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CONTINUOUS COHOMOLOGY FOR COMPACTLY SUPPORTED VECTORFIELDS ON $\mathbf{R}^n$

BY

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ABSTRACT. In this paper we study the Gelfand-Fuks cohomology of the Lie algebra of compactly supported vectorfields on $\mathbf{R}^n$ and establish the degeneracy of a certain spectral sequence at the E_1 level. We apply this result to the study of another spectral sequence introduced by Resetnikov for the cohomology of the algebra of vectorfields on S^n.

Let L be the Lie algebra of compactly supported smooth vectorfields on a manifold M. For U a precompact open subset of M let L_U be the set of vectorfields supported in U with the C^∞ topology, then $L = \bigcup_{U \subset M} L_U$ and we give L the topology of a strict inductive limit. Let $C^q(L)$ be the vectorspace of all continuous skewsymmetric $\mathbf{R}$-multilinear functions from $L \times \cdots \times L$ (q times) into $\mathbf{R}$. Define

$$d^q\colon C^q(L) \longrightarrow C^{q+1}(L),$$

$$(d^q\lambda)(\xi_1, \ldots, \xi_{q+1}) = \sum (-1)^{i+j}\lambda([\xi_i, \xi_j], \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{q+1})$$

where $[\ ,\ ]$ denotes the Lie bracket of vectorfields and $\hat{}$ indicates omission. Then $d^{q+1} \circ d^q = 0$ and $C^*(L) = \bigoplus_{q=0,\ldots,\infty} C^q(L)$ is a differential complex with differential $d = \bigoplus d^q$. The cohomology of $(C^*(L), d)$ is known as the Gelfand-Fuks cohomology of L with coefficients in $\mathbf{R}$.

Let $\mathrm{pr}_i\colon M^q \longrightarrow M$ be the projection on the ith factor of the q-fold cartesian product of M and let pr_i^*T be the pull-back of the tangent bundle to M along pr_i. Define $T^q = \mathrm{pr}_1^*T \otimes \cdots \otimes \mathrm{pr}_q^*T$ as a bundle over M^q. A vectorfield ξ on M defines a section pr_i^*T in a natural way and a q-tuple $(\xi_1, \ldots, \xi_q)$ of vectorfields defines a section $\mathrm{pr}_1^*\xi_1 \otimes \cdots \otimes \mathrm{pr}_q^*\xi_q$ of T^q over M^q. Linear combinations of sections of this type are dense in the space of compactly supported sections of T^q, denoted $[T^q]_C$, with the inductive limit topology defined similarly to that on $L = [T]_C$. Thus an element $\lambda \in C^q(L)$ defines a continuous function

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$\tilde{\lambda}$: $[T^q]_C \longrightarrow \mathbf{R}$. If $\mathrm{Hom}_{\mathbf{R}}([T^q]_C, \mathbf{R})$ denotes the continuous $\mathbf{R}$ multilinear functions, then we have a map $C^q(L) \longrightarrow \mathrm{Hom}_{\mathbf{R}}([T^q]_C, \mathbf{R})$. If we let $B^q(L)$ denote the set of not necessarily skewsymmetric continuous $\mathbf{R}$-multilinear functions $L \times \cdots \times L \longrightarrow \mathbf{R}$, then we have an isomorphism:

$$B^q(L) \cong \mathrm{Hom}_{\mathbf{R}}([T^q]_C, \mathbf{R}). \tag{1}$$

Let Σ_q be the permutation group on q-letters and corresponding to $\sigma \in \Sigma_q$ and $\lambda \in B^q(L)$ let $\sigma \circ \lambda \in B^q(L)$ be defined by

$$(\sigma \circ \lambda)(\xi_1, \ldots, \xi_q) = \epsilon_\sigma \lambda(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(q)})$$

where ϵ_σ is the sign of σ as a permutation. With these definitions $C^q(L)$ is the subspace of Σ_q invariants in $B^q(L)$.

$$B^q(L)^{\Sigma_q} = C^q(L). \tag{2}$$

Let $\mathcal{D}'(M^q)$ be the space of distributions on M^q,

$$\mathcal{D}'(M^q) = \mathrm{Hom}_{\mathbf{R}}(C_0^\infty(M^q), \mathbf{R}) = \mathrm{Hom}_{\mathbf{R}}([1]_C, \mathbf{R}).$$

Consider $C_0^\infty(M^q)$ as a left $C^\infty(M^q)$ module making $\mathcal{D}'(M^q)$ a right $C^\infty(M^q)$ module. Then

$$\begin{aligned} \mathrm{Hom}_{\mathbf{R}}([T^q]_C, \mathbf{R}) &= \mathrm{Hom}_{\mathbf{R}}([T^q] \otimes_{C^\infty(M^q)} [1]_C, \mathbf{R}) \\ &= \mathrm{Hom}_{C^\infty(M^q)}([T^q], \mathrm{Hom}([1]_C, \mathbf{R})) \\ &= \mathrm{Hom}_{C^\infty(M^q)}([T^q], \mathcal{D}'(M^q)) \cong \mathcal{D}'(M^q) \otimes_{C^\infty(M^q)} [T^{q*}]. \end{aligned} \tag{3}$$

Let Σ_q act on M^q by permuting factors $\sigma(x_1, \ldots, x_q) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(q)})$. This induces an action on $C_0^\infty(M^q)$ and by duality on $\mathcal{D}'(M^q)$. Let Σ_q act on T^{q*} by permuting factors and multiplying by ϵ_σ, then for $\omega_1 \otimes \cdots \otimes \omega_q \in [T^{q*}]$, $\xi_1 \otimes \cdots \otimes \xi_q \in [T^q]_C$ and $u \in \mathcal{D}'(M^q)$,

$$\begin{aligned} &\sigma(u \otimes \omega_1 \otimes \cdots \otimes \omega_q)[\xi_1 \otimes \cdots \otimes \xi_q] \\ &\quad = \epsilon_\sigma(\sigma \circ u \otimes \omega_{\sigma^{-1}(1)} \otimes \cdots \otimes \omega_{\sigma^{-1}(q)})[\xi_1 \otimes \cdots \otimes \xi_q] \\ &\quad = \epsilon_\sigma(\sigma \circ u)[\langle \omega_{\sigma^{-1}(1)}, \xi_1 \rangle_{x_1} \circ \cdots \circ \langle \omega_{\sigma^{-1}(q)}, \xi_q \rangle_{x_q}] \\ &\quad = \epsilon_\sigma u[\langle \omega_{\sigma^{-1}(1)}, \xi_1 \rangle_{x_{\sigma^{-1}(1)}} \circ \cdots \circ \langle \omega_{\sigma^{-1}(q)}, \xi_q \rangle_{x_{\sigma^{-1}(q)}}] \\ &\quad = \epsilon_\sigma u[\langle \omega_1, \xi_{\sigma(1)} \rangle_{x_1} \circ \cdots \circ \langle \omega_q, \xi_{\sigma(q)} \rangle_{x_q}] \\ &\quad = \epsilon_\sigma u \otimes \omega_1 \otimes \cdots \otimes \omega_q [\xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(q)}]. \end{aligned}$$

