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REPRESENTATIONS OF THE EXCEPTIONAL LIE SUPERALGEBRA $E(3,6)$ I: DEGENERACY CONDITIONS

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Abstract. Recently one of the authors obtained a classification of simple infinite dimensional Lie superalgebras of vector fields which extends the well known classification of E. Cartan in the Lie algebra case. The list consists of many series defined by simple equations, and of several exceptional superalgebras, among them $E(3,6)$.

In the article we study irreducible representations of the exceptional Lie superalgebra $E(3,6)$. This superalgebra has $s\ell(3) \times s\ell(2) \times g\ell(1)$ as the zero degree component of the consistent $\mathbb{Z}$-grading which leads us to believe that its representation theory has potential for physical applications.

0. Introduction

Recently V. Kac obtained the classification of infinite dimensional simple linearly compact Lie superalgebras [K1]. Two of the exceptional superalgebras, $E(3,6)$ and $E(3,8)$, have the Lie algebra $\mathfrak{g}_0 = s\ell(3) \times s\ell(2) \times g\ell(1)$ as the zero degree component in their consistent $\mathbb{Z}$-grading. This points to the potential physical applications of representations of these algebras.

We deal with representations of $E(3,6)$ in this article having the main objective to classify and describe continuous irreducible representations.

We follow the approach developed for representations of infinite dimensional simple linearly compact Lie algebras by A. Rudakov in [R]. The problem reduces quite quickly (Proposition 1.3) to the description of the so-called degenerate modules, and for the latter we have to study singular vectors and secondary singular vectors (see definitions in Section 1).

In this first article on the topic we get an important restriction on the list of degenerate irreducible representations of $E(3,6)$ (Theorem 4.1).

In order to obtain the complete list of these representations and to get a hold on their construction and structure, more work is to be done. We describe it in subsequent articles. In particular, it turns out that the degenerate irreducible representations of $E(3,6)$ fall into four series, and we construct four complexes of $E(3,6)$-modules which lead to an explicit description of these series. In more detail, the Lie superalgebra $E(3,6)$ has a unique open subalgebra of the form $L_0 = \mathfrak{g}_0 + L_1$, where $\mathfrak{g}_0$ is as above and L_1 is

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an ideal. Denote by $M(p,q;r;y)$ the $E(3,6)$-module induced from the irreducible finite dimensional L_0-module on which L_1 acts trivially and $\mathfrak{g}_0$ has highest weight $(p,q;r;y)$, where $p,q,r \in \mathbb{Z}_+$, $y \in \mathbb{C}$. The main result of the present paper is that if $M(p,q;r;y)$ is a reducible $E(3,6)$-module, then either $p=0$ or $q=0$. Based on this, we show in a subsequent paper that a complete list of reducible $E(3,6)$-modules $M(p,q;r;y)$ consists of the following four series ($p,q,r \in \mathbb{Z}_+$):

$$M(p,0;r;\tfrac{2}{3}p-r)\,,\ M(p,0;r;\tfrac{2}{3}p+r+2)\,,\ M(0,q;r;-\tfrac{2}{3}q-r-2)\,,\ M(0,q;r;-\tfrac{2}{3}q+r)\,.$$

Let us mention that the existence of exceptional infinite dimensional Lie superalgebras was announced by I. Shchepochkina [S1] in 1983, but her construction is implicit and quite difficult to use (see [S2]). We rely on the explicit construction of $E(3,6)$ found by S.J. Cheng and V. Kac ([K1, CK1]).

For related mathematical development, see [K2, K3, CK2]. Basic properties of superalgebras can be found for example in [K4]. All vector spaces, linear maps and tensor products are considered over $\mathbb{C}$.

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1. General remarks on representations of linearly compact Lie superalgebras

The representation theory of linearly compact infinite dimensional Lie algebras was initiated by A. Rudakov some 25 years ago. We will follow the same approach in the Lie superalgebra case. It is a worthwhile undertaking because the list of linearly compact infinite dimensional simple Lie superalgebras and their irreducible modules turns out to be much richer than in the Lie algebra case and because of their potential applications to quantum physics.

It is most natural to consider continuous representations of linearly compact Lie superalgebras in linearly compact topological spaces. However, technically it is more convenient to work with the contragredient to these, which are continuous representations in spaces with discrete topology. The continuity of a representation of a linearly compact Lie superalgebra L in a vector space V with discrete topology means that the stabilizer $L_v = \{g \in L \mid gv = 0\}$ of any $v \in V$ is an open (hence of finite codimension) subalgebra of L.

Let L_0 be an open subalgebra of L. In order to avoid pathological examples we shall always assume that the L-module V is L_0-locally finite, meaning that any $v \in V$ is contained in a finite dimensional L_0-invariant subspace. We shall denote by $\mathcal{P}(L, L_0)$ the category of all continuous L-modules V, where V is a vector space with discrete topology, that are L_0-locally finite. When talking about representations of L, we shall always mean modules from $\mathcal{P}(L, L_0)$ (after a suitable choice of L_0), unless otherwise stated.

Suppose that L is a simple infinite dimensional linearly compact Lie superalgebra. It is known [K1] that L has a maximal open subalgebra L_0 called a *maximally even* subalgebra which contains all even exponentiable elements of L (in most of the cases L

has a unique such subalgebra). If L_0' is another open subalgebra of L, then, since all even elements of L_0' are exponentiable [K1], we conclude that the even part of L_0' lies in L_0. It follows that $\mathcal{P}(L, L_0') \supset \mathcal{P}(L, L_0)$, and therefore any open subalgebra of L acts locally finitely on modules from $\mathcal{P}(L, L_0)$.

Let L_- be a complementary to L_0 (finite dimensional) subspace in L. In most (but not all cases) of simple L and maximally even L_0, one can choose L_- to be a subalgebra. Choosing an ordered basis of L_- we denote by $U(L_-)$ the span of all PBW monomials in this basis. We have $U(L) = U(L_-) \otimes U(L_0)$, a vector space tensor product. (Here and further $U(L)$ stands for the universal enveloping algebra of the Lie algebra L.) It follows that any irreducible L-module V from the category $\mathcal{P}(L, L_0)$ is finitely generated over $U(L_-)$:

$$V = U(L_-)E$$

for some finite dimensional subspace E. This last property is very important in the theory of conformal modules [CK2], [K2].

Let V be an L-module from the category $\mathcal{P}(L, L_0)$. Denote by $\operatorname{Sing} V$ the sum of all irreducible L_0-submodules of V. This subspace is clearly different from zero. Its vectors are called *singular vectors* of the L-module V.

Given an L_0-module F, we may consider the associated induced L-module

$$M(F) = \operatorname{Ind}_{L_0}^{L} F = U(L) \otimes_{U(L_0)} F\,,$$

called also the *universal* L-module (associated to F). Other names used for these kinds of modules are generalized Verma modules, Weyl modules, and so on.

The L_0-module F is canonically an L_0-submodule of $M(F)$ and the sum of its irreducible submodules, that is, $\operatorname{Sing} F$ is a subspace of $\operatorname{Sing} M(F)$, called *the subspace of trivial singular vectors.*

Let us mention that if F is finite dimensional, then being a continuous L_0-module, it is annihilated by an open ideal I of L_0, so, in fact, in this case F is a module over a finite dimensional Lie superalgebra L_0/I.

The following proposition is standard.

Proposition 1.1. (a) *A finite dimensional L_0-module F is continuous if and only if $AnnF = \{g \in L_0 \mid gF = 0\}$ is an open ideal of L_0.*

(b) *If L has a filtration by open subalgebras: $L = L_{-1} \supset L_0 \supset L_1 \supset \dots$ and F is a continuous finite dimensional L_0-module, then the L-module $M(F)$ lies in $\mathcal{P}(L, L_0)$.*

Proof. (a) is trivial. Hence, if F is a continuous finite dimensional L_0-module, we have $L_jF = 0$ for $j \gg 0$. Note that $M(F) = U(L_-)F$, hence we can make an increasing filtration of $M(F)$ by finite dimensional subspaces: $F \subset L_-F + F \subset L_-^2F + L_-F + F \subset \dots$. But, clearly, each member of this filtration is annihilated by L_j for $j \gg 0$, which proves the continuity of the L-module $M(F)$. Since $\dim L_- < \infty$, we conclude also that $M(F)$ is L_0-locally finite, proving (b). $\square$

Definition 1.2. An irreducible L-module V is called nondegenerate if $V = M(F)$ for an irreducible L_0-module F. We often call F and $M(F)$ in this case nondegenerate as well.

