Holomorphic M-theory and the SU(4)-invariant twist of type IIA

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BCOV with potentials refers to a modification of minimal BCOV theory where we impose certain constraints on the fields so as to make the Poisson BV structure of the theory invertible. These constraints amount to requiring that certain fields lie in the image of the divergence operator ∂, or better yet replacing ∂-closed fields in a summand $\mathrm{PV}^{d,\bullet}$ with all of $\mathrm{PV}^{d,\bullet}$ and using a fixed choice of splitting of $\partial : \mathrm{PV}^{d,\bullet} \to \mathrm{PV}^{d-1,\bullet}$ to rewrite $\mathrm{PV}^{d,\bullet} \cong \mathrm{im}\partial \oplus \ker \partial$.

Under the conjectures of Costello-Li that describe twisted type II supergravity in terms of BCOV theory, these primitives correspond to certain components of Ramond-Ramond fields, which are chosen as potentials for Ramond-Ramond field strengths.

1 Warm-up: Kodaira–Spencer theory on a Calabi–Yau surface

On a Calabi–Yau surface X, minimal Kodaira–Spencer theory is the $\mathbb{Z}/2$-graded theory described by two fundamental sets of fields:

- An odd field given by a divergence-free holomorphic vector field μ^1.
- An even field given by a holomorphic function μ^0.

In the BV formalism, minimal Kodaira–Spencer theory on X is a (degenerate) Poisson BV theory with space of fields given by

$$\begin{array}{ccc} \underline{\text{odd}} & & \underline{\text{even}} \\ & & \mathrm{PV}^{0,\bullet} \\ \mathrm{PV}^{1,\bullet} & \xrightarrow{u\partial} & u\mathrm{PV}^{0,\bullet}. \end{array}$$

We will denote this sheaf of $\mathbb{Z}/2$-graded cochain complexes by $\mathcal{E}_{\mathrm{KS}}$.

There is a local (dg) Lie algebra structure on the parity shifted object $\Pi\mathcal{E}_{\mathrm{KS}}$. The Lie bracket is defined using the Schouten-Nijenhuis bracket $[-,-]_{\mathrm{NS}}$ on polyvector fields and is given by the formula

$$[u^k\alpha, u^\ell\beta] = u^{k+\ell}[\alpha,\beta]_{\mathrm{NS}}.$$

where $k,\ell = 0,1$. Together with the differential this equips the parity shifted sheaf of cochain complexes $\Pi\mathcal{E}_{\mathrm{KS}}$ with the structure of a local (dg) Lie algebra.

The fields of minimal Kodaira–Spencer theory $\mathcal{E}_{\mathrm{KS}}$ is equipped with an odd Poisson tensor defined by

$$\Pi_{\mathrm{KS}} = (\partial \otimes 1)\delta_{\mathrm{Diag}}.$$

We introduce another theory on the Calabi–Yau surface X that we call minimal Kodaira–Spencer theory *with potentials*. The underlying vector bundle is

$$\begin{array}{ccc} \underline{\text{odd}} & & \underline{\text{even}} \\ & & \mathrm{PV}^{0,\bullet} \\ \mathrm{PV}^{2,\bullet}. & & \end{array}$$

We will denote the resulting $\mathbb{Z}/2$-graded sheaf of cochain complexes by $\mathcal{E}_{\mathrm{pot}}$.

We interpret this as the theory of "potentials" of minimal Kodaira–Spencer theory in the following way. There is a map of sheaves of cochani complexes

$$\Phi : \mathcal{E}_{\mathrm{Pot}} \to \mathcal{E}_{\mathrm{KS}}$$

which is the identity on $\mathrm{PV}^{0,\bullet}$ and given by $\partial : \mathrm{PV}^{2,\bullet} \to \mathrm{PV}^{1,\bullet}$ on the remaining component. It is immediate to see that Φ defines a map of sheaves of cochain complexes. The theory $\mathcal{E}_{\text{pot}}$ is equipped with a non-degenerate BV pairing defined by the wedge-and-integrate pairing

$$\omega_{\text{pot}}(\alpha, \beta) = \int \alpha \wedge \beta.$$

It is immediate to verify that Φ intertwines the resulting bivector ω_{pot}^{-1} and the Kodaira–Spencer Poisson bivector Π_{KS}.

In fact, the parity shifted bundle $\Pi\mathcal{E}_{\text{pot}}$ also has the structure of a local Lie algebra, and the map Φ intertwines these local Lie algebra structures.

