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The Riemann-Roch theorem for compact Riemann surfaces

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THE RIEMANN-ROCH THEOREM FOR COMPACT RIEMANN SURFACES

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ABSTRACT

One of the basic theorems on compact Riemann surface is the Riemann-Roch theorem, which has received important developments through the recent works of Hirzebruch, Grothendieck, Atiyah and Singer. The purpose of this thesis is to present a proof of this basic theorem by using the well-known Hodge's harmonic decomposition theorem.

The first two sections contain the definitions of Riemann surfaces, holomorphic and meromorphic functions, differentials, the star operation and harmonic forms, and the statement of the Hodge's harmonic decomposition theorem.

In §§3,4 we discuss holomorphic and meromorphic differentials, and prove the residue and the existence theorems for meromorphic differentials.

§§5,6 give a brief review of the elementary combinatorial topology of surfaces, together with the normal forms of compact orientable surfaces.

In §7 we study the periods of holomorphic and meromorphic one-forms on compact Riemann surfaces.

In §8 we introduce divisors on a compact Riemann surface, and study their degrees and dimensions by considering the linear space $L(\mathfrak{a})$ of meromorphic functions associated with each fixed divisor $\mathfrak{a}$.

Finally, the promised proof of the Riemann-Roch theorem is given in §9.

INTRODUCTION

One of the basic theorems on compact Riemann surfaces is the Riemann-Roch theorem, which has received important developments through the recent works of Hirzebruch, Grothendieck, Atiyah and Singer. The purpose of this thesis is to present a proof of this basic theorem by using the well-known Hodge's harmonic decomposition theorem.

In §1 Riemann surfaces are defined. §2 contains the definitions of holomorphic and meromorphic functions on a Riemann surface R, together with some of their properties for compact R. Differentials, the star operation and harmonic forms are also introduced, and the Hodge's harmonic decomposition theorem is finally stated.

In §§3,4 holomorphic and meromorphic differentials are studied, and the residue and the existence theorems for meromorphic differentials are proved.

Triangulations of surfaces and barycentric subdivisions of simplexes are discussed in §5, and the normal forms of compact orientable surfaces are given in §6.

In §7 we obtain various properties of the periods of holomorphic and meromorphic one-forms on a compact Riemann surface.

In §8 we introduce divisors on a compact Riemann surface, and study their degrees and dimensions by considering the linear space $L(\mathfrak{a})$ of meromorphic functions associated with each fixed divisor $\mathfrak{a}$.

Finally, the promised proof of the Riemann-Roch theorem is given in §9.

§ 1. RIEMANN SURFACES

The idea of a Riemann surface has its foundations in the development of the multiple-valued behavior of certain complex-valued functions on the complex plane. In order to lead to the abstract definition of a Riemann surface, we must consider some elementary theory of analytic functions.

DEFINITION 1.1. An analytic function $w = w(z)$ of a complex variable z is called an algebraic function, if it satisfies an algebraic equation of the form

(1.1) $\quad a_0(z)w^n + a_1(z)w^{n-1} + \ldots + a_n(z) = 0$, $a_0(z) \neq 0$, where $a_i(z)$, $i = 0, 1, \ldots, n$, are polynomials in z with complex coefficients.

We note that for each value of z, we obtain a polynomial in w of degree n, for which, in general, equation (1.1) has no unique solution. Thus w is a multi-valued function of z.

We now give several examples of algebraic functions.

EXAMPLE 1. 1. Consider

$$w(z) = \sqrt{z} = \sqrt{r}\, e^{i\theta/2}$$

Pick some point z_0 with $\rho e^{i\theta}$, and consider a simple closed curve c through z_0 with the origin in its interior. Transversing the circle once, we find that

$$w(z_0) = \sqrt{\rho} e^{i(\theta + 2\pi)/2} = -\sqrt{\rho}\, e^{i\theta/2} .$$

In general, we have $w(z_0) = \sqrt{\rho}\, e^{i\theta/2}$ or $w(z_0) = -\sqrt{\rho}\, e^{i\theta/2}$ according as we transverse the circle c an even or odd number of times. Thus, $w(z) = \sqrt{z}$ is a two-valued function. To avoid this multiplicity, we form a surface of two sheets as seen in the elementary analytic function theory.

Using stereographic projection, we know that the extended complex plane is homeomorphic to a sphere. Therefore, we can consider the two sheets of the complex plane (each cut along the positive real axis) as spheres cut along a meridian circle from the south pole to the north pole. We now may deform each sphere without tearing into two hemispheres, and the two hemispheres may be pasted together to obtain a sphere. Analytically, the map $(z, \sqrt{z}) \to t = \sqrt{z}$ of the surface of two sheets onto the extended t-plane gives us this result. Hence, this particular surface is homeomorphic to a sphere.

EXAMPLE 1.2. Consider

$$w^2 = a(z - r_1)(z - r_2)(z - r_3) ,$$

where r_1, r_2, r_3 are distinct. In order to make this function single-valued, we use a surface of two sheets with a cut between r_1 and r_2 and another from r_3 to ∞. Again we see that each sheet is homeomorphic to a sphere with a cut. We may stretch each cut into a circular hole and pull the edges of the cuts outward to make tubes. By joining the tubes of the two spheres appropriately, we obtain a surface which can be topologically mapped onto a torus.

EXAMPLE 1.3. Consider

$$w^2 = a(z - r_1) \ldots (z - r_n) ,$$

where n is even, and $r_1,\ldots,r_n$ are distinct. We then see that the surface for which w is single-valued is homeomorphic to a sphere with $\frac{n}{2} - 1$ handles.

The sphere, torus, and a sphere with handles are examples of Riemann surfaces; the general definition of a Riemann surface is given as follows:

DEFINITION 1.2. A connected Hausdorff space R is called a <u>two-dimensional manifold</u>, if for each $x \in R$ and every open neighborhood N_x of x, there exists a homeomorphism ϕ of N_x onto an open set of the Euclidean plane.

DEFINITION 1.3. A two-dimensional manifold R is called a <u>Riemann surface</u> (complex analytic manifold), if

(i) for each $\chi \in R$ and each open neighborhood N_χ of χ, there exists an analytic homeomorphism of N_χ onto an open set of the Euclidean plane,

(ii) for each $\chi \in R$, there exists a local uniformizer such that can be expressed as an analytic function of the uniformizing parameter.

DEFINITION 1.4. A <u>canonical cut</u> on a Riemann surface is a simple closed curve c, which does not cut the surface into two pieces.

DEFINITION 1.5. The <u>genus</u> g of a Riemann surface is defined to be the maximum number of independent canonical cuts which can be performed on the surface without separating it.

Note that a sphere is of genus zero, since any simple closed curve deleted from the sphere produces a separated space. Also, a sphere with n handles is of genus n. The torus is of genus 1.

§ 2. FUNCTIONS AND DIFFERENTIALS

DEFINITION 2.1. Let R be an open region in the complex plane. f(z) is said to be <u>holomorphic</u> at $z_0 \in R$, if it has a derivative at every point in a certain neighborhood of z_0.

DEFINITION 2.2. Let R be an open region in the complex plane. f(z) is said to be <u>meromorphic</u> in R, if it is single-valued in R, and is either holomorphic or has a pole at every point $a \in R$.

Let R be a compact complex analytic manifold of one complex dimension (closed Riemann surface). Then we may define holomorphic and meromorphic functions on R.

LEMMA 2.1. *Let R be a closed Riemann surface and f a holomorphic function defined on R. Then f is constant.*

PROOF. Suppose there exists a non-constant holomorphic function $f: R \to C$, where C is the complex plane. Since R is compact, there exists $w \in R$ such that $|f(x)| \leq |f(w)|$ for all $x \in R$. Since there exists a neighborhood U of w, which is holomorphically equivalent to the disk $D = \{ z: |z| < 1\}$, that is, there exists a holomorphic function $\phi: D \to U$ such that $\phi(0) = w$, $g = f \circ \phi$ is holomorphic in D with a maximum at 0, contradicting the maximum modulus principle. Hence f is constant.

LEMMA 2.2. *Let R be a closed Riemann surface, and f a meromorphic function defined on R. Then f has at most a finite number of poles.*

PROOF. Assume f has infinitely many poles. By the Bolzano-Weierstrass theorem, the set W of all poles of f has a cluster point $a \in R$. f is not holomorphic at a, since every neighborhood U of a contains points of W. f does not have a pole at a, since a is not an isolated point of W. Thus f is not a meromorphic function, contradicting the hypothesis. Hence the lemma is proved.

LEMMA 2.3. *Let R be a closed Riemann surface, and f a meromorphic function defined on R with values on the Riemann sphere S. Then either f maps R onto S, or f is constant.*

PROOF. Assume f is non-constant. By Lemma 2.2, we know f is discontinuous at most at a finite number of points $P_1,\ldots,P_n$. Let $\{S_n, n \in D\}$ be a net in $R - \{P_i\}$ converging to P_i. Since $f(z) \to \infty$ as $z \to P_i$, $f(S_n) \to \infty$ as $n \to \infty$. But $\infty = f(P_i)$. Thus f is continuous at P_i, and therefore at every point of R. Since R is compact, f[R] is compact. But $f[R] \subset S$, and S is Hausdorff. Thus f[R] is closed. From elementary analytic function theory, we know that a non-constant meromorphic function is an open map. Therefore f[R] is open. Since S is connected and $f[R] \neq \emptyset$, $f[R] = S$.

In order to determine all meromorphic functions on R, we must consider also differentials. From the analytic structure of our surface, in a local co-ordinate neighborhood we have a local uniformizing variable

(2.1) $$z = x + iy,$$

where $i = \sqrt{-1}$.

DEFINITION 2.1. A <u>differential</u> <u>of</u> <u>degree</u> 1 (<u>exterior</u> <u>one-form</u>) on a Riemann surface is a linear expression of the form

(2.2) $$w = Adx + Bdy,$$

where $A = A(x,y)$ and $B = B(x,y)$ are class C^{∞} complex-valued functions.

Let $\bar{z}$ be the complex conjugate of z. Throughout this paper a bar over a letter will always denote the conjugate of the complex number defined by the letters. Then from equation (2.1) it follows that

(2.3) $$dz = dx + idy, \quad d\bar{z} = dx - idy.$$

By putting $A_1 = A/2 - i\,B/2$, $B_1 = A/2 + i\,B/2$, and using equations (2.2), (2.3) we can easily show that

$$A_1 dz + B_1 d\bar{z} = w.$$

If $t = u + iv$ is another local uniformizing variable, then $t = t(z)$ is a holomorphic function of z in the common domain of z and t, so that the Cauchy-Reimann equations

(2.4) $$u_x = v_y, \quad u_y = -v_x$$

are satisfied, where u_x denotes the partial derivative of u with respect to x, and etc. It should be noted that equations (2.4) can be written in the condensed form, namely, $\partial t/\partial \bar{z} = 0$. Thus we may extend w given by equation (2.2) to $w = A'du + B'dv$ in the t-co-ordinate system in such a manner that on the common domain of z and t, A' and B' are related to A and B by

$$(2.5)\qquad A = A'u_x + B'v_x \quad , \; B = A'u_y + B'v_y \; .$$

Due to equations (2.4), equations (2.5) are a linear transformation, which is real orthogonal up to a stretching factor $u_x^2 + u_y^2$. Hence the operations

$$(2.6)\qquad w \to \bar{w} = \bar{A}dx + \bar{B}dy \; ,$$

$$(2.7)\qquad w \to w = -\bar{B}dx + \bar{A}dy$$

are intrinsic.

LEMMA 2.4. *Let* $z = x + iy$ *be a local uniformizing variable on an open set* U *of* R. *Let* w, η *and* f, g *be, respectively, differential forms and complex-valued functions defined on* U.

Then

$$(2.8)\qquad (\overline{fw + g\eta}) = \bar{f}\,\bar{w} + \bar{g}\,\bar{\eta} \; ,$$

$$(2.9)\qquad (\overline{\bar{w}}) = w \; ,$$

$$(2.10)\qquad *(fw + g\eta) = \bar{f}(*w) + \bar{g}(*\eta) \; ,$$

$$(2.11)\qquad *(*w) = -w \; ,$$

$$(2.12)\qquad *(i*w) = iw \; ,$$

$$(2.13)\qquad *(\bar{w}) = (\overline{*w}) \; ,$$

$$(2.14)\qquad d(\bar{w}) = (\overline{dw}) \; ,$$

d *being the exterior differentiation,*

$$(2.15)\qquad *(Adz + Bdz) = -i\,\bar{B}d\bar{z} + i\,\bar{A}d\bar{z} \; ,$$

$$(2.16)\qquad w \wedge *\eta = \overline{\eta \wedge * w} \; ,$$

$$(2.17)\qquad w \wedge * w = (A^2 + B^2)\, dx \wedge dy \; ,$$

provided that $w = Adx + Bdy$.

The proof of this lemma is straightforward, and is therefore omitted here.

