the money options.

Multiple dates: The time series of Sigma0 and Volvol

The evolution of calibrated SABR parameters is given below. The shorter the time to maturity, the more volatile the calibrated model parameters become.





Comparison between Different Deltas

The SABR delta and Bartlett's delta are very close, except that Bartlett's delta incorporates the adjustment for the implied volatility skew. Both the SABR delta and the Bartlett's delta have similar patterns with the BS Delta.

$$\Delta^{mod} \approx \Delta^{BS} + \text{Vega}^{BS} \times \eta$$

$$\Delta t^{\text{mod}} = \frac{\partial B}{\partial F} + \frac{\partial B}{\partial \sigma}\left(\frac{\partial \sigma_{\text{imp}}}{\partial F} + \frac{\partial \sigma_{\text{imp}}}{\partial \sigma}\frac{\rho\alpha}{C(F_t)}\right)$$

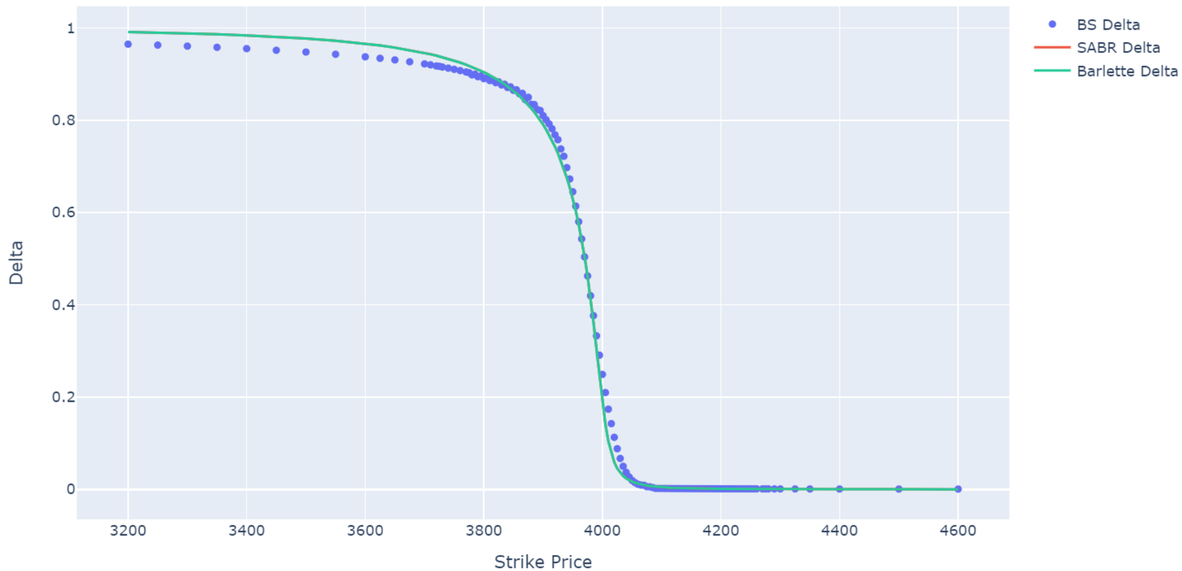


Figure 2: BS delta, SABR delta, and Bartlett's delta

Bartlett's Delta Visualization

SABR delta may lead to different conventional hedges especially near the money. Bartlett's delta is nearly independent of the beta choice and varies as the option maturity changes.