Therefore

$$(\mathcal{D}'(M^q) \otimes_{C^\infty(M^q)} [T^{q*}])^{\Sigma_q} \cong C^q(L). \tag{4}$$

To compute the cohomology of $C^q(L)$ we use the spectral sequence defined as follows. Let $\mathcal{D}'(M^q)|_{M_k^q}$ be the distributions with support on the subset $M_k^q = \{(x_1, \ldots, x_q)|$ at most k of the points $x_1 \in M\}$. Set

$$C_k^q(L) = (\mathcal{D}'(M^q)|_{M_k^q} \otimes_{C^\infty(M^q)} [T^{q*}])^{\Sigma_q},$$

then $C_k^q(L) \subset C_{k+1}^q(L)$ and $d^q C_k^q(L) \subset C_k^{q+1}(L)$. If we define $F^{-k}C^q = C_k^q$ we have a decreasing filtration preserved by the differential and thus a cohomology spectral sequence.

Note that M_k^q is a union of submanifolds. In fact if S is a partition of q elements into k sets, let M_S^q be the set of points in M^q consisting of $(x_1, \ldots, x_q)$ such that if i, j are in the same subset of the partition then $x_i = x_j$. There is an obvious diffeomorphism of M^k and M_S^q, and $M_k^q = \bigcup_{S \text{ a partition of } k} M_S^q$. Any element of $\mathcal{D}'(M^q)|_{M_S^q}$ can be written as a sum of normal derivatives of distributions on M_S^q, see Schwartz [4]. P. Trauber in his Princeton thesis [6] has used the isomorphism (4) and this fact to give a nice description of the E_0 term of the spectral sequence and then applied the methods of relative homological algebra to compute E_1. We summarize his results below, making the obvious extension to the case of compactly supported vectorfields. Let $D(M)$ be the differential operators on M, not necessarily of finite order, topologized as follows. For U a precompact open subset of M, let $D^k(U)$ be the differential operators of at most order k on smooth functions with support in U. As sections of a vector bundle $D^k(U)$ has a nuclear locally convex topology and so the inductive limit $D(U) = \varinjlim_k D^k(U)$ does also. For $U \subset V$ there is a restriction map $D(V) \longrightarrow D(U)$ and the precompact open subsets of M together with these restriction maps form a directed system. Let $D(M) = \varprojlim_{U \subset M} D(U)$, as a projective limit of nuclear spaces it is a nuclear space. If we use the cofinal family $U^q = U \times \cdots \times U$ (q times) of precompact open sets on M^q to define the topology on $D(M^q)$, then because

$$D^k(U^q) \cong D^k(U) \,\hat{\otimes} \cdots \hat{\otimes}\, D^k(U)$$

and $\hat{\otimes}$ is an exact functor we have $D(U^q) \cong D(U) \,\hat{\otimes} \cdots \hat{\otimes}\, D(U)$ and $D(M^q) \cong D(M) \,\hat{\otimes} \cdots \hat{\otimes}\, D(M)$. Similarly $[T^{q*}] \cong [T^*] \,\hat{\otimes} \cdots \hat{\otimes}\, [T^*]$. Let $D(M^q)|_{M_S^q}$ be the differential operators $C_0^\infty(M^q) \longrightarrow C_0^\infty(M_S^q)$. Composition on the left defines a left $D(M_S^q)$ module structure on $D(M^q)|_{M_S^q}$ and $C^\infty(M_S^q) \subset D(M_S^q)$. Relative to these structures we have the following

PROPOSITION (Trauber [6]).
(a) $\mathcal{D}'(M^q)|_{M_S^q} \cong \mathcal{D}'(M_S^q) \otimes_{D(M_S^q)} D(M^q)|_{M_S^q}$,
(b) $D(M^q)|_{M_S^q} \cong C^\infty(M_S^q) \otimes_{C^\infty(M^q)} D(M^q)$,
where the $C^\infty(M^q)$ module structure on $C^\infty(M_S^q)$ is restriction followed by multiplication. Using these isomorphisms we have

$$\mathcal{D}'(M^q)|_{M^q_S} \otimes_{C^\infty(M^q)} [T^{q*}]$$
$$\cong \mathcal{D}'(M^q_S) \otimes_{D(M^q_S)} C^\infty(M^q_S) \otimes_{C^\infty(M^q)} D(M^q) \otimes_{C^\infty(M^q)} [T^{q*}]$$
$$\cong \mathcal{D}'(M^q_S) \otimes_{D(M^q_S)} C^\infty(M^q_S) \otimes_{C^\infty(M^q)} (D(M) \hat{\otimes} \cdots \hat{\otimes} D(M))$$
$$\otimes_{C^\infty(M)\hat{\otimes}\cdots\hat{\otimes} C^\infty(M)} ([T^*] \hat{\otimes} \cdots \hat{\otimes} [T^*])$$
$$\cong \mathcal{D}'(M^q_S) \otimes_{D(M^q_S)} C^\infty(M^q_S) \otimes_{C^\infty(M^q)} D(M) \otimes_{C^\infty(M)} [T^*]$$
$$\hat{\otimes} \cdots \hat{\otimes} D(M) \otimes_{C^\infty(M)} [T^*].$$

Let $D \otimes T^* = D(M) \otimes_{C^\infty(M)} [T^*]$ and let X be the elements of positive degree in the exterior algebra over $C^\infty(M)$ of $D \otimes T^*$ let $X^k = X \hat{\otimes} \cdots \hat{\otimes} X$ (k times) and let $X^k(q)$ be the subspace of X^k consisting of elements with q factors of T^*. Trauber proves the following

THEOREM (Trauber [6]).

