CECH FIELD THEORY

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1. The general construction

Let E, F be two holomorphic vector bundles on X. Define the subspace

$$\Omega^{(p,q);(r,s)}(X \times X, V \boxtimes W) \subset \Omega^{p+r,q+s}(X \times X, V \boxtimes W).$$

If $k \in \Omega^{(0,q),(n,n-s)}(X \times X, E \boxtimes E^*)$, then define the operator $\underline{k} : \Omega^{0,s}_c(X, E) \to \Omega^{0,q}(X, E)$ by the formula

$$(\underline{k}\alpha)(x) = \int_{y \in X} k(x,y)\alpha(y).$$

We call k the kernel of the smoothing operator $\underline{k}$.

2. Preliminary definitions

Let (E, Q, ω) be the data of a free BV theory on a manifold X. The sheaf of local functionals $\mathcal{O}_{\rm loc}(\mathcal{E})$ is equipped with the BV bracket $\{-,-\}$ of cohomological degree $+1$ and a differential induced by Q which we denote by the same letter. We also fix an open cover $\mathcal{U}$ for X.

Consider the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{O}_{\rm loc}(\mathcal{E}))$ of the sheaf of local functionals. Recall that $\mathcal{O}_{\rm loc}(\mathcal{E})$ has an internal cohomological degree and differential Q. The complex we consider is the totalization of the bigraded complex consisting of the Čech differential δ and the differential Q.

Together with the cup product on the Čech complex, the BV bracket endows $\check{C}^\bullet(\mathcal{U}, \mathcal{O}_{\rm loc}(\mathcal{E}))$ with a bracket that we denote by $\{-,-\}^\vee$. For fixed Čech degrees k, ℓ, it is defined by the composition

$$\check{C}^k(\mathcal{U}, \mathcal{O}_{\rm loc}(\mathcal{E})) \otimes \check{C}^\ell(\mathcal{U}, \mathcal{O}_{\rm loc}(\mathcal{E})) \xrightarrow{\cup} \check{C}^{k+\ell}(\mathcal{U}, \mathcal{O}_{\rm loc}(\mathcal{E}) \otimes \mathcal{O}_{\rm loc}(\mathcal{E})) \xrightarrow{\{-,-\}} \check{C}^{k+\ell}(\mathcal{U}, \mathcal{O}_{\rm loc}(\mathcal{E}) \otimes \mathcal{O}_{\rm loc}(\mathcal{E}))$$

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Explicitly, if I is a k-cochain and J is an ℓ-cochain, then $\{I, J\}$ is the $(k+\ell)$-cochain defined by

$$\{I, J\}^{\vee}_{\alpha_0 \cdots \alpha_{k+\ell}} = \{I_{\alpha_0 \cdots \alpha_k}, J_{\alpha_k \cdots \alpha_{k+\ell}}\}$$

where the right-hand side is the usual BV bracket.

The bracket $\{-,-\}$ endows the shift of local functionals $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1]$ with the structure of a sheaf of dg Lie algebras, where the differential is given by Q. Similarly, $\{-,-\}^{\vee}$ gives $\check{\mathrm{C}}^{\bullet}(\mathcal{U}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))[-1]$ the structure of a dg Lie algebra.

Lemma 2.1. *For any open cover $\mathcal{U}$ the complex $\check{\mathrm{C}}^{\bullet}(\mathcal{U}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))[-1]$ is a dg Lie algebra where the differential is $\delta + Q$ and the bracket is $\{-,-\}^{\vee}$.*

Definition 2.2. A **Čech local functional** for an open cover $\mathcal{U}$ of X is an element

$$\check{I} \in \check{\mathrm{C}}^{\bullet}(\mathcal{U}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E})).$$

Suppose $\check{I}$ is of cohomological degree zero. The **Čech classical master equation** is the Maurer-Cartan equation

$$(\delta + Q)\check{I} + \frac{1}{2}\{\check{I}, \check{I}\}^{\vee} = 0.$$

A Čech local functional $\check{I}$ of degree zero determines a degree $+1$ element in the dg Lie algebra $\check{\mathrm{C}}^{\bullet}(\mathcal{U}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))[-1]$. The condition that this local functional satisfy the Čech classical master equation is equivalent to the condition that it is a Maurer-Cartan element.

Definition 2.3. A **Čech classical field theory on** X for the open cover $\mathcal{U}$ is a free BV theory (E, Q, ω) on X together with a degree zero Čech local functional $\check{I}$ satisfying the Čech classical master equation.

Fix a partition of unity $\rho = \{\rho_\alpha\}$ subordinate to the cover $\mathcal{U}$ of X. For any sheaf $\mathcal{F}$ on X we define the map

$$\mathcal{D}_\rho : \check{\mathrm{C}}^{\bullet}(\mathcal{U}, \mathcal{F}) \to \mathcal{F}(X)$$

by BW: ?

Lemma 2.4. *The map of cochain complexes*

$$\mathcal{D}_\rho : \check{\mathrm{C}}^{\bullet}(\mathcal{U}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))[-1] \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})(X)[-1]$$

is a map of dg Lie algebras. In particular, every Čech local functional $\check{I}$ satisfying the Čech master equation defines a local functional $I_{\mathcal{U},\rho} = \mathcal{D}_\rho(\check{I})$ which satisfies the usual classical master equation on X.

As an immediate corollary, we see that once we choose a partition of unity, every Čech classical theory $(E, Q, \omega, \check{I})$ on X defines an ordinary BV theory where the local interaction is given by $I_{\mathcal{U},\rho} = \mathcal{D}_\rho(\check{I})$.

Lemma 2.5. *BW: How does $I_{\mathcal{U},\rho}$ depend on the cover and ρ.*

In fact, for nice enough covers even more is true. BW: how to say this $\mathcal{U}$ is BW: nice enough the map $\mathcal{D}$ is a quasi-isomorphism of dg Lie algebras. In this case, we see that there is an equivalence between Čech local functionals satisfying the Čech master equation and local functionals satisfying the usual master equation.