

Holomorphic renormalization

Brian R. Williams

July 17, 2018

- 1 Introduction
- 2 The definition of a holomorphic field theory
- 3 One-loop regularization
- 4 Chiral anomalies in arbitrary dimension
- 5 Holomorphic deformations and the QME

Theorem 5.1. *Let $(V, Q^{hol}, (-, -)^{hol})$ be a free holomorphic theory on \mathbb{C}^d , and suppose I is a [BW: holomorphic deformation](#). Then, the resulting effective family $\{I[L]\}$ satisfies the quantum master equation modulo \hbar^2 if and only if $\oint d^d z I$ is a Maurer-Cartan element of $\oint \text{Obs}_V[[\hbar]]/\hbar^2$.*

Our main objective in this chapter is two-fold. First we will define the concept of a holomorphic field theory and set up notation and terminology that we will use throughout the text. Our next goal is more technical, but will provide the backbone for much of the analysis throughout the remainder of this thesis. We will show how certain holomorphic theories are surprisingly well-behaved when it comes to the problem of renormalization.

The starting point of our work is Costello's [?] mathematical formulation of the Wilsonian approach to perturbative quantum field theory. The main takeaway is that to construct a full quantum field theory it suffices to define the theory at each energy (or length) scale and to ask that these descriptions be compatible as we vary the scale. Concretely, this compatibility is through the *renormalization group (RG) flow* and is encoded by an operator $W(P_{\epsilon < L}, -)$ acting on the space of functionals. The functional $W(P_{\epsilon < L}, -)$ is defined as a sum over weights of graphs which is how Feynman diagrams appear in Costello's formalism. The infamous infinities of quantum field theory arise due to studying behavior of theories at arbitrarily high energies (or small lengths). In physics this is called the ultra-violet (UV) divergence. Classically, a theory is defined by a *local functional* I^{cl} which is a functional on the space of fields obtained by integrating a Lagrangian density. At each scale L a theory is defined by an action functional $I[L]$, which is a function on the space of fields.

In some broad generality, there are two main steps to construct a QFT in this formalism.

Renormalization: For each scale L and regulator $\epsilon > 0$ consider the RG flow from scale ϵ to L :

$$W(P_{\epsilon < L}, I). \tag{1}$$

In general, the limit $\epsilon \rightarrow 0$ will not be defined, but by Costello's main result there exists counterterms $I^{CT}(\epsilon)$ such that the $\epsilon \rightarrow 0$ limit of

$$W(P_{\epsilon < L}, I - I^{CT}(\epsilon))$$

is well-defined. Denote this limit by $I[L]$. The family $\{I[L]\}$ defines what we call a *prequantization*.

Gauge consistency: We then ask if the family $\{I[L]\}$ defines a consistent quantization. For each L we require that $I[L]$ satisfy the scale L quantum master equation

$$QI[L] + \hbar \Delta_L I[L] + \frac{1}{2} \{I[L], I[L]\}_L = 0,$$

or, equivalently, $(Q + \hbar \Delta_L) e^{I[L]/\hbar} = 0$.

In this section we are primarily concerned with the first step: renormalization. The complication here is that even very natural field theories can have a very complicated collections of counterterms. For instance, the naive quantization of Chern-Simons theory on a three-manifold has counterterms even at one-loop. For holomorphic theories, however, we will show how the situation becomes much simpler at least at the level of one-loop.

Lemma 5.2. *Let \mathcal{E} be a holomorphic theory on \mathbb{C}^d with classical interaction I^{cl} . Then, there exists a one-loop prequantization $\{I[L] \mid L > 0\}$ of I^{cl} involving no counterterms. That is, we can find a propagator*

$P_{\epsilon < L}$ for which the $\epsilon \rightarrow 0$ limit of

$$W(P_{\epsilon < L}, I) \mod \hbar^2$$

exists. Moreover, if I is holomorphically translation invariant we can pick the family $\{I[L]\}$ to be holomorphically translation invariant as well.

We will use this result repeatedly throughout this thesis. This result tells us that the analytic difficulties encountered in QFT are manageable in the case of holomorphic theories on \mathbb{C}^d . Thus, our main focus will be on obstructions to satisfying the quantum master equation. Indeed, a corollary of this result will give us a procedure for computing the one-loop obstruction explicitly in terms of Feynman diagrams. We conjecture that an extension of this result should hold to all orders in \hbar which would give a constructive way of analyzing the obstruction theory order by order in \hbar . Nevertheless, we will leverage the one-loop behavior to formulate and prove index theorems in the context of holomorphic QFT.

One surprising aspect of this comes from thinking about holomorphic theories in a different way. Any supercharge Q of a supersymmetric theory satisfying $Q^2 = 0$ allows one to construct a “twist”. In some cases, where Clifford multiplication with Q spans all translations such a twist becomes a topological theory (in the weak sense). In any case, however, such a Q defines a “holomorphic twist” [?], which results in the type of holomorphic theories we consider. We will remark on numerous examples of this phenomena throughout the thesis. Regularization in supersymmetric theories, especially gauge theories, is notoriously difficult. Our result implies that after twisting the analytic difficulties become much easier to deal with. Consequently, facets of these theories, such as their anomalies, can be cast in a more algebraic framework. We will see such an example of this in the case of the holomorphic σ -model in the next chapter.

Already, in [?] Li has proved a stronger complex one-dimensional version of this fact to all orders in \hbar . He uses this to give an elegant interpretation of the quantum master equation for two-dimensional chiral conformal field theories using vertex algebras. Although we do not make any statements in this thesis past one-loop quantizations, the higher loop behavior remains a rich and subtle problem that we hope to return to.

6 The definition of a quantum field theory

In this section we will give an expedient review of the classical and quantum Batalin-Vilkovisky formalisms. We will also set up the requisite conventions and notations that we will use throughout the thesis.

6.1 Classical field theory

Classical field theory is a formalism for describing a physical system in terms of objects called *fields*. Mathematically, the space of fields is a (most often infinite dimensional) vector space \mathcal{E} .

Classical physics is described by the critical locus of a (usually real or complex valued) linear functional on the space of fields

$$S : \mathcal{E} \rightarrow \mathbb{R} \text{ or } \mathbb{C}, \quad (2)$$

called the *action functional*. The critical locus is the locus of fields that have zero variation

$$\text{Crit}(S) := \{\varphi \in \mathcal{E} \mid dS(\varphi) = 0\}. \quad (3)$$

A field φ satisfying the equation $dS(\varphi) = 0$ is said to be a *solution to the classical equations of motion*.

Even in the finite dimensional case, if the functional S is not sufficiently well-behaved the critical locus can be still be highly singular. The starting point of the *classical Batalin-Vilkovisky* formalism is to instead consider the *derived* critical locus. To get a feel for this, we review the finite dimensional situation. Let M be a manifold, which is our ansatz for \mathcal{E} at the moment, and suppose $S : M \rightarrow \mathbb{R}$ is a smooth map. The critical locus is the intersection of the graph of dS in T^*M with the zero section $0 : M \rightarrow T^*M$. Thus, functions on the critical locus are of the form

$$\mathcal{O}(\text{Crit}(S)) = \mathcal{O}(\Gamma(dS)) \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M).$$

The derived critical locus is a derived space whose dg ring of functions is

$$\mathcal{O}(\text{Crit}^h(S)) = \mathcal{O}(\Gamma(dS)) \otimes_{\mathcal{O}(T^*M)}^{\mathbb{L}} \mathcal{O}(M).$$

We have replaced the strict tensor product with the derived one. Using the Koszul resolution of $\mathcal{O}(M)$ as a $\mathcal{O}(T^*M)$ -module one can write this derived tensor product as a complex of polyvector fields equipped with some differential:

$$\mathcal{O}(\text{Crit}^h(S)) \simeq (\text{PV}^{-*}(M), \iota_{dS}).$$

In cohomological degree $-i$ we have $\text{PV}^{-i}(M) = \Gamma(M, \wedge^i TM)$ and ι_{dS} denotes contraction with the one-form dS (which raises cohomological degree with our regrading convention). With our grading convention we have $\mathcal{O}(T^*[-1]M) = \text{PV}^{-*}(M)$. The space $\mathcal{O}(T^*[-1]M)$ has natural shifted Poisson structure, which takes the form of the familiar Schouten-Nijenhuis bracket of polyvector fields.

The takeaway is that the derived critical locus of a functional $S : M \rightarrow \mathbb{R}$ has the structure of a (-1) -symplectic space. This will be the starting point for our definition of a theory in the BV formalism in the general setting.

In all non-trivial examples the space of fields \mathcal{E} is infinite dimensional and we must be careful with what functionals S we allow. The space of fields we consider will always have a natural topology, and we will choose functionals that are continuous with respect to it. We divert for a moment to discuss these issues of infinite dimensional linear algebra.

6.1.1 Some functional analysis

Homological algebra will play a paramount role in our approach to perturbative field theory. A problem with this is that the category of topological vector spaces is not an abelian category. It is

therefore advantageous to enlarge this to the category of *differentiable vector spaces*. The details of this setup are carried out in the Appendix of [?], but we will recall some key points.

Let \mathbf{Mfld} be the site of smooth manifolds. The covers defining the Grothendieck topology are given by surjective local diffeomorphisms. There is a natural sheaf of algebras on this site given by smooth functions $C^\infty : M \mapsto C^\infty(M)$.

For any p the assignment $\Omega^p : M \mapsto \Omega^p(M)$ defines a C^∞ -module. Similarly, if F is any C^∞ -module we have the C^∞ -module of p -forms with values on F defined by the assignment

$$\Omega^1(F) : M \in \mathbf{Mfld} \mapsto \Omega^1(M, F) = \Omega^1(M) \otimes_{C^\infty(M)} F(M).$$

Definition 6.1. A *differentiable vector space* is a C^∞ -module equipped with a map of sheaves on \mathbf{Mfld}

$$\nabla : F \rightarrow \Omega^1(F)$$

such that for each M , $\nabla(M)$ defines a flat connection on the $C^\infty(M)$ -module $F(M)$. A map of differentiable vector spaces is one of C^∞ -modules that intertwines the flat connections. This defines a category that we denote \mathbf{DVS} .

Our favorite example of differentiable vector spaces are imported directly from geometry.

Example 6.2. Suppose E is a vector bundle on a manifold X . Let $\mathcal{E}(X)$ denote the space of smooth global sections. Let $C^\infty(M, \mathcal{E}(X))$ be the space of sections of the bundle $\pi_X^* E$ on $M \times X$ where $\pi_X : M \times X \rightarrow X$ is projection. The assignment $M \mapsto C^\infty(M, \mathcal{E}(X))$ is a C^∞ -module with flat connection, so defines a differentiable vector space. Similarly, the space of compactly supported sections $\mathcal{E}_c(X)$ is a \mathbf{DVS} .

Many familiar categories of topological vector spaces embed inside the category of differentiable vector spaces. Consider the category of locally convex topological vector spaces \mathbf{LCTVS} . If V is such a vector space, there is a notion of a smooth map $f : U \subset \mathbb{R}^n \rightarrow V$. One can show, Proposition B.3.0.6 of [?], that this defines a functor $\text{dif}_t : \mathbf{LCTVS} \rightarrow \mathbf{DVS}$ sending V to the C^∞ -module $M \mapsto C^\infty(M, V)$. If $\mathbf{BVS} \subset \mathbf{LCTVS}$ is the subcategory with the same objects but whose morphisms are bounded linear maps, this functor restricts to embed \mathbf{BVS} as a full subcategory $\mathbf{BVS} \subset \mathbf{DVS}$.

There is a notion of completeness that is useful when discussing tensor products. A topological vector space $V \in \mathbf{BVS}$ is *complete* if every smooth map $c : \mathbb{R} \rightarrow V$ has an anti-derivative [?]. There is a full subcategory $\mathbf{CVS} \subset \mathbf{BVS}$ of complete topological vector spaces. The most familiar example of a complete topological vector space will be the smooth sections $\mathcal{E}(X)$ of a vector bundle $E \rightarrow X$.

We let $\mathbf{Ch}(\mathbf{DVS})$ denote the category of cochain complexes in differentiable vector spaces (we will refer to objects as differentiable vector spaces). It is enriched over the category of differential graded vector spaces in the usual way. We say that a map of differentiable cochain complexes $f : V \rightarrow W$ is a quasi-isomorphism if and only if for each M the map $f : C^\infty(M, V) \rightarrow C^\infty(M, W)$ is a quasi-isomorphism.

Theorem 6.3 (Appendix B [?]). *The full subcategory $\text{dif}_c : \text{CVS} \subset \text{DVS}$ is closed under limits, countable coproducts, and sequential colimits of closed embeddings. Furthermore, CVS has the structure of a symmetric monoidal category with respect to the completed tensor product $\widehat{\otimes}_\beta$.*

We will not define the tensor product $\widehat{\otimes}_\beta$ here, but refer the reader the cited reference for a complete exposition. We will recall its key properties below. Often times we will write \otimes for $\widehat{\otimes}_\beta$ where there is no potential conflict of notation. The fundamental property of the tensor product that we use is the following. Suppose that E, F are vector bundles on manifolds X, Y respectively. Then, $\mathcal{E}(X), \mathcal{F}(Y)$ lie in CVS, so it makes sense to take their tensor product using $\widehat{\otimes}_\beta$. There is an isomorphism

$$\mathcal{E}(X) \widehat{\otimes}_\beta \mathcal{F}(Y) \cong \Gamma(X \times Y, E \boxtimes F) \quad (4)$$

where $E \boxtimes F$ denotes the external product of bundles, and Γ is smooth sections.

If E is a vector bundle on a manifold X , then the spaces $\mathcal{E}(X), \mathcal{E}_c(X)$ both lie in the subcategory $\text{CVS} \subset \text{DVS}$. The differentiable structure arises from the natural topologies on the spaces of sections.

We will denote by $\overline{\mathcal{E}}(X)$ ($\overline{\mathcal{E}}_c(X)$) the space of (compactly supported) distributional sections. It is useful to bear in mind the following inclusions

$$\begin{array}{ccc} & \overline{\mathcal{E}}_c(X) & \\ \nearrow & & \nwarrow \\ \mathcal{E}_c(X) & & \overline{\mathcal{E}}(X) \\ \searrow & & \nearrow \\ & \mathcal{E}(X) & \end{array} .$$

When X is compact the bottom left and top right arrows are equalities.

Denote by E^\vee the dual vector bundle whose fiber over $x \in X$ is the linear dual of E_x . Let $E^!$ denote the vector bundle $E^\vee \otimes \text{Dens}_X$, where Dens_X is the bundle of densities. In the case X is oriented, Dens_X is isomorphic to the top wedge power of T^*X . Let $\mathcal{E}^!(X)$ denote the space of sections of $E^!$. The natural pairing

$$\mathcal{E}_c(X) \otimes \mathcal{E}^!(X) \rightarrow \mathbb{C}$$

that pairs sections of E with the evaluation pairing and integrates the resulting compactly supported top form exhibits $\overline{\mathcal{E}}_c(X)$ as the continuous dual to $\mathcal{E}^!(X)$. Likewise, $\mathcal{E}_c(X)$ is the continuous dual to $\overline{\mathcal{E}}^!(X)$. In this way, the topological vector spaces $\overline{\mathcal{E}}(X)$ and $\overline{\mathcal{E}}_c(X)$ obtain a natural differentiable structure.

If V is any differentiable vector space then we define the space of linear functionals on V to be the space of maps $V^* = \text{Hom}_{\text{DVS}}(V, \mathbb{R})$. Since DVS is enriched over itself this is again a differentiable vector space. Similarly, we can define the polynomial functions of homogeneous degree n to be the space

$$\text{Sym}^n(V^*) = \text{Hom}_{\text{DVS}}^{\text{multi}}(V \times \cdots \times V, \mathbb{R})_{S_n}$$

where the hom-space denotes multi-linear maps, and we have taken S_n -coinvariants on the right-hand side. The algebra of functions on V is defined by

$$\mathcal{O}(V) = \prod_n \text{Sym}^n(V^*).$$

As an application of Equation (??) we have the following identification.

Lemma 6.4. *Let E be a vector bundle on X . Then, there is an isomorphism*

$$\mathcal{O}(\mathcal{E}(X)) \cong \prod_n \mathcal{D}_c(X^n, (E^\dagger)^{\boxtimes n})_{S_n}$$

where $\mathcal{D}_c(X^n, (E^\dagger)^{\boxtimes n})$ is the space of compactly supported distributional sections of the vector bundle $(E^\dagger)^{\boxtimes n}$. Again, we take S_n -coinvariants on the right hand side.

6.1.2 Local functionals

In our approach, the space of fields will always be equal to the space of smooth sections of a \mathbb{Z} -graded vector bundle $E \rightarrow X$ on a manifold $\mathcal{E} = \Gamma(X, E)$. The class of functionals $S : \mathcal{E} \rightarrow \mathbb{R}$ defining the classical theories we consider are required to be *local*, or given by the integral of a Lagrangian density. We define this concept now.

Let D_X denote the sheaf of differential operators on X . The ∞ -jet bundle $\text{Jet}(E)$ of a vector bundle E is the vector bundle whose fiber over $x \in X$ is the space of formal germs at x of sections of E . It is a standard fact that $\text{Jet}(E)$ is equipped with a flat connection giving its space of sections $J(E) = \Gamma(X, \text{Jet}(E))$ the structure of a D_X -module.

Above, we have defined the algebra of functions $\mathcal{O}(\mathcal{E}(X))$ on the space of sections $\mathcal{E}(X)$. Similarly, let $\mathcal{O}_{\text{red}}(\mathcal{E}(X)) = \mathcal{O}(\mathcal{E}(X))/\mathbb{R}$ be the quotient by the constant polynomial functions. The space $\mathcal{O}_{\text{red}}(J(E))$ inherits a natural D_X -module structure from $J(E)$. We refer to $\mathcal{O}_{\text{red}}(J(E))$ as the space of *Lagrangians* on the vector bundle E . Every element $F \in \mathcal{O}_{\text{red}}(J(E))$ can be expanded as $F = \sum_n F_n$ where each F_n is an element

$$F_n \in \text{Hom}_{C_X^\infty}(J(E)^{\otimes n}, C_X^\infty)_{S_n} \cong \text{PolyDiff}(\mathcal{E}^{\otimes n}, C^\infty(X))_{S_n}$$

where the right-hand side is the space of polydifferential operators. The proof of the isomorphism on the right-hand side can be found in Chapter 5 of [?].

A local functional is given by a Lagrangian densities modulo total derivatives. The mathematical definition is the following.

Definition 6.5. Let E be a graded vector bundle on X . Define the sheaf of *local functionals* on X to be

$$\mathcal{O}_{\text{loc}}(\mathcal{E}) = \text{Dens}_X \otimes_{D_X} \mathcal{O}_{\text{red}}(J(E)),$$

where we use the natural right D_X -module structure on densities.

Note that we always consider local functionals coming from Lagrangians modulo constants. We will not be concerned with local functions associated to constant Lagrangians.

From the expression for functionals in Lemma ?? we see that integration defines an inclusion of sheaves

$$i : \mathcal{O}_{\text{loc}}(\mathcal{E}) \hookrightarrow \mathcal{O}_{\text{red}}(\mathcal{E}_c). \quad (5)$$

Often times when we describe a local functional we will write down its value on test compactly supported sections, then check that it is given by integrating a Lagrangian density, which amounts to lifting the functional along i .

6.1.3 The definition of a classical field theory

Before giving the definition, we need to recall what the proper notion of a shifted symplectic structure is in the geometric setting that we work in.