(a) $C^q_k(L) \cong (\mathcal{D}'(M^q)|_{M^q_k} \otimes_{C^\infty(M^q)} [T^{q*}])^{\Sigma_q} \cong (\mathcal{D}'(M^k) \otimes_{D(M^k)} X^k(q))^{\Sigma_k}$,

(b)
$$\frac{F^{-k}C^*(L)}{F^{-k+1}C^*(L)} \cong \left(\frac{\mathcal{D}'(M^k)}{\mathcal{D}'(M^k)|_{M^k_{k-1}}} \otimes_{D(M^k)} X^k\right)^{\Sigma_k}.$$

He also points out the following interpretation of the isomorphism (a).

Let $J^k(T)$ be the bundle of k-jets on M, for U a precompact open set let $[J^k(T)]_U$ be the sections with support in U, this is a Fréchet nuclear space. Define $[J^\infty(T)]_C = \varinjlim_U \varprojlim_k [J^k(T)]_U$. This is a nuclear l.c.s. such that

(5) $$D \otimes T^* = \mathrm{Hom}_{C^\infty(M)}([J^\infty(T)]_C, C^\infty(M)).$$

There is a continuous function j^∞: $[T]_C \longrightarrow [J^\infty(T)]_C$ which associates to any compactly supported vectorfield its infinite jet at each point. The bundle $J^\infty(T)$ has a canonical connection ∇: $[J^\infty(T)]_C \longrightarrow [T^* \otimes J^\infty(T)]_C$ introduced by Spencer, see [2]. If $\tilde{\xi} \in [J^\infty(T)]_C$ then $\tilde{\xi} = j^\infty(\xi)$ for some $\xi \in [T]_C$ if and only if $\nabla\tilde{\xi} = 0$ in $[T^* \otimes J^\infty(T)]_C$. The connection ∇ has 0 curvature and thus gives a representation of $D(M)$ on $[J^\infty(T)]_C$.(2) The image of j^∞ is the subspace of $D(M)$ invariants in $[J^\infty(T)]_C$. Using the isomorphism $D(M^q) \cong D(M) \hat{\otimes} \cdots \hat{\otimes} D(M)$ we get a representation of $D(M^q)$ on $[J^\infty(T)]_C \hat{\otimes} \cdots \hat{\otimes} [J^\infty(T)]_C$,

(2) For any vector bundle E with connection ∇: $\underline{E} \longrightarrow \underline{T^* \otimes E}$ we write ∇_X for the germ of a differential operator $(\nabla_X S)(p) = (\nabla S)(p)(X_p) \in \underline{E}_p$ where $X \in \underline{T}_p$ and $S \in \underline{E}_p$. If $\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = 0$ we say the connection has curvature zero and we get a Lie algebra representation of $[T] \longrightarrow [\mathrm{Diff}\ E]$ = differential operators on E. This extends to a representation $D(M) \longrightarrow [\mathrm{Diff}\ E]$.

which we will also denote by ∇ also. For $\xi_1 \otimes \cdots \otimes \xi_q \in [J^\infty(T)]_C \otimes \cdots \otimes [J^\infty(T)]_C$ and $\eta_1 \otimes \cdots \otimes \eta_q \in D(M) \otimes \cdots \otimes D(M)$,

$$\nabla_{\eta_1 \otimes \cdots \otimes \eta_q} \xi_1 \otimes \cdots \otimes \xi_q = \nabla_{\eta_1}\xi_1 \otimes \cdots \otimes \nabla_{\eta_i}\xi_i \otimes \cdots \otimes \nabla_{\eta_q}\xi_q.$$

Now $L_C \xrightarrow{j^\infty} [J^\infty(T)]_C$ is a Lie algebra map; therefore there is a cochain map $C^q([J^\infty(T)]_C) \xrightarrow{(j^\infty)^*} C^q(L)$ which is the same as

$$\mathcal{D}'(M^q) \otimes_{C^\infty(M^q)} [J^\infty(T)]_C^* \hat{\otimes} \cdots \hat{\otimes} [J^\infty(T)]_C^* \xrightarrow{(j^\infty)^*} \mathcal{D}'(M^q) \otimes_{C^\infty(M^q)} [T^{q*}]$$

or equivalently

$$\mathcal{D}'(M^q) \otimes_{C^\infty(M^q)} D \otimes T^* \hat{\otimes} \cdots \hat{\otimes} D \otimes T^* \xrightarrow{(j^\infty)^*} \mathcal{D}'(M^q) \otimes_{C^\infty(M^q)} [T^{q*}]. \tag{6}$$

Since the image of j^∞ is the subspace of $D(M)$ invariants it is not hard to see that $(j^\infty)^*$ factors through the tensor product over $D(M^q)$ to give an isomorphism

$$\mathcal{D}'(M^q) \otimes_{D(M^q)} D \otimes T^* \hat{\otimes} \cdots \hat{\otimes} D \otimes T^* \longrightarrow \mathcal{D}'(M^q) \otimes_{C^\infty(M^q)} [T^{q*}].$$

This allows us to identify the differential on the complex X appearing in the previous theorem: X is the exterior algebra on $[J^\infty(T)]_C^*$ and the differential d_X on X is the usual coboundary operator in the cochain complex on the dual of a Lie algebra. We can restate the previous theorem

$$(\mathcal{D}'(M^k) \otimes_{D(M^k)} \Lambda^+ [J^\infty(T)]_C^* \hat{\otimes} \cdots \hat{\otimes} \Lambda^+ [J^\infty(T)]_C^*)^{\Sigma k} \cong F^{-k}C^*(L)/F^{-k+1}C^*(L) \tag{7}$$

as cochain complexes with the isomorphism induced by $(j^\infty)^*$.

To compute $H^*(F^{-k}/F^{-k+1})$ we note that X^k is flat as a $D(M^k)$ module since $X = \Lambda^+ D \otimes T^*$ is flat as a D module in each degree of the exterior power. Therefore the higher derived functors of $\otimes_{D(M^k)} X^k$ in the category of differential complexes vanish.

$$\begin{aligned} &\mathrm{Tor}_p^{D(M^k)}(A, X^k) = 0, \quad p > 0, \\ &\mathrm{Tor}_0^{D(M^k)}(A, X^k) = H^*(A \otimes_{D(M^k)} X^k, d_{X^k}). \end{aligned} \tag{8}$$

However we can also compute the differential derived functor by resolving X^k. Let $Y_p = D(M^k) \otimes \Lambda^p [T(M^k)]$ define $\partial_p \colon Y_p \longrightarrow Y_{p-1}$ by

$$\begin{aligned} \partial_p(u \otimes \xi_1 \wedge \cdots \wedge \xi_p) = &\sum_i (-1)^{i-1} u\xi_i \otimes \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_p \\ &\cdot \sum_{i,j} (-1)^{1+j} u \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \hat{\xi}_i \\ &\wedge \cdots \wedge \hat{\xi}_j \wedge \cdots \wedge \xi_p. \end{aligned}$$