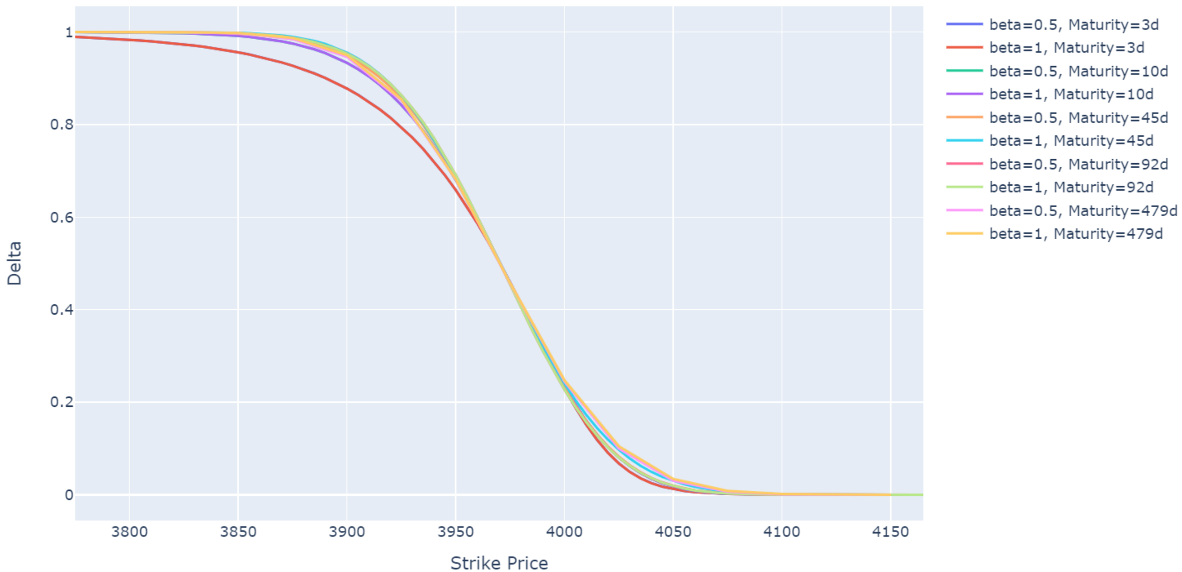


Figure 3: Bartlett's delta different T and β

Hedging Performance

Hedging Gain

In Hull, J., and White (2016), the effectiveness of a hedge is measured by the *Gain* metric, defined as the percentage reduction in the sum of squared residuals resulting from the hedge, i.e.

$$\text{Gain} = 1 - \frac{\sum(\Delta f - \delta_{\text{SABR}}\Delta S)^2}{\sum(\Delta f - \delta_{\text{BS}}\Delta S)^2}$$

where SSE denotes sum of squared errors. Using standard deviations rather than SSEs would produce a similar result but with a numerically smaller Gain.

1. Calculate adjacent SPX price changes and Option midquote changes
2. Calibrate SABR model parameters ($\beta = 1$) for each maturity and date
3. Calculate BS delta, SABR classic delta, and Bartlett's delta
4. Filter options (delta $\delta_{\text{BS}} \leq 0.05$, $\delta_{\text{BS}} \geq 0.95$, $T \leq 14$)
5. Bucketing: Create nine moneyness buckets according to δ_{BS} and seven different option maturity buckets

