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Classification of infinite-dimensional simple groups of supersymmetries and quantum field theory

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Introduction

This work was motivated by two seemingly unrelated problems:

- 1. Lie's problem of classification of "local continuous transformation groups of a finite-dimensional manifold".
- 2. The problem of classification of operator product expansions (OPE) of chiral fields in 2-dimensional conformal field theory.

I shall briefly explain in § 5 how these problems are related to each other via the theory of conformal algebras [DK], [K4]–[K6]. This connection led to the classification of finite systems of chiral bosonic fields such that in their OPE only linear combinations of these fields and their derivatives occur [DK], which is, basically, what is called a "finite conformal algebra".

It is, of course, well known that a solution to Lie's problem requires quite different methods in the cases of finite- and infinite-dimensional groups.

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The most important advance in the finite-dimensional case was made by W. Killing and E. Cartan at the end of the 19^{th} century who gave the celebrated classification of simple finite-dimensional Lie algebras over \mathbb{C} . The infinite-dimensional case was studied by E. Cartan in a series of papers written in the beginning of the 20^{th} century, which culminated in his classification of infinite-dimensional "primitive" Lie algebras [C].

The advent of supersymmetry in theoretical physics in the 1970's motivated the work on the "super" extension of Lie's problem. In the finite-dimensional case the latter problem was settled in [K2]. However, it took another 20 years before the problem was solved in the infinite-dimensional case [K7], [CK2], [CK3]. An entertaining account of the historical background of the four classifications mentioned above may be found in the review [St].

A large part of my talk (§§ 1–4) is devoted to the explanation of the fourth classification, that of simple infinite-dimensional local supergroups of transformations of a finite-dimensional supermanifold. The application of this result to the second problem, that of classification of OPE when fermionic fields are allowed as well, or, equivalently, of finite conformal superalgebras, is explained in § 5.

I am convinced, however, that the classification of infinite-dimensional supergroups may have applications to "real" physics as well. The main reason for this belief is the occurrence in my classification of certain exceptional infinite-dimensional Lie supergroups that are natural extension of the compact Lie groups $SU_3 \times SU_2 \times U_1$ and SU_5 . In § 7 I formulate a system of axioms, which, via representation theory of the corresponding Lie superalgebras (see § 6), produce precisely all the multiplets of fundamental particles of the Standard model.

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1 Lie's problem and Cartan's theorem

In his "Transformation groups" paper [L] published in 1880, Lie argues as follows. Let G be a "local continuous group of transformations" of a finite-dimensional manifold. The manifold decomposes into a union of orbits, so we should first study how the group acts on an orbit and worry later how

the orbits are put together. In other words, we should first study transitive actions of G. Furthermore, even for a transitive action it may happen that Gleaves invariant a fibration by permutting the fibers. But then we should first study how G acts on fibers and on the quotient manifold and worry later how to put these actions together. We thus arrive at the problem of classifications of transitive primitive (i.e., leaving no invariant fibrations) actions.

Next Lie establishes his famous theorems that relate the action of G on a manifold M in a neighborhood of a point p to the Lie algebra of vector fields in this neighborhood generated by this action. Actually, he talks about formal vector fields in a formal neighborhood of p, hence G gives rise to a Lie algebra L of formal vector fields and its canonical filtration by subalgebras L_i of L consisting of vector fields that vanish at p up to the order j +1. Then transitivity of the action of G is equivalent to the property that $\dim L/L_0 = m := \dim M$, and primitivity is equivalent to the property that L_0 is a maximal subalgebra of L.

The first basic example is the Lie algebra of all formal vector fields in mindeterminates:

$$W_m = \left\{ \sum_{i=1}^m P_i(x) \frac{\partial}{\partial x_i} \right\}$$

 $(P_i(x))$ are formal power series in $x=(x_1,\ldots,x_m)$, endowed with the formal topology. A subalgebra L of W_m is called transitive if dim $L/L_0 = m$, where $L_0 = (W_m)_0 \cap L$. A rigorous statement of Lie's problem is as follows:

1st formulation. Classify all closed transitive subalgebras L of W_m such that L_0 is a maximal subalgebra of L, up to a continuous automorphism of W_m .

E. Cartan published a solution to this problem in the infinite-dimensional case in 1909 [C]. The result is that a complete list over \mathbb{C} (conjectured by Lie) is as follows $(m \ge 1)$:

- 2. $S_m = \{X \in W_m | \operatorname{div} X = 0\} \ (m \ge 2),$ 2'. $CS_m = \{X \in W_m | \operatorname{div} X = \operatorname{const}\} \ (m \ge 2),$

3. $H_m = \{X \in W_m | X\omega_s = 0\} \ (m = 2k),$ where $\omega_s = \sum_{i=1}^k dx_i \wedge dx_{k+i}$ is a symplectic form,

- 3'. $CH_m = \{X \in W_m | X\omega_s = \operatorname{const} \omega_s\} (m = 2k),$
- 4. $K_m = \{X \in W_m | X\omega_c = f\omega_c\} (m = 2k + 1),$

where $\omega_c = dx_m + \sum_{i=1}^k x_i dx_{k+i}$ is a contact form and f is a formal power series (depending on X).

The work of Cartan had been virtually forgotten until the sixties. A resurgence of interest in this area began with the papers [SS] and [GS], which developed an adequate language and machinery of filtered and graded Lie algebras. The work discussed in the present talk uses heavily also the ideas from [W], [K1] and [G2].

The transitivity of the action implies that L_0 contains no non-zero ideals of L. The pair (L, L_0) is called primitive if L_0 is a (proper) maximal subalgebra of L which contains no non-zero ideals of L. Using the Guillemin-Sternberg realization theorem [GS], [B1], it is easy to show [G2] that the 1st formulation of Lie's problem is equivalent to the following, more invariant, formulation:

 2^{nd} formulation. Classify all primitive pairs (L, L_0) , where L is a linearly compact Lie algebra and L_0 is its open subalgebra.

Recall that a topological Lie algebra is called linearly compact if its underlying space is isomorphic to a topological product of discretely topologized finite-dimensional vector spaces (the basic examples are: finite-dimensional spaces with discrete topology and the space of formal power series in x with formal topology).

Any linearly compact Lie algebra L contains an open (hence of finite codimension) subalgebra L_0 [G1]. Hence, if L is simple, choosing any maximal open subalgebra L_0 , we get a primitive pair (L, L_0) . One can show that there exists a unique such L_0 , and this leads to the four series 1, 2, 3 and 4. The remaining series 2' and 3' are not simple, they actually are the Lie algebras of derivations of 2 and 3, but the choice of L_0 is again unique.