LEMMA 2.5. *A form* w *can be expressed as*

$$w = A(x,y)dz \tag{2.18}$$

in each local uniformized z, *if and only if*

$$*w = i\bar{w} \; . \tag{2.19}$$

PROOF. Lemma 2.5 follows immediately from equation (2.15). In fact, if equation (2.18) is true, then equation (2.15) implies equation (2.19) since $B = 0$. Conversely, suppose $w = Adz + Bdz$. Then equation (2.19) implies that

$$*(Adz + Bd\bar{z}) = i(\bar{A}d\bar{z} + \bar{B}dz) \; .$$

Comparison of the right side of this equation and equation (2.15) thus gives $B = 0$.

DEFINITION 2.2 A one-form w is said to be *harmonic*, if

$$dw = d(*w) = 0. \tag{2.20}$$

An important result on harmonic forms used in this paper is the following

THEOREM 2.1 (OF HODGE). *If* w *is a* C^{∞} *real one-form on a compact Riemann surface* R, *then* w *has a unique decomposition*

$$w = w_1 + w_2 + w_3 \; ,$$

where $w_1 = \theta$ *is harmonic*, $w_2 = de$ *and* $w_3 = *df$, e *and* f *being* C^{∞} *functions on* R.

By separating real and imaginary parts, Theorem 2.1 can easily be extended to complex differentials.

In particular, if $dw = 0$, from the Hodge's theorem it can be shown that

$$w = \theta + de \; ,$$

which means that each De Rham cohomology class contains a unique harmonic form.

COROLLARY 2.1. <u>The space of real harmonic forms on a compact Riemann surface</u> R <u>is a</u> 2g <u>dimensional linear space over the reals, where</u> g <u>is the genus of</u> R.

§ 3. HOLOMORPHIC DIFFERENTIALS

DEFINITION 3.1. A differential w is <u>holomorphic</u> or of <u>first kind</u>, if locally $w = f(z)dz$, where f is holomorphic.

LEMMA 3.1. <u>A one-form</u> w <u>is holomorphic if and only if</u>

$$*w = i\,\bar{w}, \tag{3.1}$$

$$dw = 0\ . \tag{3.2}$$

PROOF. By Lemma 2.5, condition (3.1) implies that $w = Adz$, and conversely. Thus $dw = (\partial A/\partial \bar{z})\, d\bar{z} \wedge dz$, and $dw = 0$ means that $\partial A/\partial \bar{z} = 0$, or A is holomorphic.

LEMMA 3.2. <u>If</u> θ <u>is harmonic, then</u> $w = \theta + i * \bar{\theta}$ <u>is holomorphic</u>.

PROOF. Since θ is harmonic, by Definition 2.2

$$dw = d\theta + id * \bar{\theta} = 0.$$

On the other hand, by equations (2.11), (2.12) we have

$$*w = *(\theta + i * \bar{\theta}) = *\theta + i\,\bar{\theta} = i(\bar{\theta} - i\,*\theta).$$

From equations (2.9), (2.13) it follows that the right side of the above equation is equal to $i\,\bar{w}$. Thus $*w = i\,\bar{w}$, and w is holomorphic due to Lemma 3.1.

Thus, if θ is real harmonic, then $w = \theta + i * \theta$ is holomorphic. Conversely,

LEMMA 3.3. If w is holomorphic, then

$$\theta = (w + \bar{w})/2 \tag{3.3}$$

<u>is real harmonic, and</u>

(3.4) $$w = \theta + i(*\theta).$$

PROOF. Since w is holomorphic, dw = 0. By equation (2.14), $d\bar{w} = (\overline{dw}) = 0$. Thus $d\theta = 0$. Moreover, from equations (3.1), (2.13), (2.14), we have

$$d(*w) = d(i\,\bar{w}) = i d\bar{w} = 0 ,$$

$$d(*\bar{w}) = d(\overline{*w}) = (\overline{d*w}) = 0.$$

Thus $d*\theta = 0$, and θ is real harmonic by Definition 2.2.

Similarly, from equations (3.3), (3.1), (2.13) we obtain

(3.5) $$*\theta = (*w + *\bar{w})/2 = (i\,\bar{w} + \overline{*w})/2 = i(\bar{w} - w)/2 .$$

Elimination of $\bar{w}$ from equations (3.3), (3.5) gives immediately equation (3.4).

THEOREM 3.1. <u>The space of holomorphic differentials on a compact Riemann surface R has complex dimensions g, where g is the genus of R.</u>

PROOF. Let θ be a real harmonic differential on R, and consider the map $C: \theta \to \theta + i*\theta$. By Lemmas 3.2 and 3.3, C is a map of the real vector space of real harmonic differentials K onto the space of holomorphic differentials K'. Since

$$C(\theta_1 + \theta_2) = C(\theta_1) + C(\theta_2),\ C(f\theta) = f\,C(\theta) ,$$

where f is a real-valued function on R, C is an isomorphism. Thus K and K' must have the same dimension. Since, by Corollary 2.1, K has complex dimension g, so does K'.

§ 4. MEROMORPHIC DIFFERENTIALS

DEFINITION 4.1. A <u>meromorphic</u> or <u>abelian differential</u> is of the form $\eta = f(z)dz$, where f(z) is a meromorphic function.

From Lemma 2.2 we know that on a compact Riemann surface R a meromorphic function f has at most a finite number of poles. Thus on R we may write a mermomorphic differential in the form

$$(4.1) \qquad \eta = (a_{-n} z^{-n} + \dots + a_{-1} z^{-1} + a_0 + a_1 z + \dots)dz .$$

DEFINITION 4.2. In equation (4.1), if $a_{-n} \neq 0$, then $-n$ is called the <u>order</u> of η at the point P where $z = 0$, and a_{-1} the <u>residue</u> of η, denoted by $a_{-1} = \operatorname{res}_p \eta$.

The residue is independent of the uniformized z, as can be seen by algebraic computation or by the evident formula

$$a_{-1} = \frac{1}{2\pi i} \oint_{\gamma} \eta ,$$

where γ is a small cycle surrounding P once.

THEOREM 4.1 (RESIDUE THEOREM). <u>If η is a meromorphic differential, then</u>

$$(4.2) \qquad \sum_{P} \operatorname{res}_p(\eta) = 0.$$

PROOF. Let $P_1, \dots, P_n$ be the poles of η, and surround each pole P_i by a circular disc D_i with boundary γ_i, $i = 1, \dots, n$. Delete the open sets $D_i - \gamma_i$, $i = 1, \dots, n$, from R, and consider the remainder of R to be Σ, so that its boundary is $\partial\Sigma = -\gamma_1 - \dots - \gamma_n$. Then by Cauchy's and Green's theorems,

$$\sum_{j=1}^{n} \operatorname{res}_{p_j} \eta = \frac{1}{2\pi i} \sum_{j=1}^{n} \oint_{\gamma_j} \eta = -\frac{1}{2\pi i} \oint_{\partial\Sigma} \eta$$

$$= -\frac{1}{2\pi i} \iint_{\Sigma} d\eta = 0,$$

since η is holomorphic on Σ, and $d\eta = 0$ on Σ by Lemma 2.1.

DEFINITION 4.3. Let f be a non-constant meromorphic function on a Riemann surface R, and at a point P(z), $f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots$, $a_n \neq 0$. Then n is called the <u>order</u> of f at P, and we write $n = \nu_p(f)$.

REMARK. If $n = 0$, f and its reciprocal are holomorphic at P. If $n > 0$, f has a zero of order n at P. If $n < 0$, f has a pole of order $(-n)$ at P.

DEFINITION 4.4. We define the logarithmic derivative of f as follows:

$$\eta = df/f = (n\, z^{-1} + \text{reg})\, dz \;,$$

so that

$$\text{res}_p \eta = \text{res}_p(df/f) = V_p(f). \tag{4.3}$$

COROLLARY 4.1. Counted with multiplicities, each non-constant meromorphic function f has the same number of zeros as poles on R. Furthermore, for each complex number a, the number of a-places of f is equal to the number of poles of f.

PROOF. Let C be a simple closed rectifiable orientable curve on R such that all the zeros and poles of f on R are in the interior of C. By a theorem in elementary analytic function theory, we then have

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz = N - P = \sum_P V_p(f) \;, \tag{4.4}$$

where N and P are, respectively, the numbers of zeros and poles of f on R with proper multiplicities. A combination of equations (4.2), (4.3), (4.4) shows immediately that $N = P$, which is the first part of our corollary.

Now, consider $g = f - a$, which is meromorphic on R. Furthermore, if $z_1 \in R$ such that $f(z_1) = a$, then $g(z_1) = 0$. By applying the first part of our corollary to g, we know that on R the number of zeros of g equals the number of poles of g. But the former equals the number of a-places of f, and the latter the number of poles of f. Hence the Corollary is proved.

It should be noted that this Corollary strengthens Lemma 2.3 in such a way that each non-constant meromorphic function f on a compace Riemann surface R not only covers the Riemann sphere S, but each point of S is covered the same number of times.

The residue theorem asserts that the principal parts of a meromorphic differential must satisfy a linear condition, namely, the residue sum must vanish. The following existence theorem tells us that once preassigned principal parts do satisfy this condition, then there exists a corresponding meromorphic differential.

THEOREM 4.2 (EXISTENCE THEOREM). *Let the following be assigned: points* $P_1, \ldots, P_k$; *corresponding local uniformizing variables* $z_1, \ldots, z_k$; *principal parts*

$$\eta_j = (a_{j,-n_j} z_j^{-n_j} + \ldots + a_{j,-1} z_j^{-1}) dz_j \quad (j = 1, \ldots, K),$$

such that $\sum_{j=1}^{K} a_{j,-1} = 0$. *Then there exists a meromorphic differential* η on R *having precisely these principal parts for its singularities*.

REMARK. The proof below uses only the Hodge theorem, a result on differentials *without* singularities. It is interesting to note that this is so strong as to imply the critical result about differentials with singularities.

PROOF. We select non-negative real functions ϕ_j ($j = 1, \ldots, k$) such that (a) ϕ_j is constant one in a neighborhood U_j of P_j, (b) vanishes outside a neighborhood of P_j, and (c) the supports of the ϕ_j are disjoint, and the supports of each ϕ_j is included in the domain of z , (support of ϕ_j: $s(\phi_j) = \{z: \phi_j(z) = 0\}^-$). Then $\alpha = \sum_{j=1}^{K} \phi_j \eta_j$

is a well defined differential on R except at the points $P_1, \ldots, P_k$. Also except at the points $P_1, \ldots, P_k$, all differentials $\eta_1, \ldots, \eta_k$ are holomorphic, and therefore by Lemma 3.1, $*\eta_j = i\,\overline{\eta_j}$ for $j = 1, \ldots, k$. Since ϕ_j are real functions, we have

(4.5) $$*\alpha = i\,\overline{\alpha}\,.$$

Enclose P_j in a disc V_j such that $V_j \subset U_j$, and set

$$\Sigma_0 = R - \bigcup_{i=1}^{n} V_i\,, \quad \Sigma_1 = \bigcup_{i=1}^{n} U_i\,.$$

Thus, our problem is to find a meromorphic differential η on R such that

(i) η is holomorphic on Σ_0,

(ii) $\eta - \alpha$ is holomorphic on Σ_1.

By Lemma 3.1, conditions (i) and (ii) are equivalent to

(i)' $*\eta = i\,\overline{\eta}$,

$d\eta = 0$, on Σ_0,

(ii)' $*(\eta-\alpha) = i(\overline{\eta-\alpha})$,

$d(\eta-\alpha) = 0$, on Σ_1.

Now we change our unknown η to $\lambda = \eta - \alpha$, so that $\eta = \lambda + \alpha$. By using equation (4.5), conditions (i)' and (ii)' then become

(i)'' $*\lambda = i\,\overline{\lambda}$,

$d\lambda = -d\alpha$, on Σ_0

(ii)'' $*\lambda = i\,\overline{\lambda}$,

$d\lambda = 0$, on Σ_1.

Define a 2-form Φ on R as follows:

$$\Phi = \begin{cases} -d\alpha\,, & \text{on } \Sigma_0\,, \\ 0\,, & \text{on } \Sigma_1\,. \end{cases}$$

This definition is possible, since $d\alpha = 0$ on $\Sigma_0 \cap \Sigma_1$. In fact, on U_j, we have $\phi_j \equiv 1$, and $\phi_i \equiv 0$ for all $i \neq j$, since the supports of $\phi_1, \ldots, \phi_k$ are disjoint. Hence, on $\Sigma_0 \cap \Sigma_1 \cap U_j$, $\alpha = \eta_j$, and $d\alpha = d\eta_j = 0$ due to equation (3.2) for η_j.

From (i)'' and (ii)'', the problem is reduced to obtaining a form λ on R without singularity satisfying

(4.6) $\quad * \lambda = i \bar{\lambda}$,

(4.7) $\quad d\lambda = \Phi$.