Definition 6.6. Let E be a graded vector bundle on X . A k -shifted symplectic structure is an isomorphism of graded vector spaces

$$E \cong_{\omega} E^{\dagger}[k] = (\text{Dens}_X \otimes E^{\vee})[k]$$

that is graded anti-symmetric.

If ω^* is the formal adjoint of the isomorphism $\omega^* : E \cong E^{\dagger}[k]$, anti-symmetry amounts to the condition $\omega^* = -\omega$. In general, ω does *not* induce a Poisson structure on the space of all functionals $\mathcal{O}(\mathcal{E})$. This is because, as we have seen above, elements of this space are given by distributional sections and hence we cannot pair elements with overlapping support. The symplectic structure does, however, induce a Poisson bracket on *local* functionals.¹ We will denote the bracket induced by a shifted symplectic structure by $\{-, -\}$.

We are now ready to give the precise definition of a classical field theory.

Definition 6.7. A *classical field theory* in the BV formalism on a smooth manifold X is a \mathbb{Z} -graded vector bundle E equipped with a (-1) -shifted symplectic structure together with a local functional $S \in \mathcal{O}_{\text{loc}}(\mathcal{E})$ such that:

1. the functional S satisfies the *classical master equation*

$$\{S, S\} = 0;$$

2. S is at least quadratic, so we can write it (in a unique way) as

$$S(\varphi) = \omega(\varphi, Q\varphi) + I(\varphi)$$

where Q is a linear differential operator such that $Q^2 = 0$, and $I \in \mathcal{O}_{\text{loc}}(\mathcal{E})$ is at least cubic;

3. the complex (\mathcal{E}, Q) is elliptic.

In the physics literature, the operator Q is known as the linearized BRST operator, and $\{S, -\} = Q + \{I, -\}$ is the full BRST operator. Ellipticity of the complex (\mathcal{E}, Q) is a technical requirement

¹Note that $\mathcal{O}_{\text{loc}}(\mathcal{E})$ is not a shifted Poisson algebra since there is no natural commutative product.

that will be very important in our approach to the issue of renormalization in perturbative quantum field theory. The classical master equation is equivalent to

$$QI + \frac{1}{2}\{I, I\} = 0.$$

A *free theory* is a classical theory with $I = 0$ in the notation above. Thus, a free theory is simply an elliptic complex equipped with a (-1) -shifted symplectic pairing where the differential in the elliptic complex is graded skew-self adjoint for the pairing.

Although the space $\mathcal{O}(\mathcal{E})$ does not have a well-defined shifted Poisson bracket induced from the symplectic pairing, the operator $\{S, -\} : \mathcal{O}(\mathcal{E}) \rightarrow \mathcal{O}(\mathcal{E})[1]$ is well-defined since S is local by assumption. By assumption, it is also square zero. The complex of global classical observables of the theory is defined by

$$\text{Obs}_{\mathcal{E}}^{\text{cl}}(X) = (\mathcal{O}(\mathcal{E}(X)), \{S, -\}).$$

This complex is the field theoretic replacement for functions on the derived locus of S from the beginning of this section. Although it does not have a P_0 -structure, there is a subspace that does. This is sometimes referred to as the *BRST* complex in the physics literature.

6.1.4 A description using L_{∞} algebras

There is a completely equivalent way to describe a classical field theory that helps to illuminate the mathematical meaningfulness of the definition given above. The requisite concept we need to introduce is that of a *local Lie algebra* (or local L_{∞} algebra).

First, recall that an L_{∞} algebra is a modest generalization of a dg Lie algebra where the Jacobi identity is only required to hold up to homotopy. The data of an L_{∞} algebra is a graded vector space V with, for each $k \geq 1$, a k -ary bracket

$$\ell_k : V^{\otimes k} \rightarrow V[2 - k]$$

of cohomological degree $2 - k$. These maps are required to satisfy a series of conditions, the first of which says $\ell_1^2 = 0$. The next says that ℓ_2 is a bracket satisfying the Jacobi identity up to a homotopy given by ℓ_3 . For a detailed definition see we refer the reader to [?, ?].

We now give the definition of a local L_{∞} algebra on a manifold X . This has appeared in Chapter 4 of [?].

Definition 6.8. A *local L_{∞} algebra* on X is the following data:

- (i) a \mathbb{Z} -graded vector bundle L on X , whose sheaf of smooth sections we denote \mathcal{L}^{sh} , and
- (ii) for each positive integer n , a polydifferential operator in n inputs

$$\ell_n : \underbrace{\mathcal{L} \times \cdots \times \mathcal{L}}_{n \text{ times}} \rightarrow \mathcal{L}[2 - n]$$

such that the collection $\{\ell_n\}_{n \in \mathbb{N}}$ satisfy the conditions of an L_{∞} algebra. In particular, \mathcal{L} is a sheaf of L_{∞} algebras.

The simplest example of a local Lie algebra starts with the data of an ordinary Lie algebra \mathfrak{g} . We can then take the constant bundle $\underline{\mathfrak{g}}_X$ with fiber \mathfrak{g} . The Lie bracket on \mathfrak{g} extends to define the structure of a local Lie algebra. In this case, the sheaf of Lie algebras is $C_X^\infty \otimes \mathfrak{g}$. Another important example of a local Lie algebra is given by the Lie algebra of vector fields $\text{Vect}(X)$ on a smooth manifold. The Lie bracket of vector fields is a bidifferential operator on the tangent bundle and this equips the sheaf of sections with the structure of a sheaf of Lie algebras. We will study the holomorphic version of this local Lie algebra in Chapter ??.

Just as in the case of an ordinary graded vector bundle, we can discuss local functionals on a local Lie algebra L . In this case, the L_∞ structure maps give this the structure of a sheaf of complexes, providing a local version of the Chevalley-Eilenberg cochain complex. Indeed, the ∞ -jet bundle JL is an L_∞ algebra object in D_X -modules and so we can define the D_X -module of reduced Chevalley-Eilenberg cochains $C_{\text{Lie,red}}^*(JL)$. Mimicking the definition above, we arrive at the following local version of Lie algebra cohomology that will come up again and again in this thesis.

Definition 6.9. Let L be a local Lie algebra. The local Chevalley-Eilenberg cochain complex is the sheaf of cochain complexes

$$C_{\text{loc}}^*(\mathcal{L}) = \text{Dens}_X \otimes_{D_X} C_{\text{Lie,red}}^*(L).$$

We denote the global sections by $C_{\text{loc}}^*(\mathcal{L}(X))$.

The *local cohomology* of a local Lie algebra is the cohomology of the local CE complex, which we will denote $H_{\text{loc}}^*(\mathcal{L}(X))$.

Remark 6.10. We have already remarked that for a graded vector bundle E there is an embedding $\mathcal{O}_{\text{loc}}(\mathcal{E}) \hookrightarrow \mathcal{O}_{\text{red}}(\mathcal{E})$. This translates to an embedding of sheaves of cochain complexes $C_{\text{loc}}^*(\mathcal{L}) \hookrightarrow C_{\text{Lie,red}}^*(\mathcal{L}_c)$ for any local Lie algebra \mathcal{L} . In the case of vector fields, there is a related cochain complex that has been studied extensively in the context of characteristic classes of foliations [?, ?, ?, ?]. Suppose, for simplicity, that X is a compact smooth manifold. If $\text{Vect}(X)$ is the Lie algebra of vector fields on X then the (reduced) *diagonal cochain complex* is the subcomplex

$$C_{\Delta,\text{red}}^*(\text{Vect}(X)) \subset C_{\text{Lie,red}}^*(\text{Vect}(X))$$

consisting of cochains $\varphi : \text{Vect}(X)^{\otimes k} \rightarrow \mathbb{C}$ satisfying $\varphi(X_1, \dots, X_k) = 0$ if $\bigcap_{i=1}^k \text{Supp}(X_i) = \emptyset$. That is, the cocycle vanishes unless all of the supports of the inputs overlap nontrivially. The inclusion of the local cochain complex $C_{\text{loc}}^*(\text{Vect}(X)) \subset C_{\text{Lie,red}}^*(\text{Vect}(X))$ factors through this subcomplex to give a sequence of inclusions

$$C_{\text{loc}}^*(\text{Vect}(X)) \hookrightarrow C_{\Delta,\text{red}}^*(\text{Vect}(X)) \hookrightarrow C_{\text{Lie,red}}^*(\text{Vect}(X)).$$

This is because the cochain of $\text{Vect}(X)$ defined from a local cochain involves the integral of local operators applied to the inputs.

It turns out that the definition of a classical field theory can be repackaged in terms of certain structures on a local L_∞ algebra. The first piece of data we need to transport to the L_∞ side is that of a symplectic pairing. The underlying data of a local L_∞ algebra L is a graded vector bundle.

In Definition ?? we have already defined a k -shifted symplectic pairing. On the local Lie algebra sign, we ask for $k = -3$ shifted symplectic structures that are also invariant for the L_∞ structure maps.

Also, an important part of a classical field theory is ellipticity. We say a local L_∞ algebra is *elliptic* if the complex $(\mathcal{L}, d = \ell_1)$ is an elliptic complex.

Proposition 6.11. *The following structures are equivalent:*

1. *a classical field theory in the BV formalism (\mathcal{E}, ω, S) ;*
2. *an elliptic local Lie algebra structure on $L = E[1]$ equipped with a (-3) -shifted symplectic pairing.*

Proof. (Sketch) The underlying graded vector bundle of the space of fields \mathcal{E} is E and we obtain the bundle underlying the local L_∞ algebra by shifting this down $L = E[1]$. The (-1) -shifted symplectic structure on E transports to a (-3) -shifted on L . The L_∞ structure maps for L come from the Taylor components of the action functional S . The exterior derivative of S is a section

$$dS \in C_{\text{loc}}^*(\mathcal{L}, \mathcal{L}^![-1]),$$

where on the right-hand side we have zero differential. The Taylor components are of the form $(dS)_n : \mathcal{L}^{\otimes n} \rightarrow \mathcal{L}^![-1]$. Using the shifted symplectic pairing we can identify these Taylor components with maps $(dS)_n : \mathcal{L}^{\otimes n} \rightarrow \mathcal{L}[2]$. Thus, dS can be viewed as a section of $C_{\text{loc}}^*(\mathcal{L}, \mathcal{L}[2])$. This is precisely the space controlling deformations of \mathcal{L} as a local Lie algebra. One checks immediately that the classical master equation is equivalent to the fact that dS is a derivation, hence it determines the structure of a local Lie algebra. The first Taylor component ℓ_1 is precisely the operator Q before, so ellipticity of (\mathcal{E}, Q) is equivalent to ellipticity of (\mathcal{L}, ℓ_1) . \square

6.1.5 Deformation theory

There is a deep relationship to the way in which we formulate perturbative classical field theories and deformation theory. We diverge a bit from our discussion of field theory to explain this.

Theorem 6.12. *There is an equivalence (of ∞ -categories) between the category of formal (pointed) moduli problems and the category of dg Lie algebras.*

Remark 6.13. Every L_∞ algebra is equivalent (in a functorial way) to a dg Lie algebra. Often times this theorem is formulated in terms of L_∞ algebras.

The ideas of deformation theory leading up to this theorem have been developed by many influential mathematicians in the past forty years including Quillen, Deligne, Kontsevich, Drinfeld, and Feigin. The modern formulation of the theorem above and its proof is due to Lurie [?].

By definition, a *formal moduli problem* over \mathbb{C} (valued in simplicial sets) is a functor (of simplicially enriched categories) of the form

$$F : \text{dgArt}_{\mathbb{C}} \rightarrow \text{sSets}$$

where $\text{dgArt}_{\mathbb{C}}$ is the category of dg Artin algebras. These are finite dimensional \mathbb{C} -linear dg algebras A concentrated in cohomological degrees ≤ 0 such that $H^0 A$ is Artinian in the usual sense. One requires that F satisfies the following conditions:

1. $F(\mathbb{C}) \simeq *$ (F is pointed);
2. F takes surjective maps of dg Artin algebras to fibrations of simplicial sets;
3. if $A \rightarrow B, C \rightarrow B$ are maps of dg Artin algebras such that one of them is surjective on H^0 , then F preserves the fiber product: $F(A \times_B C) \xrightarrow{\sim} F(A) \times_{F(B)} F(C)$.

The most important way in which formal moduli problems arise is through the *Maurer-Cartan* equation. Suppose \mathfrak{g} is an L_∞ algebra (over \mathbb{C}) and (R, \mathfrak{m}) a dg Artin algebra. Define the simplicial set

$$\mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}) \in \mathbf{sSet}$$

which assigns to an n -simplex the set of Maurer-Cartan elements $\alpha \in \mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n)$. It is a classical result that $\mathrm{MC}(\mathfrak{g}) : (A, \mathfrak{m}) \mapsto \mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m})$ defines a formal moduli problem, see [?]. In fact, the functor $\mathfrak{g} \mapsto \mathrm{MC}(\mathfrak{g})$ is the one which provides the equivalence of the main theorem of deformation theory above. Sometimes, we will use the notation $B\mathfrak{g}$ to denote this formal moduli problem associated to \mathfrak{g} . Since a Maurer-Cartan element of $\mathfrak{g} \otimes \mathfrak{m}$ is equivalent to a map of commutative dg algebras $C_{\mathrm{Lie}}^*(\mathfrak{g}) \rightarrow A$ preserving the maximal ideals, we see that $C_{\mathrm{Lie}}^*(\mathfrak{g})$ is precisely the space of functions on $B\mathfrak{g}$.

Remark 6.14. There are many natural instances of formal moduli problems. For instance, if X is any derived scheme then and $x \in X$ is a point, then the formal neighborhood X_x^\wedge is a formal moduli problem.

Local Lie algebras, as we've introduced above, provide a natural "sheafy" version of dg Lie, or L_∞ , algebras over smooth manifolds. If \mathcal{L} is a local Lie algebra on a manifold X , then for each open $U \subset X$ one can define the Maurer-Cartan space $B(\mathcal{L}(U))(A) = \mathrm{MC}(\mathcal{L}(U) \otimes \mathfrak{m})$ just as above. This defines a (homotopy) sheaf of formal moduli problems on X . Motivated by the ordinary case we can make the following definition.

Definition 6.15. An *elliptic moduli problem* is a sheaf of formal moduli problems represented by an elliptic L_∞ algebra \mathcal{L} .

According the previous section, we see that a classical theory in the BV formalism is simply a elliptic moduli problem that is equipped with the data of a shifted symplectic pairing.

6.2 Quantum field theory

We now recall the notion of a *quantum field theory* in the BV formalism. We follow the effective approach defined by Costello in [?].

6.2.1 Regularization

We have seen that part of the data of a classical field theory is that of a (-1) -shifted symplectic structure on the space of fields. If E is the graded vector bundle underlying the theory, the symplectic form determined an isomorphism of bundles $E \cong E^![-1]$. We can represent the inclusion

$\mathcal{E}_c \hookrightarrow \bar{\mathcal{E}}$ via its integral kernel $K_0 \in \bar{\mathcal{E}} \otimes \bar{\mathcal{E}}^!$. Using the symplectic pairing this is further identified with an element

$$K_0 \in \bar{\mathcal{E}} \otimes \bar{\mathcal{E}}[-1].$$

That is, K_0 is a degree one element in $\bar{\mathcal{E}} \otimes \bar{\mathcal{E}}$. The naïve BV Laplacian $\Delta = \Delta_{K_0}$ is ill-defined acting on functions on \mathcal{E} , $\mathcal{O}(\mathcal{E})$. The point of regularization is to find a replacement for this operator.

The first step in regularization is to find a replacement of K_0 as a smooth, i.e. non-distributional, section in the tensor product $\mathcal{E} \otimes \mathcal{E}$. This is a reasonable thing to ask, since by ellipticity we know that the inclusion $\mathcal{E} \otimes \mathcal{E} \hookrightarrow \bar{\mathcal{E}} \otimes \bar{\mathcal{E}}$ is a quasi-isomorphism. So, we can replace K_0 by such a smooth section up to homotopy. We refer to this as a *regularization* of the kernel.

We use a systematic way of regularization using heat kernels, which can be found in [?] or Chapter 8 of [?]. First, we fix the following data, that of a *gauge fixing* operator. This is an operator

$$Q^{GF} : \mathcal{E} \rightarrow \mathcal{E}[-1]$$

of cohomological degree -1 . We require that $D = [Q, Q^{GF}]$ is a generalized Laplacian acting on sections \mathcal{E} in the sense of [?], in addition to other conditions that can be found in Definition 5.4.0.5 in [?].

The utility of introducing the gauge fixing operator is that it allows us to introduce the operator e^{-tD} which has a kernel that we denote $K_t \in \bar{\mathcal{E}} \otimes \bar{\mathcal{E}}$ for any $t \geq 0$. This kernel satisfies the usual conditions of a heat kernel:

1. K_t satisfies the heat equation

$$\frac{\partial}{\partial t} K_t + D K_t = 0$$

2. K_0 is the kernel for the identity operator as above.

Moreover, when $t > 0$ the operator e^{-tD} is *smoothing* so that $K_t \in \mathcal{E} \otimes \mathcal{E} \subset \bar{\mathcal{E}} \otimes \bar{\mathcal{E}}$.

The point of introducing this heat kernel is that it provides a regularization of K_0 . Indeed, for any $\epsilon, L \geq 0$ introduce the *propagator*

$$P_{\epsilon < L} = \int_{t=\epsilon}^L (Q^{GF} \otimes 1) K_t dt \in \bar{\mathcal{E}} \otimes \bar{\mathcal{E}}.$$

Then, one immediately checks that

$$K_L - K_\epsilon = Q P_{\epsilon < L},$$

so that $P_{\epsilon < L}$ is a homotopy between K_L and K_ϵ . In particular, $P_{0 < L}$ provides a homotopy between the identity kernel K_0 and K_L .

Definition 6.16. The scale $L > 0$ BV Laplacian is the order two operator

$$\Delta_L = \partial_{K_L} : \mathcal{O}(\mathcal{E}) \rightarrow \mathcal{O}(\mathcal{E})$$

given by contraction with the kernel $K_L \in \mathcal{E} \otimes \mathcal{E}$.

We have already mentioned that the bracket $\{-, -\}$ is not defined on the whole space of functionals. The regularized BV operator allows us to define the following *scale L bracket*:

$$\{I, J\}_L := \Delta_L(IJ) - \Delta_L(I)J - (-1)^{|I|}I\Delta_L(J).$$

For $L > 0$ this bracket is defined on all of $\mathcal{O}(\mathcal{E})$, just as Δ_L is.

6.2.2 Effective BV quantization

Fix a free BV theory together with a gauge fixing operator. This is the data of an elliptic complex (\mathcal{E}, Q) with a (-1) -shifted symplectic form ω . In addition, let Q^{GF} be a gauge fixing operator so that the regularized heat kernels K_L and propagators $P_{\epsilon < L}$ are defined.