Using the structure results on general transitive linearly compact Lie algebras [G1], it is not difficult to reduce, in the infinite-dimensional case, the classification of primitive pairs to the classification of simple linearly compact Lie algebras (cf. [G2]). Such a reduction is possible also in the Lie superalgebra case, but it is much more complicated for two reasons:

- (a) a simple linearly compact Lie superalgebra may have several maximal open subalgebras,
- (b) construction of arbitrary primitive pairs in terms of simple primitive pairs is more complicated in the superalgebra case.

In the next sections I will discuss in some detail the classification of infinitedimensional simple linearly compact Lie superalgebras.

2 Statement of the main theorem

The "superization" basically amounts to adding anticommuting indeterminates. In other words, given an algebra (associative or Lie) \mathcal{A} we consider the Grassmann algebra $\mathcal{A}\langle n\rangle$ in n anticommuting indeterminates ξ_1, \ldots, ξ_n over \mathcal{A} . This algebra carries a canonical $\mathbb{Z}/2\mathbb{Z}$ -gradation, called parity, defined by letting

$$p(A) = \overline{0}, \quad p(\xi_i) = \overline{1}, \quad \overline{0}, \overline{1} \in \mathbb{Z}/2\mathbb{Z}.$$

For example, $\mathbb{C}\langle n\rangle$ is the Grassmann algebra in n indeterminates over \mathbb{C} . If \mathcal{O}_m denotes the algebra of formal power series over \mathbb{C} in m indeterminates, then $\mathcal{O}_m\langle n\rangle$ is the algebra over \mathbb{C} of formal power series in m commuting indeterminates $x=(x_1,\ldots,x_m)$ and n anticommuting indeterminates $\xi=(\xi_1,\ldots,\xi_n)$:

$$x_i x_j = x_j x_i$$
, $x_i \xi_j = \xi_j x_i$, $\xi_i \xi_j = -\xi_j \xi_i$.

Recall that a derivation D of parity $p(D) \in \mathbb{Z}/2\mathbb{Z}$ of a $\mathbb{Z}/2\mathbb{Z}$ -graded algebra is a vector space endomorphism satisfying condition

$$D(ab) = (Da)b + (-1)^{p(D)p(a)}a(Db).$$

Furthermore the sum of the spaces of derivations of parity $\overline{0}$ and $\overline{1}$ is closed under the "super" bracket:

$$[D, D_1] = DD_1 - (-1)^{p(D)p(D_1)}D_1D.$$

This "super" bracket satisfies "super" analogs of anticommutativity and Jacobi identity, hence defines what is called a Lie superalgebra.

For example, the algebra $\mathcal{A}\langle n\rangle$ has derivations $\frac{\partial}{\partial \xi_i}$ of parity $\overline{1}$ defined by

$$\frac{\partial}{\partial \xi_i}(a) = 0 \text{ for } a \in \mathcal{A}, \quad \frac{\partial}{\partial \xi_i}(\xi_j) = \delta_{ij},$$

and these derivations anticommute, so that $\left[\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_i}\right] = 0.$

The "super" analog of the Lie algebra W_m is the Lie superalgebra, denoted by W(m|n), of all continuous derivations of the $\mathbb{Z}/2\mathbb{Z}$ -graded algebra $\mathcal{O}_m\langle n\rangle, n\in\mathbb{Z}_+$, with the defined above "super" bracket:

$$W(m|n) = \left\{ \sum_{i=1}^{m} P_i(x,\xi) \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} Q_j(x,\xi) \frac{\partial}{\partial \xi_j} \right\},\,$$

where $P_i(x,\xi), Q_j(x,\xi) \in \mathcal{O}_m \langle n \rangle$. In a more geometric language, this is the Lie superalgebra of all formal vector fields on a supermanifold of dimension (m|n).

There is a unique way to extend divergence from W_m to W(m|n) such that the divergenceless vector fields form a subalgebra:

$$\operatorname{div}\left(\sum_{i} P_{i} \frac{\partial}{\partial x_{i}} + \sum_{j} Q_{j} \frac{\partial}{\partial \xi_{j}}\right) = \sum_{j} \frac{\partial P_{i}}{\partial x_{i}} + \sum_{j} (-1)^{p(Q_{j})} \frac{\partial Q_{j}}{\partial \xi_{j}},$$

and the "super" analog of S_m is

$$S(m|n) = \{X \in W(m|n) | \operatorname{div} X = 0\}.$$

In order to define "super" analogs of the Hamiltonian and contact Lie algebras H_m and K_m , introduce a "super" analog of the algebra of differential forms [K2]. This is an associative algebra over $\mathcal{O}_m\langle n\rangle$, denoted by $\Omega(m|n)$, on generators $dx_1,\ldots,dx_m,d\xi_1,\ldots,d\xi_n$ and defining relations:

$$dx_i dx_j = -dx_j dx_i, \quad dx_i d\xi_j = d\xi_j dx_i, \quad d\xi_i d\xi_j = d\xi_j d\xi_i,$$

and the $\mathbb{Z}/2\mathbb{Z}$ gradation defined by:

$$p(x_i) = p(d\xi_j) = \overline{0}, \quad p(\xi_j) = p(dx_i) = \overline{1}.$$

The algebra $\Omega(m|n)$ carries a unique continuous derivation d of parity $\overline{1}$ such that

$$d(x_i) = dx_i, \quad d(\xi_j) = d\xi_j, \quad d(dx_i) = 0, \quad d(d\xi_j) = 0.$$

The operator d has all the usual properties, e.g.:

$$df = (-1)^{p(f)} \sum_{i} \frac{\partial f}{\partial x_i} dx_i + \sum_{j} \frac{\partial f}{\partial \xi_j} d\xi_j \text{ for } f \in \mathcal{O}_m \langle n \rangle, \text{ and } d^2 = 0.$$

As usual, for any $X \in W(m|n)$ one defines a derivation ι_X (contraction along X) of the algebra $\Omega(m|n)$ by the properties (here x stands for x and ξ):

$$p(\iota_X) = p(X) + \overline{1}, \quad \iota_X(x_j) = 0, \quad \iota_X(dx_j) = (-1)^{p(X)} X(x_j).$$

The action of any $X \in W(m|n)$ on $\mathcal{O}_m\langle n \rangle$ extends in a unique way to the action by a derivation of $\Omega(m|n)$ such that [X,d]=0. This is called Lie's

derivative and is usually denoted by L_X , but we shall write X in place of L_X unless confusion may arise. One has the usual Cartan's formula for this action: $L_X = [d, \iota_X]$.

