Holomorphic M-theory and the SU(4)-invariant twist of type IIA

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BCOV with potentials refers to a modification of minimal BCOV theory where we impose certain constraints on the fields so as to make the Poisson BV structure of the theory invertible. These constraints amount to requiring that certain fields lie in the image of the divergence operator ∂, or better yet replacing ∂-closed fields in a summand $\mathrm{PV}^{d,\bullet}$ with all of $\mathrm{PV}^{d,\bullet}$ and using a fixed choice of splitting of $\partial : \mathrm{PV}^{d,\bullet} \to \mathrm{PV}^{d-1,\bullet}$ to rewrite $\mathrm{PV}^{d,\bullet} \cong \mathrm{im}\partial \oplus \ker \partial$.

Under the conjectures of Costello-Li that describe twisted type II supergravity in terms of BCOV theory, these primitives correspond to certain components of Ramond-Ramond fields, which are chosen as potentials for Ramond-Ramond field strengths.

1 Warm-up: Kodaira–Spencer theory on a Calabi–Yau surface

Let X be a Calabi–Yau surface. Minimal Kodaira–Spencer theory is a $\mathbb{Z}/2$-graded theory described by two fundamental sets of fields:

- An odd field given by a divergence-free holomorphic vector field μ^1.
- An even field given by a holomorphic function μ^0.

These fields combine to define a $\mathbb{Z}/2$-graded sheaf $\mathcal{E}^{\mathrm{hol}}$ on X. There is a Lie algebra structure the parity reversed sheaf $\Pi\mathcal{E}^{\mathrm{hol}}$ using the Lie bracket of holomorphic vector fields together with the natural action of holomorphic vector fields on holomorphic functions.

The sheaf $\mathcal{E}^{\mathrm{hol}}$ admits the following locally free description:

$$\begin{array}{cc} \underline{\text{odd}} & \underline{\text{even}} \\ & \\ & \mathrm{PV}^{0,\bullet} \\ & \\ \mathrm{PV}^{1,\bullet} & \xrightarrow{\partial} \mathrm{PV}^{0,\bullet}. \end{array}$$

We refer to this locally free description by $\mathcal{E}$. The Lie bracket on $\Pi\mathcal{E}^{\mathrm{hol}}$ described above extends to a Lie bracket on $\Pi\mathcal{E}$. Together with the differential this gives $\Pi\mathcal{E}$ the structure of a local dg Lie algebra.

The bundle $\mathcal{E}$ is equipped with an odd Poisson tensor defined by

$$\Pi = (\partial \otimes 1)\delta_{\mathrm{Diag}}.$$

We introduce another theory called minimal Kodaira–Spencer theory "with potentials" that we will refer to as $\mathcal{E}_{\mathrm{pot}}$. The underlying vector bundle is

$$\begin{array}{cc} \underline{\text{odd}} & \underline{\text{even}} \\ & \\ & \mathrm{PV}^{0,\bullet} \\ & \\ \mathrm{PV}^{2,\bullet}. & \end{array}$$

The parity shifted bundle $\Pi\mathcal{E}_{\mathrm{pot}}$ also has the structure of a local Lie algebra described in the following way.

First, we note that any Calabi–Yau service comes equipped with a holomorphic symplectic structure and hence there is a Poisson bracket of holomorphic functions. First, $\mathrm{PV}^{2,\bullet}$ has a Lie bracket defined by the equation

$$[\alpha, \alpha']_{\mathrm{PV}^2} = \Omega^{-1}[\Omega\alpha, \Omega\alpha'].$$

The theory $\mathcal{E}_{\mathrm{pot}}$ is a non-degenerate BV theory with BV pairing defined by the wedge-and-integrate pairing

$$\alpha, \beta \mapsto \int \alpha \wedge \beta.$$

There is a map of bundles $\Phi : \mathcal{E}_{\mathrm{Pot}} \to \mathcal{E}$ which is the identity on $\mathrm{PV}^{0,\bullet}$ and given by $\partial : \mathrm{PV}^{2,\bullet} \to \mathrm{PV}^{1,\bullet}$ on the remaining component. It is a direct calculation to check that Φ defines a map on the parity shifted local Lie algebras. In fact, we have the following.

Proposition 1.1. The map Φ determines a map of $\mathbb{P}_0$-factorization algebras on X:

$$\Phi^* : \mathrm{Obs}_{\mathcal{E}} \to \mathrm{Obs}_{\mathcal{E}_{\mathrm{Pot}}}.$$

2 BCOV theory with potentials on a CY4

Let X be a Calabi-Yau 4 fold. Minimal Kodaira-Spencer theory on X is a $\mathbb{Z}/2$-graded theory with the following fundamental fields:

- The even fields are a holomorphic function μ^0 and a ∂-closed holomorphic bivector μ^0.
- The odd fields are a divergence-free holomorphic vector field μ^1 and a ∂-closed holomorphic section of $\wedge^3 T_X$.

As before, these fields combine to define a $\mathbb{Z}/2$-graded sheaf $\mathcal{E}^{\mathrm{hol}}$ on X. The parity reversed sheaf $\Pi\mathcal{E}^{\mathrm{hol}}$ has a graded Lie algebra structure given by the Schouten-Nijenhuis bracket of holomorphic polyvector fields.