Then $Y = \bigoplus Y_p$ gives a resolution of $C^\infty(M^k)$ as a left $D(M^k)$ module and tensoring on the right over $C^\infty(M^k)$ with X^k we get a resolution:

$$\begin{array}{c} D(M^k) \otimes_{C^\infty(M^k)} \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} X^k \\ \Big\downarrow{\scriptstyle \epsilon_0} \\ X^k \end{array} \tag{9}$$

Let A be a right $D(M^k)$ module then tensoring on the left over $D(M^k)$ with A

$$\begin{array}{c} A \otimes_{C^\infty(M^k)} \Lambda^*[T(M^k)] \otimes_{C^\infty(M^k)} X^k \\ {\scriptstyle \mathrm{id}\otimes\epsilon_0}\Big\downarrow \\ A \otimes_{D(M^k)} X^k \end{array} \tag{10}$$

as an augmented complex with homology (making X^k a chain complex using negative indexing) equal to

$$\mathrm{Tor}_*^{D(M^k)}(A, X^k) = H_*(A \otimes_{D(M^k)} X^k).$$

Computing the ∂ spectral sequence of the double complex we have

$$E^1_{p,-q} \cong A \otimes \Lambda^p [T(M^k)] \otimes_{C^\infty(M^k)} H^{-q}(X^k).$$

Here we need an additional fact. Let L be the algebra of formal power series vectorfields, i.e., the fiber of $J^\infty(T)$ over a point of M, $L = \varprojlim_k J^k(T)_x$. Let $L^* = \varinjlim J^k(T)^*_x$, then $H(X) \cong C^\infty(M) \otimes_{\mathbf{R}} H(\Lambda^+ L^*)$ and the $D(M)$ module structure on $H(X)$ is trivial, see [5] or [1a, pp. 205–206]. Therefore, we have

$$H(X^k) \cong C^\infty(M^k) \otimes H(\Lambda^+ L^* \otimes \cdots \otimes \Lambda^+ L^*)$$

with $D(M^k)$ acting trivially. Hence

$$\begin{aligned} E^2_{p,-q} &\cong H(A \otimes_{C^\infty(M^k)} \Lambda[T(M^k)]) \otimes_{\mathbf{R}} H(\Lambda^+ L^* \otimes \cdots \otimes \Lambda^+ L^*), \\ E^\infty_* &\cong \mathrm{Gr}\, H_*(A \otimes_{D(M^k)} X^k). \end{aligned} \tag{11}$$

Let $\tilde{\mathcal{D}}'(M^k - M^k_{k-1})$ be the distributions on $M^k - M^k_{k-1}$ which extend to distributions on M^k. The inclusion i: $C_0^\infty(M^k - M^k_{k-1}) \longrightarrow C_0^\infty(M^k)$ induces an isomorphism

$$\mathcal{D}'(M^k)/\mathcal{D}'(M^k)|_{M^k_{k-1}} \cong \tilde{\mathcal{D}}'(M^k - M^k_{k-1}).$$

Since $\tilde{\mathcal{D}}'(M^k - M^k_{k-1})$ is dense in $\mathcal{D}'(M^k - M^k_{k-1})$ and $\mathcal{D}'(M^k - M^k_{k+1}) \otimes_{C^\infty(M^k)}$

$\Lambda[T(M^k)]$ is dual to $\Omega_C(M^k - M^k_{k-1})$ the de Rham complex of compactly supported differential forms we have a nondegenerate pairing

$$\tilde{\mathcal{D}}'(M^k - M^k_{k-1}) \otimes_{C^\infty(M^k)} \Lambda^p[T(M^k)] \times \Omega^p_C(M^k - M^k_{k-1}) \longrightarrow \mathbf{R}.$$

Moreover, the differential ∂_p on the left factor is dual to the de Rham differential. Thus if $A = \mathcal{D}'(M^k)/\mathcal{D}'(M^k_{k-1})$

$$H_p(A \otimes_{C^\infty(M^k)} \Lambda[T(M^k)]) \cong H^p_c(M^k - M^k_{k-1})^*. \tag{12}$$

Putting all this together we conclude

THEOREM 1. *Let $F^{-k}C^*(L)/F^{-k+1}C^*(L)$ be considered as a chain complex using negative indexing; then there is a homology spectral sequence with*

$$E^2_{p,-q} \cong (H^p_C(M^k - M^k_{k-1})^* \otimes H^q(\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*))^{\Sigma k}$$

and

$$E^\infty_{p,-q} = \mathrm{Gr}_p(H^{q-p}(F^{-k}/F^{-k+1})).$$

In the special case when $M = \mathbf{R}^n$ we have $X \cong C^\infty(M) \otimes_{\mathbf{R}} \Lambda^+L^*$ and $X^k \cong C^\infty(M^k) \otimes_{\mathbf{R}} \Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*$. This gives the following isomorphism

$$\begin{aligned}
&\mathcal{D}'(M^k - M^k_{k-1}) \otimes_{C^\infty(M^k)} \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} X^k \\
&\quad\cong \mathcal{D}'(M^k - M^k_{k-1}) \otimes_{C^\infty(M^k)} \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} C^\infty(M^k) \\
&\qquad \otimes_{\mathbf{R}} \Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^* \\
&\quad\cong (\mathcal{D}'(M^k - M^k_{k-1}) \otimes_{C^\infty(M^k)} \Lambda[T(M^k)]) \otimes_{\mathbf{R}} (\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*).
\end{aligned} \tag{13}$$

One can apply the Kunneth theorem to the latter complex, therefore its homology is

$$H(\mathcal{D}'(M^k - M^k_{k-1}) \otimes \Lambda[T(M^k)]) \otimes_{\mathbf{R}} H^*(\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*)$$

and we conclude that $E^2 = E^\infty$.

THEOREM 2. *If L is the Lie algebra of compactly supported vectorfields on $\mathbf{R}^n$, then with respect to the filtration defined earlier there is a spectral sequence with*

$$E_1^{-k,l+k} = H^l \frac{F^{-k}C^*(L)}{F^{-k+1}C^*(L)}$$

$$\cong \bigoplus_{q-p=l} [H^p_C((\mathbf{R}^n)^k - (\mathbf{R}^n)^k_{k-1})^* \otimes_{\mathbf{R}} H^q(\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*)]^{\Sigma k}.$$

We will give an explicit expression for this isomorphism and show that the spectral sequence collapses at E_1.