Using this action, one can define super-analogs of the Hamiltonian and contact Lie algebras for any $n \in \mathbb{Z}_+$:

$$\begin{split} H(m|n) &= \{X \in W(m|n) | X\omega_s = 0\}\,,\\ \text{where } & \omega_s = \sum_{i=1}^k dx_i \wedge dx_{k+i} + \sum_{j=1}^n (d\xi_j)^2,\\ & K(m|n) &= \{X \in W(m|n) | X\omega_c = f\omega_c\}\,,\\ \text{where } & \omega_c = dx_m + \sum_{i=1}^k x_i dx_{k+i} + \sum_{j=1}^n \xi_j \, d\xi_j, \text{ and } f \in \mathcal{O}_m \langle n \rangle. \end{split}$$

Note that W(0|n), S(0|n) and H(0|n) are finite-dimensional Lie superalgebras. The Lie superalgebras W(0|n) and S(0|n) are simple iff $n \geq 2$ and $n \geq 3$, respectively. However, H(0|n) is not simple as its derived algebra H'(0|n) has codimension 1 in H(0|n), but H'(0|n) is simple iff $n \geq 4$. Thus, in the Lie superalgebra case the lists of simple finite- and infinite-dimensional algebras are much closer related than in the Lie algebra case.

The four series of Lie superalgebras are infinite-dimensional if $m \geq 1$, in which case they are simple except for S(1|n). The derived algebra S'(1|n) has codimension 1 in S(1|n), and S'(1|n) is simple iff $n \geq 2$.

In my paper [K2] I conjectured that the constructed above four series exhaust all infinite-dimensional simple linearly compact Lie superalgebras. Remarkably, the situation turned out to be much more exciting.

As was pointed out by several mathematicians, the Schouten bracket [Sc] makes the space of polyvector fields on a *m*-dimensional manifold into a Lie superalgebra. The formal analog of this is the following fifth series of superalgebras, called by physicists the Batalin-Vilkoviski algebra:

$$HO(m|m) = \{X \in W(m|m)|X\omega_{os} = 0\},\,$$

where $\omega_{os} = \sum_{i=1}^{m} dx_i d\xi_i$ is an odd symplectic form. Furthermore, unlike in the H(m|n) case, not all vector fields of HO(m|n) have zero divergence, which gives rise to the sixth series:

$$SHO(m|m) = \{X \in HO(m|m)| \operatorname{div} X = 0\}.$$

The seventh series is the odd analog of K(m|n) [ALS]:

$$KO(m|m+1) = \{X \in W(m|m+1)|X\omega_{oc} = f\omega_{oc}\},\,$$

where $\omega_{oc} = d\xi_{m+1} + \sum_{i=1}^{m} (\xi_i dx_i + x_i d\xi_i)$ is an odd contact form. One can take again the divergence 0 vector fields in KO(m|m+1) in order to construct the eighth series, but the situation is more interesting. It turns out that for each $\beta \in \mathbb{C}$ one can define the deformed divergence $\operatorname{div}_{\beta} X$ [Ko], [K7], so that $\operatorname{div} = \operatorname{div}_0$ and

$$SKO(m|m+1;\beta) = \{X \in KO(m|m+1)| \operatorname{div}_{\beta} X = 0\}$$

is a subalgebra. The superalgebras HO(m|m) and KO(m|m+1) are simple iff $m \geq 2$ and $m \geq 1$, respectively. The derived algebra SHO'(m|m) has codimension 1 in SHO(m|m), and it is simple iff $m \geq 3$. The derived algebra $SKO'(m|m+1;\beta)$ is simple iff $m \geq 2$, and it coincides with $SKO(m|m+1;\beta)$ unless $\beta = 1$ or $\frac{m-2}{m}$ when it has codimension 1.

Some of the examples described above have simple "filtered deformations", all of which can be obtained by the following simple construction. Let L be a subalgebra of W(m|n), where n is even. Then it happens in three cases that

$$L^{\sim} := (1 + \prod_{j=1}^{n} \xi_j) L$$

is different from L, but is closed under bracket. As a result we get the following three series of superalgebras: $S^{\sim}(0|n)$ [K2], $SHO^{\sim}(m|m)$ [CK2] and $SKO^{\sim}(m|m+1;\frac{m+2}{m})$ [Ko] (the constructions in [Ko] and [CK2] were more complicated). We thus get the ninth and the tenth series of simple infinite-dimensional Lie superalgebras:

$$SHO^{\sim}(m|m), \quad m \geq 2, \ m \ \text{even} \ ,$$

$$SKO^{\sim}(m|m+1; \frac{m+2}{m}), \ m \geq 3, m \ \text{odd} \ .$$

It is appropriate to mention here that the four series W(0|n), S(0|n), $S^{\sim}(0|n)$ and H'(0|n) along with the classical series $s\ell(m|n)$ and osp(m|n), strange series p(n) and q(n), two exceptional superalgebras of dimension 40 and 31 and a family of 17-dimensional exceptional superalgebras along with the marvelous five exceptional Lie algebras, comprise a complete list of simple finite-dimensional Lie superalgebras [K2].

A surprising discovery was made in [Sh1] where the existence of three exceptional simple infinite-dimensional Lie superalgebras was announced. The proof of the existence along with one more exceptional example was given

in [Sh2]. An explicit construction of these four examples was given later in [CK3]. The fifth exceptional example was found in the work on conformal algebras [CK1] and independently in [Sh2]. (The alleged sixth exceptional example E(2|2) of [K7] turned out to be isomorphic to SK0(2|3;1) [CK3].)

Theorem 1. [K7] The complete list of simple infinite-dimensional linearly compact Lie superalgebras consists of ten series of examples described above and five exceptional examples: E(1|6), E(3|6), E(3|8), E(4|4), and E(5|10).

Now I can state the main theorem.

It happens that all infinite-dimensional simple linearly compact Lie algebras L have a unique transitive primitive action [G2]. This is certainly false in the Lie superalgebra case. However, if L is a simple linearly compact Lie superalgebra of type X(m|n), it happens that m is minimal such that L acts on a super-manifold of dimension (m|n) (i.e., $L \subset W(m|n)$), n is minimal for this m, and L has a unique action with such minimal (m|n). Incidentally, in all cases the growth of L equals m. (Recall that growth is the minimal m for which dim L/L_i is bounded by P(j), where P is a polynomial of degree m.)