In many interesting cases L has an element Y with the following properties:

(i) ad Y is diagonizable,

(ii) the spectrum of ad Y is real and discrete and eigenspaces are finite dimensional,
(iii) the number of negative eigenvalues is finite.

Such an element is called a *hypercharge operator*. It defines the *triangular decomposition*

$$L = L_- + \mathfrak{g}_0 + L_+ ,$$

where L_- is the sum of eigenspaces of ad Y with negative eigenvalues, $\mathfrak{g}_0$ is the 0-th eigenspace and L_+ is the product of eigenspaces with positive eigenvalues.

We let $L_0 = \mathfrak{g}_0 + L_+$. Both L_+ and L_0 are open subalgebras of L. Let F be an L_0-module. Suppose that $Y|_F$ is a scalar operator (which is true if F is a finite dimensional irreducible L_0-module). An eigenvector of Y in $\operatorname{Sing} M(F) \setminus \operatorname{Sing} F$ is called a *nontrivial singular vector* of $M(F)$. Denote by $V(F)$ the quotient of the L-module $M(F)$ by the submodule generated by all nontrivial singular vectors that are eigenvectors of Y. Their eigenvalues are necessarily different from those of trivial singular eigenvectors so the map of F to $V(F)$ is injective. We will often identify F with its image in $M(F)$ or $V(F)$ depending on the module under consideration.

Clearly $V(F)$ could be irreducible even if $M(F)$ is not, which often happens when $M(F)$ is degenerate, but not always. To study this we must look at singular vectors in $V(F)$.

Elements of $\operatorname{Sing} V(F)$ are called *secondary singular vectors*. The image of $\operatorname{Sing} F \subset F$ in $V(F)$ lies in $\operatorname{Sing} V(F)$ and is called the subspace of *trivial secondary singular vectors*.

Proposition 1.3. *Let L be a linearly compact Lie superalgebra with a hypercharge operator Y. Then:*

(a) *Any finite dimensional L_0-module F is continuous.*
(b) *If F is a finite dimensional L_0-module, then $M(F)$ is in $\mathcal{P}(L, L_0)$.*
(c) *In any irreducible finite dimensional L_0-module F, the subalgebra L_+ acts trivially.*
(d) *If F is an irreducible finite dimensional L_0-module, then $M(F)$ has a unique maximal submodule.*
(e) *Denote by $I(F)$ the quotient by the unique maximal submodule of $M(F)$. The map $F \mapsto I(F)$ defines a bijective correspondence between irreducible finite dimensional $\mathfrak{g}_0$-modules and irreducible L-modules from $\mathcal{P}(L, L_0)$, the inverse map being $V \mapsto \operatorname{Sing} V$.*
(f) *The L-module $M(F)$ is irreducible if and only if the L_0-module F is irreducible and $\operatorname{Sing} M(F) = F$.*
(g) *If the finite dimensional L_0-module F is irreducible, and all its secondary singular vectors are trivial, then the L-module $V(F)$ is irreducible (and coincides with $I(F)$).*
(h) *If $\widetilde{S}$ is an irreducible L_0-submodule of $M(F)$ and S is the L-submodule of $M(F)$ generated by $\widetilde{S}$, then S is irreducible iff $\operatorname{Sing} S = S \cap \operatorname{Sing} M(F) = \widetilde{S}$.*

Proof. Let F be a finite dimensional L_0-module and let v be a generalized eigenvector of Y with eigenvalue λ. If a is an eigenvector of ad Y with eigenvalue j, it follows that $a(v)$ is a generalized eigenvector with eigenvalue $\lambda + j$. Hence v is annihilated by all but finitely many eigenspaces of ad Y, proving (a).

One has a filtration of L by open subspaces given by $L_j =$ (the product of eigenspaces of ad Y with eigenvalues $\geq j$). Now (a) and the proof of Proposition 1.1b prove (b).

Similarly, one shows that all elements from L_+ act on F as nilpotent operators and therefore, by the superanalog of Engel's theorem, they annihilate a nonzero vector. Since the space spanned by these vectors is L_0-invariant, it coincides with F, which proves (c).

If F is an irreducible L_0-module, it is actually an irreducible $\mathfrak{g}_0$-module (with L_+ acting trivially) on which therefore Y acts as a scalar, let it be y_0. Then clearly Y acts diagonally on $M(F)$ in such a way that F coincides with its eigenspace for the eigenvalue y_0, and $\mathrm{Re}(y) < \mathrm{Re}(y_0)$ for any other eigenvalue y of Y on $M(F)$. This implies (d). Then (e), (f) and (g) follow.

The statement (h) follows from (f), as soon as we notice that the inclusion of L_0-modules $\widetilde{S} \subset M(F)$ induces the morphism of L-modules $M(\widetilde{S}) \longrightarrow M(F)$ and the map is injective by PBW theorem, therefore $S = M(\widetilde{S})$. $\square$

One has the following well known corollary of Proposition 1.3.

Corollary 1.4. *An L-module $M(F)$ is irreducible (hence nondegenerate) if and only if the $\mathfrak{g}_0$-module F is irreducible and $M(F)$ has no nontrivial singular vectors.*

Remark 1.5. The correspondence defined by Proposition 1.3e provides the classification of irreducible modules of the category $\mathcal{P}(L, L_0)$. For the nondegenerate ones of those modules the definition of $M(F)$ supplies the construction, and Proposition 1.3g gives a construction of the degenerate modules having only trivial secondary singular vectors, provided that one has a description of singular vectors.

2. Construction and basic properties of $E(3,6)$

One way to construct $E(3,6)$ is via its embedding into $E(5,10)$. We describe first the geometric construction of $E(5,10)$ from [CK1], Section 5.3 or [K1], Section 5.

Let $x_1, \dots, x_5$ be even variables with $\deg x_i = 2$, and let S_5 be the Lie algebra of divergence zero vector fields in these variables. Let $\mathsf{d}\Omega^1(5)$ be the space of closed differential 2-forms in these variables. A choice of degrees of the variables and the degree of d determines a $\mathbb{Z}$-grading in vector fields and differential forms. We let the degree of d be $-5/2$, so that deg $\mathsf{d}x_i = -1/2$.

The Lie superalgebra $E(5,10)$ is constructed as follows: $E(5,10)_{\bar{0}} \simeq S_5$ as a Lie algebra, $E(5,10)_{\bar{1}} \simeq \mathsf{d}\Omega^1(5)$ as an S_5-module. The bracket on $E(5,10)_{\bar{1}}$ is defined as the exterior product of differential forms which is a closed 4-form identified with the vector field whose contraction with the volume form produces this 4-form. This construction gives a $\mathbb{Z}$-grading in $E(5,10)$ that we will call the *consistent* $\mathbb{Z}$-grading (since its even and odd numbered pieces are comprised of even and odd elements, respectively).

In order to make explicit calculations we will use the following notations

$$\mathsf{d}_{jk} := \mathsf{d}x_j \wedge \mathsf{d}x_k, \quad \partial_i := \partial/\partial x_i.$$

We assume that the volume form is $\mathsf{d}x_1 \wedge \ldots \wedge \mathsf{d}x_5$. Now an element A from $E(5,10)_{\bar{0}} = S_5$ can be written as

$$A = \textstyle\sum_i a_i\partial_i\,, \quad \text{where } a_i \in \mathbb{C}[[x_1, \dots, x_5]]\,, \quad \textstyle\sum_i \partial_i a_i = 0,$$

and an element B from $E(5,10)_{\bar 1}$ is of the form

$$B=\sum_{j,k} b_{jk}\mathsf{d}_{jk}\,,\quad \text{where } b_{jk}\in\mathbb{C}[[x_1,\dots,x_5]],\ dB=0.$$

In particular the brackets in $E(5,10)_{\bar 1}$ can be computed using bilinearity and the rule

$$[a\mathsf{d}_{jk}, b\mathsf{d}_{lm}]=\varepsilon_{ijklm}ab\partial_i$$

where ε_{ijklm} is the sign of the permutation $(ijklm)$ when $\{i,j,k,l,m\}=\{1,2,3,4,5\}$ and zero otherwise.

