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A PARAMETRIX FOR $\bar{\partial}$ AND RIEMANN–ROCH IN ČECH THEORY

DOMINGO TOLEDO and YUE LIN L. TONG*

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INTRODUCTION

IN THIS paper we construct a very natural and explicit parametrix for the Dolbeault complex, i.e. a chain homotopy between the identity and a smoothing operator. Our construction is based on special properties of $\bar{\partial}$, rather than on the general theory of elliptic equations.

Its relation to other constructions and to the theory of elliptic complexes can be explained as follows. For a single elliptic operator between two bundles over a manifold one constructs a parametrix by patching together the kernels of local parametrixes. This works because the local kernels are essentially unique: any two differ by a smooth kernel. But for a complex this method fails; the difference of two local kernels is in general as singular as each of them. The usual way of getting around this difficulty is to reduce the problem to a single operator—the Laplace operator associated to a metric (cf. [2, §6]).

For $\bar{\partial}$ this procedure gives a parametrix that depends not only on the complex structure but also on a metric. And even though very simple and explicit local kernels are known for $\bar{\partial}$ (the Bochner–Martinelli kernel), the same is not true for the Laplacian of a curved metric.

Our first observation is that there is a very natural way to proceed without introducing an auxiliary Laplacian. The non-uniqueness of local kernels gives rise to a sequence of kernel-valued Čech cochains, which we call a Čech parametrix. A global parametrix can be built from these, as explained in §3 for the $\bar{\partial}$-complex. This is the purely formal part of the theory, and the same argument works for any locally acyclic elliptic complex (cf. [17, §5]).

Constructing a Čech parametrix requires solving a sequence of local equations, each of which has an infinite number of solutions. The main purpose of this paper is to construct explicit solutions of these equations which have canonical local formulas in the transition functions of the manifold. This is formulated more precisely in §4 in terms of an equivalent "universal problem". The solutions we obtain are of a very special form, which we call "regular kernels" in §2. They satisfy some strong homogeneity conditions and are much simpler than the more general pseudo-differential kernels. Indeed, the parametrix we obtain is more explicit than any that can be constructed for other elliptic systems, even the exterior derivative. This reflects the fact that even though solving these $\bar{\partial}$-equations appears to be an analytic problem, in effect it can be reduced to an algebraic one.

The reason for this is that the construction of local fundamental solutions for $\bar{\partial}$ is purely algebraic: If $f_1, \ldots, f_n$ generate the ideal of the diagonal in $\mathbf{C}^n \times \mathbf{C}^n$ and $g_1, \ldots, g_n$ are suitable smooth functions in $\mathbf{C}^n \times \mathbf{C}^n - \Delta$ such that $f_1 g_1 + \cdots + f_n g_n = 1$, then an explicit formula in f_i, g_i gives a fundamental solution. The Bochner–Martinelli kernel arises from the simplest choice of f_i, g_i; cf. [12]. Moreover this construction rests on the *coherence* of the sheaf $\mathcal{I}_\Delta$ of ideals of the diagonal, i.e., on the Koszul complex. It is then reasonable to expect that this method can be generalized to give a global solution on a manifold by exploiting the *global* implications of the coherence of $\mathcal{I}_\Delta$. This turns out to be the case, but not in an obvious way, and the precise details are rather technical. The method is outlined in some detail in §7, and the actual problem is solved in §§8 to 10. Sections 6 and 12 deal with the technicalities needed to handle the $\bar{\partial}$-complex with coefficients in a bundle. All this machinery is closely related to Grothendieck's duality theory, as will be explained elsewhere.

The Čech parametrix gives a cochain, defined by a local formula in the transition functions, that represents the Euler characteristic of the Dolbeault complex. In [1] Atiyah represents the

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Chern numbers by cochains of this type. Combining the two we obtain a direct proof of the Hirzebruch–Riemann–Roch Theorem[13] much in the spirit of Gilkey's proof of the signature theorem[10, 3]. Namely, we give conditions that characterize the Atiyah–Chern classes, Theorem II of §4 and Theorem II′ of §6, and check that these are satisfied by the (*a priori* very complicated) formula given by the parametrix. This characterization of Chern classes is very straightforward and certainly not surprising in view of Gilkey's Theorem. As far as the Riemann–Roch formula is concerned our main contribution is the derivation of an *a priori*-local formula for the Euler characteristic in the Čech context. The situation here is the opposite of the one in the Riemannian approach to index problems, where the local formulas had been known for a long time and the problem was to identify them with characteristic classes. The Čech approach to Riemann–Roch had long been advocated by Atiyah and Bott (c.f., e.g. [4]). This approach also fits in most naturally with the philosophy of Gel'fand–Fuchs cohomology. We do not elaborate further on this point, and refer the reader to [5, 6, 9].

Our presentation of this proof is self-contained, except for the following points. (1) We need the fact that the Euler characteristic can be expressed from a parametrix. This requires some knowledge of topological tensor products (Künneth formula) and the result of Atiyah and Bott quoted in Proposition (1.9) contains the relevant information. (2) Invariance theory for $GL(n, \mathbf{C})$ is used in the proofs of Theorems II, II′. (3) We assume familiarity with Hirzebruch's technique of multiplicative sequences[13] and with the Chern–Weil theory of characteristic classes.

The first direct proof of Riemann–Roch was given by Patodi[14] for Kähler manifolds; other proofs by direct analysis are given in [3, 11]. They are all based on the heat equation and Riemannian geometry. But Gilkey[10] showed that the Kähler condition, i.e. the compatibility of the metric with the complex structure, is essential to this approach. Ours is the first direct proof that is valid for any complex manifold.

We became involved in this problem through discussions with L. Sibner and R. Sibner concerning their papers[15, 16]. We are very grateful to them, and to M. Atiyah, R. Bott and A. Haefliger, for several helpful conversations.

This is a considerably revised version of our original paper. Some of the improvements announced in [18] have been incorporated here. We are very grateful to N. R. O'Brian for some suggestions on twisting cochains that led to the present §8 and further clarifications of the original argument, and to O. Gabber for his careful reading of the manuscript and the correction of many technical slips.

§1. PRELIMINARIES AND NOTATION

Throughout this paper M denotes a complex analytic manifold of complex dimension n and E a holomorphic vector bundle over M with fibre $\mathbf{C}^m$. We always assume $n \geq 2$, since the case $n = 1$ is treated in [15]. $\mathcal{U} = \{U_\alpha\}$ will always denote a locally finite open covering of M by relatively compact coordinate charts such that $E|U_\alpha$ is trivial. Thus we have holomorphic coordinate maps $\varphi_\alpha\colon U_\alpha \to \mathbf{C}^n$ and holomorphic bundle isomorphisms $\psi_\alpha\colon E|U_\alpha \to U_\alpha \times \mathbf{C}^m$. We write $\{\varphi_{\alpha\beta}\}$ for the transition functions of M and $\{\psi_{\alpha\beta}\}$ for the transition functions of E, i.e. $\varphi_{\alpha\beta} = \varphi_\alpha\varphi_\beta^{-1}$ and $\psi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(m)$ is defined by $\psi_\alpha\psi_\beta^{-1}(x, v) = (x, \psi_{\alpha\beta}(x)(v))$.

We use standard notations for forms, bundles, Čech cochains. In particular $\Lambda^{p,q}(M, E) = \Lambda^{p,q}(M) \otimes E$ is the bundle of forms of type (p, q) with coefficients in E. If U is an open subset of M, $A^{p,q}(U, E)$ denotes the smooth sections of this bundle over U and $A_c^{p,q}(U, E)$ those with compact support. We have the Dolbeault complex (or $\bar\partial$-complex) over U with coefficients in E:

$$A^{p,*}(U, E)\colon A^{p,0}(U, E) \xrightarrow{\bar\partial} A^{p,1}(U, E) \xrightarrow{\bar\partial} \cdots \longrightarrow A^{p,n}(U, E),$$

and $(\ker \bar\partial) \cap A^{p,0}(U, E)$ is the space of holomorphic p-forms with coefficients in E, $\Omega^p(U, E)$.

$L(E)$ is the bundle of linear endomorphisms of E, with fibre $L(\mathbf{C}^m)$. If $\alpha \in A^{p,q}(U, E)$ and $\beta \in A^{r,s}(U, F)$, we write $\alpha \wedge \beta \in A^{p+r,q+s}(U, E \otimes F)$ for the natural wedge product with coefficients in $E \otimes F$.

For each p we have the Čech bicomplex $C^*(\mathcal{U}, A^{p,*}(E))$ where

$$C^r(\mathcal{U}, A^{p,s}(E)) = \prod_{(\alpha_0 \ldots \alpha_r)} A^{p,s}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_r}, E) \tag{1.1}$$

(product over all r-simplices in the nerve of $\mathcal{U}$) with commuting differentials δ, $\bar\partial$ of bidegrees $(1,0)$, $(0,1)$ given respectively by the Čech coboundary and the Cauchy–Riemann operator.

Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\mathcal{U}$. Then we can define a map

$$\mathcal{D}: C^r(\mathcal{U}, A^{p,s}(E)) \to A^{p,r+s}(M, E)$$

by

$$\mathcal{D}(c^r) = \sum_{\alpha_0,\dots,\alpha_r} \rho_{\alpha_r}\bar\partial\rho_{\alpha_{r-1}}\dots\bar\partial\rho_{\alpha_0}c^r_{\alpha_0\dots\alpha_r}. \tag{1.2}$$

Note that this definition makes sense since the support of $\rho_{\alpha_r}\bar\partial\rho_{\alpha_{r-1}}\dots\bar\partial\rho_{\alpha_0}$ is a compact subset of the set $U_{\alpha_0}\cap\cdots\cap U_{\alpha_r}$ on which $c_{\alpha_0\dots\alpha_r}$ is defined. We need the following fact:

$$\bar\partial\mathcal{D}(c^r) = (-1)^{r+1}\mathcal{D}(\delta c^r) + (-1)^r\mathcal{D}(\bar\partial c^r). \tag{1.3}$$

This follows from a simple computation:

$$\begin{aligned}\mathcal{D}(\delta c^r) &= \sum_{\alpha_0\dots\alpha_{r+1}} \rho_{\alpha_{r+1}}\bar\partial\rho_{\alpha_r}\dots\bar\partial\rho_{\alpha_r}\sum_{i=0}^{r+1}(-1)^i c_{\alpha_0\dots\hat\alpha_i\dots\alpha_{r+1}}\\ &= \sum_{i=0}^{r+1}(-1)^i\sum_{\alpha_0\dots\alpha_{r+1}} \rho_{\alpha_{r+1}}\bar\partial\rho_{\alpha_r}\dots\bar\partial\rho_{\alpha_0}c_{\alpha_0\dots\hat\alpha_i\dots\alpha_{r+1}}\\ &= (-1)^{r+1}\sum_{\alpha_0\dots\alpha_r}\bar\partial\rho_{\alpha_r}\dots\bar\partial\rho_{\alpha_0}c_{\alpha_0}\dots_{\alpha_r},\end{aligned}$$

since $\Sigma\rho_{\alpha_{r+1}} = 1$ and $\Sigma\bar\partial\rho_{\alpha_i} = 0$. But

$$\bar\partial\mathcal{D}(c^r) = \sum\bar\partial\rho_{\alpha_r}\dots\bar\partial\rho_{\alpha_0}c_{\alpha_0\dots\alpha_r} + (-1)^r\sum\rho_{\alpha_r}\bar\partial\rho_{\alpha_{r-1}}\dots\bar\partial\rho_{\alpha_0}\bar\partial c_{\alpha_0\dots\alpha_r},$$

and comparing the two expressions, (1.3) follows.

$\mathcal{D}$ is related to the Dolbeault isomorphism as follows. The total complex associated to (1.1) contains both the $\bar\partial$-complex $A^{p,*}(M,E)$ and the Čech complex $C^*(\mathcal{U}, \Omega^p(E))$ as subcomplexes. It is well-known that if all the intersections $U_{\alpha_0}\cap\cdots\cap U_{\alpha_r}$ are $\bar\partial$-acyclic both these inclusions induce isomorphisms in cohomology. By composing one with the inverse of the other we obtain the Dolbeault isomorphism $H^*(M, \Omega^p(E)) \xrightarrow{\sim} H^{p,*}(M, E)$. (This may differ by sign from other definitions.) By (1.3) the map

$$\oplus(-1)^r\frac{(r+1)}{2}\mathcal{D}: C^*(\mathcal{U}, \Omega^p(E)) \to A^{p,*}(M, E)$$

is a chain map, and it is not hard to see that it induces the Dolbeault isomorphism. But we will not use this fact; we will work directly with $\mathcal{D}$, which is defined for any covering $\mathcal{U}$.

Characteristic classes. We review briefly Atiyah's description of Chern classes [1]. Let D_α be the flat holomorphic connection for $E|U_\alpha$ given by the local trivialization, i.e. $D_\alpha s = \psi_\alpha^{-1}d(\psi_\alpha s)$, where s is a section of E over U_α. Then over $U_\alpha\cap U_\beta$, $D_\alpha - D_\beta$ is a differential operator of order 0, which we denote by $\theta_{\alpha\beta}$. Explicitly $\theta_{\alpha\beta}s = \psi_\beta^{-1}(\psi_{\alpha\beta}^{-1}\,d\psi_{\alpha\beta})\psi_\beta$. We can regard $\theta_{\alpha\beta}$ as a holomorphic one-form on $U_\alpha\cap U_\beta$ with coefficients in $L(E)$. Thus we get a Čech cochain $\theta = \{\theta_{\alpha\beta}\} \in C^1(\mathcal{U}, \Omega^1(L(E)))$ which is clearly a cocycle. We can take its powers $\theta^p \in C^p(\mathcal{U}, \Omega^p(\otimes^p(L(E)))$ defined by

$$\theta^p_{\alpha_0\dots\alpha_p} = \theta_{\alpha_0\alpha_1}\wedge\cdots\wedge\theta_{\alpha_{p-1}\alpha_p} \in \Omega^p(U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}, \otimes^p L(E)).$$

Given a symmetric, $GL(m)$-invariant linear map $f: \otimes^p L(\mathbf{C}^m) \to \mathbf{C}$, we can apply f to the coefficients of θ^p and obtain a cocycle $f(\theta^p) \in C^p(\mathcal{U}, \Omega^p)$ by

$$f(\theta^p)_{\alpha_0\dots\alpha_p} = f(\theta_{\alpha_0\alpha_1}\wedge\cdots\wedge\theta_{\alpha_{p-1}\alpha_p}) \in \Omega^p(U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}). \tag{1.4}$$

Let D be the global C^∞ connection of type $(1,0)$ on E defined by $D = \Sigma\rho_\alpha D_\alpha$, and let $K \in A^2(M, L(E))$ be its curvature form. Then $K^p \in A^{2p}(M, \otimes^p L(E))$ and the Chern class of E corresponding to the invariant polynomial f is represented by $f(K^p) \in A^{2p}(M)$. The basic fact that we need is that for $p = n$, i.e., if f is a polynomial *of degree n*, then

$$f(K^n) = \mathcal{D}f(\theta^n). \tag{1.5}$$

The reason is the following. Since D is of type $(1,0)$, K has no component of type $(0,2)$, and thus we have $K = K^{1,1} + K^{2,0}$. A simple computation shows that $K^{1,1} = \mathcal{D}(\theta)$ and from this it follows easily that $f((K^{1,1})^p) = \mathcal{D}f(\theta^p)$. But for $p = n$, $(K^{1,1})^n = K^n$ and (1.5) follows.

Applying (1.5) to the holomorphic tangent bundle TM which has local trivializations $\dot\varphi_\alpha$ and transition functions $\dot\varphi_{\alpha\beta} \circ \varphi_\beta$ ($\cdot$ denotes the Jacobian), we obtain the following formula for the *Chern numbers* of M:

PROPOSITION 1.6. *Let M be compact and let $\theta \in C^1(\mathcal{U}, \Omega^1(L(TM)))$ be defined by*

$$\theta_{\alpha\beta} = \dot\varphi_\beta^{-1}(\dot\varphi_{\alpha\beta}^{-1} \circ \varphi_\beta d(\dot\varphi_{\alpha\beta} \circ \varphi_\beta))\dot\varphi_\beta.$$

Then the Chern number of M corresponding to the invariant polynomial f of degree n on $L(\mathbf{C}^n)$ is $\int_M \mathcal{D}f(\theta^n)$.

More generally we get a formula for the *mixed Chern numbers* of M, E (polynomials in Chern classes of M and E of total weight n):

If f is an invariant polynomial on $L(\mathbf{C}^n) \oplus L(\mathbf{C}^m)$, then we can write f uniquely (up to constants) as $f = \Sigma\, g_i h_i$ where g_i, h_i are invariant polynomials on $L(\mathbf{C}^n)$, $L(\mathbf{C}^m)$ respectively and $\deg g_i + \deg h_i = n$. Let $K(M)$, $K(E)$ be curvature forms on M, E and write simply $g_i(K(M))$, $h_i(K(E))$ for the corresponding characteristic forms.

PROPOSITION 1.7. *Let M be compact and let $\theta = \theta(M) \oplus \theta(E)$. Then*

$$\int_M \mathcal{D}f(\theta^n) = \sum \int_M g_i(K(M)) \wedge h_i(K(E)).$$

Products. If E, F are bundles over M, we write $E \boxtimes F$ for their exterior product over $M \times M$, whose fibre at (x, y) is $E_x \otimes F_y$. If W is open in $M \times M$, $A^{(p,q),(r,s)}(W, E \boxtimes F)$ denotes the smooth sections of $\Lambda^{p,q}(M, E) \boxtimes \Lambda^{r,s}(M, F)$ over W. This is the subspace of $A^{p+r,p+s}(W, E \boxtimes F)$ of forms of degree (p, q) in the first factor and (r, s) in the second factor. For each p, r, $A^{(p,*),(r,*)}(W, E \boxtimes F)$ is a subcomplex of the $\bar\partial$-complex $A^{p+r,*}(W, E \boxtimes F)$. Note that there is an inclusion with dense image

$$A^{p,q}(M, E) \otimes A^{r,s}(M, F) \to A^{(p,q),(r,s)}(M \times M, E \boxtimes F). \tag{1.8}$$

We write $\Omega^{p,q}(W, E \boxtimes F)$ for $\ker \bar\partial \cap A^{(p,0),(q,0)}(W, E \boxtimes F)$.

We will use the symbol "$\cdot$" for all the natural pairings obtained by combining wedge product and trace. For example

$$(\Lambda^{0,p}(M, E) \boxtimes \Lambda^{n,q}(M, E^*)) \otimes \pi_2^*\Lambda^{0,n-q}(M, E) \to \Lambda^{0,p}(M, E) \boxtimes \Lambda^{n,n}(M)$$

and

$$(\Lambda^{0,p}(M, E) \boxtimes \Lambda^{n,q}(M, E^*)) \otimes (\Lambda^{n,n-p}(M, E^*) \boxtimes \Lambda^{0,n-q}(M, E)) \to \Lambda^{n,n}(M) \boxtimes \Lambda^{n,n}(M).$$

Explicitly these are given by $(\alpha_x \otimes \beta_y) \cdot \gamma_y = \alpha_x \otimes \operatorname{tr}(\beta_y \wedge \gamma_y)$ and $(\alpha_x \otimes \beta_y) \cdot (\gamma_x \otimes \delta_y) = (-1)^{(n+q)p} \operatorname{tr}(\alpha_x \wedge \gamma_x) \otimes \operatorname{tr}(\beta_y \wedge \delta_y)$, where tr denotes the evaluation map on $E_x \otimes E_x^*$, $E_y^* \otimes E_y$. We will also use the composition

$$A^{(0,q),(n,n-q)}(M \times M, E \boxtimes E^*) \xrightarrow{\Delta^*} A^{n,n}(M, E \otimes E^*) \xrightarrow{\operatorname{tr}} A^{n,n}(M),$$

where Δ: $M \to M \times M$ is the diagonal map.

If $s \in A^{(0,p),(n,n-q)}(M \times M, E \boxtimes E^*)$, then s determines an operator S: $A_c^{0,q}(M, E) \to A^{0,p}(M, E)$ by the formula

$$(S\alpha)(x) = \int_{y \in M} s(x, y) \cdot \alpha(y).$$

An operator of this form is called a *smoothing operator* and s is called the *kernel* of S.

Assume now that M is *compact* and that $S = \oplus S_q$ is smoothing, where S_q: $A^{0,q}(M, E) \to A^{0,q}(M, E)$. If we define $\operatorname{tr} S_q = (-1)^q \int_M \operatorname{tr} \Delta^* s_q$, this agrees with the usual definition of trace on the smoothing operators of finite rank, i.e. those whose kernels lie in the image of (1.8). The following Proposition is then an immediate consequence of the elementary part of the Atiyah–Bott–Lefschetz formula (alternating sum formula for smooth endomorphisms, [2, §7]). It can also be derived in standard fashion from Serre duality and the Künneth formula for Dolbeault cohomology. $\chi(M, E)$ denotes the Euler characteristic of $A^{0,*}(M, E)$: $\chi(M, E) = \Sigma(-1)^q \dim H^{0,q}(M, E)$.