Let me now describe those linearly compact infinite-dimensional Lie superalgebras L that allow a transitive primitive action. Let S be a simple linearly compact infinite-dimensional Lie superalgebra and let $S\langle n\rangle$ denote, as before, the Grassmann algebra over S with n indeterminates. The Lie superalgebra $\mathrm{Der}(S\langle n\rangle)$ of all derivations of the Lie superalgebra $S\langle n\rangle$ is the following semi-direct sum:

$$\operatorname{Der}(S\langle n\rangle) = (\operatorname{Der} S)\langle n\rangle + W(0|n).$$

(For a description of Der S see [K7], Proposition 6.1.) Denote by $\mathcal{L}(S, n)$ the set of all open subalgebras L of $\mathrm{Der}(S\langle n\rangle)$ that contain $S\langle n\rangle$ and have the property that the canonical image of L in W(0|n) is a transitive subalgebra.

Using a description of semi-simple linearly compact Lie superalgebras similar to the one given by Theorem 6 from [K2] (cf. [Ch], [G1] and [B2]) and Proposition 4.1 from [G1], it is easy to derive the following result.

Proposition 1. If a linearly compact infinite-dimensional Lie superalgebra L allows a transitive primitive action, then L is one of the algebras of the sets $\mathcal{L}(S, n)$.

Example. Consider the semidirect sum $L = S\langle n \rangle + R$, where R is a transitive subalgebra of W(0|n). Then $(L, L_0 = S_0\langle n \rangle + R)$ is a primitive pair if S_0 is a

maximal open subalgebra of S, and these are all primitive pairs in the case when S = Der S. One can also replace in this construction S by Der S and S_0 by a maximal open subalgebra of Der S having no non-zero ideals of Der S.

3 Explanation of the proof of Theorem 1

Step 1. Introduce Weisfeiler's filtration [W] of L. For that choose a maximal open subalgebra L_0 of L and a minimal subspace L_{-1} satisfying the properties:

$$L_{-1} \supseteq L_0, \quad [L_0, L_{-1}] \subset L_{-1}.$$

Geometrically this corresponds to a choice of a primitive action of L and an invariant irreducible differential system. The pair L_{-1} , L_0 can be included in a unique filtration:

$$L = L_{-d} \supset L_{-d+1} \supset \cdots \supset L_{-1} \supset L_0 \supset L_1 \supset \cdots$$

Of course, if L leaves invariant no non-trivial differential system, then the "depth" d=1 and the Weisfeiler filtration coincide with the canonical filtration. Incidentally, in the Lie algebra case, d>1 only for K_n (when d=2), but in the Lie superalgebra case, d>1 in the majority of cases.

The associated to Weisfeiler's filtration \mathbb{Z} -graded Lie superalgebra is of the form $GrL = \prod_{j \geq -d} \mathfrak{g}_j$, and has the following properties:

- (G0) $\dim \mathfrak{g}_j < \infty \text{ (since codim } L_0 < \infty),$
- (G1) $\mathfrak{g}_{-j} = \mathfrak{g}_{-1}^j$ for $j \ge 1$ (by maximality of L_0),
- (G2) $[x, \mathfrak{g}_{-1}] = 0$ for $x \in \mathfrak{g}_j, j \ge 0 \Rightarrow x = 0$ (by simplicity of L),
- (G3) \mathfrak{g}_0 -module \mathfrak{g}_{-1} is irreducible (by choice of L_{-1}).

Weisfeiler's idea was that property (G3) is so restrictive, that it should lead to a complete classification of \mathbb{Z} -graded Lie algebras satisfying (G0)–(G3). (Incidentally, the infinite-dimensionality of L and hence of GrL, since L is simple, is needed only in order to conclude that $\mathfrak{g}_1 \neq 0$.) This indeed turned out to be the case [K1]. In fact, my idea was to replace the condition of finiteness of the depth by finiteness of the growth, which allowed one to

add to the Lie-Cartan list some new Lie algebras, called nowadays affine Kac-Moody algebras.

However, unlike in the Lie algebra case, it is impossible to classify all finite-dimensional irreducible faithful representations of Lie superalgebras. One needed a new idea to make this approach work.

Step 2. The main new idea is to choose L_0 to be invariant with respect to all inner automorphisms of L (meaning to contain all even ad-exponentiable elements of L). A non-trivial point is the existence of such L_0 . This is proved by making use of the characteristic supervariety, which involves rather difficult arguments of Guillemin [G2], that, unfortunately, I was unable to simplify.

Next, using a normalizer trick of Guillemin [G2], I prove, for the above choice of L_0 , the following very powerful restriction on the \mathfrak{g}_0 -module \mathfrak{g}_{-1} :

(G4)
$$[\mathfrak{g}_0, x] = \mathfrak{g}_{-1}$$
 for any non-zero even element x of \mathfrak{g}_{-1} .

Step 3. Consider a faithful irreducible representation of a Lie superalgebra \mathfrak{p} in a finite-dimensional vector space V. This representation is called strongly transitive if

$$\mathfrak{p} \cdot x = V$$
 for any non-zero even element $x \in V$.

Note that property (G4) along with (G0), (G2) for j = 0 and (G3), shows that the \mathfrak{g}_0 -module \mathfrak{g}_{-1} is strongly transitive.

In order to demonstrate the power of this restriction, consider first the case when \mathfrak{p} is a Lie algebra and V is purely even. Then the strong transitivity simply means that $V\setminus\{0\}$ is a single orbit of the Lie group P corresponding to \mathfrak{p} . It is rather easy to see that the only strongly transitive subalgebras \mathfrak{p} of $g\ell_V$ are $g\ell_V$, $s\ell_V$, sp_V and csp_V . These four cases lead to GrL, where $L=W_n,\ S_n,\ H_n$ and K_n , respectively.

In the super case the situation is much more complicated. First we consider the case of "inconsistent gradation", meaning that \mathfrak{g}_{-1} contains a nonzero even element. The classification of such strongly transitive modules is rather long and the answer consists of a dozen series and a half dozen exceptions (see [K7], Theorem 3.1). Using similar restrictions on $\mathfrak{g}_{-2}, \mathfrak{g}_{-3}, \ldots$, we obtain a complete list of possibilities for

$$GrL_{\leq} := \bigoplus_{j \leq 0} \mathfrak{g}_j$$

in the case when \mathfrak{g}_{-1} contains non-zero even elements. It turns out that all but one exception are not exceptions at all, but correspond to the beginning members of some series. As a result, only E(4|4) "survives" (but the infamous E(2|2) doesn't).

