Evaluation of “Renormalization for holomorphic field theories”

The current paper under evaluation presents a formulation of quantum field theories where observables vary holomorphically. This is a very important generalization of the notion of topological field theories where observables vary in a locally constant way. Mathematically rigorously, this paper presented the classical formulation of holomorphic field theories in the BV formalism, and initiate a quantum analysis when the theory is formulated on the flat $\mathbb{C}^n$. The main result of this paper is about the exact description of 1-loop quantization of holomorphic QFT, which is very useful in studying various models in the future. Besides, this paper is beautifully written and the author has made a great effort in cleaning up many technical stuff in quantum field theory. This will also be a very important resource for the community.

Reading through, I do find several small mistakes and have certain comments/questions. Those mistakes turn out to be not critical, could be corrected and I provide suggestions below. This paper surely meets the standard of Communications in Mathematical Physics, and I strongly recommend publication of this article, after the author clarifies issues listed as follows.

1. Page 8. Definition 2.4. (ii). $\mathcal{L}$ should be $\mathcal{L}^{sh}$ from the notation.

2. Page 9, Definition 2.5. I suggest spread out the definition of $C^*_{Lie,red}(L)$ explicitly in terms of polydifferential operators (if I understand correctly).

3. Page 10. Proof of Proposition 2.7, and other places describing the classical master equation. The author says ”the classical master equation is equivalent to the fact that dS is a derivation”. This is not quite true. In fact, the classical master equation is stronger. We could allow classical master equation holds up to extra central terms.

4. Page 11. Example 2.9. I don’t understand what it means ”the ∂ operator for the trivial bundle”. The author means the de Rham complex of the trivial bundle.

5. Page 12. Line 7 from bottom. ”we have $[\bar{\partial}, Q^{hol}]$”. It should be ”we have $[\bar{\partial}, Q^{hol}] = 0$”. It should be also helpful to clarify that all $[-,-]$’s used in this paper mean graded commutator.

6. Page 13. Definition/Lemma 1. “Q^{hol} is a square zero elliptic differential operator”. I think this requirement is wrong and unnecessary. Q^{hol} could be even zero though.

7. Page 13 Line 5 from bottom. The notation for expressing an action is a bit of confusing. Sometimes $S()$ means a function in the variable, sometimes it means a multi-linear map. I suggest using function notation for $S()$ all the time.

8. Page 14. Example 2.11, line 4. There should be a shift of $[d-1]$ in the second component of V.

9. Page 16. I think the bracket defined in this page is not correct. The author uses the section $(-,-)_V^{-1}$ of $V \otimes V \otimes K_V^v$ to define the bracket as described. This is not well-defined because $(-,-)_V^{-1}$ is inserted into the expression $Hom(J^{hol}(V)^k, -)$. This involves jets of the section, or differential operator. This means that if we naively insert a tensor $(-,-)_V^{-1}$ into differential operators, it will depend on the choice of the basis because differential operators are not linear. To make sense of this definition, we need to integration by part to move all those derivatives away from the insertion. In fact, Definition 2.4 for the $Lag^{hol}(V)$ is not really essentially used. Why not use Definition 2.17 for a holomorphic local function directly, and express the bracket there as what author did for the usual BV theory.

10. Page 20. Lemma 2.20. The author should mention that such formulation appeared in the reference [Li] cited in the paper when $d=1$. In fact, what the author did in this section is a straight-forward generalization of that in [Li].

11. Page 20. Line 7. ”then clearly it forgets down to an ordinary smooth D_X-module”. This is not true. A holomorphic D_X-module does not always give a smooth D_X-module. It has to be a holomorphic vector bundle first, so that $\bar{\partial}$ operator gives the other part of the flat structure.

12. Page 21. Example 2.22. The g^V in the formula for V should be $\underline{g}_X^V$ in author’s notation. Also in the formula after, about I^{hol}, the author uses the multi-linear map notation, which is a bit of confusing.

13. Page 23. The formula in the middle should be

$$(\frac{\partial}{\partial z_i} v, v')_V + (v, \frac{\partial}{\partial z_i} v')_V = L_{\partial_{z_i}}(v, v')_V.$$

14. Page 24. Bottom. The author uses $\bar{\eta}_i$ and η_i for the same thing. Similar confusion in the presentation in Page 25.

15. Page 25. Definition 2.28. I'm a bit confused about this definition. According to it, $\frac{\partial}{\partial \bar{z}_i}$ is in fact allowed. It does not break translation invariance,

and also not violating the condition $\bar{\eta}_i I = 0$. But this seems not to be the one the author wants as $C^{2d|d}$-invariance would kill both $d\bar{z}_i$ and $\frac{\partial}{\partial \bar{z}_i}$.

16. Page 25. Line 4 above Lemma 2.29. "none of the D_i's have any $d\bar{z}_j$-dependence". Do we also require that D_i does not have $\frac{\partial}{\partial \bar{z}_i}$?

17. Page 27. Line 3 above Example 2.32. The definition of $I(\beta, \gamma)$. With an ϵ inserted as indicated, I'm not sure whether $\eta_i I = 0$ still holds since ϵ has a $(0,1)$-factor. Maybe the author means a bigger space of functionals.

18. Page 29. Line 7 from bottom. It seems like the notation $O_{loc}^{hol}(V)^+$ first appears (or I might have missed it somewhere) in this paper. It seems to be those local functionals which are at least cubic. I suggest define it clearly somewhere.

19. Page 29. Line 4 from bottom. $\Omega^{d,hol} = Cd^d z$. This looks bad. LHS means all holomorphic d-forms.

20. Page 30, middle. $W(P_{L<L'}, -)$. This is only defined on those $(-)^+$.

21. Page 32, remark 3.6. This remark is a bit of confusing. Is it true that the UV finiteness holds for any other metrix?

22. Page 33. The formula for K_L^{an} and $P_{\epsilon<L}^{an}(z, w)$. The factors $(d\bar{z}_i - d\bar{z}_j)$ should be $(d\bar{z}_i - d\bar{w}_i)$.

23. Page 34. Lemma 3.8 Proof. I'm not sure where $(d^d z - d^d w)$ comes from. In the statement of Lemma 3.8, the propagator has no $d^d z$ component as ω_{BM}. But usually when we talk about Bochner-Martinelli kernel, it would have $d^d z$-factor, though this is not included in author's definition.

24. Page 36. Line 4 above Lemma 3.1. "when $k \geq d$ the weights vanish" should be $k \leq d$.

25. This is the main issue of this paper. The main statements here in Lemma 3.11., Lemma 3.12, Lemma 3.13., Prop 4.4., Lemma 4.5, Lemma 4.6. are all coming from the work

    Kevin Costello, Si Li: Quantization of open-closed BCOV theory, I

    The proofs are basically the same. But the author didn't cite this work and credit this origin.

    The trick in Lemma 3.13 and the use of D-operators come from the work

    Si Li: Feynman graph integrals and almost modular forms

    which plays an important role in reducing divergence in holomorphic field theory. The argument in Lemma 4.3. comes from [Li] as cited in this paper, which essentially also appeared in

    Qin Li, Si Li: On the B-twisted topological sigma model and Calabi-Yau geometry to do 1-loop obstruction analysis. Unfortunately these are all not mentioned and properly credited.