PROPOSITION 1.9. *Suppose* $S: A^{0,*}(M, E) \to A^{0,*}(M, E)$ *is a smoothing operator that commutes with* $\bar\partial$ *and induces the identity on* $H^{0,*}(M, E)$. *Then* $\chi(M, E) = \int_M \operatorname{tr}\Delta^* s$.

§2. THE KERNEL OF A PARAMETRIX

By a *parametrix* for $A^{0,*}(M, E)$ we mean a linear operator $K = \bigoplus K_q$, where

$$K_q: A_c^{0,q}(M, E) \to A^{0,q-1}(M, E), \qquad 1 \le q \le n$$

such that

$$\bar\partial K + K\bar\partial = 1 - S \quad \text{on } A_c^{0,*}(M, E)$$

where S is a smoothing operator.

We want to construct a parametrix of the form $(K\gamma)(x) = \int_{y \in M} k(x, y) \cdot \gamma(y)$, where $k \in A^{(0,*),(n,*)}(M \times M - \Delta, E \boxtimes E^*)$, but is *singular* on the diagonal. The singularities of k should be nice enough so that this integral converges for each $x \in M$ and gives a smooth form on M. The following definition describes the simplest type of singularities that we can allow in the construction of a parametrix (the terminology is not standard!)

Definition. Let W be open in $M \times M$, $k \in A^{(0,*),(n,*)}(W - \Delta, E \boxtimes E^*)$, and r an integer. We say that k is *regular or order r* if for each $(x_0, x_0) \in W$ there is a neighborhood U of x_0 in M and finitely many coordinate maps $\varphi_1, \ldots, \varphi_p: U \to \mathbf{C}^n$ such that $k|U \times U - \Delta$ is a finite sum of terms of the form

$$s(x, y) \prod_{i=1}^{p} \theta_i(\varphi_i(y) - \varphi_i(x)), \tag{2.1}$$

where $s \in A^{(0,*),(n,*)}(W \cap (U \times U), E \boxtimes E^*)$, each θ_i is a smooth function on $\mathbf{C}^n - 0$, homogeneous of degree $-\lambda_i$ (i.e. $\theta_i(tz) = t^{-\lambda_i}\theta_i(z)$) and

$$\sum_{i=1}^{p} \lambda_i \le 2n + r. \tag{2.2}$$

We write $\mathcal{R}^r(W, E \boxtimes E^*)$ for the regular forms of order r. Then $\mathcal{R}^{r+1}(W, E \boxtimes E^*) \supset \mathcal{R}^r(W, E \boxtimes E^*)$. We write simply $\mathcal{R}(W)$ for $\bigcup_r \mathcal{R}^r(W, E \boxtimes E^*)$. Note that $\bar\partial: \mathcal{R}^r(W, E \boxtimes E^*) \to \mathcal{R}^{r+1}(W, E \boxtimes E^*)$.

Remark. It will be clear to the reader familiar with pseudo-differential operators that if $k \in \mathcal{R}^r(M \times M, E \boxtimes E^*)$, $r < 0$, then k is the kernel of a pseudo-differential operator $K: A_c^{0,*}(M, E) \to A^{0,*}(M, E)$ or order ≤ 0. The following two lemmas are easy consequences of this fact.

LEMMA 2.3. *Let* $U \subset \mathbf{C}^n$ *be open and* $k \in \mathcal{R}^r(U \times U) \otimes L(\mathbf{C}^m)$. *Then*

(i) *For each open subset with compact closure* V *of* U *there is a constant* C_V *such that*

$$|k(z, \zeta)| \le C_V / |z - \zeta|^{2n+r} \quad \textit{for } (z, \zeta) \in V.$$

(ii) *Let* (x, y) *be real coordinates on* $\mathbf{C}^n \times \mathbf{C}^n$. *Then all "derivatives parallel to the diagonal" are regular of order* r:

$$(\partial/\partial x + \partial/\partial y)^\alpha k(x, y) \in \mathcal{R}^r(U \times U) \otimes L(\mathbf{C}^m).$$

Proof. Let φ be a holomorphic diffeomorphism from U to $\mathbf{C}^n$ and θ a smooth function on $\mathbf{C}^n - 0$, homogeneous of degree $-\lambda$.

(i) Since $|\theta(z)| \le C_1 |z|^{-\lambda}$ and on compact subsets of U, $|\varphi(z) - \varphi(\zeta)| \ge C|z - \zeta|$, we get $|\theta(\varphi(z) - \varphi(\zeta))| \le C|z - \zeta|^{-\lambda}$. Applying this estimate to each factor of each term (2.1), the assertion follows from (2.2).

(ii)

$$\left(\frac{\partial}{\partial x^j} + \frac{\partial}{\partial y^j}\right)\theta(\varphi(y) - \varphi(x)) = \sum_i \frac{\partial \theta}{\partial x^i}(\varphi(y) - \varphi(x))\left(\frac{\partial \varphi^i}{\partial x^j}(y) - \frac{\partial \varphi^i}{\partial x^j}(x)\right).$$

But $\partial\theta/\partial x^i$ is homogeneous of degree $-(1+\lambda)$ and locally we can write

$$\frac{\partial\varphi^i}{\partial x^j}(y) - \frac{\partial\varphi^i}{\partial x^j}(x) = \sum_k a_k^{ij}(x,y)(y^k - x^k)$$

for suitable smooth functions a_k^{ij}. Differentiating each term (2.1), this shows that

$$\left(\frac{\partial}{\partial x^j} + \frac{\partial}{\partial y^j}\right)k(x,y) \in \mathcal{R}^r(U\times U)\otimes L(\mathbf{C}^m)$$

and by iteration the second assertion follows.

LEMMA 2.4. *If* $k \in \mathcal{R}^r(M\times M, E\boxtimes E^*)$, $r<0$, *let*

$$(K\gamma)(x) = \int_{Y\in M} k(x,y)\cdot\gamma(y). \tag{2.5}$$

Then $K\gamma \in A^{0,*}(M,E)$ *for all* $\gamma \in A_c^{0,*}(M,E)$.

Proof. Since this is a local problem, we may assume that M is an open set U in $\mathbf{C}^n$ and E is the trivial bundle with fibre $\mathbf{C}^m$.

First note that by Lemma (2.3) (i), k is locally integrable and hence the integral (2.5) is convergent.

Since k is smooth off the diagonal, to show that $K\gamma$ is smooth at x_0 it is enough to check this for γ supported near x_0. We may in fact assume that $x_0 = 0$ and that spt γ is small enough so that $x+y\in U$ whenever $x, y\in$ spt γ. We can then write

$$\int_{y\in M} k(x,y)\cdot\gamma(y) = \int_{y\in U} k(x,x+y)\cdot\gamma(x+y). \tag{2.6}$$

But any derivative $(\partial/\partial x)^\alpha\{k(x,x+y)\cdot\gamma(x+y)\}$ is a sum of derivatives $(\partial/\partial x+\partial/\partial y)^\beta k(x,x+y)\left(\frac{\partial}{\partial x}\right)^{\beta'}\gamma(x+y)$, and by Lemma (2.3) these are all $\leq$ const (depending on $\beta,\beta',\gamma)|x-y|^{-r-2n}$. This estimates allows to differentiate under the integral sign in the right hand side of (2.6), thus showing that $K\gamma$ is smooth.

If $k\in\mathcal{R}^r(M\times M, E\boxtimes E^*)$, $r<0$, and K is defined by (2.5), we call k the *kernel* of the operator K.

It will be convenient to express the fact that K is a parametrix directly in terms of its kernel. In the language of distributions, K is a parametrix if and only if $\bar\partial k = \Delta - s$, where Δ is the Dirac function on the diagonal. More generally we have the following lemma ($M\times M$ is given here the product orientation):

LEMMA 2.7. *Suppose* $k, l\in\mathcal{R}^{-1}(M\times M, E\boxtimes E^*)$, *let* K, L *be the corresponding operators and assume* $k=\Sigma k_q$, $l=\Sigma lq$ *where* k_q *has degree* $(0,q-1)$, $(n,n-q)$ *and* l_q *has degree* $(0,q)$, $(n,n-q)$. *Then*

$$\int_{M\times M} k\cdot\bar\partial\gamma = \int_M \mathrm{tr}\Delta^*\gamma - \int_{M\times M} l\cdot\gamma \tag{2.8}$$

for all $\gamma\in A_c^{(n,*),(0,*)}(M\times M, E^*\boxtimes E)$ *if and only if*

$$\bar\partial K + K\bar\partial = 1 - L \quad \text{on } A_c^{0,*}(M,E). \tag{2.9}$$

Proof. Clearly (2.8) holds for all γ if and only if it holds on the dense subspace $A_c^{n,*}(M,E^*)\otimes A_c^{0,*}(m,E)$. Suppose $\gamma=\alpha\otimes\beta\in A_c^{n,n-q}(M,E^*)\otimes A_c^{0,q}(M,E)$. Then the left hand side of (2.8) reads

$$\int_{M\times M} k_q(x,y)\cdot(\bar\partial\alpha(x)\otimes\beta(y)) + (-1)^q\int_{M\times M} k_{q+1}(x,y)\cdot(\alpha(x)\otimes\bar\partial\beta(y)),$$

which can be rewritten as

$$\int_M \alpha(x)\cdot\bar\partial\int_{y\in M} k(x,y)\cdot\beta(y) + \int_M\alpha(x)\cdot\int_{y\in M} k_{q+1}(x,y)\cdot\bar\partial\beta(y).$$

Thus, by rewriting the right hand side, (2.8) reads

$$\int_M \alpha \cdot \bar\partial K\beta + \int_M \alpha \cdot K\bar\partial\beta = \int_M \alpha \cdot \beta - \int_M \alpha \cdot L\beta.$$

But this holds for all α, β if and only if (2.9) holds.

Example. The Bochner–Martinelli kernel. On $\mathbf{C}^n \times \mathbf{C}^n - \Delta$, let $\omega = \omega(z, \zeta)$ be the form

$$c_n \frac{\Sigma(-1)^i(\bar\zeta^i - \bar z^i)(d\bar\zeta^1 - d\bar z^1)\ldots\widehat{(d\bar\zeta^i - d\bar z^i)}\ldots(d\bar\zeta^n - d\bar z^n)d\zeta^1\ldots d\zeta^n}{|\zeta - z|^{2n}}, \tag{2.10}$$

where the normalizing constant $c_n = -(-1)^{(n(n-1))/2}\,(n-1)!\,(2\pi i)^{-n}$ is chosen so that, if $i_z(\zeta) = (z, \zeta)$, then

$$\int_{S^{2n-1}} i_0^*\omega = 1. \tag{2.11}$$

ω is called the *Bochner–Martinelli kernel.*

Clearly $\omega \in \mathcal{R}^{-1}(\mathbf{C}^n \times \mathbf{C}^n)$ since its coefficients are homogeneous of degree $1-2n$. If we write $\omega = \Sigma\omega_q$, where $\omega_q \in A^{(0,q-1),(n,n-q)}(\mathbf{C}^n \times \mathbf{C}^n - \Delta)$, then we get a corresponding operator $\Omega_q\colon A_c^{0,q}(\mathbf{C}^n) \to A^{0,q-1}(\mathbf{C}^n)$. It is easy to check that ω is $\bar\partial$ closed, and hence, for each $z \in \mathbf{C}^n$, $i_z^*\omega$ is a closed form in $\mathbf{C}^n - z$. If $\rho \in C_c^\infty(\mathbf{C}^n)$, (2.11) implies that

$$\lim_{\epsilon \to 0} \int_{|z-\zeta|=\epsilon} (i_z^*\omega)(\zeta)\rho(\zeta) = \rho(z). \tag{2.12}$$

With this information it is easy to check that

$$\int_{\mathbf{C}^n \times \mathbf{C}^n} \omega \wedge \bar\partial\gamma = \int_{\mathbf{C}^n} \Delta^*\gamma \quad \text{for all } \gamma \in A_c^{(n,*),(0,*)}(\mathbf{C}^n \times \mathbf{C}^n). \tag{2.13}$$

To prove this note that

$$\int_{\mathbf{C}^n \times \mathbf{C}^n} \omega \wedge \bar\partial\gamma = -\lim_{\epsilon \to 0} \int_{|z-\zeta|\geq\epsilon} \bar\partial(\omega \wedge \gamma),$$

and this limit can be evaluated by making the change of variable $\zeta = \xi + z$ and evaluating first the integral over the fibre $|\xi| = \epsilon$ using (2.12).

By Lemma (2.7) we get the well-known fact that Ω is a *fundamental solution* for $\bar\partial$ on $\mathbf{C}^n$, i.e., a parametrix in which the smooth error term vanishes: $\bar\partial\Omega + \Omega\bar\partial = 1$, on $A_c^{0,*}(\mathbf{C}^n)$. More generally, if U is open in $\mathbf{C}^n$, then

$$\omega \otimes 1 \in \mathcal{R}^{-1}(U \times U) \otimes L(\mathbf{C}^m) \tag{2.14}$$

and the corresponding operator is a fundamental solution for $A_c^{0,*}(U) \otimes \mathbf{C}^m$.

§3. THE ČECH PARAMETRIX

The last example gives an explicit fundamental solution for the trivial bundle over an open set in $\mathbf{C}^n$. Thus for arbitrary M, E we always have local fundamental solutions on $A_c^{0,*}(U_\alpha, E)$ with kernel $\omega_\alpha \in \mathcal{R}^{-1}(U_\alpha \times U_\alpha, E \boxtimes E^*)$ given by $\omega_\alpha(x, y) = (\varphi_\alpha \times \varphi_\alpha)^*\omega(x, y) \otimes \psi_\alpha^{-1}(x)\psi_\alpha(y)$.

We could try to construct a global parametrix by patching these together as follows. Take functions $\rho_\alpha \in C_c^\infty(U_\alpha \times U_\alpha)$ such that

$$\sum \rho_\alpha = 1 \text{ on some neighborhood of } \Delta \text{ in } M \times M. \tag{3.1}$$

Then let $k^0 = \Sigma\rho_\alpha\omega_\alpha \in \mathcal{R}^{-1}(M \times M, E \boxtimes E^*)$. Applying (2.13) to $\rho_\alpha\gamma$ over each $U_\alpha \times U_\alpha$ and summing over α we get

$$\int_{M \times M} k^0 \cdot \bar\partial\gamma = \int_M \operatorname{tr} \Delta^*\gamma + \int_{M \times M} (\Sigma\bar\partial\rho_\alpha\omega_\alpha) \cdot \gamma, \tag{3.2}$$

for all $\gamma \in A_c^{(n,*),(0,*)}(M \times M, E^* \boxtimes E)$. By Lemma (2.7) the corresponding operator K^0 satisfies $\bar\partial K^0 + K^0\bar\partial = 1 - L^0$, where the kernel of L^0 is $l^0 = -\Sigma\bar\partial\rho_\alpha\omega_\alpha$. K^0 is *not* a parametrix since l^0 is not smooth. But on the neighborhood where (3.1) holds we have $-\sum_\alpha \bar\partial\rho_\alpha\omega_\alpha = \sum_{\alpha,\beta} \rho_\beta\bar\partial\rho_\alpha(\omega_\beta - \omega_\alpha)$, and since the ω_α are smooth outside the diagonal, this holds on $M\times M$ up to a smooth form vanishing near Δ. Thus the only singularities of l^0 are in those points of Δ lying in double intersections $U_\alpha \cap U_\beta$ and are caused by the fact that $\omega_\beta - \omega_\alpha$ is in general not smooth. But $\omega_\beta - \omega_\alpha$ is $\bar\partial$-exact, as is known from general principles and will be shown explicitly in §10.

This suggests that we may obtain a better approximation to a parametrix by solving

$$\omega_\beta - \omega_\alpha = \bar\partial\omega_{\alpha\beta} \quad \text{for} \quad \omega_{\alpha\beta} \in \mathcal{R}^{-2}(W_{\alpha\beta}, E\boxtimes E^*)$$

where $W_{\alpha\beta}$ is a neighborhood of Δ in $(U_\alpha\cap U_\beta)\times(U_\alpha \cap U_\beta)$, and adding a correction term to k^0. In fact we can find ρ_α with the further property that $\operatorname{spt}\rho_\alpha\rho_\beta \subset W_{\alpha\beta}$ and define $k^1 = k^0 - \sum_{\alpha,\beta} \rho_\beta\bar\partial\rho_\alpha\omega_{\alpha\beta}$. A computation shows that the new error term L^1 has kernel l^1 which satisfies

$$l^1 = \sum_{\alpha,\beta,\gamma} \rho_\gamma\bar\partial\rho_\beta\bar\partial\rho_\alpha(\omega_{\beta\gamma} - \omega_{\alpha\gamma} + \omega_{\alpha\beta}) \quad \text{near } \Delta.$$

It is now clear how to proceed. Let $W_\alpha = U_\alpha \times U_\alpha$ and $\omega_\alpha{}^0 = \omega_\alpha$. We want, for $q = 1,\ldots,n-1$, to inductively solve the equations

$$(\delta\omega^{q-1})_{\alpha_0\ldots\alpha_q} \overset{\text{def}}{=} \sum (-1)^i \omega^{q-1}_{\alpha_0\ldots\hat\alpha_i\ldots\alpha_n}\big| W_{\alpha_0\ldots\alpha_q} = \bar\partial\omega^q_{\alpha_0\ldots\alpha_q} \tag{3.3}$$

for $\omega^q_{\alpha_0\ldots\alpha_q} \in \mathcal{R}^{-q-1}(W_{\alpha_0\ldots\alpha_q}, E\boxtimes E^*)$ where $W_{\alpha_0\ldots\alpha_q}$ is some neighborhood of Δ in $\cap W_{\alpha_0\ldots\hat\alpha_i\ldots\alpha_q}$. A sequence $\check\omega = (\omega^0, \omega^1, \ldots, \omega^{n-1})$ satisfying these conditions will be called a *Čech parametrix.*

Note that $\omega^q_{\alpha_0\ldots\alpha_q} \in A^{n,n-q-1}(W_{\alpha_0\ldots\alpha_q} - \Delta, E\boxtimes E^*)$. In particular if we let $W_{\alpha_0\ldots\alpha_n} = \cap W_{\alpha_0\ldots\hat\alpha_i\ldots\alpha_n}$ and $\omega^n = \delta\omega^{n-1}$, then $\omega^n_{\alpha_0\ldots\alpha_n} \in A^{n,0}(W_{\alpha_0\ldots\alpha_n} - \Delta, E\boxtimes E^*)$. By (3.3) we get $\bar\partial\omega^n_{\alpha_0\ldots\alpha_n} = 0$, i.e., $\omega^n_{\alpha_0\ldots\alpha_n}$ is *holomorphic*, and since $n \geq 2$ (and from the elementary fact that a holomorphic functions has removable singularities on a submanifold of codimension ≥ 2) it extends over Δ to a holomorphic form $\omega^n_{\alpha_0\ldots\alpha_n} \in \Omega^{0,n}(W_{\alpha_0\ldots\alpha_n}, E\boxtimes E^*)$. Thus we can define a cochain κ by $\kappa = \operatorname{tr}\Delta^*\omega^n \in C^n(\mathcal{U}, \Omega^n)$ which is clearly a cocycle.

Now it is easy to find $\rho_\alpha \in C_c^\infty(W_\alpha)$ satisfying (3.1) and such that for each $\alpha_0\ldots,\alpha_q$, $\operatorname{spt}\rho_{\alpha_0}\ldots\rho_{\alpha_q} \subset W_{\alpha_0\ldots\alpha_q}$. We can then define a map

$$\mathcal{D}\colon \prod_{\alpha_0,\ldots,\alpha_q} A^{n,*}(W_{\alpha_0\ldots\alpha_q} - \Delta) \to A^{n,*+q}(M\times M - \Delta)$$

by the same formula (1.2):

$$\mathcal{D}(c^q) = \sum_{\alpha_0,\ldots,\alpha_q} \rho_{\alpha_q}\bar\partial\rho_{\alpha_{q-1}}\ldots\bar\partial\rho_{\alpha_0}c^q_{\alpha_0\ldots\alpha_q}.$$

Note that if each $c^q_{\alpha_0\ldots\alpha_q} \in \mathcal{R}^p(W_{\alpha_0\ldots\alpha_p}, E\boxtimes E^*)$ then $\mathcal{D}c^q \in \mathcal{R}^p(M\times M, E\boxtimes E^*)$. Moreover (1.3) holds on the neighborhood of Δ where (3.1) holds, and therefore

$$\bar\partial\mathcal{D}(c^q) = (-1)^{q+1}\mathcal{D}(\delta c^q) + (-1)^q\mathcal{D}(\bar\partial c^q) + \sigma \text{ on } M\times M, \tag{3.4}$$

where σ is a smooth form on $M\times M$ (depending on c) which vanishes near Δ. Using $\mathcal{D}$ we can construct a global kernel k from $\check\omega$ by

$$k = \sum_{q=0}^{n-1} (-1)^q\mathcal{D}(\omega^q) \in \mathcal{R}^{-1}(M\times M, E\boxtimes E^*),$$

and we let K be the corresponding operator. This is the desired parametrix.