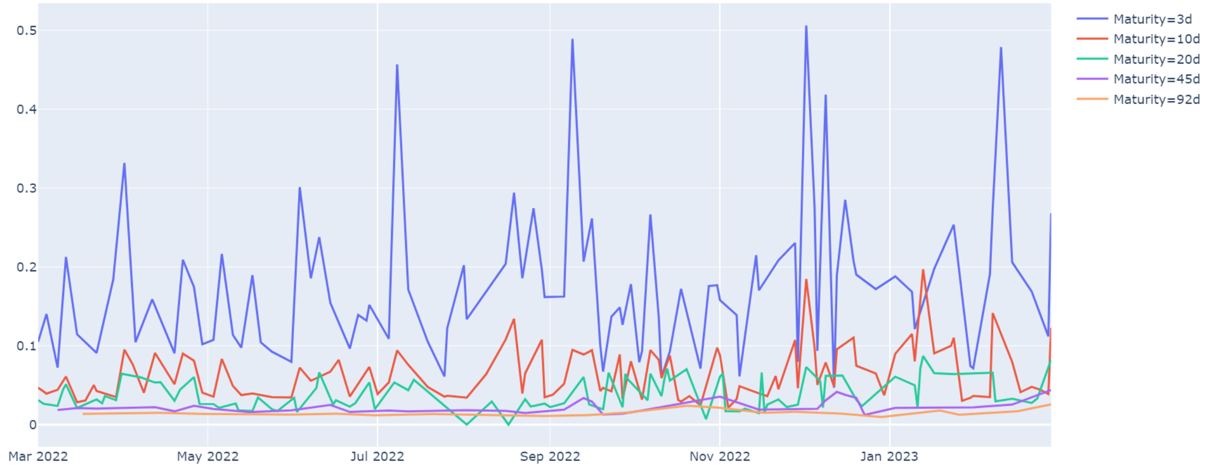


Figure 4: Vol of vol parameter evolution

Bucketing Results

- For call options, the Gain is largest for OTM options (22% for the highest strike options) and smallest (around 0.1%) for ITM options
- Positive gains are achieved for all 63 moneyness and maturity buckets
- Smaller the delta bucket, larger the Gain in general

Buckets	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0.12%	6.33%	13.78%	13.56%	15.78%	20.37%	22.72%
1	6.25%	9.98%	16.05%	15.16%	15.88%	16.80%	20.71%
2	6.95%	9.23%	14.09%	12.87%	14.76%	16.74%	17.23%
3	5.35%	6.99%	10.86%	09.32%	11.33%	12.23%	14.20%
4	3.06%	4.56%	6.66%	6.68%	8.11%	8.72%	9.76%
5	1.28%	2.67%	3.75%	4.20%	5.41%	5.96%	6.07%
6	0.35%	1.63%	2.44%	2.88%	3.46%	3.97%	4.24%
7	0.21%	1.17%	1.81%	1.92%	2.29%	2.76%	2.72%
8	0.59%	0.91%	1.38%	1.19%	1.32%	1.62%	1.27%

Table 1: Hedging Gains for Bartlett's Delta

Bucketing Results

- Using standard deviations in measuring gains produce similar yet numerically smaller results
- Include very short-term maturity options slightly worsens the gains due to large near the money gamma

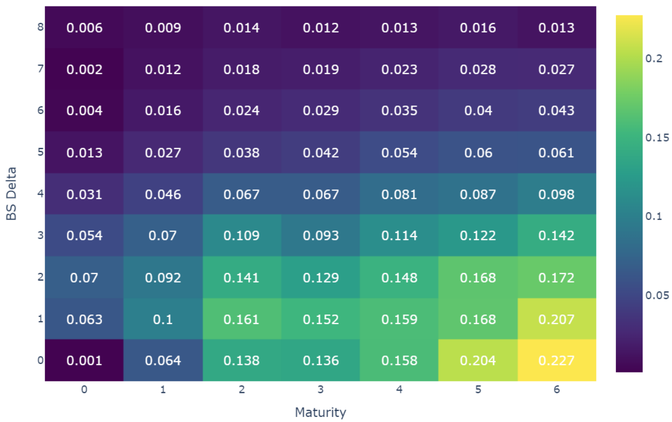


Figure 5: Bartlett's delta Gain (SSE)

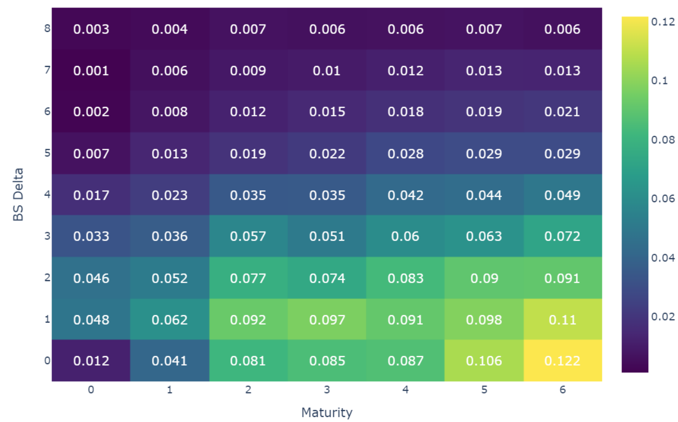


Figure 6: Bartlett's delta Gain (std)

SABR Delta vs Bartlett's Delta

SABR delta may lead to different conventional hedges especially near the money, yet Bartlett's delta is nearly independent of β and tends to provide more robust hedges especially for near the money options.