We introduce the formal variable \hbar and consider \hbar -dependent functionals $\mathcal{O}(\mathcal{E})[[\hbar]]$. Let $\mathcal{O}^+(\mathcal{E})[[\hbar]] \subset \mathcal{O}(\mathcal{E})[[\hbar]]$ be the subset of functionals that are at least cubic modulo \hbar . We define a map

$$W(P_{\epsilon < L}, -) : \mathcal{O}^+(\mathcal{E})[[\hbar]] \rightarrow \mathcal{O}^+(\mathcal{E})[[\hbar]],$$

renormalization group flow. Formally, $W(P_{\epsilon < L}, I)$ is defined by the formula

$$e^{W(P_{\epsilon < L}, I)/\hbar} = e^{\hbar \partial_{P_{\epsilon < L}}} e^{I/\hbar}.$$

Concretely, $W(P_{\epsilon < L}, I)$ can be written as a sum over graphs Γ

$$W(P_{\epsilon < L}, I) = \sum_{\Gamma} \frac{\hbar^{g(\Gamma)}}{|\text{Aut}(\Gamma)|} W_{\Gamma}(P_{\epsilon < L}, I),$$

where $W_{\Gamma}(P_{\epsilon < L}, I)$ is the weight of the graph Γ whose edges are labeled by $P_{\epsilon < L}$ and vertices labeled by I . This is our mathematical definition of the Feynman weight of the graph Γ , and the precise definition can be found in Chapter 2 of [?].

In the BV formalism, as developed in [?, ?, ?], one has the following definition of a quantum field theory.

Definition 6.17. A *quantum field theory* in the BV formalism consists of a free BV theory (\mathcal{E}, Q, ω) and an effective family of functionals

$$\{I[L]\}_{L \in (0, \infty)} \subset \mathcal{O}_{P, sm}^+(\mathcal{E})[[\hbar]]$$

that satisfy:

- (a) the exact renormalization group (RG) flow equation

$$I[L'] = W(P_{L < L'}, I[L]);$$

- (b) the scale L quantum master equation (QME) at every length scale L :

$$(Q + \hbar \Delta_L) e^{I[L]/\hbar} = 0.$$

Equivalently,

$$QI[L] + \hbar \Delta_L I[L] + \frac{1}{2} \{I[L], I[L]\}_L = 0;$$

(c) as $L \rightarrow 0$, the functional $S[L]$ has an asymptotic expansion that is local.

The subspace $\mathcal{O}_{P,sm}^+(\mathcal{E})[[\hbar]] \subset \mathcal{O}(\mathcal{E})[[\hbar]]$ is the one of smooth and proper functionals that are at least cubic modulo \hbar . Smooth and properness is a technical condition that we will not delve into in this section, but refer to the original reference of Costello-Gwilliam.

Condition (a) ensures that the scale L action functional $S[L]$ determines the functional at every other scale. Condition (b) can be interpreted as saying that we have a proper path integral measure at scale L (i.e., the QME can be seen as a definition of the measure). Finally, condition (c) implies that the effective action is a quantization of a classical field theory, since a defining property of a classical theory is that its action functional is local. (A full definition is available in Section 8.2 of [?].)

Remark 6.18. The length scale is often associated with a choice of Riemannian metric on the underlying manifold, but the formalism of [?] keeps track of how the space of quantum BV theories depends upon such a choice (and other choices that might go into issues like renormalization). Hence, when the choices should not be essential — such as with a topological field theory — one can typically show rigorously that different choices give equivalent answers. The length scale is also connected with the use of heat kernels in [?], but one can work with more general parametrices (and hence more general notions of “scale”), as explained in Chapter 8 of [?]. We use a natural length scale in this section; when it becomes relevant, in the context of factorization algebras, one must switch to general parametrices.

The locality condition ensures that the limit $I^{cl} = \lim_{L \rightarrow 0} I[L] \bmod \hbar$ exists and is a local functional. The QME modulo \hbar implies that I^{cl} satisfying the CME, so that $(\mathcal{E}, Q, \omega, I^{cl})$ defines a classical theory in the BV formalism.

6.2.3 Deformation theory for quantizations

There is a well-established deformation theory for studying quantizations of a fixed classical field theory. If $(\mathcal{E}, Q, \omega, I)$ is a fixed classical theory, one would like to study the problem of finding quantizations which modulo \hbar are equal to this classical theory.

According to the definition of a QFT there are two main steps.

1. Find an effective family $\{I[L]\}$ which, modulo \hbar , agrees with the classical theory I , and satisfies the RG flow equation. The main result of [?] is that this step always has a solution. Naively, the proposed family is of the form $I[L] = W(P_{0<L}, I)$, but since $P_{0<L}$ is distributional this functional may not be well-defined. (This is the problem of UV divergence in QFT) The key fact is that there exists a family of counterterms $I^{CT}(\epsilon) \in \mathcal{O}(\mathcal{E})[[\hbar]]$ such that the limit

$$I[L] = \lim_{\epsilon \rightarrow 0} W(P_{\epsilon < L}, I - I^{CT}(\epsilon))$$

does exist. Moreover, it automatically satisfies the RG flow equation.

2. Once we have the effective family $\{I[L]\}$, the remaining condition to defining a QFT is the quantum master equation. In general this equation is not satisfied, and there may in fact be

obstructions to having a solution.

For holomorphic theories we will study both problems above. We will show that the analysis involved in renormalization for holomorphic theories is extremely well-behaved. In fact, the counterterms for holomorphic theories on \mathbb{C}^d can be taken to be zero. Using this, we will show how solving the QME for holomorphic theories can be done a systematic way.

To study the problem of solving the quantum master equation in general, we work order by order in the formal parameter \hbar . Suppose that $I[L]$ is defined modulo \hbar^{n+2} and satisfies the QME modulo \hbar^{n+1} . The obstruction to satisfying the QME at scale L modulo \hbar^{n+2} is the functional

$$\Theta_{n+1}[L] = \hbar^{-n-1}(QI[L] + \frac{1}{2}\{I[L], I[L]\}_L + \hbar\Delta_L).$$

The obstruction $\Theta[L]$ satisfies the classical master equation and hence the limit $\Theta_{n+1} = \lim_{L \rightarrow 0} \Theta[L]$ is a local functional and is closed for the differential $Q + \{I, -\}$. It is thus a closed element of degree one of the *deformation complex*

$$\text{Def}_{\mathcal{E}} = (\mathcal{O}_{\text{loc}}(\mathcal{E}), Q + \{I, -\}).$$

If Θ_{n+1} is cohomologically trivial in $H^1(\text{Def}_{\mathcal{E}})$, the space of possible lifts of $\{I[L]\}$ to a solution of the QME modulo \hbar^{n+2} is a torsor for $H^0(\text{Def}_{\mathcal{E}})$.

7 Holomorphic field theories

The goal of this section is to define the notion of a holomorphic field theory. This is a variant of Costello's definition of a BV theory, see the previous section, and we will take for granted that the reader is familiar with the general format. In summary, we modify the definition of a theory by inserting the word "holomorphic" in front of most objects (bundles, differential operators, etc.). By applying the Dolbeault complex in appropriate locations, we will recover Costello's definition of a theory, but with a holomorphic flavor, see Table ??.

There are many references in the physics literature to codify the concept of a holomorphic field theory. See, most closely related to our approach, special cases of this in the work of Nekrasov and collaborators in [?, ?, ?]. We will discuss in more detail the relationship of our analysis of holomorphic theories to this work in Chapters ?? and ??.

7.1 The definition of a holomorphic theory

We give a general definition of a classical holomorphic theory on a general complex manifold X of complex dimension d . We start with the definition of a *free* holomorphic field theory. After that we will go on to define what an interacting holomorphic theory is.

7.1.1 Free holomorphic theories

The fields of any theory are always expressed as sections of some \mathbb{Z} -graded vector bundle. Here, the \mathbb{Z} -grading is the cohomological, or BRST, grading of the theory. For a holomorphic theory we take this graded vector bundle to be holomorphic. By a *holomorphic* \mathbb{Z} -graded vector bundle we mean a \mathbb{Z} -graded vector bundle $V = \oplus_i V^i$ such that each graded piece V^i is a holomorphic vector bundle. Thus, the data we start with is the following:

- (1) a \mathbb{Z} -graded holomorphic vector bundle $V^* = \oplus_i V^i[-i]$, so that the finite dimensional holomorphic vector bundle V^i is in cohomological degree i .

A free classical theory is made up of a space of fields as above together with the data of a linearized BRST differential Q^{BRST} and a symplectic pairing. Ordinarily, the BRST operator is a differential operator on the vector bundle defining the fields. For the class of theories we are considering, we want this operator to be holomorphic.

If E and F are two holomorphic vector bundles on X , we can speak of holomorphic differential operators between E and F . First, note that the Hom-bundle $\text{Hom}(E, F)$ inherits a natural holomorphic structure. By definition, a holomorphic differential operator of order m is a linear map

$$D : \Gamma^{hol}(X; E) \rightarrow \Gamma^{hol}(X; F)$$

such that, with respect to a holomorphic coordinate chart $\{z_i\}$ on X , D can be written as

$$D|_{\{z_i\}} = \sum_{|I| \leq m} a_I(z) \frac{\partial^{|I|}}{\partial z_I} \quad (6)$$

where $a_I(z)$ is a local holomorphic section of $\text{Hom}(E, F)$. Here, the sum is over all multi-indices $I = (i_1, \dots, i_d)$ and

$$\frac{\partial^{|I|}}{\partial z_I} := \prod_{k=1}^d \frac{\partial^{i_k}}{\partial z_k^{i_k}}.$$

The length is defined by $|I| := i_1 + \dots + i_d$.

Example 7.1. The most basic example of a holomorphic differential operator is the ∂ operator for the trivial vector bundle. For each $1 \leq \ell \leq d = \dim_{\mathbb{C}}(X)$, it is a holomorphic differential operator from $E = \wedge^{\ell} T^{1,0*} X$ to $F = \wedge^{\ell+1} T^{1,0*} X$ which on sections is

$$\partial : \Omega^{\ell, hol}(X) \rightarrow \Omega^{\ell+1, hol}(X).$$

Locally, of course, it has the form

$$\partial = \sum_{i=1}^d (dz_i \wedge (-)) \frac{\partial}{\partial z_i},$$

where $dz_i \wedge (-)$ is the vector bundle homomorphism $\wedge^{\ell} T^{1,0*} X \rightarrow \wedge^{\ell+1} T^{1,0*} X$ sending $\alpha \mapsto dz_i \wedge \alpha$.

The next piece of data we fix is:

(2) a square-zero holomorphic differential operator

$$Q^{hol} : \mathcal{V}^{hol} \rightarrow \mathcal{V}^{hol}$$

of cohomological degree +1. Here \mathcal{V}^{hol} denotes the holomorphic sections of V .

Finally, to define a free theory we need the data of a symplectic pairing. For reasons to become clear in a moment, we must choose this pairing to have a strange cohomological degree. The last piece of data we fix is:

(3) an invertible bundle map

$$(-, -)_V : V \otimes V \rightarrow K_X[d-1]$$

Here, K_X is the canonical bundle on X .

The definition of the fields of an ordinary field theory are the *smooth* sections of the vector bundle V . In our situation this is a silly thing to do since we lose all of the data of the complex structure we used to define the objects above. The more natural thing to do is to take the *holomorphic* sections of the vector bundle V . By construction, the operator Q^{hol} and the pairing $(-, -)_V$ are defined on holomorphic sections, so on the surface this seems reasonable. We will take a natural resolution of holomorphic sections in order to relate to the usual definition of a classical BV theory.

Given any holomorphic vector bundle V we can define its *Dolbeault complex* $\Omega^{0,*}(X, V)$ with its Dolbeault operator

$$\bar{\partial} : \Omega^{0,p}(X, V) \rightarrow \Omega^{0,p+1}(X, V).$$

Here, $\Omega^{0,p}(X, V)$ denotes smooth sections of the vector bundle $\wedge^p T^{0,1*}X \otimes V$. The fundamental property of the Dolbeault complex is that it provides a resolution for the sheaf of holomorphic sections $\mathcal{V}^{hol} \simeq \Omega_X^{0,*}(V)$.

We now take a graded holomorphic vector bundle V as above, equipped with the differential operator Q^{hol} . We can then define the totalization of the Dolbeault complex with the operator Q^{hol} :

$$\mathcal{E}_V = \left(\Omega^{0,*}(X, E), \bar{\partial} + Q^{hol} \right).$$

The operator $\bar{\partial} + Q^{hol}$ will be the linearized BRST operator of our theory. By assumption, we have $\bar{\partial}Q^{hol} = Q^{hol}\bar{\partial}$ so that $(\bar{\partial} + Q^{hol})^2 = 0$ and hence the fields still define a complex. The (-1) -shifted symplectic pairing is obtained by composition of the pairing $(-, -)_V$ with integration on $\Omega_X^{d,hol}$. The thing to observe here is that $(-, -)_V$ extends to the Dolbeault complex in a natural way: we simply combine the wedge product of forms with the pairing on V . The (-1) -shifted pairing ω_V on \mathcal{E} is defined by the diagram

$$\begin{array}{ccc} \mathcal{E}_V \otimes \mathcal{E}_V & \xrightarrow{(-, -)_V} & \Omega^{0,*}(X, K_X)[d-1] \\ & \searrow \omega_V & \downarrow \int_X \\ & & \mathbb{C}[-1]. \end{array}$$

We note that the top Dolbeault forms with values in the canonical bundle K_X are precisely the top forms on the smooth manifold X , so integration makes sense.

We arrive at the following definition.

Definition/Lemma 1. A free holomorphic theory on a complex manifold X is the data $(V, Q^{hol}, (-, -)_V)$ as in (1), (2), (3) above such that Q^{hol} is a square zero elliptic differential operator that is graded skew self-adjoint for the pairing $(-, -)_V$. The triple $(\mathcal{E}_V, Q_V = \bar{\partial} + Q^{hol}, \omega_V)$ defines a free BV theory in the usual sense.

The usual prescription for writing down the associated action functional holds in this case. If $\varphi \in \Omega^{0,*}(X, V)$ denotes a field the action is

$$S(\varphi) = \int_X \left(\varphi, (\bar{\partial} + Q^{hol})\varphi \right)_V.$$

The first example we explain is related to the subject of Chapter ?? and will serve as the fundamental example of a holomorphic theory.

Example 7.2. The free $\beta\gamma$ system. Suppose that

$$V = \underline{\mathbb{C}} \oplus K_X[d-1].$$

Let $(-, -)_V$ be the pairing

$$(\underline{\mathbb{C}} \oplus K_X) \otimes (\underline{\mathbb{C}} \oplus K_X) \rightarrow K_X \oplus K_X \rightarrow K_X$$

sending $(\lambda, \mu) \otimes (\lambda', \mu') \mapsto (\lambda\mu', \lambda'\mu) \mapsto \lambda\mu' + \lambda'\mu$. In this example we set $Q^{hol} = 0$. One immediately checks that this is a holomorphic free theory as above. The space of fields can be written as

$$\mathcal{E}_V = \Omega^{0,*}(X) \oplus \Omega^{d,*}(X)[d-1].$$

We write $\gamma \in \Omega^{0,*}(X)$ for a field in the first component, and $\beta \in \Omega^{d,*}(X)[d-1]$ for a field in the second component. The action functional reads

$$S(\gamma + \beta, \gamma' + \beta') = \int_X \beta \wedge \bar{\partial}\gamma' + \beta' \wedge \bar{\partial}\gamma.$$

When $d = 1$ this reduces to the ordinary chiral $\beta\gamma$ system from conformal field theory. The $\beta\gamma$ system is a bosonic version of the ghost bc system that appears in the quantization of the bosonic string, see Chapter 6 of [?]. We will study this higher dimensional version further in Chapter ?. For instance, we will see how this theory is the starting block for constructing general holomorphic σ -models.

Of course, there are many variants of the $\beta\gamma$ system that we can consider. For instance, if E is any holomorphic vector bundle on X we can take

$$V = E \oplus K_{\mathbb{C}^d} \otimes E^\vee$$

where E^\vee is the linear dual bundle. The pairing is constructed as in the case above where we also use the evaluation pairing between E and E^\vee . In this case, the fields are $\gamma \in \Omega^{0,*}(X, E)$ and $\beta \in \Omega^{d,*}(X, E^\vee)[d-1]$. The action functional is simply

$$S(\gamma, \beta) = \int \text{ev}_E(\beta \wedge \bar{\partial}\gamma).$$

When E is a tensor bundle of type (r, s) this theory is a bosonic version of the bc ghost system of spin (r, s) . For a general bundle E we will refer to it as the $\beta\gamma$ system with coefficients in the bundle E .

Remark 7.3. We will only work with a holomorphic theory prescribed by the data $(V, (-, -)_V, Q^{hol})$ through its associated BV theory. One might propose a definition of a BV theory in the analytic category based off of holomorphic sections of holomorphic vector bundles. There are numerous technical reason why this approach fails in our approach to QFT. In particular, the sheaf of holomorphic sections of a holomorphic bundle is not fine, and there do not exists partitions of unity in general.

7.1.2 Interacting holomorphic theories

We now define what an interacting holomorphic theory is. In general, an interacting field theory on a manifold M is prescribed by the data of a free theory plus a local functional $I \in \mathcal{O}_{\text{loc}}(\mathcal{E})$ that satisfies the classical master equation. Recall, the sheaf of local functionals on $\mathcal{E} = \Gamma(E)$ is defined as the sheaf of Lagrangian densities

$$\mathcal{O}_{\text{loc}}(\mathcal{E}) = \text{Dens}_M \otimes_{D_M} \mathcal{O}_{\text{red}}(JE).$$

In the expression above JE stands for the sheaf of smooth sections of the ∞ -jet bundle $\text{Jet}(E)$ which has the structure of a D_X -module.

If V is a holomorphic vector bundle we can define the bundle of holomorphic ∞ -jets $\text{Jet}^{hol}(V)$, $[?, ?]$. This is a pro-vector bundle that is holomorphic in a natural way. The fibers of this infinite rank bundle $\text{Jet}^{hol}(V)$ are isomorphic to

$$\text{Jet}^{hol}(V)|_w = V_w \otimes \mathbb{C}[[z_1, \dots, z_d]]$$

where $w \in X$ and where $\{z_i\}$ is the choice of a formal coordinate near w . We denote by $J^{hol}V$ denote the sheaf of holomorphic sections of this jet bundle. The sheaf $J^{hol}V$ has the structure of a D_X^{hol} -module, that is, it is equipped with a holomorphic flat connection ∇^{hol} . This is completely analogous to the smooth case. Locally, the holomorphic flat connection is of the form

$$\nabla^{hol}|_w = \sum_{i=1}^d dw_i \left(\frac{\partial}{\partial w_i} - \frac{\partial}{\partial z_i} \right),$$

where $\{w_i\}$ is the local coordinate on X near w and z_i is the fiber coordinate labeling the holomorphic jet expansion. Using holomorphic jets we can make a completely analogous definition in our setting.