THEOREM 1. *Let $\check\omega$ be a Čech parametrix and K, ω^n as above. Then $\bar\partial K + K\bar\partial = 1 - S$, where S is smoothing and its kernel s satisfies $s = \mathcal{D}(\omega^n)$ in some neighborhood of Δ.*

In view of Proposition (1.9) we have the immediate corollary:

COROLLARY 3.5. *If M is compact and $\kappa = \operatorname{tr}\Delta^*\omega^n$ as above, then $\chi(M, E) = \int_M \mathcal{D}(\kappa)$.*

Proof of Theorem 1. We have to compute for all $\gamma \in A_c^{(n,*),(0,*)}(M\times M, E^*\boxtimes E)$

$$\int_{M\times M} k\cdot\bar\partial\gamma = \int_{M\times M}\mathcal{D}(\omega^0)\cdot\bar\partial\gamma + \int_{M\times M}\left(\sum_{q=1}^{n-1}(-1)^q\mathcal{D}(\omega^q)\right)\cdot\bar\partial\gamma.$$

The first term of the right hand side is computed in (3.2), and for the second note that by Lemma (2.3) (i), $\mathscr{D}(\omega^q)$ is $O(|x-y|^{q+1-2n})$, so for $q \geq 1$ we can integrate by parts $\int_{M\times M}\mathscr{D}(\omega^q)\cdot\bar\partial\gamma = \int_{M\times M}(\bar\partial\mathscr{D}(\omega^q))\cdot\gamma$. Therefore

$$\int_{M\times M} k\cdot\bar\partial\gamma = \int_M \operatorname{tr}\Delta^*\gamma + \int_{M\times M}\left(\bar\partial\sum_{q=0}^{n-1}(-1)^q\mathscr{D}(\omega^q)\right)\cdot\gamma.$$

By Lemma (2.7), to prove the first assertion we have to show that $\bar\partial\Sigma(-1)^q\mathscr{D}(\omega^q)$ is smooth. Computing each term by (3.4) and substituting the relations (3.3) we get the usual cancellations in the alternating sum and $\bar\partial(\sum_{q=0}^{n-1}(-1)^q\mathscr{D}(\omega^q)) = -\mathscr{D}(\omega^n)+\sigma$, where σ is smooth and vanishes near Δ. But $\mathscr{D}(\omega^n)$ is smooth since ω^n is holomorphic, and this proves the second assertion as well.

Remark. In the lenguage of distributions, a parametrix satisfies the equation $\bar\partial k = \Delta - s$, where Δ is the Dirac measure on the diagonal. A Čech parametrix satisfies $D\check\omega = \Delta - \omega^n$, where $D = \delta \pm \bar\partial$ is the total differential of the appropriate Čech bicomplex.

§4. THE UNIVERSAL PROBLEM

We need a canonical procedure for solving the equations (3.3) explicitly in terms of the transition functions of M and E. The best way of stating what we mean by this is in terms of the following universal problem. For simplicity of exposition we describe it first for the case in which E is always the trivial line bundle.

Let $\mathscr{A}_n$ be the set (actually groupoid) of holomorphic automorphisms between open sets in $\mathbf{C}^n$, i.e., the set of all possible transition functions arising from complex structures on manifolds. Let $(\mathscr{A}_n)^q_{\text{comp}}$ be the subset of $(\mathscr{A}_n)^q$ of composable maps. Thus $(\varphi_1,\dots,\varphi_q)\in(\mathscr{A}_n)^q_{\text{comp}}$ if and only if each $\varphi_i\in\mathscr{A}_n$ and $\operatorname{dom}\varphi_1\dots\varphi_q\neq\emptyset$.

Let U be open in $\mathbf{C}^n$. Then the collection of spaces $\mathscr{R}^r(W)$, for all neighborhoods W of Δ in $U\times U$, forms a direct system of vector spaces (under restriction) and we define $\mathscr{R}_\Delta{}^r(U) = \varinjlim \mathscr{R}^r(W)$. The elements of $\mathscr{R}_\Delta{}^r(U)$ are the germs at Δ of regular kernels of order r in $U\times U$.

Note that if $k\in\mathscr{R}^r(W)$ and $\varphi\in\mathscr{A}_n$, then $(\varphi\times\varphi)^*k\in\mathscr{R}^r((\varphi\times\varphi)^{-1}W)$. Thus if $\varphi\colon U\to V$ we get a map $(\varphi\times\varphi)^*\colon\mathscr{R}_\Delta{}^r(V)\to\mathscr{R}_\Delta{}^r(U)$.

We define a complex $C^*(\mathscr{A}_n,\mathscr{R}_\Delta{}^r)$ as follows. $C^0(\mathscr{A}_n,\mathscr{R}_\Delta{}^r) = \mathscr{R}_\Delta{}^r(\mathbf{C}^n)$ and for $q>0$

$$C^q(\mathscr{A}_n,\mathscr{R}_\Delta{}^r) = \prod_{(\varphi_1,\dots,\varphi)}\mathscr{R}_\Delta{}^r(\operatorname{dom}\varphi_1\dots\varphi_q) \tag{4.1}$$

(product over all $(\varphi_1,\dots,\varphi_q)\subset(\mathscr{A}_n)^q\text{comp}$). We define a coboundary $\delta\colon C^q(\mathscr{A}_n,\mathscr{R}_\Delta{}^r)\to C^{q+1}(\mathscr{A}_n,\mathscr{R}_\Delta{}^r)$ by the formula

$$(\delta c)(\varphi_0,\dots,\varphi_q) = c(\varphi_1,\dots,\varphi_q) + \sum_{i=1}^{q}(-1)^i c(\varphi_0,\dots,\varphi_{i-1}\varphi_i,\dots,\varphi_q) + (-1)^{q+1}(\varphi_q\times\varphi_q)^*c(\varphi_0,\dots,\varphi_{q-1}). \tag{4.2}$$

We also have the operator $\bar\partial\colon C^*(\mathscr{A}_n,\mathscr{R}_\Delta{}^r)\to C^*(\mathscr{A}_n,\mathscr{R}_\Delta{}^{r+1})$ defined by $(\bar\partial c)(\varphi_1,\dots,\varphi_q) = \bar\partial(c(\varphi_1,\dots,\varphi_q))$. Then $\delta^2=0$, $\bar\partial^2=0$, and $\delta\bar\partial=\bar\partial\delta$.

Note that the Bochner–Martinelli kernel (2.10) defines an element $\omega\in\mathscr{R}_\Delta{}^{-1}(\mathbf{c}^n) = C^0(\mathscr{A}_n,\mathscr{R}_\Delta{}^{-1})$.

Definition 4.3. A *universal parametrix* is a sequence $\{\omega^0,\omega^1,\dots,\omega^{n-1}\}$ such that $\omega^q\in C^q(\mathscr{A}_n,\mathscr{R}_\Delta{}^{-q-1})$ and is of type $(n,n-q-1)$, ω^0 is the Bochner–Martinelli kernel and

$$\delta\omega^q = \bar\partial\omega^{q+1}. \tag{4.4}$$

The reason for this terminology is that $\{\omega^q\}$ is a universal Čech parametric in the following sense. For each $(\varphi_1,\dots,\varphi_q)\in(\mathscr{A}_n)^q_{\text{comp}}$ choose a neighborhood $W(\varphi_1,\dots,\varphi_q)$ of Δ in $(\operatorname{dom}\varphi_1\dots\varphi_q)^2$ such that the germ $\omega^q(\varphi_1,\dots,\varphi_q)$ has a representative defined on $W(\varphi_1,\dots,\varphi_q)$. We may assume that

$$W(\varphi_0,\dots,\varphi_{q+1})\subset W(\varphi_1,\dots,\varphi_q)\cap\cap\ W(\varphi_0,\dots,\varphi_{i-1}\varphi_i,\dots,\varphi_q) \cap(\varphi_q\times\varphi_q)^{-1}W(\varphi_0,\dots,\varphi_{q-}$$

and that (4.4) holds on each $W(\varphi_0, \ldots, \varphi_q)$. Given any manifold M, let

$$\check{\omega}^q_{\alpha_0\ldots\alpha_q} = (\varphi_{\alpha_q} \times \varphi_{\alpha_q})^* \omega^q(\varphi_{\alpha_0\alpha_1}, \ldots, \varphi_{\alpha_{q-1}\alpha_q}) \in \mathscr{R}^{-q-1}(W_{\alpha_0\ldots\alpha_q})$$

where $W_{\alpha_0\ldots\alpha_q} = (\varphi_{\alpha_q} \times \varphi_{\alpha_q})^{-1} W(\varphi_{\alpha_0\alpha_1}, \ldots, \varphi_{\alpha_{q-1}\alpha_q})$. Then $\check{\omega} = \{\omega^q\}$ is a Čech parametrix for M. Moreover, if we let $\omega^n = \delta\omega^{n-1}$, then we also have $\check{\kappa}_{\alpha^0\ldots\alpha_n} = \varphi^*_{\alpha_n}\kappa(\varphi_{\alpha_0\alpha_1}, \ldots, \varphi_{\alpha_{n-1}\alpha_n})$ where $\kappa(\varphi_1, \ldots, \varphi_n) = \Delta^*\omega^n(\varphi_1, \ldots, \varphi_n) \in \Omega^n$ (dom $\varphi_1 \ldots \varphi_n$). (Restriction to Δ makes sense since $\omega^n(\varphi_1, \ldots, \varphi_n)$ is *holomorphic* and hence extends to Δ.) κ is naturally an n-cocycle in the complex $C^*(\mathscr{A}_n, \Omega^n)$ defined in analogy to (4.1) by $C^q(\mathscr{A}_n, \Omega^n) = \Pi\Omega^n(\text{dom}\, \varphi_1 \ldots \varphi_n)$ with δ defined by the same formula (4.2) with $(\varphi_q \times \varphi_q)^*$ replaced by φ^*_q.

Now the Atiyah–Chern classes also have universal representatives in $C^n(\mathscr{A}_n, \Omega^n)$. If $\varphi \in \mathscr{A}_n$, let $\theta(\varphi)$ be the matrix of one forms defined by $\theta(\varphi) = \dot{\varphi}^{-1} d\dot{\varphi} \in \Omega^1(\text{dom}\, \varphi) \otimes L(\mathbf{C}^n)$. If $\varphi(U) \subset V$ we have a map $\varphi^{\#}: \Omega^1(V) \otimes L(\mathbf{C}^n) \to \Omega^1(U) \otimes L(\mathbf{C}^n)$ given by $\varphi^{\#}(\alpha \otimes a)(x) = (\varphi^*\alpha)(x) \otimes \dot{\varphi}^{-1}(x) a \dot{\varphi}(x)$. If f is a symmetric invariant polynomial as in §1, we can define a cocycle $f(\theta^n) \in C^n(\mathscr{A}_n, \Omega^n)$ by

$$f(\theta^n)(\varphi_1, \ldots, \varphi_n) = f((\varphi_2 \ldots \varphi_n)^{\#}\theta(\varphi_1) \wedge \cdots \wedge \varphi_n^{\#}\theta(\varphi_{n-1}) \wedge \theta(\varphi_n)). \tag{4.5}$$

On a given manifold M, the class $f(\theta^n)$ defined by (1.4) is $f(\theta^n)_{\alpha_0\ldots\alpha_n} = \varphi^*_{\alpha_n} f(\theta^n)(\varphi_{\alpha_0\alpha_1}, \ldots, \varphi_{\alpha_{n-1}\alpha_n})$.

Our goal is to construct a universal parametrix that has the property that the cocycle $\kappa = f(\theta^n)$ for some invariant polynomial f. In view of Proposition (1.6) and Corollary (3.5) this would imply that for all compact M, $\chi(M)$ is the Chern number of M corresponding to f. But using the multiplicative properties of χ, this determines f completely (Cf. [13, §10] or the similar argument for the signature in [3, §5]): $f = \mathscr{T}$ where $\mathscr{T}$ is the polynomial that gives the Todd genus of M. In this way we obtain a very strong form of the Hirzebruch–Riemann–Roch theorem, which may be called the *universal Riemann–Roch theorem*: $\kappa = \mathscr{T}(\theta^n)$.

To this end we need a characterization of the cocycles $f(\theta^n)$. We use the following terminology. The symmetric group S_{q+1} acts on $(\mathscr{A}_n)^q_{\text{comp}}$; if $\sigma \in S_{q+1}$, let $(\varphi_1{}^\sigma, \ldots, \varphi_q{}^\sigma)$ be defined by

$$\varphi_i{}^\sigma = \begin{cases} \varphi_{\sigma(i-1)+1} \ldots \varphi_{\sigma(i)} & \text{if } \sigma(i-1) < \sigma(i) \\ \varphi^{-1}_{\sigma(i-1)} \ldots \varphi^{-1}_{\sigma(i)+1} & \text{if } \sigma(i-1) > \sigma(i) \end{cases} \tag{4.6}$$

(This is the action induced on the transition functions by permuting the coordinate maps $\varphi_{\alpha_0}, \ldots, \varphi_{\alpha_q}$ by σ.) A cochain $c \in C^q(\mathscr{A}_n, \Omega^n)$ is called *skew* if

$$(\varphi_{\sigma(q)+1} \ldots \varphi_q)^* c(\varphi_1{}^\sigma, \ldots, \varphi_q{}^\sigma) = \operatorname{sign} \sigma\, c(\varphi_1, \ldots, \varphi_q) \text{ for all } \sigma \in S_{q+1}. \tag{4.7}$$

$G_n{}^N$ denotes the set of invertible N-jets from $\mathbf{C}^n$ to $\mathbf{C}^n$, i.e., N-jets with invertible linear part. If $\varphi \in \mathscr{A}_n$ and $x \in \text{dom}\, \varphi$, then the N-jet of φ at x, $j^N\varphi(x)$, is in $G_n{}^N$. $G_n{}^N$ is a bundle over $\mathbf{C}^n \times \mathbf{C}^n$ under the projection $g \to$ (target (g), source (g)). Let $(G_n{}^N)^n_{\text{comp}}$ be the bundle over $(\mathbf{C}^n)^{n+1}$ of sequences $(g_1, \ldots, g_n) \in (G_n{}^N)^n$ such that source (g_i) = target (g_{i+1}).

THEOREM II. *Suppose $c \in C^n(\mathscr{A}_n, \Omega^n)$ is a skew cocycle satisfying:*

(i) *There exists an N and a holomorphic bundle map F*

$$\begin{array}{ccc} (G_n{}^N)^n_{\text{comp}} & \xrightarrow{F} & \Lambda^n T\mathbf{C}^n \\ \downarrow & & \downarrow \\ (\mathbf{C}^n)^{n+1} & \xrightarrow{\pi_{n+1}} & \mathbf{C}^n \end{array}$$

such that for all $x \in \mathbf{C}^n$

$$c(\varphi_1, \ldots, \varphi_n)(x) = F(j^N\varphi_1(\varphi_2 \ldots \varphi_n x), \ldots, j^N\varphi_{n-1}(\varphi_n x), j^N\varphi_n(x)).$$

(ii) $c(\varphi_1, \ldots, \varphi_n) = 0$ *if one of the φ_i is an affine map. Then $c = f(\theta^n)$ for some $GL(n)$-invariant symmetric linear map $f: \otimes^n L(\mathbf{C}^n) \to \mathbf{C}$.*

Note that all the hypothesis are satisfied by the Atiyah–Chern classes. Condition (i) just says that c is a *differential* operator in $\varphi_1, \ldots, \varphi_n$. It alone implies, via the work of Bott and Haefliger on continuous cohomology [5], [6], than on any M, $\{\varphi^*_{\alpha_n} c(\varphi_{\alpha_0\alpha_1}, \ldots, \varphi_{\alpha_{n-1}\alpha_n})\}$ is *cohomologous* to a Chern class. Adding the very natural condition (ii) gives a direct characterization of the *cocycles* themselves.

The same characterization (and proof) holds for the cocycles $f(\theta^p)$, $p < n$, but we do not need this fact.

§5. PROOF OF THEOREM II

Condition (ii) says roughly that c is essentially a function of derivatives of the φ_i of order at least 2. We begin by making this precise and deriving some simple consequences of (ii).

LEMMA 5.1. *Let A be an affine map and $(\varphi_1, \dots, \varphi_n) \in (\mathscr{A}_n)^n_{\text{comp}}$. Then*

$$c(A\varphi_1, \varphi_2, \dots, \varphi_n) = c(\varphi_1, \dots, \varphi_n), \qquad c(\varphi_1, \dots, \varphi_n A) = A^* c(\varphi_1, \dots, \varphi_n).$$

Moreover, if $(\varphi_1, \dots \varphi_i, A, \varphi_{i+1}, \dots, \varphi_n) \in (\mathscr{A}_n)^{n+1}_{\text{comp}}$, *we also have*

$$c(\varphi_1, \dots, \varphi_i A, \varphi_{i+1}, \dots, \varphi_n) = c(\varphi_1, \dots, \varphi_i, A\varphi_{i+1}, \dots, \varphi_n), \qquad 1 \le i < n.$$

Proof. Evaluate the expression $\delta c = 0$ on each of the sequences $(A, \varphi_1, \dots, \varphi_n)$, $(\varphi_1, \dots, \varphi_i, A, \varphi_{i+1}, \dots, \varphi_n)$, $(\varphi_1, \dots, \varphi_n, A)$.

An immediate consequence of this is that c is invariant under affine maps:

COROLLARY 5.2. *If $A_0, \dots, A_n$ are affine maps, then*

$$c(A_0\varphi_1 A_1^{-1}, A_1\varphi_2 A_2^{-1}, \dots, A_{n-1}\varphi_n A_n^{-1}) = (A_n^{-1})^* c(\varphi_1, \dots, \varphi_n).$$

Let $\mathscr{A}_n^+$ be the set of all $\varphi \in \mathscr{A}_n$ such that $\varphi(0) = 0$ and $\dot\varphi(0) = 1$. Given $(\varphi_1, \dots, \varphi_n) \in (\mathscr{A}_n)^n_{\text{comp}}$ and $x \in \operatorname{dom} \varphi_1 \dots \varphi_n$ we obtain a sequence in $\mathscr{A}_n^+$ as follows. Let $x_n = x$ and for $0 \le i < n$, let $x_i = \varphi_{i+1} \dots \varphi_n(x)$, and T_{x_i} translation by x_i. Define affine maps A_i (depending on $(\varphi_1, \dots, \varphi_n)$ and x) by

$$A_n = T_{x_n}, \qquad A_i = \dot\varphi_{i+1}(x_{i+1}) \dots \dot\varphi_n(x_n) T_{x_i}, \qquad 0 \le i < n.$$

Then we have that $A_{i-1}^{-1}\varphi_i A_i \in \mathscr{A}_n^+$, $1 \le i \le n$ and by Corollary (5.2) $c(\varphi_1, \dots, \varphi_n) = (A_n^{-1})^* c(A_0^{-1}\varphi_1 A_1, A_1^{-1}\varphi_2 A_2, \dots, A_{n-1}^{-1}\varphi_n A_n)$ and thus by (i)

$$c(\varphi_1, \dots, \varphi_n)(x) = (A_n^{-1})^* F(j^N(A_0^{-1}\varphi_1 A_1)(0), \dots, j^N(A_{n-1}^{-1}\varphi_n A_n)(0)). \tag{5.3}$$

We write $(G_n^N)_+$ for the group of N-jets at 0 of elements of $\mathscr{A}_n^+$. Then $(G_n^N)_+^n$ is contained in the fibre over 0 of $(G_n^N)^n_{\text{comp}}$ and by (5.3) c is determined by F_+, the restriction of F to $(G_n^N)_+^n$.