To describe the local Lie algebra structure on minimal Kodaira–Spencer theory with potentials we use the Calabi–Yau form Ω to identify $\mathcal{E}_{\text{Pot}}$ with the sheaf of cochain complexes

$$\begin{array}{cc} \underline{\text{odd}} & \underline{\text{even}} \\ & \\ & \Omega^{2,\bullet} \\ & \\ \Omega^{0,\bullet}. & \end{array}$$

Now, note that any Calabi–Yau surface comes equipped with a holomorphic symplectic structure and there is a Poisson bracket defined on the sheaf of holomorphic functions. Since the bracket is defined in terms of holomorphic differential operators, it extends to a bracket on the Dobleault complex $\Omega^{0,\bullet}(X)$.

This further extends to a local Lie algebra structure on the semi-direct product

$$\Omega^{0,\bullet}(X) \ltimes \Pi\Omega^{2,\bullet}(X)$$

which describes the local Lie structure on $\Pi\mathcal{E}_{\text{Pot}}$. It is immediate to verify that the map $\Phi : \mathcal{E}_{\text{pot}} \to \mathcal{E}$ intertwines the two L_∞-structures.

This equipped $\mathcal{E}_{\text{pot}}$ with the structure of an interacting (non-degenerate) BV theory. It's relationship to (minimal) Kodaira–Spencer theory can be summarized as follows.

Proposition 1.1. The map $\Phi : \mathcal{E}_{\text{pot}} \to \mathcal{E}_{\text{KS}}$ determines a map of $\mathbb{P}_0$-factorization algebras on X:

$$\Phi^* : \text{Obs}_{\mathcal{E}_{\text{KS}}} \to \text{Obs}_{\mathcal{E}_{\text{Pot}}}.$$

2 BCOV theory with potentials on a CY4

Let X be a Calabi-Yau 4 fold. Minimal Kodaira-Spencer theory on X is a $\mathbb{Z}/2$-graded theory with the following fundamental fields:

- The even fields are a holomorphic function μ^0 and a ∂-closed holomorphic bivector μ^2.
- The odd fields are a divergence-free holomorphic vector field μ^1 and a ∂-closed holomorphic section μ^3 of $\wedge^3 T_X$.

The space of fields admits a locally free description obtained by including the "descendants". The descendants of the field μ^j will be denoted $u^k\mu^j$ where $k = 1, \ldots, j$. Here, $u^k\mu^j$ is a section of $\mathrm{PV}^{j-k,\bullet}$. The sheaf of cochain complexes $\mathcal{E}$ underlying minimal Kodaira–Spencer theory on X is

$$\begin{array}{ccccccc}
\underline{\text{odd}} & & \underline{\text{even}} & & \underline{\text{odd}} & & \underline{\text{even}} \\
& & & & & & \mu^0 \in \mathrm{PV}^{0,\bullet} \\
& & & & \sum u^k\mu^1 \in \mathrm{PV}^{1,\bullet} & \xrightarrow{u\partial} & u\mathrm{PV}^{0,\bullet} \\
& & \sum u^k\mu^2 \in \mathrm{PV}^{2,\bullet} & \xrightarrow{u\partial} & u\mathrm{PV}^{1,\bullet} & \xrightarrow{u\partial} & u^2\mathrm{PV}^{0,\bullet} \\
\sum u^k\mu^3 \in \mathrm{PV}^{3,\bullet} & \xrightarrow{u\partial} & u\mathrm{PV}^{2,\bullet} & \xrightarrow{u\partial} & u^2\mathrm{PV}^{1,\bullet} & \xrightarrow{u\partial} & u^3\mathrm{PV}^{0,\bullet}
\end{array}$$

The differential on this sheaf of cochain complexes is given by $\overline{\partial} + u\partial$.

There is a local Lie algebra structure on $\Pi\mathcal{E}$ using the Schouten-Nijenhuis bracket $[-,-]_{\mathrm{Sch}}$ on polyvector fields. On the fields (including the descendants) it is defined by the formula

$$[u^k\mu^i, u^\ell\mu^j] = u^{k+\ell}[\mu^i, \mu^j]_{\mathrm{NS}}.$$

The space of fields of minimal Kodaira–Spencer theory $\mathcal{E}_{\mathrm{KS}}$ is equipped with an odd Poisson tensor defined by

$$\Pi_{\mathrm{KS}} = (\partial \otimes 1)\delta_{\mathrm{Diag}}.$$

Together with the local Lie algebra structure, this data equips $\mathcal{E}$ with the structure of a $\mathbb{Z}/2$-graded Poisson BV theory.