By the two-dimensional De Rham Theorem, a necessary and sufficient condition that equation (4.7) alone has a solution is that

$$\iint_R \Phi = 0.$$

This indeed is the case, since

$$\iint_R \Phi = -\iint_{\Sigma_0} d\alpha = -\oint_{\partial\Sigma_0} \alpha = \sum_j \oint_{\partial V_j} \alpha$$

$$= \sum_j \oint_{\partial V_j} \eta_j = 2\pi i \sum_j \operatorname{res}_{p_j} \eta_j = 0,$$

the last step being obtained by applying Theorem 4.1 to η_j. Thus there exists a one-form β on R such that $d\beta = \Phi$. The proof of our Theorem will be complete when we establish

LEMMA 4.1. <u>If β is a one-form on R, then there exists a one-form satisfying</u>

(4.8) $\quad * \lambda = i \bar{\lambda}$,

$\quad d\lambda = d\beta$.

PROOF. Before writing down the solution λ, we can motivate the choice by the following consideration. From equations (4.8), (4.9) we have

(4.10) $$d * \lambda = i\, d\bar{\lambda} = i\, d\bar{\beta},$$

so that if λ_1 and λ_2 are two solutions, then $\lambda_1 - \lambda_2$ is harmonic, that is,

$$d(\lambda_1 - \lambda_2) = d*(\lambda_1 - \lambda_2) = 0.$$

By Hodge's theorem,

(4.11) $$\beta = \mu + *df,$$

where $d\mu = 0$, and f is a C^∞ function on R. Equations (4.10), (4.11) suggest us to consider

(4.12) $$\lambda = *df - i\, d\bar{f}.$$

Then

$$*\lambda = -df + i * d\bar{f}, \quad \bar{\lambda} = *d\bar{f} + i\, df,$$

from which and equations (4.11), (4.12), we can easily verify conditions (4.8), (4.9). Hence λ is the required form.

§ 5. COMBINATORIAL TOPOLOGY

DEFINITION 5.1. A Euclidean 0-simplex is a point, a Euclidean 1-simplex is a closed line segment, and a Euclidean 2-simplex is a closed triangular plane region.

DEFINITION 5.2. An n-simplex S^n, n = 0,1,2, on a manifold M is a pair $[e^n, \phi]$, where e^n is a Euclidean n-simplex, and ϕ is a 1-1 bicontinuous mapping of e^n into M. We shall denote $\phi(e^n) = |S^n|$.

DEFINITION 5.3. Let M be a two-dimensional manifold, and Δ a collection of 2-simplexes defined on M such that each point $P \in M$ belongs to at least one member of Δ. Δ is called a triangulation of M if the following conditions are satisfied:

(i) if $P \in S^2 \in \Delta$, P does not belong to the boundary ∂S^2 of S^2, then S^2 is the only triangle containing P, and $|S^2|$ is a neighborhood of P,

(ii) if P belongs to an edge S' of $S_1^2 \in \Delta$, but is not a vertex of S_1^2, then there exists exactly one other triangle $S_2^2 \in \Delta$ such that $|S_1^2| \cap |S_2^2| = |S'|$, S_1^2 and S_2^2 are the only triangles containing P, and $|S_1^2| \cup |S_2^2|$ is a neighborhood of P,

(iii) if P is a vertex of S_1^2, there is a finite number of triangles $S_1^2, \ldots, S_k^2$ of Δ, each having P as a vertex such that each successive pair of triangles S_j^2, S_{j+1}^2 ($j = 1, \ldots, k-1$) have only one edge in common, S_k^2 has one edge in common with S_1^2, $S_1^2, \ldots, S_k^2$ are the only triangles containing P, and $|S_1^2| \cup \ldots \cup |S_k^2|$ forms a neighborhood of P. The triangles $S_1^2, \ldots, S_k^2$ are said to form a <u>star</u> of triangles in Δ.

DEFINITION 5.4. If a triangulation Δ of a two-dimensional manifold M exists, then M is said to be <u>triangulable</u>, and can be denoted by M_Δ.

THEOREM 5.1. <u>Every closed surface is triangulable</u>.

The proof due to T. Rado, can be found in the paper of Szeged Acta (1925), pp. 101-121.

THEOREM 5.2. <u>Any compact set on a triangulated surface S_Δ meets only a finite number of triangles in Δ</u>.

PROOF. Assume a compact set K on S_Δ meets infinitely many triangles in Δ. Choose a point of K in each triangle. By the compactness of K, this set G has a cluster point P, and therefore any open set about P has infinitely many points of G. But, by Definition 5.3, P is contained in an open set, which meets only a finite number of triangles in Δ. Thus we obtain a contradiction, and the theorem is proved.

THEOREM 5.3. A surface S is compact if and only if it has a finite triangulation.

PROOF. Assume S is compact. By Theorems 5.1, 5.2, S has a triangulation Δ, and meets only a finite number of triangles in Δ. Hence Δ is finite.

Conversely, assume S has a finite triangulation Δ. Since each triangle of Δ is a compact set, the union of a finite number of triangles is compact, and therefore S is compact.

DEFINITION 5.5. A chain of triangles is a finite collection of triangles $s_1, \ldots, s_n$, of which each pair of successive triangles has a common edge. s_1 and s_n are said to be joined by this chain.

DEFINITION 5.6. In the Euclidean plane E^2, let $P_o, P_1, \ldots, P_n$ be vertices of a Euclidean n-simplex e^n, $n = 0,1,2$, and select a coordinate system (x_1, x_2) so that P_k has coordinates (x_{1k}, x_{2k}), $0 \leq k \leq n$. On each vertex P_k assume a point mass μ_k such that $\mu_k \geq 0$ and $\sum_{k=o}^{n} \mu_k = 1$. This distribution has a centroid $P = (x_1, x_2)$ given by $x_j = \sum_{k=o}^{n} \mu_k \, x_{ik}$, $j = 1,2$.

$\mu_o, \mu_1, \ldots, \mu_n$ are called the barycentric coordinates of the point P in e^n.

REMARK. The barycentric coordinates are independent of the coordinate system used.

DEFINITION 5.7. A barycentric mapping of an n-simplex e_1^n onto an r-simplex e_2^r, $r \leq n$, is a mapping, which takes each vertex of e_1^n into a vertex of e_2^r, and makes each vertex of e_2^r the image of at least one vertex of e_1^n, such that if the masses $\mu_1, \ldots, \mu_n$ are placed at the vertices of e_1^n, and the same masses are placed at the image vertices in e_2^n, then the centroids of the two systems e_1^n and e_2^n correspond to each other.

REMARK. The above mapping is said to be degenerate if $r < n$, and is a homeomorphism if $r = n$.

EXAMPLE 5.1. The vertices P_o, P_1, P_2 of a 2-simplex have barycentric coordinates (1,0,0), (0,1,0), and (0,0,1), respectively. A point P lying on the segment $\overline{P_oP_1}$ has barycentric coordinates $(\mu_o, \mu_1, 0)$, and the locus of points, for which $\mu_o = \mu_1$, is the median line from P_2 to the edge $\overline{P_oP_1}$.

DEFINITION 5.8. The <u>barycentric</u> <u>subdivision</u> of an n-simplex e^n, n = 0,1,2, is the collection of n-simplexes obtained by considering the locus of points for which any two of the barycentric coordinates are equal.

EXAMPLE 5.2. (i) If e^n is the 0-simplex, the barycentric subdivision is the same 0-simplex.

(ii) If e^n is a 1-simplex, the mid-point is the only point with two equal barycentric coordinates. This mid-point divides the line segment e^1 into two equal segments.

(iii) The 2-simplex is divided into six triangles by the three medians.

DEFINITION 5.9. An n-simplex s^n, n = 0, 1,2 on a manifold M is said to be <u>oriented</u>, if its n − 1 vertices are specified in a definite order.

EXAMPLE 5.3. The 0-simplex with one vertex P_o has two orientations, denoted by $< P_o >$ and $- < P_o >$. The 1-simplex with vertices P_o and P_1 has two orientations $< P_o,P_1 >$ and $< P_1,P_o >$. A 2-simplex with vertices

P_o, P_1, P_2 has two orientations:

$$< P_o,P_1,P_2 > = < P_1,P_2,P_o > = < P_2,P_o,P_1 > \ ,$$
$$< P_2,P_1,P_o > = < P_1,P_o,P_2 > = < P_o,P_2,P_1 > \ .$$

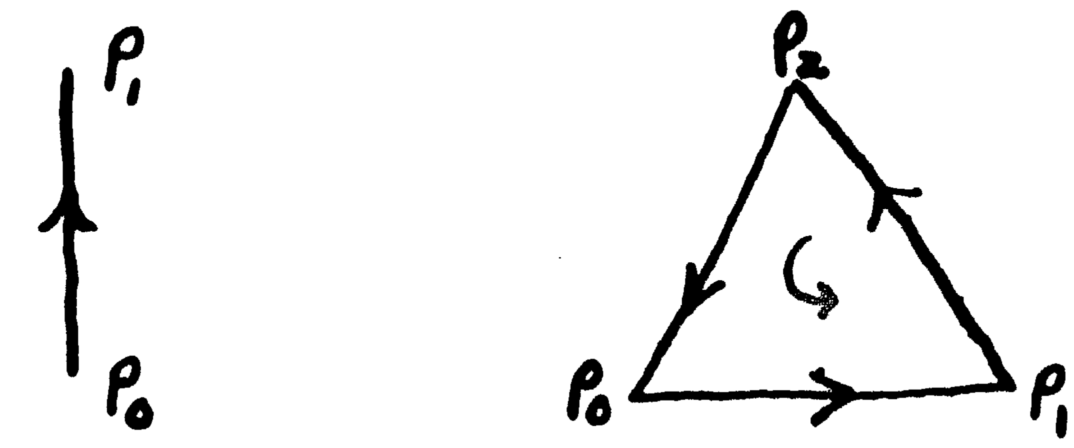


Fig. 1

Note that an orientation of a 2-simplex induces an orientation in each of the 1-simplexes forming its edges.

DEFINITION 5.10. Two adjacent oriented triangles are said to be oriented <u>coherently</u>, if they induce opposite orientations in their common edge.

DEFINITION 5.11. A simple chain of triangles s_1^2, ..., s_n^2, $n > 2$, is said to be <u>closed</u> if s_1^2 and s_n^2 have a common edge.

DEFINITION 5.12. A closed chain of triangles is said to be <u>coherently oriented</u>, if each triangle is oriented.

DEFINITION 5.13. A manifold is orientable, if every simple closed chain of triangles on the manifold can be coherently oriented.

REMARK. It can be shown that every Riemann surface is orientable.

§ 6. NORMAL FORMS OF COMPACT ORIENTABLE SURFACES

Many topological problems about a surface, when reduced to those concerning a model which is homeomorphic to the given surface, are readily clarified.

On a given compact Riemann surface R we fix a triangulation Δ with a coherent orientation. Since R is compact, Δ is a finite set. Now, mapping the triangles of Δ into the Euclidean plane E^2, we obtain a regular polygon π in E^2 having a given orientation in its boundary induced by the orientations in the individual triangles of R. This orientation is called the positive direction when traversing the boundary of π.

LEMMA 6.1. Let R be a compact Riemann surface, and Δ a triangulation of R consisting of n triangles. Then the regular polygon π in the Euclidean plane E^2 obtained from Δ has $n + 2$ sides.

PROOF. For $n = 1$, the conclusion is trivially true. Assume it is true for $n = k$. Then for the case where $n = k + 1$, the $(k + 1)$th triangle has one side adjacent to the kth triangle. By the hypothesis of induction, the polygon π_k for k triangles would have $k + 2$ sides; one of these sides disappears when the $(k + 1)$th triangle is affixed. However, the two remaining sides of the $(k + 1)$th triangle now become sides of the polygon π_{k+1} for $k + 1$ triangles. Hence π_{k+1} has $k + 2 - 1 + 2 = k + 3$ sides, which completes the proof.

In Δ, each edge belongs to exactly two triangles, and exactly two edges of π correspond to the same edge in Δ. Hence π has an even number of sides.

A topological model of a compact Riemann surface R is obtained by identifying the pairs of sides of π, which are adjacent on R_Δ.