By definition ([K1], Example 5.4) the algebra $E(3,6)$ is a consistently $\mathbb{Z}$-graded simple linearly compact Lie superalgebra such that

$$\mathfrak{g}_0\simeq s\ell(3)+s\ell(2)+g\ell(1),\quad \mathfrak{g}_{-1}\simeq\mathbb{C}^3\otimes\mathbb{C}^2\otimes\mathbb{C}(-1),\quad \mathfrak{g}_{-2}\simeq\mathbb{C}^3\otimes 1\otimes\mathbb{C}(-2),$$
$$\mathfrak{g}_{-3}\simeq 0,\quad \mathfrak{g}_1\simeq S^2\mathbb{C}^3\otimes\mathbb{C}^2\otimes\mathbb{C}(1)+\mathbb{C}^{3^*}\otimes\mathbb{C}^2\otimes\mathbb{C}(1).$$

We may construct $E(3,6)$ as a subalgebra of $E(5,10)$ as follows (we slightly modify the construction from [CK1]). Consider the *secondary grading in* $E(5,10)$ defined by the conditions:

$$\begin{array}{ll}\deg x_1=\deg x_2=\deg x_3=0, & \deg\partial_1=\deg\partial_2=\deg\partial_3=0,\\ \deg x_4=\deg x_5=1, & \deg\partial_4=\deg\partial_5=-1\,,\ \deg\mathsf{d}=-1/2.\end{array}\tag{2.1}$$

Proposition 2.1. ([CK1]) *For the secondary grading of* $E(5,10)$, *the zero-degree subalgebra is the Lie superalgebra* $E(3,6)$. *The consistent* $\mathbb{Z}$*-grading in* $E(3,6)$ *is induced by the consistent grading of* $E(5,10)$.

As a result we have for $L=E(3,6)$ the following description of the first three pieces of its consistent $\mathbb{Z}$-grading $L=\Pi_{j\geq -2}\mathfrak{g}_j$:

$$\mathfrak{g}_{-2}=\langle\partial_i, i=1,2,3\rangle,\quad \mathfrak{g}_{-1}=\langle\mathsf{d}_{ij}, i=1,2,3, j=4,5\rangle.$$

We shall use the following basis of $\mathfrak{g}_0=s\ell(3)+s\ell(s)+g\ell(1)$:

$$\begin{array}{llllll} h_1=x_1\partial_1-x_2\partial_2, & h_2=x_2\partial_2-x_3\partial_3, & e_1=x_1\partial_2, & e_{12}=x_2\partial_3, & e_3=x_1\partial_3, & \\ f_1=x_2\partial_1, & f_2=x_3\partial_2, & f_{12}=x_3\partial_1, & h_3=x_4\partial_4-x_5\partial_5, & e_3=x_4\partial_5, & f_3=x_5\partial_4,\\ \multicolumn{6}{l}{Y=\tfrac{2}{3}(x_1\partial_1+x_2\partial_2+x_3\partial_3)-(x_4\partial_4+x_5\partial_5).}\end{array}$$

Here $s\ell(3)$ (resp. $s\ell(2)$) is spanned by elements involving indeterminates x_i with $i=1,2,3$ (resp. $4,5$) and $g\ell(1)=\mathbb{C}Y$. We use the element Y as the hypercharge operator and we fix the standard Cartan subalgebra $\mathcal{H}=\langle h_1,h_2,h_3,Y\rangle$ and the standard Borel subalgebra $\mathcal{B}=\mathcal{H}\oplus\mathcal{N}$, where $\mathcal{N}=\langle e_i\ (i=1,2,3), e_{12}\rangle$, of $\mathfrak{g}_0$. Note that the eigenspace decomposition of $3Y$ coincides with the consistent $\mathbb{Z}$-grading of $E(3,6)$. (Incidentally, $E(5,10)$ has no hypercharge operators.)

The algebra $E(3,6)$ is generated by $\mathfrak{g}_{-1},\mathfrak{g}_0,\mathfrak{g}_1$ [CK1]; moreover it is generated by $\mathfrak{g}_0$ and the following three elements e_0,e_0',f_0:

$$f_0=\mathsf{d}_{14},\quad e_0'=x_3\mathsf{d}_{35},\quad e_0=x_3\mathsf{d}_{25}-x_2\mathsf{d}_{35}+2x_5\mathsf{d}_{23},$$

where the element f_0 is the highest weight vector of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$, while e_0', e_0 are the lowest weight vectors of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$, and one has

$$[e_0', f_0] = f_2, \tag{2.2}$$

$$[e_0, f_0] = \tfrac{2}{3}h_1 + \tfrac{1}{3}h_2 - h_3 - Y =: h_0. \tag{2.3}$$

So the elements $\{h_i, e_i, f_i\,(i=0,1,2,3), e_0'\}$ generate $E(3,6)$. We call them the generalized Chevalley generators of $E(3,6)$ (since apart from (2.2) they satisfy the relations satisfied by the ordinary Chevalley generator of a semisimple Lie algebra).

The above observations give the following proposition.

Proposition 2.2. *The elements $e_i\,(i = 0,1,2,3)$ and e_0' generate $\mathcal{N} + L_+$. Consequently, a $\mathfrak{g}_0$-highest weight vector v of a $E(3,6)$-module is singular iff $e_0 \cdot v = 0, e_0' \cdot v = 0$.*

In order to see the action of $\mathfrak{g}_0$ on the space $\mathfrak{g}_{-1} = \langle \mathsf{d}_{ij},\, i = 1,2,3, j = 4,5\rangle$ more clearly we write

$$\mathsf{d}_i^+ := \mathsf{d}_{i4} \qquad \mathsf{d}_i^- := \mathsf{d}_{i5},$$

and we define $\mathfrak{g}_{-1}^\pm = \langle \mathsf{d}_1^\pm, \mathsf{d}_2^\pm, \mathsf{d}_3^\pm\rangle$. We also use the following shorthand notations for the elements from $\Lambda\langle \mathsf{d}_1^-, \mathsf{d}_2^-, \mathsf{d}_3^-\rangle$:

$$\mathsf{d}_{ij}^- := \mathsf{d}_i^- \cdot \mathsf{d}_j^-, \qquad \mathsf{d}_{ijk}^- := \mathsf{d}_i^- \cdot \mathsf{d}_j^- \cdot \mathsf{d}_k^-,$$

and similarly for the "+"-type. We let $\Lambda^\pm := \Lambda\langle \mathsf{d}_1^\pm, \mathsf{d}_2^\pm, \mathsf{d}_3^\pm\rangle$.

Consider the following abelian subalgebras of $\mathfrak{g}_1$, normalized by $s\ell(3)$:

$$\mathfrak{g}_1^+ = \langle x_i d_{j5} + x_j d_{i5} \mid i,j = 1,2,3\rangle\,,\ \mathfrak{g}_1^- = \langle x_i d_{j4} + x_j d_{i4} \mid i,j = 1,2,3\rangle\,,$$

and let

$$S(3)^\pm = \mathfrak{g}_{-1}^\pm + s\ell(3) + \mathfrak{g}_1^\pm\,.$$

It is easy to see that $S(3)^\pm$ are subalgebras of $\mathfrak{g}$ isomorphic to the simple Lie superalgebra $S(3)$ of divergenceless vector fields in three anticommuting indeterminates. Note that

$$[\mathfrak{g}_{-1}^\pm, \mathfrak{g}_1^\mp] = 0\,. \tag{2.4}$$

One can check that $E(3,6)_{\bar 0} \simeq W_3 + \Omega^0(3) \otimes s\ell(2)$ and $E(3,6)_{\bar 1} \simeq \Omega^1(3) \otimes \mathbb{C}^2$. Here the first isomorphism maps $D \in W_3$ to $D - \frac{1}{2}\operatorname{div}(D)(x_4\partial_4 + x_5\partial_5)$ and is identical on the second summand. The second isomorphism could be chosen according to the following formula (which differs from the one in [CK1])

$$f\mathsf{d}x_i \cdot \varepsilon_a \longrightarrow -\mathsf{d}(f\mathsf{d}x_i \cdot x_{a+3}), \quad i = 1,2,3, \quad a = 1,2\,,$$

where $\varepsilon_a, a = 1,2$, is the standard basis in $\mathbb{C}^2$. Of course it is possible to define brackets in $E(3,6)$ in terms of these isomorphisms and this gives the construction of $E(3,6)$ from [CK1].

