Renormalization for holomorphic field theories

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1 Introduction

2 The definition of a holomorphic field theory

The goal of this section is to define the notion of a holomorphic field theory. This is a variant of Costello's definition of a BV theory, see the previous section, and we will take for granted that the reader is familiar with the general format. In summary, we modify the definition of a theory by inserting the word "holomorphic" in front of most objects (bundles, differential operators, etc..). By applying the Dolbeault complex in appropriate locations, we will recover Costello's definition of a theory, but with a holomorphic flavor, see Table 1.

There are many references in the physics literature to codify the concept of a holomorphic field theory. See, most closely related to our approach, special cases of this in the work of Nekrasov and collaborators in [**?**, **?**, **?**]. We will discuss in more detail the relationship of our analysis of holomorphic theories to this work in Chapters **??** and **??**.

2.1 The definition of a holomorphic theory

We give a general definition of a classical holomorphic theory on a general complex manifold X of complex dimension d. We start with the definition of a *free* holomorphic field theory. After that we will go on to define what an interacting holomorphic theory is.

2.1.1 Free holomorphic theories

The essential information that govern a classical field theory are its equations of motion. For a free theory, the equations of motion are linear in the space of fields. Thus, at least classically, the setting of free theories can essentially be reduced to the study linear partial differential equations. BW: finish...

In the formalism for field theory developed in [**?**] and [**?**, **?**] the fields of a theory on a manifold X are always expressed as sections of some $\mathbb{Z}$-graded vector bundle on X. Here, the $\mathbb{Z}$-grading is

the cohomological, or BRST [1], grading of the theory. For a *holomorphic theory* we will impose that this graded vector bundle be holomorphic. By a holomorphic $\mathbb{Z}$-graded vector bundle we mean a $\mathbb{Z}$-graded vector bundle $V^\bullet = \oplus_i V^i[-i]$ (which we will usually abbreviate simply as V) such that each graded piece V^i is a holomorphic vector bundle (here V^i is in cohomological degree $+i$). Thus, in order to define a holomorphic field theory on a complex manifold X we start with the data:

(1) a $\mathbb{Z}$-graded holomorphic vector bundle $V^\bullet = \oplus_i V^i[-i]$ on X, so that the finite dimensional holomorphic vector bundle V^i is in cohomological degree i.

Remark 2.1. For supersymmetric theories it may be desirable to include an additional $\mathbb{Z}/2$, or fermionic, grading into the data of the space of fields, but we do not do that here.

A free classical theory is made up of a space of fields as above together with the data of a linearized BRST differential Q^{BRST} and a shifted symplectic pairing of cohomological degree -1. Ordinarily, the BRST operator is simply a differential operator on the underlying vector bundle defining the fields. For the class of theories we are considering, we require this operator be holomorphic. For completeness, we briefly recall this notion.

Suppose that E and F are two holomorphic vector bundles on X. Note that the Hom-bundle $\mathrm{Hom}(E, F)$ inherits a natural holomorphic structure. By definition, a *holomorphic differential operator of order m* is a linear map

$$D : \Gamma^{hol}(X; E) \to \Gamma^{hol}(X; F)$$

such that, with respect to a holomorphic coordinate chart $\{z_i\}$ on X, D can be written as

$$D|_{\{z_i\}} = \sum_{|I| \leq m} a_I(z) \frac{\partial^{|I|}}{\partial z_I} \tag{1}$$

where $a_I(z)$ is a local holomorphic section of $\mathrm{Hom}(E, F)$. Here, the sum is over all multi-indices $I = (i_1, \ldots, i_d)$ and

$$\frac{\partial^{|I|}}{\partial z_I} := \prod_{k=1}^{d} \frac{\partial^{i_k}}{\partial z_k^{i_k}}.$$

The length of the multi-index I is defined by $|I| := i_1 + \cdots + i_d$.

Example 2.2. The most basic example of a holomorphic differential operator is the ∂ operator for the trivial vector bundle. For each $1 \leq \ell \leq d = \dim_{\mathbb{C}}(X)$, it is a holomorphic differential operator from $E = \wedge^\ell T^{1,0*}X$ to $F = \wedge^{\ell+1} T^{1,0*}X$ which on sections is

$$\partial : \Omega^{\ell,hol}(X) \to \Omega^{\ell+1,hol}(X).$$

Locally, of course, it has the form

$$\partial = \sum_{i=1}^{d} (\mathrm{d}z_i \wedge (-)) \frac{\partial}{\partial z_i},$$

where $\mathrm{d}z_i \wedge (-)$ is the vector bundle homomorphism $\wedge^\ell T^{1,0*}X \to \wedge^{\ell+1} T^{1,0*}X$ sending $\alpha \mapsto \mathrm{d}z_i \wedge \alpha$.

[1] Named after Becchi, Rouet, Stora, Tyutin, for which our approach to field theory is greatly influenced by their original mathematical approach to quantization.

The next piece of data we fix is:

(2) a square-zero holomorphic differential operator

$$Q^{hol} : \mathcal{V}^{hol} \to \mathcal{V}^{hol}$$

of cohomological degree $+1$. Here $\mathcal{V}^{hol}$ denotes the holomorphic sections of V.

Finally, to define a free theory we need the data of a shifted symplectic pairing. For reasons to become clear in a moment, we must choose this pairing to have a strange cohomological degree. The last piece of data we fix is:

(3) an invertible bundle map

$$(-,-)_V : V \otimes V \to K_X[d-1]$$

Here, K_X is the canonical bundle on X.

The definition of the fields of an ordinary field theory are the *smooth* sections of the vector bundle V. In our situation this is a silly thing to do since we lose all of the data of the complex structure we used to define the objects above. The more natural thing to do is to take the *holomorphic* sections of the vector bundle V. By construction, the operator Q^{hol} and the pairing $(-,-)_V$ are defined on holomorphic sections, so on the surface this seems reasonable. The technical caveat that the sheaf of holomorphic sections does not satisfy certain conditions necessary to study observables in our approach to QFT. For more details on this see Remark 2.4. We will take a natural resolution of holomorphic sections in order to relate to the usual definition of a classical BV theory.

Given any holomorphic vector bundle V we can define its *Dolbeault complex* $\Omega^{0,*}(X,V)$ with its Dolbeault operator

$$\overline{\partial} : \Omega^{0,p}(X,V) \to \Omega^{0,p+1}(X,V).$$

Here, $\Omega^{0,p}(X,V)$ denotes smooth sections of the vector bundle $\bigwedge^p T^{0,1*}X \otimes V$. The fundamental property of the Dolbeault complex is that it provides a resolution for the sheaf of holomorphic sections $\mathcal{V}^{hol} \simeq \Omega^{0,*}_X(V)$.

We now take a graded holomorphic vector bundle V as above, equipped with the differential operator Q^{hol}. We can then define the totalization of the Dolbeault complex with the operator Q^{hol}:

$$\mathcal{E}_V = \left(\Omega^{0,*}(X,E), \overline{\partial} + Q^{hol}\right).$$

The operator $\overline{\partial} + Q^{hol}$ will be the linearized BRST operator of our theory. By assumption, we have $\overline{\partial} Q^{hol} = Q^{hol}\overline{\partial}$ so that $(\overline{\partial} + Q^{hol})^2 = 0$ and hence the fields still define a complex. The (-1)-shifted symplectic pairing is obtained by composition of the pairing $(-,-)_V$ with integration on $\Omega^{d,hol}_X$. The thing to observe here is that $(-,-)_V$ extends to the Dolbeault complex in a natural way: we simply combine the wedge product of forms with the pairing on V. The (-1)-shifted pairing ω_V on $\mathcal{E}$ is defined by the diagram

$$\begin{tikzcd} \mathcal{E}_V \otimes \mathcal{E}_V \arrow[r, "(-,-)_V"] \arrow[dr, dotted, "\omega_V"'] & \Omega^{0,*}(X,K_X)[d-1] \arrow[d, "\int_X"] \\ & \mathbb{C}[-1]. \end{tikzcd}$$

We note that the top Dolbeault forms with values in the canonical bundle K_X are precisely the top forms on the smooth manifold X, so integration makes sense.

We arrive at the following definition.

Definition/Lemma 1. A *free holomorphic theory* on a complex manifold X is the data $(V, Q^{hol}, (-,-)_V)$ as in (1), (2), (3) above such that Q^{hol} is a square zero elliptic differential operator that is graded skew self-adjoint for the pairing $(-,-)_V$. The triple $(\mathcal{E}_V, Q_V = \overline{\partial} + Q^{hol}, \omega_V)$ defines a free BV theory in the usual sense.

The usual prescription for writing down the associated action functional holds in this case. If $\varphi \in \Omega^{0,*}(X, V)$ denotes a field the action is

$$S(\varphi) = \int_X \left(\varphi, (\overline{\partial} + Q^{hol})\varphi\right)_V .$$

The first example we explain is related to the subject of Chapter **??** and will serve as the fundamental example of a holomorphic theory.

Example 2.3. *The free $\beta\gamma$ system*. Suppose that

$$V = \underline{\mathbb{C}} \oplus K_X[d-1].$$

Let $(-,-)_V$ be the pairing

$$(\underline{\mathbb{C}} \oplus K_X) \otimes (\underline{\mathbb{C}} \oplus K_X) \to K_X \oplus K_X \to K_X$$

sending $(\lambda, \mu) \otimes (\lambda', \mu') \mapsto (\lambda\mu', \lambda'\mu) \mapsto \lambda\mu' + \lambda'\mu$. In this example we set $Q^{hol} = 0$. One immediately checks that this is a holomorphic free theory as above. The space of fields can be written as

$$\mathcal{E}_V = \Omega^{0,*}(X) \oplus \Omega^{d,*}(X)[d-1].$$

We write $\gamma \in \Omega^{0,*}(X)$ for a field in the first component, and $\beta \in \Omega^{d,*}(X)[d-1]$ for a field in the second component. The action functional reads

$$S(\gamma + \beta, \gamma' + \beta') = \int_X \beta \wedge \overline{\partial}\gamma' + \beta' \wedge \overline{\partial}\gamma.$$

When $d = 1$ this reduces to the ordinary chiral $\beta\gamma$ system from conformal field theory. The $\beta\gamma$ system is a bosonic version of the ghost bc system that appears in the quantization of the bosonic string, see Chapter 6 of [**?**]. We will study this higher dimensional version further in Chapter **??**. For instance, we will see how this theory is the starting block for constructing general holomorphic σ-models.

Of course, there are many variants of the $\beta\gamma$ system that we can consider. For instance, if E is *any* holomorphic vector bundle on X we can take

$$V = E \oplus K_{\mathbb{C}^d} \otimes E^\vee$$

where $E^\vee$ is the linear dual bundle. The pairing is constructed as in the case above where we also use the evaluation pairing between E and $E^\vee$. In thise case, the fields are $\gamma \in \Omega^{0,*}(X, E)$ and $\beta \in \Omega^{d,*}(X, E^\vee)[d-1]$. The action functional is simply

$$S(\gamma, \beta) = \int \mathrm{ev}_E(\beta \wedge \overline{\partial}\gamma).$$

When E is a tensor bundle of type (r,s) this theory is a bosonic version of the bc ghost system of spin (r,s). For a general bundle E we will refer to it as the $\beta\gamma$ system with coefficients in the bundle E.

Remark 2.4. We will only work with a holomorphic theory prescribed by the data $(V, (-,-)_V, Q^{hol})$ through its associated BV theory. One might propose a definition of a BV theory in the analytic category based off of holomorphic sections of holomorphic vector bundles. There are numerous technical reason why this approach fails in our approach to QFT. In particular, the sheaf of holomorphic sections of a holomorphic bundle is not fine, and there do not exists partitions of unity in general. BW: why is this bad?

2.1.2 Interacting holomorphic theories

We proceed to define what an interacting holomorphic theory is. Roughly, a general interacting field theory with space of fields $\mathcal{E}$ is prescribed by a functional

$$S : \mathcal{E} \to \mathbb{C}$$

that satisfies the *classical master equation*. The key technical condition is that this functional must, in addition, be *local*. We proceed to briefly recall this notion.

First off, consider the algebra of functions $\mathcal{O}(\mathcal{E}(X))$ on the space of sections $\mathcal{E}(X)$. Similarly, let $\mathcal{O}_{red}(\mathcal{E}(X)) = \mathcal{O}(\mathcal{E}(X))/\mathbb{R}$ be the quotient by the constant polynomial functions. For recollections on our conventions for these spaces of functions, see Appendix BW: ref.

If E is any graded vector bundle on X let $\mathrm{Jet}(E)$ denote its bundle of ∞-jets. This is a smooth vector bundle, albeit infinite rank, on X whose fiber over $w \in X$ can be identified with

$$E_w \times \mathbb{C}[[z_1, \ldots, z_d, \bar{z}_1, \ldots, \bar{z}_d]].$$

This object can be given the natural structure of a pro object in the category of vector bundles. See, for example, Section BW: .... We let $J(E)$ denotes the associated sheaf of smooth sections. It is well-known that $\mathrm{Jet}(E)$ is equipped with a natural flat connection rendering $J(E)$ with the structure of a smooth D_X-module.