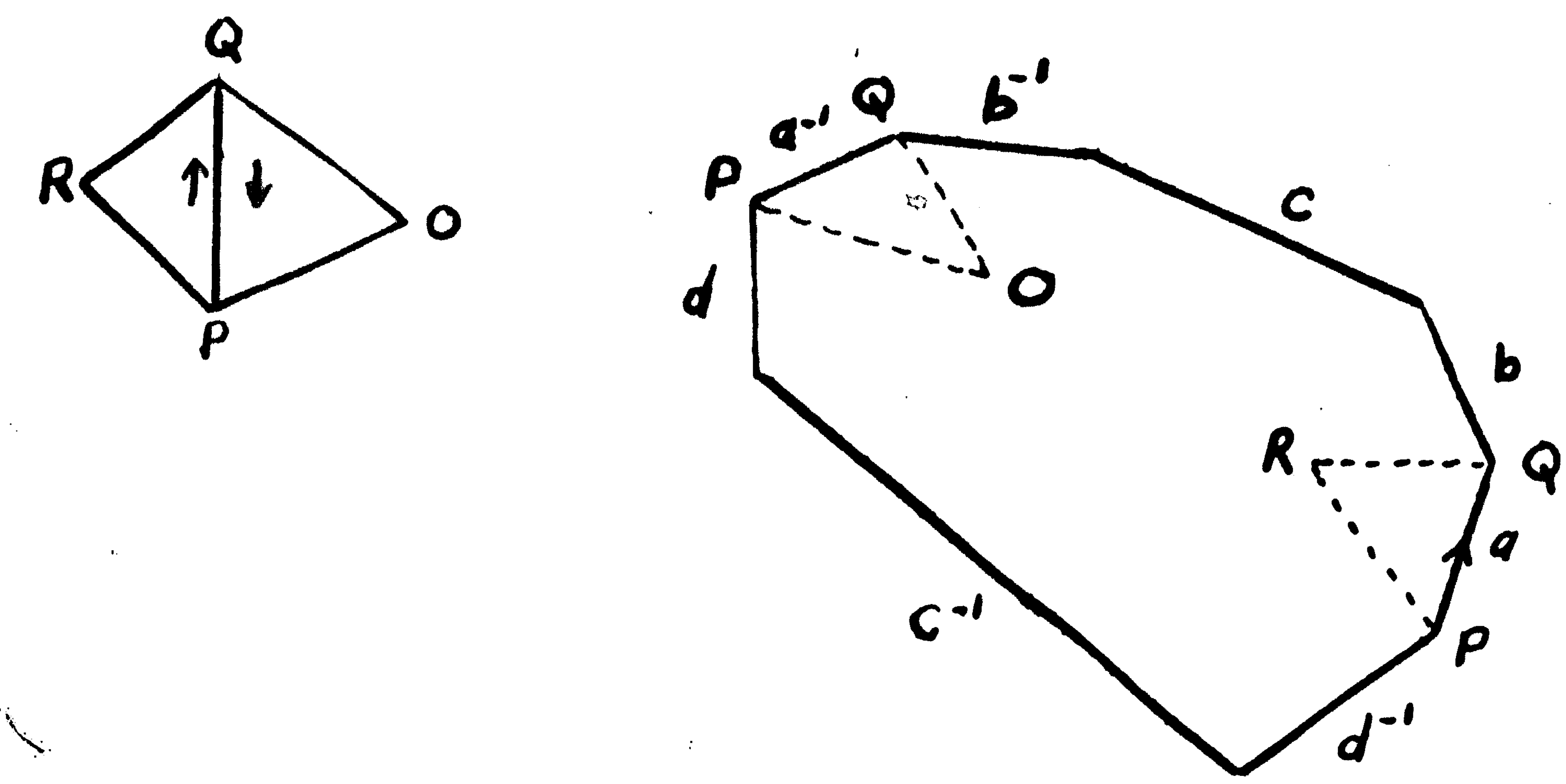


Fig. 2

Let $S' = \langle p, q \rangle$ be an edge of Δ which corresponds to two edges of π. Then the vertex p corresponds to two vertices of π denoted by P. Likewise, the vertex q corresponds to two vertices of π denoted by Q. Transversing the boundary of π in the positive direction, we cross the edge $\overline{PQ}$ and then the edge $\overline{QP}$. This is easily seen to be the case, since triangles PRQ and PQO are coherently oriented. Denote $\overline{PQ}$ by a, and $\overline{QP}$ by a^{-1}. In like manner, associate a letter with each edge of π.

DEFINITION 6.1. A <u>symbol</u> for π is obtained by writing the associated letter in order, in which they are encountered by traversing the boundary of π in the positive direction.

For example, in figure 2, π may be denoted by $abcb^{-1}a^{-1}dc^{-1}d^{-1}$.

We may simplify π in the following steps: First of all, note that if π is cut into two polygons along a line joining two of its vertices, the two parts being attached along a pair of identified edges, and also if we identify the two edges obtained by the cut, we have a new polygon π'. However, it is clearly seen that π and π' represent the same surface.

Note also that a cyclic interchange of the letters in the symbol of π merely gives a new symbol of π.

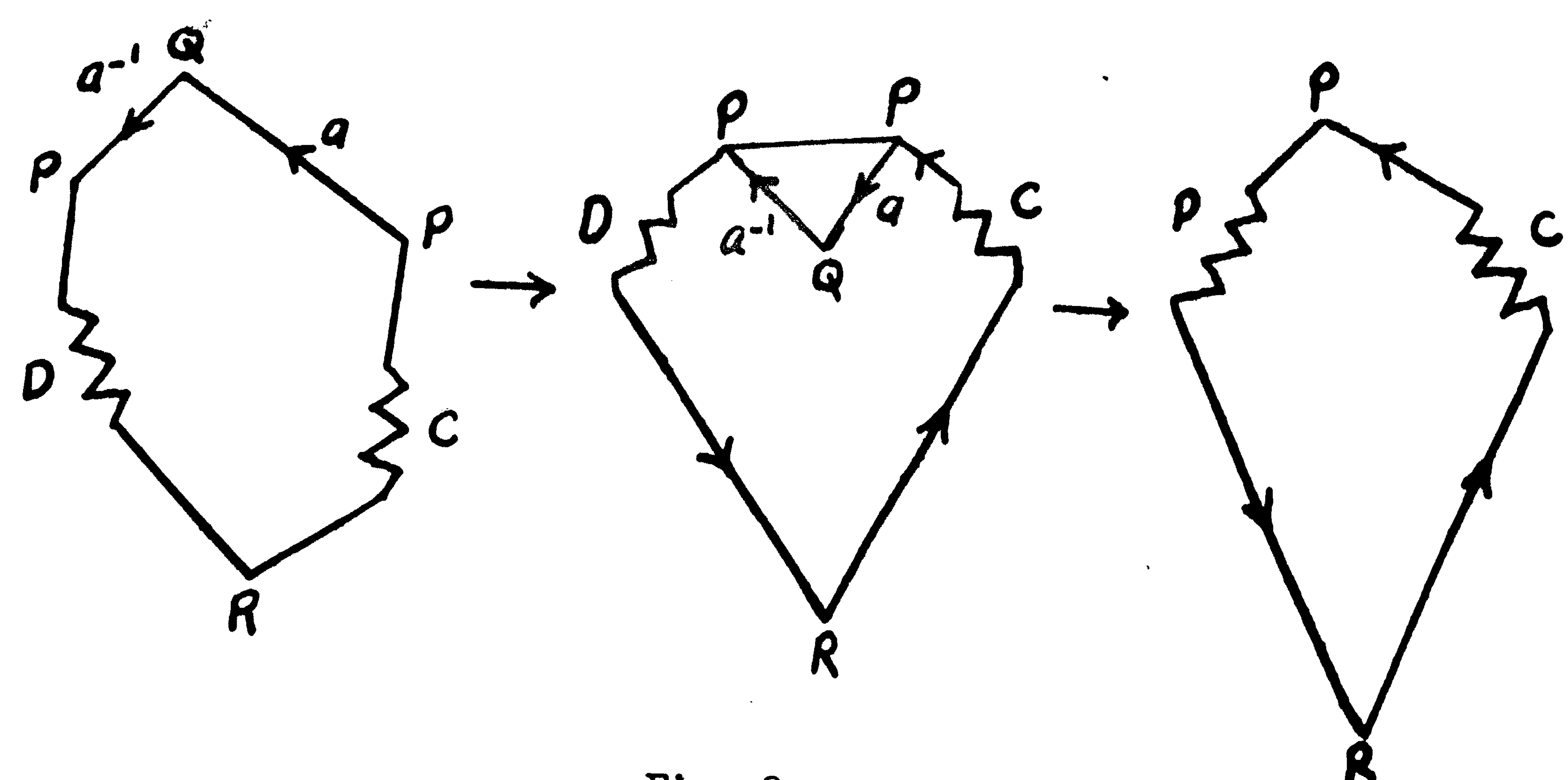


Fig. 3

In figure 3, we see that if aa^{-1} appears in a symbol having at least two letters, we may eliminate the letter a from the symbol.

Now, let a given vertex of π be called P. Then we also label by P all other vertices of π, which correspond to the same point on R. If there exists an edge a of π with one vertex unlabeled, call it Q along with all other vertices equivalent to Q. Let $a = \overline{PQ}$, and b be the edge of π, which has the vertex Q in common with a. Then we know that $b = a^{-1}$. Join the other vertex R of b to the vertex P of a by a diagonal c to form a triangle Δ with edges a,b,c as in

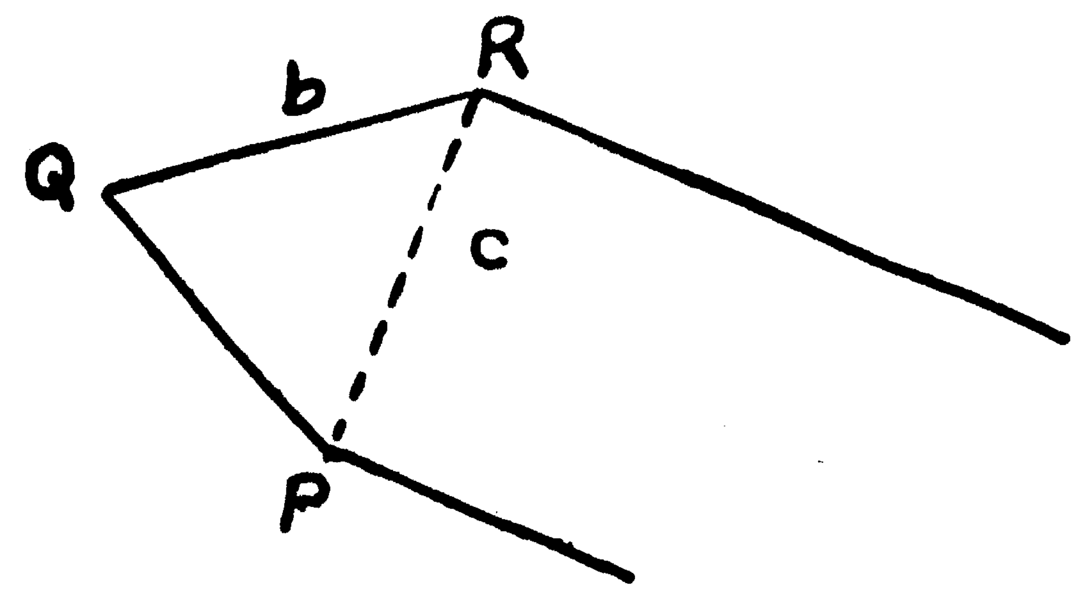


Fig. 4

figure 4. Cut Δ out of π along c, and attach Δ to π along the edge b of Δ and b^{-1} of π . Thus, we obtain a new polygon π', which has one more P vertex and one less Q vertex than π. By continuing this process, we obtain a polygon, in which all vertices are equivalent.

DEFINITION 6.2. A pair of edges a and b of π are called linked, if they appear in the symbol of π in the following order:

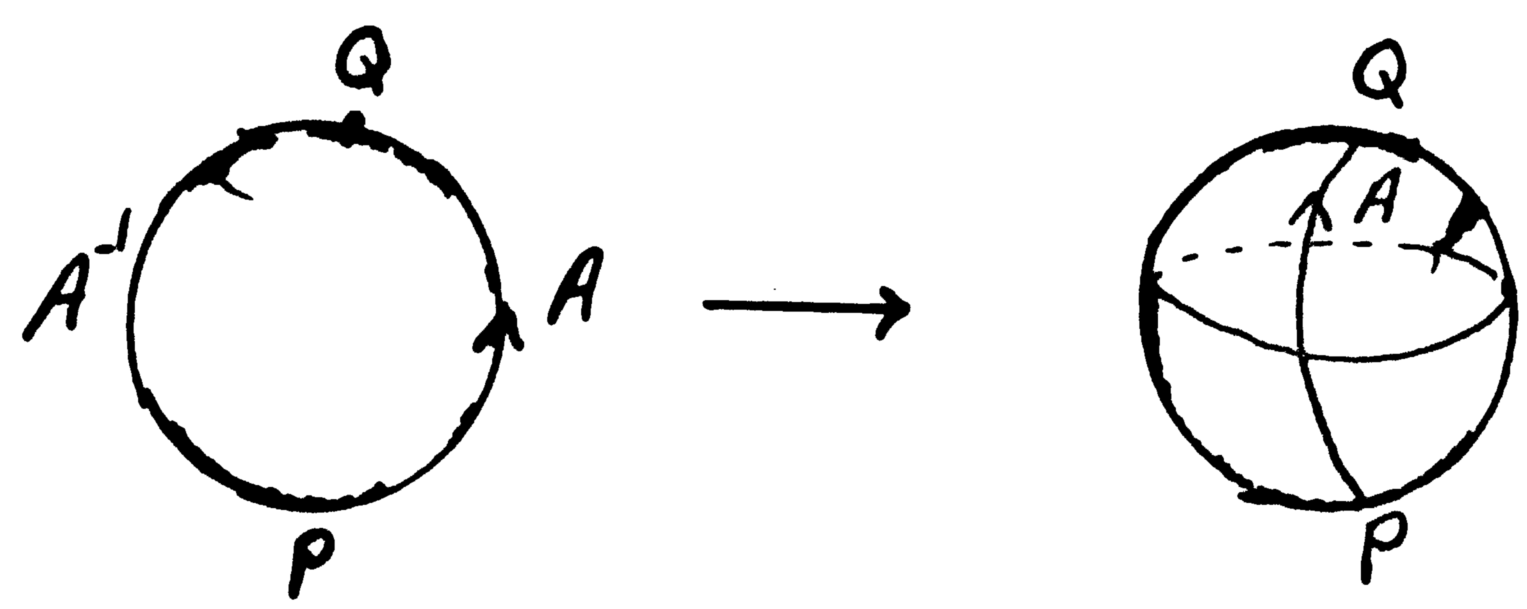
$$\ldots a \ldots b \ldots a^{-1} \ldots b^{-1} \ldots .$$

We now show that each edge of π is linked with some other edge. If this is not the case, there exists an edge c such that all the letters between c and c^{-1} are identified among themselves. Now select a point P on the edge c (not a vertex), and join it by a line segment d in π to the equivalent point on c^{-1}. Thus, we have divided into two parts π_1 and π_2, which have the point P and the points on d identified. Now one vertex P of c lies in π_1 and the other vertex P lies in π_2. Thus P would not have an open neighborhood on the surface, contradicting the definition of a surface. Hence each edge of π is linked with another.