Differential operators between holomorphic bundles are the same as bundle maps between the associated jet bundles. Suppose V, W are holomorphic vector bundles with spaces of holomorphic sections given by $\mathcal{V}^{hol}, \mathcal{W}^{hol}$ respectively. Then we can express polydifferential operators from V to W as

$$\text{PolyDiff}^{hol}(\mathcal{V}^{hol} \times \dots \times \mathcal{V}^{hol}, \mathcal{W}^{hol}) \cong \text{Hom}(\text{Jet}^{hol}(V) \otimes \dots \otimes \text{Jet}^{hol}(V), W).$$

Definition 7.4. Let V be a vector bundle. The space of *holomorphic Lagrangian densities* on V is

$$\mathcal{O}_{red}^{hol}(V) = \prod_{n \geq 0} \text{Hom}(\text{Jet}^{hol}(V)^{\otimes n}, K_X)_{S_n},$$

where Hom is taken in the category of holomorphic vector bundles.² Equivalently, a holomorphic Lagrangian density is of the form $F = \sum_n F_n \in \mathcal{O}_{red}^{hol}(V)$ where each F_n is a holomorphic polydifferential operator

$$F_n : \mathcal{V}^{hol} \times \dots \times \mathcal{V}^{hol} \rightarrow \Omega_X^{d, hol}.$$

Suppose that V is part of the data of a free holomorphic theory $(V, Q^{hol}, (-, -)_V)$. The pairing $(-, -)_V$ endows the space of holomorphic Lagrangians with a sort of bracket. Indeed, suppose $F, F' \in \mathcal{O}_{red}^{hol}(V)$. For simplicity suppose F, F' are of homogenous symmetric degree k, k' respectively. Then, their product $F \otimes F'$ is an element in the homomorphism space

$$\text{Hom}(\text{Jet}^{hol}(V)^{\otimes(k+k')}, K_X \otimes K_X).$$

Now, the bundle map $(-, -)_V : V \otimes V \rightarrow K_X[d-1]$ is invertible, hence it determines an element $(-, -)_V^{-1} \in V \otimes V \otimes K_X^*$, where K_X^* is the dual bundle. We can then consider the composition

$$\text{Hom}(\text{Jet}^{hol}(V)^{\otimes n}, K_X \otimes K_X) \xrightarrow{(-, -)_V^{-1}} \text{Hom}(\text{Jet}^{hol}(V)^{\otimes(k+k'-2)}, K_X^* \otimes K_X \otimes K_X) \longrightarrow \text{Hom}(\text{Jet}^{hol}(V)^{\otimes(k+k'-2)}, K_X).$$

In the first arrow we have evaluated $(-, -)_V^{-1}$ on the first two factors and the second arrow is simply the evaluation pairing. We symmetrize this to obtain an element $\{F, F'\}^{hol} \in \text{Sym}^{k+k'-2}(\text{Jet}^{hol}(V)^{\otimes n}, K_X)_{S_{k+k'-2}}$. In this way, we have produced a map

$$\{-, -\}^{hol} : \mathcal{O}_{red}^{hol}(V) \times \mathcal{O}_{red}^{hol}(V) \rightarrow \mathcal{O}_{red}^{hol}(V)[d-1].$$

Note that this bracket is of cohomological degree $-d+1$.

We can now state the definition of a classical holomorphic theory.

Definition 7.5. A *classical holomorphic theory* on a complex manifold X is the data of a free holomorphic theory $(V, Q^{hol}, (-, -)_V)$ plus a functional

$$I^{hol} \in \prod_{n \geq 3} \text{Hom}(\text{Jet}^{hol}(V)^{\otimes n}, K_X)_{S_n} \subset \mathcal{O}_{red}^{hol}(V)$$

of cohomological degree d such that $Q^{hol} I^{hol} + \frac{1}{2} \{I^{hol}, I^{hol}\}^{hol} = 0$.

Just as in the free case, we see that classical holomorphic theories define ordinary classical BV theories with interactions. The underlying space of fields, as we have already seen is $\mathcal{E}_V = \Omega^{0,*}(X, V)$. We will write $I^{hol} = \sum_k I_k^{hol}$ where I_k^{hol} is symmetric degree k . Now, we know that I_k^{hol} is a $\Omega_X^{d, hol}$ -valued functional of the form

$$I_k^{hol} : (\varphi_1, \dots, \varphi_k) \mapsto D_1(\varphi_1) \cdots D_k(\varphi_k) \in \Omega_X^{d, hol}$$

²The holomorphic vector bundle $\text{Jet}^{hol}(V)$ is infinite dimensional and can be expressed as a pro-object in the category of holomorphic vector bundles. We require the bundle maps to be continuous with respect to the natural adic topology.

Holomorphic theory	BV theory
Holomorphic bundle V	Space of fields $\mathcal{E}_V = \Omega^{0,*}(X, V)$
Holomorphic differential operator Q^{hol}	Linear BRST operator $\bar{\partial} + Q^{hol}$
Non-degenerate pairing $(-, -)_V$	(-1) -symplectic structure ω_V
Holomorphic Lagrangian I^{hol}	Local functional $I^{\Omega^{0,*}} \in \mathcal{O}_{loc}(\mathcal{E}_V)$

Table 1: From holomorphic to BV

where $\varphi_i \in \mathcal{V} = \Gamma^{hol}(X, V)$ and D_i is a holomorphic differential operator on \mathcal{V} . Every holomorphic differential operator on the holomorphic vector bundle V extends to a differential operator on its Dolbeault complex $\mathcal{E}_V = \Omega^{0,*}(X, V)$. Thus, we can define the functional

$$I_k^{\Omega^{0,*}} : (\varphi_1, \dots, \varphi_k) \mapsto \int D_1(\varphi_1) \cdots D_k(\varphi_k)$$

where, now φ_i is a section of the Dolbeault complex $\Omega^{0,*}(X, V)$. The symbol \int reminds us that we are working modulo total derivatives, so that the above expression defines an element of $\mathcal{O}_{loc}(\mathcal{E}_V)$. This defines a linear map $\mathcal{O}_{loc}^{hol}(V) \rightarrow \mathcal{O}_{loc}(\mathcal{E}_V)$ that we denote $I^{hol} \mapsto I^{\Omega^{0,*}}$. Note that since I^{hol} is cohomological degree d , the local functional $I^{\Omega^{0,*}}$ is degree zero.

Lemma 7.6. *Every classical holomorphic theory $(V, Q^{hol}, (-, -)_V, I^{hol})$ determines the structure of a classical BV theory. The underlying free BV theory is given in Definition/Lemma ?? $(\mathcal{E}_V, Q, \omega_V)$ and the interaction is $I^{\Omega^{0,*}}$.*

Proof. We must show that $Q^{hol} I^{hol} + \frac{1}{2} \{I^{hol}, I^{hol}\}^{hol} = 0$ implies the ordinary classical master equation for $I^{\Omega^{0,*}}$:

$$\bar{\partial} I^{\Omega^{0,*}} + Q^{hol} I^{\Omega^{0,*}} + \frac{1}{2} \{I^{\Omega^{0,*}}, I^{\Omega^{0,*}}\} = 0.$$

Since $I^{\Omega^{0,*}}$ is defined using holomorphic differential operators, the first term vanishes. The fact that $Q^{hol} I^{hol} + \frac{1}{2} \{I^{hol}, I^{hol}\}^{hol} = 0$ implies $Q^{hol} I^{\Omega^{0,*}} + \frac{1}{2} \{I^{\Omega^{0,*}}, I^{\Omega^{0,*}}\} = 0$ follows immediately from our definitions. \square

Table ?? is a useful summary showing how we are producing a BV theory from a holomorphic theory.

Example 7.7. Holomorphic BF-theory Let \mathfrak{g} be a Lie algebra and X any complex manifold. Consider the following holomorphic vector bundle on X :

$$V = \underline{\mathfrak{g}}_X[1] \oplus K_X \otimes \mathfrak{g}^*[d-2].$$

The notation $\underline{\mathfrak{g}}_X$ denotes the constant bundle with fiber \mathfrak{g} . The pairing $V \otimes V \rightarrow K_X[d-1]$ is similar to the pairing for the $\beta\gamma$ system, except we use the evaluation pairing $\langle -, - \rangle_{\mathfrak{g}}$ between \mathfrak{g} and its dual. In this example, $Q^{hol} = 0$. Write $f \in \mathcal{O}_X^{hol}$ and $\beta \in K_X$ and consider

$$I^{hol}(f_1 \otimes X_1, f_2 \otimes X_2, \beta \otimes X^\vee) = f_1 f_2 \beta \langle X^\vee, [X_1, X_2] \rangle_{\mathfrak{g}} + \cdots$$

where the \cdots means that we symmetrize the inputs. This defines an element $I^{hol} \in \mathcal{O}_{loc}^{hol}(V)^+$ and the Jacobi identity ensures $\{I^{hol}, I^{hol}\}^{hol} = 0$. The fields of the corresponding BV theory are

$$\mathcal{E}_V = \Omega^{0,*}(X, \mathfrak{g})[1] \oplus \Omega^{d,*}(X, \mathfrak{g}^*)[d-2].$$

The induced local functional $I^{\Omega^{0,*}}$ on \mathcal{E}_V is

$$I^{\Omega^{0,*}}(\alpha, \beta) = \int_X \langle \beta, [\alpha, \alpha] \rangle_{\mathfrak{g}}.$$

The total action is $S(\alpha, \beta) = \int \langle \beta, \bar{\partial}\alpha \rangle + \langle \beta, [\alpha, \alpha] \rangle_{\mathfrak{g}}$. This is formally similar to *BF* theory (see below) and for that reason we refer to it as *holomorphic BF* theory. The moduli problem this describes is the cotangent theory to the moduli space of holomorphic connections on the trivial G -bundle near the trivial bundle. There is an obvious enhancement that works near any holomorphic principal bundle. When $d = 2$, in [?], or for a more mathematical treatment see [?], it is shown that this theory is a twist of $N = 1$ supersymmetric Yang-Mills on \mathbb{R}^4 .

Example 7.8. Topological BF-theory This is a deformation of the previous example that has appeared throughout the physics literature. Suppose we take as our graded holomorphic vector bundle

$$V = \left(\mathfrak{g}_X \otimes \left(\oplus_{k=0}^d \wedge^k T^{*1,0} X [1-k] \right) \right) \oplus \left(\mathfrak{g}_X^* \otimes \left(\oplus_{k=0}^d \wedge^k T^{*1,0} X [2(d-1)-k] \right) \right).$$

Here $\wedge^0 T^{*1,0} X$ is understood as the trivial bundle $\underline{\mathbb{C}}_X$. The pairing is given by combining the evaluation pairing between \mathfrak{g} and \mathfrak{g}^* and taking the wedge product and projecting onto the components isomorphic to K_X . Explicitly, the pairing is equal to the sum of bundle maps of the form

$$\text{ev}_{\mathfrak{g}} \otimes \wedge : \left(\mathfrak{g}_X \otimes \wedge^k T^{*1,0} X [1-k] \right) \otimes \left(\mathfrak{g}_X^* \otimes \wedge^{d-k} T^{*1,0} X [d-1+k] \right) \rightarrow K_X [d-1].$$

The differential is of the form

$$Q^{hol} = \text{id}_{\mathfrak{g}} \otimes \partial + \text{id}_{\mathfrak{g}^*} \otimes \bar{\partial},$$

where ∂ is the holomorphic de Rham differential. The holomorphic interaction is given by combining the Lie algebra structure on \mathfrak{g} with the wedge product of the holomorphic bundles $\wedge^k T^{*1,0} X$. We observe that the associated BV theory has classical space of fields given by

$$(A, B) \in \mathcal{E}_V = \Omega^*(X, \mathfrak{g}[1] \oplus \mathfrak{g}^*[2d-2])$$

where Ω^* is now the *full* de Rham complex. The action functional is

$$S = \int_X \langle B, dA \rangle_{\mathfrak{g}} + \frac{1}{3} \langle B, [A, A] \rangle_{\mathfrak{g}}.$$

As above, $\langle -, - \rangle_{\mathfrak{g}}$ denotes the pairing between \mathfrak{g} and its dual. This is the well-known topological BF theory on the even dimensional *real* manifold X (of real dimension $2d$). It might seem silly that we have used the formalism of holomorphic field theory to describe a very simple topological theory. We will discuss advantages of this approach at the end of the next section. In particular, the theory of regularization for holomorphic theories we will employ has peculiar consequences for renormalizing certain classes of topological theories such as Topological BF theory.

When we construct a BV theory from a holomorphic theory $V \rightsquigarrow \mathcal{E}_V$ it is natural to expect that deformations of the theory must come from holomorphic data. In the special case that $Q^{hol} = 0$ we have the following result which relates the deformation complex of the classical theory \mathcal{E}_V to a sheaf built from holomorphic differential operators.

Lemma 7.9. *Suppose $(V, 0, (-, -)_V, I^{hol})$ is the data of a holomorphic theory with $Q^{hol} = 0$. Let $(\mathcal{E}_V, Q = \bar{\partial}, \omega_V, I)$ be the corresponding BV theory. Then, there is a quasi-isomorphism of sheaves*

$$\text{Def}_{\mathcal{E}_V} \simeq \Omega_X^{d, hol} \otimes_{D_X^{hol}}^{\mathbb{L}} \mathcal{O}_{red}(J^{hol} V)[d]$$

that is compatible with the brackets and $\{-, -\}$ and $\{-, -\}^{hol}$ on both sides.

Proof. This follows from the following observation about D -modules. If M is a holomorphic D_X^{hol} -module, then clearly it forgets down to an ordinary smooth D_X -module that we denote M^{C^∞} . Moreover, there is a quasi-isomorphism of D -modules

$$\Omega_X^{d, d} \otimes_{D_X}^{\mathbb{L}} M^{C^\infty} \simeq \Omega_X^{d, hol} \otimes_{D_X^{hol}} M[d].$$

We apply this to the case $M = \mathcal{O}_{red}(J^{hol} V)$, where V is a holomorphic vector bundle. To complete the proof, we need to show that for any holomorphic vector bundle V that there is a quasi-isomorphism of D_X -modules between $\mathcal{O}_{red}(J\Omega^{0,*}(X, V))$ and $\mathcal{O}_{red}(J^{hol} V)$. For this, it suffices to show that the space of linear functionals are quasi-isomorphic. For any vector bundle E there always exists a (non-canonical) splitting $JE \cong \mathcal{E} \otimes_{C_X^\infty} J_X$, where \mathcal{E} is the sheaf of sections and J_X is the sheaf of ∞ -jets of the trivial bundle. Thus, we can assume that V is the trivial vector bundle, where the claim is now $(J\Omega^{0,*}(X))^\vee \simeq (J_X^{hol})^\vee$. Both sides are quasi-isomorphic to the smooth sections of the bundle of holomorphic differential operators D^{hol} , so we are done. \square

Remark 7.10. Just as in the ordinary case we can formulate the data of a classical holomorphic theory in terms of sheaves of L_∞ algebras. We will not do that here, but hope the idea of how to do so is clear.

7.2 Holomorphically translation invariant theories

When working on affine space \mathbb{R}^n one can ask for a theory to be invariant with respect to translations. In this section, we consider the affine manifold $\mathbb{C}^d = \mathbb{R}^{2d}$ equipped with its standard complex structure and define what a *holomorphically translation invariant* theory is on it. It will be a very special case of a general holomorphic theory as defined above.

Let V be a holomorphic vector bundle on \mathbb{C}^n and suppose we fix an identification of bundles

$$V \cong \mathbb{C}^d \times V_0$$

where V_0 is the fiber of V at $0 \in \mathbb{C}^d$. We want to consider a classical theory with space of fields given by $\Omega^{0,*}(\mathbb{C}^d, V) \cong \Omega^{0,*}(\mathbb{C}^d) \otimes_{\mathbb{C}} V_0$. Moreover, we want this theory to be invariant with respect to the group of translations on \mathbb{C}^d . Per usual, it is best to work with the corresponding Lie algebra of translations. Using the complex structure, we choose a presentation for the complex Lie algebra of translations given by

$$\mathbb{C}^{2d} \cong \text{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right\}_{1 \leq i \leq d}.$$

To define a theory, we need to fix a non-degenerate pairing on V . Moreover, we want this to be translation invariant. So, suppose

$$(-, -)_V : V \otimes V \rightarrow K_{\mathbb{C}^d}[d-1] \quad (7)$$

is a skew-symmetric bundle map that is equivariant for the Lie algebra of translations. The shift is so that the resulting pairing on the Dolbeault complex is of the appropriate degree. Here, equivariance means that for sections v, v' we have

$$\left(\frac{\partial}{\partial z_i} v, v'\right)_V = L_{\partial_{z_i}}(v, v')_V$$

where the right-hand side denotes the Lie derivative applied to $(v, v')_V \in \Omega_{\mathbb{C}^d}^{d, hol}$. There is a similar relation for the anti-holomorphic derivatives. We obtain a \mathbb{C} -valued pairing on $\Omega_c^{0,*}(\mathbb{C}^d, V)$ via integration:

$$\int_{\mathbb{C}^d} \circ (-, -)_V : \Omega_c^{0,*}(\mathbb{C}^d, V) \otimes \Omega_c^{0,*}(\mathbb{C}^d, V) \xrightarrow{\wedge \cdot (-, -)_V} \Omega^{d,*}(\mathbb{C}^d) \xrightarrow{\int} \mathbb{C}. \quad (8)$$

The first arrow is the wedge product of forms combined with the pairing on V . The second arrow is only nonzero on forms of type $\Omega^{d,d}$. Clearly, integration is translation invariant, so that the composition is as well.

The pairing (??) together with the differential $\bar{\partial}$ are enough to define a free theory. However, it is convenient to consider a slightly generalized version of this situation. We want to allow deformations of the differential $\bar{\partial}$ on Dolbeault forms of the form

$$Q = \bar{\partial} + Q^{hol}$$

where Q^{hol} is a holomorphic differential operator of the form

$$Q^{hol} = \sum_I \frac{\partial}{\partial \bar{z}^I} \mu_I \quad (9)$$

where I is some multi-index and $\mu_I : V \rightarrow V$ is a linear map of cohomological degree $+1$. Note that we have automatically written Q^{hol} in a way that it is translation invariant. Of course, for this differential to define a free theory there needs to be some compatibility with the pairing on V .

We can summarize this in the following definition, which should be viewed as a slight modification of a free theory to this translation invariant holomorphic setting.

Definition 7.11. *A holomorphically translation invariant free BV theory is the data of a holomorphic vector bundle V together with*

1. an identification $V \cong \mathbb{C}^d \times V_0$;
2. a translation invariant skew-symmetric pairing $(-, -)_V$ as in (??);
3. a holomorphic differential operator Q^{hol} as in (??);

such that the following conditions hold

1. the induced \mathbb{C} -valued pairing $\int \circ (-, -)_V$ is non-degenerate;
2. the operator Q^{hol} satisfies $(\bar{\partial} + Q^{hol})^2 = 0$ and is skew self-adjoint for the pairing:

$$\int (Q^{hol} v, v')_V = \pm \int (v, Q^{hol} v').$$

The first condition is required so that we obtain an actual (-1) -shifted symplectic structure on $\Omega^{0,*}(\mathbb{C}^d, V)$. The second condition implies that the derivation $Q = \bar{\partial} + Q^{hol}$ defines a cochain complex

$$\mathcal{E}_V = (\Omega^{0,*}(\mathbb{C}^d, V), \bar{\partial} + Q^{hol}),$$

and that Q is skew self-adjoint for the symplectic structure. Thus, in particular, \mathcal{E}_V together with the pairing define a free BV theory in the ordinary sense. In the usual way, we obtain the action functional via

$$S(\varphi) = \int (\varphi, (\bar{\partial} + Q^{hol})\varphi)_V.$$

Before going further, we will give a familiar example from the last section.

Example 7.12. The free $\beta\gamma$ system on \mathbb{C}^d . Consider the $\beta\gamma$ system with coefficients in any holomorphic vector bundle from Example ?? (and the remarks after it) specialized to the manifold $X = \mathbb{C}^d$. One immediately checks that this is a holomorphically translation invariant free theory.

7.2.1 Translation invariant interactions

Let's fix a general free holomorphically translation invariant theory $(V, (-, -)_V, Q^{hol})$ as above. We now define what a holomorphically translation invariant interacting theory is. Recall, translations span a $2d$ -dimensional abelian Lie algebra $\mathbb{C}^{2d} = \mathbb{C} \left\{ \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right\}$. The first condition that an interaction be holomorphically translation invariant is that it be translation invariant, so invariant for this Lie algebra. The additional condition is a bit more involved.

