The dg Lie algebra $\mathcal{T}(\mathbb{C}^n) = \Omega^{0,\bullet}(\mathbb{C}^n, T)$ is a resolution for the Lie algebra $\mathcal{T}^{hol}(\mathbb{C}^n)$ of holomorphic vector fields on $\mathbb{C}^n$. There is a quasi-isomorphism

$$p : \mathcal{T}(\mathbb{C}^n) \xrightarrow{\simeq} H^0(\mathcal{T}(\mathbb{C}^n)) = \mathcal{T}^{hol}(\mathbb{C}^n).$$

induced by the projection onto the zeroeth cohomology of $\mathcal{T}(\mathbb{C}^n)$. Furthermore, there is a map

$$j_0^\infty : \mathcal{T}^{hol}(\mathbb{C}^n) \to \mathfrak{w}_n$$

which takes the Taylor expansion of a holomorphic vector field at $0 \in \mathbb{C}^n$. The composition $j \overset{\text{def}}{=} j_0^\infty \circ p : \mathcal{T}(\mathbb{C}^n) \to \mathfrak{w}_n$ is a map of dg Lie algebras.

The map j defines a map on (continuous) Chevalley–Eilenberg complexes

$$j^* : \mathrm{C}^\bullet(\mathfrak{w}_n) \to \mathrm{C}^\bullet\left(\Omega^{0,\bullet}(\mathcal{T}(\mathbb{C}^n))\right).$$

Lemma 0.1. *The map j^* factors through $\mathrm{C}^\bullet\left(J\mathcal{T}(\mathbb{C}^n)\right) \hookrightarrow \mathrm{C}^\bullet(\mathcal{T}(\mathbb{C}^n))$.*

As a graded vector space $\mathrm{C}^\#(J\mathcal{T}(\mathbb{C}^n))$ is the space of global sections of a graded (infinite rank) vector bundle on $\mathbb{C}^n$ that we denote by $\mathrm{C}^\#(J\mathcal{T})$. Equipped with the Chevalley–Eilenberg differential $\mathrm{C}^\bullet(J\mathcal{T})$ becomes a complex of vector bundles.

By the lemma, we obtain for each $\phi \in \mathrm{C}^\bullet(\mathfrak{w}_n)$ a global section $j^*\phi$ of the vector bundle $\mathrm{C}^\bullet(J\mathcal{T})$.

Example 0.2. Suppose $n = 1$ and consider the 1-cochain $\phi : f\frac{\partial}{\partial z} \mapsto f'(0)$ of $\mathfrak{w}_1$. The value of the section $j^*\phi$ at the point $z_0 \in \mathbb{C}$ is the cochain for $\mathcal{T} = \Omega^{0,\bullet}(\mathbb{C}, T_{\mathbb{C}})$ defined by

$$a(z, \overline{z})\frac{\partial}{\partial z} + b(z, \overline{z})\mathrm{d}\overline{z}\frac{\partial}{\partial z} \mapsto \frac{\partial}{\partial z}a(z, \overline{z})|_{z=z_0}.$$

The sheaf of sections of the bundle $\mathrm{C}^\bullet(J\mathcal{T})$ is a sheaf of commutative dg algebras. In fact, it is a commutative dg algebra in the category of $D_{\mathbb{C}^n}$-modules.

Consider the de Rham complex of the $D_{\mathbb{C}^n}$-algebra $\mathrm{C}^\bullet(J\mathcal{T})$

$$\Omega^\bullet\left(\mathbb{C}^n, \mathrm{C}^\bullet(J\mathcal{T})\right).$$

Theorem 0.3. *Suppose $\phi \in \mathrm{C}^\bullet(\mathfrak{w}_n)$ and let $\phi^0 = j^*\phi$. Then, there exists $\phi^{i,j} \in \Omega^{i,j}(\mathbb{C}^n, \mathrm{C}^\bullet(J\mathcal{T}))$, $1 \leq i, j \leq n$ such that the element*

$$\Phi \overset{\text{def}}{=} \sum_{i,j} \phi^{i,j}$$

satisfies the equation $(\mathrm{d}_{\mathrm{dR}} + \mathrm{d}_{\mathcal{T}})\Phi = 0$.

Using the Hodge decomposition of the Rham differential $\mathrm{d}_{\mathrm{dR}} = \overline{\partial} + \partial$, we we will actually show that the elements $\phi^{i,j}$ satisfy a pair of descent equations:

- Holomorphic descent:

$$\overline{\partial}\phi^{i,j} = \overline{\partial}_{\mathcal{T}}\phi^{i,j+1}$$

for $0 \leq i, j \leq n$ and

- Cartan descent:

$$\partial\phi^{i,j} = \mathrm{d}_{\mathrm{CE}}\phi^{i+1,j}$$

for $0 \leq i, j \leq n$.