By further cutting and identification, two linked edges c, d occur in the symbol in the form $cdc^{-1}d^{-1}$. Whence, the normal form of π is either aa^{-1} or

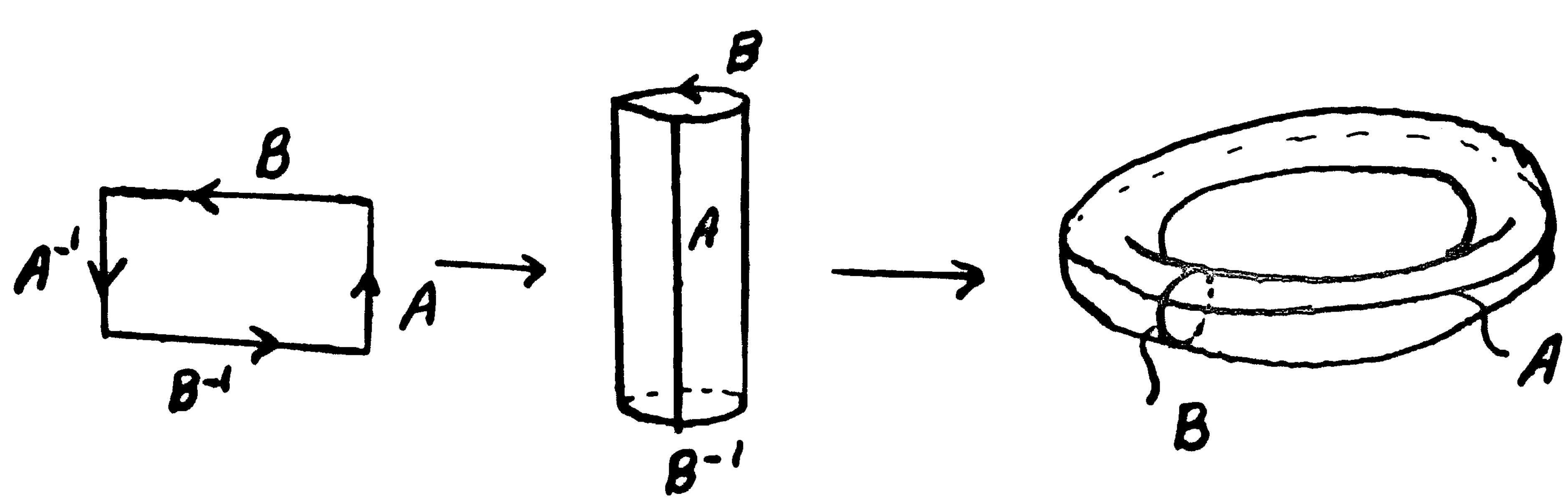
$$a_1b_1a_1^{-1}b_1^{-1}a_2n_2a_2^{-1}b_2^{-1} \ldots a_gb_ga_g^{-1}b_g^{-1} .$$

EXAMPLE 6.1. Consider the form AA^{-1}.



Upon pasting together the edges A^{-1} and A, we obtain a surface homeomorphic to a sphere.

EXAMPLE 6.2. Consider the form $ABA^{-1}B^{-1}$.



We obtain a cyclinder by identifying the edges A and A^{-1}, and hence a torus by further identifying B and B^{-1}.

In general, a normal form with 2g letters in its symbol is homeomorphic to a sphere with g handles attached. Note that in Example 6.2, the edges AA^{-1} and BB^{-1} become closed curves in the torus. This is true in general. It is sometimes convenient to think of the edges as closed curves (or cycles) on the surface.

§ 7. PERIOD RELATIONS

As seen in the last section, a compact Riemann surface R with genus g may be represented topologically by a polygon of 4g sides with cycles $A_1, \ldots, A_g, B_1, \ldots, B_g$.

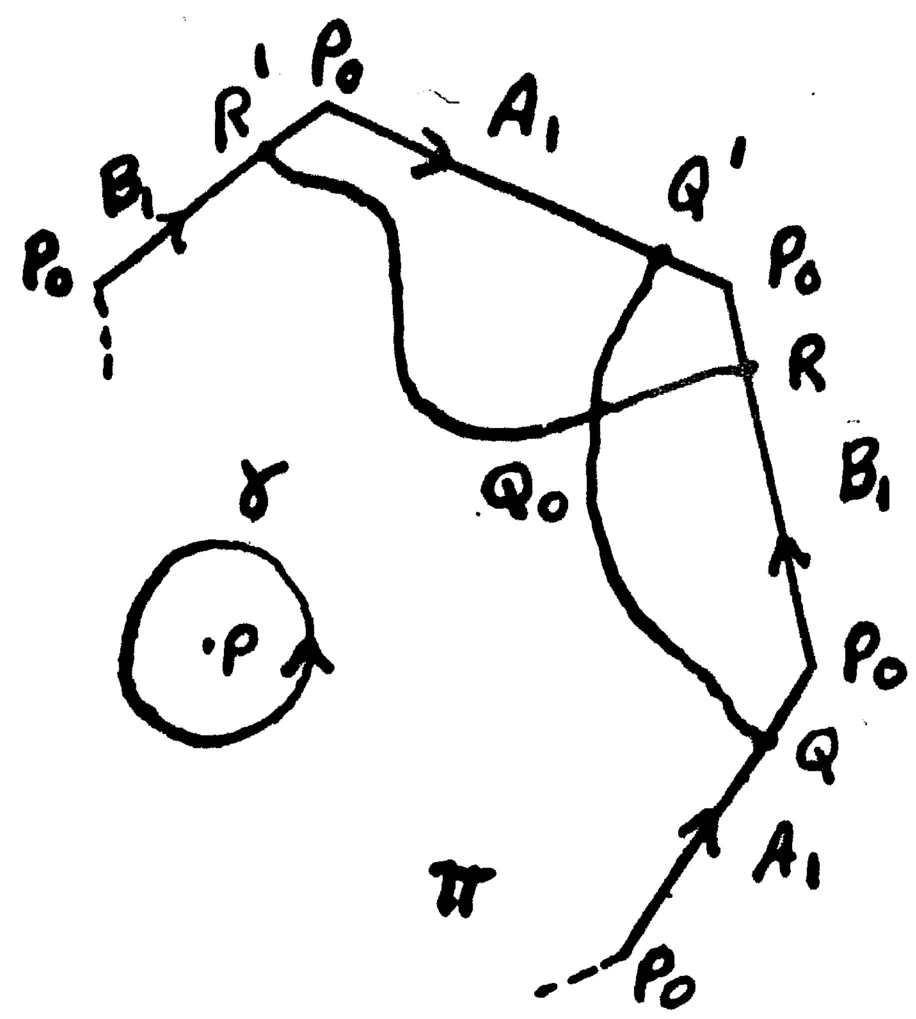
DEFINITION 7.1. If is a closed one-form on R, the periods of α are the numbers

$$a_j = \oint_{A_j} \alpha \quad , \quad b_j = \oint_{B_j} \alpha \quad , \tag{7.1}$$

which will be referred to as A- and B- periods respectively.

If $\alpha = \eta$ is a meromorphic differential, we apply the same terminology provided no pole of η lies on one of the fundamental cycles. We may remark here that if η is a differential of the second kind (by which we mean the residues of η all vanish), then the definition is valid without exception, for if a pole of η lies (say) on A_1, we may displace A_1 slightly to either side with the same resulting period a_1.

Relations between the periods of meromorphic differentials all derive from one basic idea. To illustrate the method let us take a closed differential α and a second differential β, which is defined on the whole R with the possible exceptions of a finite set of points P, and which is closed where it is defined. Inside the fundamental polygon π we have



$\alpha = df$,

where for example we may take

(7.2) $$f(\cdot) = \int_{Q_0}^{\cdot} \alpha ,$$

after a base point Q_0 is fixed.

If Q and Q' are equivalent points of A_1, then the path $Q_0QP_0B_1P_0Q'Q_0$ is contractible, and therefore

(7.3) $$\left(\int_{Q_0}^{Q} + \int_{Q}^{P_0} + \oint_{B_1} + \int_{P_0}^{Q'} - \int_{Q_0}^{Q'} \right)(\alpha) = 0.$$

It is clear that

$$\int_{Q}^{P_0} \alpha = - \int_{P_0}^{Q'} \alpha .$$

By equation (7.2) we thus obtain, from equation (7.3),

(7.4) $$f(Q) - f(Q') = \int_{Q_0}^{Q} \alpha - \int_{Q_0}^{Q'} \alpha = - \oint_{B_1} \alpha .$$

Similarly, if R and R' are equivalent points of B, then

$$\left(\int_{Q_0}^{R} + \int_{R}^{P_0} - \oint_{A_1} + \int_{P_0}^{R'} - \int_{Q_0}^{R'} \right)(\alpha) = 0,$$

and therefore

(7.5) $$f(R) - f(R') = \int_{Q_0}^{R} \alpha - \int_{Q_0}^{R'} \alpha = \oint_{A_1} \alpha .$$

To continue, we delete a small disc with boundary γ around each of the singularities P of β, and let Σ denote the resulting region. In Σ we have

$$\alpha \wedge \beta = df \wedge \beta = d(f\beta),$$

and therefore

$$\iint_{\Sigma} \alpha \wedge \beta = \iint_{\Sigma} d(f\beta) = \oint_{\partial(\Sigma)} f\beta \quad . \tag{7.6}$$

The boundary of Σ splits naturally into two parts, the boundary of and the sum of the cycles γ. The first contribution is evaluated, by means of equations (7.4),(7.5), as follows:

$$\oint_{\partial(\pi)} f\beta = \int_{A_1} f(Q)\beta\Big|_Q + \int_{B_1} f(R)\beta\Big|_R$$

$$- \int_{A_1} f(Q')\beta\Big|_{Q'} - \int_{B_1} f(R')\beta\Big|_{R'} + \cdots$$

$$= \int_{A_1} [f(Q) - f(Q')]\beta\Big|_Q + \int_{B_1} [f(R) - f(R')]\beta\Big|_R + \cdots$$

$$= \int_{A_1} \Big[- \oint_{B_1} \alpha \Big] \beta + \int_{B_1} \Big[\oint_{A_1} \alpha \Big] \beta + \cdots$$

$$= \Big(\oint_{A_1} \alpha \Big)\Big(\oint_{B_1} \alpha \Big) - \Big(\oint_{A_1} \beta \Big)\Big(\oint_{B_1} \alpha \Big) + \cdots$$

Thus, if we denote the periods of α and β by

$$a_j = \oint_{A_j} \alpha \quad , \quad b_j = \oint_{B_j} \alpha \quad ,$$

$$a'_j = \oint_{A_j} \beta \quad , \quad b'_j = \oint_{B_j} \beta \quad ,$$

then we have

$$\oint_{\partial(\pi)} f\beta = \sum_{j=1}^{g} (a_j b'_j - a'_j b_j) \, ,$$

and therefore our final formula, in consequence of equation (7.6),

$$(7.7) \quad \iint_{\Sigma} \alpha \wedge \beta = \sum_{j=1}^{g} (a_j b'_j - a'_j b_j) + \sum_{P} \oint_{\gamma} f\beta \, .$$

The first application of the formula (7.7) gives

THEOREM 7.1 (RIEMANN BILINEAR RELATION). If w and w' are holomorphic differentials with periods a_j , b_j , a'_j , b'_j respectively, then

$$(7.8) \quad \sum_{j=1}^{g} (a_j b'_j - a'_j b_j) = 0.$$

PROOF. Locally, $w = h(z)dz$, $w' = h'(z)dz$, so that

$$w \wedge w' = h(x,y)(dx + idy) \wedge h'(xy)(dx + idy)$$
$$= hh'(i - i) \, dx \wedge dy = 0.$$

By putting $\alpha = w$, $\beta = w'$ in equation (7.7) we thus obtain equation (7.8), since the second term on the right side of equation (7.7) vanishes because of the fact that w' has no singularities.