Let $\bar{\eta}_i$ denote the operator on Dolbeault forms given by contraction with the antiholomorphic vector field $\frac{\partial}{\partial \bar{z}_i}$. Note that η_i acts on the Dolbeault complex on \mathbb{C}^d with values in any vector bundle. In particular it acts on the fields of a free holomorphically translation invariant theory as above, in addition to functionals on fields.

Definition 7.13. A holomorphically translation invariant local functional is a translation invariant local functional $I \in \mathcal{O}_{\text{loc}}(\mathcal{E}_V)^{\mathbb{C}^{2d}}$ such that $\bar{\eta}_i I = 0$ for all $1 \leq i \leq d$.

There is a succinct way of expressing holomorphic translation invariance as the Lie algebra invariants of a certain *dg Lie algebra*. Denote by $\mathbb{C}^d[1]$ the abelian d -dimensional graded Lie algebra in concentrated in degree -1 by the elements $\{\bar{\eta}_i\}$. We want to consider deformations that are invariant for the action by the total *dg Lie algebra* $\mathbb{C}^{2d|d} = \mathbb{C}^{2d} \oplus \mathbb{C}^d[1]$. The differential sends $\eta_i \mapsto \frac{\partial}{\partial \bar{z}_i}$. The space of holomorphically translation invariant local functionals are denoted by $\mathcal{O}_{\text{loc}}(\mathcal{E}_V)^{\mathbb{C}^{2d|d}}$. The enveloping algebra of $\mathbb{C}^{2d|d}$ is of the form

$$U(\mathbb{C}^{2d|d}) = \mathbb{C} \left[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}, \eta_i \right]$$

with differential induced from that in $\mathbb{C}^{2d|d}$. Note that this algebra is quasi-isomorphic to the algebra of constant coefficient holomorphic differential operators $\mathbb{C}[\partial/\partial z_i] \xrightarrow{\simeq} U(\mathbb{C}^{2d|d})$.

Any translation invariant local functional is a sum of functionals of the form

$$\varphi \mapsto \int_{\mathbb{C}^d} F(D_1 \varphi, \dots, D_k \varphi)$$

where $F : V^{\otimes k} \rightarrow \mathbb{C} \cdot d^d z$ is a linear map and each D_α is an operator in the space

$$\mathbb{C} \left[d\bar{z}_i, \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}, \eta_i \right].$$

The condition $\eta_i I = 0$ means that none of the D_i 's have any $d\bar{z}_j$ -dependence. Using this description we can exhibit the space of holomorphically translation functionals as follows. Note that if E is any vector bundle on \mathbb{C}^d we can consider the fiber at zero of its jet bundle that we denote $J_0 E$.

Lemma 7.14. *Let V be a holomorphic vector bundle on \mathbb{C}^d and denote $\mathcal{E}_V = \Omega^{0,*}(X, V)$. Then*

$$\mathcal{O}_{\text{loc}}(\mathcal{E}_V)^{\mathbb{C}^{2d|d}} \cong \mathbb{C} \cdot d^d z \otimes_{U(\mathbb{C}^{2d|d})} \mathcal{O}_{\text{red}}(J_0 E_V)$$

where E_V is the vector bundle on \mathbb{C}^d such that $\mathcal{E}_V = \Gamma(E_V)$.

This description of holomorphically translation invariant local functionals allows us to give a convenient description of deformations of holomorphically translation invariant theories. Suppose $(V, Q^{\text{hol}}, (-, -)_V, I)$ be the data of an interacting holomorphically translation invariant theory on \mathbb{C}^d . We have already encountered the space of local functionals $\mathcal{O}_{\text{loc}}(\mathcal{E}_V)$ and the deformation complex of the interacting BV theory is

$$\text{Def}_{\mathcal{E}_V} = \left(\mathcal{O}_{\text{loc}}(\mathcal{E}_V), \bar{\partial} + Q^{\text{hol}} + \{I, -\} \right).$$

We'd like to characterize deformations that preserve holomorphically translation invariance.

Recall that in the holomorphic case there is the holomorphic jet bundle $J^{\text{hol}} V$. The fiber at zero of this jet bundle may be identified as $J_0^{\text{hol}} V = V_0[[z_1, \dots, z_d]]$ where the z_i 's denote the formal jet coordinate.

Corollary 7.15. *Suppose that $Q^{\text{hol}} = 0$. Then, there is a quasi-isomorphism*

$$(\text{Def}_{\mathcal{E}_V})^{\mathbb{C}^{2d|d}} \simeq \mathbb{C} \cdot d^d z \otimes_{\mathbb{C}[\partial_{z_1}, \dots, \partial_{z_d}]}^{\mathbb{L}} \mathcal{O}_{\text{red}}(V_0[[z_1, \dots, z_d]])[d].$$

Equipped with differential $\{I^{\text{hol}}, -\}$ where I^{hol} only depends on holomorphic differential operators. Here, $\partial_{z_i} = \frac{\partial}{\partial z_i}$ and $\mathbb{C} \cdot d^d z$ denotes the trivial right $\mathbb{C}[\partial_{z_i}]$ -module.

The local functional I defining the classical holomorphic theory endows $J^{\text{hol}} V[-1]$ the structure of a L_∞ algebra in $D_{\mathbb{C}^d}$ -modules. Repackaging the statement using Lie algebraic data we can rewrite the equivalence in the lemma as

$$(\text{Def}_{\mathcal{E}_V})^{\mathbb{C}^{2d|d}} \simeq \mathbb{C} \cdot d^d z \otimes_{\mathbb{C}[\partial_{z_1}, \dots, \partial_{z_d}]}^{\mathbb{L}} \mathbb{C}_{\text{Lie, red}}^*(V_0[[z]][-1])[d].$$

Proof. By Lemma ?? we have an expression for the holomorphically translation local functionals

$$(\text{Def}_{\mathcal{E}_V})^{\mathbb{C}^{2d|d}} = \left(\mathbb{C} \cdot d^d z \otimes_{U(\mathbb{C}^{2d|d})} \mathcal{O}_{\text{red}}(J_0 E_V)[d], \bar{\partial} + \{I, -\} \right).$$

Since $\mathcal{O}_{red}(J_0 E_V)$ is flat as a $U(\mathbb{C}^{2d|d})$ -module, it follows that we can replace the tensor product by the derived tensor product $\otimes^{\mathbb{L}}$ up to quasi-isomorphism so that

$$(\text{Def}_{\mathcal{E}_V})^{\mathbb{C}^{2d|d}} \simeq \left(\mathbb{C} \cdot d^d z \otimes_{U(\mathbb{C}^{2d|d})}^{\mathbb{L}} \mathcal{O}_{red}(J_0 E_V)[d], \bar{\partial} + \{I, -\} \right).$$

Consider the complex $(\mathcal{O}_{red}(J_0 E_V), \bar{\partial} + \{I, -\})$. This complex is graded by symmetric degree, and the associated spectral sequence has first page the associated graded of $\mathcal{O}_{red}(J_0 E_V)$ equipped with the $\bar{\partial}$ differential. Moreover, at the E_1 -page, we have the quasi-isomorphism

$$(\mathcal{O}(J_0 E_V), \bar{\partial}) = (\mathcal{O}_{red}(V_0[[z_i, \bar{z}_i]][d\bar{z}_i]), \bar{\partial}) \simeq \mathcal{O}_{red}(V_0[[z_i]]).$$

Finally, we have already remarked that there is a quasi-isomorphism of algebras $U(\mathbb{C}^{2d|d}) \simeq U(\mathbb{C}^d)$ where the right-hand site is generated by the constant holomorphic vector fields. The proof of the claim follows. □

8 Renormalization of holomorphic theories

In this section we study the renormalization of holomorphically translation invariant field theories on \mathbb{C}^d for any $d \geq 1$. We start with a classical interacting holomorphic theory on \mathbb{C}^d and consider one-loop homotopy RG flow from some finite scale ϵ to scale L . That is, we consider the sum over graphs of genus zero and one where at each vertex we place the holomorphic interaction. To obtain a prequantization of a classical theory one must make sense of the $\epsilon \rightarrow 0$ limit of this construction. In general, this involves introducing a family of counterterms. Our main result is that for a holomorphic theory no such counterterms are required, and one obtains a well-defined $\epsilon \rightarrow 0$ limit.

We can write the fields of a holomorphic theory on \mathbb{C}^d as

$$\mathcal{E}_V = \left(\Omega^{0,*}(\mathbb{C}^d, V), \bar{\partial} + Q^{hol} \right)$$

where V is a graded holomorphic vector bundle and Q^{hol} is a holomorphic differential operator.

Since the theory is holomorphically translation invariant we have an identification $\Omega^{0,*}(\mathbb{C}^d, V) \cong \Omega^{0,*}(\mathbb{C}^d) \otimes_{\mathbb{C}} V_0$ where V_0 is the fiber of V over $0 \in \mathbb{C}^d$. Further, we can write the (-1) -shifted symplectic structure defining the classical BV theory in the form

$$\omega_V(\alpha \otimes v, \beta \otimes w) = (v, w)_{V_0} \int d^d z (\alpha \wedge \beta)$$

where $(-, -)_{V_0}$ is a degree $(d-1)$ -shifted pairing on the finite dimensional vector space V_0 .

We will assume that the holomorphic Lagrangian I^{hol} is also translation invariant and so defines an interaction of the form $I = \sum_k I_k$ where I_k is symmetric degree k and

$$I_k = \int I_k^{hol}(\varphi) = \int D_{k,1}(\varphi) \cdots D_{k,k}(\varphi) d^d z$$

where each $D_{i,j}$ is a translation invariant holomorphic differential operator $D_{i,j} \in \mathbb{C} \left[\frac{\partial}{\partial z_i} \right]$.

8.1 Holomorphic gauge fixing

To begin the process of renormalization we must fix the data of a gauge fixing operator. Recall, a gauge fixing operator is an operator on fields

$$Q^{GF} : \mathcal{E}_V \rightarrow \mathcal{E}_V[-1]$$

of cohomological degree -1 such that $[Q, Q^{GF}]$ is a generalized Laplacian on \mathcal{E} where Q is the linearized BRST operator.

For holomorphic theories there is a convenient choice for a gauge fixing operator. To construct it we fix the standard flat metric on \mathbb{C}^d . Doing this, we let $\bar{\partial}^*$ be the adjoint of the operator $\bar{\partial}$. Using the coordinates on $(z_1, \dots, z_d) \in \mathbb{C}^d$ we can write this operator as

$$\bar{\partial}^* = \sum_{i=1}^d \frac{\partial}{\partial(\bar{d}\bar{z}_i)} \frac{\partial}{\partial z_i}.$$

Equivalently $\frac{\partial}{\partial(\bar{d}\bar{z}_i)}$ is equal to contraction with the anti-holomorphic vector field $\frac{\partial}{\partial \bar{z}_i}$. The operator $\bar{\partial}^*$ extends to the complex of fields via the formula

$$Q^{GF} = \bar{\partial}^* \otimes \text{id}_V : \Omega^{0,*}(X, V) \rightarrow \Omega^{0,*-1}(X, V),$$

We claim that this is a gauge fixing operator for our holomorphic theory. Indeed, since Q^{hol} is a translation invariant holomorphic differential operator we have

$$[\bar{\partial} + Q^{hol}, Q^{GF}] = [\bar{\partial}, \bar{\partial}^*] \otimes \text{id}_V.$$

The operator $[\bar{\partial}, \bar{\partial}^*]$ is simply the Dolbeault Laplacian on \mathbb{C}^d , which is certainly a generalized Laplacian. In coordinates it is

$$[\bar{\partial}, \bar{\partial}^*] = - \sum_{i=1}^d \frac{\partial}{\partial \bar{z}_i} \frac{\partial}{\partial z_i}$$

By definition, the scale $L > 0$ heat kernel $K_L^V \in \mathcal{E}_V(\mathbb{C}^d) \otimes \mathcal{E}_V(\mathbb{C}^d)$ satisfies

$$\omega_V(K_L, \varphi) = e^{-L[Q, Q^{GF}]} \varphi$$

for any field $\varphi \in \mathcal{E}_V$. Pick a basis $\{e_i\}$ of V_0 and let

$$\mathbf{C}_{V_0} = \sum_{i,j} \omega_{ij}(e_i \otimes e_j) \in V_0 \otimes V_0$$

be the quadratic Casimir. Here, (ω_{ij}) is the inverse matrix to the pairing $(-, -)_{V_0}$. We see that for the holomorphic theory we can write this regularized heat kernel as

$$K_L^V(z, w) = K_L^{an}(z, w) \cdot \mathbf{C}_{V_0}$$

where the analytic part is independent of V and equal to

$$K_L^{an}(z, w) = \frac{1}{(4\pi L)^d} e^{-|z-w|^2/4L} \prod_{i=1}^d (d\bar{z}_i - d\bar{z}_j) \in \Omega^{0,*}(\mathbb{C}^d) \otimes \Omega^{0,*}(\mathbb{C}^d) \cong \Omega^{0,*}(\mathbb{C}^d \times \mathbb{C}^d).$$

The propagator is defined by

$$P_{\epsilon < L}^V(z, w) = \int_{t=\epsilon}^L dt (Q^{GF} \otimes 1) K_L^V(z, w).$$

Since \mathbf{C}_{V_0} is independent of the coordinate on \mathbb{C} this propagator is of the form $P_{\epsilon < L}^V(z, w) = P_{\epsilon < L}^{an}(z, w) \cdot \mathbf{C}_{V_0}$ where

$$\begin{aligned} P_{\epsilon < L}^{an}(z, w) &= \int_{t=\epsilon}^L dt (\bar{\partial}^* \otimes 1) K_L^V(z, w) \\ &= \int_{t=\epsilon}^L dt \frac{1}{(4\pi t)^d} \sum_{j=1}^d \left(\frac{z_j - w_j}{4t} \right) e^{-|z-w|^2/4t} \prod_{i \neq j}^d (d\bar{z}_i - d\bar{z}_j). \end{aligned}$$

Our goal in this section is to show that one-loop RG flow produces a prequantization modulo \hbar^2 that requires no counterterms. The one-loop RG flow from ϵ to L is defined by the weight expansion

$$W(P_{\epsilon < L}^V, I) = \sum_{\Gamma} \frac{\hbar^{g(\Gamma)}}{|\text{Aut}(\Gamma)|} W_{\Gamma}(P_{\epsilon < L}^V, I)$$

where the sum is over graphs of genus ≤ 1 and W_{Γ} is the weight associated to the graph Γ .

For the genus zero graphs, or trees, we do not have any analytic difficulties to worry about. The propagator $P_{\epsilon < L}^V$ is smooth so long as $\epsilon, L > 0$ but when $\epsilon \rightarrow 0$ it inherits a singularity along the diagonal $z = w$. But, if Γ is a tree the weight $W_{\Gamma}(P_{\epsilon < L}^V, I)$ only involves multiplication of distributions with transverse singular support, so is well-defined. Thus we have observed the following.

Lemma 8.1. *If Γ is a tree then $\lim_{\epsilon \rightarrow 0} W_{\Gamma}(P_{\epsilon < L}^V, I)$ exists.*

8.2 One-loop weights

The only possible divergences in the $\epsilon \rightarrow 0$ limit, then, must come from graphs of genus one. Every graph of genus one is a wheel with some trees protruding from the external edges of the tree. Thus, we can write the weight of a genus one graph as a product of weights associated to trees times the weight associated to a wheel. As we just saw, the weights associated to trees are automatically convergent in the $\epsilon \rightarrow 0$ limit, thus it suffices to focus on genus one graphs that are purely wheels with some number of external edges.

The definition of the weight of the wheel involves placing the propagator at each internal edge and the interaction I at each vertex. The weights are evaluated by placing compactly supported fields $\varphi \in \mathcal{E}_{V, \epsilon} = \Omega_c^{0,*}(\mathbb{C}^d, V)$ at each of the external edges. We will make two simplifications:

1. the only ϵ dependence appears in the analytic part of the propagator $P_{\epsilon < L}^{an}$, so we can forget about the combinatorial factor \mathbf{C}_{V_0} and assume all external edges are labeled by compactly supported Dolbeault forms in $\Omega_c^{0,*}(\mathbb{C}^d)$;
2. each vertex labeled by I is a sum of interactions of the form

$$\int_{\mathbb{C}^d} D_1(\varphi) \cdots D_k(\varphi) d^d z$$

where D_i is a translation invariant differential operator. Some of the differential operators will hit the compactly supported Dolbeault forms placed on the external edges of the graph. The remaining operators will hit the internal edges labeled by the propagators. Since a holomorphic differential operator preserves the space of compactly supported Dolbeault forms that is independent of ϵ , we replace each input by an arbitrary compactly supported Dolbeault form.

Thus, for the $\epsilon \rightarrow 0$ behavior it suffices to look at weights of wheels with arbitrary compactly supported functions as inputs where each of the internal edges are labeled by some translation invariant holomorphic differential operator

$$D = \sum_{n_1, \dots, n_d} \frac{\partial^{n_1}}{\partial z_1^{n_1}} \cdots \frac{\partial^{n_d}}{\partial z_d^{n_d}}$$

applied to the propagator $P_{\epsilon < L}^{an}$. This motivates the following definition.

Definition 8.2. Let $\epsilon, L > 0$. In addition, fix the following data.

- (a) An integer $k \geq 1$ that will be the number of vertices of the graph.
- (b) For each $\alpha = 1, \dots, k$ a sequence of integers

$$\vec{n}^\alpha = (n_1^\alpha, \dots, n_d^\alpha).$$

We denote by $(\vec{n}) = (n_i^j)$ the corresponding $d \times k$ matrix of integers.

The analytic weight associated to the pair $(k, (\vec{n}))$ is the smooth distribution

$$W_{\epsilon < L}^{k, (n)} : C_c^\infty((\mathbb{C}^d)^k) \rightarrow \mathbb{C},$$

that sends a smooth compactly supported function $\Phi \in C_c^\infty((\mathbb{C}^d)^k) = C_c^\infty(\mathbb{C}^{dk})$ to

$$W_{\epsilon < L}^{k, (n)}(\Phi) = \int_{(z^1, \dots, z^k) \in (\mathbb{C}^d)^k} \prod_{\alpha=1}^k d^d z^\alpha \Phi(z^1, \dots, z^k) \prod_{\alpha=1}^k \left(\frac{\partial}{\partial z^\alpha} \right)^{\vec{n}^\alpha} P_{\epsilon < L}^{an}(z^\alpha, z^{\alpha+1}). \quad (10)$$

In the above expression, we use the convention that $z^{k+1} = z^1$.

The coordinate on $(\mathbb{C}^d)^k$ is given by $\{z_i^\alpha\}$ where $\alpha = 1, \dots, k$ and $i = 1, \dots, d$. For each α , $\{z_1^\alpha, \dots, z_d^\alpha\}$ is the coordinate for the space \mathbb{C}^d sitting at the vertex labeled by α . We have also used the shorthand notation

$$\left(\frac{\partial}{\partial z^\alpha} \right)^{\vec{n}^\alpha} = \frac{\partial^{n_1^\alpha}}{\partial z_1^{n_1^\alpha}} \cdots \frac{\partial^{n_d^\alpha}}{\partial z_d^{n_d^\alpha}}.$$

We will refer to the collection of data $(k, (\vec{n}))$ in the definition as *wheel data*. The motivation for this is that the weight $W_{\epsilon < L}^{k, (n)}$ is the analytic part of the full weight $W_\Gamma(P_{\epsilon < L}^V, I)$ where Γ is a wheel with k vertices.