The space $\mathcal{O}_{red}(J(E))$ inherits a natural D_X-module structure from $J(E)$. We refer to $\mathcal{O}_{red}(J(E))$ as the space of *Lagrangians* on the vector bundle E. Every element $F \in \mathcal{O}_{red}(J(E))$ can be expanded as $F = \sum_n F_n$ where each F_n is an element

$$F_n \in \mathrm{Hom}_{C^\infty_X}(J(E)^{\otimes n}, C^\infty_X)_{S_n} \cong \mathrm{PolyDiff}(\mathcal{E}^{\otimes n}, C^\infty(X))_{S_n}$$

where the right-hand side is the space of polydifferential operators. The proof of the isomorphism on the right-hand side can be found in Chapter 5 of [**?**].

A local functional is prescribed by the data of a Lagrangian modulo terms involving total derivatives. The mathematical definition is the following.

Definition 2.5. Let E be a graded vector bundle on X. Define the sheaf of *local functionals* on X to be

$$\mathcal{O}_{\mathrm{loc}}(\mathcal{E}) = \mathrm{Dens}_X \otimes_{D_X} \mathcal{O}_{red}(J(E)),$$

where we use the natural right D_X-module structure on densities.

Suppose that E is the graded vector bundle underlying the data of a free theory in the BV formalism. It is shown in BW: ref [**?**] that the inverse of the (-1)-shifted symplectic form induces a bracket $\{-,-\}$ of cohomological degree $+1$ on local functionals $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$. This bracket satisfies a graded version of the Jacobi identity, and is compatible with the BV differential Q. In summary, the shift of local functionals $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1]$ inherits the structure of a dg Lie algebra.

If V is a holomorphic vector bundle we can define the bundle of holomorphic ∞-jets $\mathrm{Jet}^{hol}(V)$, [**?**, **?**]. This is a pro-vector bundle that is holomorphic in a natural way. The fibers of this infinite rank bundle $\mathrm{Jet}^{hol}(V)$ are isomorphic to

$$\mathrm{Jet}^{hol}(V)|_w = V_w \otimes \mathbb{C}[[z_1, \ldots, z_d]],$$

where $w \in X$ and where $\{z_i\}$ is the choice of a holomorphic formal coordinate near w. We denote by $J^{hol}V$ the sheaf of holomorphic sections of this jet bundle. The sheaf $J^{hol}V$ has the structure of a D_X^{hol}-module, that is, it is equipped with a holomorphic flat connection ∇^{hol}. This situation is completely analogous to the smooth case. Locally, the holomorphic flat connection on $\mathrm{Jet}^{hol}(V)$ is of the form

$$\nabla^{hol}|_w = \sum_{i=1}^{d} \mathrm{d}w_i \left(\frac{\partial}{\partial w_i} - \frac{\partial}{\partial z_i} \right),$$

where $\{w_i\}$ is the local coordinate on X near w and z_i is the fiber coordinate labeling the holomorphic jet expansion.

One natural appearance of the bundle of holomorphic jets is in providing an explicit description of holomorphic differential operators. The statement in the smooth category is simply that a differential operator between vector bundles is equivalent to the data of a map of D-modules between the associated ∞-jet bundles. In a completely analogous way, holomorphic differential operators are the same as bundle maps between the associated holomorphic jet bundles. A similar result holds for *poly*differential operators, which we also state.

Lemma 2.6. *Suppose V, W are holomorphic vector bundles with spaces of holomorphic sections given by $\mathcal{V}^{hol}, \mathcal{W}^{hol}$ respectively. There is an isomorphism of sheaves on X*

$$\mathrm{Diff}^{hol}(\mathcal{V}^{hol}, \mathcal{W}^{hol}) \cong \mathrm{Hom}_{D_X^{hol}}(J^{hol}(V), J^{hol}(W)).$$

Similarly, if $V_1, \ldots, V_n, W$ are holomorphic bundles on X, there is an isomorphism

$$\mathrm{PolyDiff}^{hol}(\mathcal{V}_1^{hol} \times \cdots \times \mathcal{V}_n^{hol}, \mathcal{W}^{hol}) \cong \mathrm{Hom}(J^{\mathrm{hol}}(V_1) \otimes \ldots \otimes J^{\mathrm{hol}}(V_n), W).$$

In both cases, the right-hand side denotes the space of homomorphisms of holomorphic D-modules that are compatible with the adic topology on jets.

We will utlize this intepretation of holomorphic jet bundles momentarily.

In ordinary field theory, local functionals are defined as integrals of Lagrangian densities. By definition, a Lagrangian density is a density valued functional on the fields that only depends on the fields through it's partial derivatives. In the holomorphic setting we have the following definition.

Definition 2.7. Let V be a vector bundle. The space of *holomorphic Lagrangian densities* on V is

$$\mathrm{Lag}^{hol}(V) = \Omega_X^{d,hol} \otimes_{\mathcal{O}_X^{hol}} \left(\prod_{n>0} \mathrm{Hom}_{\mathcal{O}_X^{hol}}(J^{hol}(V)^{\otimes n}, \mathcal{O}_X^{hol})_{S_n} \right).$$

The hom space inside the parentheses denotes maps of holomorphic vector bundles respecting the natural filtration on jets. That is, we require the bundle maps to be continuous with respect to the natural adic topology. We also take coinvariants for the symmetric group S_n.

Note that we take the product over $n > 0$. We do not want to consider Lagrangians that are constant in the fields.

Equivalently, a holomorphic Lagrangian density is of the form $\omega \otimes F$ where ω is a top holomorphic form and F is a functional $F = \sum_k F_k$ where, for each k, the multilinear map

$$F_k : \mathcal{V}^{hol} \times \cdots \times \mathcal{V}^{hol} \to \mathcal{O}_X^{hol}$$

depends only on the holomorphic ∞-jet of sections of V.

Next, suppose that V is part of the data of a free holomorphic theory $(V, Q^{hol}, (-,-)_V)$. The pairing $(-,-)_V$ endows the space of holomorphic Lagrangians with a sort of bracket that we now construct.

Suppose $\omega \otimes F, \omega' \otimes F' \in \mathrm{Lag}^{hol}(V)$. For simplicity, we assume F, F' are of homogenous symmetric degree k, k' respectively. Consider the natural map

$$\begin{array}{c} \left(\Omega_X^{d,hol} \otimes \mathrm{Hom}(J^{hol}(V)^{\otimes k}, \mathcal{O}_X^{hol})\right) \otimes \left(\Omega_X^{d,hol} \otimes \mathrm{Hom}(J^{hol}(V)^{\otimes k'}, \mathcal{O}_X^{hol})\right) \\ \downarrow \\ \left(\Omega_X^{d,hol} \otimes \Omega_X^{d,hol}\right) \otimes \mathrm{Hom}(J^{hol}(V)^{\otimes (k+k')}, \mathcal{O}_X^{hol}) \end{array} \tag{2}$$

for which the product $(\omega \otimes F) \otimes (\omega' \otimes F')$ maps to $(\omega \otimes \omega') \otimes (F \otimes F')$. The bundle map $(-,-)_V : V \otimes V \to K_X[d-1]$ is invertible, hence it determines a section $(-,-)_V^{-1}$ of $V \otimes V \otimes K_X^\vee$ of cohomological degree $d-1$, where $K_X^\vee$ is the dual bundle to K_X. Sections of $K_X^\vee$ will be denoted PV_X^d (the PV is stands for polyvector fields). The element $(-,-)_V^{-1}$ determines a map

$$\mathrm{Hom}(J^{hol}(V)^{\otimes (k+k')}, \mathcal{O}_X^{hol}) \xrightarrow{(-,-)_V^{-1}} \mathrm{Hom}(J^{hol}(V)^{\otimes (k+k'-2)}, \mathrm{PV}_X^d). \tag{3}$$

Explicitly, this sends the element $F \otimes F'$ to the functional

$$\varphi_1 \otimes \cdots \varphi_{k+k'-2} \mapsto (F \otimes F' \otimes \mathrm{id}_{K_X^\vee})(\varphi_1 \otimes \cdots \varphi_{k+k'-2} \otimes (-,-)_V^{-1}) \in \mathrm{PV}_X^d.$$

Finally, using the natural pairing between top polyvector fields and top holomorphic forms, we obtain a map

$$\left(\Omega_X^{d,hol} \otimes \Omega_X^{d,hol}\right) \otimes \mathrm{Hom}(J^{hol}(V)^{\otimes (k+k'-2)}, \mathrm{PV}_X^d) \to \Omega_X^{d,hol} \otimes \mathrm{Hom}(J^{hol}(V)^{\otimes (k+k')}, \mathcal{O}_X^{hol}). \tag{4}$$

Upon symmetrizing the inputs, we see that the image of $(\omega \otimes F) \otimes (\omega' \otimes F')$ under the composition of the maps (2), (3), and (4) determines an element $\{F, F'\}^{hol} \in \mathrm{Lag}^{hol}(V)$. We extend to non-homogenous functionals in the obvious way.

In conclusion, we have produced a bilinear map

$$\{-,-\}^{hol} : \mathrm{Lag}^{hol}(V) \times \mathrm{Lag}^{hol}(V) \to \mathrm{Lag}^{hol}(V)[d-1].$$

Note that this bracket is of cohomological degree $-d+1$ and lowers the polynomial degree by two.

We can now state the definition of a classical holomorphic theory. Note that top holomorphic forms have a natural action by the Lie algebra of holomorphic vector fields $\mathcal{T}_X^{hol}$ via Lie derivative. This induces an action of holomorphic vector fields on the space of holomorphic Lagrangians.

Definition 2.8. A *classical holomorphic theory* on a complex manifold X is the data of a free holomorphic theory $(V, Q^{hol}, (-,-)_V)$ plus a holomorphic Lagrangian

$$I^{hol} \in \Omega_X^{d,hol} \otimes_{\mathcal{O}_X^{hol}} \prod_{n \geq 3} \mathrm{Hom}(J^{hol}(V)^{\otimes n}, \mathcal{O}_X^{hol})_{S_n} \subset \mathrm{Lag}^{hol}(V)$$

of cohomological degree d such that the following holomorphic Lagrangian

$$Q^{hol} I^{hol} + \frac{1}{2}\{I^{hol}, I^{hol}\}^{hol}$$

is in the image of a some holomorphic vector field. That is, there exists some $\xi \in \mathcal{T}_X^{hol}$ and $F \in \mathrm{Lag}^{hol}(V)$ such that $Q^{hol} I^{hol} + \frac{1}{2}\{I^{hol}, I^{hol}\}^{hol} = L_\xi F$.

Remark 2.9. Note that in the definition we require that the functional I^{hol} be at least cubic. For brevity, we will denote the subspace $\prod_{n \geq 3} \mathrm{Hom}(\mathrm{Jet}^{hol}(V)^{\otimes n}, K_X)_{S_n}$ by $\mathrm{Lag}^{hol,+}(V)$.

There is an alternative way to understand the condition on the functional I^{hol} to define a classical holomorphic theory. To define it, we introduce the notion of a holomorphic local functional. A holomorphic local functional is, by definition, a holomorphic Lagrangian defined up to a total holomorphic derivative. Precisely, we have the following definition.

Definition 2.10. The sheaf of holomorphic local functionals is defined to be the quotient

$$\mathcal{O}_{\mathrm{loc}}^{hol}(V) := \mathrm{Lag}^{hol}(V) / \mathcal{T}_X^{hol} \cdot \mathrm{Lag}^{hol}(V),$$

where $\mathcal{T}_X \cdot \mathrm{Lag}^{hol}(V)$ denotes the subspace of holomorphic Lagrangians that are in the image of the Lie derivative by some holomorphic vector field.

Equivalently, we may express this quotient using holomorphic D-modules in the following way. The left D_X^{hol}-module structure on $J^{hol}(V)$ carries over to a left D_X^{hol}-module structure on the product

$$\prod_{n>0} \mathrm{Hom}_{\mathcal{O}_X^{hol}}(J^{hol}(V)^{\otimes n}, \mathcal{O}_X^{hol}).$$

Moreover, we have already mentioned that there is the natural structure of a right D_X^{hol}-module structure on $\Omega_X^{d,hol}$.

Lemma 2.11. *There is an isomorphism*

$$\mathcal{O}^{hol}_{\mathrm{loc}}(V) = \Omega^{d,hol}_X \otimes_{D^{hol}_X} \prod_{n>0} \mathrm{Hom}_{\mathcal{O}^{hol}_X}(J^{hol}(V)^{\otimes n}, \mathcal{O}^{hol}_X).$$

Compare this to the definition of ordinary local functionals that we recalled in Definition 2.5.

Next, we have the following observation.

Lemma 2.12. *Both the bracket $\{-,-\}^{hol}$ and the differential Q^{hol} on holomorphic Lagrangians descends to a bracket and differential on $\mathcal{O}^{hol}_{\mathrm{loc}}(V)$ (that we denote by the same names). This yields the structure of a dg Lie algebra*

$$(\mathcal{O}^{hol}_{\mathrm{loc}}(V)[d-1], Q^{hol}, \{-,-\}^{hol}).$$

An immediate corollary of this lemma is that the condition for a holomorphic Lagrangian I^{hol} to define a classical theory in Definition 2.10 is equivalent to the condition that I^{hol} is a Maurer-Cartan element in $\mathcal{O}^{hol}_{\mathrm{loc}}(V)[d-1]$.