When $M = \mathbf{R}^n$ we can find a global basis $[T(M^k)]$ as a $C^\infty(M^k)$ module which consists of commuting vectorfields; then

$$[T(M^k)] \cong C^\infty(\mathbf{R}^{nk}) \otimes \mathbf{R}^{nk}, \qquad \Lambda[T(M^k)] \cong C^\infty(\mathbf{R}^{nk}) \otimes_{\mathbf{R}} \Lambda\mathbf{R}^{nk}.$$

Let $\widetilde{X}^k = C^\infty(\mathbf{R}^{nk}) \otimes \Lambda(L \oplus \cdots \oplus L)^*$, i.e., the full exterior algebra. It is clear that X^k is a direct summand of $\widetilde{X}^k$ as a $D(M^k)$ module. Let j be the inclusion and π the projection $X^k \xrightarrow{j} \widetilde{X}^k \xrightarrow{\pi} X^k$. Both i and π are cochain maps. Since $L \cong \mathbf{R}^n \oplus L^0$ we have $L \oplus \cdots \oplus L \cong \mathbf{R}^{nk} \otimes L^0 \oplus \cdots \oplus L^0$ and there is an obvious interior product $\Lambda\mathbf{R}^{nk} \otimes_{\mathbf{R}} \widetilde{X}^k \longrightarrow \widetilde{X}^k$. Using the isomorphisms given above we get a map

$$\tilde{i}: \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} \widetilde{X}^k \longrightarrow \widetilde{X}^k.$$

Composing on the right with $\mathrm{id} \otimes j$ and on the left with π we get

$$i: \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} X^k \longrightarrow X^k$$

which we will denote

$$i: \xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha \longmapsto \xi_1 \wedge \cdots \wedge \xi_p \lrcorner\, \alpha.$$

Tensoring on the left over $C^\infty(M^k)$ with $D(M^k)$

$$\mathrm{id} \otimes i: D(M^k) \otimes_{C^\infty(M^k)} \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} X^k \longrightarrow D(M^k) \otimes_{C^\infty(M^k)} X^k.$$

Composition with the left module structure on X^k with $D(M^k) \otimes X^k \longrightarrow X^k$ gives

$$\psi: D(M^k) \otimes_{C^\infty(M^k)} \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} X^k \longrightarrow X^k,$$
$$u \otimes \xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha \longmapsto u(\xi_1 \wedge \cdots \wedge \xi_p \lrcorner\, \alpha).$$

We will show that ψ is a cochain map. Passing to Σ_k invariants we get an explicit isomorphism for the E^1 term of the spectral sequence given in the previous theorem.

The map ψ is defined with respect to a fixed parallelisation of $T(M^k)$, with respect to which we have

$$D(M^k) \otimes_{C^\infty(M^k)} \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} X_k$$
$$\cong D(\mathbf{R}^{nk}) \otimes_{\mathbf{R}} \Lambda\mathbf{R}^{nk} \otimes_{\mathbf{R}} \Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*.$$

The differential is given by

$$d(u \otimes \xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha) = \sum (-1)^{i-1} u\xi_i \otimes \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_p \otimes \alpha$$
$$+ (-1)^p u \otimes \xi_1 \wedge \cdots \wedge \xi_p \otimes d_L\alpha$$

where d_L is the differential in $\Lambda L^* \otimes \cdots \otimes \Lambda L^*$,

$$\begin{aligned} d\psi(u \otimes \xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha) \\ &= d(u(\xi_1 \wedge \cdots \wedge \xi_p \lrcorner \alpha)) = ud(\xi_1 \wedge \cdots \wedge \xi_p \lrcorner \alpha) \\ &= u\Big(\sum (-1)^{i-1}(\xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_p \lrcorner \operatorname{ad} \xi_i \alpha) \\ &\qquad + (-1)^p(\xi_1 \wedge \cdots \wedge \xi_p \lrcorner d_L \alpha)\Big). \end{aligned}$$

By definition ad is the adjoint representation of $L \oplus \cdots \oplus L$ on $\Lambda(L \oplus \cdots \oplus L)^*$ dual to the adjoint representation of $L \oplus \cdots \oplus L$ on $\Lambda(L \oplus \cdots \oplus L)$. For $\alpha \in \Lambda(L \oplus \cdots \oplus L)^*$ and $\xi_1, \ldots, \xi_p \in \mathbf{R}^{nk}$ we have ad $\xi_i \alpha = \xi_i \cdot \alpha$ where $\cdot$ indicates the module structure and ξ_i are considered as constant coefficient differential operators. Furthermore $(\xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_p \lrcorner \operatorname{ad} \xi_i \alpha) =$ ad $\xi_i(\xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_p \lrcorner \alpha)$, thus ψ is a cochain map.

We can represent the induced map on cohomology

$$[H_c^p(\mathbf{R}^{nk} - (\mathbf{R}^n)_{n-1}^k) \otimes_{\mathbf{R}} H^q(\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*)]^{\Sigma k} \longrightarrow H^{q-p}(F^{-k}/F^{-k+1})$$

more conveniently as follows. For $\eta \in L, j^\infty(\eta) \in C_0^\infty(M) \otimes L$ so if $\alpha \in \Lambda L^*$ we can form $j^\infty(\eta) \lrcorner \alpha \in C_0^\infty(M) \otimes \Lambda L^*$. For $\alpha = \Sigma \alpha_1^i \otimes \cdots \otimes \alpha_k^i \in \Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*$ and for S a partition $(a_1, \ldots, a_s)(a_1, \ldots, b_{s_2}) \cdots (c_1, \ldots, c_{s_k})$ of q into k sets it makes sense to partition a set of q vectorfield $\eta_1, \ldots, \eta_q$ into $\eta_{a_1}, \ldots, \eta_{a_{s_1}}, \ldots, \eta_{b_1}, \ldots, \eta_{b_{s_2}}, \ldots, \eta_{c_1}, \ldots, \eta_{c_{s_k}}$ and form

$$\sum_i (j^\infty(\eta_{a_1}) \wedge \cdots \wedge j^\infty(\eta_{a_{s_1}}) \lrcorner \alpha_1^i) \wedge (j^\infty(\eta_{b_1}) \wedge \cdots \wedge j^\infty(\eta_{b_{s_2}}) \lrcorner \alpha_2^i)$$

$$\wedge \cdots \wedge (j^\infty(\eta_{c_1}) \wedge \cdots \wedge j^\infty(\eta_{c_{s_k}}) \lrcorner \alpha_k^i).$$