As explained above the proof of Lemma ?? has been reduced to showing that the $\epsilon \rightarrow 0$ limit of the analytic weight $W_{\epsilon < L}^{k, (n)}(\Phi)$ exists for any choice of wheel data $(k, (\vec{n}))$. To do this, there are two steps. First, we show a vanishing result that says when $k \geq d$ the weights vanish for purely algebraic reasons. The second part is the most technical aspect of the chapter where we show that for $k > d$ the weights have nice asymptotic behavior as a function of ϵ .

Lemma 8.3. Let $(k, (\vec{n}))$ be a pair of wheel data. If the number of vertices k satisfies $k \leq d$ then

$$W_{\epsilon < L}^{k, (n)} = 0$$

as a distribution on \mathbb{C}^{dk} for any $\epsilon, L > 0$.

Proof. In the integral expression for the weight (??) there is the following factor involving the product over the edges of the propagators:

$$\prod_{\alpha=1}^k \left(\frac{\partial}{\partial z^\alpha} \right)^{\vec{n}^\alpha} P_{\epsilon < L}^{an}(z^\alpha, z^\alpha). \quad (11)$$

We will show that this expression is identically zero. To simplify the expression we first make the following change of coordinates on \mathbb{C}^{dk} :

$$w^\alpha = z^{\alpha+1} - z^\alpha, \quad 1 \leq \alpha < k \quad (12)$$

$$w^k = z^k. \quad (13)$$

Introduce the following operators

$$\eta^\alpha = \sum_{i=1}^d \bar{w}_i^\alpha \frac{\partial}{\partial (\bar{w}_i^\alpha)}$$

acting on differential forms on \mathbb{C}^{dk} . The operator η^α lowers the anti-holomorphic Dolbeault type by one : $\eta : (p, q) \rightarrow (p, q - 1)$. Equivalently, η^α is contraction with the anti-holomorphic Euler vector field $\bar{w}_i^\alpha \partial / \partial \bar{w}_i^\alpha$.

Once we do this, we see that the expression (??) can be written as

$$\left(\left(\sum_{\alpha=1}^{k-1} \eta^\alpha \right) \prod_{i=1}^d \left(\sum_{\alpha=1}^{k-1} d\bar{w}_i^\alpha \right) \right) \prod_{\alpha=1}^{k-1} \left(\eta^\alpha \prod_{i=1}^d d\bar{w}_i^\alpha \right).$$

Note that only the variables \bar{w}_i^α for $i = 1, \dots, d$ and $\alpha = 1, \dots, k - 1$ appear. Thus we can consider it as a form on $\mathbb{C}^{d(k-1)}$. As such a form it is of Dolbeault type $(0, (d-1) + (k-1)(d-1)) = (0, (d-1)k)$. If $k < d$ then clearly $(d-1)k > d(k-1)$ so the form has greater degree than the dimension of the manifold and hence it vanishes.

The case left to consider is when $k = d$. In this case, the expression in (??) can be written as

$$\left(\left(\sum_{\alpha=1}^{d-1} \eta^\alpha \right) \prod_{i=1}^d \left(\sum_{\alpha=1}^{d-1} d\bar{w}_i^\alpha \right) \right) \prod_{\alpha=1}^{d-1} \left(\eta^\alpha \prod_{i=1}^d d\bar{w}_i^\alpha \right). \quad (14)$$

Again, since only the variables \bar{w}_i^α for $i = 1, \dots, d$ and $\alpha = 1, \dots, d - 1$ appear, we can view this as a differential form on $\mathbb{C}^{d(d-1)}$. Furthermore, it is a form of type $(0, d(d-1))$. For any vector field X on $\mathbb{C}^{d(d-1)}$ the interior derivative i_X is a graded derivation. Suppose ω_1, ω_2 are two $(0, *)$ forms on $\mathbb{C}^{d(d-1)}$ such that the sum of their degrees is equal to d^2 . Then, $\omega_1 \iota_X \omega_2$ is a top form for any vector field on $\mathbb{C}^{d(d-1)}$. Since $\omega_1 \omega_2 = 0$ for form type reasons, we conclude that $\omega_1 \iota_X \omega_2 = \pm (i_X \omega_1) \omega_2$ with sign depending on the dimension d . Applied to the vector field $\bar{z}_i^1 \partial / \partial \bar{w}_i^1$ in (??) we see that the expression can be written (up to a sign) as

$$\eta^1 \left(\sum_{\alpha=1}^{d-1} \eta^\alpha \prod_{i=1}^d \left(\sum_{\alpha=1}^{d-1} d\bar{w}_i^\alpha \right) \right) \left(\prod_{i=1}^d d\bar{w}_i^1 \right) \prod_{\alpha=2}^{d-1} \left(\eta^\alpha \prod_{i=1}^d d\bar{w}_i^\alpha \right).$$

Repeating this, for $\alpha = 2, \dots, k-1$ we can write this expression (up to a sign) as

$$\left(\eta_{k-1} \cdots \eta_2 \eta_1 \sum_{\alpha=1}^{k-1} \eta^\alpha \prod_{i=1}^d \left(\sum_{\alpha=1}^{k-1} d\bar{w}_i^\alpha \right) \right) \prod_{\alpha=1}^{k-1} \prod_{i=1}^d d\bar{w}_i^\alpha$$

The expression inside the parentheses is zero since each term in the sum over α involves a term like $\eta^\beta \eta^\beta = 0$. This completes the proof for $k = d$. \square

Lemma 8.4. *Let $(k, (\vec{n}))$ be a pair of wheel data such that $k > d$. Then the $\epsilon \rightarrow 0$ limit of the analytic weight*

$$\lim_{\epsilon \rightarrow 0} W_{\epsilon < L}^{k, (n)}$$

exists as a distribution on \mathbb{C}^{dk} .

Proof. We will bound the absolute value of the weight in Equation (??) and show that it has a well-defined $\epsilon \rightarrow 0$ limit. First, consider the change of coordinates as in Equations (??), (??). For any compactly supported function Φ we see that $W_{\epsilon < L}^{k, (n)}(\Phi)$ has the form

$$\int_{w^k \in \mathbb{C}^d} d^d w^k \int_{(w_1, \dots, w_{k-1}) \in (\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} d^d w^\alpha \right) \Phi(w^1, \dots, w^k) \left(\prod_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial w^\alpha} \right)^{\vec{n}^\alpha} P_{\epsilon < L}^{an}(w^\alpha) \right) \sum_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial w^\alpha} \right)^{\vec{n}^k} P^{an} \left(\sum_{\alpha=1}^{k-1} w^\alpha \right). \quad (15)$$

For $\alpha = 1, \dots, k-1$ the notation $P_{\epsilon < L}^{an}(w^\alpha)$ makes sense since $P_{\epsilon < L}^{an}(z^\alpha, z^{\alpha+1})$ is only a function of $w^\alpha = z^{\alpha+1} - z^\alpha$. Similarly $P_{\epsilon < L}^{an}(z^{k+1}, z^1)$ is a function of

$$z^k - z^1 = \sum_{\alpha=1}^{k-1} w^\alpha.$$

Expanding out the propagators the weight takes the form

$$\begin{aligned} & \int_{w^k \in \mathbb{C}^d} d^{2d} w^k \int_{(w_1, \dots, w_{k-1}) \in (\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} d^{2d} w^\alpha \right) \Phi(w^1, \dots, w^k) \int_{(t_1, \dots, t_k) \in [\epsilon, L]^k} \prod_{\alpha=1}^k \frac{dt_\alpha}{(4\pi t_\alpha)^d} \\ & \times \sum_{i_1, \dots, i_{k-1}=1}^d \epsilon_{i_1 \dots i_k} \left(\frac{\bar{w}_{i_1}^1}{4t_1} \frac{(\bar{w}^1)^{n^1}}{4t_1 |n^1|} \right) \cdots \left(\frac{\bar{w}_{i_{k-1}}^{k-1}}{4t_{k-1}} \frac{(\bar{w}^{k-1})^{n^{k-1}}}{4t_{k-1} |n^{k-1}|} \right) \left(\sum_{\alpha=1}^{k-1} \frac{\bar{w}_{i_k}^\alpha}{4t_k} \cdot \frac{1}{t |n^k|} \left(\sum_{\alpha=1}^{k-1} \bar{w}^\alpha \right)^{n^k} \right) \\ & \times \exp \left(- \sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{4t_\alpha} - \frac{1}{4t_k} \left| \sum_{\alpha=1}^{k-1} w^\alpha \right|^2 \right) \end{aligned}$$

The notation used above warrants some explanation. Recall, for each α the vector of integers is defined as $n^\alpha = (n_1^\alpha, \dots, n_d^\alpha)$. We use the notation

$$(\bar{w}^\alpha)^{n^\alpha} = \bar{w}_1^{n_1^\alpha} \cdots \bar{w}_d^{n_d^\alpha}.$$

Furthermore, $|n^\alpha| = n_1^\alpha + \cdots + n_d^\alpha$. Each factor of the form $\frac{\bar{w}_{i_\alpha}^\alpha}{t_\alpha}$ comes from the application of the operator $\frac{\partial}{\partial z_i}$ in $\bar{\partial}^*$ applied to the propagator. The factor $\frac{(\bar{w}^\alpha)^{n^\alpha}}{t |n^\alpha|}$ comes from applying the operator $\left(\frac{\partial}{\partial w} \right)^{n^\alpha}$ to the propagator. Note that $\bar{\partial}^*$ commutes with any translation invariant holomorphic differential operator, so it doesn't matter which order we do this.

To bound this integral we will recognize each of the factors

$$\frac{\overline{w}_{i_\alpha}^\alpha (\overline{w}^\alpha)^{n^\alpha}}{4t_\alpha |4t|^{n^\alpha}}$$

as coming from the application of a certain holomorphic differential operator to the exponential in the last line. We will then integrate by parts to obtain a simple Gaussian integral which will give us the necessary bounds in the t -variables. Let us denote this Gaussian factor by

$$E(w, t) := \exp \left(- \sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{4t_\alpha} - \frac{1}{4t_k} \left| \sum_{\alpha=1}^{k-1} w^\alpha \right|^2 \right)$$

For each α, i_α introduce the $t = (t_1, \dots, t_k)$ -dependent holomorphic differential operator

$$D_{\alpha, i_\alpha}(t) := \left(\frac{\partial}{\partial w_{i_\alpha}^\alpha} - \sum_{\beta=1}^{k-1} \frac{t_\beta}{t_1 + \dots + t_k} \frac{\partial}{\partial w_{i_\alpha}^\beta} \right) \prod_{j=1}^d \left(\frac{\partial}{\partial w_j^\alpha} - \sum_{\beta=1}^{k-1} \frac{t_\beta}{t_1 + \dots + t_k} \frac{\partial}{\partial w_j^\beta} \right)^{n_j^\alpha}.$$

The following lemma is an immediate calculation

Lemma 8.5. *One has*

$$D_{\alpha, i_\alpha} E(w, t) = \frac{\overline{w}_{i_\alpha}^\alpha (\overline{w}^\alpha)^{n^\alpha}}{4t_\alpha |t|^{n^\alpha}} E(w, t).$$

Note that all of the D_{α, i_α} operators mutually commute. Thus, we can integrate by parts iteratively to obtain the following expression for the weight:

$$\begin{aligned} & \pm \int_{w^k \in \mathbb{C}^d} d^{2d} w^k \int_{(w^1, \dots, w^{k-1}) \in (\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} d^{2d} w^\alpha \right) \int_{(t_1, \dots, t_k) \in [\epsilon, L]^k} \prod_{\alpha=1}^k \frac{dt_\alpha}{(4\pi t_\alpha)^d} \\ & \times \left(\sum_{i_1, \dots, i_k} \epsilon_{i_1, \dots, i_d} D_{1, i_1} \dots D_{k-1, i_{k-1}} \sum_{\alpha=1}^{k-1} D_{\alpha, i_\alpha} \Phi(w^1, \dots, w^k) \right) \times \exp \left(- \sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{t_\alpha} - \frac{1}{t_k} \left| \sum_{\alpha=1}^{k-1} w^\alpha \right|^2 \right). \end{aligned}$$

Thus, the absolute value of the weight is bounded by

$$|W_{\epsilon < L}^{k, (n)}(\Phi)| \leq C \int_{w^k \in \mathbb{C}^d} d^{2d} w^k \int_{(w^1, \dots, w^{k-1}) \in (\mathbb{C}^d)^{k-1}} \prod_{\alpha=1}^{k-1} d^{2d} w^\alpha \Psi(w^1, \dots, w^{k-1}, w^k) \int_{(t_1, \dots, t_k) \in [\epsilon, L]^k} dt_1 \dots dt_k \frac{1}{(4\pi)^{dk}} \frac{1}{t_1^d \dots t_k^d} \times E(w, t) \quad (16)$$

where Ψ is some compactly supported function on \mathbb{C}^{dk} that is independent of t .

To compute the right hand side we will perform a Gaussian integration with respect to the variables (w^1, \dots, w^{k-1}) . To this end, notice that the exponential can be written as

$$E(w, t) = \exp \left(- \frac{1}{4} M_{\alpha\beta}(w^\alpha, w^\beta) \right)$$

where $(M_{\alpha\beta})$ is the $(k-1) \times (k-1)$ matrix given by

$$\begin{pmatrix} a_1 & b & b & \dots & b \\ b & a_2 & b & \dots & b \\ b & b & a_3 & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a_{k-1} \end{pmatrix}$$

where $a_\alpha = t_\alpha^{-1} + t_k^{-1}$ and $b = t_k^{-1}$. The pairing (w^α, w^β) is the usual Hermitian pairing on \mathbb{C}^d , $(w^\alpha, w^\beta) = \sum_i w_i^\alpha \bar{w}_i^\beta$. After some straightforward linear algebra we find that

$$\det(M_{\alpha\beta})^{-1} = \frac{t_1 \cdots t_k}{t_1 + \cdots + t_k}.$$

We now perform a Wick expansion for the Gaussian integral in the variables (w^1, \dots, w^{k-1}) . For a reference similar to the notation used here see the Appendix of our work in [?]. The inequality in (??) becomes

$$|W_{\epsilon < L}^{k,(n)}(\Phi)| \leq C' \int_{w^k \in \mathbb{C}^d} d^{2d} w^k \Psi(0, \dots, 0, w^k) \int_{(t_1, \dots, t_k) \in [\epsilon, L]^k} dt_1 \dots dt_k \frac{1}{(4\pi)^{dk}} \frac{1}{(t_1 \cdots t_k)^d} \left(\frac{t_1 \cdots t_k}{t_1 + \cdots + t_k} \right)^d + O(\epsilon) \quad (17)$$

$$= C' \int_{w^k \in \mathbb{C}^d} d^{2d} w^k \Psi(0, \dots, 0, w^k) \int_{(t_1, \dots, t_k) \in [\epsilon, L]^k} dt_1 \dots dt_k \frac{1}{(4\pi)^{dk}} \frac{1}{(t_1 + \cdots + t_k)^d} + O(\epsilon). \quad (18)$$

The first term in the Wick expansion is written out explicitly. The $O(\epsilon)$ refers to higher terms in the Wick expansion, which one can show all have order ϵ , so disappear in the $\epsilon \rightarrow 0$ limit. The expression $\Psi(0, \dots, 0, w^k)$ means that we have evaluate the function $\Psi(w^1, \dots, w^k)$ at $w^1 = \dots = w^{k-1} = 0$ leaving it as a function only of w^k . In the original coordinates this is equivalent to setting $z^1 = \dots = z^{k-1} = z^k$.

Our goal is to show that $\epsilon \rightarrow 0$ limit of the right-hand side exists. The only ϵ dependence on the right hand side of (??) is in the integral over the regulation parameters t_1, \dots, t_k . Thus, it suffices to show that the $\epsilon \rightarrow 0$ limit of

$$\int_{(t_1, \dots, t_k) \in [\epsilon, L]^k} \frac{dt_1 \dots dt_k}{(t_1 + \cdots + t_k)^d}$$

exists. By the AM/GM inequality we have $(t_1 + \cdots + t_k)^d \geq (t_1 \cdots t_k)^{d/k}$. So, the integral is bounded by

$$\int_{(t_1, \dots, t_k) \in [\epsilon, L]^k} \frac{dt_1 \dots dt_k}{(t_1 + \cdots + t_k)^d} \leq \int_{(t_1, \dots, t_k) \in [\epsilon, L]^k} \frac{dt_1 \dots dt_k}{(t_1 \cdots t_k)^{d/k}} = \frac{1}{(1 - d/k)^k} \left(\epsilon^{1-d/k} - L^{1-d/k} \right)^k.$$

By assumption, $d < k$, so the right hand side has a well-defined $\epsilon \rightarrow 0$ limit. This concludes the proof. □

8.3 A general result about chiral anomalies

Once any theory has been renormalized, the next step to constructing a quantization is to solve the quantum master equation. In general, there may be an obstruction to solving this equation. Such obstructions in the physics literature are known as *anomalies*. In general, it may be difficult to characterize such anomalies, but in the case of holomorphic theories on \mathbb{C}^d our result in the previous section makes this problem much easier. Indeed, since there are no counterterms

requires, we can plug in the RG flow of the classical action functional and study the quantum master equation directly. As is usual in perturbation theory, we work order by order in \hbar to construct a quantization. In this section we will study the first step, which is to promote a classical theory to a solution of the quantum master equation modulo \hbar^2 .

As above, \mathcal{E} will be a holomorphically translation invariant theory on \mathbb{C}^d and I will be the holomorphic interaction. The linearized BRST operator is of the form $Q = \bar{\partial} + Q^{hol}$ where Q^{hol} is a holomorphic differential operator. For this section, it will be most convenient to set $Q^{hol} = 0$.

Define $I[L] = W(P_{\epsilon < L}, I) \mod \hbar^2$ as in the last section. Recall, from Section ?? that the regularized quantum master equation at scale L is

$$QI[L] + \hbar \Delta_L I[L] + \frac{1}{2} \{I[L], I[L]\}_L = 0.$$

This is equivalent to the equation $(Q + \hbar \Delta_L) e^{I[L]/\hbar} = 0$. Therefore, the *obstruction* to satisfying the quantum master equation modulo \hbar^2 at scale L is

$$\Theta[L] = \hbar^{-1} \left(QI[L] + \hbar \Delta_L I[L] + \frac{1}{2} \{I[L], I[L]\}_L \right),$$

or, equivalently $\Theta[L] = e^{-I[L]/\hbar} (Q + \hbar \Delta_L) e^{I[L]/\hbar}$. By definition, $I[L] = \lim_{\epsilon \rightarrow 0} W(P_{\epsilon < L}, I) \mod \hbar^2$ which is equivalent to $e^{I[L]/\hbar} = \lim_{\epsilon \rightarrow 0} e^{\hbar \partial_{P_{\epsilon < L}}} e^{I/\hbar} \mod \hbar^2$ as a formal series in \hbar . Thus, we can rewrite

$$(Q + \hbar \Delta_L) e^{I[L]/\hbar} = \lim_{\epsilon \rightarrow 0} (Q + \hbar \Delta_L) \left(e^{\hbar \partial_{P_{\epsilon < L}}} e^{I/\hbar} \right).$$

The operator Q commutes with $e^{\hbar \partial_{P_{\epsilon < L}}}$, and one has $\Delta_L e^{\hbar \partial_{P_{\epsilon < L}}} = e^{\hbar \partial_{P_{\epsilon < L}}} \Delta_\epsilon$ acting on functionals. Thus,

$$\lim_{\epsilon \rightarrow 0} (Q + \hbar \Delta_L) \left(e^{\hbar \partial_{P_{\epsilon < L}}} e^{I/\hbar} \right) = \lim_{\epsilon \rightarrow 0} e^{\hbar \partial_{P_{\epsilon < L}}} (Q + \hbar \Delta_\epsilon) e^{I/\hbar}.$$

Since Δ_ϵ is a BV operator for the bracket $\{-, -\}_\epsilon$, we can rewrite the right-hand side as

$$\frac{1}{\hbar} \lim_{\epsilon \rightarrow 0} e^{\hbar \partial_{P_{\epsilon < L}}} (QI + \hbar \Delta_\epsilon I + \frac{1}{2} \{I, I\}_\epsilon) e^{I/\hbar}.$$

For every $\epsilon > 0$ we have $\Delta_\epsilon I = 0$. Moreover, since I is holomorphic (and since $Q^{hol} = 0$) we have $QI = \bar{\partial}I = 0$.