Step 4. Next, we turn to the case of a consistent gradation, i.e., when \mathfrak{g}_{-1} is purely odd. But then \mathfrak{g}_0 is an "honest" Lie algebra, having a faithful irreducible representation in \mathfrak{g}_{-1} (condition (G4) becomes vacuous). An explicit description of such representations is given by the classical Cartan-Jacobson theorem. In this case I use the "growth" method developed in [K1] and [K2] to determine a complete list of possibilities for GrL_{\leq} . This case produces mainly the (remaining four) exceptions.

Step 5 is rather long and tedious [CK3] . For each GrL_{\leq} obtained in Steps 3 and 4 we determine all possible "prolongations", i.e., infinite-dimensional \mathbb{Z} -graded Lie superalgebras satisfying (G2), whose negative part is the given GrL_{\leq} .

Step 6. It remains to reconstruct L from GrL, i.e., to find all possible filtered simple linearly compact Lie superalgebras L with given GrL (such an L is called a simple filtered deformation of GrL). Of course, there is a trivial filtered deformation: $GrL := \prod_{j \geq -d} \mathfrak{g}_j$, which is simple iff GrL is.

It is proved in [CK2] by a long and tedious calculation that only SHO(m|m) for m even ≥ 2 and $SKO(m|m+1;\frac{m+2}{m})$ for m odd ≥ 3 have a non-trivial simple filtered deformation, which are the ninth and tenth series. It would be nice to have a more conceptual proof. Recall that SHO(m|m) is not simple, though it does have a simple filtered deformation. Note also that in the Lie algebra case all filtered deformations are trivial.

4 Construction of exceptional linearly compact Lie superalgebras

In order to describe the construction of the exceptional infinite-dimensional Lie superalgebras (given in [CK3]), I need to make some remarks. Let $\Omega_m = \Omega(m|0)$ be the algebra of differential forms over \mathcal{O}_m , let Ω_m^k denote the space of forms of degree k, and $\Omega_{m,c\ell}^k$ the subspace of closed forms. For any $\lambda \in \mathbb{C}$ the representation of W_m on Ω_m^k can be "twisted" by letting

$$X \mapsto L_X + \lambda \operatorname{div} X, \quad X \in W_m$$

to get a new W_m -module, denoted by $\Omega_m^k(\lambda)$ (the same can be done for $W_{m,n}$). Obviously, $\Omega_m^k(\lambda) = \Omega_m^k$ when restricted to S_m . Then we have the following obvious W_m -module isomorphisms: $\Omega_m^0 \simeq \Omega_m^m(-1)$ and $\Omega_m^0(1) \simeq \Omega_m^m$. Furthermore, the map $X \mapsto \iota_X(dx_1 \wedge \ldots \wedge dx_m)$ gives the following W_m -module and S_m -module isomorphisms:

$$W_m \simeq \Omega_m^{m-1}(-1), \quad S_m \simeq \Omega_{m,c\ell}^{m-1}.$$

We shall identify the representation spaces via these isomorphisms.

The simplest is the construction of the largest exceptional Lie superalgebra E(5|10). Its even part is the Lie algebra S_5 , its odd part is the space of closed 2-forms $\Omega^2_{5,c\ell}$. The remaining commutators are defined as follows for $X \in S_5$, $\omega, \omega' \in \Omega^2_{5,c\ell}$:

$$[X, \omega] = L_X \omega, \quad [\omega, \omega'] = \omega \wedge \omega' \in \Omega^4_{5,c\ell} = S_5.$$

Each quintuple of integers (a_1, a_2, \ldots, a_5) such that $a = \sum_i a_i$ is even, defines a \mathbb{Z} -gradation of E(5|10) by letting:

$$\deg x_i = -\frac{\partial}{\partial x_i} = a_i, \quad \deg dx_i = a_i - \frac{1}{4}a.$$

The quintuple (2, 2, ..., 2) defines the (only) consistent \mathbb{Z} -gradation, which has depth 2: $E(5|10) = \prod_{j \geq -2} \mathfrak{g}_j$, and one has:

$$\mathfrak{g}_0 \simeq s\ell_5$$
 and $\mathfrak{g}_{-1} \simeq \Lambda^2 \mathbb{C}^5$, $\mathfrak{g}_{-2} \simeq \mathbb{C}^{5*}$ as \mathfrak{g}_0 -modules.

Furthermore, $\Pi_{j\geq 0}\mathfrak{g}_j$ is a maximal open subalgebra of E(5|10) (the only one which is invariant with respect to all automorphisms). There are three other maximal open subalgebras in E(5|10), associated to \mathbb{Z} -gradations corresponding to quintuples (1,1,1,1,2), (2,2,2,1,1) and (3,3,2,2,2), and one can show that these four are all, up to conjugacy, maximal open subalgebras (cf. [CK3]).

Another important \mathbb{Z} -gradation of E(5|10), which is, unlike the previous four, by infinite-dimensional subspaces, corresponds to the quintuple (0,0,0,1,1) and has depth 1: $E(5|10) = \Pi_{\lambda \geq -1} \mathfrak{g}^{\lambda}$. One has: $\mathfrak{g}^{0} \simeq E(3|6)$ and the \mathfrak{g}^{λ} form an important family of irreducible E(3|6)-modules [KR]. The consistent \mathbb{Z} -gradation of E(5|10) induces that of $\mathfrak{g}^{0}: E(3|6) = \Pi_{j \geq -2} \mathfrak{a}_{j}$, where

$$\mathfrak{a}_0 \simeq s\ell_3 \oplus s\ell_2 \oplus g\ell_1, \quad \mathfrak{a}_{-1} \simeq \mathbb{C}^3 \boxtimes \mathbb{C}^2 \boxtimes \mathbb{C}, \quad \mathfrak{a}_{-2} \simeq \mathbb{C}^3 \boxtimes \mathbb{C} \boxtimes \mathbb{C}.$$

A more explicit construction of E(3|6) is as follows [CK3]: the even part is $W_3 + \Omega_3^0 \otimes s\ell_2$, the odd part is $\Omega_3^1(-\frac{1}{2}) \otimes \mathbb{C}^2$ with the obvious action of the even part, and the bracket of two odd elements is defined as follows:

$$[\omega \otimes u, \omega' \otimes v] = (\omega \wedge \omega') \otimes (u \wedge v) + (d\omega \wedge \omega' + \omega \wedge d\omega') \otimes (u \cdot v).$$

Here the identifications $\Omega_3^2(-1) = W_3$ and $\Omega_3^0 = \Omega_3^3(-1)$ are used.