As in the surface case, there is a closely related BV theory describing the "potentials" of minimal Kodaira–Spencer theory. The underlying sheaf of cochain complexes is

$$
\begin{array}{cccc}
\underline{\text{odd}} & \underline{\text{even}} & \underline{\text{odd}} & \underline{\text{even}} \\
 & & & \mu^0 \in \mathrm{PV}^{0,\bullet} \\
 & & \sum u^k \mu^1 \in \mathrm{PV}^{1,\bullet} \xrightarrow{u\partial} & u\mathrm{PV}^{0,\bullet} \\
u^{-1}\gamma^3 + \gamma^3 \in u^{-1}\mathrm{PV}^{4,\bullet} \xrightarrow{u\partial} & \mathrm{PV}^{3,\bullet} & & \\
\gamma^4 \in \mathrm{PV}^{4,\bullet} & & &
\end{array}
$$

We will again refer to this sheaf as $\mathcal{E}_{\text{pot}}$.

There is a local Lie algebra structure described as follows.

- There is a self bracket between the μ-fields defined by

$$[u^k \mu^i, u^\ell \tilde{\mu}^j] = u^{k+j}[\mu^i, \tilde{\mu}^j]_{\mathrm{NS}}.$$

  This gives brackets $\mathrm{PV}^{1,\bullet} \times \mathrm{PV}^{0,\bullet} \to \mathrm{PV}^{0,\bullet}$ and $\mathrm{PV}^{1,\bullet} \times \mathrm{PV}^{1,\bullet} \to \mathrm{PV}^{1,\bullet}$.

- Next, there is a self-bracket between the γ-fields defined by

$$[\gamma^3, \tilde{\gamma}^3] = [\partial\gamma^3, \tilde{\gamma}^3]_{\mathrm{NS}} \pm [\gamma^3, \partial\tilde{\gamma}^3]_{\mathrm{NS}}$$

  This gives a bracket $\mathrm{PV}^{3,\bullet} \times \mathrm{PV}^{3,\bullet} \to \mathrm{PV}^{4,\bullet}$.

- Finally, there are mixed terms in the bracket defined by

$$
\begin{aligned}
\mathrm{PV}^{4,\bullet} \times \mathrm{PV}^{0,\bullet} \to \mathrm{PV}^{3,\bullet}, &\quad [\gamma^4, \mu^0] = [\gamma^4, \mu^0]_{\mathrm{NS}} \\
\mathrm{PV}^{3,\bullet} \times \mathrm{PV}^{0,\bullet} \to \mathrm{PV}^{1,\bullet}, &\quad [\gamma^3, \mu^0] = [\partial\gamma^3, \mu^0]_{\mathrm{NS}} \\
\mathrm{PV}^{3,\bullet} \times \mathrm{PV}^{1,\bullet} \to \mathrm{PV}^{3,\bullet}, &\quad [\gamma^3, \mu^1] = [\gamma^3, \mu^1]_{\mathrm{NS}}.
\end{aligned}
$$

Together with the wedge and integrate pairing, $\mathcal{E}_{\text{pot}}$ has the structure of a nondegenerate BV theory.

Like in the case of Kodaira–Spencer theory on a complex surface, there is a map of sheaves of cochain complexes

$$\Phi : \mathcal{E}_{\text{pot}} \to \mathcal{E}_{\text{KS}}.$$

It is given by the identity map on $\text{PV}^{1,\bullet}$ and $\text{PV}^{0,\bullet}$ and the ∂ operator on $\text{PV}^{3,\bullet}$ and $\text{PV}^{4,\bullet}$. Explicitly, in formulas

$$\begin{aligned}
\Phi(u^k \mu^i) &= u^k \mu^i \in u^k \text{PV}^{i,\bullet} \; i = 0, 1 \\
\Phi(u^{-1}\gamma^3) &= \; 0 \\
\Phi(\gamma^3) &= \partial \gamma^3 \in \; \text{PV}^{2,\bullet} \\
\Phi(\gamma^4) &= \partial \gamma^4 \in \; \text{PV}^{3,\bullet}.
\end{aligned}$$

In fact, we have the following result, in analogy with the case of a Calabi-Yau surface.