As in the free case, we proceed to verify that a holomorphic theory defines an interacting classical BV theory in the sense of Definition BW: ref?? in [**?**, **?**].

The underlying space of fields, as we have already seen in the free case, is $\mathcal{E}_V = \Omega^{0,*}(X, V)$. We show how to extend a holomorphic Lagrangian to a functional on this Dolbeualt complex.

Recall, a holomorphic Lagrangian can be written as $I^{hol} = \sum_k I^{hol}_k$ where $I^{hol}_k = \omega \otimes F_k$ for $\omega \in \Omega^{d,hol}$ and $F_k : \mathcal{V}^{hol} \times \cdots \times \mathcal{V}^{hol} \to \mathcal{O}^{hol}$ is of the form

$$F_k(\varphi_1, \ldots, \varphi_k) = \sum_{i_1,\ldots,i_k} D_{i_1}(\varphi_1) \cdots D_{i_k}(\varphi_k) \in \mathcal{O}^{hol}_X.$$

Here, $\varphi_i \in \mathcal{V}^{hol}$ is a holomorphic section, and each D_{i_j} is a holomorphic differential operator $D_{i_j} : \mathcal{V}^{hol} \to \mathcal{O}^{hol}$.

In general, suppose V, W are holomorphic vector bundles. Every holomorphic differential operator $D : \mathcal{V} \to \mathcal{W}$ extends to a *smooth* differential operator on the associated Dolbeualt complexes with the property that it is compatible with the $\overline{\partial}$-operator on both sides.

To see how this works, suppose $D : \mathcal{V}^{hol} \to \mathcal{W}^{hol}$ is locally of the form $D = \sum_{m_1,\ldots,m_d} a_{m_1\cdots m_d}(z) \frac{\partial^{m_1}}{\partial z_1^{m_1}} \cdots \frac{\partial^{m_d}}{\partial z^{m_d}}$, where $a_{m_1\cdots m_d}(z)$ denotes a local holomorphic section of $\mathrm{Hom}(V, W)$. Then, if $\alpha = s_I(z, \overline{z}) \mathrm{d}\overline{z}_I \in \Omega^{0,*}(X, V)$, where s_I is a local *smooth* section of V, we define

$$D^{\Omega^{0,*}}\alpha = \sum_{m_1,\ldots,m_d} a_{m_1\cdots m_d}(z) \left(\frac{\partial^{m_1}}{\partial z_1^{m_1}} \cdots \frac{\partial^{m_d}}{\partial z^{m_d}} s_I(z, \overline{z}) \right) \mathrm{d}\overline{z}_I \in \Omega^{0,*}(X, W).$$

In this way, D extends to a differential operator

$$D^{\Omega^{0,*}} : \Omega^{0,*}(X, V) \to \Omega^{0,*}(X, W).$$

Since D is holomorphic, it is immediate that $D^{\Omega^{0,*}}\overline{\partial}_V = \overline{\partial}_W D^{\Omega^{0,*}}$ where $\overline{\partial}_V, \overline{\partial}_W$ are the $(0,1)$-connections on V, W respectively. Thus, $D^{\Omega^{0,*}}$ is a map of sheaves of cochain complexes.

Via this construction, we extend F_k to a $\Omega^{0,*}(X)$-valued functional on $\Omega^{0,*}(X,V)$ by the formula

$$F_k^{\Omega^{0,*}} : (\alpha_1, \dots, \alpha_k) \mapsto \sum_{i_1,\dots,i_k} D_{i_1}^{\Omega^{0,*}}(\alpha_1) \wedge \cdots \wedge D_{i_k}^{\Omega^{0,*}}(\alpha_k) \in \Omega^{0,*}(X).$$

We have thus produced a map

$$(-)^{\Omega^{0,*}} : \mathrm{Hom}_{\mathcal{O}^{hol}}(J^{hol}V, \mathcal{O}^{hol}) \to \mathrm{Hom}_{C^\infty}(J\Omega^{0,*}(X,V), \Omega^{0,*}(X))$$

where $J\Omega^{0,*}(X,V)$ denotes the sheaf of smooth jets of the graded vector bundle underlying the Dolbeault complex. Taking direct products and tensoring with $\Omega_X^{d,hol}$ we have a map

$$\mathrm{Lag}^{hol}(V) \to \Omega^{d,hol} \otimes_{C^\infty} \prod_{k>0} \mathrm{Hom}_{C^\infty}(J\Omega^{0,*}(X,V)^{\otimes k}, \Omega^{0,*}(X)) \cong \Omega^{d,*} \otimes_{C^\infty} \prod_{k>0} \mathrm{Hom}(J\Omega^{0,*}(X,V)^{\otimes k}, C^\infty).$$

We have already mentioned that this map is compatible with the $\overline{\partial}$-operator on the right-hand side. Moreover, the holomorphic differential operator Q^{hol} also extends to a differential operator on the right-hand side in a way compatible with $\overline{\partial}$. Thus, $(-)^{\Omega^{0,*}}$ is a map of cochain complexes, where $\mathrm{Lag}^{hol}(X)$ is equipped with the differential Q^{hol} and the right-hand side has differential $\overline{\partial} + Q^{hol}$.

The right-hand side admits a map of degree $-d$ to $\Omega^{d,d} \otimes_{C^\infty} \prod_{k>0} \mathrm{Hom}(J\Omega^{0,*}(X,V)^{\otimes k}, C^\infty)$ by projecting onto the (d,d)-component of $\Omega^{d,*}$. Note that this map is only graded linear, it does not preserve the $\overline{\partial}$-differential. However, once we quotient by the action of vector fields we do get a well-defined map

$$\mathcal{O}^{hol}_{\mathrm{loc}}(V) \to \Omega^{d,d} \otimes_D \prod_{k>0} \mathrm{Hom}(J\Omega^{0,*}(X,V)^{\otimes k}, C^\infty)_{S_k}[-d].$$

Note that we have accounted for the shift of d coming from $\Omega^{d,*} \to \Omega^{d,d}[-d]$. The right-hand side is precisely the (shifted) space of ordinary local functionals for the sheaf $\mathcal{E}_V = \Omega^{0,*}(X,V)$ defined in Definition 2.5.

In conclusion, we have obtained the following map of sheaves of cochain complexes

$$\int (-) : \mathcal{O}^{hol}_{\mathrm{loc}}(V) \to \mathcal{O}_{\mathrm{loc}}(\Omega^{0,*}(X,V))[-d].$$

In fact, we have the following stronger result, that this map is compatible with the brackets on both sides.

Lemma 2.13. *The map $\int(-)$ defines a map of dg Lie algebras*

$$\int (-) : \mathcal{O}^{hol}_{\mathrm{loc}}(V)[d-1] \to \mathcal{O}_{\mathrm{loc}}(\Omega^{0,*}(X,V))[-1]$$

As an immediate corollary, we have the following.

Corollary 2.14. *Every classical holomorphic theory $(V, Q^{hol}, (-,-)_V, I^{hol})$ determines the structure of a classical BV theory. The underlying free BV theory is given in Definition/Lemma 1 $(\mathcal{E}_V, Q, \omega_V)$ and the interaction is $I = \int I^{\Omega^{0,*}}$.*

Holomorphic theory	BV theory
Holomorphic bundle V	Space of fields $\mathcal{E}_V = \Omega^{0,*}(X, V)$
Holomorphic differential operator Q^{hol}	Linear BRST operator $\bar{\partial} + Q^{hol}$
Non-degenerate pairing $(-,-)_V$	(-1)-symplectic structure ω_V
Holomorphic Lagrangian I^{hol}	Local functional $I = \int I^{\Omega^{0,*}} \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_V)$

Table 1: From holomorphic to BV

Proof. We must show that $Q^{hol} I^{hol} + \frac{1}{2}\{I^{hol}, I^{hol}\}^{hol}$ exact implies the ordinary classical master equation for I:

$$\bar{\partial} I + Q^{hol} I + \frac{1}{2}\{I, I\} = 0.$$

Since $I^{\Omega^{0,*}}$ is defined using holomorphic differential operators, we see that $(\bar{\partial} I^{\Omega^{0,*}})(\alpha) = \bar{\partial}(I^{\Omega^{0,*}}(\alpha))$. Thus, upon integration, we see that the first term is identically zero.

Now, since $Q^{hol} I^{hol} + \frac{1}{2}\{I^{hol}, I^{hol}\}^{hol}$ is ∂-exact, $Q^{hol} I^{\Omega^{0,*}} + \frac{1}{2}\{I^{\Omega^{0,*}}, I^{\Omega^{0,*}}\}$ is also ∂-exact. By integration we obtain the result. □

BW: This is now repetitive Furthermore, via integration we can define the $\mathbb{C}$-valued functional on compactly supported sections

$$I_k = \int_X I_k^{0,*} : \Omega_c^{0,*}(X, V)^{\otimes k} \to \mathbb{C}.$$

Performing this for each homogenous piece, we obtain the functional $I = \sum_k \int I_k^{\Omega^{0,*}}$. BW: This too The symbol $\int_X$ reminds us that we are working modulo total derivatives. Moreover, since we began with a functional involving differential operators, we see that the above expression defines an element of $\mathcal{O}_{\mathrm{loc}}(\mathcal{E}_V)$. In conclusion, our construction thus outlined determines a linear map $\mathcal{O}_{\mathrm{loc}}^{hol}(V) \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_V)$ obtained via the composition $I^{hol} \mapsto I^{\Omega^{0,*}} \mapsto I = \int I^{\Omega^{0,*}}$. Note that since I^{hol} is cohomological degree d, the local functional $I^{\Omega^{0,*}}$ is degree zero since integration is degree $-d$.

Table 1 is a useful summary showing how we are producing a BV theory from a holomorphic theory.

Example 2.15. *Holomorphic BF-theory* Let $\mathfrak{g}$ be a Lie algebra and X any complex manifold. Consider the following holomorphic vector bundle on X:

$$V = \underline{\mathfrak{g}}_X[1] \oplus K_X \otimes \mathfrak{g}^\vee[d-2].$$

The notation $\underline{\mathfrak{g}}_X$ denotes the trivial bundle with fiber $\mathfrak{g}$. The pairing $V \otimes V \to K_X[d-1]$ is similar to the pairing for the $\beta\gamma$ system, except we use the evaluation pairing $\langle -.- \rangle_{\mathfrak{g}}$ between $\mathfrak{g}$ and its dual $\mathfrak{g}^\vee$. In this example, $Q^{hol} = 0$.

We describe the holomorphic Lagrangian. If $f_i : X \to \mathbb{C}, i = 1, 2$ are holomorphic functions and $\beta \in K_X$, consider the trilinear functional

$$I^{hol}(f_1 \otimes X_1, f_2 \otimes X_2, \beta \otimes X^\vee) = f_1 f_2 \beta \langle X^\vee, [X_1, X_2] \rangle_{\mathfrak{g}} + \cdots$$

where the $\cdots$ means that we symmetrize the inputs. This defines an element $I^{hol} \in \mathcal{O}^{hol}_{\mathrm{loc}}(V)^+$ and the Jacobi identity for $\mathfrak{g}$ guarantees $\{I^{hol}, I^{hol}\}^{hol} = 0$. The fields of the corresponding BV theory are

$$\mathcal{E}_V = \Omega^{0,*}(X, \mathfrak{g})[1] \oplus \Omega^{d,*}(X, \mathfrak{g}^*)[d-2].$$

The induced local functional $I^{\Omega^{0,*}}$ on $\mathcal{E}_V$ is

$$I^{\Omega^{0,*}}(\alpha, \beta) = \int_X \langle \beta, [\alpha, \alpha] \rangle_{\mathfrak{g}}.$$

The total action is $S(\alpha, \beta) = \int \langle \beta, \overline{\partial}\alpha \rangle + \langle \beta, [\alpha, \alpha] \rangle_{\mathfrak{g}}$. This is formally similar to *BF* theory (see below) and for that reason we refer to it as *holomorphic* BF theory. The moduli problem this describes is the cotangent theory to the moduli space of holomorphic connections on the trivial G-bundle near the trivial bundle. There is an obvious enhancement that works near any holomorphic principal bundle. When $d = 2$, in [**?**], or for a more mathematical treatment see [**?**], it is shown that this theory is a twist of $N = 1$ supersymmetric Yang-Mills on $\mathbb{R}^4$.