The gradation of E(5|10) corresponding to the quintuple (0, 1, 1, 1, 1) has depth 1 and its 0th component is isomorphic to E(1|6) (cf. [CK3]).

The construction of E(4|4) is also very simple [CK3]: The even part is W_4 , the odd part is $\Omega_4^1(-\frac{1}{2})$ and the bracket of two odd elements is:

$$[\omega, \omega'] = d\omega \wedge \omega' + \omega \wedge d\omega' \in \Omega_4^3(-1) = W_4.$$

The construction of E(3|8) is slightly more complicated, and we refer to [CK3] for details.

5 Classification of superconformal algebras

Superconformal algebras have been playing an important role in superstring theory and in conformal field theory. Here I will explain how to apply Theorem 1 to the classification of "linear" superconformal algebras. By a ("linear") superconformal algebra I mean a Lie superalgebra $\mathfrak g$ spanned by coefficients of a finite family F of pairwise local fields such that the following two properties hold:

(1) for $a, b \in F$ the singular part of OPE is finite, i.e.,

$$[a(z), b(w)] = \sum_{j} c_{j}(w) \partial_{w}^{j} \delta(z - w) \quad \text{(a finite sum)},$$

where all $c_j(w) \in \mathbb{C}[\partial_w]F$,

(2) \mathfrak{g} contains no non-trivial ideals spanned by coefficients of fields from a $\mathbb{C}[\partial_w]$ -submodule of $\mathbb{C}[\partial_w]F$.

This problem goes back to the physics paper [RS], some progress in its solution was made in [K6] and a complete solution was stated in [K4], [K5]. (A complete classification even in the "quadratic" case seems to be a much

harder problem, see [FL] for some very interesting examples.) The simplest example is the loop algebra $\widetilde{\mathfrak{g}}=\mathbb{C}[x,x^{-1}]\otimes\mathfrak{g}$ (= centerless affine Kac-Moody (super)algebra), where \mathfrak{g} is a simple finite-dimensional Lie (super)algebra. Then $F=\{a(z)=\sum_{x\in\mathbb{Z}}(x^n\otimes a)z^{-n-1}\}_{a\in\mathfrak{g}},$ and $[a(z),b(w)]=[a,b](w)\delta(z-w).$

The next example is the Lie algebra $\operatorname{Vect} \mathbb{C}^{\times}$ of regular vector fields on \mathbb{C}^{\times} (= centerless Virasoro algebra); F consists of one field, the Virasoro field $L(z) = -\sum_{n \in \mathbb{Z}} (x^n \frac{d}{dx}) z^{-n-2}$, and $[L(z), L(w)] = \partial_w L(w) \delta(z-w) + 2L(w) \delta'_w(z-w)$. One of the main theorems of $[\operatorname{DK}]$ states that these are all examples in

One of the main theorems of [DK] states that these are all examples in the Lie algebra case. The strategy of the proof is the following. Let $\partial = \partial_z$ and consider the (finitely generated) $\mathbb{C}[\partial]$ -module $R = \mathbb{C}[\partial]F$. Define the " λ -bracket" $R \otimes R \to \mathbb{C}[\lambda] \otimes R$ by the formula:

$$[a_{\lambda}b] = \sum_{j} \lambda^{j} c_{j} .$$

This satisfies the axioms of a conformal (super)algebra (see [DK], [K4]), similar to the Lie (super)algebra axioms:

(i)
$$[\partial a_{\lambda}b] = -\lambda[a_{\lambda}b], [a_{\lambda}\partial b] = (\partial + \lambda)[a_{\lambda}b],$$

(ii)
$$[a_{\lambda}b] = -[b_{-\lambda-\partial}a],$$

(iii)
$$[a_{\lambda}[b_{\mu}c]] = [[a_{\lambda}b]_{\lambda+\mu}c] + (-1)^{p(a)p(b)}[b_{\mu}[a_{\lambda}c]].$$

The main observation of [DK] is that a conformal (super)algebra is completely determined by the Lie (super)algebra spanned by all coefficients of negative powers of z of the fields a(z), called the annihilation algebra, along with an even surjective derivation of the annihilation algebra. Furthermore, apart from the case of current algebras, the completed annihilation algebra turns out to be an infinite-dimensional simple linearly compact Lie (super)algebra of growth 1. Since in the Lie algebra case the only such example is W_1 , the proof is finished.

In the superalgebra case the situation is much more interesting since there are many infinite-dimensional simple linearly compact Lie superalgebras of growth 1. By Theorem 1, the complete list is as follows:

$$W(1|n)$$
, $S'(1|n)$, $K(1|n)$ and $E(1|6)$.

The corresponding superconformal algebras in the first three cases are defined in the same way, except that we replace $\mathcal{O}_1\langle n \rangle$ by $\mathbb{C}((x))\langle n \rangle$; denote them

by $W_{(n)}$, $S_{(n)}$ and $K_{(n)}$, respectively. The superconformal algebras $W_{(n)}$ and $K_{(n)}$ are simple for $n \geq 0$, except for $K_{(4)}$ which should be replaced by its derived algebra $K'_{(4)}$, and $S'_{(n)}$ is simple for $n \geq 2$. The unique superconformal algebra corresponding to E(1|6) is denoted by $CK_{(6)}$. Its construction is more difficult and may be found in [CK3] or [K6].

However, the superconformal algebra with the annihilation algebra S'(1|n) is not unique since there are two up to conjugacy even surjective derivations ∂ of S'(1|n). In order to show this, we may assume that both ∂ and S'(1|n) are in W(1|n). Using a change of indeterminates, we may assume that $\partial = \frac{\partial}{\partial x}$, but then the standard volume form v that defines div, may change to $P(\xi)v$ (since it must be annihilated by $\frac{\partial}{\partial x}$). Further change of indeterminates brings this form to $(1 + \epsilon \xi_1 \dots \xi_n)v$, where $\epsilon = 0$ or 1. This gives us deformations of $S'_{(n)}$ with the annihilation algebra S'(1|n), which are derived algebras of

$$S_{(n),\epsilon,a} = \{X \in W_{(n)} | \operatorname{div}(e^{ax}(1 + \epsilon \xi_1 \dots \xi_n)X) = 0\}, \ a \in \mathbb{C}.$$

(The situation is more interesting in the case n=2, since the algebra of outer derivations of S'(1|2) is 3-dimensional [P], but this gives no new superconformal algebras.) One argues similarly in the case K(1|n). The case E(1|6) is checked directly. We thus have arrived at the following theorem.

