FACTORIZATION AND VERTEX ALGEBRAS FROM HOLOMORPHIC FIBRATIONS

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ABSTRACT. Let X be a complex manifold, $\pi : Y \to X$ a locally trivial holomorphic fibration with fiber F, and $(\mathfrak{g}, \langle \bullet, \bullet \rangle)$ a Lie algebra with an invariant symmetric form. We show how to associate to this data a holomorphic factorization algebra $\mathcal{F}_{Y/X}(\mathfrak{g})$ on X in the formalism of Costello-Gwilliam via a type of pushforward operation. When $X = \mathbb{C}$, this construction produces a vertex algebra which is a vacuum module for the universal central extension of $\mathfrak{g} \otimes H^0(F, \mathcal{O})[z, z^{-1}]$. In particular, when F is a torus $(\mathbb{C}^*)^n$, we recover a vertex algebra naturally associated to an $n+1$–toroidal algebra. We give an explicit description of the chiral homology of $\mathcal{F}_{Y/X}(\mathfrak{g})$.

1. INTRODUCTION

Suppose $\mathfrak{g}$ is a Lie algebra. By definition, a universal central extension of $\mathfrak{g}$ is a central extension of $\mathfrak{g}$ such that all other central extensions are pulled back from it. Universal central extensions are unique if they exist. As an example, consider the (infinite dimensional) Lie algebra $\mathfrak{g}[z, z^{-1}] = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ where $\mathfrak{g}$ is a complex simple Lie algebra. Here, $\mathbb{C}[z, z^{-1}]$ is the commutative algebra of Laurent polynomials and the resulting Lie algebra $\mathfrak{g}[z, z^{-1}]$ is known as the *loop algebra* of $\mathfrak{g}$. . In [?Garland] BW: is this reference correct? it is shown that there exists a universal central extension of $\mathfrak{g}[z, z^{-1}]$ of the form

$$\mathbb{C} \to \widehat{\mathfrak{g}} \to \mathfrak{g}[z, z^{-1}].$$

These extensions are known as *affine algebras* and are BW: ....

In [?Kassel], a generalization of this universal central extension is considered where $\mathbb{C}[z, z^{-1}]$ is replaced by an arbitrary commutative ring R. That is, one considers the Lie algebra $\mathfrak{g}_R = \mathfrak{g} \otimes R$ where the Lie bracket is defined by

$$[X \otimes r, Y \otimes s] = [X, Y] \otimes rs.$$

It is shown that there exists a universal central extension of the form

$$H_2^{\mathrm{Lie}}(\mathfrak{g}_R) \to \widehat{\mathfrak{g}}_R \to \mathfrak{g}_R.$$

Furthermore, when $\mathfrak{g}$ is simple there is an isomorphism of the Lie algebra homology $H_2(\mathfrak{g}_R) \cong \Omega^1_R/\mathrm{d}R$ where Ω^1_R is the R-module of Kähler differentials and the quotient is by all exact differentials. We will review the precise form of the cocycle defining this central extension below.

In this paper, we start with the data of a finite dimensional Lie algebra $\mathfrak{g}$ equipped with an invariant symmetric form and a holomorphic fibration $\pi : Y \to X$. To such data we will associate an extension of local Lie algebras BW: ??. In the case that $Y = X$ is a Riemann surface and π is the identity, the extension of local Lie algebras is modest enhancement of the usual affine algebra.

2. Preliminaries

2.1. **Lie algebras and universal central extensions.** Given a complex Lie algebra $\mathfrak{g}$ with invariant bilinear form $\langle \bullet, \bullet \rangle$, and a $\mathbb{C}$-algebra $\mathcal{R}$, $\mathfrak{g}_{\mathcal{R}} := \mathfrak{g} \otimes \mathcal{R}$ carries a natural Lie algebra structure with bracket

$$[X \otimes r, Y \otimes s] = [X, Y] \otimes rs.$$

The universal central extension $\widehat{\mathfrak{g}}_{\mathcal{R}}$ of $\mathfrak{g}_{\mathcal{R}}$ fits into a short exact sequence

$$0 \to \Omega^1_{\mathcal{R}}/\mathrm{d}\mathcal{R} \to \widehat{\mathfrak{g}}_{\mathcal{R}} \to \mathfrak{g}_{\mathcal{R}} \to 0$$

where $\Omega^1_{\mathcal{R}}$ denotes the Kähler differentials of $\mathcal{R}/\mathbb{C}$ and $\mathrm{d} : \mathcal{R} \to \Omega^1_{\mathcal{R}}$ is the universal derivation. The bracket on $\widehat{\mathfrak{g}}_{\mathcal{R}}$ is given by

$$[X \otimes r, Y \otimes s] = [X, Y] \otimes rs + \overline{\langle X, Y \rangle r ds}$$

where the second term lands in the quotient $\Omega^1_{\mathcal{R}}/\mathrm{d}\mathcal{R}$.