3. Lemmas about $s\ell(3)$-modules

From now on we let $L = E(3,6)$. As before, we use notation $L_- = \oplus_{j<0}\, \mathfrak{g}_j$, $L_+ = \Pi_{j>0}\, \mathfrak{g}_j$, $L_0 = \mathfrak{g}_0 + L_+$. We shall use the realization of this Lie superalgebra as a subalgebra of $E(5,10)$ described in Section 2. As explained in Section 1, our first main objective is to study the irreducibility of the induced $\mathfrak{g}$-modules

$$M(V) = U(L) \otimes_{U(L_0)} V \cong U(L_-) \otimes V, \tag{3.1}$$

where V is a finite dimensional irreducible $\mathfrak{g}_0$-module extended to L_0 by letting $\mathfrak{g}_j$ for $j > 0$ acting trivially. The isomorphism in (3.1) is an isomorphism of $\mathfrak{g}_0$-modules, which can be used to define the action of L on $U(L_-) \otimes V$ (in particular L_- acts by left multiplication).

Recall that $\mathfrak{g}_0 = s\ell(3) \oplus s\ell(2) \oplus g\ell(1)$, where

$$s\ell(3) = \langle x_i \partial_j \mid 1 \le i,j \le 3 \rangle \cap \mathfrak{g}_0, \quad s\ell(2) = \langle x_i \partial_j \mid i,j = 4,5 \rangle \cap \mathfrak{g}_0\,.$$

Hence it is important to have a *model* for $s\ell(3)$, i.e., an $s\ell(3)$-module in which every finite dimensional irreducible $s\ell(3)$-module appears exactly once. Note that $s\ell(3)$ acts on the polynomial algebra $\mathbb{C}[\partial_1, \partial_2, \partial_3, x_1, x_2, x_3]$ in a natural way (by derivations $x_i\partial_j(x_k) = [x_i\partial_j, x_k] = \delta_{jk} x_i$, $x_i\partial_j(\partial_k) = [x_i\partial_j, \partial_k] = -\delta_{ik}\partial_j$), so that the element $P := \partial_1 x_1 + \partial_2 x_2 + \partial_3 x_3$ is annihilated. Denote by $\mathcal{M}$ the quotient of this polynomial algebra by the ideal generated by P, with the induced action of $s\ell(3)$.

Lemma 3.1. *The $s\ell(3)$-module $\mathcal{M}$ is a model. The irreducible $s\ell(3)$-module with highest weight (m,n) appears in $\mathcal{M}$ as the bigraded component*

$$\left\langle \partial_1^{a_1} \partial_2^{a_2} \partial_3^{a_3} x_1^{b_1} x_2^{b_2} x_3^{b_3} \;\Big|\; \sum a_i = m,\, \sum b_i = n \right\rangle .$$

The highest weight vector of this submodule is $\partial_3^n x_1^m$.

Proof. Let U denote the subgroup of the group $G = SL(3,\mathbb{C})$ consisting of upper triangular matrices with 1's on the diagonal. It is well known that in the space $\mathbb{C}[G/U]$ of regular functions on G/U, all irreducible finite dimensional G-modules occur exactly once. On the other hand, G/U is isomorphic to the orbit of the sum of highest weight vectors in the G-module $\mathbb{C}^3 \oplus {\mathbb{C}^3}^*$, and this orbit is the complement to 0 in the quadric $\sum_i x_i \partial_i = 0$, where x_i (resp. ∂_i) are standard coordinates on $\mathbb{C}^3$ (resp. ${\mathbb{C}^3}^*$, the dual to $\mathbb{C}^3$).

Since this quadric is a normal variety, we conclude that the G-module $\mathbb{C}[G/U]$ is isomorphic to $\mathcal{M}$. The lemma follows. $\square$

Thus, every L-module $M(V)$ is contained in $U(L_-) \otimes \mathcal{M} \otimes T$, where T is a (finite dimensional irreducible) $s\ell(2)$-module. We shall use the following shorthand notation:

$$u[m]t = u \otimes m \otimes t \in U(L_-) \otimes \mathcal{M} \otimes T\,.$$

(This notation also reminds one that elements of $\mathcal{M}$ are cosets.)

We shall mark the elements $\partial_i \in \mathfrak{g}_{-2}$ by a hat in order to distinguish them from the elements ∂_i used in the construction of $\mathcal{M}$. We let $\mathtt{S} = \mathbb{C}[\hat\partial_1, \hat\partial_2, \hat\partial_3]$.

We shall consider the tensor product $U(L_-)\otimes\mathcal{M}$ of associative algebras. This is a $U(s\ell(3))$-module with the usual action on the tensor product. Hence we may consider the smash product

$$U=(U(L_-)\otimes\mathcal{M})\#U(s\ell(3))\,.$$

This is an associative algebra which acts on $U(L_-)\otimes\mathcal{M}$ in the obvious way (elements from $U(L_-)\mathcal{M}$ act by left multiplication).

The algebra $U(L_-)\otimes\mathcal{M}$ contains a commutative $U(s\ell(3))$-invariant subalgebra $\mathtt{S}\otimes\mathcal{M}$. In the following proposition and further we shall denote by $\mathrm{wt}_3 v$ the $s\ell(3)$-weight of a vector v.

Proposition 3.2. *Consider the following elements of* $\mathtt{S}\otimes\mathcal{M}$*:*

$$\bar{D}_1=\hat{\partial}_1[x_1]+\hat{\partial}_2[x_2]+\hat{\partial}_3[x_3],\quad \bar{D}_2=\hat{\partial}_2[\partial_3]-\hat{\partial}_3[\partial_2],\quad \bar{D}_3=\hat{\partial}_3[1]=\hat{\partial}_3.$$

Any $s\ell(3)$*-highest weight vector* $\bar{w}$ *in* $\mathtt{S}\otimes\mathcal{M}$ *can be uniquely written as*

$$\bar{w}=\textstyle\sum_{\alpha,m,n}\bar{c}_{\alpha;m,n}\bar{D}_1^{\alpha_1}\bar{D}_2^{\alpha_2}\bar{D}_3^{\alpha_3}[\partial_3^n x_1^m].$$

If $\mathrm{wt}_3\bar{w}=(a,b)$*, then* $(a,b)=(m,n)+(\alpha_2,\,\alpha_3)$ *(for nonzero* $\bar{c}_{\alpha;m,n}$*).*

Proof. It follows from [Sh] that the algebra of U-invariants for the action of $SL(3)$ on the algebra $A:=\mathbb{C}[\hat{\partial}_1,\hat{\partial}_2,\hat{\partial}_3,\partial_1,\partial_2,\partial_3,x_1,x_2,x_3]$ is generated by six algebraically independent elements: $\bar{D}_0:=\sum_i\partial_i x_i,\bar{D}_1,\bar{D}_2,\bar{D}_3,\partial_3$ and x_1. We notice that $\mathrm{wt}_3\bar{D}_0=\mathrm{wt}_3\bar{D}_1=(0,0)$, $\mathrm{wt}_3\bar{D}_2=(1,0)$, $\mathrm{wt}_3\bar{D}_3=(0,1)$. Since $\mathtt{S}\otimes\mathcal{M}\cong A/(\bar{D}_0)$ by Lemma 3.1, the proposition follows. □

Suppose $\bar{w}\in\mathtt{S}\otimes M$, where $M\subset\mathcal{M}$ is an irreducible $s\ell(3)$-module generated by the highest vector $[\partial_3^{\nu_2}x_1^{\nu_1}]\in\mathcal{M}$. Then in the above formula we should have $(\nu_1,\nu_2)=(m,n)+(\alpha_1,\alpha_2)$. So the weights of $s\ell(3)$-highest weight vectors in $\mathtt{S}\otimes M$ are

$$(a,b)=(\nu_1,\nu_2)-(\alpha_1,\,\alpha_2)+(\alpha_2,\,\alpha_3)$$

where $\alpha_i\le\nu_i$, $i=1,2$.

Suppose now that an $s\ell(3)$-module M is given as an abstract finite dimensional module. We would like to extend our description of the highest weight vectors in $\mathtt{S}\otimes\mathcal{M}$ to $\mathtt{S}\otimes M$. Note that $\mathtt{S}\otimes M$ is an $\mathtt{S}$-module via the left multiplication, and also a $U(s\ell(3))$-module with the usual action on tensor product, so that we have the action of $\mathtt{S}\#U(s\ell(3))$ on $\mathtt{S}\otimes M$.