Example 2.16. *Topological BF-theory* This is a deformation of the previous example that has appeared throughout the physics literature. Suppose we take as our graded holomorphic vector bundle

$$V = \left(\underline{\mathfrak{g}}_X \otimes \left(\oplus_{k=0}^d \wedge^k T^{*1,0} X[1-k] \right) \right) \oplus \left(\underline{\mathfrak{g}^*}_X \otimes \left(\oplus_{k=0}^d \wedge^k T^{*1,0} X[2(d-1)-k] \right) \right).$$

Here $\wedge^0 T^{*1,0} X$ is understood as the trivial bundle $\underline{\mathbb{C}}_X$. The pairing is given by combining the evaluation pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ and taking the wedge product and projecting onto the components isomorphic to K_X. Explicitly, the pairing is equal to the sum of bundle maps of the form

$$\mathrm{ev}_{\mathfrak{g}} \otimes \wedge : \left(\underline{\mathfrak{g}}_X \otimes \wedge^k T^{*1,0} X[1-k] \right) \otimes \left(\underline{\mathfrak{g}^*}_X \otimes \wedge^{d-k} T^{*1,0} X[d-1+k] \right) \to K_X[d-1].$$

The differential is of the form

$$Q^{hol} = \mathrm{id}_{\mathfrak{g}} \otimes \partial + \mathrm{id}_{\mathfrak{g}^*} \otimes \partial,$$

where ∂ is the holomorphic de Rham differential. The holomorphic interaction is given by combining the Lie algebra structure on $\mathfrak{g}$ with the wedge product of the holomorphic bundles $\wedge^k T^{*1,0} X$. We observe that the associated BV theory has classical space of fields given by

$$(A, B) \in \mathcal{E}_V = \Omega^*(X, \mathfrak{g}[1] \oplus \mathfrak{g}^*[2d-2])$$

where Ω^* is now the *full* de Rham complex. The action functional is

$$S = \int_X \langle B, \mathrm{d}A \rangle_{\mathfrak{g}} + \frac{1}{3} \langle B, [A, A] \rangle_{\mathfrak{g}}.$$

As above, $\langle -, - \rangle_{\mathfrak{g}}$ denotes the pairing between $\mathfrak{g}$ and its dual. This is the well-known topological BF theory on the even dimensional *real* manifold X (of real dimension $2d$). It might seem silly that we have used the formalism of holomorphic field theory to describe a very simple topological theory. We will discuss advantages of this approach at the send of the next section. In particular, the theory of regularization for holomorphic theories we will employ has peculiar consequences for renormalizing certain classes of topological theories such as Topological BF theory.

When we construct a BV theory from a holomorphic theory $V \rightsquigarrow \mathcal{E}_V$ it is natural to expect that deformations of the theory must come from holomorphic data. In the special case that $Q^{hol} = 0$ we have the following result which relates the deformation complex of the classical theory $\mathcal{E}_V$ to a sheaf built from holomorphic differential operators.

Lemma 2.17. *Suppose $(V, 0, (-,-)_V, I^{hol})$ is the data of a holomorphic theory with $Q^{hol} = 0$. Let $(\mathcal{E}_V, Q = \bar{\partial}, \omega_V, I)$ be the corresponding BV theory. Then, there is a quasi-isomorphism of sheaves*

$$\mathrm{Def}_{\mathcal{E}_V} \simeq \Omega_X^{d,hol} \otimes^{\mathbb{L}}_{D_X^{hol}} \mathcal{O}_{red}(J^{hol}V)[d]$$

that is compatible with the brackets and $\{-,-\}$ and $\{-,-\}^{hol}$ on both sides.

Proof. By definition, the deformation complex is equal to

$$\mathrm{Def}_{\mathcal{E}_V} = \mathrm{Dens}_X \otimes_{D_X} \mathcal{O}_{red}(J\mathcal{E}_V).$$

Since $\mathcal{O}_{red}(J\mathcal{E}_V)$ is flat as a D_X-module [**?**], we can replace the tensor product $\otimes_{D_X}$ with the derived tensor product $\otimes^{\mathbb{L}}_{D_X}$.

The remainder of the proof uses the following observation about D-modules. If M is a holomorphic D_X^{hol}-module, then clearly it forgets down to an ordinary smooth D_X-module (with the same underlying C_X^∞-module structure) that we denote M^{C^∞}. Moreover, there is a quasi-isomorphism of D-modules

$$\Omega_X^{d,d} \otimes^{\mathbb{L}}_{D_X} M^{C^\infty} \simeq \Omega_X^{d,hol} \otimes^{\mathbb{L}}_{D_X^{hol}} M[d].$$

To prove this we use the following "anti-holomorphic" Spenser resolution. First, notice that $\Omega_X^{d,d} = \Omega_X^{d,0} \otimes_{C_X^\infty} \Omega_X^{0,d}$. Next, consider the free resolution of $(0,d)$-forms as a...BW: finish

We apply this to the case $M = \mathcal{O}_{red}(J^{hol}V)$, where V is a holomorphic vector bundle. To complete the proof, we need to show that for any holomorphic vector bundle V that there is a quasi-isomorphism of D_X-modules between $\mathcal{O}_{red}(J\Omega^{0,*}(X,V))$ and $\mathcal{O}_{red}(J^{hol}V)$. For this, it suffices to show that the space of linear functionals are quasi-isomorphic. For any vector bundle E there always exists a (non-canonical) splitting $JE \cong \mathcal{E} \otimes_{C_X^\infty} J_X$, where $\mathcal{E}$ is the sheaf of sections and J_X is the sheaf of ∞-jets of the trivial bundle. Thus, we can assume that V is the trivial vector bundle, where the claim is now $(J\Omega^{0,*}(X))^\vee \simeq (J_X^{hol})^\vee$. Both sides are quasi-isomorphic to the smooth sections of the bundle of holomorphic differential operators D^{hol}, so we are done. □

Remark 2.18. There is an alternative formulation of classical field theory in terms of sheaves of L_∞ algebras, see Chapter BW: ref in [**?**]. Just as in the ordinary case we can formulate the data of a classical holomorphic theory in terms of sheaves of L_∞ algebras. We will not do that here, but hope the idea of how to do so is clear.

Remark 2.19. BW: relate to Si Li's, Dijkraaf notion of chiral theory.

3 One-loop regularization for theories on $\mathbb{C}^d$

In this section we study the renormalization of holomorphically field theories on $\mathbb{C}^d$ for general $d \geq 1$. We start with a classical holomorphic theory on $\mathbb{C}^d$ and consider its one-loop homotopy

renormalization group flow from some finite scale ϵ to scale L. Explicitly, this flow manifests as a sum over weights of graphs; that is, *Feynman diagrams*. In terms of diagrams we consider the sum over graphs of genus at most one where at each vertex we place the holomorphic interaction defining the classical theory. To obtain a quantization of a classical theory one must make sense of the $\epsilon \to 0$ limit of this construction. In general, this involves introducing a family of *counterterms*. The presence of counterterms can be a BW: ref. For instance, some theories of gravity require the introduction of infinitely many such counterterms. Our main result is that for a holomorphic theory on $\mathbb{C}^d$ no counterterms are required, and one obtains a well-defined $\epsilon \to 0$ limit.

The model for holomorphic theories on $\mathbb{C}^d$ we use follows our setup in the previous section. Suppose $(V, Q^{hol}, (-,-)_V)$ prescribes the data of a free holomorphic theory on $\mathbb{C}^d$. This means that V is a holomorphic bundle on $\mathbb{C}^d$, $Q^{hol} : \mathcal{V}^{hol} \to \mathcal{V}^{hol}$ is a holomorphic differential operator, and $(-,-)_V$ is a (shifted) $K_{\mathbb{C}^d}$-valued pairing on V. We assume, in addition, that Q^{hol} is translation invariant. Concretely, this means that

$$Q^{hol} \in \mathbb{C}\left[\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d}\right].$$

We can write the fields, in the BV formalism, as the following deformed Dolbeault complex

$$\mathcal{E}_V = \left(\Omega^{0,*}(\mathbb{C}^d, V), \bar{\partial} + Q^{hol}\right).$$

We will fix a trivialization for the holomorphic vector bundle $V = \mathbb{C}^d \times V_0$, where V_0 is the fiber over $0 \in \mathbb{C}^d$. This leads to an identification $\Omega^{0,*}(\mathbb{C}^d, V) = \Omega^{0,*}(\mathbb{C}^d) \otimes_{\mathbb{C}} V_0$. Further, we write the (-1)-shifted symplectic structure defining the classical BV theory in the form

$$\omega_V(\alpha \otimes v, \beta \otimes w) = (v, w)_{V_0} \int \mathrm{d}^d z (\alpha \wedge \beta)$$

where $(-,-)_{V_0}$ is a degree $(d-1)$-shifted pairing on the finite dimensional vector space V_0.

A holomorphic interacting theory is prescribed by a holomorphic Lagrangian $I^{hol} \in \mathcal{O}^{hol}_{\mathrm{loc}}(V)^+$. As we have seen in Section 2.1.2 any holomorphic Lagrangian determines a local functional on its Dolbeualt complex via integration $I = \int_X I^{\Omega^{0,*}}$. Here, as above, the notation $I^{\Omega^{0,*}}$ denotes the canonical extension of I^{hol} to the Dolbeualt complex for V. Using the trivialization $V = \mathbb{C}^d \times V_0$ and $\Omega^{d,hol} = \mathbb{C} \cdot \mathrm{d}^d z$, we can express the local functional as

$$I_k(\alpha) = \int I_k^{hol}(\alpha) = \int D_{k,1}(\phi_{k,1}(\alpha)) \cdots D_{k,k}(\phi_{k,k}(\alpha)) \mathrm{d}^d z$$

where each $D_{i,j}$ is a holomorphic differential operator $D_{i,j} \in \mathbb{C}\left[\frac{\partial}{\partial z_i}\right]$, and $\phi_{i,j} \in V_0^\vee$.

3.1 Homotopy RG flow

As we've already mentioned, the main goal of this section is to show that for holomorphic theories on $\mathbb{C}^d$ the one-loop renormalization group flow produces a prequantization modulo $\hbar^2$. Recall, a prequantization is a an effective family of functionals satisfying renormalization group flow

but not necessarily the quantum master equation. We study the consequences for solving the quantum master equation modulo $\hbar^2$ in the next section.

With the requisite notation set up in the previous section, we move towards the main calculation, which will amount to producing an explicit bound on certain one-loop Feynman diagrams. Before proceeding with the core analysis, we recall the definition of homotopy renormalization group flow, as defined in [**?**], which used to define the prequantization.

The building block in Costello's approach to renormalization is an effective family of functionals $\{I[L]\}$ parametrized by a *length scale* $L > 0$. For each $L > 0$ the functional $I[L] \in \mathcal{O}(\mathcal{E})[[\hbar]]$ must satisfy various conditions. The first of which is a compatibility between the functionals as one changes the length scale; this is referred to as *homotopy renormalization group (RG) flow*. The flow from scale L to L' is encoded by a linear map

$$W(P_{L<L'}, -) : \mathcal{O}(\mathcal{E})[[\hbar]] \to \mathcal{O}(\mathcal{E})[[\hbar]]$$

defined as a sum over weights of graphs $W(P_{L<L'}, I) = \sum_\Gamma W_\Gamma(P_{L<L'}, I)$. Here, Γ denotes a graph [2], and the weight W_Γ is defined as follows.

BW: finish this

The family of functionals $\{I[L]\}$ defining a quantization must satisfy the *RG flow equation*

$$I[L'] = W(P_{L<L'}, I[L])$$

for all $L < L'$. Given a classical interaction $I \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$, there is a natural way to attempt construct an effective family of functionals satisfying the RG flow equations. Indeed, it follows from elementary properties of the homotopy RG flow operator $W(P_{L<L'}, -)$ that *if* the functional

$$I[L] \ \text{“}=\text{”} \ W(P_{0<L}, I)$$

were to be well-defined for each $L > 0$, then the RG flow equations would automatically be satisfied for the collection $\{I[L]\}$. The problem is that this naive guess is ill-defined due to the distributional nature of the propagator $P_{0<L}$. The approach of Costello is to introduce a small parameter $\epsilon > 0$ and to consider the limit of the functionals $W(P_{\epsilon<L}, I)$ as $\epsilon \to 0$. For most theories, this $\epsilon \to 0$ limit is ill-defined, but there always exist ϵ-dependent *counterterms* $I^{CT}(\epsilon)$ rendering the existence of the $\epsilon \to 0$ limit of $W(P_{\epsilon<L}, I - I^{CT}(\epsilon))$.

Our main goal in this section amounts to showing that the naive $\epsilon \to 0$ limit exists without the necessity to introduce counterterms. This is a salient feature of holomorphic theories on $\mathbb{C}^d$ that we will take advantage of to characterize anomalies, for instance.

We will only consider quantizations defined modulo $\hbar^2$. In this case, the homotopy RG flow takes the explicit form:

$$W(P^V_{\epsilon<L}, I) = \sum_\Gamma \frac{\hbar^{g(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} W_\Gamma(P^V_{\epsilon<L}, I).$$

The sum is over graphs of genus ≤ 1 and W_Γ is the weight associated to the graph Γ.

We can now state the main result of this section.

[2] For our purposes, a graph...BW: finish

Proposition 3.1. *Let $\mathcal{E}$ be a holomorphic theory on $\mathbb{C}^d$ with classical interaction I^{cl}. Then, there exists a one-loop prequantization $\{I[L] \mid L > 0\}$ of I^{cl} involving no counterterms. That is, the $\epsilon \to 0$ limit of*

$$W(P_{\epsilon<L}, I) \mod \hbar^2 \in \mathcal{O}(\mathcal{E})[[\hbar]]/\hbar^2$$

exisits. Moreover, if I is holomorphically translation invariant we can pick the family $\{I[L]\}$ to be holomorphically translation invariant as well.

3.2 Holomorphic gauge fixing

We follow the mathematically precise approach to renormalization developed by Costello [**?**] which is largely motivated by the original length based regularization due to Wilson. BW: Give citation

In this formalism, to begin the process of renormalization we must first fix some auxiliary data, that of a *gauge fixing operator*. A gauge fixing operator is a square-zero operator on fields

$$Q^{GF} : \mathcal{E}_V \to \mathcal{E}_V[-1],$$

of cohomological degree -1 such that $[Q, Q^{GF}]$ is a generalized Laplacian on $\mathcal{E}$ where Q is the linearized BRST operator. For a complete definition see Section 8.2.1 of [**?**].