We will write $j^\infty(\eta_1) \wedge \cdots \wedge j^\infty(\eta_q) \lrcorner_s \alpha$ to mean the interior product just defined. Let i: $\mathbf{R}^{nk} \longrightarrow L \oplus \cdots \oplus L$ be the injection defined earlier and $\Lambda(L \oplus \cdots \oplus L)^* \xrightarrow{i^*} \Lambda \mathbf{R}^{nk*}$ the extension of the dual map to exterior algebras. Let ϕ be the isomorphism

$$C_0^\infty(\mathbf{R}^{nk}) \otimes \Lambda \mathbf{R}^{nk*} \xrightarrow{\phi} \Omega_c(\mathbf{R}^{nk})$$

given by the choice of a parallelism. Finally for S, the partition above, let ϵ_S be the sign of the permutation

$$\begin{pmatrix} 1 & \cdots & S_1 & \cdots & k - S_k + 1 & \cdots & k \\ a_1 & \cdots & a_{S_1} & \cdots & c_1 & \cdots & c_{S_k} \end{pmatrix}.$$

Then for $\lambda \in \widetilde{\mathcal{D}}'(\mathbf{R}^{nk} - (\mathbf{R}^n)_{k-1}^k) \otimes \Lambda^p[T(M^k)]\, \alpha \in (\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*)^q$ we have

$$\begin{aligned} &\psi(\lambda \otimes \alpha)(\eta_1, \dots, \eta_{q-p}) \\ &\qquad = \sum \epsilon_S \lambda [\phi i^*(j^\infty(\eta_1) \wedge \cdots \wedge j^\infty(\eta_{q-p}) \lrcorner_S \alpha)] \end{aligned} \tag{14}$$

and

$$\psi(d\lambda \otimes \alpha) + (-1)^{q-p} \psi(\lambda \otimes d_L \alpha) = d_0(\psi(\lambda \otimes \alpha)) \tag{15}$$

where d is the differential in $\widetilde{\mathcal{D}}'(M^k - M^k_{k-1}) \otimes \Lambda[T(M^k)]$, d_L is the differential in $\Lambda^+ L^* \otimes \cdots \otimes \Lambda^+ L^*$ and d_0 is the differential in $F^k C^*(L)/F^{-k+1} C^*(L)$.

Let $v_i \in \mathbf{R}^n$ and $(v_1, \dots, v_k) \in \mathbf{R}^{nk}$ and let $\mathbf{R}^{nk-n}_{(i,j)} = \{(v_1, \dots, v_k) | v_i = v_j\}$ then $(\mathbf{R}^n)^k_{k-1} = \bigcup_{i<j} \mathbf{R}^{nk-n}_{(i,j)}$. Let $\mathbf{R}^{nk} \cup \{\infty\} = S^{nk}$ and $\mathbf{R}^{nk-n}_{(i,j)} \cup \{\infty\} = S^{nk-n}_{(i,j)}$, then

$$\begin{aligned} H^p_c(\mathbf{R}^{nk} - (\mathbf{R}^n)^k_{k-1}) &= H^p_c\Big(\mathbf{R}^{nk} - \bigcup_{i<j\leq k} \mathbf{R}^{nk-n}_{(i,j)}\Big) \\ &= H^p_c\Big(S^{nk} - \bigcup_{i<j\leq k} S^{nk-n}_{(i,j)}\Big) \\ &\cong H^p\Big(S^{nk}, \bigcup_{i<j\leq k} S^{nk-n}_{(i,j)}\Big). \end{aligned}$$

Hence

$$H^p_c(\mathbf{R}^{nk} - (\mathbf{R}^n)^k_{k-1})^* \cong H_p\Big(S^{nk}, \bigcup_{i<j\leq k} S^{nk-n}_{(i,j)}\Big)$$

and composing these isomorphisms with ψ we have

$$\begin{aligned} \Phi\colon &\Big(H_p\Big(S^{nk}, \bigcup_{i<j} S^{nk-n}_{(i,j)}\Big) \otimes H^q(\Lambda^+ L^* \otimes \cdots \otimes \Lambda^+ L^*)\Big)^{\Sigma_k} \\ &\qquad \longrightarrow E_1^{-k, q-p+k}. \end{aligned} \tag{16}$$

For $\Sigma_{l=1}^m [\sigma_l] \otimes [\alpha_l]$ an element of the left-hand side if we choose representative cycles σ_i and representative cocycles α_i we get a representative element of $\Phi(\Sigma_{l=1}^m [\sigma_l] \otimes [\alpha_l])$.

$$\begin{aligned} &(\eta_1, \dots, \eta_{q-p}) \\ &\qquad \mapsto \sum_{j=1}^m \int_{\sigma_i} \sum_{\text{partitions}} \epsilon_S \phi i^*(j^\infty(\eta_1) \wedge \cdots \wedge j^\infty(\eta_{q-p}) \lrcorner_S \alpha_i). \end{aligned} \tag{17}$$

If we pull back $d_1\colon E^{-k,h+k} \longrightarrow E^{-k+1,h+k}$ by the isomorphism Φ we get a mapping for $q - p = h$,

$$\begin{array}{c}\left[H_p\left(S^{nk}, \bigcup S^{nk-n}_{(1,j)}\right) \otimes H^q(\underbrace{\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*}_{k})\right]^{\Sigma_k}\\ \Big\downarrow \bar{d}_1 \\ \left[H_{p-1}\left(S^{nk-n}, \bigcup_{i<j\leqslant k-1} S^{nk-2n}_{(i,j)}\right) \otimes H^q(\underbrace{\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*}_{k-1})\right]^{\Sigma_{k-1}}.\end{array}$$