Let $h=h_1+h_2+1$. Instead of $\bar{D}_i$, we define the operators D_i from $\mathtt{S}\#U(s\ell(3))\subset U$ as follows (as before, we drop the tensor signs):

$$D_1=\hat{\partial}_1\,h_1h+\hat{\partial}_2\,f_{12}h+\hat{\partial}_3\,(f_3h_1+f_2f_1),\quad D_2=\hat{\partial}_2\,h_2+\hat{\partial}_3\,f_2,\quad D_3=\hat{\partial}_3 \tag{3.2}$$

The action of the operators D_i on $\mathtt{S}\otimes\mathcal{M}$ is related to the left multiplication by the $\bar{D}_i$ by the formula

$$\begin{aligned} D_1(s[\partial_3{}^q x_1^p])&=p(p+q+1)\bar{D}_1(s[\partial_3{}^q x_1^{p-1}]),\\ D_2(s[\partial_3{}^q x_1^p])&=q\;\bar{D}_2(s[\partial_3{}^{q-1}x_1^p]),\\ D_3(s[\partial_3{}^q x_1^p])&=\bar{D}_3(s[\partial_3{}^q x_1^p]).\end{aligned} \tag{3.3}$$

Note that equations (3.2) represent the defining property of the operators D_i, and (3.1) is a solution of these equations.

Proposition 3.3. *The operators D_i commute with each other, and while acting on $\mathtt{S} \otimes \mathcal{M}$ the operator D_i commutes with $\bar{D}_j$ for $j < i$.*

Proof. This is not difficult to check by a straightforward calculation (see below). $\square$

In the following $A^{[n]} := A(A-1)\dots(A-n+1)$.

Proposition 3.4. *Let M be an irreducible $s\ell(3)$-module with the highest weight vector m_0. Any highest weight vector in $\mathtt{S} \otimes M$ can be written uniquely as a linear combination of the form $w = \sum_\alpha c_\alpha D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} m_0$.*

Proof. Pick a monomorphism of $s\ell(3)$-modules $\mu : M \longrightarrow \mathcal{M}$ such that $\mu(m_0) = [\partial_3^{\nu_2} x_1^{\nu_1}] \in \mathcal{M}$ for a highest weight vector m_0 of M. Let us denote also by μ its extension to the monomorphism $\mathtt{S} \otimes M \longrightarrow \mathtt{S} \otimes \mathcal{M}$. Given a highest weight vector $w \in \mathtt{S} \otimes M$, its image $\mu(w) = \bar{w}$ is also a highest weight vector. Clearly, by Proposition 3.3 and equation (3.3), from the expression for $\bar{w}$ given by Proposition 3.2 we get

$$w = \textstyle\sum_{\alpha,m,n} c_{\alpha;m,n} D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} m_0, \tag{3.4}$$

where

$$(\nu_1)^{[\alpha_1]}(\nu_1+\nu_2+1)^{[\alpha_1]}(\nu_2)^{[\alpha_2]}\, c_{\alpha;m,n} = \bar{c}_{\alpha;m,n}. \quad \square \tag{3.5}$$

Any element v of $\mathtt{S} \otimes \mathcal{M}$ can be written uniquely in the form

$$v = \textstyle\sum_{\alpha\in\mathbb{Z}_+^3} \hat{\partial}_1^{\alpha_1}\hat{\partial}_2^{\alpha_2}\hat{\partial}_3^{\alpha_3} t_\alpha \equiv \sum_\alpha \hat{\partial}^\alpha t_\alpha\,, \text{ where } t_\alpha \in \mathcal{M}.$$

We define $\ell ht\, v = \hat{\partial}^\sigma t_\sigma$, where σ is the lexicographically highest element of the set $\{\alpha \in \mathbb{Z}_+^3 \mid t_\alpha \neq 0\}$. It is immediate to see

$$\ell ht\, \bar{D}^\alpha [\partial_3^n x_1^m] = \hat{\partial}^\alpha [\partial_3^{n+\alpha_2} x_1^{m+\alpha_1}]\,. \tag{3.6}$$

Proposition 3.5. $\ell ht\, D^\alpha = \hat{\partial}^\alpha h^{[\alpha_1]} h_1^{[\alpha_1]} h_2^{[\alpha_2]}$.

Using (3.6), we get the following corollary.

Corollary 3.6. $\ell ht\, D^\alpha [\partial_3^q x_1^p] = p^{[\alpha_1]}(p+q+1)^{[\alpha_1]} q^{[\alpha_2]} \ell ht\, \bar{D}^\alpha [\partial_3^{q-\alpha_2} x_1^{p-\alpha_1}]$.

The proof of Proposition 3.5 is based on several lemmas, which also establish some properties of the operators D_i used in the sequel.

Lemma 3.7. *D_i commute with each other.*

Proof. Let $A = \hat{\partial}_1 h_1 + \hat{\partial}_2 f_1$, $B = f_{12}h_1 + f_2 f_1$ so that $D_1 = Ah + \hat{\partial}_3 B$. We see that $[A, \hat{\partial}_2] = 0$, $[A, h_2] = 0, [A, \hat{\partial}_3] = 0$ and $[A, f_2] = \hat{\partial}_1 f_2 - \hat{\partial}_2 f_{12}$. Therefore $[A, D_2] = \hat{\partial}_1\hat{\partial}_3 f_2 - \hat{\partial}_2\hat{\partial}_3 f_{12}$. Also $[h, D_2] = 0$, $[\hat{\partial}_3, D_2] = 0$ and $[B, \hat{\partial}_2] = \hat{\partial}_2 f_{12} - \hat{\partial}_1 f_2$, $[B, h_2] = B$, $[B, \hat{\partial}_3] = -\hat{\partial}_1 h_1 - \hat{\partial}_2 f_1 = -A$, $[B, f_2] = [f_{12}, f_2] = 0$, thus $[B, D_3] = (\hat{\partial}_2 f_{12} - \hat{\partial}_1 f_2)h_2 + \hat{\partial}_2 B - A f_2 = (\hat{\partial}_2 f_{12} - \hat{\partial}_1 f_2)h$. We conclude that $[D_1, D_2] = 0$. We have computed that $[A, \partial_3] = 0$, $[B, \partial_3] = -A$, and $[h, \partial_3] = \partial_3$, so $[D_1, D_3] = 0$. Clearly $[D_2, D_3] = 0$ as well. $\square$

The following lemma could be applied either to D_1 or D_2. It provides quite a nice expression for D_i^k (here we will need only the lexicographically highest term of the sum but later we will also use the second one).

Lemma 3.8. *Let $D = ah + \partial b$ where $[a,b] = 0$, $[\partial, b] = +a$, $[\partial, a] = 0$, $[h,b] = -2b$, $[h,a] = -a$, $[h,\partial] = \partial$. Then $D^k = \sum_{m=0}^{k} \binom{k}{m} a^{k-m}\partial^m b^m (h-m)^{[k-m]}$.*

Proof. The formula is easily proven by induction on k. $\square$

One can easily check that D_1 with $a = A, b = B, \partial = \hat\partial_3$ satisfies the above lemma, as well as D_2 with $a = \hat\partial_2, b = f_2, \partial = \hat\partial_3$ and A with $a = \hat\partial_1, b = f_1, \partial = \hat\partial_2$. This makes it easy to compute the following lexicographically highest terms.

Corollary 3.9. $\ell ht\, D_1^k = \hat\partial_1^k h^{[k]} h_1^{[k]}, \ell ht\, D_2^\ell = \hat\partial_2^\ell h_2^{[\ell]}$.

Lemma 3.10. *Let $D_2\{+m\} = \hat\partial_2(h_2+m) + \hat\partial_3 f_2$. Then $[D_2, \hat\partial_3] = [D_2, \hat\partial_1] = 0$ and $D_2\hat\partial_2 = \hat\partial_2 D_2\{+1\}$.*

Proof. This is a straightforward calculation. $\square$

Corollary 3.11. *If $\ell ht\, f = \hat\partial^\alpha u$, then*

$$\ell ht(D_2 f) = \ell ht(\hat\partial^\alpha D_2\{+\alpha_2\}u) = \hat\partial^\alpha(\ell ht D_2\{+\alpha_2\})u.$$

Proof of Proposition 3.5. We can apply Corollary 3.11 to the situation of the proposition: $\ell ht(D_2^\ell D_1^k) = \ell ht(\hat\partial_1^k (D_2)^\ell h^{[k]} h_1^{[k]}) = \hat\partial_1^k \hat\partial_2^\ell h_2^{[\ell]} h^{[k]} h_1^{[k]}$. The formula of the proposition follows. $\square$

Now we describe the $s\ell(3)$-highest weight vectors in $\Lambda^\pm \otimes F$. We here omit $\pm$ because the results are exactly the same for "+" and for "−".