Proposition 2.1. The map Φ intertwines the local Lie algebra structures and Poisson BV structures on $\mathcal{E}_{\text{KS}}$ and $\mathcal{E}_{\text{pot}}$. So, it induces a map of $\mathbb{P}_0$-factorization algebras:

$$\Phi^* : \text{Obs}_{\mathcal{E}} \to \text{Obs}_{\mathcal{E}_{\text{Pot}}}.$$

3 Dimensional Reduction

Let's consider the 11-dimensional theory on the manifold

$$\mathbb{R} \times \mathbb{C}^\times \times \mathbb{C}^4.$$

The fields decompose as follows. For the "base" direction, fields we labeled by μ, we have

$$\mu = \begin{pmatrix} \mu_{01} \\ \mu_{10} \end{pmatrix} + u\mu^0_{11d} \in \left(\begin{matrix} \text{PV}^{0,\bullet}(\mathbb{C}^\times) \otimes \text{PV}^{1,\bullet}(\mathbb{C}^4) \\ \text{PV}^{1,\bullet}(\mathbb{C}^\times) \otimes \text{PV}^{0,\bullet}(\mathbb{C}^4) \end{matrix} \to u\text{PV}^{0,\bullet}(\mathbb{C}^\times \times \mathbb{C}^4) \right) \widehat{\otimes} \Omega^\bullet(\mathbb{R})[1].$$

For the "fiber" direction, fields we labeled by γ, we have

$$\gamma = u^{-1}\gamma^0_{11d} + \begin{pmatrix} \gamma_{01} \\ \gamma_{10} \end{pmatrix} \in \left(u^{-1}\Omega^{0,\bullet}(\mathbb{C}^\times \times \mathbb{C}^4) \to \begin{matrix} \Omega^{0,\bullet}(\mathbb{C}^\times) \otimes \Omega^{1,\bullet}(\mathbb{C}^4) \\ \Omega^{1,\bullet}(\mathbb{C}^\times) \otimes \Omega^{0,\bullet}(\mathbb{C}^4) \end{matrix} \right) \widehat{\otimes} \Omega^\bullet(\mathbb{R}).$$

We consider the dimensional reduction of the theory along the circle $S^1 \subset \mathbb{C}^\times$. The dimensional reduction has the affect of replacing $\mathrm{PV}^{i,\bullet}(\mathbb{C}^\times)$ and $\Omega^{1,\bullet}(\mathbb{C}^\times)$ by the de Rham along the radial direction in $\mathbb{C}^\times$, see Proposition 1.59 of [?]. Denote the space of fields of the dimensional reduction by $\oint_{S^1} \mathcal{E}_{11d}$.

We will use the same symbols for the fields in the dimensionally reduced ten-dimensional theory: for the dimensionally reduced μ-fields we have

$$\mu = \begin{pmatrix} \mu_{01} \\ \mu_{10} \end{pmatrix} + u\mu^0_{11d} \in \begin{pmatrix} \mathrm{PV}^{1,\bullet}(\mathbb{C}^4) & \\ & \to u\mathrm{PV}^{0,\bullet}(\mathbb{C}^4) \\ \mathrm{PV}^{0,\bullet}(\mathbb{C}^4) & \end{pmatrix} \widehat{\otimes} \Omega^\bullet(\mathbb{R}^2)[1].$$

and for the dimensionally reduced γ-fields we have

$$\gamma = u^{-1}\gamma^0_{11d} + \begin{pmatrix} \gamma_{01} \\ \gamma_{10} \end{pmatrix} \in \begin{pmatrix} & \Omega^{1,\bullet}(\mathbb{C}^4) \\ u^{-1}\Omega^{0,\bullet}(\mathbb{C}^4) \to & \\ & \Omega^{0,\bullet}(\mathbb{C}^4) \end{pmatrix} \widehat{\otimes} \Omega^\bullet(\mathbb{R}^2).$$

So far, we have only described how the fields of the 11-dimensional theory behave upon dimensional reduction. One can rename the fields of the dimensionally reduced theory to match precisely with the fields of the conjectural minimal twist of Type IIA supergravity with potentials $\mathcal{E}_{\mathrm{pot}}(\mathbb{C}^4)\widehat{\otimes}\Omega^\bullet(\mathbb{R}^2)$. In fact, we have the following stronger result, which identifies the interacting BV theories of the dimensionally reduced theory with the conjectural twist of Type IIA supergravity with potentials.