It is computed as follows. For $\eta_1, \eta_2, \ldots, \eta_{h+1} \in L$,

$$\begin{aligned}\Phi\left(\bar{d}_1 \sum_{l=1}^{m} [\sigma_l] \otimes [\alpha_l]\right)&(\eta_1, \ldots, \eta_{h+1})\\ &= \sum_{i<j\leqslant h+1} (-1)^{i+j}\Phi\left(\sum_{l=1}^{m} [\sigma_l] \otimes [\alpha_l]\right)\\ &\qquad\qquad \cdot([\eta_i, \eta_j], \eta_1, \ldots, \hat{\eta}_i, \ldots, \hat{\eta}_j, \ldots, \eta_{h+1})\\ &= \sum_{i<j\leqslant h+1} \sum_{l} \sum_{S} \int_{\sigma_l} \epsilon_S \phi i^*(j^\infty([\eta_i, \eta_j]) \wedge j^\infty(\eta_1) \wedge \cdots \wedge \widehat{j^\infty(\eta_i)}\\ &\qquad\qquad \wedge \cdots \wedge \widehat{j^\infty(\eta_j)} \wedge \cdots \wedge j^\infty(\eta_{h+1}) \lrcorner_S \alpha_l)\\ &= \sum_{i<j\leqslant h+1} \sum_{l=1}^{m} \sum_{S} \int_{\sigma_l} \epsilon_S \phi i^*([j^\infty(\eta_i), j^\infty(\eta_j)] \wedge j^\infty(\eta_1) \wedge \cdots \wedge \widehat{j^\infty(\eta_i)}\\ &\qquad\qquad \wedge \cdots \wedge \widehat{j^\infty(\eta_j)} \wedge \cdots \wedge j^\infty(\eta_{h+1}) \lrcorner_S \alpha_l).\end{aligned}$$

Now α_l is a tensor product of k cycles $\alpha_{l,j} \in Z(\Lambda^+L^*)$. To compute the last term we see what is happening to each $\alpha_{l,j}$. For $\alpha \in Z^t(\Lambda L^*)$ and $\eta_1, \ldots, \eta_S \in L$

$$\begin{aligned}\sum_{i<j\leqslant s}^{i+j} &\phi i^*([j^\infty(\eta_i), j^\infty(\eta_j)] \wedge j^\infty(\eta_1) \wedge \cdots \wedge \widehat{j^\infty(\eta_i)}\\ &\qquad\qquad \wedge \cdots \wedge \widehat{j^\infty(\eta_j)} \wedge \cdots \wedge j^\infty(\eta_{s+1}) \lrcorner \alpha)\\ &= \sum_{i<j\leqslant s} \sum_{i_1<i_2<\cdots<i_{t-s}\leqslant n} (-1)^{i+j}\alpha([j^\infty(\eta_i), j^\infty(\eta_j)], j^\infty(\eta_1) \cdots \widehat{j^\infty(\eta_i)}\\ &\qquad\qquad \cdots \widehat{j^\infty(\eta_j)} \cdots j^\infty(\eta_s), e_{i_1} \cdots e_{i_{t-s}})\\ &\qquad\qquad dx^{i_1} \wedge \cdots \wedge dx^{i_{t-s}}\end{aligned}$$

$$+ \sum_{r,j} \sum_{i_1<i_2<\cdots<i_{t-s}\leq n} (-1)^{r+s+j}\alpha([e_{i_r}, j^\infty(\eta_j)], j^\infty(\eta_1)\cdots\widehat{j^\infty(\eta_j)}$$

$$\cdots j^\infty(\eta_s), e_{i_1}\cdots\hat{e}_{i_r}\cdots e_{i_{t-s}})$$

$$dx^{i_1}\wedge\cdots\wedge dx^{i_{t-s}}$$

$$+ \sum_{r,j} \sum_{i_1<i_2<\cdots<i_{t-S}\leq n} \frac{\partial}{\partial x^{i_r}}\alpha(j^\infty(\eta_1),\ldots,j^\infty(\eta_S), e_{i_1}\cdots\hat{e}_{i_r}\cdots e_{i_{t-S}})$$

$$dx^{i_1}\wedge\cdots\wedge dx^{i_{t-S}}$$

$$= d\phi i^*(j^\infty(\eta_1)\wedge\cdots\wedge j^\infty(\eta_S)\lrcorner\,\alpha).$$

This shows what happens to each factor of α_l; hence the end product is

$$\Phi(\bar{d}_1 \sum[\sigma_l]\otimes[\alpha_l])(\eta_1,\ldots,\eta_{h+1})$$

$$= \sum_l \sum_{S'} \int_{\sigma_l} \epsilon_{S'} d\phi i^*(j^\infty(\eta_1)\wedge\cdots\wedge j^\infty(\eta_{h+1})\lrcorner_{S'}\alpha)$$

$$= \sum_l \sum_{S'} \int_{\partial\sigma_l} \epsilon_{S'} \phi i^*(j^\infty(\eta_1)\wedge\cdots\wedge j^\infty(\eta_{h+1})\lrcorner_{S'}\alpha)$$

where S' ranges over partitions of $h+1$ elements into k sets. We can decompose $\partial\sigma_l$ into a sum of $\partial_{(i,j)}\sigma_l$ where $|\partial_{(i,j)}\sigma_l| \subset S^{nk-n}_{(i,j)}$. When $\phi i^*(j^\infty(\eta_1)\wedge\cdots\wedge j^\infty(\eta_{h+1})\lrcorner_S\alpha)$ is integrated over $S^{nk-n}_{(i,j)}$, the ith and jth factors are identified by restricting to the diagonal in the product of the ith and jth factors. This gives a mapping

$$\begin{array}{ccc} H(\underbrace{\Lambda^+L^*\otimes\cdots\otimes\Lambda^+L^*}_{k}) \cong & H(\underbrace{\Lambda^+L^*)\otimes\cdots\otimes H(\Lambda^+L^*}_{k}) \\ & \big\downarrow & \\ H(\underbrace{\Lambda^+L^*\otimes\cdots\otimes\Lambda^+L^*}_{k-1}) \cong & H(\underbrace{\Lambda^+L^*)\otimes\cdots\otimes H(\Lambda^+L^*}_{k-1}) \end{array}$$

by multiplying the ith and jth factors, just as restriction to the diagonal induces the cup product in singular cohomology. Therefore the $\bar{d}_1$ operator involves multiplication in the cohomology algebra of the formal Lie algebra. It is known that this multiplication is trivial [5], [7], so $\bar{d}_1 = 0$. In a similar way one can see that all the higher differentials involve multiplication in the formal algebra so we have

THEOREM 3. *There is a spectral sequence for the continuous cohomology of the algebra of compactly supported vectorfields on* $\mathbf{R}^n$ *which collapses at the* E_1 *level.*

$$E^{-k,l+k} \cong \left[\bigoplus_{q-p=l} H_p\left(S^{nk}, \bigcup_{i<j} S^{nk-n}_{i,j}\right)\otimes \bigotimes^k H^+(L)\right]^{\Sigma_k}.$$