From now on we write V for the vector space $\mathbf{C}^n$. $GL(V)$ acts on $(G_n^N)_+$ by conjugation and by (5.2) and (ii) we have that the holomorphic map $F_+ : (G_n^N)_+^n \to \Lambda^n V^*$ satisfies

$$F_+(Ag_1A^{-1}, \dots, Ag_nA^{-1}) = (A^{-1})^* F_+(g_1, \dots, g_n) \text{ for all } A \in GL(V), \tag{5.4}$$

$$F_+(g_1, \dots, g_n) = 0 \text{ if one } g_i \text{ is the identity.} \tag{5.5}$$

The underlying space of $(G_n^N)_+$ is a vector space, namely we have an isomorphism, compatible with the natural actions of $GL(V)$,

$$(G_n^N)_+ \approx \sum_{r=2}^{N} S^r V^* \otimes V.$$

This gives us coordinates X_J^i on $(G_n^N)_+$: in the usual notation $J = (j_1, \dots, j_n)$, $|J| = j_1 + \dots + j_n$, if $g \in (G_n^N)_+$ then we can write uniquely

$$g^i(z) = z^i + \sum_{2 \le |J| \le N} X^i_{j_1 \dots j_n}(g)(z^1)^{j_1} \dots (z^n)^{j_n}.$$

In this way we obtain coordinates $\{X_J^{i,k}\}$ for $(C_n^N)_+^n$, i.e. $X_J^{i,k}(g_1, \dots, g_n) = X_J^i(g_k)$. We then have $F_+(g_1, \dots, g_n) = h(\{X_J^{i,k}\}) dz^1 \dots dz^n$ for some holomorphic function h of these coordinates. Note that if $\lambda \in \mathbf{C}$, $\lambda \ne 0$, then $X_J^i(\lambda^{-1} g \lambda) = \lambda^{|J|-1} X_J^i(g)$, and hence by (5.4) (with $A = \lambda^{-1}$) we get

$$h(\{\lambda^{|J|-1} X_J^{i,k}\}) = \lambda^n h(\{X_J^{i,k}\}). \tag{5.6}$$

LEMMA. *Suppose $f: \mathbf{C}^l \to \mathbf{C}$ is holomorphic and for some positive integers $a, a_1, \dots, a_l$ satisfies $f(\lambda^{a_1} y_1, \dots, \lambda^{a_l} y_l) = \lambda^a f(y_1, \dots, y_l)$ for all $\lambda \in \mathbf{C}$. Then f is a weighted homogeneous polynomial of degree a, i.e., $f(y_1, \dots, y_l) = \Sigma b_{\alpha_1 \dots \alpha_l} y_1^{\alpha_1} \dots y_l^{\alpha_l}$, where the sum is finite and $a_1\alpha_1 + \dots + a_l\alpha_l = a$.*

Proof. $(\partial^\alpha f/\partial y^\alpha)(0) \neq 0$ only if $a_1\alpha_1 + \cdots + a_l a_l = a$, and since $a_1, \ldots, a_l > 0$ only finitely many α satisfy this.

Since all $|J| \geq 2$ in (5.6), applying this lemma we see that h is uniquely a linear combination of monomials of the form

$$(X_{J_1}^{i_1,k_1})^{\alpha_1} \ldots (X_{J_r}^{i_r,k_r})^{\alpha_r}, \qquad \alpha_1, \ldots, \alpha_r > 0, \tag{5.7}$$

where

$$\alpha_1(|J_1| - 1) + \cdots + \alpha_r(|J_r| - 1) = n. \tag{5.8}$$

First we observe that h can only involve monomials in which

$$\{k_1, \ldots, k_r\} = \{1, \ldots, n\}. \tag{5.9}$$

For suppose that some k_0 did not appear in a monomial (5.7). Let $p(X_{J_1}^{i_1,k_1}, \ldots, X_{J_r}^{i_r,k_r})$ be the sum of all terms of h that contain only the variables that appear in this monomial. Then $p = 0$ because $p(X_{J_1}^{i_1,k_1}, \ldots, X_{J_r}^{i_r,k_r})$ is the evaluation of F_+ on the set of $(g_1, \ldots, g_n)$ for which all the other coordinates $X_J^{i,k}$ vanish. But in this set g_{k_0} is always the identity, and by (5.5) F_+ vanishes on this set. Since $p \equiv 0$, the monomial we started with must have had coefficient zero in h.

We see immediately that the only monomials that satisfy both (5.8) and (5.9) are those for which $|J_i| = 2$, $\alpha_i = 1$, $r = n$, $k_1, \ldots, k_n$ are distinct. In other words, the polynomial h is actually a *multilinear* function of the second derivatives of the g_i. Thus F_+ is induced by a linear map F_2: $\otimes^n(S^2V^* \otimes V) \to \Lambda^n V^*$ which, by (5.4), is $GL(V)$ equivariant.

At this point we must quote the following fact from the theory of invariants for $GL(n, \mathbf{C})$:

THEOREM 5.10. *The $GL(V)$-invariant linear maps $\otimes^p V^* \otimes^p V \to \mathbf{C}$ are spanned by the maps $v_1^* \otimes \cdots \otimes v_p^* \otimes v_1 \otimes \cdots \otimes v_p \to v_{\sigma(1)}^*(v_1) \ldots v_{\sigma(p)}^*(v_p)$, where σ is any permutation of $\{1, \ldots, p\}$.*

A proof of this theorem is given in Appendix I of [3]; see also [8], particularly Appendix, Part IV. An immediate consequence of this theorem is:

COROLLARY 5.11. *The $GL(V)$-equivariant linear maps $\otimes^{p+n} V^* \otimes^p V \to \Lambda^n V^*$ are spanned by the maps*

$$v_1^* \otimes \cdots \otimes v_{p+n}^* \otimes v_1 \otimes \cdots \otimes v_p \to v_{\sigma(1)}^*(v_1) \ldots v_{\sigma(p)}^*(v_p) v_{\sigma(p+1)}^* \wedge \cdots \wedge v_{\sigma(p+n)}^*$$

for σ a permutation of $\{1, \ldots, p+n\}$.

Regarding $S^2V^* \otimes V$ as a subspace of $\otimes^2 V^* \otimes V$, our map F_2 can be extended to a $GL(V)$-equivariant map defined on all of $\otimes^n(\otimes^2 V^* \otimes V)$. Let $X_{j_1,k_1,\ldots,j_n,k_n}^{i_1,\ldots,i_n}$ be coordinates in $\otimes^n(\otimes^2 V^* \otimes V)$ with respect to a basis $e^{j_1} \otimes e^{k_1} \otimes e_{i_1} \otimes \cdots \otimes e^{j_n} \otimes e^{k_n} \otimes e_{i_n}$. Then $\otimes^n(S^2V^* \otimes V)$ is the subspace in which

$$X_{j_1,k_1,\ldots,j_r,k_r,\ldots,j_n,k_n}^{i_1,\ldots,i_n} = X_{j_1,k_1,\ldots,k_r,j_r,\ldots,j_n,k_n}^{i_1,\ldots,i_n} \quad \text{for } 1 \leq r \leq n. \tag{5.12}$$

In terms of these coordinates Corollary (5.11) asserts that the extension of F_2 must be a linear combination of operations of the form: contract the upper indices $i_1, \ldots, i_n$ with half of the lower indices $j_1, k_1, \ldots, j_n, k_n$ and skew-symmetrize the remaining lower indices. But if the skew-symmetrization involves both j_r, k_r, then its restriction to the subspace (5.12) vanishes. It follows that (by changing the extension of F_2 if necessary) all the operations are of the form: contract $i_1, \ldots, i_n$ with a permutation of $k_1, \ldots, k_n$ and skew-symmetrize $j_1, \ldots, j_n$.

Thus F_2 is the restriction to $\otimes^n(S^2V^* \otimes V)$ of the composition

$$\otimes^n(\otimes^2 V^* \otimes V) \to \otimes^n(V^* \otimes V) \otimes \Lambda^n V^* \xrightarrow{f \otimes 1} \Lambda^n V^*, \tag{5.13}$$

where, writing $x_i = y_i \otimes w_i$, $y_i \in V^*$, $w_i \in V^* \otimes V$, the first map is given by

$$x_1 \otimes \cdots \otimes x_n \to (w_i \otimes \cdots \otimes w_n) \otimes (y_1 \wedge \cdots \wedge y_n) \tag{5.14}$$

and f is a $GL(V)$-invariant map f: $\otimes^n(V^* \otimes V) \to \mathbf{C}$. But in terms of $g_1, \ldots, g_n \in (G_n^N)_+$ this says precisely that $F_+(g_1, \ldots, g_n) = f((d\dot{g}_1)(0) \wedge \cdots \wedge (d\dot{g}_n)(0))$.

To conclude that c is a Chern class it only remains to check that f is *symmetric*. First we observe that f is symmetric if and only if the composition (5.13) is skew in $x_1, \ldots, x_n$. Next we observe that the restriction of (5.14) to $\otimes^n(S^2V^* \otimes V)$ is surjective. Hence (5.13) is skew if and only if F_2 is skew, and for this we only have to show that $F_2(s_1 \otimes \cdots \otimes s_n) = 0$ if $s_i = s_{i+1}$, $1 \le i < n$.

It is at this point that we need the assumption that c is *skew*. Since for $\varphi_1, \varphi_2 \in \mathscr{A}_n^+$ the second derivatives satisfy $D^2(\varphi_1\varphi_2)(0) = D^2\varphi_1(0) + D^2\varphi(0)$, the action induced on $\otimes^n(S^2V^* \otimes V)$ via (4.6) by the transposition that interchanges i and $i+1$ is

$$s_1 \otimes \cdots \otimes s_n \to s_1 \otimes \cdots \otimes s_{i-1} \otimes (s_i + s_{i+1}) \otimes (-s_{i+1}) \otimes (s_{i+1} + s_{i+2}) \otimes \cdots \otimes s_n.$$

Hence the skewness of c implies the relation

$$\begin{aligned} &F_2(s_1 \otimes \cdots \otimes s_{i-1} \otimes s_i \otimes s_{i+1} \otimes s_{i+1} \otimes \cdots \otimes s_n) \\ &\quad + F_2(s_1 \otimes \cdots \otimes s_{i-1} \otimes s_{i+1} \otimes s_{i+1} \otimes s_{i+1} \otimes \cdots \otimes s_n) \\ &\quad + F_2(s_1 \otimes \cdots \otimes s_{i-1} \otimes s_{i+1} \otimes s_{i+1} \otimes s_{i+2} \otimes \cdots \otimes s_n) = 0. \end{aligned}$$

Setting first $s_i = s_{i+1} = s_{i+2}$ we see that the middle term is always zero, hence if we set $s_{i+1} = s_i$ the first two terms vanish and we get $F_2(s_1 \otimes \cdots \otimes s_n) = 0$ if $s_i = s_{i+1}$ as desired. This completes the proof of Theorem II.

§6. THE UNIVERSAL PROBLEM WITH COEFFICIENTS IN A BUNDLE

We discuss briefly the universal problem for the Dolbeault complex with coefficients in a holomorphic vector bundle. We write $\mathscr{A}_{n,m}$ for the set of all pairs (φ, ψ) where $\varphi \in \mathscr{A}_n$ and ψ: dom $\varphi \to GL(m)$ is a holomorphic map. $(\mathscr{A}_{n,m})^q_{\text{comp}}$ denotes the set of all $((\varphi_1, \psi_1), \ldots, (\varphi_q, \psi_q)) \in (\mathscr{A}_{n,m})^q$ such that $(\varphi_1, \ldots, \varphi_q) \in (\mathscr{A}_n)^q_{\text{comp}}$. There is a composition $(\mathscr{A}_{n,m})^2_{\text{comp}} \to \mathscr{A}_{n,m}$ defined by $(\varphi_1, \psi_1) \cdot (\varphi_2, \psi_2) = (\varphi_1\varphi_2, (\psi_1 \circ \varphi_2)\psi_2)$ (the juxtaposition $(\psi_1 \circ \varphi_2)\psi_2$ denotes matrix multiplication. Note that for given M, E, $(\varphi_{\alpha\beta}, \psi_{\alpha\beta} \circ \varphi_\beta^{-1}) \in \mathscr{A}_{n,m}$ whenever $U_\alpha \cap U_\beta = \emptyset$, and the composition corresponds to the composition of transition functions of M and E over triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$

$$\mathscr{A}_{n,m} \text{ acts on } \mathscr{R}_\Delta \otimes L(\mathbf{C}^m)\text{: if } \varphi(U) \subset V$$

we have $(\varphi, \psi)^\#: \mathscr{R}_\Delta(V) \otimes L(\mathbf{C}^m) \to \mathscr{R}_\Delta(U) \otimes L(\mathbf{C}^m)$, where $(\varphi, \psi)^\#(k \otimes a)(z, \zeta) = (\varphi \times \varphi)^* k(z, \zeta) \otimes \psi^{-1}(z) a \psi(\zeta)$. Hence we can define, in analogy with (4.1)

$$C^q(\mathscr{A}_{n,m}, \mathscr{R}_\Delta{}^r \otimes L(\mathbf{C}^m)) = \prod_{(\varphi_1,\psi_1)\ldots(\varphi_q,\psi_q)} \mathscr{R}_\Delta{}^r(\text{dom}\, \varphi_1 \ldots \varphi_q) \otimes L(\mathbf{C}^m)$$

with coboundary δ given by the same formula (4.2), replacing φ_i by (φ_i, ψ_i), compositions $\varphi_{i-1}\varphi_i$ by $(\varphi_{i-1}, \psi_{i-1}) \cdot (\varphi_i, \psi_i)$ and $(\varphi_q \times \varphi_q)^*$ by $(\varphi_q, \psi_q)^\#$.

A *universal parametrix* for bundles with fibre $\mathbf{C}^m$ over n-dimensional manifolds is then a sequence $\{\omega^0, \ldots, \omega^{n-1}\}$ with $\omega^q \in C^q(\mathscr{A}_{n,m}, \mathscr{R}_\Delta{}^{-q-1} \otimes L(\mathbf{C}^m))$ such that equations (4.4) hold and $\omega^0 = \omega \otimes 1$, the Bochner–Martinelli kernel (2.14). For given M, E we get a Čech parametrix $\{\check\omega^q\}$ where the value $\check\omega_{\alpha_0\ldots\alpha_q}(x, y)$ is

$$\psi_{\alpha_q}^{-1}(x)\{(\varphi_{\alpha_q} \times \varphi_{\alpha_q})^* \omega^q((\varphi_{\alpha_0\alpha_1}, \psi_{\alpha_0\alpha_1} \circ \varphi_{\alpha_1}^{-1}), \ldots, (\varphi_{\alpha_{q-1}\alpha_q}, \psi_{\alpha_{q-1}\alpha_q} \circ \varphi_{\alpha_q}^{-1}))\}\psi_{\alpha_q}(y).$$

We get as before that $\kappa_m = \operatorname{tr} \Delta^*(\delta\omega^{n-1}) \in C^n(\mathscr{A}_{n,m}, \Omega^n)$ is a universal representative for $\chi(M, E)$.

The mixed Chern numbers of Proposition (1.7) also have universal representatives in $C^n(\mathscr{A}_{n,m}, \Omega^n)$: if $(\varphi, \psi) \in \mathscr{A}_{n,m}$, let

$$\theta_m(\varphi, \psi) = \dot\varphi^{-1} d\dot\varphi \oplus \psi^{-1} d\psi \in \Omega^1(\text{dom}\, \varphi) \otimes (L(\mathbf{C}^n) \oplus L(\mathbf{C}^m)).$$

Given an invariant symmetric polynomial f on $L(\mathbf{C}^n) \oplus L(\mathbf{C}^m)$ we can define in analogy with (4.5) a cocycle $f(\theta_m{}^n) \in C^n(\mathscr{A}_{n,m}, \Omega^n)$ which represents the corresponding Chern number of $TM \oplus E$.

Let $G^N_{n,m}$ be the set of all pairs (g, γ) where $g \in G_n{}^N$, γ is an N-jet from $\mathbf{C}^n$ to $GL(m)$ and source $(g) =$ source (γ). This is a bundle over $\mathbf{C}^n \times \mathbf{C}^n$ and we get as before a bundle $(G^N_{n,m})^n_{\text{comp}}$ over $(\mathbf{C}^n)^{n+1}$. Also S_{q+1} acts on $(\mathscr{A}_{n,m})^q_{\text{comp}}$ in analogy with (4.6) and we get a corresponding notion of skew cochains. Finally let $ch_k\colon \otimes^k L(\mathbf{C}^m) \to \mathbf{C}$ be the invariant symmetric linear map defined by

$$ch_k(X_1 \otimes \cdots \otimes X_k) = \left(\frac{-1}{2\pi i}\right)^k \left(\frac{1}{k!}\right)^2 \sum_{\sigma \in S_k} \operatorname{tr}(X_{\sigma(1)} \ldots X_{\sigma(k)}).$$

With this notation we have the following extension of Theorem II:

THEOREM II′. *Suppose $c_m \in C^n(\mathscr{A}_{n,m}, \Omega^n)$ is a skew cocycle satisfying:*

(i) *There exists an N and a holomorphic bundle map F*

$$\begin{array}{ccc} (G^N_{n,m})^n_{\text{comp}} & \xrightarrow{F} & \Lambda^n T\mathbf{C}^n \\ \downarrow & & \downarrow \\ (\mathbf{C}^n)^{n+1} & \xrightarrow{\pi_{n+1}} & \mathbf{C}^n \end{array}$$

such that for all $x \in \mathbf{C}^n$

$$c_m((\varphi_1, \psi_1), \ldots, (\varphi_n, \psi_n))(x) = F(j^N(\varphi_1, \psi_1)(\varphi_2 \ldots \varphi_n x), \ldots, j^N(\varphi_n, \psi_n)(x)).$$

(ii) $c_m((\varphi_1, \psi_1), \ldots, (\varphi_n, \psi_n)) = 0$ *if for some i, φ_i is affine and ψ_i is constant.*

Then $c_m = f(\theta_m{}^n)$ for some $GL(n) \times GL(m)$ invariant symmetric linear map $f: \otimes^n(L(\mathbf{C}^n) \oplus L(\mathbf{C}^m)) \to \mathbf{C}$. Moreover, if for all m, m' we have that

(iii) $c_{m+m'}((\varphi_1, \psi_1 \oplus \psi'_1), \ldots, (\varphi_n, \psi_n \oplus \psi'_n)) = c_m((\varphi_1, \psi_1), \ldots, (\varphi_n, \psi_n)) + c_{m'}((\varphi_1, \psi'_1), \ldots, (\varphi_n, \psi')),$ *then $f = \sum_{i=0}^{n} t_i ch_{n-i}$ for some invariant symmetric linear maps $t_i: \otimes^i L(\mathbf{C}^n) \to \mathbf{C}$, independent of m.*

The proof is identical to that of Theorem II. We have the analogue of (5.1) for pairs (A, B) with A affine and B constant. This gives that c_m is determined by the restriction of F to $(G^N_{n,m})_+{}^n$, where $(G^N_{n,m})_+ = \{(g, \gamma) \in G^N_{n,m}: g \in (G_n{}^N)_+ \text{ and } \gamma(0) = 1\}$. Letting W stand for the fibre $\mathbf{C}^m$,

$$(G^N_{n,m})_+ \approx \sum_{r=2}^{N} S^r V^* \otimes V \oplus \sum_{s=1}^{N} \otimes^s V^* \otimes W \otimes W^*.$$

Studying the action of conjugation by $(\lambda, 1)$, $\lambda \in \mathbf{C}^*$, on the induced coordinates we get that F_+ is a multilinear function of second derivatives of the g_i and first derivatives of the γ_i, i.e., F_+ is induced by a linear map $F_2: \otimes^n(S^2 V^* \otimes V \oplus V^* \otimes W \otimes W^*) \to \Lambda^n V^*$, which is equivariant with respect to the actions of $GL(V) \times GL(W)$ and $GL(V)$. Applying (5.10) (with V replaced by W) and (5.11) we get easily that

$$F_+((g_1, \gamma_1), \ldots, (g_n, \gamma_n)) = f((d\dot{g}_1 \otimes d\gamma_1)(0) \wedge \cdots \wedge (d\dot{g}_n + d\gamma_n)(0))$$

for some invariant f, which must be symmetric by the skewness of c_m.

The final assertion follows by standard additivity arguments (cf. [3, §6]).

§7. OUTLINE OF THE CONSTRUCTION

We now proceed to the construction of a universal parametrix. The problem is to inductively find solutions $\omega^q(\varphi_1, \ldots, \varphi_q)$ of

$$(\delta\omega^{q-1})(\varphi_1, \ldots, \varphi_q) = \bar{\partial}\omega^q(\varphi_1, \ldots, \varphi_q) \tag{7.1}$$

which in some sense *depend analytically on* $\varphi_1, \ldots, \varphi_q$. One's first inclination would be to formulate precisely what this analytic dependence means, and then to prove a Poincaré lemma for the action of $\bar{\partial}$ on the cochains that depend analytically on parameters. Both these problems are however quite subtle, and our approach is to reduce (7.1) to a composition of a large number of simpler acyclicity statements. Each individual step will be quite trivial, and the meaning of analytic dependence will be quite clear at each step.