For holomorphic theories there is a convenient choice for a gauge fixing operator. To construct it we fix the standard flat metric on $\mathbb{C}^d$. Doing this, we let $\overline{\partial}^*$ be the adjoint of the operator $\overline{\partial}$. Using the coordinates on $(z_1, \ldots, z_d) \in \mathbb{C}^d$ we can write this operator as

$$\overline{\partial}^* = \sum_{i=1}^{d} \frac{\partial}{\partial(\mathrm{d}\overline{z}_i)} \frac{\partial}{\partial z_i}.$$

Equivalently $\frac{\partial}{\partial(\mathrm{d}\overline{z}_i)}$ is equal to contraction with the anti-holomorphic vector field $\frac{\partial}{\partial \overline{z}_i}$. The operator $\overline{\partial}^*$ extends to the complex of fields via the formula

$$Q^{GF} = \overline{\partial}^* \otimes \mathrm{id}_V : \Omega^{0,*}(X, V) \to \Omega^{0,*-1}(X, V),$$

Lemma 3.2. *The operator $Q^{GF} = \overline{\partial}^* \otimes \mathrm{id}_V$ is a gauge fixing operator for the free theory $(\mathcal{E}_V, \overline{\partial} + Q, \omega_V)$.*

Proof. Clearly, Q^{GF} is square zero since $(\overline{\partial}^*)^2 = 0$. Since Q^{hol} is a translation invariant holomorphic differential operator we have

$$[\overline{\partial} + Q^{hol}, Q^{GF}] = [\overline{\partial}, \overline{\partial}^*] \otimes \mathrm{id}_V.$$

The operator $[\overline{\partial}, \overline{\partial}^*]$ is simply the Dolbeault Laplacian on $\mathbb{C}^d$, which in coordinates is

$$[\overline{\partial}, \overline{\partial}^*] = -\sum_{i=1}^{d} \frac{\partial}{\partial \overline{z}_i} \frac{\partial}{\partial z_i}.$$

In particular, the operator $[\overline{\partial}, \overline{\partial}^*] \otimes \mathrm{id}_V$ is a generalized Laplacian.

Finally, we must show that Q^{GF} is (graded) self-adjoint for the shifted symplectic pairing ω_V. This follows from the fact about Dolbeualt forms on $\mathbb{C}^d$. If $\alpha, \beta \in \Omega_c^{0,*}(\mathbb{C}^d)$ then

$$\int_{\mathbb{C}^d} (\bar{\partial}^* \alpha) \wedge \beta \wedge \mathrm{d}^d z = \pm \int_{\mathbb{C}^d} \alpha \wedge (\bar{\partial}^* \beta) \wedge \mathrm{d}^d z.$$

□

3.3 The propagator on $\mathbb{C}^d$

The gauge fixing operator determines a generalized Laplaican, which for us is essentially the ordinary Hodge Laplacian on $\mathbb{C}^d$. Our regularization scheme utilizes the heat kernel associated to the Laplacian, for which we recall the explicit form below.

By definition, the scale $L > 0$ heat kernel is a symmetric element $K_L^V \in \mathcal{E}_V(\mathbb{C}^d) \otimes \mathcal{E}_V(\mathbb{C}^d)$ that satisfies

$$\omega_V(K_L, \varphi) = e^{-L[Q, Q^{GF}]}\varphi$$

for any field $\varphi \in \mathcal{E}_V$. For a more detailed definition see Section 8.2.3 in **??**.

The tensor square of $\mathcal{E}_V(\mathbb{C}^d)$ decomposes as

$$\mathcal{E}_V(\mathbb{C}^d) \otimes \mathcal{E}_V(\mathbb{C}^d) = \left(\Omega^{0,*}(\mathbb{C}^d) \otimes \Omega^{0,*}(\mathbb{C}^d)\right) \otimes (V_0 \otimes V_0). \tag{5}$$

We will decompose the heat kernel accordingly.

Pick a basis $\{e_i\}$ of V_0 and let

$$\mathbf{C}_{V_0} = \sum_{i,j} \omega_{ij}(e_i \otimes e_j) \in V_0 \otimes V_0$$

be the quadratic Casimir. Here, (ω_{ij}) is the inverse matrix to the pairing $(-,-)_{V_0}$.

The heat kernel splits with respect to the decomposition in Equation 5 as

$$K_L^V(z,w) = K_L^{an}(z,w) \cdot \mathbf{C}_{V_0}$$

where the analytic part is independent of V and equal to

$$K_L^{an}(z,w) = \frac{1}{(4\pi L)^d} e^{-|z-w|^2/4L} \prod_{i=1}^d (\mathrm{d}\bar{z}_i - \mathrm{d}\bar{z}_j) \in \Omega^{0,*}(\mathbb{C}^d) \otimes \Omega^{0,*}(\mathbb{C}^d) \cong \Omega^{0,*}(\mathbb{C}^d \times \mathbb{C}^d).$$

The propagator for the holomorphic theory $\mathcal{E}_V$ is defined by

$$P_{\epsilon<L}^V(z,w) = \int_{t=\epsilon}^L \mathrm{d}t (Q^{GF} \otimes 1) K_L^V(z,w).$$

Since the element $\mathbf{C}_{V_0}$ is independent of the coordinate on $\mathbb{C}^d$, the propagator simplifies as $P_{\epsilon<L}^V(z,w) = P_{\epsilon<L}^{an}(z,w) \cdot \mathbf{C}_{V_0}$ where

$$\begin{aligned} P_{\epsilon<L}^{an}(z,w) &= \int_{t=\epsilon}^L \mathrm{d}t (\bar{\partial}^* \otimes 1) K_L^V(z,w) \\ &= \int_{t=\epsilon}^L \mathrm{d}t \frac{1}{(4\pi t)^d} \sum_{j=1}^d \left(\frac{z_j - w_j}{4t}\right) e^{-|z-w|^2/4t} \prod_{i \neq j}^d (\mathrm{d}\bar{z}_i - \mathrm{d}\bar{z}_j). \end{aligned}$$

3.4 Trees

For the genus zero graphs, or trees, we do not have any analytic difficulties to worry about. The propagator $P^V_{\epsilon<L}$ is smooth so long as $\epsilon, L > 0$ but when $\epsilon \to 0$ it inherits a singularity along the diagonal $z = w$. This is what contributes to the divergences in the naive definition of RG flow $W(P_{0<L}, -)$. But, if Γ is a tree the weight $W_\Gamma(P^V_{0<L}, I)$ only involves multiplication of distributions with transverse singular support, so is well-defined. Thus we have observed the following.

Lemma 3.3. *If Γ is a tree then $\lim_{\epsilon\to 0} W_\Gamma(P_{\epsilon<L}, I)$ exists.*

The only possible divergences in the $\epsilon \to 0$ limit, then, must come from graphs of genus one, which we now direct our attention to.

3.5 A simplification for one-loop weights

Every graph of genus one is a wheel with some trees protruding from the external edges of the tree. Thus, we can write the weight of a genus one graph as a product of weights associated to trees times the weight associated to a wheel. We have just observed that the weights associated to trees are automatically convergent in the $\epsilon \to 0$ limit, thus it suffices to focus on genus one graphs that are purely wheels with some number of external edges.

The definition of the weight of the wheel involves placing the propagator at each internal edge and the interaction I at each vertex. The weights are evaluated by placing compactly supported fields $\varphi \in \mathcal{E}_{V,c} = \Omega^{0,*}_c(\mathbb{C}^d, V)$ at each of the external edges. We will make two simplifications:

1. the only ϵ dependence appears in the analytic part of the propagator $P^{an}_{\epsilon<L}$, so we can forget about the combinatorial factor $\mathbf{C}_{V_0}$ and assume all external edges are labeled by compactly supported Dolbeault forms in $\Omega^{0,*}_c(\mathbb{C}^d)$;

2. each vertex labeled by I is a sum of interactions of the form

$$\int_{\mathbb{C}^d} D_1(\varphi) \cdots D_k(\varphi) \mathrm{d}^d z$$

   where D_i is a holomorphic differential operator (only involves $\frac{\partial}{\partial z_i}$-derivatives). Some of the differential operators will hit the compactly supported Dolbeault forms placed on the external edges of the graph. The remaining operators will hit the internal edges labeled by the propagators. Since a holomorphic differential operator preserves the space of compactly supported Dolbeault forms that is independent of ϵ, we replace each input by an arbitrary compactly supported Dolbeault form.

Thus, for the $\epsilon \to 0$ behavior it suffices to look at weights of wheels with arbitrary compactly supported functions as inputs where each of the internal edges are labeled by some translation invariant holomorphic differential operator

$$D = \sum_{n_1,\ldots n_d} \frac{\partial^{n_1}}{\partial z_1^{n_1}} \cdots \frac{\partial^{n_d}}{\partial z_d^{n_d}}$$

applied to the propagator $P^{an}_{\epsilon<L}$. This motivates the following definition.

Definition 3.4. Let $\epsilon, L > 0$. In addition, fix the following data.

(a) An integer $k \geq 1$ that will be the number of vertices of the graph.

(b) For each $\alpha = 1, \ldots, k$ a sequence of integers

$$\vec{n}^\alpha = (n_1^\alpha, \ldots, n_d^\alpha).$$

We denote by $(\vec{n}) = (n_i^j)$ the corresponding $d \times k$ matrix of integers.

The analytic weight associated to the pair $(k, (\vec{n}))$ is the smooth distribution

$$W_{\epsilon<L}^{k,(n)} : C_c^\infty((\mathbb{C}^d)^k) \to \mathbb{C},$$

that sends a smooth compactly supported function $\Phi \in C_c^\infty((\mathbb{C}^d)^k) = C_c^\infty(\mathbb{C}^{dk})$ to

$$W_{\epsilon<L}^{k,(n)}(\Phi) = \int_{(z^1,\ldots,z^k)\in(\mathbb{C}^d)^k} \prod_{\alpha=1}^k \mathrm{d}^d z^\alpha \Phi(z^1, \ldots, z^k) \prod_{\alpha=1}^k \left(\frac{\partial}{\partial z^\alpha}\right)^{\vec{n}^\alpha} P_{\epsilon<L}^{an}(z^\alpha, z^{\alpha+1}). \tag{6}$$

In the above expression, we use the convention that $z^{k+1} = z^1$.

The coordinate on $(\mathbb{C}^d)^k$ is given by $\{z_i^\alpha\}$ where $\alpha = 1, \ldots, k$ and $i = 1, \ldots, d$. For each α, $\{z_1^\alpha, \ldots, z_d^\alpha\}$ is the coordinate for the space $\mathbb{C}^d$ sitting at the vertex labeled by α. We have also used the shorthand notation

$$\left(\frac{\partial}{\partial z^\alpha}\right)^{\vec{n}^\alpha} = \frac{\partial^{n_1^\alpha}}{\partial z_1^\alpha} \cdots \frac{\partial^{n_d^\alpha}}{\partial z_d^\alpha}.$$

We will refer to the collection of data $(k, (\vec{n}))$ in the definition as *wheel data*. The motivation for this is that the weight $W_{\epsilon<L}^{k,(n)}$ is the analytic part of the full weight $W_\Gamma(P_{\epsilon<L}^V, I)$ where Γ is a wheel with k vertices.

We have reduced the proof of Proposition 3.1 to showing that the $\epsilon \to 0$ limit of the analytic weight $W_{\epsilon<L}^{k,(\vec{n})}(\Phi)$ exists for any choice of wheel data $(k, (\vec{n}))$. To do this, there are two steps. First, we show a vanishing result that says when $k \geq d$ the weights vanish for purely algebraic reasons. The second part is the most technical aspect of the chapter where we show that for $k > d$ the weights have nice asymptotic behavior as a function of ϵ.

Lemma 3.5. *Let $(k, (\vec{n}))$ be a pair of wheel data. If the number of vertices k satisfies $k \leq d$ then*

$$W_{\epsilon<L}^{k,(n)} = 0$$

as a distribution on $\mathbb{C}^{dk}$ for any $\epsilon, L > 0$.

Proof. In the integral expression for the weight (6) there is the following factor involving the product over the edges of the propagators:

$$\prod_{\alpha=1}^k \left(\frac{\partial}{\partial z^\alpha}\right)^{\vec{n}^\alpha} P_{\epsilon<L}^{an}(z^\alpha, z^\alpha). \tag{7}$$

We will show that this expression is identically zero. To simplify the expression we first make the following change of coordinates on $\mathbb{C}^{dk}$:

$$w^\alpha = z^{\alpha+1} - z^\alpha \quad , \quad 1 \leq \alpha < k \tag{8}$$

$$w^k = z^k. \tag{9}$$

Introduce the following operators

$$\eta^\alpha = \sum_{i=1}^{d} \overline{w}_i^\alpha \frac{\partial}{\partial(\mathrm{d}\overline{w}_i^\alpha)}$$

acting on differential forms on $\mathbb{C}^{dk}$. The operator η^α lowers the anti-holomorphic Dolbuealt type by one : $\eta : (p,q) \to (p,q-1)$. Equivalently, η^α is contraction with the anti-holomorphic Euler vector field $\overline{w}_i^\alpha \partial/\partial\overline{w}_i^\alpha$.