The sheaf $\mathcal{E}^{\mathrm{hol}}$ admits the following locally free description:

$$\begin{array}{cccc}
\underline{\text{odd}} & \underline{\text{even}} & \underline{\text{odd}} & \underline{\text{even}} \\
 & & & \mathrm{PV}^{0,\bullet} \\
 & & \mathrm{PV}^{1,\bullet} \xrightarrow{\partial} & \mathrm{PV}^{0,\bullet} \\
 & \mathrm{PV}^{2,\bullet} \xrightarrow{\partial} & \mathrm{PV}^{1,\bullet} \xrightarrow{\partial} & \mathrm{PV}^{0,\bullet} \\
\mathrm{PV}^{3,\bullet} \xrightarrow{\partial} & \mathrm{PV}^{2,\bullet} \xrightarrow{\partial} & \mathrm{PV}^{1,\bullet} \xrightarrow{\partial} & \mathrm{PV}^{0,\bullet}
\end{array}$$

We denote this locally free description as $\mathcal{E}$. As before, the Lie bracket induced from the one on $\Pi\mathcal{E}^{\mathrm{hol}}$ and the differntial conspire to give $\Pi\mathcal{E}$ the structure of a local dg Lie algebra. Furthermore, the sheaf $\mathcal{E}$ is equipped with an odd Poisson tensor given by $\Pi = (\partial \otimes 1)\delta_{\mathrm{Diag}}$. Together, this data equips $\mathcal{E}$ with the structure of a $\mathbb{Z}/2$-graded Poisson BV theory.

As in the surface case, there is a closely related BV theory gotten by adding "potentials" - we will again refer to this as $\mathcal{E}_{\text{pot}}$. The underlying vector bundle is

$$
\begin{array}{cccc}
\underline{\text{odd}} & \underline{\text{even}} & \underline{\text{odd}} & \underline{\text{even}} \\
 & & & \mathrm{PV}^{0,\bullet} \\
 & & \mathrm{PV}^{1,\bullet} \xrightarrow{\partial} & \mathrm{PV}^{0,\bullet} \\
 & \mathrm{PV}^{3,\bullet} & & \\
\mathrm{PV}^{4,\bullet} & & &
\end{array}
$$

There is a bundle map $\Phi : \mathcal{E}_{\text{pot}} \to \mathcal{E}$ given by the identity map on $\mathrm{PV}^{1,\bullet}$ and $\mathrm{PV}^{0,\bullet}$ and the ∂ operator on $\mathrm{PV}^{3,\bullet}$ and $\mathrm{PV}^{4,\bullet}$. We may equip the parity shifted bundle $\Pi\mathcal{E}_{\text{pot}}$ with the structure of a local dg Lie algebra such that Φ is a map of Lie algebras. (Surya: Will add explicit description of the brackets). Together with the wedge and integrate pairing, $\mathcal{E}_{\text{pot}}$ has the structure of a nondegenerate BV theory.

In fact, we have the following, analogous to the case of a Calabi-Yau surface:

Proposition 2.1. The map Φ determines a map of $\mathbb{P}_0$-factorization algebras on X:

$$\Phi^* : \mathrm{Obs}_{\mathcal{E}} \to \mathrm{Obs}_{\mathcal{E}_{\mathrm{Pot}}}.$$

3 Dimensional Reduction

Let's consider the holomorphic twist of M-theory on $\mathbb{R} \times \mathbb{C}^\times \times \mathbb{C}^4$. We may decompose the fields as

- $$\mu = \mu_{01} + \mu_{10} \in \Omega^\bullet(\mathbb{R}) \otimes \left(\begin{matrix} (\mathrm{PV}^{0,\bullet}(\mathbb{C}^\times) \otimes \mathrm{PV}^{1,\bullet}(\mathbb{C}^4)) \\ \oplus (\mathrm{PV}^{1,\bullet}(\mathbb{C}^\times) \otimes \mathrm{PV}^{0,\bullet}(\mathbb{C}^4)) \end{matrix} \to \mathrm{PV}^{0,\bullet}(\mathbb{C}^\times \times \mathbb{C}^4) \right).$$

- $\gamma = \gamma_{01} + \gamma_{10} \in \Omega^\bullet(\mathbb{R}) \otimes (\Omega^{0,\bullet}(\mathbb{C}^\times) \otimes \Omega^{1,\bullet}(\mathbb{C}^4) \oplus \Omega^{1,\bullet}(\mathbb{C}^\times) \otimes \Omega^{0,\bullet}(\mathbb{C}^4))$.

Proposition 3.1. There is a homomorphism of L_∞-algebras from the $\bar{\partial}_{\mathbb{C}^\times}$-cohomology of M theory on $\mathbb{R} \times \mathbb{C}^\times \times \mathbb{C}$ to $\Omega^\bullet(\mathbb{R}^2) \times \mathcal{E}^{C_1,C_2}_{mBCOV}$ given by

- $[\mu_{01}] \mapsto \mu^1 \in \ker \partial \subset \mathrm{PV}^{1,\bullet}(\mathbb{C}^4) \subset \mathcal{E}^{C_1,C_2}_{mBCOV}$
- $[\mu_{10}] \mapsto \mu^3 = \partial_{\mathbb{C}^4}(\mu_{10}\Omega^{-1}_{\mathbb{C}^4}) \subset \mathrm{im}\partial \subset \mathrm{PV}^{3,\bullet} \subset \mathcal{E}^{C_1,C_2}_{mBCOV}$ where $\Omega_{\mathbb{C}^4}$ denotes the holomorphic volume form on $\mathbb{C}^4$.
- $[\gamma_{01}] \mapsto \mu^2 = \partial_{C^4}(\gamma_{01} \vee \Omega^{-1}_{\mathbb{C}^4}) \subset \mathrm{im}\partial \subset \mathrm{PV}^{2,\bullet} \subset \mathcal{E}^{C_1,C_2}_{mBCOV}$.
- $[\gamma_{10}] \mapsto \mu^0 \in \mathrm{PV}^0 \subset \mathcal{E}^{C_1,C_2}_{mBCOV}$

preserving the relevant pairings.

That is, the reduction of the holomorphic M theory on a holomorphic circle should be the SU(4) invariant twist of IIA.