To find $\omega^1(\varphi)$ we can proceed as follows. First, if $W \subset \mathbf{C}^n \times \mathbf{C}^n$, let $\mathscr{W} = \{W_i\}$ be the covering of $W - \Delta$ by $W_i = \{\zeta^i - z^i \neq 0\}$. Let $C(\mathscr{W}, \mathscr{R})$ be the corresponding Čech bicomplex, with total differential $D =$ Čech coboundary $\pm\bar{\partial}$. As is well known, the Bochner–Martinelli kernel ω^0 and the Cauchy kernel

$$\gamma^n = \frac{1}{(2\pi i)^n} \frac{d\zeta^1 \ldots d\zeta^n}{(\zeta^1 - z^1) \ldots (\zeta^n - z^n)} \in C^{n-1}(\mathscr{W}, \Omega^{0,n})$$

are homologous in this complex, i.e., there is a cochain c (Harvey's "connecting sequence" [12]) such that $Dc = \omega^0 - \gamma^n$. Thus finding $\omega^1(\varphi)$ should be equivalent to solving a combinatorial equation involving the Cauchy kernel, namely:

$$d\gamma^{n-1} = \gamma^n - \frac{1}{(2\pi i)^n}(\det A(z,\zeta))^{-1}\frac{\det \dot\varphi(\zeta)d\zeta^1 \wedge \cdots \wedge d\zeta^n}{(\zeta^1 - z^1)\dots(\zeta^n - z^n)} \tag{7.2}$$

for some cochain $\gamma^{n-1} \in C^{n-2}(\mathcal{W}, \Omega^{0,n})$ (d = Čech coboundary), where $A(z,\zeta)$ is an invertible matrix of functions satisfying

$$\varphi(\zeta) - \varphi(z) = A(z,\zeta)(\zeta - z). \tag{7.4}$$

The two equations are equivalent for the following reason. Let $K^*(W) = \Lambda^*(\mathcal{O}(W)e^1 \oplus \cdots \oplus \mathcal{O}(W)e^n)$ be the Koszul complex (cf. §9) with differential $\epsilon: K^r \to K^{r+1}$ given by exterior multiplication with $(\zeta^1 - z^1)e^1 + \cdots + (\zeta^n - z^n)e^n \in K^1(W)$. Then the map

$$K^* \otimes \mathcal{R} \to C^{*-1}(\mathcal{W}, \mathcal{R})$$
$$e^{i_1} \wedge \cdots \wedge e^{i_p} \otimes k \to \frac{k}{(\zeta^{i_1} - z^{i_1})\dots(\zeta^{i_p} - z^{i_p})}$$

is an injective chain map whose image is the subcomplex of skew cochains which have the simplest possible poles on the planes $\zeta^i - z^i = 0$. Now ω^0, γ^n and c all lie in this subcomplex and hence can be equivalently interpreted in the Koszul complex. Next the map

$$K^* \otimes \mathcal{R} \xrightarrow{\varphi^\#} K^* \otimes \mathcal{R}$$
$$e^{i_1} \wedge \cdots \wedge e^{i_p} \otimes k \to A^{-1}e^{i_1} \wedge \cdots \wedge A^{-1}e^{i_p} \otimes (\varphi \times \varphi)^* k \tag{7.4}$$

(with A as in (7.3) interpreted as an endomorphism of K^1) is a chain map which coincides with $(\varphi \times \varphi)^*$ on $K^0 \otimes \mathcal{R} \approx \mathcal{R}$. It follows that $\omega^0 - (\varphi \times \varphi)^* \omega^0 = D(c - \varphi^* c + \gamma^{n-1})$ and that $\omega^1(\varphi)$ can be constructed from $c - \varphi^\# c + \gamma^{n-1}$ via a suitable partition of unity in much the same way that we constructed a global parametrix from a Čech parametrix in §3.

We generalize this construction to solve all the other equations (7.1). First we need a complex in which combinatorial equations generalizing (7.2) can be formulated and solved. It is clear that the underlying space of this complex should be $C^*(\mathcal{A}_n, K^* \otimes \Omega^{0,n})$: the cochains with values in the Koszul complex (cf. 9.5). The Koszul differential ϵ acting on the coefficients gives a differential of bidegree (0, 1) and the problem is to complete this to a total differential. The natural thing to do is to define a map δ of bidegree (1, 0) by using the "action" of $\mathcal{A}_n$ on $K^* \otimes \Omega^{0,n}$ given by (7.4) in analogy with formula (4.2). Unfortunately (7.4) does not define an action: $(\varphi\psi)^\# \neq \psi^\# \varphi^\#$ in general, and consequently $\delta^2 \neq 0$. Moreover, this is not an accident due to a bad choice in the definitions; in fact, it turns out that the complicated nature of the formula for the Todd class in terms of the Chern classes reflects the fact that it is impossible to define an action of $\mathcal{A}_n$ of $K^* \otimes \Omega^{0,n}$ commuting with ϵ.

However, (7.4) defines an action "up to homotopy" and by introducing a whole sequence of correcting homotopies we will be able to complete $\delta \pm \epsilon$ to a differential of total degree one. The general setting for this construction is explained in §8, and the explicit details of the case at hand are worked out in §9.

In this context the generalization of (7.2) is to find a cycle $\tau = \tau^{0,n} + \tau^{1,n-1} + \cdots + \tau^{n,0}$ in $C^*(\mathcal{A}_n, K^* \otimes \Omega^{0,n})$ of total degree n such that $\tau^{0,n} = \gamma^n$, the Cauchy kernel:

$$\gamma^n = \frac{1}{(2\pi i)^n} e^1 \wedge \cdots \wedge e^n \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n \in K^n \otimes \Omega^{0,n} = C^0(\mathcal{A}_n, K^n \otimes \Omega^{0,n}).$$

This problem is easily disposed of in §10.

This "extended Cauchy kernel" τ plays a similar role with respect to the universal parametrix as the Cauchy kernel γ^n does with respect to the Bochner–Martinelli kernel ω^0. More precisely, let $f: K^* \otimes \Omega^{0,n} \to \Omega^{0,n} \subset \mathcal{R}$ be the projection on $K^0 \otimes \Omega^{0,n} \approx \Omega^{0,n}$ followed by inclusion into $\mathcal{R}$ (i.e., $f(K^p \otimes \Omega^{0,n}) = 0$ for $p > 0$ and $f|K^0 \otimes \Omega^{0,n}$ = inclusion). Then f is clearly a chain map: $f\epsilon = \bar\partial f$, and it is not hard to show (cf. 10.4) that there exists a map $g: K^* \otimes \Omega^{0,n} \to \mathcal{R}$ which:

(i) is a chain homotopy of f to zero: $g\epsilon + \tilde{\partial} g = f$,

(ii) $g(\gamma^n) = \omega^0$.

This is a slight reformulation of Harvey's connecting sequence mentioned above. Operating on the coefficients, f extends to a map

$$f\colon C^*(\mathscr{A}_n, K^* \otimes \Omega^{0,n}) \to C^*(\mathscr{A}_n, \Omega^{0,n}) \subset C^*(\mathscr{A}_n, \mathscr{R}_\Delta)$$

and it turns out that f is still a chain map with respect to the total differentials on both complexes. Suppose then that we can find a map $\tilde{g}\colon C^*(\mathscr{A}_n, K^* \otimes \Omega^{0,n}) \to C^*(\mathscr{A}_n, \mathscr{R}_\Delta)$ which satisfies:

(i) $\tilde{g}$ is a chain homotopy of f to zero;

(ii) $\tilde{g}|(C^0(\mathscr{A}_n, K^* \otimes \Omega^{0,n}) \approx K \otimes \Omega^{0,n})$ is the above map g.

Then it follows immediately that (up to sign conventions) $g(\tau)$ *is a universal parametrix* and that its corresponding holomorphic cochain ω^n is just $f(\tau) = \tau^{n,0}$.

The map $\tilde{g}$, and hence the universal parametrix, is constructed in §10. In §11 we show that $\kappa^n = \Delta^*\omega^n = \Delta^*\tau^{n,0}$ satisfies the hypothesis of Theorem II. In fact it turns out (although we do not prove it explicitly here) that the restriction (properly interpreted) of τ to Δ is the *total Todd class* $\mathscr{T}$.

§8. TWISTED COMPLEXES

Our constructions are all special cases of the following situation. We are given a bigraded associative C-algebra $A = \bigoplus_{pq} A^{pq}$ and a bigraded C-module $M = \bigoplus_{pq} M^{pq}$ which is also an A-module. Thus we have pairings

$$A \otimes_{\mathbf{C}} A \to A, \qquad A \otimes_{\mathbf{C}} M \to M. \tag{8.1}$$

We assume that $A^{pq} = M^{pq} = 0$ if $p < 0$ and that A^{pq}, M^{pq} are non-zero only for finitely many (possibly negative) q. We are also given C-linear maps

$$D = D' + D'' \text{ on } A, \qquad \nabla = \nabla' + \nabla'' \text{ on } M,$$

where D', ∇' have bidegree $(1, 0)$ and D'', ∇'' have bidegree $(0, 1)$.

We make the following basic assumptions:

$$M \text{ is a } \textit{faithful } A\text{-module, i.e., if } a \cdot m = 0 \text{ for all } m, \text{ then } a = 0. \tag{8.2}$$

Equivalently, A is identified with a subalgebra of the C-endomorphisms of M.

The maps ∇, D are *derivations* with respect to the pairings (8.1):

$$D(a \cdot b) = (Da) \cdot b + (-1)^{|a|} a \cdot Db, \qquad \nabla(a \cdot m) = (Da) \cdot m + (-1)^{|a|} a \cdot \nabla m, \tag{8.3}$$

where $|a| = p + q$ if $a \in A^{pq}$.

$$(\nabla'')^2 = (D'')^2 = 0 \text{ and } \nabla'\nabla'' + \nabla''\nabla' = D'D'' + D''D' = 0. \tag{8.4}$$

However, we *do not assume* that either $(D')^2 = 0$ or $(\nabla')^2 = 0$. In general $(\nabla')^2$ will be a non-zero endomorphism of M of bidegree $(2, 0)$ and we assume:

$$\text{There is an element } k \in A^{2,0} \text{ such that } (\nabla')^2 m = k \cdot m \text{ for all } m \in M. \tag{8.5}$$

It follows that $\nabla^2 m = k \cdot m$ for all $m \in M$. Also a brief computation using (8.3) gives that $(D^2 a) \cdot m = \nabla^2(a \cdot m) - a \cdot \nabla^2 m = (k \cdot a - a \cdot k) \cdot m$ holds for all $m \in M$, and hence by (8.2)

$$D^2 a = k \cdot a - a \cdot k \quad \text{for all } a \in A. \tag{8.6}$$

Writing $\nabla^3 m$ in two ways via (8.4) and (8.3) gives $k \cdot \nabla m = Dk \cdot m + k \cdot \nabla m$ and hence by (8.2) that

$$Dk = 0. \tag{8.7}$$

In this situation we want to modify ∇ *by an element* of A to obtain an honest differential. More precisely, we want to find an element

$$a = \sum_{i \geq 1} a^{i+1,-i}, \qquad a^{i+1,-i} \in A^{i+1,-i} \tag{8.8}$$

such that if we define ∇_a by

$$\nabla_a m = \nabla m + a \cdot m, \tag{8.9}$$

then $\nabla_a^2 = 0$.

Now $\nabla_a^2 m = \nabla^2 m + a \cdot \nabla m + \nabla(a \cdot m) + a \cdot a \cdot m$, and from (8.5) we get that the first term is $k \cdot m$, from (8.3) we get that the sum of the second and third is $(Da) \cdot m$. Therefore $\nabla_a^2 m = (Da + a \cdot a + k) \cdot m$ and hence by (8.2):

$$\nabla_a^2 = 0 \text{ if and only if } Da + a \cdot a + k = 0. \tag{8.10}$$

Thus we are led to:

Definition 8.11. A *twisting cochain* for (M, ∇) is an element $a \in A$ as in (8.8) satisfying

$$Da + a \cdot a = -k. \tag{8.12}$$

The *twisted complex* associated to a is the complex (M, ∇_a) where ∇_a is defined by (8.9).

We borrow the terminology from the twisted tensor products of [7], where the same identity (8.12) appears (with $k = 0$). Note also the analogy with the following differential-geometric situation: If we think of ∇ as a connection with curvature k, then ∇_a corresponds to a *flat* connection obtained by modifying ∇ by an operator a *of order zero.*

The twisted complex is the natural generalization of the bicomplex (i.e., the case $k = a = 0$) that we need. Its differential ∇_a still has degree one with respect to the total grading $p + q$, but it preserves only one of the two filtrations of the bigraded module M. Thus if $F^s M = \bigoplus_{p \geq s} M^{p,*}$ then $\nabla_a : F^s M \to F^s M$. But the filtration by the second component is not preserved. We define the filtration $F^s A$ similarly.

First of all, we need to know that under favorable conditions twisting cochains exist. Roughly speaking, if A is D''-acyclic, then a twisting cochain can be constructed by "successive approximations" using a contracting homotopy for D'':

LEMMA 8.13. *Let (A,D), (M, ∇) and k be as above, and let h'', ρ'': $A \to A$ be* C*-linear maps of bidegree* $(0, -1)$, $(0,0)$ *respectively which satisfy*

$$D''h'' + h''D'' = 1 - \rho'', \qquad \rho''(A^{p,q}) = 0 \quad \text{if } q < 0, \qquad \rho'' k = 0.$$

For $i \geq 1$, define $a^{i+1,-i}$ inductively by $a^{2,-1} = -h''k$ and

$$a^{i+1,-i} = -h''(D' a^{i,-i+1} + \sum_{r+s=i-1} a^{r+1,-r} \cdot a^{s+1,-s}) \text{ for } i > 1.$$

Then $a = \sum_{i \geq 1} a^{i+1,-i}$ is a twisting cochain for (M, ∇).

Proof. Writing down explicitly the component of bidegree $(i+1, -i+1)$ of $Da + a \cdot a + k$ we see that a is a twisting cochain if and only if its components satisfy the sequence of equations

$$D'' a^{2,-1} + k = 0$$

and

$$D'' a^{i+1,-i} + D' a^{i,-i+1} + \sum_{r+s=i-1} a^{r+1,-r} \cdot a^{s+1,-s} = 0 \text{ for } i > 1. \tag{8.14}$$

In view of our hypothesis and (8.7) the first equation is satisfied, and the other equations hold provided that

$$D''\left(D' a^{i,-i+1} + \sum_{r+s=i-1} a^{r+1,-r} \cdot a^{s+1,-s}\right) = 0. \tag{8.15}$$

We prove this by induction:

Let $a_j = \sum_{i=1}^{j} a^{i+1,-i}$ and suppose that (8.15), and hence (8.14), holds for all $i \leq j$. This means that equation (8.12) is satisfied "to sufficiently high order" with respect to the filtration FA. Precisely we have:

$$Da_j + a_j \cdot a_j + k = D' a^{j+1,-j} + \sum_{r+s=j} a^{r+1,-r} \cdot a^{s+1,-s} + \text{terms in } F^{j+3}A.$$

In particular, since the first term is in $A^{j+2,-j-1}$,

$$Da_j + a_j \cdot a_j + k \in F^{j+2}A, \tag{8.16}$$

and since D'' preserves the filtration and D' *increases* the filtration degree by one, we see that (8.15) holds for $i = j+1$ if and only if $D(Da_j + a_j \cdot a_j + k) \in F^{j+3}A$. In view of (8.6) and the derivation rule for D,

$$D(Da_j + a_j \cdot a_j + k) = k \cdot a_j - a_j \cdot k + (Da_j) \cdot a_j - a_j \cdot Da_j.$$

The last two terms can be rewritten using (8.16) and the fact that $a_j \in F^2A$:

$$(Da_j) \cdot a_j \equiv -k \cdot a_j - a_j \cdot a_j \cdot a_j \bmod F^{j+4}A, \qquad a_j \cdot Da_j \equiv -a_j \cdot k - a_j \cdot a_j \cdot a_j \bmod F^{j-4}A.$$

Thus everything cancels $\bmod F^{j+4}$: $D(Da_j + a_j \cdot a_j + k) \in F^{j+4}A \subset F^{j+3}A$, and the proof is complete.

Our arguments require acyclicity results which are obtained from repeated applications of one basic construction: Given a chain homotopy for ∇'', we need to know how to construct a chain homotopy for ∇_a. We consider only the simplest situation that covers our applications, namely, the case in which the ∇''-homology of M is concentrated in the maximal q-degree.

LEMMA 8.17. *Let (M, ∇_a) be a twisted chain complex such that $M^{pq} = 0$ for $q > q_0$ and all p. Let h'', ρ'' be* C*-linear endomorphisms of M of bidegree $(0,-1)$ and $(0,0)$ respectively satisfying*

$$\nabla'' h'' + h'' \nabla'' = 1 - \rho'', \qquad \rho''(M^{pq}) = 0 \text{ for } q \neq q_0.$$

Let $\nabla'_a = \nabla' + a \cdot$ (so that $\nabla_a = \nabla'_a + \nabla''$) and define C*-linear maps h, ρ by*

$$h = \sum_{i\geq 0} (-1)^i h''(\nabla'_a h'')^i, \qquad \rho = \sum_{i \geq 0} (-1)^i (h''\nabla'_a)^i \rho''.$$

Then $h(F^sM) \subset F^s(M)$ and $\nabla_a h + h\nabla_a = 1 - \rho$.

Proof. The first assertion is immediate, because h'' preserves the filtrations and ∇'_a increases the filtration. The second assertion is a straightforward computation that we briefly outline.

Let $h_i = h''(\nabla'_a h'')^i$, $\rho_i = (h''\nabla'_a)^i \rho''$, $a_i = (\nabla'_a h'')^i$, $b_i = (h''\nabla'_a)^i$. From the assumptions on h'', ρ'' it follows easily that $\nabla_a h_0 + h_0 \nabla_a = 1 - \rho_0 + a_1 + b_1$ and for $i > 0$, $\nabla_a h_i + h_i \nabla_a = -\rho_i + a_{i+1} + a_i + b_{i+1} + b_i + c_i$, where $c_i = -(h''\nabla''(\nabla'_a h'')^i + (h''\nabla'_a)^i \nabla h'')$. (Here we have used $\rho''(\nabla'_a h'')^i = 0$ for $i > 0$ because h'' decreases the second bidegree.) Taking alternating sums as in the definition of h, ρ we get

$$\nabla_a h + h \nabla_a = 1 - \rho + \sum_{i \geq 1} (-1)^i c_i. \tag{8.18}$$

Next, $\nabla_a{}^2 = (\nabla'')^2 = 0$ gives the identity $\nabla''\nabla'_a = -\nabla'_a \nabla'' - \nabla'_a \nabla'_a$ and combining this with the properties of h'', ρ'' we get

$$h''\nabla''(\nabla'_a h'')^i = -(h''\nabla'_a)(\nabla'_a h'')^{i-1} - (h''\nabla'_a)(\nabla'_a h'')^i + (h''\nabla'_a) h''\nabla''(\nabla'_a h'')^{i-1} + (h''\nabla'_a)\rho''(\nabla'_a h'')^{i-1}.$$

Computing the third term on the right by iteration of the same identity and taking into account that $\rho''(\nabla'_a h'')^j = 0$ for $j > 0$ we obtain

$$h''\nabla''(\nabla'_a h'')^i = -\sum_{j=1}^{i} (h''\nabla'_a)^j (\nabla'_a h'')^{i-j} - \sum_{j=1}^{i} (h''\nabla'_a)^j (\nabla'_a h'')^{i-j+1} + (h''\nabla'_a)^i h''\nabla'' + (h''\nabla'_a)^i \rho''.$$

Substituting this in the definition of c_i we get easily that $c_i = d_i + d_{i+1}$ where $d_i = \Sigma_{j=1}^{i-1} (h''\nabla'_a)^j (\nabla'_a h'')^{i-j}$, and hence that $\Sigma(-1)^i c_i = 0$ and (8.18) gives the second assertion.

COROLLARY 8.19. *Suppose that $m^{0,q} \in M^{0q}$ satisfies $\nabla'' m^{0q} = \rho''\nabla' m^{0q} = 0$, and let $x = m^{0q} - h\nabla'_a m^{0q}$. Then $x^{0q} = m^{0q}$ and $\nabla_a x = 0$.*

COROLLARY 8.20. *Suppose $\rho'' = 0$ and suppose that $y^{0q} \in M^{0,q}$, $m^{0,q-1} \in M^{0,q-1}$ satisfy $\nabla_a y^{0q} = 0$ and $\nabla'' m^{0,q-1} = y^{0,q}$. Let $x = m^{0,q-1} - h(\nabla_a m^{0,q-1} - y^{0q})$. Then $x^{0,q-1} = m^{0,q-1}$ and $\nabla_a x = y^{0q}$.*

Proof. In both cases it is clear that $\nabla_a x$ is as asserted, and x has the prescribed "initial value" because h is applied to an element of F^1M.

§9. KOSZUL COMPLEXES

In this section we show that the module $M = C^*(\mathscr{A}_n, K^* \otimes \Omega^{0,n})$ mentioned in §7 can be made into a twisted complex. The twisting cochain lies in the algebra A of cochains on $\mathscr{A}_n$ with values in the endomorphisms of K^*. The action of A on M is the natural one which extends (7.4), except that some care has to be taken in sign conventions to make the pairing a derivation. We begin by giving the precise definition of these objects.