We conclude that the one-loop anomaly is

$$\Theta = \lim_{L \rightarrow 0} \Theta[L] = \frac{1}{2} \lim_{L \rightarrow 0} \lim_{\epsilon \rightarrow 0} e^{-I/\hbar} e^{\hbar \partial_{P_{\epsilon < L}}} \left(\{I, I\}_\epsilon e^{I/\hbar} \right) \mod \hbar^2$$

The main result of this section is the following.

Lemma 8.6. *The obstruction $\Theta = \lim_{L \rightarrow 0} \Theta[L]$ to satisfying the one-loop quantum master equation is given by the expression*

$$\Theta = \lim_{L \rightarrow 0} \lim_{\epsilon \rightarrow 0} \sum_{\Gamma \in \text{Wheel}_{d+1}} W_\Gamma(P_{\epsilon < L}, K_\epsilon, I)$$

where the sum is over all wheels with $(d+1)$ -vertices.

Proof. Like the proof of the non-existence of counterterms for holomorphic theories, the proof of this result will be the consequence of an explicit calculations and bounds of certain Feynman diagrams.

Consider the quantity

$$\lim_{\epsilon \rightarrow 0} e^{-I/\hbar} e^{\hbar \partial_{P_{\epsilon < L}}} \left(\{I, I\}_{\epsilon} e^{I/\hbar} \right) \mod \hbar^2 \quad (19)$$

which can be thought of as a sum over graphs. By assumption, we are only looking at graphs of genus one which look like wheels with possible trees attach. Graphically, the quantity $\{I, I\}_{\epsilon}$ is the graph of two vertices with a separating edge labeled by the heat kernel K_{ϵ} . Thus, all weights appearing in the expansion of (??) attach the propagator $P_{\epsilon < L}$ to all edges besides a single distinguished edge e , which is labeled by K_{ϵ} . Thus, as a over a sum of graphs, we see that the following two types of weights occur in the expansion of (??).

- (a) the distinguished edge e is separating;
- (b) the distinguished edge e is *not* separating, and so appears as the internal edge of the wheel portion of the graph.

By the classical master equation, we see that the $\epsilon \rightarrow 0$ limit of weights of Type (a) go to zero. Thus, we must only consider the weights of Type (b).

The result will follow from two steps. These should seem familiar from the proof of the main result about the existence of no counterterms.

1. If Γ is a wheel with $k < d + 1$ vertices, then $W_{\Gamma}(P_{\epsilon < L}, K_{\epsilon}, I) = 0$ identically.
2. If Γ is a wheel with $k > d + 1$ vertices, then $\lim_{\epsilon \rightarrow 0} W_{\Gamma}(P_{\epsilon < L}, K_{\epsilon}, I) = 0$.

The proof of both of these facts is only dependent on the analytic part of the weights. Thus, it suffices to make the same reduction as we did in the previous section. To extract that analytic part of the graph we proceed as in Definition ?? . If $(k, (\vec{n}))$ is a pair of wheel data (recall k labels the number of vertices and \vec{n} labels the derivatives at each vertex) define the smooth distribution

$$\tilde{W}_{\epsilon < L}^{k, (n)} : C_c^{\infty}((\mathbb{C}^d)^k) \rightarrow \mathbb{C},$$

that sends a smooth compactly supported function $\Phi \in C_c^{\infty}((\mathbb{C}^d)^k) = C_c^{\infty}(\mathbb{C}^{dk})$ to

$$\tilde{W}_{\epsilon < L}^{k, (n)}(\Phi) = \int_{(z^1, \dots, z^k) \in (\mathbb{C}^d)^k} \prod_{\alpha=1}^k d^d z^{\alpha} \Phi(z^1, \dots, z^k) \left(\frac{\partial}{\partial z^k} \right)^{\vec{n}^k} K_{\epsilon}(z^1, z^k) \prod_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial z^{\alpha}} \right)^{\vec{n}^{\alpha}} P_{\epsilon < L}^{an}(z^{\alpha}, z^{\alpha+1}). \quad (20)$$

Item (1) follows from the following observation.

Lemma 8.7. *Let $(k, (\vec{n}))$ be a pair of wheel data. If the number of vertices k satisfies $k \leq d$ then*

$$\tilde{W}_{\epsilon < L}^{k, (n)} = 0$$

as a distribution on \mathbb{C}^{dk} for any $\epsilon, L > 0$.

Proof. In fact, the integrand of (??) is identically zero provided $k \leq d$ by a simple observation of the differential form type. Consider the factor in the integrand of $\tilde{W}_{\epsilon < L}^{k,(n)}$ given by

$$\left(\frac{\partial}{\partial z^k}\right)^{\vec{n}^k} K_\epsilon(z^1, z^k) \prod_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial z^\alpha}\right)^{\vec{n}^\alpha} P_{\epsilon < L}^{an}(z^\alpha, z^{\alpha+1}).$$

Making the usual change of coordinates $w^\alpha = z^{\alpha+1} - z^\alpha$ and $w^k = z^k$ we see that this factor is proportional to the following constant coefficient differential form

$$\left(\prod_{i=1}^d \left(\sum_{\alpha=1}^{k-1} d\bar{w}_i^\alpha\right)\right) \prod_{\alpha=1}^{k-1} \left(\eta^\alpha \prod_{i=1}^d d\bar{w}_i^\alpha\right).$$

Note that this differential form only involves the coordinates (w_i^α) for $\alpha = 1, \dots, k-1$. Thus, we may consider it as a Dolbeault form on $\mathbb{C}^{d(k-1)}$. As such, it is of the type $(0, d + (k-1)(d-1)) = (0, (d-k+1) + d(k-1))$. Clearly, $(d-k+1) + d(k-1) > d(k-1)$ provided $k \leq d$. Thus, the weight is identically zero provided $k \leq d$, as desired. \square

Item (2) follows from the following technical lemma that the analytic weight associated to the wheels of valency $k > d+1$ vanish in the limit $\epsilon \rightarrow 0$.

Lemma 8.8. *Let $(k, (\vec{n}))$ be a pair of wheel data such that $k > d+1$. Then the $\epsilon \rightarrow 0$ limit of the analytic weight*

$$\lim_{\epsilon \rightarrow 0} \tilde{W}_{\epsilon < L}^{k,(n)} = 0$$

is identically zero as a distribution on \mathbb{C}^{dk} .

Proof. The proof is very similar to the argument we gave in the proof of Lemma ??, so we will be a bit more concise. First, we make the familiar change of coordinates as in Equations (??),(??). Using the explicit form the heat kernel and propagator we see that for any $\Phi \in C_c^\infty(\mathbb{C}^{dk})$ the weight is

$$\begin{aligned} \tilde{W}_{\epsilon < L}^{k,(n)}(\Phi) &= \int_{w^k \in \mathbb{C}^d} d^{2d} w^k \int_{(w_1, \dots, w_{k-1}) \in (\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} d^{2d} w^\alpha\right) \Phi(w^1, \dots, w^k) \int_{(t_1, \dots, t_k) \in [\epsilon, L]^{k-1}} \frac{1}{(4\pi\epsilon)^d} \prod_{\alpha=1}^{k-1} \frac{dt_\alpha}{(4\pi t_\alpha)^d} \\ &\times \sum_{i_1, \dots, i_{k-1}=1}^d \epsilon_{i_1, \dots, i_d} \left(\frac{\bar{w}_{i_1}^1 (\bar{w}^1)^{n^1}}{t_1 4t|n^1|}\right) \dots \left(\frac{\bar{w}_{i_{k-1}}^{k-1} (\bar{w}^{k-1})^{n^{k-1}}}{4t_{k-1} 4t|n^{k-1}|}\right) \left(\frac{1}{4t|n^k|} \left(\sum_{\alpha=1}^{k-1} \bar{w}^\alpha\right)^{n^k}\right) \\ &\times \exp\left(-\sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{4t_\alpha} - \frac{1}{4\epsilon} \left|\sum_{\alpha=1}^{k-1} w^\alpha\right|^2\right). \end{aligned}$$

We will integrate by parts to eliminate the factors of \bar{w}_i^α .

For each $1 \leq \alpha < k$ and i_α , define the ϵ and $t = (t_1, \dots, t_{k-1})$ -dependent holomorphic differential operator

$$D_{\alpha, i_\alpha}(t) := \left(\frac{\partial}{\partial w_{i_\alpha}^\alpha} - \sum_{\beta=1}^{k-1} \frac{t_\beta}{t_1 + \dots + t_{k-1} + \epsilon} \frac{\partial}{\partial w_{i_\alpha}^\beta}\right) \prod_{j=1}^d \left(\frac{\partial}{\partial w_j^\alpha} - \sum_{\beta=1}^{k-1} \frac{t_\beta}{t_1 + \dots + t_{k-1} + \epsilon} \frac{\partial}{\partial w_j^\beta}\right)^{n_j^\alpha}.$$

And the ϵ, t -dependent holomorphic differential operator

$$D_k(t) = \prod_{j=1}^d \left(\frac{\partial}{\partial w_j^k} - \sum_{\beta=1}^{k-1} \frac{t_\beta}{t_1 + \dots + t_{k-1} + \epsilon} \frac{\partial}{\partial w_j^\beta} \right)^{n_j^k}.$$

By a completely analogous version of Lemma the operators above allow us to integrate by parts and express the weight in the form

$$\begin{aligned} \tilde{W}_{\epsilon < L}^{k,(n)}(\Phi) &= \pm \int_{w^k \in \mathbb{C}^d} d^{2d} w^k \int_{(w_1, \dots, w_{k-1}) \in (\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} d^{2d} w^\alpha \right) \int_{(t_1, \dots, t_{k-1}) \in [\epsilon, L]^{k-1}} \frac{1}{(4\pi\epsilon)^d} \prod_{\alpha=1}^{k-1} \frac{dt_\alpha}{(4\pi t_\alpha)^d} \\ &\quad \times \left(\sum_{i_1, \dots, i_{k-1}} \epsilon_{i_1 \dots i_d} D_{1, i_1}(t) \dots D_{k-1, i_{k-1}}(t) D_k(t) \Phi(w^1, \dots, w^k) \right) \times \exp \left(- \sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{4t_\alpha} - \frac{1}{4\epsilon} \left| \sum_{\alpha=1}^{k-1} w^\alpha \right|^2 \right). \end{aligned}$$

Observe that the operators $D_{\alpha, i_\alpha}(t), D_k(t)$ are uniformly bounded in t . Thus, there exists a constant $C = C(\Phi) > 0$ depending only on the function Φ such that we can bound the weight as

$$\begin{aligned} |\tilde{W}_{\epsilon < L}^{k,(n)}(\Phi)| &\leq C \int_{(w^1, \dots, w^{k-1})} \prod_{\alpha=1}^{k-1} d^{2d} w^\alpha \int_{(t_1, \dots, t_{k-1}) \in [\epsilon, L]^{k-1}} dt_1 \dots dt_k \frac{1}{\epsilon^d t_1^d \dots t_{k-1}^d} \\ &\quad \times \exp \left(- \sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{4t_\alpha} - \frac{1}{4\epsilon} \left| \sum_{\alpha=1}^{k-1} w^\alpha \right|^2 \right). \end{aligned} \quad (21)$$

Thus, to show that the limit $\lim_{L \rightarrow 0} \lim_{\epsilon \rightarrow 0} \tilde{W}_{\epsilon < L}^{k,(n)}(\Phi) = 0$ it suffices to show that the limit of the right-hand side vanishes.

The Gaussian integral over the variables w_i^α contributes the following factor

$$\int_{(w^1, \dots, w^{k-1})} \prod_{\alpha=1}^{k-1} d^{2d} w^\alpha \exp \left(- \sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{4t_\alpha} - \frac{1}{4\epsilon} \left| \sum_{\alpha=1}^{k-1} w^\alpha \right|^2 \right) = C' \left(\frac{\epsilon t_1 \dots t_{k-1}}{\epsilon + t_1 + \dots + t_{k-1}} \right)^d.$$

Where C' involves factors of 2 and π . Plugging this back in to the right-hand side of (??) we see that

$$|\tilde{W}_{\epsilon < L}^{k,(n)}(\Phi)| \leq CC' \int_{[\epsilon, L]^{k-1}} \frac{dt_1 \dots dt_{k-1}}{(\epsilon + t_1 + \dots + t_{k-1})^d} \leq CC' \prod_{\alpha=1}^{k-1} \int_{t_\alpha=\epsilon}^L dt_\alpha t_\alpha^{-d/(k-1)}.$$

In the second inequality we have used the fact that $\epsilon > 0$ and the AM-GM inequality. It is immediate to see that the $\epsilon \rightarrow 0$ limit of the above exists provided $k > d + 1$, which is the situation we are in, and that the $L \rightarrow 0$ limit vanishes. \square

This completes the proof of Lemma ?? \square

8.3.1 Relation to the ABJ anomaly

The lemma we have just proved implies that for holomorphic theories on \mathbb{C}^d the anomaly is given by evaluating a collection of wheel diagrams with exactly $d + 1$ vertices. This expression for the obstruction fits into a generic class of one-loop anomalies from gauge theory called the

Adler-Bell-Jackiw (ABJ) anomaly [?, ?]. This anomaly is most commonly associated with four dimensional gauge theory.

We recall the basic setup for the ABJ anomaly. Consider a free Dirac fermion Ψ on \mathbb{R}^4 coupled to a background gauge field $A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{g}$. For this to make sense, Ψ is taken to be valued in a representation V of the Lie algebra \mathfrak{g} so we may think of it as an element $\Psi \in \mathcal{S}(\mathbb{R}^4) \otimes V$. Here, $\mathcal{S}(\mathbb{R}^4)$ is the space of sections of the full spinor bundle on \mathbb{R}^4 . The action functional is

$$S(A, \Psi) = \int \langle \Psi, \not{D}_A \Psi \rangle_V$$

where $\not{D}_A = \not{D} + [A, -]$ is the A -coupled Dirac operator. We are implicitly using the canonical spin invariant symplectic pairing $\mathcal{S} \otimes \mathcal{S} \rightarrow \Omega^4(\mathbb{R}^4) = C^\infty(\mathbb{R}^4) d^4x$ and a \mathfrak{g} -invariant pairing $\langle -, - \rangle_V : V \otimes V \rightarrow \mathbb{C}$, to obtain a local functional.

For any smooth map $\alpha : \mathbb{R}^4 \rightarrow \mathfrak{g}$, the infinitesimal transformation $\Psi \rightarrow \Psi + \epsilon[\alpha, \Psi]$ (where ϵ is an even parameter of square zero) is a classical symmetry of $S(A, \Psi)$. Quantum mechanically, there is a one-loop anomaly which measures the failure of the path integral to be invariant with respect to this symmetry. It is a well-known calculation, see for instance [?], that this anomaly is measured by the following local functional

$$\int \text{Tr}_V (\alpha F_A F_A). \quad (22)$$

The trace is taken in the representation V . The fundamental calculation is the infamous “triangle diagram”, where two vertices are labeled by the gauge field and the third by α . In practice, physicists express the anomaly as a failure for the Noether current associated to the symmetry α to be divergenceless.

There is the following holomorphic version of this anomaly. Again, let V be a \mathfrak{g} representation. Consider the following action functional on \mathbb{C}^2 :

$$S(A, \beta, \gamma) = \int \langle \beta, \bar{\partial}_A \gamma \rangle_V$$

where $\gamma : \mathbb{C}^2 \rightarrow V$, $\beta \in \Omega^{2,1}(\mathbb{C}^2, V)$, and $A \in \Omega^{0,1}(\mathbb{C}^2, \mathfrak{g})$. Since A is a $(0,1)$ form it defines a deformation of the trivial holomorphic G -bundle. Although we have not put this theory in the BV formalism, there is a natural way to do so. The infinitesimal symmetry we contemplate is of the form $\gamma \rightarrow \gamma + \epsilon[\alpha, \gamma]$ where $\alpha : \mathbb{C}^2 \rightarrow \mathfrak{g}$. We study the anomaly to quantizing this symmetry to one-loop. Following the result for the anomaly given in the previous section, one sees that it is computed by a wheel with three vertices. For type reasons, one vertex is labeled α and the other two are labeled by the gauge fields A . A special case of a general calculation performed later in Chapter ?? of this thesis computes the value of the diagram as

$$\int \text{Tr}_V (\alpha \partial A \bar{\partial} A).$$

This is the holomorphic version of ABJ anomaly (?). Note that there are no terms of order A^3 or above. In fact, the functional $\int \text{Tr}(\alpha F_A F_A)$ is cohomologous to the expression above in the local deformation complex.

Remark 8.9. In the next section, using the concept of the “equivariant” BV formalism, we will make coupling background fields to a classical theory precise. Then, Lemma ?? above applies rigorously to give the form of anomaly we have given. We will see a precise statement of this for the holomorphic current algebra in our proof of the Grothendieck-Riemann-Roch theorem in Chapter ??.

Remark 8.10. We have already shown how familiar topological theories can be cast in a holomorphic language. For instance, topological BF theory is a holomorphic deformation of holomorphic BF theory. It is a peculiar consequence of the above result that such topological theories also admit a simple regularization procedure. Without much more difficulty, one can extend this to certain topological theories to odd dimensional manifolds of the form $X \times S$, where X is a complex manifold and S is a real one-dimensional manifold. We consider the theory as a product of a holomorphic theory on X and a one-dimensional topological theory on S . This can be further extended to transversely holomorphic foliations [?, ?], which we will study in a future publication. Further, often topological BF theory further deforms to Yang-Mills. It would be interesting to apply our analysis above to such gauge theories.

9 Equivariant BV quantization

Equivariant BV quantization is an enhancement of ordinary BV quantization where one takes into account an additional action of a group or Lie algebra. We will heavily rely on techniques of equivariant BV quantization throughout this thesis, notably in the construction of the holomorphic σ -model in Chapter ?? and in the proof of a local version of the Grothendieck-Riemann-Roch theorem in Chapter ?? using Feynman diagrammatic expansions.

We motivate the definition of the equivariant BV formalism through a central result in classical field theory: Noether’s theorem. Roughly speaking, this states that symmetries of a theory are encoded by a conserved quantity. For instance, a symmetry by translations gives rise to conservation of energy through the stress-energy momentum tensor. There is an enhancement of Noether’s theorem using the language of factorization algebras proved in [?] that we will not review here, but will recall in Chapter ?. For us, the manifestation of Noether’s theorem will come from a description of a symmetry through a functional satisfying a certain equivariant version of the classical or quantum master equations.