Once we do this, we see that the expression (7) can be written as

$$\left(\left(\sum_{\alpha=1}^{k-1} \eta^\alpha\right) \prod_{i=1}^{d} \left(\sum_{\alpha=1}^{k-1} \mathrm{d}\overline{w}_i^\alpha\right)\right) \prod_{\alpha=1}^{k-1} \left(\eta^\alpha \prod_{i=1}^{d} \mathrm{d}\overline{w}_i^\alpha\right).$$

Note that only the variables $\overline{w}_i^\alpha$ for $i = 1,\ldots,d$ and $\alpha = 1,\ldots,k-1$ appear. Thus we can consider it as a form on $\mathbb{C}^{d(k-1)}$. As such a form it is of Dolbeault type $(0,(d-1)+(k-1)(d-1)) = (0,(d-1)k)$. If $k < d$ then clearly $(d-1)k > d(k-1)$ so the form has greater degree than the dimension of the manifold and hence it vanishes.

The case left to consider is when $k = d$. In this case, the expression in (7) can be written as

$$\left(\left(\sum_{\alpha=1}^{d-1} \eta^\alpha\right) \prod_{i=1}^{d} \left(\sum_{\alpha=1}^{d-1} \mathrm{d}\overline{w}_i^\alpha\right)\right) \prod_{\alpha=1}^{d-1} \left(\eta^\alpha \prod_{i=1}^{d} \mathrm{d}\overline{w}_i^\alpha\right). \tag{10}$$

Again, since only the variables $\overline{w}_i^\alpha$ for $i = 1,\ldots,d$ and $\alpha = 1,\ldots,d-1$ appear, we can view this as a differential form on $\mathbb{C}^{d(d-1)}$. Furthermore, it is a form of type $(0,d(d-1))$. For any vector field X on $\mathbb{C}^{d(d-1)}$ the interior derivative i_X is a graded derivation. Suppose ω_1,ω_2 are two $(0,*)$ forms on $\mathbb{C}^{d(d-1)}$ such that the sum of their degrees is equal to d^2. Then, $\omega_1 \iota_X \omega_2$ is a top form for any vector field on $\mathbb{C}^{d(d-1)}$. Since $\omega_1\omega_2 = 0$ for form type reasons, we conclude that $\omega_1 \iota_X \omega_2 = \pm(i_X\omega_1)\omega_2$ with sign depending on the dimension d. Applied to the vector field $\overline{z}_i^1 \partial/\partial\overline{w}_i^1$ in (10) we see that the expression can be written (up to a sign) as

$$\eta^1 \left(\sum_{\alpha=1}^{d-1} \eta^\alpha \prod_{i=1}^{d} \left(\sum_{\alpha=1}^{d-1} \mathrm{d}\overline{w}_i^\alpha\right)\right) \left(\prod_{i=1}^{d} \mathrm{d}\overline{w}_i^1\right) \prod_{\alpha=2}^{d-1} \left(\eta^\alpha \prod_{i=1}^{d} \mathrm{d}\overline{w}_i^\alpha\right).$$

Repeating this, for $\alpha = 2,\ldots,k-1$ we can write this expression (up to a sign) as

$$\left(\eta_{k-1}\cdots\eta_2\eta_1 \sum_{\alpha=1}^{k-1} \eta^\alpha \prod_{i=1}^{d} \left(\sum_{\alpha=1}^{k-1} \mathrm{d}\overline{w}_i^\alpha\right)\right) \prod_{\alpha=1}^{k-1}\prod_{i=1}^{d} \mathrm{d}\overline{w}_i^\alpha$$

The expression inside the parentheses is zero since each term in the sum over α involves a term like $\eta^\beta\eta^\beta = 0$. This completes the proof for $k = d$. □

Lemma 3.6. *Let $(k,(\vec{n}))$ be a pair of wheel data such that $k > d$. Then the $\epsilon \to 0$ limit of the analytic weight*

$$\lim_{\epsilon\to 0} W_{\epsilon<L}^{k,(n)}$$

exists as a distribution on $\mathbb{C}^{dk}$.

Proof. We will bound the absolute value of the weight in Equation (6) and show that it has a well-defined $\epsilon \to 0$ limit. First, consider the change of coordinates as in Equations (8),(9). For any compactly supported function Φ we see that $W^{k,(n)}_{\epsilon<L}(\Phi)$ has the form

$$\int_{w^k\in\mathbb{C}^d} \mathrm{d}^d w^k \int_{(w_1,\dots,w_{k-1})\in(\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} \mathrm{d}^d w^\alpha\right) \Phi(w^1,\dots,w^k) \left(\prod_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial w^\alpha}\right)^{\vec{n}^\alpha} P^{an}_{\epsilon<L}(w^\alpha)\right) \sum_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial w^\alpha}\right)^{\vec{n}^k} P^{an}_{\epsilon<L}\left(\sum_{\alpha=1}^{k-1} w^\alpha\right). \tag{11}$$

For $\alpha = 1,\dots,k-1$ the notation $P^{an}_{\epsilon<L}(w^\alpha)$ makes sense since $P^{an}_{\epsilon<L}(z^\alpha, z^{\alpha+1})$ is only a function of $w^\alpha = z^{\alpha+1} - z^\alpha$. Similarly $P^{an}_{\epsilon<L}(z^{k+1}, z^1)$ is a function of

$$z^k - z^1 = \sum_{\alpha=1}^{k-1} w^\alpha.$$

Expanding out the propagators the weight takes the form

$$\begin{aligned}
&\int_{w^k\in\mathbb{C}^d} \mathrm{d}^{2d} w^k \int_{(w_1,\dots,w_{k-1})\in(\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} \mathrm{d}^{2d} w^\alpha\right) \Phi(w^1,\dots,w^k) \int_{(t_1,\dots,t_k)\in[\epsilon,L]^k} \prod_{\alpha=1}^{k} \frac{\mathrm{d}t_\alpha}{(4\pi t_\alpha)^d} \\
&\times \sum_{i_1,\dots,i_{k-1}=1}^{d} \epsilon_{i_1\cdots,i_k} \left(\frac{\overline{w}^1_{i_1}}{4t_1} \frac{(\overline{w}^1)^{n^1}}{4t^{|n^1|}}\right) \cdots \left(\frac{\overline{w}^{k-1}_{i_{k-1}}}{4t_{k-1}} \frac{(\overline{w}^{k-1})^{n^{k-1}}}{4t^{|n^{k-1}|}}\right) \left(\sum_{\alpha=1}^{k-1} \frac{\overline{w}^\alpha_{i_k}}{4t_k} \cdot \frac{1}{t^{|n^k|}} \left(\sum_{\alpha=1}^{k-1} \overline{w}^\alpha\right)^{n^k}\right) \\
&\times \exp\left(-\sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{4t_\alpha} - \frac{1}{4t_k}\left|\sum_{\alpha=1}^{k-1} w^\alpha\right|^2\right)
\end{aligned}$$

The notation used above warrants some explanation. Recall, for each α the vector of integers is defined as $n^\alpha = (n^\alpha_1,\dots,n^\alpha_d)$. We use the notation

$$(\overline{w}^\alpha)^{n^\alpha} = \overline{w}_1^{n^\alpha_1} \cdots \overline{w}_d^{n^\alpha_d}.$$

Furthermore, $|n^\alpha| = n^\alpha_1 + \cdots + n^\alpha_d$. Each factor of the form $\frac{\overline{w}^\alpha_{i_\alpha}}{t_\alpha}$ comes from the application of the operator $\frac{\partial}{\partial z_i}$ in $\overline{\partial}^*$ applied to the propagator. The factor $\frac{(\overline{w}^\alpha)^{n^\alpha}}{t^{|n^\alpha|}}$ comes from applying the operator $\left(\frac{\partial}{\partial w}\right)^{n^\alpha}$ to the propagator. Note that $\overline{\partial}^*$ commutes with any translation invariant holomorphic differential operator, so it doesn't matter which order we do this.

To bound this integral we will recognize each of the factors

$$\frac{\overline{w}^\alpha_{i_\alpha}}{4t_\alpha} \frac{(\overline{w}^\alpha)^{n^\alpha}}{4t^{|n^\alpha|}}$$

as coming from the application of a certain holomorphic differential operator to the exponential in the last line. We will then integrate by parts to obtain a simple Gaussian integral which will give us the necessary bounds in the t-variables. Let us denote this Gaussian factor by

$$E(w,t) := \exp\left(-\sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{4t_\alpha} - \frac{1}{4t_k}\left|\sum_{\alpha=1}^{k-1} w^\alpha\right|^2\right)$$

For each α, i_α introduce the $t = (t_1,\dots,t_k)$-dependent holomorphic differential operator

$$D_{\alpha,i_\alpha}(t) := \left(\frac{\partial}{\partial w^\alpha_{i_\alpha}} - \sum_{\beta=1}^{k-1} \frac{t_\beta}{t_1+\cdots+t_k} \frac{\partial}{\partial w^\beta_{i_\alpha}}\right) \prod_{j=1}^{d} \left(\frac{\partial}{\partial w^\alpha_j} - \sum_{\beta=1}^{k-1} \frac{t_\beta}{t_1+\cdots+t_k} \frac{\partial}{\partial w^\beta_j}\right)^{n^\alpha_j}.$$

The following lemma is an immediate calculation

Lemma 3.7. *One has*

$$D_{\alpha,i_\alpha} E(w,t) = \frac{\overline{w}^\alpha_{i_\alpha}}{4t_\alpha} \frac{(\overline{w}^\alpha)^{n^\alpha}}{t^{|n^\alpha|}} E(w,t).$$

Note that all of the D_{α,i_α} operators mutually commute. Thus, we can integrate by parts iteratively to obtain the following expression for the weight:

$$\pm \int_{w^k \in \mathbb{C}^d} \mathrm{d}^{2d} w^k \int_{(w_1,\dots,w_{k-1}) \in (\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1} \mathrm{d}^{2d} w^\alpha \right) \int_{(t_1,\dots,t_k)\in[\epsilon,L]^k} \prod_{\alpha=1}^{k} \frac{\mathrm{d}t_\alpha}{(4\pi t_\alpha)^d}$$
$$\times \left(\sum_{i_1,\dots,i_k} \epsilon_{i_1\cdots,i_d} D_{1,i_1} \cdots D_{k-1,i_{k-1}} \sum_{\alpha=1}^{k-1} D_{\alpha,i_k} \Phi(w^1,\dots,w^k) \right) \times \exp\left(-\sum_{\alpha=1}^{k-1} \frac{|w^\alpha|^2}{t_\alpha} - \frac{1}{t_k} \left| \sum_{\alpha=1}^{k-1} w^\alpha \right|^2 \right).$$

Thus, the absolute value of the weight is bounded by

$$|W^{k,(n)}_{\epsilon<L}(\Phi)| \le C \int_{w^k\in\mathbb{C}^d} \mathrm{d}^{2d} w^k \int_{(w^1,\dots,w^{k-1}} \prod_{\alpha=1}^{k-1} \mathrm{d}^{2d} w^\alpha \Psi(w^1,\dots,w^{k-1},w^k) \int_{(t_1,\dots,t_k)\in[\epsilon,L]^k} \mathrm{d}t_1 \dots \mathrm{d}t_k \frac{1}{(4\pi)^{dk}} \frac{1}{t_1^d \cdots t_k^d} \times E(w,t) \tag{12}$$

where Ψ is some compactly supported functnio on $\mathbb{C}^{dk}$ that is independent of t.

To compute the right hand side we will perform a Gaussian integration with respect to the variables $(w^1,\dots,w^{k-1})$. To this end, notice that the exponential can be written as

$$E(w,t) = \exp\left(-\frac{1}{4} M_{\alpha\beta}(w^\alpha, w^\beta) \right)$$

where $(M_{\alpha\beta})$ is the $(k-1)\times(k-1)$ matrix given by

$$\begin{pmatrix} a_1 & b & b & \cdots & b \\ b & a_2 & b & \cdots & b \\ b & b & a_3 & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a_{k-1} \end{pmatrix}$$

where $a_\alpha = t_\alpha^{-1} + t_k^{-1}$ and $b = t_k^{-1}$. The pairing (w^α, w^β) is the usual Hermitian pairing on $\mathbb{C}^d$, $(w^\alpha, w^\beta) = \sum_i w_i^\alpha \overline{w}_i^\beta$. After some straightforward linear algebra we find that

$$\det(M_{\alpha\beta})^{-1} = \frac{t_1 \cdots t_k}{t_1 + \cdots + t_k}.$$

We now perform a Wick expansion for the Gaussian integral in the variables $(w^1,\dots,w^{k-1})$. For a reference similar to the notation used here see the Appendix of our work in [**?**]. The inequality in (12) becomes

$$|W^{k,(n)}_{\epsilon<L}(\Phi)| \le C' \int_{w^k\in\mathbb{C}^d} \mathrm{d}^{2d} w^k \Psi(0,\dots,0,w^k) \int_{(t_1,\dots,t_k)\in[\epsilon,L]^k} \mathrm{d}t_1 \dots \mathrm{d}t_k \frac{1}{(4\pi)^{dk}} \frac{1}{(t_1\cdots t_k)^d} \left(\frac{t_1\cdots t_k}{t_1+\cdots+t_k} \right)^d + O(\epsilon) \tag{13}$$

$$= C' \int_{w^k\in\mathbb{C}^d} \mathrm{d}^{2d} w^k \Psi(0,\dots,0,w^k) \int_{(t_1,\dots,t_k)\in[\epsilon,L]^k} \mathrm{d}t_1 \dots \mathrm{d}t_k \frac{1}{(4\pi)^{dk}} \frac{1}{(t_1+\cdots+t_k)^d} + O(\epsilon). \tag{14}$$

The first term in the Wick expansion is written out explicitly. The $O(\epsilon)$ refers to higher terms in the Wick expansion, which one can show all have order ϵ, so disappear in the $\epsilon \to 0$ limit. The expression $\Psi(0, \ldots, 0, w^k)$ means that we have evaluate the function $\Psi(w^1, \ldots, w^k)$ at $w^1 = \ldots = w^{k-1} = 0$ leaving it as a function only of w^k. In the original coordinates this is equivalent to setting $z^1 = \cdots = z^{k-1} = z^k$.