Proposition 3.1. There is an equivalence of interacting classical BV theories on $\mathbb{C}^2 \times \mathbb{R}^2$

$$\Psi : \oint_{S^1} \mathcal{E}_{11d} \to \mathcal{E}_{\mathrm{pot}}(\mathbb{C}^4)\widehat{\otimes}\Omega^\bullet(\mathbb{R}^2)$$

defined on the fields by

$$\begin{array}{rcccl}
\Psi(\mu_{01}) & = & \mu_{01} & \in & \mathrm{PV}^{1,\bullet}(\mathbb{C}^4)\widehat{\otimes}\Omega^\bullet(\mathbb{R}^2) \\
\Psi(\mu_{10}) & = & \Omega^{-1}_{\mathbb{C}^4} \vee \mu_{10} & \in & \mathrm{PV}^{4,\bullet}(\mathbb{C}^4)\widehat{\otimes}\Omega^\bullet(\mathbb{R}^2) \\
\Psi(u\mu^0_{11d}) & = & u\mu^0_{11d} & \in & u\mathrm{PV}^{0,\bullet}(\mathbb{C}^4)\widehat{\otimes}\Omega^\bullet(\mathbb{R}^4) \\
\Psi(u^{-1}\gamma^0_{11d}) & = & u^{-1}\Omega^{-1}_{\mathbb{C}^4} \vee \gamma^0_{11d} & \in & u^{-1}\mathrm{PV}^{4,\bullet}(\mathbb{C}^4)\widehat{\otimes}\Omega^\bullet(\mathbb{R}^2) \\
\Psi(\gamma_{01}) & = & \Omega^{-1}_{\mathbb{C}^4} \vee \gamma_{01} & \in & \mathrm{PV}^{3,\bullet}(\mathbb{C}^4)\widehat{\otimes}\Omega^\bullet(\mathbb{R}^2) \\
\Psi(\gamma_{10}) & = & \gamma_{10} & \in & \mathrm{PV}^{0,\bullet}(\mathbb{C}^4)\widehat{\otimes}\Omega^\bullet(\mathbb{R}^2).
\end{array}.$$

That is, the reduction of the holomorphic M theory on a holomorphic circle should be the SU(4) invariant twist of IIA.

4 Compactification to five dimensions

Now, consider placing the 11-dimensional theory on the manifold

$$X \times Y \times \mathbb{R}$$

where X is compact Calabi–Yau three-fold and Y is a Calabi–Yau surface. We consider the compactification of the theory along the projection map

$$\pi : X \times Y \times \mathbb{R} \to Y \times \mathbb{R}.$$

Recall, the space of fields of the 11-dimensional theory is

$$T^*[-1]\left(\mathrm{PV}^{\leq 1,\bullet}(X \times Y)\widehat{\otimes}\Omega^\bullet(\mathbb{R})[1]\right). \tag{1}$$

Since X is compact and Calabi–Yau, we have a sequence of quasi-isomorphisms

$$\mathrm{PV}^{j,\bullet}(X) \cong_{\Omega_X} \Omega^{3-j,\bullet}(X) \simeq H^{3-j,\bullet}(X).$$

The first isomorphism is simply contraction with the Calabi–Yau form $\Omega_X \in \Omega^{3,hol}(X)$ and the second quasi-isomorphism follows from formality of X.

In particular, at the level of sheaves one has a quasi-isomorphism

$$\pi_*\left(\mathrm{PV}^{\leq 1,\bullet}(X \times Y)\widehat{\otimes}\Omega^\bullet(\mathbb{R})[1]\right) \simeq H^{\geq 2,\bullet}(X) \otimes \mathrm{PV}^{\leq 1,\bullet}(Y)\widehat{\otimes}\Omega^\bullet(\mathbb{R})[1].$$

This describes the "base" direction, fields we labeled by μ, of the space of fields (1) upon compactification.

We extract the piece of the above sheaf which involves the top cohomology $H^{3,3}(X) \cong H^6(X) \cong \mathbb{C}$. This is the sheaf

$$\mathrm{PV}^{\leq 1,\bullet}(Y)\widehat{\otimes}\Omega^\bullet(\mathbb{R})[1]. \tag{2}$$

(Recall, we are only working with sheaves of $\mathbb{Z}/2$-graded cochain complexes.)

Likewise, the "fiber" direction of (1) becomes, after compactification:

$$\pi_*\left(\Omega^{\leq 1,\bullet}(X \times Y)\widehat{\otimes}\Omega^\bullet(\mathbb{R})\right) \simeq H^{\leq 1,\bullet}(X) \otimes \Omega^{\leq 1,\bullet}(X)\widehat{\otimes}\Omega^\bullet(\mathbb{R}).$$

Under the BV pairing, the piece of this sheaf of cochain complexes with (2) is the part involving $H^0(X) \cong \mathbb{C}$. This is precisely

$$\Omega^{\leq 1}(X)\widehat{\otimes}\Omega^\bullet(\mathbb{R}).$$