Next we have

THEOREM 7.2 (RIEMANN INEQUALITY). If w is holomorphic with periods a_j and b_j, then

$$i \sum_{j=1}^{g} (a_j \bar{b}_j - \bar{a}_j b_j) \geqslant 0, \tag{7.9}$$

where the equality holds if and only if $w = 0$.

PROOF. It is evident that if w has periods a_j, b_j, then $\bar{w}$ has periods $\bar{a}_j$, $\bar{b}_j$. By putting $\alpha = w$, $\beta = \bar{w}$ in equation (7.7) we thus have

$$\iint_R w \wedge \bar{w} = \sum_{j=1}^{g} (a_j \bar{b}_j - \bar{a}_j b_j). \tag{7.10}$$

Locally, $w = h(z)dz$, h being holomorphic, so that

$$\begin{aligned} w \wedge \bar{w} &= |h|^2\, dz \wedge d\bar{z} \\ &= -2i\, |h|^2\, dx \wedge dy, \end{aligned} \tag{7.11}$$

where $z = x + iy$. Thus

$$i \iint_R w \wedge \bar{w} = 2 \iint_R |h|^2\, dx \wedge dy \geqslant 0,$$

which and equation (7.10) give (7.9) immediately.

For the equality of (7.9) we first assume $w = 0$. Then $w \wedge \bar{w} = 0$, and therefore, from equation (7.10), equality holds in (7.9). Conversely, assume that the equality holds in (7.9). Then equations (7.10), (7.11) imply that $h = 0$ so that $w = 0$.

COROLLARY 7.2.1. If w is holomorphic with zero A- periods, then $w = 0$.

PROOF. If $a_j = 0$, then $\bar{a}_j = 0$, so that the equality holds in (7.9). Hence $w = 0$ by Theorem 7.2.

COROLLARY 7.2.2. <u>There exists a unique holomorphic differential with prescribed A-periods.</u>

PROOF. Since by Theorem 3.1, the space of holomorphic differentials has (complex) dimension g, and by Corollary 7.2.1 the mapping $w \to (a_1, \dots, a_g)$ onto this space is one-one; this mapping is an isomorphism onto the g-dimensional complex coordinate space.

By Corollary 7.2.2 we may select a basis $w_1, \dots, w_g$ of holomorphic differentials according to

$$(7.12) \qquad \oint_{A_i} w_j = \delta_{ij} = \begin{cases} 1 \text{ for } i = j, \\ 0 \text{ for } i \neq j. \end{cases}$$

This basis $w_1, \dots, w_g$ is called a <u>normalized</u> basis of holomorphic differentials. For these differentials $w_1, \dots, w_g$ we set

$$b_{jk} = \oint_{B_k} \omega_j ,$$

and write

$$(7.13) \qquad b_{jk} = b'_{jk} + i\, b''_{jk} .$$

THEOREM 7.3. <u>The matrix</u> $\| b_{jk} \|$ <u>of B- periods is symmetric, and the matrix</u> $\| b''_{jk} \|$ <u>of imaginary parts of</u> b_{jk} <u>is positive definite.</u>

PROOF. The first part is easy, for simply applying Theorem 7.1 to the differentials w_j and w_k we obtain

$$\sum_{\ell=1}^{g} (\delta_{\ell j} b_k - \delta_{\ell k} b_{j\ell}) = 0 ,$$

so that $b_{kj} = b_{jk}$.

For the second part, let $w = \sum_{j=1}^{g} a_j w_j$ be an arbitrary holomorphic differential. Then

$$\oint_{A_k} w = \sum_{j=1}^{g} a_j \oint_{A_k} w_j = \sum_{j=1}^{g} a_j \delta_{kj} = a_k ,$$

$$\oint_{B_k} w = \sum_{j=1}^{g} a_j \oint_{B_k} w_j = \sum_{j=1}^{g} a_j b_{jk} ,$$

and therefore the inequality (7.9) becomes

$$(7.14) \qquad i \sum_{j,k=1}^{g} (a_k \bar{b}_{jk} \bar{a}_j - \bar{a}_k a_j b_{jk}) \geq 0 ,$$

where the equality holds if and only if $w = 0$, that is, if and only if $(a_1, \ldots, a_g) = 0$. Substituting equation (7.13) in the inequality (7.14) and simplifying the result we obtain

$$(7.15) \qquad i \Big(\sum_{j,k=1}^{g} a_k b'_{jk} \bar{a}_j - \sum_{j,k=1}^{g} \bar{a}_k a_j b'_{jk} \Big) + \sum_{j,k=1}^{g} a_k b''_{jk} \bar{a}_j + \sum_{j,k=1}^{g} \bar{a}_k a_j b''_{jk} \geq 0 .$$

Due to the symmetry of b'_{jk} and b''_{jk} in the indices j, k, (7.15) is equivalent to

$$\sum_{j,k=1}^{g} a_j b''_{jk} \bar{a}_k \geq 0,$$

where the equality holds if and only if $(a_1, \ldots, a_g) = 0$. Thus, by definition, $\| b''_{jk} \|$ is positive definite.

REMARK. Conversely, we can easily verify that Theorem 7.3 implies Theorem 7.2.

DEFINITION 7.2. A meromorphic differential η is said to be of the <u>second kind</u>, if $\mathrm{res}_P(\eta) = 0$ for all P.

Given such an η we may <u>normalize</u> it to one with all A-periods zero simply by adding a suitable holomorphic differential; this leaves unchanged the principal parts of η.

THEOREM 7.4. (PERIOD RELATIONS FOR DIFFERENTIALS OF THE SECOND KIND). If η is a normalized differential of the second kind with poles at points P and B- periods b_j', and w is a holomorphic differential with $w = df$ on π and with periods a_j, b_j, then

$$(7.16) \qquad \sum_{j=1}^{g} a_j b_j' = 2\pi i \sum_P \operatorname{res}_p (fh).$$

PROOF. By putting $\alpha = w$, $\beta = \eta$ in equation (7.7), and noticing that the A-periods α_i' of η vanish, we obtain

$$(7.17) \qquad \iint_\Sigma w \wedge \eta = \sum_{j=1}^{g} a_j b_j' - \sum_P \oint_\gamma f\eta.$$

Since $w \wedge \eta = 0$ in Σ by local expansion, and each of the last terms of equation (7.17) is a residue, that is,

$$\oint_\gamma (fh) = 2\pi i \operatorname{res}_p (fh),$$

we thus obtain equation (7.16).

In particular, if we take $w = w_j$ from the normalized basis, then $w_j = df_j$, and from equations (7.12) we have $a_j = 1$, $a_i = 0$ if $i \neq j$. Thus formula (7.16) is reduced to:

$$(7.18) \qquad b_j' = \oint_{\beta_j} \eta = 2\pi i \sum_P \operatorname{res}_p (f_i \eta).$$

By way of general philosophy here we may remark that once the principal parts and the A- periods of η are prescribed, the B- periods an uniquely determined, and this formula (7.18) gives the answer. Later we shall evaluate the right hand side of the formula (7.18) more explicitly.

§ 8. DIVISORS

Our basic goal is the determination of all meromorphic functions on R. Given such a function f, we see that df is a meromorphic differential such that

$$\oint_{\gamma} df = 0$$

over each closed curve γ, which avoids the poles of f. Thus $\eta = df$ is a differential of the second kind, all of whose periods vanish. Of course, the singular parts of f completely determine those of η, and since we already know how to construct normalized η with prescribed singular parts, the natural procedure in constructing a function with prescribed singular parts seems to be as follows. We construct the with corresponding singular parts, normalize it so that all A- periods vanish, and then use the period relations to determine if the B- periods also vanish. We shall carry through this program to the end of this paper. In this section we give some preliminary definitions and results needed for the precise formulation of the Riemann-Roch theorem.

DEFINITION 8.1. A <u>divisor</u> on R (also called a polygox) is a finite formal linear combination

$$\sigma = \sum n_k P_k$$

of points P_k on R with integral coefficients n_k, in the language of topology, a finite integral (singular) zero-chain. Thus, if γ is an integral one-chain, then $\sigma = \partial\gamma$ is a divisor

DEFINITION 8.2. Let f be a non-zero meromorphic function on R. Then

$$(f) = \sum V_p(f)P$$

is called the <u>divisor of zeros and poles</u> of f, where $V_p(f)$ is the order of f at P (see § 6).

LEMMA 8.1. <u>Let f, g be non-zero meromorphic functions on R. Then</u>

(8.1) $$(fg) = (f) + (g) .$$

PROOF. Let

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots , \; a_n = 0,$$

$$g(z) = b_m z^m + b_{m+1} z^{m+1} + \cdots , \; b_m = 0.$$

Then

$$fg(z) = a_n b_m z^{n+m} + (a_{n+1} b_m + a_n b_{m+1}) z^{n+m+1} + \cdots ,$$

and therefore $V_p(fg) = n + m$. Hence equation (8.1).

LEMMA 8.2. The set K of divisors on R forms an abelian group.

PROOF. Let $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ be any divisors on R so that

$$\mathfrak{a}_1 = \sum n_{k_1} P_{k_1}, \; \mathfrak{a}_2 = \sum n_{k_2} P_{k_2}, \; \mathfrak{a}_3 = \sum n_{k_3} P_{k_3} .$$

It is easy to verify the following:

(i) $\mathfrak{a}_1 + \mathfrak{a}_2 = \sum n_k P_k \in K$,

where $P_k = P_{k_1} \cup P_{k_2}$.

(ii) $(\mathfrak{a}_1 + \mathfrak{a}_2) + \mathfrak{a}_3 = \mathfrak{a}_1 + (\mathfrak{a}_2 + \mathfrak{a}_3)$.

(iii) The unit element of K is $\mathfrak{a}_0 = \sum 0 P_k$ so that $\mathfrak{a} + \mathfrak{a}_0 = \mathfrak{a}$, where $\mathfrak{a} \in K$.

(iv) The inverse element of $\mathfrak{a} = \sum n_k P_k$ is $\mathfrak{a}^{-1} = \sum (-n_k) P_k \in K$, so that $\mathfrak{a} + \mathfrak{a}^{-1} = \mathfrak{a}_0$.

(v) $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{a}_2 + \mathfrak{a}_1$.

DEFINITION 8.3. Let η be a non-zero meromorphic differential defined on R locally at P by

$$\eta = (a_n z^n + a_{n+1} z^{n+1} + \ldots) \, dz$$

in terms of a local uniformizing variable z. Set $V_p(\eta) = n$ if $a_n = 0$. Then $(\eta) = \sum V_p(\eta) P$ is called the divisor of zeros and poles of η.

The divisor of a differential is often called a <u>canonical divisor</u>.

DEFINITION 8.4. The <u>degree</u> of a divisor $\mathfrak{A} = \sum n_p P$ is defined by $\deg \mathfrak{A} = \partial_0 \mathfrak{A} = \sum n_p$.

It is easily seen that the map

$$\partial_0 : \sum n_p P \to \sum n_p$$

is a homomorphism on the abelian group of divisors into the integers. In fact, ∂_0 is single-valued and onto the set of integers, and

$$\partial_0(\mathfrak{A}_1 + \mathfrak{A}_2) = \partial_0 \mathfrak{A}_1 + \partial_0 \mathfrak{A}_2 ,$$

where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are any two divisors.

Moreover, from Corollary 4.1 we have known that a non-constant meromorphic function has the same number of zeros as poles. But now this statement becomes that $\deg(f) = 0$ for any non-constant meromorphic function f.

We remark that if γ is a one-chain and $\mathfrak{A} = \partial \gamma$, then $\deg \mathfrak{A} = 0$. Conversely, if $\deg \mathfrak{A} = 0$, then the connectivity of R implies $\mathfrak{A} = \partial \gamma$ for some γ.

DEFINITION 8.5. $\sum n_p P \geqslant 0$, if $n_p \geqslant 0$ for all P.
$\sum n_{k_1} P_{k_1} \geqslant \sum n_{k_2} P_{k_2}$, if $\sum (n_{k_1} P_{k_1} - n_{k_2} P_{k_2}) \geqslant 0$.

LEMMA 8.3. "$\geqslant$" <u>is a partial order relation on the group of divisors</u>.

PROOF. Let $\mathfrak{A}_i = \sum n_{k_i} P_{k_i}$, $i = 1, 2, 3$.

(i) $\mathfrak{A}_i \geqslant \mathfrak{A}_i$, since $\sum (n_{k_i} P_i - n_{k_i} P_i) = \sum 0 P_{k_i} \geqslant 0$.

(ii) Suppose $\mathfrak{A}_1 \geqslant \mathfrak{A}_2$, $\mathfrak{A}_2 \geqslant \mathfrak{A}_1$. Then
$\sum (n_{k_1} P_{k_1} - n_{k_2} P_{k_2}) \geqslant 0$, $\sum (n_{k_2} P_{k_2} - n_{k_1} P_{k_1}) \geqslant 0$.