First, since we will constantly be forced to shrink the neighborhoods of the diagonal in which our objects are defined, we pass to germs on Δ. Thus, if U is an open set in $\mathbf{C}^n$, we denote by $\mathcal{O}_\Delta(U)$ the ring of germs on Δ of holomorphic functions in $U \times U$: $\mathcal{O}_\Delta(U) = \varinjlim \mathcal{O}(W)$, where W runs over the neighborhoods of Δ in $U \times U$. Similarly we get $\Omega_\Delta{}^{0,n}(U) = \varinjlim \Omega^{0,n}(W)$. We write z, ζ for the coordinates in $U \times U$, and in general we do not distinguish in our notation a function or form in W from its germ on Δ.

Over U we consider the free $\mathcal{O}_\Delta(U)$-module $\mathcal{O}_\Delta(U)e_1 \oplus \cdots \oplus \mathcal{O}_\Delta(U)e_n$ with basis $e_1, \ldots, e_n$, and its dual $\mathcal{O}_\Delta(U)$-module with dual basis $e^1, \ldots, e^n$. If $\varphi: U \to V$ is biholomorphic, let $x_\varphi = (\varphi^1(\zeta) - \varphi^1(z))e^1 + \cdots + (\varphi^n(\zeta) - \varphi^n(z))e^n$ be regarded as an element of $\mathcal{O}_\Delta(U)e^1 \oplus \cdots \oplus \mathcal{O}_\Delta(U)e^n$. The *Koszul complex* $K_*(U, \varphi)$ is defined by $K_p(U, \varphi) = \Lambda^p(\mathcal{O}_\Delta(U)e_1 \oplus \cdots \oplus \mathcal{O}_\Delta(U)e_n)$ with differential $\iota_\varphi: K_p(U, \varphi) \to K_{p-1}(U, \varphi)$ being interior multiplication by x_φ. Explicitly,

$$\iota_\varphi(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum (-1)^{r-1}(\varphi^{i_r}(\zeta) - \varphi^{i_r}(z))e_{i_1} \wedge \cdots \wedge \hat{e}_{i_r} \wedge \cdots \wedge e_{i_p}.$$

Similarly, we have the *dual Koszul complex*

$$K^*(U, \varphi) = \operatorname{Hom}_{\mathcal{O}_\Delta}(U)(K_*(U, \varphi), \mathcal{O}_\Delta(U)) = \Lambda^*(\mathcal{O}_\Delta(U)e^1 \oplus \cdots \oplus \mathcal{O}_\Delta(U)e^n)$$

with adjoint differential $\epsilon_\varphi: K^p(U, \varphi) \to K^{p+1}(U, \varphi)$ which is just exterior multiplication with x_φ.

Of particular interest to us is the complex $K^*(U, \varphi) \otimes \Omega_\Delta{}^{0,n}$ with differential $\epsilon_\varphi \otimes 1$. We denote this complex simply by $K^*(U, \varphi, \Omega^{0,n})$ and its differential simply by ϵ_φ. $K^*(U, \mathrm{id}, \Omega^{0,n})$ is the complex of §7. Note that the pull-back $(\varphi \times \varphi)^*$ gives a *chain map* $(\varphi \times \varphi)^*: K^*(V, \mathrm{id}, \Omega^{0,n}) \to K^*(U, \varphi, \Omega^{0,n})$.

In order to have a convenient complex that contains maps of the form (7.4), we define the complex $L^*(U, \varphi)$ of $\mathcal{O}_\Delta(U)$-linear maps from $K^*(U, \varphi)$ to $K^*(U, \mathrm{id})$ as follows. First, let

$$L^{pq}(U, \phi) = \operatorname{Hom}_{\mathcal{O}_\Delta(U)}(K^{-q}(U, \phi), K^p(I, \mathrm{id}) \approx K^p(U, \mathrm{id}) \otimes \mathrm{K}_{-q}(\mathrm{U}, \phi)$$

(thus L^{pq} lies in the fourth quadrant), and then let $L^r(U, \phi) = \bigoplus_{p+q=r} L^{pq}(U, \phi)$ with differential $d_\phi: L^r(U, \phi) \to L^{r+1}(U, \phi)$ given by $d_\phi = \epsilon_{\mathrm{id}} \otimes 1 + (-1)^{p+q+1} 1 \otimes \iota_\phi$ on $L^{pq}(U, \phi)$. $L^*(U, \phi)$ acts naturally (by evaluation) not only on $K^*(U, \phi)$ but also on $K^*(U, \phi, \Omega^{0,n})$, and we have a natural pairing

$$L^r(U, \phi) \otimes K^s(V, \mathrm{id}, \Omega^{0,n}) \to K^{r+s}(U, \mathrm{id}, \Omega^{0,n}) \tag{9.1}$$

given by $l^{r+s,-s} \otimes k^s \to l^{r+s,-s}((\phi \times \phi)^* k^s)$ on $L^{r+s,-s}$. Note that the map (7.4) is precisely of this form: We first pull back by $(\phi \times \phi)^*$ to obtain an element of $K^*(U, \phi, \Omega^{0,n})$, and then apply an element of $L^*(U, \phi)$ to end up in $K^*(U, \mathrm{id}, \Omega^{0,n})$.

Similarly, if $\psi: V \to W$ is also biholomorphic, we also have a pairing

$$L^r(U, \phi) \otimes L^s(V, \psi) \to L^{r+s}(U, \psi\phi) \tag{9.2}$$

which on $L^{p+r,-p} \otimes L^{p,s-p}$ is defined by $l^{p+r,-p} \otimes \tilde{l}^{p,s-p} \to l^{p+r,-p}((\phi \times \phi)^* \tilde{l}^{p,s-p})$.
Here $\tilde{l}$ and l are interpreted as matrices of functions defined near Δ in $U \times U$, $V \times V$ respectively, and the pairing is the only natural one connecting the spaces concerned: matrix multiplication of l and $(\phi \times \phi)^* \tilde{l}$.

We denote both pairings simply by juxtaposition. The signs in the definitions have been chosen precisely so that the natural derivation rules hold:

$$\epsilon_{\mathrm{id}}(l^r k^s) = (d_\phi l^r)k^s + (-1)^r l^r(\epsilon_{\mathrm{id}} k^s),$$

$$d_{\psi\phi}(l^r \tilde{l}^s) = (d_\phi l^r)\tilde{l}^s + (-1)^r l^r(d_\psi \tilde{l}^s). \tag{9.3}$$

The first identity has the following immediate consequence:

LEMMA 9.4. *If r is even, then $d_\phi l^r = 0$ if and only if the mapping $k \to l^r k$ given by (9.1) is a chain map, and $d_\phi l^{r-1} = l^r$ if and only if the mapping $k \to l^{r-1}k$ is a chain homotopy of l^r to zero.*

Similar statements hold up to sign if r is odd.

Next we can define cochains on $\mathcal{A}_n$ with values in the Koszul complexes as functions that assign to a composable sequence an element of the corresponding complex in the domain of the composition. More precisely, we define,

$$C^q(\mathcal{A}_n, K^*(\Omega^{0,n})) = \Pi K^*(\operatorname{dom} \phi_1 \dots \phi_q, \operatorname{id}, \Omega^{0,n}),$$

$$C^q(\mathcal{A}_n, L^*) = \Pi L^*(\operatorname{dom} \phi_1 \dots \phi_q, \phi_1 \dots \phi_q)$$

(product over all composable sequences). Now these spaces of cochains are too big for our purposes. To ensure good dependence on parameters in (7.1) we restrict ourselves to the subspace of cochains which depend *holomorphically* on $\phi_1, \dots, \phi_q$ in the sense that all normal derivatives on Δ are holomorphic differential operators in $\phi_1, \dots, \phi_q$.

To state our condition precisely, we introduce some notation. If $x \in K^*(U, \operatorname{id}, \Omega^{0,n})$, let $j_\zeta^r x(z)$ denote the Taylor polynomial

$$j_\zeta^r x(z) = \sum_{|\alpha| \le r} \frac{1}{|\alpha|!} \frac{\partial^\alpha x}{\partial \zeta^\alpha}(z, z)(\zeta - z)^\alpha,$$

where differentiation of x means differentiation of its component functions relative to the basis $e^{i_1} \wedge \cdots \wedge e^{i_p} \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n$. Let $J_\zeta^r(K^*(\Omega^{0,n}))$ denote the bundle over C^n whose fibre at z consists of all polynomials $j_\zeta^r x(z)$, i.e., the bundle of normal r-jets of elements of $K^*(\Omega^{0,n})$. Define the bundle J_ζ^r associated to the other Koszul complexes in the same way.

Using this notation, and the notation of §4, our condition is the following:

Definition 9.5. $\mathscr{C}^q(\mathcal{A}_n, K^*(\Omega^{0,n}))$ denotes the subspace of all cochains $c \in C^q(\mathcal{A}_n, K^*(\Omega^{0,n}))$ which satisfy:

For each non-negative integer r there exists an integer N_r and a holomorphic bundle map F_r:

$$\begin{array}{ccc} (G^{N_r})^q_{\text{comp}} & \xrightarrow{F_r} & J_\zeta^r(K^*(\Omega^{0,n})) \\ \downarrow & & \downarrow \\ (C^n)^{q+1} & \xrightarrow{\pi_{q+1}} & C^n \end{array}$$

such that

$$J_\zeta^r c(\phi_1, \dots, \phi_q)(z) = F_r(j^{N_r}\phi_1(\phi_2 \dots \phi_q z), \dots, j^{N_r}\phi_{q-1}(\phi_q z), j^{N_r}\phi_q(z))$$

holds for all $(\phi_1, \dots, \phi_q) \in (\mathcal{A}_n)^q$comp. We define $\mathscr{C}^q(\mathcal{A}_n, L^*)$ in the same way, using $J_\zeta^r(L^*)$ in the above diagram.

We note first that $\mathscr{C}^*(\mathcal{A}_n, L^*)$ is an algebra under the "cup product" pairing

$$\mathscr{C}^p(\mathcal{A}_n, L^q) \otimes \mathscr{C}^r(\alpha_n, L^s) \to \mathscr{C}^{p+r}(\mathcal{A}_n, L^{q+s}) \tag{9.6}$$

defined by

$$(c^{pq} \cdot \tilde{c}^{rs})(\phi_1, \dots, \phi_{p+r}) = (-1)^{(p+q)r} c^{pq}(\phi_{r+1}, \dots, \phi_{p+r})\tilde{c}^{rs}(\phi_1, \dots, \phi_r),$$

where the product on the coefficients in the right hand side is the pairing (9.2). To make this assertion we need to know that this cup product is associative and preserves the defining condition (9.5) of holomorphic cochains. Both of these are straightforward verifications. For the associativity one has to check that the sign introduced in (9.6) preserves the associativity. The reason for this sign will be clear later.

In the same way, $\mathscr{C}^*(\mathcal{A}_n, K^*(\Omega^{0,n}))$ is a $\mathscr{C}^*(\mathcal{A}_n, L^*)$-module under the action

$$\mathscr{C}^p(\mathcal{A}_n, L^q) \otimes \mathscr{C}^r(\mathcal{A}_n, K^s(\Omega^{0,n})) \to \mathscr{C}^{p+r}(\mathcal{A}_n, K^{q+s}(\Omega^{0,n})) \tag{9.7}$$

defined by

$$(a^{pq}\cdot c^{rs})(\phi_1,\dots,\phi_{p+r})=(-1)^{(p+q)r}a^{pq}(\phi_{r+1},\dots,\phi_{p+r})c^{rs}(\phi_1,\dots,\phi_r),$$

where the action on the coefficients is the one given in (9.1). Again one has to check that this defines an action, and that it preserves holomorphic cochains.

Now the differentials ϵ_{id}, d_ϕ on the coefficients induce differentials ϵ, d on $\mathscr{C}^*(\mathscr{A}_n, K^*(\Omega^{0,n}))$, $\mathscr{C}^*(\mathscr{A}_n, L^*)$ of bidegree (0, 1) as follows:

$$(\epsilon c)(\phi_1,\dots,\phi_p)=\epsilon_{\mathrm{id}}c(\phi_1,\dots,\phi_p),\qquad (dc)(\phi_1,\dots,\phi_p)=d_{\phi_1\dots\phi_p}c(\phi_1,\dots,\phi_p).$$

To define an operator of bidegree (1, 0) we need an "action" of $\mathscr{A}_n$ on $K^*(\Omega^{0,n})$, L^*. To this end, let W be a neighborhood of Δ in $U\times U$ with the property that whenever $(z,\zeta)\in W$, the straight line segment from z to ζ lies in U. Define a matrix $A(\phi)$ of functions in W by $A(\phi)(z,\zeta)=\int_0^1\dot\phi(z+t(\zeta-z))dt$ so that $\phi(\zeta)-\phi(z)=A(\phi)(z,\zeta)(\zeta-z)$ for all $(z,\zeta)\in W$. (Thus $A(\phi)$ is a canonical choice of the matrix required in (7.3).) Regarding this matrix as an endomorphism of $\mathcal{O}_\Delta(U)e^1\oplus\cdots\oplus\mathcal{O}_\Delta(U)e^n$, we can rewrite this identity as

$$x_\phi=A(\phi)x_{\mathrm{id}}.\tag{9.8}$$

Moreover, for each multi-index α, differentiation under the integral sign gives

$$\left(\left(\frac{\partial}{\partial\zeta}\right)^\alpha A(\phi)\right)(z,z)=\frac{1}{|\alpha|+1}\left(\frac{\partial}{\partial z}\right)^\alpha\dot\phi(z).\tag{9.9}$$

In particular $A(\phi)(z,z)=\dot\phi(z)$ is invertible, so the germ $A(\phi)$ is invertible and $A(\phi)^{-1}$ also defines an endomorphism of $\mathcal{O}_\Delta(U)e^1\oplus\cdots\oplus\mathcal{O}_\Delta(U)e^n$.

We need the following properties of $A(\phi)$:

LEMMA 9.10. *The maps*

$$K^*(V,\mathrm{id},\Omega^{0,n})\xrightarrow{(\phi\times\phi)^*}K^*(U,\phi,\Omega^{0,n})\xrightarrow{\Lambda^*A(\phi)^{-1}}K^*(U,\mathrm{id},\Omega^{0,n})$$

are chain maps.

Proof. We know that $(\phi\times\phi)^*$ is a chain map. $\Lambda^*A(\phi)^{-1}$ is a chain map because of (9.8) and the definition of ϵ_ϕ, ϵ_{id}:

$$\begin{aligned}\epsilon_{\mathrm{id}}\Lambda^pA(\phi)^{-1}(x^1\wedge\cdots\wedge x^p)&=x_{\mathrm{id}}\wedge A(\phi)^{-1}x^1\wedge\cdots\wedge A(\phi)^{-1}x^p\\&=\Lambda^{p+1}A(\phi)^{-1}(x_\phi\wedge x^1\wedge\cdots\wedge x^p)=\Lambda^{p+1}A(\phi)^{-1}(\epsilon_\phi(x^1\wedge\cdots\wedge x^p)).\end{aligned}$$

LEMMA 9.11. *Suppose that* (ϕ_1,ϕ_2) *is a composable pair. Then*

(i) $A(\phi_1\phi_2)(z,z)=((\phi_2\times\phi_2)^*A(\phi_1))(z,z)A(\phi_2)(z,z)$.

(ii) *If either* ϕ_1 *or* ϕ_2 *is an* affine *map, then* $A(\phi_1\phi_2)=(\phi_2\times\phi_2)^*A(\phi_1))A\phi_2)$.

Proof. Since $A(\phi)(z,z)=\dot\phi(z)$, (i) is just the statement of the chain rule. (ii) is obvious if ϕ_1 is affine, and follows from a simple change of variables if ϕ_2 is affine.

We remark that a statement stronger than (i) actually holds, namely $A(\phi_1\phi_2)\equiv((\phi_2\times\phi_2)^*A(\phi_1))A(\phi_2)\bmod\mathscr{I}_\Delta^2$ where $\mathscr{I}_\Delta$ is the ideal of the diagonal, generated by $\zeta^1-z^1,\dots,\zeta^n-z^n$. ((i) just asserts congruence mod $\mathscr{I}_\Delta$.) Although we will not use this fact here, it is of considerable geometric significance for other applications. We also note that simple computations show that this congruence does not hold mod $\mathscr{I}_\Delta^3$, and this naturally introduces a "curvature" into the situation.

To be able to apply the previous machinery we make the following

Definition 9.12. $a_0\in C^1(\mathscr{A}_n,L^0)$ is the cochain defined by $a_0(\phi)=\Lambda A(\phi)^{-1}\in L^0(\mathrm{dom}\,\phi,\phi)$.

LEMMA 9.13. $a_0\in\mathscr{C}^1(\mathscr{A}_n,L^0)$ *and* $da_0=0$.

Proof. The first statement means that $a_0(\phi)$ depends holomorphically on ϕ in the sense of (9.5). This is clear from (9.9) and the fact that (9.5) is preserved under inversion: The coefficients of the Taylor expansion of $A(\phi)^{-1}$ in terms $(\zeta-z)^\alpha$ are rational functions in those of $A(\phi)$ (inversion of formal power series). The second statement is equivalent, via Lemma (9.4), to Lemma (9.10).

Using the action of a_0 we can now define operators of bidegree $(1,0)$ as follows. First $\delta: \mathscr{C}^p(\mathscr{A}_n, K^*(\Omega^{0,n})) \to \mathscr{C}^{p+1}(\mathscr{A}_n, K^*(\Omega^{0,n}))$ is defined in analogy with (4.2) by

$$(\delta c)(\phi_0, \ldots, \phi_p) = c(\phi_1, \ldots, \phi_p) + \sum_{i=1}^{p} (-1)^i c(\phi_0, \ldots, \phi_{i-1}\phi_i, \ldots, \phi_p) + (-1)^{p+1} a_0(\phi_p) c(\phi_0, \ldots, \phi_{p-1}).$$

Similarly we define $\delta_L: \mathscr{C}^p(\Delta_n, L^*) \to \mathscr{C}^{p+1}(\Delta_n, L^*)$ by

$$(\delta_L c)(\phi_0, \ldots, \phi_p) = c(\phi_1, \ldots, \phi_p) a_0(\phi_0) + \sum_{i=1}^{p} (-1)^i c(\phi_0, \ldots, \phi_{i-1}\phi_i, \ldots, \phi_p) + (-1)^{p+1} a_0(\phi_p) c(\phi_0, \ldots, \phi_{p-1}).$$

Note first of all that the definitions make sense: δ, δ_L map holomorphic cochains to themselves in view of the first assertion of Lemma (9.13). Moreover the second assertion of (9.13), $da_0 = 0$, together with the derivation rules (9.3) give at once that δ, δ_L commute with the corresponding differentials of bidegree (0, 1): $\delta\epsilon = \epsilon\delta$, $\delta_L d = d\delta_L$. However δ^2 and $\delta_L{}^2$ are not zero. In fact if we define $k \in \mathscr{C}^2(\mathscr{A}_n, L^0)$ by

$$k(\phi_1, \phi_2) = a_0(\phi_1\phi_2) - a_0(\phi_2)a_0(\phi_1) \tag{9.14}$$

(or equivalently by the more suggestive formula $-k = \delta_L a_0 + a_0 \cdot a_0$), then we obtain easily the basic identity $\delta^2 c = k \cdot c$, for all $c \in \mathscr{C}^*(\mathscr{A}_n, K^*(\Omega^{0,n}))$. Finally if we define operators ∇, D of total degree one by

$$\begin{aligned} \nabla &= \nabla' + \nabla'' = \delta + (-1)^p \epsilon \quad &&\text{on } \mathscr{C}^p(\mathscr{A}_n, K^q(\Omega^{0,n})), \\ D &= D' + D'' = \delta_L + (-1)^p d \quad &&\text{on } \mathscr{C}^p(\mathscr{A}_n, L^q), \end{aligned}$$

then simple computations based on the identities (9.3) and the definitions (9.6), (9.7) show that ∇, D are *derivations* with respect to the pairings (9.6), (9.7). Thus we are in the situation considered in §8, since we have just checked that the basic assumptions (8.2-5) hold with $A = \mathscr{C}^*(\mathscr{A}_n, L^*)$ and $M = \mathscr{C}^*(\mathscr{A}_n, K^*(\Omega^{0,n}))$.

It remains to show that ∇ can be completed to a twisted differential ∇_a. For this we need to construct a suitable chain homotopy for D'', as required by Lemma (8.13). We do this in two steps.