Let L be the algebra of vectorfields on the n sphere S^n, let $p \in S^n$ and let $\widetilde{L}$ be the ideal of vectorfields flat at p in some, hence any, coordinate system. Let $C^*(L)$ be the Gelfand-Fuks complex for the continuous cohomology of L, and define a filtration

$$F^k C^q(L) = \{\lambda \in C^q(L) \mid \lambda(\xi_1, \ldots, \xi_q) = 0 \text{ if } q - k + 1 \text{ of } \xi_i \text{ are in } \widetilde{L}\},$$

then $F^k \supset F^{k+1}$ and $dF^k \subset F^k$. This is the filtration defining the Hochschild-Serre spectral sequence for $H(L)$ with respect to the ideal $\widetilde{L}$.

$$E_2^{p,q} \cong H^p(L/\widetilde{L}, H^q(\widetilde{L})), \qquad E_\infty^{p,q} \simeq \mathrm{Gr}_p(H^{p+q}(L)).$$

There is an exact sequence of Lie algebras

$$0 \longrightarrow \widetilde{L} \longrightarrow L \longrightarrow L \longrightarrow 0.$$

Thus $E_2^{p,q} \cong H^p(L, H^q(\widetilde{L}))$. The action of L on $H^q(\widetilde{L})$ is defined as follows: for $\eta \in L$ let $\bar{\eta} \in L$ be a vectorfield such that $j^\infty(\bar{\eta})_p = \eta$ then Lie derivation with respect to $\bar{\eta}$ defines a map $D_{\bar{\eta}}$: $\widetilde{L} \longrightarrow \widetilde{L}$ which in turn defines a cochain map $D_{\bar{\eta}}$: $C^*(\widetilde{L}) \longrightarrow C^*(\widetilde{L})$ and therefore a map $D_{\bar{\eta}}^*$: $H^*(\widetilde{L}) \longrightarrow H^*(\widetilde{L})$. If $j^\infty(\bar{\eta}_1)_p = j^\infty(\bar{\eta}_2)_p$ then $\bar{\eta}_1 - \bar{\eta}_2 \in \widetilde{L}$ and as is well known $D_{\bar{\eta}_1 - \bar{\eta}_2}$ induces the trivial map in cohomology, so $D_{\bar{\eta}_1}^* = D_{\bar{\eta}_2}^*$. Resetnikov [3] has stated the following theorem for arbitrary M but it is not clear to us that his proof is correct.

THEOREM. *Since L acts trivially on $H^*(\widetilde{L})$, the E_2 term of the previous spectral sequence is $E_2^{p,q} \cong H^p(L) \otimes H^q(\widetilde{L})$. Furthermore if L_C is the algebra of compactly supported vectorfields on $\mathbf{R}^n$, then $H^q(\widetilde{L}) \cong H^q(L_C)$.*

PROOF. Let $\{U_i\}$ be a decreasing sequence of open sets which form a neighborhood basis at p. Let $K_i = S^n - U_i$; then K is compact, and if we define ϕ: $S^n - \{p\} \longrightarrow \mathbf{R}^n$ by stereographic projection with p as north pole then the $\phi(K_i)$ form a compact exhaustion of $\mathbf{R}^n$. Let L_i be the algebra of vectorfields on S^n with support in K_i, there are inclusions τ_j^i: $L_i \longrightarrow L_j$; therefore, we can define $L_\infty = \varinjlim L_i$. Clearly $L_\infty \cong L_C$, compactly supported vectorfields. Let ψ^i: $L_i \longrightarrow \widetilde{L}$ be the inclusion; then $\psi^j \cdot \tau_j^i = \psi^i$ so we can define ψ: $L_\infty \longrightarrow \widetilde{L}$. This induces ψ^*: $H(\widetilde{L}) \longrightarrow H(L_\infty)$. For $\eta \in L$ let $\bar{\eta}_i \in L$ be a vectorfield such that $j^\infty(\bar{\eta}_i)_p = \eta$ and supp $\bar{\eta}_i \subset U_i$; then for $\lambda \in H^*(\widetilde{L})$ we have $\eta \cdot [\lambda) = [D_{\bar{\eta}_i}^* \lambda]$ for any i. Clearly $\psi^{i*}[D_{\bar{\eta}_i}^*\lambda] = 0$ and from the fact that $\psi^*\eta \cdot [\lambda] = 0$ if and only if $(\psi^i)^*\eta \cdot [\lambda] = 0$ for all i we conclude $\psi^*\eta \cdot [\lambda] = 0$. To conclude the proof it is sufficient to show that ψ^* is injective. In fact, ψ^* is an isomorphism. To see this, look at the spectral sequences defined at the beginning of the paper. Since $\widetilde{L}$ can be thought of as rapidly decreasing vectorfields on $\mathbf{R}^n$, the space that arises in defining $C^*(\widetilde{L})$ is $S'(\mathbf{R}^n)$. From this observation we see that the spectral sequence converging to $H^*(F^{-k}C^*(\widetilde{L})/F^{-k+1}C^*(\widetilde{L}))$, which is E_1 of another spectral sequence, has $E^2_{p,-q}$,

$$[H_p(S'(\mathbf{R}^{nk}/S'(\mathbf{R}^{nk})|_{(\mathbf{R}^n)^k_{k-1}}) \otimes \Lambda[T(\mathbf{R}^{nk})]) \otimes H^q(\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*)]^{\Sigma_k}.$$

We can identify the factor on the left from the following exact sequences (see Schwartz [4]). If p is the north pole of S^{nk},

$$0 \longrightarrow E'(S^{nk})|_p \longrightarrow E'(S^{nk}) \longrightarrow S'(\mathbf{R}^{nk}) \longrightarrow 0$$

$$0 \longrightarrow E'(S^{nk})|_p \longrightarrow E'(S^{nk})|_{\bigcup\limits_{i<j\leq k} S^{nk-n}_{(i,j)}} \longrightarrow S'(\mathbf{R}^{nk})|_{(\mathbf{R}^n)^k_{k-1}} \longrightarrow 0.$$

Thus

$$H_p(S'(\mathbf{R}^{nk}/S'(\mathbf{R}^{nk})|_{(\mathbf{R}^n)^k_{k-1}}) \otimes \Lambda[T(\mathbf{R}^{nk})]) \cong H_p(S^{nk}, \bigcup S^{k-n}_{(i,j)})$$

and $E^2_{p,-q}(\widetilde{L}) \cong E^2_{p,-q}(L_\infty)$.

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