Thus, if P_{j_1} appears in only one of the divisors, then $n_{j_1} \geq 0$ and $n_{j_1} \leq 0$, which imply $n_{j_1} = 0$. Furthermore, if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ have a common point $P_{j_1} = P_{i_2}$, then $n_{j_1} - n_{i_2} \geq 0$ and $n_{i_2} - n_{j_1} \geq 0$, and therefore $n_{j_1} = n_{i_2}$. Hence $\mathfrak{a}_1 = \mathfrak{a}_2$.

(iii) Suppose $\mathfrak{a}_1 \geq \mathfrak{a}_2$, $\mathfrak{a}_2 \geq \mathfrak{a}_3$. Then $\sum (n_{k_1} P_{k_1} - n_{k_2} P_{k_2}) \geq 0$, $\sum (n_{k_2} P_{k_2} - n_{k_3} P_{k_3}) \geq 0$. If $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{a}_3$ have a common point $P_{j_1} = P_{j_2} = P_{j_3}$, then $n_{j_1} - n_{j_2} \geq 0$, $n_{j_2} - n_{j_3} \geq 0$, which imply $n_{j_1} - n_{j_3} \geq 0$. If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ have a common point $P_{j_1} = P_{j_2}$, which is not in $\mathfrak{a}_3$, then $n_{j_1} - n_{j_2} > 0$, $n_{j_2} > 0$, and therefore $n_{j_1} > 0$. If $\mathfrak{a}_1$ and $\mathfrak{a}_3$ have a common point $P_{j_1} = P_{j_3}$, which is not in $\mathfrak{a}_2$, then $n_{j_1} \geq 0$, $-n_{j_3} \geq 0$, which imply $n_{j_1} - n_{j_3} \geq 0$. Finally, if P_{j_1} is in $\mathfrak{a}_1$ but not in $\mathfrak{a}_2$ and $\mathfrak{a}_3$, then $n_{j_1} \geq 0$. Hence $\mathfrak{a}_1 \geq \mathfrak{a}_3$.

DEFINITION 8.6. Let $\mathfrak{a}$ be a divisor on R. Define

$$L(\mathfrak{a}) = \{f : (f) + \mathfrak{a} \geq 0\},$$

where f is a meromorphic function on R. By convention, $0 \in L(\mathfrak{a})$.

We can best understand the meaning of $L(\mathfrak{a})$ by considering an example. Suppose $\mathfrak{a} = 7P - 3Q$. $f \in L(\mathfrak{a})$, then $(f) + 7P - 3Q \geq 0$. This means that f must be holomorphic at each point R distinct from P and Q, that f has a possible singularity only at P and this is a pole of order at most 7, and that f has a zero of order at least 3 at Q.

LEMMA 8.4. <u>$L(\mathfrak{a})$ is a linear space.</u>

PROOF. Let $f, g \in L(\mathfrak{a})$. It is clear that under addition the commutative and associative laws hold. Also, by definition, $0 \in L(\mathfrak{a})$. Now, $(f) + \mathfrak{a} \geq 0$, $(g) + \mathfrak{a} \geq 0$, where $(f) = \sum V_{p_1}(f) P_1$, $(g) = \sum V_{p_2}(f) P_2$, $\mathfrak{a} = \sum n_{p_3} P_3$. Consider $(f + g) = \sum V_p(f + g) P$. Then at P,

$V_p(f+g) \geqslant \min\{V_p(f), V_p(g)\}$. Since $V_p(f) + n_p \geqslant 0$,

and $V_p(g) + n_p \geqslant 0$, $V_p(f+g) + n_p \geqslant 0$, which implies that $f + g \in L(\mathfrak{a})$.
It is obvious that if α and β are complex numbers, then

$\alpha(f+g) = \alpha f + \alpha g$, $(\alpha+\beta)f = \alpha f + \beta f$, $\alpha(\beta f) = (\alpha\beta)f$, $1\cdot f = f$, $0\cdot f = 0$.
Hence the lemma is proved.

We shall shortly see that these spaces $L(\mathfrak{a})$ are always finite dimensional, and we shall set

$$\dim \mathfrak{a} = \dim L(\mathfrak{a}),$$

where we always take complex dimension.

DEFINITION 8.7. It is convenient to define an equivalence relation among divisors: $\mathfrak{a} \sim \mathfrak{b}$, if $\mathfrak{a} = \mathfrak{b} + (f)$ for some (f), where f is a meromorphic function on R.

LEMMA 8.5. <u>If</u> $\mathfrak{a} \sim \mathfrak{b}$, <u>then</u>

(8.2) $\deg \mathfrak{a} = \deg \mathfrak{b}$,

(8.3) $\dim \mathfrak{a} = \dim \mathfrak{b}$.

PROOF. By assumption there exists a meromorphic function f on R such that $\mathfrak{a} = \mathfrak{b} + (f)$. Therefore

$$\deg \mathfrak{a} = \deg \mathfrak{b} + \deg (f) = \deg \mathfrak{b},$$

since $\deg (f) = 0$.

Consider the map $\Gamma: g \to fg$, where $g \in L(\mathfrak{a})$, so that $\mathfrak{a} + (g) \geqslant 0$.
Let

(8.4) $\mathfrak{a} = \mathfrak{b} + (f)$.

Then, by equation (8.1),

$\mathfrak{a} + (g) = \mathfrak{b} + (fg) \geqslant 0$, sothat $fg \in L(\mathfrak{b})$. Thus $\Gamma: L(\mathfrak{a}) \to L(\mathfrak{b})$.

Γ is 1-1. In fact, suppose $h \in L(\mathfrak{b})$, and $g \neq h$.

Then $fg \neq fh$, that is, $\Gamma(g) \neq \Gamma(h)$.

Γ is onto. In fact, let $h \in L(\mathfrak{B})$. Then $\mathfrak{B} + (h) \geqslant 0$, which and equation (8.4) imply $\mathfrak{A} - (f) + (h) \geqslant 0$. Since $-(f) = (1/f)$, by equation (8.1) we have $-(f) + (h) = (h/f)$, and therefore $\mathfrak{A} + (h/f) \geqslant 0$, that is, $h/f \in L(\mathfrak{A})$. Hence $h = \Gamma(h/f)$.

Now let $h, g \in L(\mathfrak{A})$. Then

$$\Gamma(h + g) = f(h + g) = fh + fg = \Gamma(h) + \Gamma(g).$$

Let c be a complex number, and $g \in L(\mathfrak{A})$. Then

$$\Gamma(cg) = f(cg) = c(fg) = c\,\Gamma(g).$$

Thus Γ is an isomorphism of $L(\mathfrak{A})$ onto $L(\mathfrak{B})$, and hence equation (8.3).

LEMMA 8.6. <u>If</u> $\mathfrak{A} \leqslant \mathfrak{B}$, <u>then</u>

$$L(\mathfrak{A}) \subset L(\mathfrak{B}). \tag{8.6}$$

<u>If in addition</u> $\dim \mathfrak{A} < \infty$, <u>then</u> $\dim \mathfrak{B} < \infty$, <u>and</u>

$$\deg \mathfrak{A} - \dim \mathfrak{A} \leqslant \deg \mathfrak{B} - \dim \mathfrak{B}. \tag{8.7}$$

PROOF. Let $f \in L(\mathfrak{A})$. Then $(f) + \mathfrak{A} \geqslant 0$. Since, by equation (8.5), $(f) + \mathfrak{B} \geqslant (f) + \mathfrak{A}$, we have $(f) + \mathfrak{B} \geqslant 0$, so that $f \in L(\mathfrak{B})$. Hence equation (8.6).

Next, assume $\dim \mathfrak{A} < \infty$, and for each P set

$$\mathfrak{A} = \sum m_p P, \quad \mathfrak{B} = \sum n_p P, \quad m_p \leqslant n_p .$$

At each P, select a uniformizing variable z, so that if $f \in L(\mathfrak{B})$, then $V_p(f) + n_p \geqslant 0$. Thus at each P

$$f(z) = a_{-n_p} z^{-n_p} + a_{-n_p+1} z^{-n_p+1} + \cdots + a_{-m_p-1} z^{-m_p-1} + a_{-m_p} z^{-m_p} + \cdots,$$

and $f \in L(\mathfrak{A})$ if and only if $a_{-n_p} = \cdots = a_{-m_p-1} = 0$ for all P. In other words, the subspace $L(\mathfrak{A})$ of $L(\mathfrak{B})$ is determined by the vanishing of a certain finite set of linear functionals. Each point P contributes $(n_p - m_p)$

of these, so there are in total

$$\sum_P (n_p - m_p) = \deg \mathfrak{b} - \deg \mathfrak{a}$$

such functionals. Since by assumption $\dim \mathfrak{a} < \infty$, the above relation already implies that $\dim \mathfrak{b} < \infty$. Furthermore, if we remember that these functionals are not necessarily linearly independent, we conclude that

$$\dim \mathfrak{a} = \dim \mathfrak{b} - [(\deg \mathfrak{b} - \deg \mathfrak{a}) - \text{relations}]$$
$$\geqslant \dim \mathfrak{b} - \deg \mathfrak{b} + \deg \mathfrak{a} .$$

We remark that the Riemann-Roch theorem is proved, once these relations are determined.

COROLLARY 8.6.1. <u>If</u> $\mathfrak{a} \geqslant 0$, <u>then</u>

(8.8) $$\dim \mathfrak{a} \leqslant \deg \mathfrak{a} + 1.$$

PROOF. By Lemma 8.6,

(8.9) $$\deg 0 - \dim 0 < \deg \mathfrak{a} - \dim \mathfrak{a}.$$

But $\deg 0 = 0$, and $L(0) = \{f : (f) \geqslant 0\}$. Since $(f) \geqslant 0$ means that $V_p(f) \geqslant 0$ for each $P \in R$, f is holomorphic. Thus by Lemma 2.1, f is necessarily constant. Therefore $L(0)$ is the field of constants, and $\dim 0 = \dim$ the set of complex numbers $= 1$. Hence equation (8.8) follows immediately from equation (8.9).

COROLLARY 8.6.2. <u>For each</u> $\mathfrak{a}$, $\dim \mathfrak{a} < \infty$.

PROOF. Let $\mathfrak{a} = \sum n_p P$, and define $\mathfrak{b}$ by merely deleting the negative terms in $\mathfrak{a}$, that is,

$$\mathfrak{b} = \sum m_p P$$

where

$$m_p = \begin{cases} 0, & \text{if } n_p < 0, \\ n_p, & \text{if } n_p \geqslant 0. \end{cases}$$

Then $\mathfrak{b} \geqslant 0$ and $\mathfrak{a} < \mathfrak{b}$. Therefore by Lemma 8.6, $L(\mathfrak{a}) \subset L(\mathfrak{b})$, from which it follows, in consequence of Corollary 8.6.1,

$$\dim \mathfrak{a} \leqslant \dim \mathfrak{b} < \deg \mathfrak{b} + 1 < \infty .$$

REMARK. Due to Corollary 8.6.2, equation (8.7) of Lemma 8.6 is always valid.

LEMMA 8.7. <u>If</u> η_1 <u>and</u> η_2 <u>are non-zero differentials, then</u> $\eta_2 = f\,\eta_1$, <u>and</u> $(\eta_2) \sim (\eta_1)$, <u>where</u> f <u>is a meromorphic function on</u> R.

PROOF. Locally, $\eta_1 = h_1(z)dz$, $\eta_2 = h_2(z)dz$, h_1 and h_2 being non-zero meromorphic functions, and we can easily see that $h_2(z)/\, h_1(z) = f(z)$ is a meromorphic function. Hence $\eta_2 = f\,\eta_1$, and

$$(\eta_2) = (f\,\eta_1) = (f) + (\eta_1) \sim (\eta_1).$$

§ 9. RIEMANN-ROCH THEOREM

Let $\mathfrak{a}$ be a divisor, and η a differential, and consider the space $L(\,(\eta) - \mathfrak{a}\,)$. If $f \in L(\,(\eta) - \mathfrak{a}\,)$, then $(f) + (\eta) - \mathfrak{a} \geqslant 0$, $(f\eta) \geqslant \mathfrak{a}$, since $(f) + (\eta) = (f\eta)$. By Lemma 8.7, for every differential γ we have $\gamma = g\eta$, where g is a meromorphic function. Thus $L(\,(\eta) - \mathfrak{a}\,)$ has the same dimension as $\{\gamma : (\gamma) \geqslant \mathfrak{a}\}$, since in this case $g \in L(\,(\eta) - \mathfrak{a}\,)$. With this preliminary remark we can state the Riemann-Roch theorem as follows:

THEOREM 9.1 (RIEMANN-ROCH THEOREM). <u>Let</u> $\mathfrak{a}$ <u>be</u> a <u>divisor</u> <u>on</u> a <u>compact</u> <u>Riemann</u> <u>surface</u> R <u>with</u> <u>genus</u> g. <u>Then</u>

(9.1) $\underline{\dim}\, \mathfrak{a} = \underline{\deg}\, \mathfrak{a} + 1 - g + \underline{\dim}\, [\,(\eta) - \mathfrak{a}\,]$, <u>where</u> (η) <u>is any canonical divisor</u>.

PROOF. We divide our proof into several cases.

CASE 1. $\mathfrak{a} = 0$.