The symmetries of BV theories that we consider is a direct analog of symmetries in ordinary Hamiltonian mechanics, which we briefly recall. Suppose that \mathfrak{h} is a Lie algebra on (M, ω) is an ordinary symplectic manifold. A symplectic action of \mathfrak{h} on X is a map of Lie algebras

$$\rho : \mathfrak{h} \rightarrow \text{SympVect}(M)$$

where $\text{SympVect}(M)$ is the Lie algebra of symplectic vector fields; those vector fields X which preserve the symplectic form $L_X \omega = 0$. On any symplectic manifold, the Poisson algebra of functions admits a Lie algebra map $\mathcal{O}(M) \rightarrow \text{SympVect}(M)$ sending a function f to its Hamiltonian vector field $X_f = \{f, -\}$, where $\{-, -\}$ is the Poisson bracket. An action ρ is said to be *inner* if it

lifts to a map of Lie algebras $\tilde{\rho} : \mathfrak{h} \rightarrow \mathcal{O}(M)$. Recall that on any symplectic manifold the kernel of $f \mapsto X_f$ is precisely the constant functions.

Classical theories in the BV formalism arise as (-1) -shifted symplectic formal moduli problems. Hence, suppose we replace the symplectic manifold M by a formal moduli problem $B\mathfrak{g}$, where \mathfrak{g} is some dg Lie (or L_∞ algebra), equipped with a (-1) -shifted symplectic structure. Functions on $B\mathfrak{g}$ are precisely the Chevalley-Eilenberg cochains $\mathcal{O}(B\mathfrak{g}) = C_{\text{Lie}}^*(\mathfrak{g})$. The (-1) -shifted symplectic structure equips $C_{\text{Lie}}^*(\mathfrak{g})[-1]$ with the structure of a dg Lie algebra. Since all symplectic vector fields are Hamiltonian in this case we see that

$$\text{SympVect}(B\mathfrak{g}) = C_{\text{Lie}}^*(\mathfrak{g})[-1]/\mathbb{C} = C_{\text{Lie,red}}^*(\mathfrak{g})[-1]$$

where we have taken the quotient by the constants, which by definition is the reduced cochains. We modify the notion of a symplectic action slightly to allow for more general maps of Lie algebras. A symplectic action of \mathfrak{h} on the (-1) -shifted symplectic formal moduli space $B\mathfrak{g}$ is a map of L_∞ algebras, or a homotopy coherent map of dg Lie algebras

$$\rho : \mathfrak{h} \rightsquigarrow C_{\text{Lie,red}}^*(\mathfrak{g})[-1].$$

Such a map ρ is equivalent to a Maurer-Cartan element in the dg Lie algebra

$$I^{\mathfrak{h}} \in C_{\text{Lie}}^*(\mathfrak{h}) \otimes C_{\text{Lie,red}}^*(\mathfrak{g})[-1].$$

This is a cohomological degree $+1$ element $I^{\mathfrak{h}}$ such that $dI^{\mathfrak{h}} + \frac{1}{2}\{I^{\mathfrak{h}}, I^{\mathfrak{h}}\} = 0$. Here $\{-, -\}$ is the bracket on $C_{\text{Lie,red}}^*(\mathfrak{g})$ and d is the sum of the Chevalley-Eilenberg differentials on \mathfrak{h} and \mathfrak{g} . This is a version of the classical master equation over the base ring $C_{\text{Lie}}^*(\mathfrak{h})$.

9.1 Classical equivariance

We proceed to mimic the above discussion to define the notion of equivariance for a general classical BV theory. Again, let \mathfrak{h} be an L_∞ algebra. A classical field theory, in the BV formalism, is given by an elliptic formal moduli problem satisfying some conditions. In the beginning of this chapter, we saw that this is encoded by a space of fields \mathcal{E} , an action functional $S \in \mathcal{O}_{\text{loc}}(\mathcal{E})$, and a (-1) -shifted symplectic structure.

We have just seen that we can express an action of \mathfrak{h} using a Maurer-Cartan element that is a functional of both \mathfrak{h} and \mathfrak{g} . The additional requirement in this situation is that we require our functionals to be *local* with respect to their dependence on the fields \mathcal{E} . Recall that the shifted symplectic structure induced a P_0 -bracket on local functionals $\mathcal{O}_{\text{loc}}(\mathcal{E})$. Thus, $\mathcal{O}_{\text{loc}}(\mathcal{E})[-1]$ has the structure of a dg Lie algebra with differential given by $\{S, -\}$.

Definition 9.1. An action of a Lie algebra \mathfrak{h} on a classical theory (\mathcal{E}, S, ω) is a Maurer-Cartan element of the dg Lie algebra

$$I^{\mathfrak{h}} \in C_{\text{Lie,red}}^*(\mathfrak{h}) \otimes \mathcal{O}_{\text{loc}}(\mathcal{E})[-1].$$

In other words, $I^{\mathfrak{h}}$ satisfies the *equivariant classical master equation*:

$$d_{\mathfrak{h}} I^{\mathfrak{h}} + \{S, I^{\mathfrak{h}}\} + \frac{1}{2}\{I^{\mathfrak{h}}, I^{\mathfrak{h}}\} = 0.$$

Analogous to the manipulations above, we see that such an I^\hbar defines a sequence of maps

$$\mathfrak{h}^{\otimes m} \otimes \mathcal{E}(X)^{\otimes n} \rightarrow \mathcal{E}(X)$$

combining to give $\mathcal{E}(X)$ the structure of an L_∞ -module over \mathfrak{h} . The equivariant classical master equation exhibits I^\hbar as a conserved quantity encoding the symmetry by the Lie algebra \mathfrak{h} . This is the fundamental idea of Noether's theorem.

Remark 9.2. There is a natural map $C_{\text{Lie}}^*(\mathfrak{h}) \rightarrow C_{\text{Lie,red}}^*(\mathfrak{h})$. An *inner* action of \mathfrak{h} on \mathcal{E} is a lift of an action I^\hbar to an Maurer-Cartan element of the dg Lie algebra $C_{\text{Lie}}^*(\mathfrak{h}) \otimes \mathcal{O}_{\text{loc}}(\mathcal{E})[-1]$. Note that there is, in general, an obstruction to lifting which lives in the cohomology $H_{\text{Lie}}^1(\mathfrak{h})$. Thus, if \mathfrak{h} is semi-simple we see that actions always lift to inner actions. We will be more interested in this problem in the case that \mathfrak{h} is a *local* Lie algebra, where the obstruction theory is more interesting.

9.2 Quantum equivariance

If we start with an \mathfrak{h} -equivariant classical BV theory with fields \mathcal{E} with action functional S — so that \mathfrak{h} has an L_∞ action on the fields that preserves the pairing and the action functional S — then we can encode the action of \mathfrak{h} as a Maurer-Cartan element I^\hbar in $C_{\text{Lie}}^*(\mathfrak{h}) \otimes \mathcal{O}_{\text{loc}}(\mathcal{E})$. We then view the sum $S + I^\hbar$ as the *equivariant* action functional: the operator $\{S + I^\hbar, -\}$ is the twisted differential on $C_{\text{Lie}}^*(\mathfrak{h}) \otimes \mathcal{O}_{\text{loc}}(\mathcal{E})$ with I^\hbar as the twisting cocycle, and this operator is square-zero because $\{S + I^\hbar, S + I^\hbar\}$ is a “constant” (i.e., lives in $C_{\text{Lie}}^*(\mathfrak{h})$ and hence is annihilated by the BV bracket).

This perspective suggests the following definition of an equivariant quantum BV theory. The starting data is two-fold: an \mathfrak{h} -equivariant classical BV theory with equivariant action functional $S + I^\hbar$, and a BV quantization $\{S[L]\}$ of the non-equivariant action functional S . Following Costello, it is convenient to write S as $S_{\text{free}} + I$, where the first “free” term is a quadratic functional and the second “interaction” term is cubic and higher. In this situation, the effective action $S[L] = S_{\text{free}} + I[L]$, i.e., only the interaction changes with the length scale.

As in Section ?? we let $\mathcal{O}_{P,sm}^+(\mathcal{E})$ be the functionals that are at least cubic, have proper support, and have smooth first derivative.

Definition 9.3. An \mathfrak{h} -equivariant BV quantization is a collection of effective interactions $\{I^\hbar[L]\}_{L \in (0,\infty)} \subset C_{\text{Lie,red}}^*(\mathfrak{h}) \otimes \mathcal{O}_{P,sm}^+(\mathcal{E})[[\hbar]]$ satisfying

(a) the RG flow equation

$$W(P_\epsilon^L, I[\epsilon] + I^\hbar[\epsilon]) = I[L] + I^\hbar[L]$$

for all $0 < \epsilon < L$,

(b) the equivariant scale L quantum master equation, which is that

$$Q(I[L] + I^\hbar[L]) + \mathbf{d}_\hbar I^\hbar[L] + \frac{1}{2} \{I[L] + I^\hbar[L], I[L] + I^\hbar[L]\}_L + \hbar \Delta_L(I[L] + I^\hbar[L])$$

lives in $C_{\text{Lie}}^*(\mathfrak{h})$ for every scale L , and

- (c) the locality axiom, with the additional condition that as $L \rightarrow 0$, we recover the equivariant classical action functional $S + I^{\mathfrak{h}}$ modulo \hbar .

In other words, we simply follow the constructions of [?] working over the base ring $C_{\text{Lie}}^*(\mathfrak{h})$. A careful reading of those texts shows that the freedom to work over interesting dg commutative algebras is built into the formalism.

9.3 The case of a local Lie algebra

The above formalism works equally well, with some slight modifications, if we replace the Lie algebra \mathfrak{h} by a *local Lie algebra* \mathcal{L} on the manifold where the theory \mathcal{E} lives. This is done in detail in Chapter 11 of [?], and we refer the reader there for more details.

For the classical case, the first thing we must define is where the classical Noether current $I^{\mathfrak{h}} \leftrightarrow I^{\mathcal{L}}$ lives. Naively, we expect this to live in the space

$$C_{\text{Lie,red}}^*(\mathcal{L}_c(X)) \otimes \mathcal{O}_{\text{loc}}(\mathcal{E})[-1]. \quad (23)$$

This is not quite good enough for our purposes since we have not taken into account the *locality* in the Lie algebra direction. Note that (??) is still a dg Lie algebra, just as above. The inclusion (??) determines an inclusion of vector spaces

$$\mathcal{O}_{\text{loc}}(\mathcal{L}[1] \oplus \mathcal{E}) \hookrightarrow C_{\text{Lie,red}}^*(\mathcal{L}_c(X)) \otimes \mathcal{O}_{\text{loc}}(\mathcal{E})$$

We can further quotient this subspace by $\mathcal{O}_{\text{loc}}(\mathcal{L}[1]) \oplus \mathcal{O}_{\text{loc}}(\mathcal{E}) \subset \mathcal{O}_{\text{loc}}(\mathcal{L}[1] \oplus \mathcal{E})$ consisting of those local functionals that depend solely on \mathcal{L} or \mathcal{E} to obtain an inclusion of vector spaces

$$\text{Act}(\mathcal{L}, \mathcal{E}) := \mathcal{O}_{\text{loc}}(\mathcal{L}[1] \oplus \mathcal{E}) / \mathcal{O}_{\text{loc}}(\mathcal{L}[1]) \oplus \mathcal{O}_{\text{loc}}(\mathcal{E}) \hookrightarrow C_{\text{Lie,red}}^*(\mathcal{L}_c(X)) \otimes \mathcal{O}_{\text{loc}}(\mathcal{E}).$$

Thus, $\text{Act}(\mathcal{L}, \mathcal{E})$ consists of functionals on $\mathcal{L}[1] \oplus \mathcal{E}$ that are local as both a function of $\mathcal{L}[1]$ and \mathcal{E} and do not depend solely on $\mathcal{L}[1]$ and \mathcal{E} .

Lemma 9.4 (Chapter 12 [?]). *The differential and bracket defining the dg Lie algebra $C_{\text{Lie,red}}^*(\mathcal{L}_c(X)) \otimes \mathcal{O}_{\text{loc}}(\mathcal{E})[-1]$ in (??) restricts to give a dg Lie algebra structure on the subspace $\text{Act}(\mathcal{L}, \mathcal{E})[-1]$.*

Using this lemma, the following definition is well-posed.

Definition 9.5. Let \mathcal{L} be a local Lie algebra and \mathcal{E} a classical field theory. An \mathcal{L} action on \mathcal{E} is a Maurer-Cartan element

$$I^{\mathcal{L}} \in \text{Act}(\mathcal{L}, \mathcal{E})[-1].$$

In other words, $I^{\mathcal{L}}$ satisfies the equivariant classical master equation

$$d_{\mathcal{L}} I^{\mathcal{L}} + \{S, I^{\mathcal{L}}\} + \frac{1}{2} \{I^{\mathcal{L}}, I^{\mathcal{L}}\} = 0.$$

Remark 9.6. Given any L_{∞} algebra \mathfrak{h} one can define the local Lie algebra $\Omega_X^* \otimes \mathfrak{h}$ on X . The data of an action of \mathfrak{h} on a theory as in Definition ?? is equivalent (up to homotopy) to the data of an action of the local Lie algebra $\Omega_X^* \otimes \mathfrak{h}$ in the definition above. In fact, there is an equivalence of dg Lie algebras $C_{\text{Lie,red}}^*(\mathfrak{h}) \otimes \mathcal{O}_{\text{loc}}(\mathcal{E})[-1] \simeq \text{Act}(\Omega_X^* \otimes \mathfrak{h}, \mathcal{E})$.

9.3.1 The action of a local Lie algebra on a QFT

The quantum story for an action by a local Lie algebra is also similar to the case of an ordinary Lie algebra. There are two spaces of functionals that appear when discussing actions of a local Lie algebra \mathcal{L} on a quantum field theory. We will fix the data of a quantum field theory as in Definition ?? . This is the data of a free BV theory (\mathcal{E}, Q, ω) together with a family of functionals $\{I[L]\}$ satisfying RG flow and the QME (plus a locality condition).

Definition 9.7. An \mathcal{L} -action on the quantum field theory $(\mathcal{E}, Q, \omega, \{I[L]\})$ is the data of a family of functionals

$$\{I^\mathcal{L}[L]\} \subset \mathcal{O}_{P,sm}^+(\mathcal{L}[1] \oplus \mathcal{E}) / \mathcal{O}_{P,sm}(\mathcal{L}[1])[[\hbar]]$$

satisfying the following properties:

- (a) The RG equation $W(P_{L < L'}, I^\mathcal{L}[L]) = I^\mathcal{L}[L']$;
- (b) The equivariant quantum master equation at scale L :

$$d_\mathcal{L} I^\mathcal{L}[L] + Q I^\mathcal{L}[L] + \frac{1}{2} \{I^\mathcal{L}[L], I^\mathcal{L}[L]\}_L + \hbar \Delta_L I^\mathcal{L}[L] = 0$$

where $d_\mathcal{L}$ is the Chevalley-Eilenberg differential on \mathcal{L} ;

- (c) the locality axiom as in Definition ??;
- (d) under the natural quotient map

$$\mathcal{O}_{P,sm}^+(\mathcal{L}[1] \oplus \mathcal{E}) / \mathcal{O}_{P,sm}(\mathcal{L}[1])[[\hbar]] \rightarrow \mathcal{O}_{P,sm}^+(\mathcal{E})[[\hbar]]$$

sends $I^\mathcal{L}[L] \mapsto I[L]$ for each $L > 0$.

In the definition above we require $I^\mathcal{L}[L]$ to be an element in $\mathcal{O}_{P,sm}^+(\mathcal{L}[1] \oplus \mathcal{E}) / \mathcal{O}_{P,sm}(\mathcal{L}[1])[[\hbar]]$, which is the space of smooth and proper functionals on $\mathcal{L}[1] \oplus \mathcal{E}$ that are at least cubic modulo \hbar and do not depend solely on \mathcal{L} . A stricter definition is that of an *inner* action, where we allow the functionals that depend solely on $\mathcal{L}[1]$.

Definition 9.8. An *inner* action of \mathcal{L} on the QFT $(\mathcal{E}, Q, \omega, \{I[L]\})$ is an effective family

$$\{I^\mathcal{L}[L]\} \subset \mathcal{O}_{P,sm}^+(\mathcal{L}[1] \oplus \mathcal{E})$$

satisfying conditions (a)-(c) above and under the natural map

$$\mathcal{O}_{P,sm}^+(\mathcal{L}[1] \oplus \mathcal{E})[[\hbar]] \rightarrow \mathcal{O}_{P,sm}^+(\mathcal{E})[[\hbar]]$$

we have $I^\mathcal{L}[L] \mapsto I[L]$ for each $L > 0$.

Every inner action clearly defines an ordinary action on a QFT. In practice, we will study the problem of *lifting* an ordinary action to an inner action. Just as in the obstruction theory discussed in Section ?? there is a deformation complex controlling this lifting problem. Indeed, suppose $I^\mathcal{L}[L] \in \mathcal{O}_{P,sm}^+(\mathcal{L}[1] \oplus \mathcal{E}) / \mathcal{O}_{P,sm}(\mathcal{L}[1])[[\hbar]]$ is a family satisfying the condition of having an action by \mathcal{L} . We can lift this to a family of functionals

$$\tilde{I}^\mathcal{L}[L] \in \mathcal{O}_{P,sm}^+(\mathcal{L}[1] \oplus \mathcal{E})[[\hbar]]$$

that satisfy RG flow and the locality axioms, but in general they do not satisfy the equivariant quantum master equation. The obstruction is an element

$$\Theta[L] = d_{\mathcal{L}} \tilde{I}^{\mathcal{L}}[L] + Q \tilde{I}^{\mathcal{L}}[L] + \frac{1}{2} \{ \tilde{I}^{\mathcal{L}}[L], \tilde{I}^{\mathcal{L}}[L] \}_L + \hbar \Delta_L \tilde{I}^{\mathcal{L}}[L].$$

Since the right-hand side is zero modulo $\mathcal{O}(\mathcal{L}[1])[[\hbar]]$, by assumption, we must have $\Theta[L] \in \mathcal{O}(\mathcal{L}[1])[[\hbar]]$. By homotopy RG flow it suffices to solve this equation at any scale L . Moreover, by the locality axiom the limit $\lim_{L \rightarrow 0} \Theta[L]$ exists and is a local functional of $\mathcal{L}[1]$. Thus we arrive at the following.

Lemma 9.9. *Suppose $\{I^{\mathcal{L}}[L]\}$ is an effective family defining an action of \mathcal{L} on a QFT. Then, the obstruction to lifting this action to an inner action, that is the anomaly to solving the equivariant quantum master equation, is the degree +1 cocycle in $\Theta = \lim_{L \rightarrow 0} \Theta[L] \in C_{\text{loc}}^*(\mathcal{L})$.*

Remark 9.10. Equivariant quantization is essentially a version of the background field method in QFT. One treats elements of \mathcal{L} as background fields and the interaction terms $I^{\mathcal{L}}[L]$ encode the variation of the path integral measure with respect to these background fields. (Solving the QME is our definition of well-posedness of the measure.) This should not be confused with *gauging* a theory by \mathcal{L} , which involves putting the elements of \mathcal{L} in the theory as propagating fields.