Our goal is to show that $\epsilon \to 0$ limit of the right-hand side exists. The only ϵ dependence on the right hand side of (13) is in the integral over the regulation parameters $t_1, \ldots, t_k$. Thus, it suffices to show that the $\epsilon \to 0$ limit of

$$\int_{(t_1,\ldots,t_k)\in[\epsilon,L]^k} \frac{\mathrm{d}t_1 \ldots \mathrm{d}t_k}{(t_1 + \cdots + t_k)^d}$$

exists. By the AM/GM inequality we have $(t_1 + \cdots + t_k)^d \geq (t_1 \cdots t_d)^{d/k}$. So, the integral is bounded by

$$\int_{(t_1,\ldots,t_k)\in[\epsilon,L]^k} \frac{\mathrm{d}t_1 \ldots \mathrm{d}t_k}{(t_1 + \cdots + t_k)^d} \leq \int_{(t_1,\ldots,t_k)\in[\epsilon,L]^k} \frac{\mathrm{d}t_1 \ldots \mathrm{d}t_k}{(t_1 \cdots t_k)^{d/k}} = \frac{1}{(1-d/k)^k} \left(\epsilon^{1-d/k} - L^{1-d/k}\right)^k.$$

By assumption, $d < k$, so the right hand side has a well-defined $\epsilon \to 0$ limit. This concludes the proof.

□

4 Chiral anomalies in arbitrary dimensions

Renormalization is an important step in constructing a quantum field theory. In the context of gauge theory, however, a consistent quantization requires that this renormalization behaves appropriately with respect to gauge symmetries present in the classical theory. This formalism for studying quantizations of gauge theories is due to Batalin-Vilkovisky [**?**], and has been made mathematically rigorous in the work of Costello [**?**]. The precise consistency of gauge symmetry with renormalization is encoded by the *quantum master equation*. Heuristically, one can think of the quantum master equation as a closedness condition on the path integral measure defined by the quantum action functional.

The key idea is the following: once a classical theory has been renormalized, so that we have a $\hbar$-linear effective family of functionals $\{I[L]\}$ whose $\hbar = 0$ limit is the classical action, the next step to constructing a quantization is to solve the quantum master equation (QME) for each functional $I[L]$. (In fact, once the QME holds at a single positive length $L > 0$ it holds for every other length by RG flow.) Often, the QME is not satisfied by the functional $I[L]$, but there exists a "correction" to $I[L]$ that does satisfy the QME. On the other hand, there may be unavoidable obstructions to solving this quantum master equation. These are known as *anomalies* in the physics literature. Since our method for solving the QME is deformation-theoretic in nature, these anomalies appear as cohomology classes in the cochain complex of local functionals.

In general, it is difficult to characterize such anomalies, but in the case of holomorphic theories on $\mathbb{C}^d$ our result of one-loop finiteness from the previous section makes this problem much more

tractable. Indeed, since there are no counterterms required, we can plug in the RG flow of the classical action functional and study the quantum master equation directly. As is usual in perturbation theory, one works order by order in $\hbar$ to construct a quantization. However, in this section we continue to work linearly in $\hbar$, which is to say we study solutions to the quantum master equation modulo $\hbar^2$.

4.1 The quantum master equation

The regularized quantum master equation at scale L is

$$QI[L] + \hbar\Delta_L I[L] + \frac{1}{2}\{I[L], I[L]\}_L = 0.$$

This is equivalent to the equation $(Q + \hbar\Delta_L)e^{I[L]/\hbar} = 0$. Therefore, the *obstruction* to satisfying the quantum master equation modulo $\hbar^2$ at scale L is

$$\Theta[L] = \hbar^{-1}\left(QI[L] + \hbar\Delta_L I[L] + \frac{1}{2}\{I[L], I[L]\}_L\right),$$

or, equivalently $\Theta[L] = e^{-I[L]/\hbar}(Q + \hbar\Delta_L)e^{I[L]/\hbar}$.

4.2

We return to the holomorphic setting. Fix a holomorphic theory $\mathcal{E}_V$ on $\mathbb{C}^d$ with classical action I. The linearized BRST operator is of the form $Q = \overline{\partial} + Q^{hol}$ where Q^{hol} is a holomorphic differential operator. Let $I[L] = W(P_{\epsilon<L}, I) \mod \hbar^2$ to be the one-loop renormalization group flow using the propagator defined in Section **??**

Before we begin with the main calculation of this section, we state the following lemma, which is a simplification of the QME given our assumptions.

Lemma 4.1. *Let $\Theta[L]$ be the one-loop obstruction to the QME. Then, one has*

$$\Theta[L] = \lim_{\epsilon\to 0} W(P_{\epsilon<L}, Q^{hol}I) + \frac{1}{2}\lim_{\epsilon\to 0} e^{-I/\hbar}e^{\hbar\partial_{P_{\epsilon<L}}}\left(\{I, I\}_\epsilon e^{I/\hbar}\right) \mod \hbar^2.$$

Proof. Formally, the RG flow can equivalently be written as $e^{I[L]/\hbar} = \lim_{\epsilon\to 0} e^{\hbar\partial_{P_{\epsilon<L}}}e^{I/\hbar} \mod \hbar^2$. Thus, we can write

$$(Q + \hbar\Delta_L)e^{I[L]/\hbar} = \lim_{\epsilon\to 0}(Q + \hbar\Delta_L)\left(e^{\hbar\partial_{P_{\epsilon<L}}}e^{I/\hbar}\right).$$

The left-hand side is related

The operator Q commutes with $e^{\hbar\partial_{P_{\epsilon<L}}}$. Moreover, one has $\Delta_L e^{\hbar\partial_{P_{\epsilon<L}}} = e^{\hbar\partial_{P_{\epsilon<L}}}\Delta_\epsilon$ as operators acting on functionals. Thus,

$$\lim_{\epsilon\to 0}(Q + \hbar\Delta_L)\left(e^{\hbar\partial_{P_{\epsilon<L}}}e^{I/\hbar}\right) = \lim_{\epsilon\to 0} e^{\hbar\partial_{P_{\epsilon<L}}}(Q + \hbar\Delta_\epsilon)e^{I/\hbar}.$$

Since Δ_ϵ is a BV operator for the bracket $\{-,-\}_\epsilon$, we can rewrite the right-hand side as

$$\frac{1}{\hbar}\lim_{\epsilon\to 0} e^{\hbar P_{\epsilon<L}}(QI + \hbar\Delta_\epsilon I + \frac{1}{2}\{I, I\}_\epsilon)e^{I/\hbar}.$$

For every $\epsilon > 0$ we have $\Delta_\epsilon I = 0$. Moreover, since I is holomorphic (and since $Q^{hol} = 0$) we have $QI = \bar{\partial} I = 0$.

We conclude that the one-loop anomaly is

$$\Theta[L] = \frac{1}{2} \lim_{\epsilon \to 0} e^{-I/\hbar} e^{\hbar \partial_{P_{\epsilon<L}}} \left(\{I, I\}_\epsilon e^{I/\hbar} \right) \mod \hbar^2$$

as desired. □

As we saw above, the anomaly $\Theta[L]$ has a well-defined $L \to 0$ limit as a local functional and it is closed for the classical differential. The main result of this section is the following.

Proposition 4.2. *The obstruction $\Theta = \lim_{L \to 0} \Theta[L]$ to satisfying the one-loop quantum master equation is given by the expression*

$$\Theta = \lim_{L \to 0} \lim_{\epsilon \to 0} W(P_{\epsilon<L}, Q^{hol} I) + \frac{1}{2} \lim_{L \to 0} \lim_{\epsilon \to 0} \sum_{\Gamma \in \mathrm{Wheel}_{d+1}} W_\Gamma(P_{\epsilon<L}, K_\epsilon, I)$$

where the sum is over all wheels with $(d+1)$-vertices. In particular, when $Q^{hol} = 0$ (so that the first term vanishes), the anomaly is expressed as the sum over wheels with exactly $(d+1)$-vertices.

Proof. Like the proof of the non-existence of counterterms for holomorphic theories, the proof of this result will be the consequence of an explicit calculations and bounds of certain Feynman diagrams.

Consider the quantity

$$\lim_{\epsilon \to 0} e^{-I/\hbar} e^{\hbar \partial_{P_{\epsilon<L}}} \left(\{I, I\}_\epsilon e^{I/\hbar} \right) \mod \hbar^2 \tag{15}$$

which can be thought of as a sum over graphs. By assumption, we are only looking at graphs of genus one which look like wheels with possible trees attach. Graphically, the quantity $\{I, I\}_\epsilon$ is the graph of two vertices with a separating edge labeled by the heat kernel K_ϵ. Thus, all weights appearing in the expansion of (15) attach the propagator $P_{\epsilon<L}$ to all edges besides a single distinguished edge e, which is labeled by K_ϵ. Thus, as a over a sum of graphs, we see that the following two types of weights occur in the expansion of (15).

(a) the distinguished edge e is separating;

(b) the distinguished edge e is *not* separating, and so appears as the internal edge of the wheel portion of the graph.

By the classical master equation, we see that the $\epsilon \to 0$ limit of weights of Type (a) go to zero. Thus, we must only consider the weights of Type (b).

The result will follow from two steps. These should seem familiar from the proof of the main result about the existence of no counterterms.

1. If Γ is a wheel with $k < d+1$ vertices, then $W_\Gamma(P_{\epsilon<L}, K_\epsilon, I) = 0$ identically.
2. If Γ is a wheel with $k > d+1$ vertices, then $\lim_{\epsilon \to 0} W_\Gamma(P_{\epsilon<L}, K_\epsilon, I) = 0$.

The proof of both of these facts is only dependent on the analytic part of the weights. Thus, it suffices to make the same reduction as we did in the previous section. To extract that analytic part of the graph we proceed as in Definition 3.4. If $(k, (\vec{n}))$ is a pair of wheel data (recall k labels the number of vertices and $\vec{n}$ labels the derivatives at each vertex) define the smooth distribution

$$\widetilde{W}^{k,(n)}_{\epsilon<L} : C^\infty_c((\mathbb{C}^d)^k) \to \mathbb{C},$$

that sends a smooth compactly supported function $\Phi \in C^\infty_c((\mathbb{C}^d)^k) = C^\infty_c(\mathbb{C}^{dk})$ to

$$\widetilde{W}^{k,(n)}_{\epsilon<L}(\Phi) = \int_{(z^1,\ldots,z^k)\in(\mathbb{C}^d)^k} \prod_{\alpha=1}^{k} \mathrm{d}^d z^\alpha \Phi(z^1,\ldots,z^\alpha) \left(\frac{\partial}{\partial z^k}\right)^{\vec{n}^k} K_\epsilon(z^1,z^k) \prod_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial z^\alpha}\right)^{\vec{n}^\alpha} P^{an}_{\epsilon<L}(z^\alpha, z^{\alpha+1}). \tag{16}$$

Item (1) follows from the following observation.

Lemma 4.3. *Let $(k, (\vec{n}))$ be a pair of wheel data. If the number of vertices k satisfies $k \leq d$ then*

$$\widetilde{W}^{k,(n)}_{\epsilon<L} = 0$$

as a distribution on $\mathbb{C}^{dk}$ for any $\epsilon, L > 0$.

Proof. In fact, the integrand of (16) is identically zero provided $k \leq d$ by a simple observation of the differential form type. Consider the factor in the integrand of $\widetilde{W}^{k,(n)}_{\epsilon<L}$ given by

$$\left(\frac{\partial}{\partial z^k}\right)^{\vec{n}^k} K_\epsilon(z^1,z^k) \prod_{\alpha=1}^{k-1} \left(\frac{\partial}{\partial z^\alpha}\right)^{\vec{n}^\alpha} P^{an}_{\epsilon<L}(z^\alpha, z^{\alpha+1}).$$

Making the usual change of coordinates $w^\alpha = z^{\alpha+1} - z^\alpha$ and $w^k = z^k$ we see that this factor is proportional to the following constant coefficient differential form

$$\left(\prod_{i=1}^{d}\left(\sum_{\alpha=1}^{k-1} \mathrm{d}\overline{w}^\alpha_i\right)\right) \prod_{\alpha=1}^{k-1}\left(\eta^\alpha \prod_{i=1}^{d} \mathrm{d}\overline{w}^\alpha_i\right).$$

Note that this differential form only involves the coordinates (w^α_i) for $\alpha = 1,\ldots,k-1$. Thus, we may consider it as a Dolbeualt form on $\mathbb{C}^{d(k-1)}$. As such, it is of the type $(0, d + (k-1)(d-1)) = (0, (d-k+1) + d(k-1))$. Clearly, $(d-k+1) + d(k-1) > d(k-1)$ provided $k \leq d$. Thus, the weight is identically zero provided $k \leq d$, as desired. □

Item (2) follows from the following technical lemma that the analytic weight associated to the wheels of valency $k > d+1$ vanish in the limit $\epsilon \to 0$.