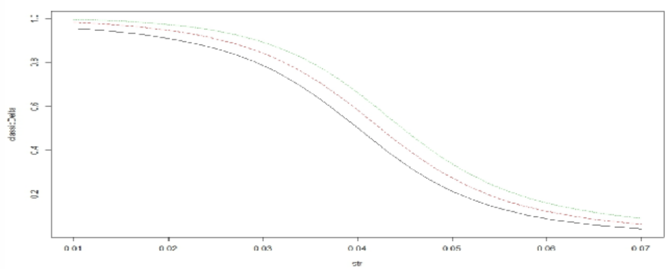


Figure 7: Classic SABR delta

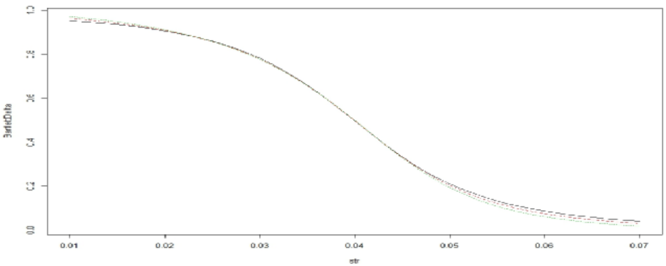


Figure 8: Bartlett's SABR delta

SABR Delta vs Bartlett's Delta

Empirical results show:

1. Bartlett's delta performs slightly but consistently better than SABR delta.
2. Bartlett's delta performs uniformly better especially for near the money options across all maturities, as shown in the relative gain graph below.

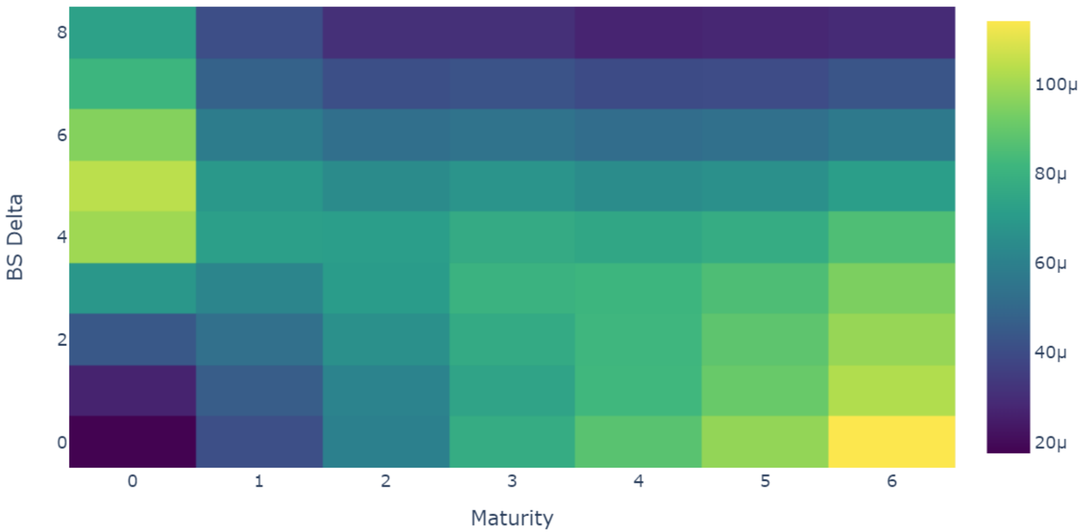


Figure 9: Relative Gain for Bartlett's delta