Lemma 3.12. *Let $F \subset \mathcal{M}$ be an irreducible finite dimensional $s\ell(3)$-module with highest weight (p,q). For the $s\ell(3)$-highest weight element $u \in \Lambda\langle \mathsf{d}_1, \mathsf{d}_2, \mathsf{d}_3\rangle \otimes F$ of weight $(m,n) = (p,q)+\delta$, there are the following possibilities (up to a constant factor):*

- $(00)'$: $\delta = (0,0)$ *and* $u = [{\partial_3}^n x_1^m]$,
- $(+0)$: $\delta = (+1,0)$ *and* $u = \mathsf{d}_1[{\partial_3}^n x_1^{m-1}]$,
- $(-+)$: $\delta = (-1,1)$ *and* $u = (\mathsf{d}_1[x_2] - \mathsf{d}_2[x_1])\,[{\partial_3}^{n-1} x_1^m]$,
- $(0-)$: $\delta = (0,-1)$ *and* $u = (\mathsf{d}_1[\partial_1] + \mathsf{d}_2[\partial_2] + \mathsf{d}_3[\partial_3])\,[{\partial_3}^n x_1^m]$,
- $(0+)$: $\delta = (0,+1)$ *and* $u = \mathsf{d}_{12}[{\partial_3}^{n-1} x_1^m]$,
- (-0): $\delta = (-1,0)$ *and* $u = (\mathsf{d}_{12}[x_3] + \mathsf{d}_{31}[x_2] + \mathsf{d}_{23}[x_1])\,[{\partial_3}^n x_1^m]$,
- $(+-)$: $\delta = (1,-1)$ *and* $u = \mathsf{d}_1(\mathsf{d}_2[\partial_2] + \mathsf{d}_3[\partial_3])\,[{\partial_3}^n x_1^{m-1}]$,
- $(00)''$: $\delta = (0,0)$ *and* $u = \mathsf{d}_{123}[{\partial_3}^n x_1^m]$.

Proof. This is standard and we leave the proof to the reader. $\square$

In the following let $\Delta^\pm := \mathsf{d}_1^\pm[\partial_1] + \mathsf{d}_2^\pm[\partial_2] + \mathsf{d}_3^\pm[\partial_3]$.

Lemma 3.13. *Let F be an irreducible $s\ell(3)$-module with highest weight (p,q) and such that the action of L_+ on F is trivial. If $u \in \Lambda^+ \otimes F$ is an $s\ell(3)$-highest weight vector of weight (m,n) and $e_0' \cdot u = 0$, then there are the following possibilities for u (up to a constant factor):*

- (T0): $(m,n) = (p,q)$ *and* $u = [{\partial_3}^q x_1^p] \in F$,
- (T1): $p \geq 0$, $q = 0$, $(m,n) = (p+1,0)$ *and* $u = \mathsf{d}_1^+[x_1^p]$,
- (T2): $p = 0$, $q \geq 1$, $(m,n) = (0,q-1)$ *and*
  $u = (\mathsf{d}_1^+[\partial_1] + \mathsf{d}_2^+[\partial_2] + \mathsf{d}_3^+[\partial_3])[{\partial_3}^n] = \Delta^+[{\partial_3}^{q-1}]$,
- (T3): $(p,q) = (0,1)$, $(m,n) = (1,0)$ *and* $u = \mathsf{d}_{12}^+[\partial_2] + \mathsf{d}_{13}^+[\partial_3] = \mathsf{d}_1^+\Delta^+$,
- (T4): $(p,q) = (m,n) = (0,0)$ *and* $u = \mathsf{d}_{123}^+[1]$.

In particular in all cases except for (T0), *either* $p=0$ *or* $q=0$.

Proof. We will write u in the form provided by Lemma 3.12 and calculate $e'_0\cdot u$. We are to remember that $e'_0(F)=0$ and the following relations for the action of e'_0 on $\langle \mathsf{d}_1^+,\mathsf{d}_2^+,\mathsf{d}_3^+\rangle$ are important to have in mind:

$$e'_0\cdot \mathsf{d}_1^+ = f_2,\quad e'_0\cdot \mathsf{d}_2^+ = -f_{12},\quad e'_0\cdot \mathsf{d}_3^+ = 0. \tag{3.7}$$

Case (00): $(p,q)=(m,n)$ and $u=(c_0+c_1\mathsf{d}_{123}^+)\,[\partial_3{}^n x_1^m]$, hence

$$\begin{aligned}0 = e'_0u &= 0 + c_1\left(f_2\mathsf{d}_2^+\mathsf{d}_3^+\,[\partial_3{}^n x_1^m] - \mathsf{d}_1^+(-f_{12})\mathsf{d}_3^+\,[\partial_3{}^n x_1^m]\right) = \\ &= c_1\left(0 - n\,\mathsf{d}_2^+\mathsf{d}_3^+\,[\partial_2\partial_3{}^{n-1}x_1^m] - n\,\mathsf{d}_1^+\mathsf{d}_3^+\,[\partial_1\partial_3{}^{n-1}x_1^m] + m\,\mathsf{d}_1^+\mathsf{d}_3^+\,[\partial_3{}^n x_1^{m-1}x_3]\right).\end{aligned}$$

We see that either $c_1=0$, which gives us (T0), or $m=n=0$, which gives (T4).

Case (+0): $(p,q)=(m-1,n)$ and $u=\mathsf{d}_1^+\,[\partial_3{}^n x_1^{m-1}]$, so

$$0 \quad = \quad e'_0u = f_2\,[\partial_3{}^n x_1^{m-1}] = -n\,[\partial_2\partial_3{}^{n-1}x_1^{m-1}].$$

The solution exists for $n=0$, $m\geq 1$. This is (T1).

Case (−+): $(p,q)=(m+1,n-1)$ and $u=(\mathsf{d}_1^+[x_2]-\mathsf{d}_2^+[x_1])\,[\partial_3{}^{n-1}x_1^m]$. We have

$$\begin{aligned}0 = e'_0u &= f_2\,[\partial_3{}^{n-1}x_1^m x_2] - (-f_{12})\,[\partial_3{}^{n-1}x_1^{m+1}] = -(n-1)\,[\partial_2\partial_3{}^{n-2}x_1^m x_2] \\ &+ [\partial_3{}^{n-1}x_1^m x_3] - (n-1)\,[\partial_1\partial_3{}^{n-2}x_1^{m+1}] + (m+1)\,[\partial_3{}^{n-1}x_1^m x_3].\end{aligned}$$

This implies $-(n-1)=m+2$, but both m,n are nonnegative hence no solution is possible.

Case (0–): $(p,q)=(m,n+1)$ and $u=\Delta^+\,[\partial_3{}^n x_1^m]$. We have

$$\begin{aligned}0 = e'_0u &= f_2\,[\partial_1\partial_3{}^n x_1^m] - f_{12}\,[\partial_2\partial_3{}^n x_1^m] \\ &= -n\,[\partial_1\partial_2\partial_3{}^{n-1}x_1^m] + n\,[\partial_1\partial_2\partial_3{}^{n-1}x_1^m] - m\,[\partial_2\partial_3{}^n x_1^{m-1}x_3] \\ &= -m\,[\partial_2\partial_3{}^n x_1^{m-1}x_3].\end{aligned}$$

We conclude that $m=0$ and this gives (T2).

Case (0+): $(p,q)=(m,n-1)$ and $u=\mathsf{d}_{12}^+\,[\partial_3{}^{n-1}x_1^m]$; therefore

$$\begin{aligned}0 = e'_0u &= f_2\mathsf{d}_2^+\,[\partial_3{}^{n-1}x_1^m] - \mathsf{d}_1^+(-f_{12})\,[\partial_3{}^{n-1}x_1^m] \\ &= \mathsf{d}_3^+\,[\partial_3{}^{n-1}x_1^m] - (n-1)\mathsf{d}_2^+\,[\partial_2\partial_3{}^{n-2}x_1^m] \\ &\quad - (n-1)\mathsf{d}_1^+\,[\partial_1\partial_3{}^{n-2}x_1^m] + m\,\mathsf{d}_1^+\,[\partial_3{}^{n-1}x_1^{m-1}x_3].\end{aligned}$$

The d_3^+-term shows that there are no solutions here.