Note first that given any $\phi \in \mathscr{A}_n$ with dom $\phi = U$, we can define a C-linear (not $\mathcal{O}_\Delta(U)$-linear) map

$$P^*: K^*(U, \mathrm{id}) \to K^{*-1}(U, \mathrm{id}) \tag{9.15}$$

by

$$P^*(fe^I) = \sum_{j=1}^{n} \left(\int_0^1 t^{n-|I|} \frac{\partial f}{\partial \zeta^j}(z, z + t(\zeta - z))\, \mathrm{d}t \right) \iota(e_j) e^I,$$

where $I = (i_1, \ldots, i_p)$ is an increasing multi-index and ι denotes as usual interior multiplication. The (z, ζ) range over a small enough neighborhood of Δ in $U \times U$ so that the integral is defined. We also define a C-linear map

$$\mathscr{R}es^*: K^*(U, \mathrm{id}) \to K^*(U, \mathrm{id}) \tag{9.16}$$

by $\mathscr{R}es^p = 0$ for $p < n$ and $\mathscr{R}es^n(fe^1 \wedge \cdots \wedge e^n)(z, \zeta) = f(z, z) e^1 \wedge \cdots \wedge e^n$. If we note the analogy between P^* and the usual chain homotopy for the Poincaré lemma, it is not surprising that P^* turns out to be a chain homotopy between the identity and $\mathscr{R}es^*$.

We will need a similar chain homotopy for $K_*(U, \phi)$. This is constructed in analogy with (9.15), except that, since we have now to solve a division problem with respect to the functions $\phi^i\zeta - \phi^i z$, we have to differentiate with respect to these variables. Precisely, let

$$P_*(\phi): K_*(U, \phi) \to K_{*+1}(U, \phi) \tag{9.17}$$

be defined by

$$P_*(\phi)(fe_I) = (\phi \times \phi)^* \sum_{j=1}^{n} \left(\int_0^1 t^{|I|} \left(\frac{\partial}{\partial \zeta^j} (\phi^{-1} \times \phi^{-1})^* f \right)(z, z + t(\zeta - z))\, \mathrm{d}t \right) e_j \wedge e_I,$$

using again the multi-index notation and the same understanding of the range of (z, ζ) as in

(9.15). Then let

$$\mathcal{R}es_*: K_*(U,\phi) \to K_*(U,\phi) \tag{9.18}$$

be defined by $\mathcal{R}es_p = 0$ for $p > 0$ and $\mathcal{R}es_0(f)(z,\zeta) = f(z,z)$ for $f \in K_0(U,\phi) \approx \mathcal{O}_\Delta(U)$. We summarize the relevant properties of these operators in the following lemma:

LEMMA 9.19. (i) $\epsilon_{\mathrm{id}}P^* + P^*\epsilon_{\mathrm{id}} = 1 - \mathcal{R}es^*$; $\iota_\phi P_*(\phi) + P_*(\phi)\iota_\phi = 1 - \mathcal{R}es_*$.

(ii) $(P^*)^2 = P_*^2 = 0$.

(iii) *P^* and P_* are equivariant with respect to affine maps, i.e., if ψ is* affine, *then*

$$P^*(a_0(\psi)x) = a_0(\psi)P^*x \quad \text{for all } x \in K^*(U,\mathrm{id})$$
$$P_*(\phi\psi)(xa_0(\psi)) = (P_*(\phi)(x))a_0(\psi) \quad \text{for all } x \in K_*(U,\phi).$$

(here $a_0(\psi)$ acts on the right on K_* either by the transpose of the pairing (9.1) *or equivalently by regarding $K_* \approx 1 \otimes K_* \subset K^* \otimes K_* \approx L$ and applying the pairing* (9.2).)

(iv) *For each r there exists a linear bundle map $L_r: J_\zeta^{r+1}(K^*) \to J_\zeta^r(K^{*-1})$ such that*

$$j_\zeta^r(P^*x)(z) = L_r(j_\zeta^{r+1}(x)(z)) \text{ for all } x \in K^*(U,\mathrm{id})$$

and there exists a holomorphic bundle map $B_r: G^r \times J_\zeta^{r+1}(K_*) \to J_\zeta^r(K_{*+1})$ such that

$$j_\zeta^r(P_*(\phi)x)(z) = B_r(j^r\phi(z), j_\zeta^{r+1}x(z))$$

holds for all $\phi \in \mathcal{A}_n$ and all $x \in K_*(\mathrm{dom}\,\phi, \phi)$.

Proof. (i) is a straightforward computation and (ii) is an immediate consequence of the symmetry of second partial derivatives. (iii) follows from a simple change of variable. Differentiation under the integral sign in the definitions gives (iv), with L_r being actually a differential operator with constant rational coefficients, and B_r a differential operator which is linear in x and nonlinear but rational in the derivatives of ϕ.

We can now build a chain homotopy for the bicomplex $(\bigoplus L^{pq}(U,\phi), \epsilon \otimes 1 \pm 1 \otimes \iota_\phi)$. First by "tensoring" with the identity we obtain a chain homotopy $P''_L(\phi)$ for $\pm 1 \otimes \iota_\phi$

$$P''_L(\phi)(fe^I \otimes e_J) = (-1)^{|I|+|J|}e^I \otimes P_*(\phi)(fe_J).$$

(Since $P_*(\phi)$ is not $\mathcal{O}_\Delta(U)$-linear, we have to define $P''_L(\phi)$ explicitly with respect to the given basis.) Let $\mathcal{R}es''_L(\phi) = 1 \otimes \mathcal{R}es_*(\phi)$. Then if we define $P_L(\phi)$, $\mathcal{R}es_L(\phi)$ from $P''_L(\phi)$, $\mathcal{R}es''_L(\phi)$ by the formulas of Lemma (8.17), we obtain at once:

LEMMA 9.20. (i) $d_\phi P_L(\phi) + P_L(\phi)d_\phi = 1 - \mathcal{R}es_L(\phi)$, $\mathcal{R}es_L(\phi)(L^r(U,\phi)) = 0$ *for* $r < 0$ *and* $P_L(\phi)$ *preserves the filtration* $F^s = \bigoplus_{p \ge s} L^{p*}$ of $L^*(U,\phi)$.

(ii) $P_L(\phi)^2 = 0$.

(iii) *If ψ is an* affine *map then*

$$P_L(\phi\psi)(a_0(\psi)l) = a_0(\psi)P_L(\phi)l \text{ and } P_L(\psi\phi)(la_0(\psi)) = (P_L(\phi)l)a_0(\psi)$$

for all $l \in L^*(U,\phi)$.

(iv) *For each $r \ge 0$ there exists a holomorphic bundle map $F_r: G^r \times J_\zeta^{r+1}(L^*) \to J_\zeta^r(L^{*-1})$ such that*

$$j_\zeta^r(l)(z) = F_r(j^r\phi(z), j^{r+1}l(z))$$

for all $\phi \in \mathcal{A}_n$ and all $l \in L^*(\mathrm{dom}\,\phi, \psi)$.

Proof. (i) Follows from Lemma (8.17) and (i) of Lemma (9.19). The remaining assertions follow easily from the definitions and the corresponding assertions in (9.19).

Letting P_L act on the coefficients, we get a chain homotopy P for $D'' = (-1)^p d$:

$$(Pc)(\phi_1, \ldots, \phi_p) = (-1)^p P_L(\phi_1 \ldots \phi_p)c(\phi_1, \ldots, \phi_p), \qquad c \in \mathcal{C}^p(\mathcal{A}_n, L^*).$$

The only point to check is that $P: \mathcal{C}^*(\mathcal{A}_n, L^*) \to \mathcal{C}^*(\mathcal{A}_n, L^{*-1})$, i.e. that P preserves *holomorphic* cochains. But this is guaranteed by (iv) of Lemma (9.20). We then get $D''P + PD'' = 1 - \mathcal{R}es$, where $\mathcal{R}es$ denotes $\mathcal{R}es_L$ acting on the coefficients. Define then a cochain $a = \sum_{i \ge 1} a^{i+1,-i}$ by the formula of (8.13):

$$a^{2,-1} = -Pk$$
$$a^{i+1,-i} = -P(D'a^{i,-i+1} + \sum_{r+s=i-1} a^{r+1,-r} \cdot a^{s+1,-s}), \qquad i > 1. \tag{9.21}$$

The end result is:

THEOREM 9.22. *The cochain* $a \in \mathscr{C}^*(\mathscr{A}_n, L^*)$ *defined by* (9.21) *is a twisting cochain for* $(\mathscr{C}^*(\mathscr{A}_n, K^*(\Omega^{0,n})), \nabla)$ *and the natural projection*

$$\mathscr{C}^*(\mathscr{A}_n, K^*(\Omega^{0,n})) \to \mathscr{C}^*(\mathscr{A}_n, \Omega_\Delta{}^{0,n})$$

induced by the projection $K^*(\Omega^{0,n}) \to K^0(\Omega^{0,n}) \approx \Omega_\Delta{}^{0,n}$ *is a* chain map *for the corresponding twisted differential* ∇_a.

Proof. We apply Lemma (8.13) with $h'' = P$, $\rho'' = \mathscr{R}es$. Because of (9.20), all the hypotheses have been verified except for $\mathscr{R}es\ k = 0$. Looking at the definition of $\mathscr{R}es_L$, this means that $1 \otimes \mathscr{R}es_0(\phi_1\phi_2)k(\phi_1, \phi_2) = 0$, i.e. that the component of $k(\phi_1, \phi_2)$ in $L^{00}(\phi_1\phi_2)$ vanishes on Δ.

But from the definition (9.14) of k and the fact that the component of $a_0(\phi)$ in L^{00} is 1, we see that the L^{00}-component of k vanishes. This implies the residue condition and hence the first assertion of the theorem. It also implies that k has filtration F^1, and since by (9.20) (i), P preserves the filtration, that each $a^{i+1,-i}$ has filtration F^1, i.e. $a^{i+1,-i} \cdot |K^i(\Omega^{0,n})\colon K^i(\Omega^{0,n}) \to K^0(\Omega^{0,n})$ vanishes. Thus if $c = \Sigma c^{p-i,i} \in \mathscr{C}^*(\mathscr{A}_n, K^*(\Omega^{0,n}))$, the projection of $\nabla_a c$ is $\delta c^{p,0} + \Sigma_i a^{i+1,-i} \cdot c^{p-i,i} = \delta c^{p,0}$, hence the second assertion.

§10. CONSTRUCTION OF THE UNIVERSAL PARAMETRIX

We now complete the program outlined in §7. Two problems remain to be solved. The first is to complete the Cauchy kernel

$$\gamma^n = \frac{1}{(2\pi i)^n} e^1 \wedge \cdots \wedge e^n \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n \in K^n(C^n, \mathrm{id}, \Omega^{0,n}) = \mathscr{C}^0(\mathscr{A}_n, K^n(\Omega^{0,n}))$$

to a ∇_a-cycle τ in $\mathscr{C}^*(\mathscr{A}_n, K^*(\Omega^{0,n}))$, i.e. we have to construct a chain homotopy for ∇_a.

Let P^* and $\mathscr{R}es^*$ be as in (9.15), (9.16) and define C-linear maps P_Ω, $\mathscr{R}es_\Omega$ on $K^*(U, \mathrm{id}, \Omega^{0,n})$ by "tensoring with the identity", i.e.

$$P_\Omega(fe^I \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n) = P^*(fe^I) \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n$$
$$\mathscr{R}es_\Omega(fe^I \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n) = \mathscr{R}es^*(fe^I) \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n.$$

Remark 10.1. Lemma (9.19) holds for P_Ω, $\mathscr{R}es_\Omega$. In particular, if we define an operator Q'' on $\mathscr{C}^*(\mathscr{A}_n, K^*(\Omega^{0,n}))$ by $(Q''c)(\phi_1, \ldots, \phi_p) = (-1)^p P_\Omega(c(\phi_1, \ldots, \phi_p))$, then Q'' *preserves holomorphic cochains* and is a chain homotopy for $\nabla'' = (-1)^p \epsilon$: $\nabla'' Q'' + Q'' \nabla'' = 1 - \mathscr{R}es_\Omega$. Thus if we define an operator Q by the formula of Lemma (8.17), we obtain a chain homotopy for ∇_a.

LEMMA 10.2. $\mathscr{R}es_\Omega(\delta\gamma^n) = 0$.

Proof. $(\delta\gamma^n)(\phi) = \dfrac{1}{(2\pi i)^n}\left(1 - \dfrac{\det \dot\phi(\zeta)}{\det A(\phi)(z, \zeta)}\right) e^1 \wedge \cdots \wedge e^n \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n$, and the residue vanishes because $A(\phi)(z, z) = \dot\phi(z)$. Corollary (8.19) then gives the desired formula for τ:

PROPOSITION 10.3. *Let* $\tau = \tau^{0,n} + \cdots + \tau^{n,0} \in \mathscr{C}^*(\mathscr{A}_n, K^*(\Omega^{0,n}))$ *be defined by* $\tau = \gamma^n - Q\nabla'_a\gamma^n$. *Then* τ *is a* ∇_a-*cycle and* $\tau^{0,n} = \gamma^n$. *Explicitly,* $\tau^{i,n-i}$ *is given inductively by*

$$\tau^{0,n} = \gamma^n,$$
$$\tau^{i,n-i} = -Q''(\delta\tau^{i-1,n-i+1} + \sum_{1 \le j < i} a^{i-j+1,j-i} \cdot \tau^{j-1,n-j+1}), \qquad i > 0.$$

The second problem is to find a map from $\mathscr{C}^*(\mathscr{A}_n, K^*(\Omega^{0,n}))$ to $C^*(\mathscr{A}_n, \mathscr{R}_\Delta)$ that applied to τ gives a universal parametrix. The natural maps to consider are the ones obtained by cup product with cochains of homomorphisms of the coefficients, in analogy with the formalism of §9. To define the appropriate homomorphisms we need some further technicalities.

Let $R_r{}^{0,p}(U)$ denote the space of forms that satisfy the "regularity" condition of §2 and which have total degree 0 in the holomorphic coordinates z, ζ and total degree p in $\bar z$, $\bar\zeta$. Thus $\mathscr{R}_\Delta{}^r = R_r{}^{0,*} \otimes \Omega_\Delta{}^{0,n}$. If $\phi \in \mathscr{A}_n$, let

$$\mathcal{L}_r^{p,q}(\phi) = R_r^{0,p}(\operatorname{dom}\phi)\otimes K_{-q}(\operatorname{dom}\phi,\phi) = \operatorname{Hom}(K^{-q}(\operatorname{dom}\phi,\phi), R_r^{0,p}(\operatorname{dom}\phi))$$

(Hom and tensor over $\mathcal{O}_\Delta$), let $\mathcal{L}_r^s(\phi) = \bigoplus_{p+q=s} \mathcal{L}_r^{p,q}(\phi)$, and let $\mathcal{L}^s(\phi) = \cup_r \mathcal{L}_r^s(\phi)$. $\mathcal{L}^*(\phi)$ is a bicomplex with differential $\hat d_\phi = \bar\partial\otimes 1 + (-1)^{s+1}1\otimes\iota_\phi$, which we write simply as $\bar\partial \pm \iota_\phi$. Let

$$y_\phi = \sum_i \frac{\overline{\phi^i(\zeta)-\phi^i(z)}}{|\phi(\zeta)-\phi(z)|^2} e_i \in L_1^{0-1}(\phi),$$

Then $\langle y_\phi, x_\phi\rangle = |\phi(\zeta)-\phi(z)|^{-2}\Sigma(\overline{\phi^i(\zeta)-\phi^i(z)})(\phi^i(\zeta)-\phi^i(z)) = 1$, thus exterior multiplication by y_ϕ, $\epsilon(y_\phi)$, is a chain homotopy for $\iota_\phi = \iota(x_\phi)$: $\epsilon(y_\phi)\iota_\phi + \iota_\phi\epsilon(y_\phi) = 1$. Observe that

$$\bar\partial: \mathcal{L}_r^{p,q}(\phi)\to\mathcal{L}_{r+1}^{p+1,q}(\phi),\qquad \iota_\phi: \mathcal{L}_r^{p,q}(\phi)\to\mathcal{L}_{r-1}^{p,q+1}(\phi),\qquad \epsilon(y_\phi): \mathcal{L}_r^{p,q}(\phi)\to\mathcal{L}_{r+1}^{p,q-1}(\phi).$$

Thus if we let $\hat{\mathcal{L}}(\phi) = \bigoplus_{pq}\mathcal{L}^{p,q}_{p-q-2n}$, then $\hat{\mathcal{L}}(\phi)$ is stable under $\bar\partial$, $\iota\phi$, $\epsilon(y_\phi)$; i.e. $\hat{\mathcal{L}}(\phi)$ is a *subcomplex* of $\mathcal{L}(\phi)$ and, because of Lemma (8.17), it is *chain contractible*. We denote by $\hat\epsilon(y_\phi)$ the chain homotopy for $\hat d_\phi$ obtained via (8.17) starting from the chain homotopy $h'' = (-1)^s\epsilon(y_\phi)$ for $\hat d''_\phi = (-1)^{s+1}\iota_\phi$. Then

$$\hat\epsilon(y_\phi)\hat d_\phi + \hat d_\phi\hat\epsilon(y_\phi) = 1 \quad \text{on } \hat{\mathcal{L}}(\phi). \tag{10.4}$$

The elements of $\hat{\mathcal{L}}$ give homomorphisms of $K^*(\Omega^{0,n})$ to $\mathcal{R}_\Delta$ via the pairing

$$\begin{gathered}\hat{\mathcal{L}}^s(\phi)\otimes K^t(\operatorname{im}\phi,\operatorname{id},\Omega^{0,n})\to(R^{0,s+t}_{2t+s-2n}\otimes\Omega_\Delta^{0n})(\operatorname{dom}\phi),\\ l^{t+s,-t}\otimes(k^t\otimes\omega)\to(l^{t+s,-t}\otimes 1)(\phi\times\phi)^*(k^t\otimes\omega).\end{gathered}\tag{10.5}$$

This pairing is a derivation: $\bar\partial(l^sk^t) = (\hat d_\phi l^s)d^t + (-1)^s l^s(\epsilon_{\mathrm{id}}k^t)$; in particular the interpretation of cycles, boundaries of Lemma (9.4) holds in this context.

We define cochains on $\mathcal{A}_n$ with coefficients in $\hat{\mathcal{L}}$ in the usual way:

$$C^p(\mathcal{A}_n\hat{\mathcal{L}}^q) = \prod_{\langle\phi_1,\dots,\phi_p\rangle}\hat{\mathcal{L}}^q(\phi_1\dots\phi_p);$$

and also the cup product pairing

$$C^p(\mathcal{A}_n,\hat{\mathcal{L}}^q)\otimes\mathcal{C}^{p'}(\mathcal{A}_n,K^{q'}(\Omega^{0,n})\to C^{p+p'}(\mathcal{A}_n,R^{0,q+q'}_{q+2q'-2n}\otimes\Omega_\Delta^{0,n})\tag{10.6}$$

by

$$(l^{pq}\cdot c^{p'q'})(\phi_1\dots\phi_{p+p'}) = (-1)^{(p+q)p'}l(\phi_{p'+1},\dots,\phi_{p+p'})c^{p'q'}(\phi_1,\dots,\phi_{p'}),$$

where the pairing on the coefficients is (10.5).

For the rest of this section we use the notation $(A,D) = (\mathcal{C}^*(\mathcal{A}_n,L^*),D)$, $(M,\nabla) = (\mathcal{C}^*(\mathcal{A}_n,K^*(\Omega^{0,n})),\nabla)$, and $(\bar M,\bar\nabla) = (C^*(\mathcal{A}_n,\mathcal{R}_\Delta),\delta\pm\bar\partial)$. We also write $\hat M$ for $C^*(\mathcal{A}_n,\hat{\mathcal{L}}^*)$, and A^- for the subalgebra $\bigoplus_{r\le 0}\mathcal{C}^*(\mathcal{A}_n,L^r)$ of A. (A^-,D) is a *subcomplex* of (A,D) which contains the twisting cochain a of M. The cup-product (10.6) gives a representation of $\hat M$ in the module of homomorphisms from M to $\bar M$, hence $\hat M$ should also be a twisted complex with a twisting cochain simply related to that of M.

To see this, note first that A acts naturally on the right on $\mathcal{C}^*(\mathcal{A}_n,K_{-*})$ by regarding $K_{-*} = 1\otimes K_{-*} = L^{0*}\subset L^*$ and restricting the pairing (9.6) to K_{-*} in the first variable. This is the natural "transpose" of the action of A on K^*-valued cochains. Tensoring this action with the identity we obtain a right action of A on $\mathcal{L}$-valued cochains, and A^- leaves the $\hat{\mathcal{L}}$-valued cochains invariant. Thus we have a right action of A^- on $\hat M$: $\hat M\otimes A^-\to\hat M$. We convert this to a left action $A^-\otimes\hat M\xrightarrow{*}\hat M$ by the formula $a*l = (-1)^{|a||l|}l\cdot a$, $\|$ = total degree. This becomes left action of the "transposed" algebra $\hat A^-$, which is just A^- with the product $*$ redefined by $a*b = (-1)^{|a||b|}b\cdot a$, i.e. $\hat A^-$ is associative and $(a*b)*l = a*(b*l)$ holds for all a, $b\in A^-$ and all $l\in\hat M$.