Theorem 2. A complete list of superconformal algebras consists of loop algebras $\widetilde{\mathfrak{g}}$, where \mathfrak{g} is a simple finite-dimensional Lie superalgebra and of Lie superalgebras $(n \in \mathbb{Z}_+)$: $W_{(n)}$, $S'_{(n+2),\epsilon,a}$ $(n \text{ even and } a = 0 \text{ if } \epsilon = 1)$, $K_{(n)}(n \neq 4)$, $K'_{(4)}$, and $CK_{(6)}$.

Note that the first members of the above series are well-known superalgebras: $W_{(0)} \simeq K_{(0)}$ is the Virasoro algebra, $K_{(1)}$ is the Neveu-Schwarz algebra, $K_{(2)} \simeq W_{(1)}$ is the N=2 algebra, $K_{(3)}$ is the N=3 algebra, $S'_{(2)}=S'_{(2),0,0}$ is the N=4 algebra, $K'_{(4)}$ is the big N=4 algebra. These algebras, along with $W_{(2)}$ and $CK_{(6)}$ are the only superconformal algebras for which all fields are primary with positive conformal weights [K6]. It is interesting to note that all of them are contained in $CK_{(6)}$, which consists of 32 fields, the even ones are the Virasoro fields and 15 currents that form \widetilde{so}_6 , and the odd ones are 6 and 10 fields of conformal weight 3/2 and 1/2, respectively. Here is the table of inclusions, where in square brackets the number of fields is indicated:

$$\begin{array}{cccc} CK_{(6)}[32] & \supset & W_{(2)}[12] & \supset W_{(1)} = K_{(2)}[4] \supset K_{(1)}[2] \supset \mathrm{Vir} \\ & \cup & & \cup \\ K_{(3)}[8] \subset K'_{(4)}[16] & & S'_{(2),\epsilon,a}[8] \end{array}.$$

All of these Lie superalgebras have a unique non-trivial central extension, except for K'_4 that has three [KL] and $CK_{(6)}$ that has none. All other superalgebras listed by Theorem 2 have no non-trivial central extensions. (The presence of a central term is necessary for the construction of an interesting conformal field theory.)

6 Representations of linearly compact Lie superalgebras

By a representation of a linearly compact Lie superalgebra L we shall mean a continuous representation in a vector space V with discrete topology (then the contragredient representation is a continuous representation in a linearly compact space V^*). Fix an open subalgebra L_0 of L. We shall assume that V is locally L_0 -finite, meaning that any vector of V is contained in a finite-dimensional L_0 -invariant subspace (this property actually often implies that V is continuous). These kinds of representations were studied in the Lie algebra case by Rudakov [R].

It is easy to show that such an irreducible L-module V is a quotient of an induced module $\operatorname{Ind}_{L_0}^L U = U(L) \otimes_{U(L_0)} U$, where U is a finite-dimensional irreducible L_0 -module, by a (unique in good cases) maximal submodule. The induced module $\operatorname{Ind}_{L_0}^L U$ is called degenerate if it is not irreducible. An irreducible quotient of a degenerate induced module is called a degenerate irreducible module.

One of the most important problems of representation theory is to determine all degenerate representations. I will state here the result for L = E(3|6) with $L_0 = \Pi_{j\geq 0}\mathfrak{a}_j$ (see §4), so that the finite-dimensional irreducible L_0 -modules are actually $\mathfrak{a}_0 = s\ell_3 \oplus s\ell_2 \oplus g\ell_1$ -modules (with $\Pi_{j>0}\mathfrak{a}_j$ acting trivially). We shall normalize the generator Y of $g\ell_1$ by the condition that its eigenvalue on \mathfrak{a}_{-1} is -1/3. The finite-dimensional irreducible \mathfrak{a}_0 -modules are labeled by triples (mn, b, Y), where mn (resp. b) are labels of the highest weight of an irreducible representation of $s\ell_3$ (resp. $s\ell_2$), so that m0 and 0m label $S^m\mathbb{C}^3$ and $S^m\mathbb{C}^{3*}$ (resp. b labels $S^b\mathbb{C}^2$), and Y is the eigenvalue of the central element Y. Since irreducible E(3|6)-modules are unique quotients of induced modules, they can be labeled by the above triples as well.

Theorem 3. [KR] The complete list of irreducible degenerate E(3|6)-modules

is as follows $(m, b \in \mathbb{Z}_+)$:

$$(0m, b, -b - \frac{2}{3}m - 2), (0m, b, b - \frac{2}{3}m), (m0, b, -b + \frac{2}{3}m), (m0, b, b + \frac{2}{3}m + 2).$$

7 Fundamental particle multiplets

In order to explain the connection of representation theory of linearly compact Lie superalgebras to particle physics, let me propose the following axiomatics of fundamental particles:

- A. The algebra of symmetries is a linearly compact Lie superalgebra L with an element Y, called the hypercharge operator, such that
 - (i) ad Y is diagonalizable and normalized such that its spectrum is bounded below, $\subset \frac{1}{3}\mathbb{Z}$ and $\not\subset \mathbb{Z}$,
 - (ii) the centralizer of Y in L is $\mathfrak{a}_0 = s\ell_3 + s\ell_2 + \mathbb{C}Y$ (one may weaken this by requiring \supset in place of =).
- B. A particle multiplet is an irreducible subrepresentation of \mathfrak{a}_0 in a degenerate irreducible representation of L. Particles in a multiplet are linearly independent eigenvectors of the $s\ell_2$ generator $I_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Charge Q of a particle is given by the Gell-Mann-Nishijima formula:

$$Q = (I_3 \text{ eigenvalue}) + \frac{1}{2} \text{ (hypercharge)}.$$

- C. Fundamental particle multiplet is a particle multiplet such that
 - (i) $|Q| \le 1$ for all particles of the multiplet,
 - (ii) only the 1-dimensional, the two fundamental representations or the adjoint representation of $s\ell_3$ occur.