Case (–0): $(p,q)=(m+1,n)$ and $u=(\mathsf{d}_{12}^+[x_3]-\mathsf{d}_{13}^+[x_2]+\mathsf{d}_{23}^+[x_1])\,[\partial_3{}^n x_1^m]$. Then

$$\begin{aligned}0 = e'_0u &= f_2\mathsf{d}_2^+\,[\partial_3{}^n x_1^m x_3] - \mathsf{d}_1^+(-f_{12})\,[\partial_3{}^n x_1^m x_3] \\ &\quad - f_2\mathsf{d}_3^+\,[\partial_3{}^n x_1^m x_2] + (-f_{12})\mathsf{d}_3^+\,[\partial_3{}^n x_1^{m+1}] \\ &= \mathsf{d}_3^+\,[\partial_3{}^n x_1^m x_3] - n\mathsf{d}_2^+\,[\partial_2\partial_3{}^{n-1}x_1^m x_3] - n\,\mathsf{d}_1^+\,[\partial_1\partial_3{}^{n-1}x_1^m x_3] \\ &\quad + m\,\mathsf{d}_1^+\,[\partial_3{}^n x_1^{m-1}x_3^2] + n\,\mathsf{d}_3^+\,[\partial_2\partial_3{}^{n-1}x_1^m x_2] - \mathsf{d}_3^+\,[\partial_3{}^n x_1^{m-1}x_3] \\ &\quad + n\,\mathsf{d}_3^+\,[\partial_1\partial_3{}^{n-1}x_1^{m+1}] - (m+1)\,\mathsf{d}_3^+\,[\partial_3{}^n x_1^m x_3].\end{aligned}$$

Now there is only one d_2^+-term and this gives $n=0$. Then we are left with only one d_1^+-term and it gives $m=0$. Then we are left with a nonzero d_3^+-term, so there are no solutions in this case.

Case $(+-)$: $(p,q)=(m-1,n+1)$ and $u=(\mathsf{d}_{12}^+[\partial_2]+\mathsf{d}_{13}^+[\partial_3])\,[\partial_3{}^n x_1^{m-1}]$. Then

$$\begin{aligned}0=e_0'u&=f_2\mathsf{d}_2^+\,[\partial_2\partial_3{}^n x_1^{m-1}]-\mathsf{d}_1^+(-f_{12})\,[\partial_2\partial_3{}^n x_1^{m-1}]+f_2\mathsf{d}_3^+\,[\partial_3{}^{n+1}x_1^{m-1}]\\&=\mathsf{d}_3^+\,[\partial_2\partial_3{}^n x_1^{m-1}]-n\,\mathsf{d}_2^+\,[\partial_2{}^2\partial_3{}^{n-1}x_1^{m-1}]-n\,\mathsf{d}_1^+\,[\partial_1\partial_2\partial_3{}^{n-1}x_1^{m-1}]\\&\quad+(m-1)\,\mathsf{d}_1^+\,[\partial_2\partial_3{}^n x_1^{m-2}x_3]-(n+1)\,\mathsf{d}_3^+\,[\partial_2\partial_3{}^n x_1^{m-1}]\,.\end{aligned}$$

Again there is only one d_2^+-term and it gives $n=0$. Then d_1^+-term gives $m=1$, and this gives (T3). □

Lemma 3.14. *Let $e_1'=x_3d_{34}\in\mathfrak{g}_1$. Let F be an irreducible $s\ell(3)$-module with highest weight (p,q) and such that the action of L_+ on F is trivial. If $u\in\Lambda^-\otimes F$ is an $s\ell(3)$-highest weight vector of weight (m,n) and $e_1'\cdot u=0$, then there are the following possibilities for u (up to a constant factor):*

(T0): $(m,n)=(p,q)$ *and* $u=[\partial_3{}^q x_1^p]\in F$,
(T1): $p\geq 0$, $q=0$, $(m,n)=(p+1,0)$ *and* $u=\mathsf{d}_1^-\,[x_1^p]$,
(T2): $p=0$, $q\geq 1$, $(m,n)=(0,q-1)$ *and* $u=\Delta^-[\partial_3{}^{q-1}]$,
(T3): $(p,q)=(0,1)$, $(m,n)=(1,0)$ *and* $u=(\mathsf{d}_{12}^-[\partial_2]+\mathsf{d}_{13}^-[\partial_3])=\mathsf{d}_1^-\Delta^-$,
(T4): $(p,q)=(m,n)=(0,0)$ *and* $u=\mathsf{d}_{123}^-\,[1]$.

Proof. As relations for e_1'

$$e_1'\cdot\mathsf{d}_1^-=-f_2,\quad e_1'\cdot\mathsf{d}_2^-=+f_{12},\quad e_1'\cdot\mathsf{d}_3^-=0 \tag{3.8}$$

differ only in sign from the corresponding relations (3.7) for e_0', the calculations above provide the proof. □

Remark 3.15. In Lemmas 3.6 and 3.7 we actually describe singular vectors of induced $S(3)^\pm$-modules.

4. The highest $s\ell(3)$-weights of degenerate modules

We keep $L=E(3,6)$. Let V be a finite dimensional irreducible $\mathfrak{g}_0$-module. We are concerned with singular vectors in $M(V)$ (see (3.1)) that are also the highest weight vectors with respect to the standard Cartan and Borel subalgebras $\mathcal{H}$ and $\mathcal{B}$ of $\mathfrak{g}_0$.

We shall use the following notation:

$$\Lambda_i^\pm:=\Lambda^i(\mathfrak{g}_{-1}^\pm),\quad \Lambda^\pm:=\sum_{i\geq 0}\Lambda_i^\pm\,,\quad \mathsf{S}^k:=\mathrm{Sym}^k(\mathfrak{g}_{-2})\,,\quad \mathsf{S}=\sum_{k\geq 0}\mathsf{S}^k.$$

We know that $M(V)=U(L_-)\otimes V$ and by the PBW theorem we have the isomorphisms of vector spaces (where, as before, we drop the tensor product signs):

$$M(V)=\mathsf{S}\,\Lambda^-\Lambda^+\,V,\qquad M(V)=\mathsf{S}\,\Lambda^+\Lambda^-\,V.$$

When we use the first isomorphism, we say that the $(-+)$-order (for elements of $\mathfrak{g}_{-1}$) is chosen and when we use the second, we speak of the $(+-)$-order.

Theorem 4.1. *If an $E(3,6)$-module $M(V)$ is degenerate, then the $sl(3)$-highest weight of V is either $(p,0)$ or $(0,q)$.*

Proof. Suppose that the $s\ell(3)$-highest weight of V is (p,q) and $pq \neq 0$ and that the module $M(V)$ is degenerate. We have to show that this is impossible. Let w be a nontrivial singular vector, which is a $\mathfrak{g}_0$-highest weight vector.

Using the $(-+)$ order we write $M(V) = \sum_{m,i,j} \mathsf{S}^m \Lambda_i^- \Lambda_j^+ V$, where the summands are $s\ell(3)$-modules, and let $w = \sum w_{m;i,j}$ be the corresponding decomposition of w. Similarly $M(V) = \sum_{m,i,j} \mathsf{S}^m \Lambda_j^+ \Lambda_i^- V$, and $w = \sum \widetilde{w}_{m;j,i}$ is the decomposition for $(+-)$order.

Let n be the maximum value of m such that there exists $w_{m;i,j} \neq 0$, and let n' be the similar maximum for $\widetilde{w}_{m;j,i}$.