Theorem 0.4. *The assignment $\phi \mapsto \phi^{n,n}$ defines a quasi-isomorphism $\mathrm{C}^\bullet(\mathfrak{w}_n)[2n] \simeq \mathrm{C}^\bullet_{\mathrm{loc}}(\mathcal{T})$. In particular, if $\phi \in \mathrm{C}^\bullet(\mathfrak{w}_n)$ is a cocycle then $\phi^{n,n} \in \mathrm{C}^\bullet_{\mathrm{loc}}(\mathcal{T})$ is a local cocycle and up to equivalence all such local cocycles are obtained in this way.*

Example 0.5. Consider the Gelfand–Fuks cocycle $\phi \in \mathrm{C}^3(\mathfrak{w}_n)$ defined by

$$\phi\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, g(x)\frac{\mathrm{d}}{\mathrm{d}x}, h(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) = \textit{BW :youknowthedeterminant}.$$

The section $\phi^0 = j^*\phi$ of $\mathrm{C}^\bullet(J\mathcal{T})$ is

$$\phi^0\left(\alpha(z,\overline{z})\frac{\partial}{\partial z}, \beta(z,\overline{z})\frac{\partial}{\partial z}, \gamma(z,\overline{z})\frac{\partial}{\partial z}\right) =$$

We first solve for the descent element $\phi^{0,1}$ which satisfies

$$\overline{\partial}\phi^0 = \mathrm{d}_{\mathcal{T}}\phi^{0,1} = (\mathrm{d}_{\mathrm{CE}} + \overline{\partial}_{\mathcal{T}})\phi^{0,1}$$

which has the general formula

$$\phi^{0,1} = \mathrm{d}\overline{z}\frac{\partial}{\partial(\mathrm{d}\overline{z})}\phi^0.$$

Then, automatically $\mathrm{d}_{\mathrm{CE}}\phi^{0,1} = 0$.

Appendix A. Smooth version

Let's first consider the smooth case. Suppose M is an n-dimensional manifold and let $\mathfrak{X}(M)$ denote the associated Lie algebra of vector fields. For each $x \in M$ we have a cochain map

$$j_x^* : \mathrm{C}^\bullet(\mathfrak{w}_n) \to \mathrm{C}^\bullet(\mathfrak{X}(M))$$

which sends a cochain α to $j_x^*\alpha$ where

$$j_x : \mathfrak{X}(M) \to \mathfrak{w}_n$$

takes a vector field and computes its ∞-jet at $x \in M$.

Lemma A.1. *Let $\alpha \in \mathrm{C}^k(\mathfrak{w}_n)$ be a cochain. For any $x, y \in M$ the cochains $j_x^*\alpha$ and $j_y^*\alpha$ are cohomologous. In particular, there exists a $\alpha^{(1)} \in \Omega^1(M) \otimes \mathrm{C}^{k-1}(\mathfrak{w}_n)$ such that $(\mathrm{d}_{dR} \otimes 1)\alpha = (1 \otimes \mathrm{d}_{\mathfrak{X}})\alpha^{(1)}$.*

Inductively, we obtain a sequence of cochains $(j_x^*\alpha, \alpha^{(1)}, \ldots, \alpha^{(n)})$ satisfying

$$(\mathrm{d}_{dR} \otimes 1)\alpha^{(j)} = (1 \otimes \mathrm{d}_{\mathfrak{X}})\alpha^{(j+1)}.$$

It follows immediately that for any ℓ-cycle $N \subset M$ one obtains a cochain of $\mathfrak{X}(M)$ with *trivial* coefficients:

$$\int_N \alpha^{(\ell)} \in \mathrm{C}^{k-\ell}(\mathfrak{X}(M)).$$

Lemma A.2. *If $\alpha \in \mathrm{C}^k(\mathfrak{w}_n)$ is a cocycle then $\int_N \alpha^{(\ell)}$ is a cocycle.*

Example A.3. Consider the cocycle $\alpha \in \mathrm{C}^3(\mathfrak{w}_1)$ which is dual to the homology 3-cycle

$$L_{-1} \wedge L_0 \wedge L_1 \in \mathrm{C}_3(\mathfrak{w}_1).$$

Consider $M = S^1$. For $x \in S^1$ one has

$$j_x^*\alpha\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, g(x)\frac{\mathrm{d}}{\mathrm{d}x}, h(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) = BW : you know the determinant.$$

An explicit form of a cochain $\alpha^{(1)} \in \Omega^1(S^1) \otimes \mathrm{C}^2(\mathfrak{X}(S^1))$ satisfying $\mathrm{d}_{\mathfrak{X}}\alpha^{(1)} = \mathrm{d}_{dR} j_x^*\alpha$ is

$$\alpha^{(1)}\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, g(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) \overset{\mathrm{def}}{=} f'(x)\mathrm{d}_{dR}(g'(x)) - g'(x)\mathrm{d}_{dR}(f'(x)) \in \Omega^1(S^1).$$

As one can immediately check, $\int_{S^1} \alpha^{(1)} \in \mathrm{C}^2(\mathfrak{X}(S^1))$ is the usual Virasoro cocycle.