Using Theorem 3, it is easy to classify all fundamental multiplets when the algebra of symmetries L = E(3|6) [KR]. The answer is given in the left half of Table 1. The right half contains all the fundamental particles of the Standard model (see e.g. [O]): the upper part is comprised of three generations of quarks and the middle part of three generations of leptons (these are all fundamental fermions from which matter is built), and the lower part is comprised of fundamental bosons (which mediate the strong and electro-weak interactions). Except for the last line, the match is perfect.

	r	Table 1:		
multiplets	charges		particles	
(01, 1, 1/3)	2/3, -1/3	$\binom{u_L}{d_L}$	$\binom{c_L}{s_L}$	${t_L \choose b_L}$
(10, 1, -1/3)	-2/3, 1/3	${\widetilde{\widetilde{u}}_R \choose \widetilde{d}_R}$	${\widetilde{c}_R \choose \widetilde{s}_R}$	${\widetilde{t}_R \choose \widetilde{b}_R}$
(10, 0, -4/3)	-2/3	\widetilde{u}_L	\widetilde{c}_L	\widetilde{t}_R
(01, 0, 4/3)	2/3	u_R	c_R	t_R
(01, 0, -2/3)	-1/3	d_R	s_R	b_R
(10, 0, 2/3)	1/3	\widetilde{d}_L	\widetilde{s}_L	\widetilde{b}_L
(00, 1, -1)	0, -1	$\binom{ u_L}{e_L}$	$egin{pmatrix} u_{\mu L} \\ \mu_L \end{pmatrix}$	$egin{pmatrix} u_{ au L} \\ au_L \end{pmatrix}$
(00, 1, 1)	0, 1	$egin{pmatrix} \widetilde{ u}_R \ \widetilde{e}_R \end{pmatrix}$	$inom{\widetilde{ u}_{\mu R}}{\widetilde{\mu}_R}$	$inom{\widetilde{ u}_{ au R}}{\widetilde{ au}_R}$
(00, 0, 2)	1	\widetilde{e}_L	$\widetilde{\mu}_L$	$\widetilde{ au}_L$
(00,0,-2)	-1	e_R	μ_R	$ au_R$
(11, 0, 0)	0	gluons		
(00, 2, 0)	1, -1, 0	W^+, W^-, Z	(gauge bosons)	
(00, 0, 0)	0	γ	(photon)	
$(11, 0, \pm 2)$	±1	_		

8 Speculations and visions

As the title of the conference suggests, each speaker is expected to propose his (or her) visions in mathematics for the 21st century. This is an obvious invitation to be irresponsibly speculative. Some of the items proposed below are of this nature, but some others are less so.

- 1. It is certainly impossible to classify all simple infinite-dimensional Lie algebras or superalgebras. The most popular types of conditions that have emerged in the past 30 years and that I like most are these:
 - (a) existence of a gradation by finite-dimensional subspaces and finiteness of growth [K1], [M].
 - (b) topological conditions [G2], [K7], §§ 1–3,
 - (c) the condition of locality [DK], [K4], [K5], § 5.

Problem (a) in the Lie algebra case has been completely solved in [M], but an analogous conjecture in the Lie superalgebra case [KL] is apparently much harder.

Concerning (b), let me state a concrete problem. Let $L = \mathbb{C}((x))^n$, where $\mathbb{C}((x))$ is the space of formal Laurent series in x with formal topology. Examples of simple topological Lie algebras with the underlying space L are the completed (centerless) affine and Virasoro algebras. Are there any other examples?

Incidentally, after going to the dual, Theorem 1 gives a complete classification of simple Lie co-superalgebras.

- 2. In §5 I explained how to use classification of simple linearly compact Lie superalgebras of growth 1 in order to classify simple "linear" OPE of chiral fields in 2-dimensional conformal field theory. Will CK_6 play a role in physics or is it just an exotic animal? Are the linearly compact Lie superalgebras of growth > 1 in any way related to OPE of higher dimensional quantum field theories?
- 3. Each of the four types W, S, H, K of simple primitive Lie algebras (L, L_0) correspond to the four most important types of geometries of manifolds: all manifolds, oriented manifolds, symplectic and contact manifolds. Since every smooth supermanifold of dimension (m|n)

comes from a rank n vector bundle on a m-dimensional manifold, it is natural to expect that each of the simple primitive Lie superalgebras corresponds to one of the most important types of geometries of vector bundles on manifolds. For example, the five exceptional superalgebras have altogether, up to conjugacy, 15 maximal open subalgebras. They correspond to irreducible \mathbb{Z} -gradations listed in [CK3]: 4 for E(1|6), 3 for E(3|6), 3 for E(3|8), 1 for E(4|4) and 4 for E(5|10) (as Shchepochkina pointed out, we missed two \mathbb{Z} -gradations of E(1|6): (1,0,0,0,1,1,1) and (2,2,0,1,1,1,1) in notation of [CK3]) . There should be therefore 15 exceptional types of geometries of vector bundles on manifolds which are especially important.

4. The main message of § 7 of my talk is the following principle:

Nature likes degenerate representations.

There are several theories where this principle works very well. First, it is the theory of 2-dimensional statistical lattice models, especially the minimal models of [BPZ], which are (for 0 < c < 1) nothing else but the top degenerate representations of the Virasoro algebra, and the WZW models, including the case of fractional levels, which are based on degenerate top modules over the affine Kac-Moody algebras (see [K3] for a review on these modules). Second, it is the theoretical explanation of the quantum Hall effect by [CTZ] based on degenerate top modules of $W_{1+\infty}$ (see [KRa]).

5. In view of the discussion in §§ 4 and 7, it is natural to suggest that the algebra $su_3 + su_2 + u_1$ of internal symmetries of the Weinberg-Salam-Glashow Standard model extends to E(3|6). I am hopeful that representation theory will shed new light on various features of the Standard model (including the Kobayashi-Maskawa matrix). It turns out [KR] that all degenerate E(3|6) Verma modules have a unique nontrivial singular vector. This should lead to some canonical differential equations on the correlation functions (cf. [BPZ]).

I find it quite remarkable that the SU_5 Grand unified model of Georgi-Glashow combines the left multiplets of fundamental fermions in precisely the negative part of the consistent gradation of E(5|10) (see § 4). This is perhaps an indication of the possibility that an extension from

- su_5 to E(5|10) algebra of internal symmetries may resolve the difficulties with the proton decay.
- One, of course, may try other finite- or infinite-dimensional Lie superalgebras. For example, J. van der Jeugt has tried recently $L = s\ell(3|2)$ and it worked rather nicely, but osp(6|2) has been ruled out.
- 6. Let me end with the most irresponsible suggestion. Since W_4 is, on the one hand, the algebra of symmetries of Einstein's gravity theory, and, on the other hand, the even part of E(4|4), it is a natural guess that E(4|4) is the algebra of symmetries of a nice super extension of general relativity. One knows that the algebra of symmetries of the minimal N=1 supergravity theory is S(4|2) [OS].

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