The natural "coboundary" operator $\hat\delta$ on $\hat M$ is given by

$$\begin{aligned}(\hat\delta l)(\phi_0,\dots,\phi_p) = l(\phi_1,\dots,\phi_p)a_0(\phi_0)+\sum_{i=1}^p(-1)^i l(\phi_0,\dots,\phi_{i-1}\phi_i,\dots,\phi_p)\\ +(-1)^{p+1}(\phi_p\times\phi_p)^*l(\phi_0,\dots,\phi_{p-1}).\end{aligned}$$

The action of a_0 is interpreted by the remarks above. We then write $\hat{\nabla} = \hat{\delta} \pm \hat{d}$ for the operator in $\hat{M}$ of total degree one.

PROPOSITION 10.7. *$(\hat{A}^-, D)$, $(\hat{M}, \hat{\nabla})$ and the pairings $*$ satisfy the assumptions* (8.2)–(8.5). *$\hat{M}$ has curvature $\hat{k} = -k$ and twisting cochain $\hat{a} = -a$, where k, a are the curvature and twisting cochain of M. Moreover, the pairing* (10.6) *and the differentials $\bar{\nabla}$, ∇_a, $\hat{\nabla}_{\hat{a}}$ satisfy the derivation rule $\bar{\nabla}(l \cdot c) = (\hat{\nabla}_{\hat{a}} l) \cdot c + (-1)^{|l|} l \cdot (\nabla_a c)$, $l \in \hat{M}$, $c \in M$.*

Proof. All the assertions are straightforward verifications.

Let $f^{\infty} \in \hat{M}^{\infty} = C^0(\mathscr{A}_n, \hat{\mathscr{L}}^0) \approx \hat{\mathscr{L}}^0(\mathrm{id})$ be the homomorphism $K^* \to R^{0,*}$ defined by

$$f^{\infty}(K^p) = 0 \quad \textit{for } p > 0, \qquad f^{\infty}|K^0 = (-1)^{[(n-1)(n-2)]/2} \text{ inclusion } K^0 \approx \mathcal{O}_\Delta \subset R^{\infty},$$

where f^{∞} is trivially a chain map with respect to ϵ_{id}, $\bar{\partial}$, and hence a $\hat{d}_{\mathrm{id}}$-cycle. Let $g^{0,-1} = \hat{\epsilon}(y_{\mathrm{id}}) f^{\infty} \in \hat{\mathscr{L}}^{-1}(\mathrm{id})$, so that, by (10.4), $\hat{d}_{\mathrm{id}} g^{0,-1} = f^{\infty}$. The sign $(-1)^{[(n-1)(n-2)]/2}$ has been chosen so that the following lemma holds without sign factors:

LEMMA 10.8. *$g^{0,-1}\tau^{0,n} = \omega^0$, where $\tau^{0,n} = \gamma^n$, the Cauchy kernel, and ω^0 is the Bochner–Martinelli kernel.*

Proof. Using the duality between exterior and interior multiplication and the explicit formula for $\hat{\epsilon}(y_{\mathrm{id}})$ via (8.17), a short computation gives

$$g^{0,-1}\tau^{0,n} = (-1)^{[n(n-1)]/2}\iota(y_{\mathrm{id}})(\bar{\partial}\iota(y_{\mathrm{id}}))^{n-1}\tau^{0,n}.$$

If we identify $R^{0,*} \otimes K^*$ with a subcomplex of the Cech bicomplex $C(\{\zeta^i - z^i\} \neq 0, R^{0,*} \otimes \Omega^{0,n})$, then $\iota(y_{\mathrm{id}})$ corresponds to the chain homotopy used in the proof of [12, Theorem 2.2], and the assertion is equivalent to the computation there.

Let h'' be the chain homotopy for $\hat{\nabla}''_{\hat{a}} = \hat{\nabla}'' = (-1)^p \hat{d}$ obtained by applying $(-1)^p\hat{\epsilon}$ to the coefficients in $\hat{M}^{p,*}$:

$$(h''l)(\phi_1, \ldots, \phi_p) = (-1)^p \hat{\epsilon}(y_{\phi_1\ldots\phi_p}) l(\phi_1, \ldots, \phi_p);$$

and let h be the chain homotopy for $\hat{\nabla}_{\hat{a}}$ given by (8.17). Then h is a chain contraction of $\hat{M}$. Define an element $g = g^{0,-1} + g^{1,-2} + \cdots + g^{m-1,-n} \in \hat{M}$ of total degree -1 by the formula of Corollary (8.20): $g = g^{0,-1} - h(\hat{\nabla}_{\hat{a}} g^{0,-1} - f^{\infty})$. Then $g \cdot \tau$ is essentially the desired parametrix:

THEOREM 10.9. *Let $\omega^q = (-1)^{[q(q-1)]/2}$ component of bidegree $(q, n-q-1)$ of $g \cdot \tau$, i.e. $\omega^q = (-1)^{[q(q-1)]/2} \sum_i g^{i,-i-1} \cdot \tau^{q-i,n-q+i}$. Then $\{\omega^q\}_{q=0}^{n-1}$ is a universal parametrix, and the corresponding holomorphic cochain is*

$$\omega^n = \tau^{n,0} \in \mathscr{C}^n(\mathscr{A}_n, K^0(\Omega^{0,n})) = \mathscr{C}^n(\mathscr{A}_n, \Omega_\Delta{}^{0,n}).$$

Proof. We note first that the definition of $\hat{\mathscr{L}}$ has been rigged up so that the appropriate "regularity" condition holds automatically: $\omega^q \in C^q(\mathscr{A}_n, \mathscr{R}_\Delta{}^{-q-1})$. Also the recursive relations $\delta\omega^q = \bar{\partial}\omega^{q+1}$ are equivalent to $\bar{\nabla}(g \cdot \tau) = (-1)^{[(n-1)(n-2)]/2}\omega^n = f^{\infty} \cdot \tau^{n,0}$. Hence it is enough to check that $\omega^0 = g^{0,-1} \cdot \tau^{0,n}$ is the Bochner–Martinelli kernel, and, because of the derivation rule of (10.7), that $\hat{\nabla}_{\hat{a}} g = f^{\infty}$. The first assertion is Lemma (10.8) and the second would follow from Corollary (8.20) provided that the necessary "integrability conditions" $\hat{\nabla}'' g^{0,-1} = f^{\infty}$ and $\hat{\nabla}_{\hat{a}} f^{\infty} = 0$ hold. The first is true by definition of $g^{0,-1}$ the second is precisely the second assertion of Theorem (9.22): f^{∞} is a chain map.

§11. PROPERTIES OF THE LOCAL FORMULA

To prove the Riemann–Roch theorem from Theorem (10.9) it remains to check that the hypothesis of Theorem II can be satisfied.

LEMMA 11.1. *Let a be the twisting cochain given by* (9.21). *Then $a^{p+1,-p}(\phi_1, \ldots, \phi_{p+1}) = 0$ if one of the ϕ_i is an affine map.*

Proof. Lemma (9.11) (ii) gives that $k(\phi_1, \phi_2) = a_0(\phi_1\phi_2) - a_0(\phi_2)a_0(\phi_1)$ vanishes if either ϕ_1 or ϕ_2 is affine, and hence the same holds for $a^{2,-1} = -Pk$. We proceed by induction, assuming that the conclusion holds up to $p-1$. Then the second term in (9.21), $\Sigma a^{r+1,-r} \cdot a^{s+1,-s}$, certainly vanishes when one ϕ_i is affine. Since $a^{p,-p+1}$ is by definition in the image of P, and $P^2=0$ by (9.20), $Pa^{p,-p+1}=0$. In particular, looking at the definition of δ_L following (9.13), this means that the second term, $-P(\delta_L a^{p,-p+1})$ reduces (up to sign) to

$$P_L(\phi_1\ldots\phi_{p+1})(a^{p,-p+1}(\phi_2,\ldots,\phi_{p+1})a_0(\phi_1)\pm a_0(\phi_{p+1})a^{p,-p+1}(\phi_1,\ldots,\phi_p)) \tag{11.2}$$

Thus: (i) If ϕ_i, $1<i<p+1$ is affine, then (11.2) vanishes by the induction hypothesis.

(ii) If ϕ_1 is affine, the second term in (11.2) vanishes and by the equivariance of P with respect to affine maps (9.20 iii), the first term is $(P_L(\phi_2\ldots\phi_{p+1})a^{p,-p+1}(\phi_2\ldots\phi_{p+1}))a_0(\phi_1)$ which vanishes because $Pa^{p,-p+1}=0$.

(iii) If ϕ_{p+1} is affine, (11.2) vanishes for the same reason as in (ii).

LEMMA 11.3. *Let τ be as in Proposition* (10.3). *Then for $1\le p\le n$, $\tau^{p,n-p}(\phi_1,\ldots,\phi_p)=0$ if one of the ϕ_i is an affine map.*

Proof. If ϕ is affine,

$$(\delta\tau^{0,n})(\phi)=\left(\frac{1}{2\pi i}\right)^n\left(1-\frac{\det\dot\phi(\zeta)}{\det A(\phi)(z,\zeta)}\right)e^1\wedge\cdots\wedge e^n\otimes d\zeta^1\wedge\cdots\wedge d\zeta^n=0$$

because $A(\phi)(z,\zeta)=\dot\phi$ which is independent of z, ζ, hence $\tau^{1,n-1}=-Q''(\delta\tau^{0,n})$ vanishes on affine maps. We proceed by induction, assuming the conclusion holds up to $p-1$ and using the explicit formula for $\tau^{p,n-p}$ given in (10.3). By Lemma (11.1) and the induction hypothesis the terms involving the twisting coching vanish. The term $-Q''(\delta\tau^{p-1,n-p+1})=(-1)^pP_\Omega(\delta\tau^{p-1,n-p+1})$ vanishes by the same reasoning as in (11.1) using the equivaraince of P_Ω and $P_\Omega{}^2=0$ (Remark (10.1)).

Let $\{\omega^q\}$ be the universal parametrix of Theorem (10.9). Since $\omega^n\in\mathscr{C}^n(\mathscr{A}_n,\Omega_\Delta{}^{0,n})$, $\kappa^n=\Delta^*\omega^n$ satisfies (i) of Theorem II, and since it agrees up to sign with $\tau^{n,0}$, (11.3) says that (ii) is also satisfied. It would remain to verify that κ^n is *skew*. To avoid such a verification, it is expedient to skew-symmetrize the whole construction. Let $C_s^*(\mathscr{A}_n,\mathscr{R}_\Delta)\subset C^*(\mathscr{A}_n,\mathscr{R}_\Delta)$ be the subspace of skew cochains, where skewness is defined as in (4.7). Let $\mathscr{S}\colon C^q(\mathscr{A}_n,\mathscr{R}_\Delta)\to C_s{}^q(\mathscr{A}_n,\mathscr{R}_\Delta)$ be the skew-symmetrization:

$$(\mathscr{S}c)(\phi_1,\ldots,\phi_q)=\frac{1}{(q+1)!}\sum_{\sigma\in S_{q+1}}\operatorname{sgn}\sigma(\phi_{\sigma(q)+1}\ldots\phi_q\times\phi_{\sigma(q)+1}\ldots\phi_q)^*c(\phi_1{}^\sigma,\ldots,\phi_q{}^\sigma).$$

Then $\mathscr{S}$ is a chain map: $\delta\mathscr{S}=\mathscr{S}\delta$, $\bar\partial\mathscr{S}=\mathscr{S}\bar\partial$, hence $(\mathscr{S}\omega^q)$ is a skew universal parametrix. We need to know that $\mathscr{S}$ preserves cocycles which vanish on affine maps.

LEMMA 11.4. *If $u\in C^n(\mathscr{A}_n,\Omega^n)$ is a δ cocycle and $u(\phi_1,\ldots,\phi_n)=0$ when one ϕ_i is affine, then the same is true of $\mathscr{S}u$.*

Proof. It will be convenient to give the proof in terms of the associated "homogeneous cochain" u' defined by

$$u'(x_0,\ldots,x_n)=x_n^*u(x_0x_1{}^{-1},\ldots,x_{n-1}x_n{}^{-1})\in\Omega^n(\operatorname{dom}x_0\cap\cdots\cap\operatorname{dom}x_n).$$

Note that u is uniquely determined by u' by the formula $u(\phi_1,\ldots,\phi_n)=u'(\phi_1\ldots\phi_n,\ldots,\phi_{n-1}\phi_n,\phi_n,1)$. u vanishes when ϕ_i is an affine map A if and only if u' vanishes when $x_{i+1}=Ax_i$. Moreover, Lemma (5.1) implies that

$$u'(x_0,\ldots,Ax_i,\ldots,x_n)=u'(x_0,\ldots,x_n). \tag{11.18}$$

Finally, u is skew if and only if u' is skew in the usual sense.

Thus, to prove the lemma it suffices to show that

$$(\mathscr{S}u')(x_0,\ldots,x_n)=\frac{1}{(n+1)!}\sum_{\sigma\in S_{n+1}}(\operatorname{sign}\sigma)u'(x_{\sigma(0)},\ldots,x_{\sigma(n)})=0$$

whenever $x_{i+1}=Ax_i$. We show that the summands above are pairwise cancelled. Let $\sigma\in S_{n+1}$ such that $\sigma(s)=i$ and $\sigma(t)=i+1$, we choose $\sigma'\in S_{n+1}$ satisfying $\sigma'(s)=i+1$, $\sigma'(t)=i$ and

$\sigma'(j) = \sigma(j)$, $j \neq s, t$. We have sign $\sigma' = -\text{sign}\,\sigma$, and by (11.8) above

$$\begin{aligned}u'(x_{\sigma(0)}, \ldots, x_{\sigma(s)}, \ldots, x_{\sigma(t)}, \ldots, x_{\sigma(n)}) \\ = u'(x_{\sigma(0)}, \ldots, x_i, \ldots, Ax_i, \ldots, x_{\sigma(n)}) \\ = u'(x_{\sigma(0)}, \ldots, x_i, \ldots, x_i, \ldots, x_{\sigma(n)}) = u'(x_{\sigma'(0)}, \ldots, x_{\sigma'(n)}).\end{aligned}$$

Collecting our results, we can now state the following:

THEOREM III. *Let $\bar{\omega}^q = \mathcal{S}\omega^q$, then $\{\bar{\omega}^q\}$ is a skew universal parametrix and $\bar{\kappa} = \Delta^*\bar{\omega}^n$ satisfies the hypotheses of Theorem II.*

Finally, if $c_i\colon \otimes^i L(C^n) \to C$ denotes the ith Chern polynomial then by the remarks in §4 we have

COROLLARY 11.5. Universal Riemann–Roch Theorem: $\bar{\kappa} = \mathcal{T}_n(\theta^n)$ where $\mathcal{T}_n = \text{Todd}_n(c_1, \ldots, c_n)$ is the nth Todd polynomial.

§12. THE UNIVERSAL PARAMETRIX WITH COEFFICIENTS IN A BUNDLE

We consider briefly the straightforward generalization of preceding sections to construct a universal parametrix for the Dolbeault complex with coefficients in a holomorphic C^m bundle. We now have cochains on $\mathcal{A}_{n,m}$ with values in L^*, $K^*(\Omega^{0,n}) \otimes L(C^m)$, and $\mathcal{L}^*$, in the usual way, and holomorphic cochains with values in the first two complexes by the natural extension of definition (9.5). Then $\mathcal{C}^*(\mathcal{A}_{n,m}, K^*(\Omega^{0,n}) \otimes L(C^m))$ is a module over $\mathcal{C}(\mathcal{A}_{n,m}, L^*)$ via the action analogous to (9.7):

$$(a \cdot c)((\phi_1, \psi_1), \ldots (\phi_{p+r}\psi_{p+r})) = (-1)^{|a|\cdot r} a((\phi_{r+1}, \psi_{r+1}), \ldots (\phi_{p+r}, \psi_{p+r}))c((\phi_1, \psi_1), \ldots (\phi_r, \psi_r))$$

where the action on the coefficients is obtained by combining (9.1) and $(\phi, \psi)^{*}$, namely

$$\pm a((\phi_{r+1}, \psi_{r+1}), \ldots, (\phi_{p+r}, \psi_{p+r}))((\phi_{r+1}, \psi_{r+1}) \ldots (\phi_{p+r}, \psi_{p+r}))^{*} c((\phi_1, \psi_1) \ldots (\phi_r, \psi_r)).$$

We define $\nabla_m = \nabla'_m + \nabla''_m$ on $\mathcal{C} = (\mathcal{A}_{n,m}, K^*(\Omega^{0,n}) \otimes L(C^m))$ by $\nabla''_m = \pm \epsilon \otimes 1$ and $\nabla'_m =$ the usual operator δ defined via the action of $a_0{}^m \in \mathcal{C}^1(\mathcal{A}_{n,m}, L^0)$ $a_0{}^m(\phi, \psi) = a_0(\phi)$. It follows easily that the cochain $a_m \in \mathcal{C}^*(\mathcal{A}_{n,m}, L^*)$ defined by $a_m{}^{i+1,-i}(\phi, \psi) = a^{i+1,-i}(\phi)$, where a, as in (9.21), is a twisting cochain for ∇_m.

If we define a chain homotopy $P_\Omega{}^m$ for $K^*(\Omega^{0,n}) \otimes L(C^m)$ by choosing a basis $\{w_j\}$ for C^m and letting

$$P_\Omega{}^m(fe^I \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n \otimes w_j \boxtimes w_k^*) = P_*(fe^I) \otimes d\zeta^1 \wedge \cdots \wedge d\}^n \otimes w_j \boxtimes w_k^*$$

then lemma (9.20) holds for $P_\Omega{}^m$, where in (iii) the equivariance is with respect to pairs (ϕ, ψ) when ϕ is affine and ψ is constant. A chain homotopy Q_m for the twisted differential ∇_{a_m} now follows as in (10.1). If we let $\tau_m{}^{0,n} = \dfrac{1}{(2\pi i)^n} e^1 \wedge \cdots \wedge e^n \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n \otimes 1_{C^m}$, then $(\delta\tau_m{}^{0,n})(\phi, \psi)$ is

$$\frac{1}{(2\pi i)^n}\Big\{e^1 \wedge \cdots \wedge e^n \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n \otimes 1 - \frac{\det \dot{\phi}(\zeta)}{\det A(\phi)(z, \zeta)} e^1 \wedge \cdots \wedge e^n \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n \otimes \psi^{-1}(z)\psi(\zeta)\Big\}$$

which vanishes on Δ so again we can complete $\tau_m{}^{0,n}$ to a ∇_{a_m} cycle τ_m, as in (10.3). We note that by our construction τ_* is additive in the coefficients:

$$\begin{aligned}\tau_{m+m'}^{p,n-p}((\phi_1, \psi_1 \oplus \psi'_1), \ldots, (\phi_p, \psi_p \oplus \psi'_p)) \\ = \tau_m^{p,n-p}((\phi_1, \psi_1) \cdots (\phi_p, \psi_p)) + \tau_{m'}^{p,n-p}((\phi_1, \psi'_1) \cdots (\phi_p, \psi'_p)).\end{aligned}$$

Finally define $g_m \in C^*(\mathcal{A}_{n,m}, \hat{\mathcal{L}}^*)$ by $g_m{}^{p,-p-1}((\phi_1, \psi_1), \ldots, (\phi_p, \psi_p)) = g^{p,-p-1}(\phi_1, \ldots, \phi_p)$, where g is defined in §10. Then by the same formula as in Theorem (10.9) using the natural representation of $C^*(a_{n,m}, \hat{\mathcal{L}}^*)$ in $\text{Hom}(\mathcal{C}^*(\mathcal{A}_{n,m}, K^*(\Omega^{0,n}) \otimes L(C^m), C^*(\mathcal{A}_{n,m}, \mathcal{R}_\Delta \otimes L(C^m)))$ we get a universal parametrix $\{\omega_m{}^q\}_{q=0}^{n-1}$. Applying to this the skew symmetrization $\mathcal{S}$ we have obtained:

THEOREM III′. *Let $\bar{\omega}_m{}^q = \mathcal{S}\omega_m{}^q$, then $\{\bar{\omega}_n{}^q\}$ is a skew universal parametrix for C^m vector* *bundles over n-manifolds, and* $\tilde{\kappa}_m = \operatorname{tr} \Delta^* \tilde{\omega}_m{}^n$ *satisfies the hypotheses of Theorem* II′.

COROLLARY 12.1. *Universal Riemann–Roch Theorem for vector bundles*:

$$\tilde{\kappa}_m = \left(\sum_{i=0}^{n} \mathcal{T}_i \mathrm{ch}_{n-i}\right)(\theta_m)^n$$

where $\mathcal{T}_i$ represents the ith Todd polynomial.

Proof. By Theorem II′, it only remains to identify t_i with $\mathcal{T}_i$; but this is standard (cf. similar argument for L_i in [3, §6]).

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Institute for Advanced Study
Columbia University
Purdue University