Example 2.1. An important class of examples is obtained by taking

$$\mathcal{R} = \mathcal{R}_n := \mathbb{C}[w_0^{\pm 1}, \cdots, w_n^{\pm -1}]$$

to be the algebra of functions on the complex $n+1$–dimensional torus $(\mathbb{C}^\times)^{n+1}$. $\widehat{\mathfrak{g}}_{\mathcal{R}_n}$ is called the $n+1$–toroidal Lie algebra for $n > 0$ and an affine Kac-Moody algebra for $n = 0$.

We may resolve $\widehat{\mathfrak{g}}_{\mathcal{R}}$ by an L_∞ algebra $\widetilde{\mathfrak{g}}_{\mathcal{R}}$ as follows. Let

$$\mathcal{K}_{\mathcal{R}} = \mathcal{R}[1] \xrightarrow{\mathrm{d}} \Omega^1_{\mathcal{R}}$$

and let

$$\phi^{(1)} : (\mathfrak{g}_{\mathcal{R}})^{\otimes 2} \to \Omega^1_{\mathcal{R}}$$

$$\phi^{(1)}((X \otimes r) \otimes (Y \otimes s)) = \langle X, Y \rangle (r\mathrm{d}s - s\mathrm{d}r)$$

and

$$\phi^{(0)} : (\mathfrak{g}_{\mathcal{R}})^{\otimes 3} \to R[1]$$

$$\phi^{(0)}((X \otimes r) \otimes (Y \otimes s) \otimes (Z \otimes t)) = \langle [X, Y], Z \rangle rst$$

We may view $\phi = \phi^{(0)} + \phi^{(1)}$ as an cochain in the cohomological Chevalley-Eilenberg complex

$$\mathcal{C}^*(\mathfrak{g}_{\mathcal{R}}, \mathcal{K}_{\mathcal{R}})$$

of total degree 2.

Lemma 2.2. *ϕ defines a cocycle in $\mathcal{C}^*(\mathfrak{g}_{\mathcal{R}}, \mathcal{K}_{\mathcal{R}})$ of total degree 2.*

Proof. One readily checks that $d\phi^0 + \mathrm{d}_{CE}\phi^{(1)} = 0$ and $\mathrm{d}_{CE}\phi^0 = \mathrm{d}\phi^{(1)} = 0$, which implies that $(\mathrm{d}_{CE} + \mathrm{d})\phi = 0$. □

SIGNS ! We may now use ϕ to define an L_∞ central extension $\widetilde{\mathfrak{g}}_{\mathcal{R}}$ of $\mathfrak{g}_{\mathcal{R}}$. As an $\mathcal{R}$-module, $\widetilde{\mathfrak{g}}_{\mathcal{R}} = \mathcal{K} \oplus \mathfrak{g}_{\mathcal{R}}$, and Taylor coefficients $l_1 = d, l_2 = [,] + \phi^{(1)}$, and $l_3 = \phi^{(0)}$. The following is immediate:

Proposition 2.3. $H^*(\widetilde{\mathfrak{g}}_{\mathcal{R}}, l_1) = \widehat{\mathfrak{g}}_{\mathcal{R}}$

2.2. **Vertex algebras.**

(1) generalities

(2) Construction and structure of V_F as a vertex algebra

2.3. **(Pre)-factorization algebras.**

(1) Pre-factorization algebras

(2) translation-invariant pre-fact algebras

(3) holomorphically translation-invariant pre-factorization algebras

(4) relations with vertex algebras in one dimension.

3. Factorization algebras from holomorphic fibrations

3.1. $dim(X) = 1$ **and vertex algebras.**

4. Factorization homology

(1) Discuss factorization algebra with base a compact complex manifold and potentially non-trivial torus bundle

(2) What is factorization homology or equivalently conformal blocks

(3) Compute factorization homology for compact base and tirivial/non-trivial torus bundle

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