Lemma 4.4. *Let $(k, (\vec{n}))$ be a pair of wheel data such that $k > d+1$. Then the $\epsilon \to 0$ limit of the analytic weight*

$$\lim_{\epsilon\to 0} \widetilde{W}^{k,(n)}_{\epsilon<L} = 0$$

is identically zero as a distribution on $\mathbb{C}^{dk}$.

Proof. The proof is very similar to the argument we gave in the proof of Lemma 3.6, so we will be a bit more concise. First, we make the familiar change of coordinates as in Equations (8),(9). Using the explicit form the heat kernel and propagator we see that for any $\Phi \in C_c^\infty(\mathbb{C}^{dk})$ the weight is

$$\begin{aligned}\widetilde{W}^{k,(n)}_{\epsilon<L}(\Phi) &= \int_{w^k\in\mathbb{C}^d} \mathrm{d}^{2d}w^k \int_{(w_1,\dots,w_{k-1})\in(\mathbb{C}^d)^{k-1}} \left(\prod_{\alpha=1}^{k-1}\mathrm{d}^{2d}w^\alpha\right)\Phi(w^1,\dots,w^k)\int_{(t_1,\dots,t_k)\in[\epsilon,L]^{k-1}} \frac{1}{(4\pi\epsilon)^d}\prod_{\alpha=1}^{k-1}\frac{\mathrm{d}t_\alpha}{(4\pi t_\alpha)^d}\\ &\times \sum_{i_1,\dots,i_{k-1}=1}^{d}\epsilon_{i_1,\dots,i_d}\left(\frac{\overline{w}^1_{i_1}}{t_1}\frac{(\overline{w}^1)^{n^1}}{4t^{|n^1|}}\right)\cdots\left(\frac{\overline{w}^{k-1}_{i_{k-1}}}{4t_{k-1}}\frac{(\overline{w}^{k-1})^{n^{k-1}}}{4t^{|n^{k-1}|}}\right)\left(\frac{1}{4t^{|n^k|}}\left(\sum_{\alpha=1}^{k-1}\overline{w}^\alpha\right)^{n^k}\right)\\ &\times\exp\left(-\sum_{\alpha=1}^{k-1}\frac{|w^\alpha|^2}{4t_\alpha}-\frac{1}{4\epsilon}\left|\sum_{\alpha=1}^{k-1}w^\alpha\right|^2\right).\end{aligned}$$

We will integrate by parts to eliminate the factors of $\overline{w}^\alpha_i$.

For each $1\le\alpha<k$ and i_α, define the ϵ and $t=(t_1,\dots,t_{k-1})$-dependent holomorphic differential operator

$$D_{\alpha,i_\alpha}(t) := \left(\frac{\partial}{\partial w^\alpha_{i_\alpha}} - \sum_{\beta=1}^{k-1}\frac{t_\beta}{t_1+\cdots+t_{k-1}+\epsilon}\frac{\partial}{\partial w^\beta_{i_\alpha}}\right)\prod_{j=1}^{d}\left(\frac{\partial}{\partial w^\alpha_j} - \sum_{\beta=1}^{k-1}\frac{t_\beta}{t_1+\cdots+t_{k-1}+\epsilon}\frac{\partial}{\partial w^\beta_j}\right)^{n^\alpha_j}.$$

And the ϵ,t-dependent holomorphic differential operator

$$D_k(t) = \prod_{j=1}^{d}\left(\frac{\partial}{\partial w^k_j} - \sum_{\beta=1}^{k-1}\frac{t_\beta}{t_1+\cdots+t_{k-1}+\epsilon}\frac{\partial}{\partial w^\beta_j}\right)^{n^k_j}.$$

By a completely analogous version of Lemma the operators above allow us to integrate by parts and express the weight in the form

$$\begin{aligned}\widetilde{W}^{k,(n)}_{\epsilon<L}(\Phi) &= \pm\int_{w^k\in\mathbb{C}^d}\mathrm{d}^{2d}w^k\int_{(w_1,\dots,w_{k-1})\in(\mathbb{C}^d)^{k-1}}\left(\prod_{\alpha=1}^{k-1}\mathrm{d}^{2d}w^\alpha\right)\int_{(t_1,\dots,t_{k-1})\in[\epsilon,L]^{k-1}}\frac{1}{(4\pi\epsilon)^d}\prod_{\alpha=1}^{k-1}\frac{\mathrm{d}t_\alpha}{(4\pi t_\alpha)^d}\\ &\times\left(\sum_{i_1,\dots,i_{k-1}}\epsilon_{i_1\cdots,i_d}D_{1,i_1}(t)\cdots D_{k-1,i_{k-1}}(t)D_k(t)\Phi(w^1,\dots,w^k)\right)\times\exp\left(-\sum_{\alpha=1}^{k-1}\frac{|w^\alpha|^2}{4t_\alpha}-\frac{1}{4\epsilon}\left|\sum_{\alpha=1}^{k-1}w^\alpha\right|^2\right).\end{aligned}$$

Observe that the operators $D_{\alpha,i_\alpha}(t), D_k(t)$ are uniformly bounded in t. Thus, there exists a constant $C=C(\Phi)>0$ depending only on the function Φ such that we can bound the weight as

$$\begin{aligned}|\widetilde{W}^{k,(n)}_{\epsilon<L}(\Phi)| &\le C\int_{(w^1,\dots,w^{k-1}}\prod_{\alpha=1}^{k-1}\mathrm{d}^{2d}w^\alpha\int_{(t_1,\dots,t_{k-1})\in[\epsilon,L]^{k-1}}\mathrm{d}t_1\dots\mathrm{d}t_k\frac{1}{\epsilon^d t_1^d\cdots t_{k-1}^d}\\ &\times\exp\left(-\sum_{\alpha=1}^{k-1}\frac{|w^\alpha|^2}{4t_\alpha}-\frac{1}{4\epsilon}\left|\sum_{\alpha=1}^{k-1}w^\alpha\right|^2\right).\end{aligned}\tag{17}$$

Thus, to show that the limit $\lim_{L\to0}\lim_{\epsilon\to0}\widetilde{W}^{k,(n)}_{\epsilon<L}(\Phi)=0$ it suffices to show that the limit of the right-hand side vanishes.

The Gaussian integral over the variables w^α_i contributes the following factor

$$\int_{(w^1,\dots,w^{k-1}}\prod_{\alpha=1}^{k-1}\mathrm{d}^{2d}w^\alpha\exp\left(-\sum_{\alpha=1}^{k-1}\frac{|w^\alpha|^2}{4t_\alpha}-\frac{1}{4\epsilon}\left|\sum_{\alpha=1}^{k-1}w^\alpha\right|^2\right) = C'\left(\frac{\epsilon t_1\cdots t_{k-1}}{\epsilon+t_1+\cdots+t_{k-1}}\right)^d.$$

Where C' involves factors of 2 and π. Plugging this back in to the right-hand side of (17) we see that

$$|\widetilde{W}^{k,(n)}_{\epsilon<L}(\Phi)| \leq CC' \int_{[\epsilon,L]^{k-1}} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_{k-1}}{(\epsilon + t_1 + \cdots + t_{k-1})^d} \leq CC' \prod_{\alpha=1}^{k-1} \int_{t_\alpha=\epsilon}^{L} \mathrm{d}t_\alpha t_\alpha^{-d/(k-1)}.$$

In the second inequality we have used the fact that $\epsilon > 0$ and the AM-GM inequality. It is immediate to see that the $\epsilon \to 0$ limit of the above exists provided $k > d+1$, which is the situation we are in, and that the $L \to 0$ limit vanishes. □

This completes the proof of Lemma 4.2 □

4.2.1 Relation to the ABJ anomaly

The lemma we have just proved implies that for holomorphic theories on $\mathbb{C}^d$ the anomaly is given by evaluating a collection of wheel diagrams with exactly $d+1$ vertices. This expression for the obstruction fits into a generic class of of one-loop anomalies from gauge theory called the Adler-Bell-Jackiw (ABJ) anomaly [**?**, **?**]. This anomaly is most commonly associated with four dimensional gauge theory.

We recall the basic setup for the ABJ anomaly. Consider a free Dirac fermion Ψ on $\mathbb{R}^4$ coupled to a background gauge field $A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{g}$. For this to make sense, Ψ is taken to be in valued in a representation V of the Lie algebra $\mathfrak{g}$ so we may think of it as an element $\Psi \in \mathcal{S}(\mathbb{R}^4) \otimes V$. Here, $\mathcal{S}(\mathbb{R}^4)$ is the space of sections of the full spinor bundle on $\mathbb{R}^4$. The action functional is

$$S(A, \Psi) = \int \langle \Psi, \not{\partial}_A \Psi \rangle_V$$

where $\not{\partial}_A = \not{\partial} + [A, -]$ is the A-coupled Dirac operator. We are implicitly using the canonical spin invariant symplectic pairing $\mathcal{S} \otimes \mathcal{S} \to \Omega^4(\mathbb{R}^4) = C^\infty(\mathbb{R}^4)\mathrm{d}^4x$ and a $\mathfrak{g}$-invariant pairing $\langle -, - \rangle_V : V \otimes V \to \mathbb{C}$, to obtain a local functional.

For any smooth map $\alpha : \mathbb{R}^4 \to \mathfrak{g}$, the infinitesimal transformation $\Psi \to \Psi + \epsilon[\alpha, \Psi]$ (where ϵ is an even parameter of square zero) is a classical symmetry of $S(A, \Psi)$. Quantum mechanically, there is a one-loop anomaly which measures the failure of the path integral to be invariant with respect to this symmetry. It is a well-known calculation, see for instance [**?**], that this anomaly is measured by the following local functional

$$\int \mathrm{Tr}_V \left(\alpha F_A F_A \right). \tag{18}$$

The trace is taken in the representation V. The fundamental calculation is the infamous "triangle diagram", where two vertices are labeled by the gauge field and the third by α. In practice, physicists express the anomaly as a failure for the Noether current associated to the symmetry α to be divergenceless.

There is the following holomorphic version of this anomaly. Again, let V be a $\mathfrak{g}$ representation. Consider the following action functional on $\mathbb{C}^2$:

$$S(A, \beta, \gamma) = \int \langle \beta, \bar{\partial}_A \gamma \rangle_V$$

where $\gamma : \mathbb{C}^2 \to V$, $\beta \in \Omega^{2,1}(\mathbb{C}^2, V)$, and $A \in \Omega^{0,1}(\mathbb{C}^2, \mathfrak{g})$. Since A is a $(0,1)$ form it defines a deformation of the trivial holomorphic G-bundle. Although we have not put this theory in the BV formalism, there is a natural way to do so. The infinitesimal symmetry we contemplate is of the form $\gamma \to \gamma + \epsilon[\alpha, \gamma]$ where $\alpha : \mathbb{C}^2 \to \mathfrak{g}$. We study the anomaly to quantizing this symmetry to one-loop. Following the result for the anomaly given in the previous section, one sees that it is computed by a wheel with three vertices. For type reasons, one vertex is labeled α and the other two are labeled by the gauge fields A. A special case of a general calculation performed later in Chapter **??** of this thesis computes the value of the diagram as

$$\int \mathrm{Tr}_V(\alpha \partial A \partial A).$$

This is the holomorphic version of ABJ anomaly (18). Note that there are no terms of order A^3 or above. In fact, the functional $\int \mathrm{Tr}(\alpha F_A F_A)$ is cohomologous to the expression above in the local deformation complex.

Remark 4.5. In the next section, using the concept of the "equivariant" BV formalism, we will make coupling background fields to a classical theory precise. Then, Lemma 4.2 above applies rigorously to give the form of anomaly we have given. We will see a precise statement of this for the holomorphic current algebra in our proof of the Grothendieck-Riemann-Roch theorem in Chapter **??**.

Remark 4.6. We have already shown how familiar topological theories can be cast in a holomorphic language. For instance, topological BF theory is a holomorphic deformation of holomorphic BF theory. It is a peculiar consequence of the above result that such topological theories also admit a simple regularization procedure. Without much more difficulty, one can extend this to certain topological theories to odd dimensional manifolds of the form $X \times S$, where X is a complex manifold and S is a real one-dimensional manifold. We consider the theory as a product of a holomorphic theory on X and a one-dimensional topological theory on S. This can be further extended to transversely holomorphic foliations [**?**, **?**], which we will study in a future publication. Further, often topological BF theory further deforms to Yang-Mills. It would be interesting to apply our analysis above to such gauge theories.