In this case

$$\dim(\eta) = \dim L((\eta) - \mathfrak{a}) = \dim \{w: (w) \geqslant 0\}$$
$$= \dim \{w: w \text{ holomorphic }\} = g$$

by Theorem 3.1. Since $\dim 0 = 1$, and $\deg 0 = 0$, equation (9.1) is true.

CASE 2. $\mathfrak{a} > 0$.

For the way this proof is organized, this will be the critical case. Let $\mathfrak{a} = \sum n_p P$ with $n_p > 0$ for all P in the summation. For each of these P select a uniformizing variable $z = z_p$. Set

$$\mathcal{K}(\mathfrak{a}) = \{\eta : \eta \text{ is of 2nd kind, } V_p(\eta) \geqslant -n_p - 1, \text{ and all A- periods of } \eta \text{ vanish}\}.$$

If $\eta \in \mathcal{K}(\mathfrak{a})$, then locally

$$\eta = (c_{-n_p-1} z^{-n_p-1} + \dots + c_{-2} z^{-2} + \text{reg}) \, dz,$$

and this expansion is uniquely determined by its principal parts according to Theorem 4.2 (The Existence Theorem) and the fact that its A- periods vanish. At each point P, the constants $C_{-2}, \dots, C_{-n_p-1}$ are arbitrary, and altogether there are $\sum n_p$ of such constants. Whence

(9.2) $$\dim \mathcal{K}(\mathfrak{a}) = \deg \mathfrak{a}.$$

Consider the mapping $f \to df$ of $L(\mathfrak{a})$ into $\mathcal{K}(\mathfrak{a})$ by noting that $df \in \mathcal{K}(\mathfrak{a})$. It is evident that the kernel $N = \{f: df = 0\}$ of the mapping is the field of complex numbers, so that $\dim N = 1$. Thus

(9.3) $$\dim \mathfrak{a} = 1 - \dim d[L(\mathfrak{a})].$$

The image $d[L(\mathfrak{a})]$ consists of all elements η of $\mathcal{K}(\mathfrak{a})$ whose B- periods vanish. For, if $\eta \in \mathcal{K}(\mathfrak{a})$ and the B- periods of η vanish, then

$$\oint_{\gamma} \eta = 0$$

for each cycle γ on R avoiding the poles of η, since this is true for the A- and B- cycles, and the residues of η vanish. Hence $f = \int \eta$ is a well-defined meromorphic function on R with $df = \eta$.

Now let $\eta \in \mathcal{L}(\sigma)$, and w be a holomorphic differential with A- periods $a_1, \cdots, a_g$. By using the formula (7.16), we have

$$\sum a_j \oint_{B_j} \eta \cdot = 2\pi i \sum \operatorname{res}_p (f\eta),$$

where $w = df$ on the polygon π. Since a_j are arbitrarily prescribed, we can set $a_j = 0$ so that $\sum \operatorname{res}_p (f\eta) = 0$. Conversely, if $\sum \operatorname{res}_p (f\eta) = 0$ for each $w = df$ of the first kind, then

$$b_j' = \oint_{B_j} \eta = 0 ,$$

or

$$\eta \in d[L(\sigma)].$$

Denote by D the g-dimensional space of holomorphic differentials $w = df$, and consider the pairing

$$< w, \eta > = \sum \operatorname{res}_p (f\eta)$$

on $(D, \mathcal{L}(\sigma))$ to the complex numbers. This induces a mapping of D into the conjugate space $\mathcal{L}(\sigma)^*$ of $\mathcal{L}(\sigma)$:

$$w \rightarrow < w, \cdot > .$$

Denote the image of this induced mapping by D^*. Since $\eta \in d[L(\sigma)]$ if and only if $\sum \operatorname{res}_p (f\eta) = 0$, $d[L(\sigma)]$ is precisely the subspace of $\mathcal{L}(\mathfrak{A})$ annihilated by the space D^* of linear functionals, that is, $<w, \eta> = 0$ for $\eta \in L(\mathfrak{A})$. Thus, in consequence of equation (9.2),

(9.4) $\dim d[L(\mathfrak{A})] = \dim L(\mathfrak{A}) - \dim D^*$

$$= \deg \mathfrak{A} - \dim D^*.$$

It remains to determine $\dim D^*$. Since by Theorem 3.1 the space D has complex dimension g, we have

(9.5) $\dim D^* = g - \dim \{\text{kernel } [w \rightarrow <w, \cdot>]\}$. The kernel in equation (9.5) can be found as follows. Suppose $w \in D$, and $<w, \eta> = 0$ for all $\eta \in \mathcal{L}(\mathfrak{A})$. Set

(9.6) $w = df = (e_0 + e_1 z + \dots)\, dz.$

Then

$$f = f(P) = e_0 z + \tfrac{1}{2} e_1 z^2 + \dots .$$

Thus, locally, if

$$= (c_{-n_p-1} z^{-n_p-1} + \dots + c_{-2} z^{-2} + \text{reg}) dz,$$

then,

$$\text{res}_p(f\eta) = e_0 c_{-2} + \tfrac{1}{2} e_1 c_{-3} + \dots + e_{n_p-1}\, c_{-n_p-1}/n_p \, .$$

Since the principal parts of η may be prescribed arbitrarily, we conclude that $\sum \text{res}_p(f\eta) = 0$ for <u>all</u> η if and only if

(9.7) $e_0 = e_1 = \dots = e_{n_p-1} = 0$ for all P.

But equation (9.7) reduces equation (9.6) to

$$w = e_{n_p} z^{n_p} + e_{n_p-1}\, z^{n_p+1} + \dots ,$$

so that $V_p(w) \geqslant n_p$ for all P, or $V_p(w) - n_p \geqslant 0$ for all P, that is $(w) \geqslant \mathfrak{A}$. Thus

(9.8) $\text{kernel } [w \rightarrow <w, \cdot>] = \{w : (w) \geqslant \mathfrak{A}\},$

and therefore this kernel has dimension

(9.9) $\dim \{w : (w) \geqslant \mathfrak{A}\} = \dim [(w) - \mathfrak{A}].$

By combining equations (9.3),(9.4),(9.5),(9.8),(9.9), and making use of the remark at the beginning of this section we hence obtain the required formula (9.1) for Case 2.

CASE 3. $\dim \mathfrak{A} > 0$.

In this case, there exists a non-zero function f such that

$$\mathfrak{B} = (f) + \mathfrak{A} \geqslant 0.$$

For, if there is no such f, then $L(\mathfrak{A}) = 0$, and whence $\dim \mathfrak{A} = 0$, which is a contradiction. By applying Cases 1 and 2 of the Riemann-Roch theorem to $\mathfrak{B}$ and making use of equations (8.2),(8.3) we hence complete the proof of this case.

COROLLARY 9.1. <u>If</u> (η) <u>is canonical, then</u>

(9.10) $\deg(\eta) = 2g - 2.$

PROOF. First we assume $g > 0$. Then by Theorem 3.1, $\dim(\eta) = g > 0$. Application of Case 2 of the Riemann-Roch Theorem to $\mathfrak{A} = (\eta)$ gives

$$g = \dim(\eta) = \deg(\eta) + 1 - g - \dim 0,$$

which is equivalent to equation (9.10).

Next we consider the case $g = 0$. Applying Case 1 of the Riemann-Roch Theorem to $\mathfrak{A} = P$, a single point, we then obtain

$$\dim \mathfrak{A} = 1 + 1 - 0 + \dim [(\eta) - P] \geqslant 2,$$

since $\dim [(\eta) - P] \geqslant 0$. Thus

(9.11) $\dim L(P) = \dim \{f : (f) + P > 0\} > 2,$

which means that there exists at least two linearly independent functions belonging to $L(P)$. From equation (9.11) it also follows that if $z \in L(P)$, then $V_P(z) \geqslant -1$, which implies that z has at worst a first order pole at P. Furthermore, z is holomorphic at every other point on R; for, if $Q \neq P$ and $V_Q(z) < 0$, then $z \notin L(P)$. Thus by Corollary 4.1, for each complex number a there is a single a-place Q_a (of the first order) of z. The mapping $Q \rightarrow z(Q)$ is therefore a conformal equivalence of R onto the z-sphere. We compute (dz). At P, $t = 1/z$ is a uniformizing variable, and therefore $dz = -t^{-2}\,dt$. At Q_a, $t = z - a$ is a uniformizing variable, and so $dz = dt$. Hence

$$(dz) = -2P, \ \deg\,(dz) = -2 = 2g - 2,$$

since $g = 0$. Hence Corollary 9.1 is proved.

In view of this information we may present the Riemann-Roch Theorem in a symmetric form called the

BRILL-NOETHER RECIPROCITY LAW. <u>If $\mathfrak{a}$ and $\mathfrak{b}$ are divisors such that</u>

$$\mathfrak{a} + \mathfrak{b} = (\eta) \tag{9.12}$$

<u>is canonical, then</u>

$$\dim \mathfrak{a} - \tfrac{1}{2}\deg \mathfrak{a} = \dim \mathfrak{b} - \tfrac{1}{2}\deg \mathfrak{b}\ . \tag{9.13}$$

PROOF. By using equation (9.12) and Corollary 9.1 we obtain

$$\deg \mathfrak{b} = \deg \eta - \deg \mathfrak{a} = 2g - 2 - \deg \mathfrak{a}\ . \tag{9.14}$$

substitution of equation (9.14). Conversely, by substituting equation (9.14) in equation (9.1) we may obtain equation (9.13).

CASE 4, dun $[(\eta) - \mathfrak{a}] > 0$, where (η) is any canonical divisor.

Set $\mathfrak{b} = (\eta) - \mathfrak{a}$. Then by applying Case 3 of the Riemann-Roch Theorem to $\mathfrak{b}$ and making use of equation (9.10) we can immediately obtain equation (9.1) for this case.

CASE 5. $\mathfrak{a} + \mathfrak{b} = (\eta)$, and $\dim \mathfrak{a} = \dim \mathfrak{b} = 0$, where (η) is any canonical divisor.

At first we assert that

$$\deg \mathfrak{a} \leq g - 1. \tag{9.15}$$

For otherwise, suppose

$$\deg \mathfrak{a} \geq g. \tag{9.16}$$

Select any divisor $\mathfrak{a}_1$ such that $\mathfrak{a} \leqslant \mathfrak{a}_1$ and $\mathfrak{a}_1 \geqslant 0$. By applying Cases 1 and 2 of the Riemann-Roch Theorem to $\mathfrak{a}_1$ we obtain

(9.17) $$\dim \mathfrak{a}_1 \geqslant \deg \mathfrak{a}_1 + 1 - g$$

since $\dim [(\eta) - \mathfrak{a}_1]$ is always nonnegative. On the other hand, by means of equation (8.7) for the relation $\mathfrak{a} \leqslant \mathfrak{a}_1$ we have

(9.18) $$\dim \mathfrak{a} \geqslant \dim \mathfrak{a}_1 - \deg \mathfrak{a}_1 + \deg \mathfrak{a} \quad .$$

Combination of equations (9.16),(9.17),(9.18) thus yields

$$\dim \mathfrak{a} \geqslant 1 - g + \deg \mathfrak{a} \geqslant 1 - g + g = 1,$$

which contradicts our hypothesis $\dim \mathfrak{a} = 0$. Thus we have (9.15). Similarly, we can obtain

(9.19) $$\deg \mathfrak{b} \leqslant g - 1.$$

But $\deg \mathfrak{a} + \deg \mathfrak{b} = \deg (\eta) = 2g - 2$ by Corollary 9.1, and therefore we have, in consequence of equations (9.15) and (9.19),

$$\deg \mathfrak{a} = \deg \mathfrak{b} = g - 1,$$

which is the Riemann-Roch Theorem for Case 5 according to Brill-Noether Reciprocity Law.

REMARK. The method used in Case 2 may be used to prove the Riemann-Roch Theorem in general, but it is technically complicated.

LEMMA 9.1. <u>If</u> $\deg \mathfrak{a} < 0$, <u>then</u> $\dim \mathfrak{a} = 0$.

PROOF. Suppose there exists $f \in L(\mathfrak{a})$ so that $\mathfrak{a} + (f) \geqslant 0$, and let $\mathfrak{b} = \mathfrak{a} + (f)$. Then by equation (8.2) we have

$$\deg \mathfrak{a} = \deg \mathfrak{b} \geqslant 0,$$

which contradicts our hypothesis $\deg \mathfrak{a} < 0$. Thus $L(\mathfrak{a}) = 0$, $\dim \mathfrak{a} = 0$.

COROLLARY 9.2. <u>If</u>

(9.20) $$\deg \mathfrak{a} \geqslant 2g - 1,$$

then

(9.21) $$\dim \mathfrak{A} = \deg \mathfrak{A} + 1 - g.$$

PROOF. From equations (9.10), (9.20) it follows that

$$\deg [(\eta) - \mathfrak{A}] = 2g - 2 - \deg \mathfrak{A} \leq -1 < 0.$$

Thus, by Lemma 9.1 we obtain

$$\dim [(\eta) - \mathfrak{A}] = 0,$$

which reduces equation (9.1) immediately to equation (9.21).

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VITA

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