Lemma 4.2. *If $j \neq 0$, then $w_{n;i,j} = 0$, and if $i \neq 0$ then $\widetilde{w}_{n';j,i} = 0$.*

Proof. Notice that

$$e_0' \cdot \mathsf{S}^m \Lambda_i^- \Lambda_j^+ V \subset \mathsf{S}^{m-1} \Lambda_{i+1}^- \Lambda_j^+ V + \mathsf{S}^m \Lambda_i^- \Lambda_{j-1}^+ V\,. \tag{4.1}$$

This follows from the commutation relations

$$\begin{array}{lll}
[e_0', \hat{\partial}_1] = 0, & [e_0', \mathsf{d}_1^-] = 0, & [e_0', \mathsf{d}_1^+] = +f_2, \\
[e_0', \hat{\partial}_2] = 0, & [e_0', \mathsf{d}_2^-] = 0, & [e_0', \mathsf{d}_2^+] = -f_{12}, \\
[e_0', \hat{\partial}_3] = -\mathsf{d}_3^-, & [e_0', \mathsf{d}_3^-] = 0, & [e_0', \mathsf{d}_3^+] = 0.
\end{array}$$

We denote by $P_{(m;i,j)}$ the projection onto $\mathsf{S}^m \Lambda_i^- \Lambda_j^+ V$ (in the $(-+)$ decomposition), and we see from (4.1) that for any $i \geq 0$ and $j \geq 1$ we have $0 = P_{(n;i,j-1)} e_0' w = P_{(n;i,j-1)} e_0' w_{n;i,j}$. Let us write $w_{n;i,j} = \sum \hat{\partial}^a l_-^I l_+^J w_a^{IJ}$ where a, I, J are multiindices and $|a| = n$, $|I| = i$, $|J| = j$. We get $P_{(n;i,j-1)} e_0' w_{n;i,j} = \sum \hat{\partial}^a l_-^I (e_0' (l_+^J w_a^{IJ})) = 0$. So we conclude that for any given a, I, $\sum_{|J|=j} e_0' l_+^J w_a^{IJ} = 0$. Since each $w_{n;i,j}$ is a highest weight vector for $s\ell(3)$, the coefficient $\sum_J \ell_+^J w_a^{IJ}$ of $\hat{\partial}^a \ell_-^I$ of lowest weight in the expression for $w_{n;i,j}$ is an $s\ell(3)$-highest weight vector. Hence, by Lemma 3.13, $w_{n;i,j} = 0$ for $j > 0$.

In a similar way the commutation relations for e_1' and Lemma 3.14 imply the second statement of the lemma. $\square$

Lemma 4.3.

(a) *$w_{n-k;i,j} = 0$ for $j > k$ and $\widetilde{w}_{n'-k;j,i} = 0$ for $i > k$.*
(b) *$n = n'$.*
(c) *If $w_{n-k;i,j} \neq 0$ or $\widetilde{w}_{n-k;j,i} \neq 0$, then $i + j = 2k$.*
(d) *If $w_{n-k;i,j} \neq 0$, then $j \leq k \leq i$, and if $\widetilde{w}_{n-k;j,i} \neq 0$, then $i \leq k \leq j$.*
(e) *If $w_{n-k;i,j} \neq 0$, then $i = j = k$.*
(f) *$sl(2)$ acts trivially on V.*

Corollary 4.4. *$w = w_{n;0,0} + w_{n-1;1,1} + w_{n-2;2,2} + \ldots$ and $w_{n;0,0} \neq 0$.*

Proof. (a) Let us use induction on k. The case $k = 0$ follows from Lemma 4.2. Equation (4.1) shows that

$$0 = P_{(n-k;i,j-1)} e_0' w = P_{(n-k;i,j-1)} e_0' w_{n-k;i,j} + P_{(n-k;i,j-1)} e_0' w_{n-k+1;i-1,j-1}\,,$$

but for $j > k$ the last summand is zero by induction. Now we can apply Lemma 3.13 as we did above and conclude that $w_{n-k;i,j} = 0$. The other statement follows in the same way from Lemma 3.14.

To prove (b) let us notice first that

$$\Lambda_j^+ \Lambda_i^- V \subset \Lambda_i^- \Lambda_j^+ V + \mathtt{S}^1 \Lambda_{i-1}^- \Lambda_{j-1}^+ V + \mathtt{S}^2 \Lambda_{i-2}^- \Lambda_{j-2}^+ V + \dots . \tag{4.2}$$

We know that if $\widetilde{w}_{(n'-k);j,i} \neq 0$, then $i \leq k$, and therefore from (4.2) it follows that

$$\widetilde{w}_{(n'-k);j,i} \in \textstyle\sum_{s \leq k} \mathtt{S}^{n'-k+s} \Lambda_{i-s}^- \Lambda_{j-s}^+ V.$$

As $s \leq k$, this implies $n \leq n'$, but the arguments can be reversed so $n = n'$.

For (c) let us notice that the Y-eigenvalue of $w_{n-k;i,j}$ and of $\widetilde{w}_{n-k;i,j}$ is equal to $y_V - (i+j)/3 - 2(n-k)/3$ where y_V is defined by $Y|_V = y_V \,\mathrm{Id}_V$. But the eigenvalues are all the same whatever i, j, k, so (c) follows. Now (d) follows immediately from (a-c).

To get (e) let us consider $\mathbb{P}(w)$ where

$$\mathbb{P} = \textstyle\sum_{m, i \leq j} P_{(m;i,j)}.$$

If $\widetilde{w}_{n-k;j,i} \neq 0$, then $i \leq k \leq j$ by (d) and from (4.2) it follows that $\mathbb{P}\, \widetilde{w}_{n-k;j,i} = \widetilde{w}_{n-k;j,i}$. We conclude that $\mathbb{P}\, w = w$. But at the same time $w = \sum w_{n-k;i,j}$ and because $i > j$ implies $\mathbb{P}\, w_{n-k;i,j} = 0$, it follows that $w_{n-k;i,j} = 0$ for $i > j$. This proves (e).

Corollary 4.4 follows from (e). To establish (f) we need the following lemma.

Lemma 4.5. *Let $h \in \mathbb{C}[x_1, x_2, x_3]$ be of degree n and $g = hx_5\partial_4$. Then $g(\mathtt{S}^{n-k}\Lambda_k^- \Lambda_k^+ V) = 0$ for $k > 0$.*

Proof. One has to check it for $n = k = 1, 2, 3$; then the relation $[hx_5\partial_4, \partial_i] = -(\partial_i h)(x_5\partial_4)$ makes it easy to organize induction on $n - k$. □

Now from the lemma it follows that $f_3 w_{n;0,0} = (x_5\partial_4) w_{n;0,0} = 0$. On the other hand, $e_3 w = 0$, and using the expression for w from Corollary 4.4, we conclude that $e_3 w_{n;0,0} = 0$, but $w_{n;0,0} \neq 0$.

As a result, because e_3 and f_3 act trivially on $\mathfrak{g}_{-2}$, we conclude that they act trivially on all coefficients in $w_{n;0,0}$, which are elements of V. But we know that V is isomorphic to the tensor product of irreducible representations of $s\ell(2)$ and $s\ell(3)$. Therefore the existence of a trivial $s\ell(2)$ submodule in V means that $s\ell(2)$ acts trivially on V, which gives (f). □

Unless otherwise stated, we *use the* $(-+)$*-order*. In the following we can suppose that V is realized as a submodule in $\mathcal{M}$, i.e., that elements of V are linear combinations of monomials $\left[\prod_{i,j} \partial_j{}^{n_j} x_i^{m_i}\right]$, because the action of $s\ell(2)$ is trivial due to Lemma 4.3f. For $\alpha \in \mathbb{Z}_+^3$ we let, as before, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}$.

According to Proposition 3.4 one has

$$w = \textstyle\sum_\alpha D^\alpha T_\alpha, \tag{4.3}$$

where T_α are highest weight vectors in $\Lambda^-\Lambda^+ V$, and for their weights we have the relation

$$\mathrm{wt}\,_3 w = (-\alpha_1 + \alpha_2,\, -\alpha_2 + \alpha_3) + \mathrm{wt}\,_3 T_\alpha.$$

If $|\sigma| = n$ and $T_\sigma \neq 0$, then, because of Corollary 4.4, $T_\sigma \in V$, so $\mathrm{wt}\,_3 T_\sigma = (p,\,q)$; thus

$$\mathrm{wt}\,_3 w = (-\sigma_1 + \sigma_2,\ -\sigma_2 + \sigma_3) + (p,\,q). \tag{4.4}$$

This means that given n, $\mathrm{wt}\,_3 w$ and (p,q) we have a unique choice for σ, and we can write $T_\sigma = [\partial_3{}^q x_1^p]s$, where s is a nonzero scalar. Therefore $w_n = w_{n;0,0} = D^\sigma[\partial_3^q x_1^p]s,\ s \in \mathbb{C}$. At the same time, due to (3.3)–(3.5) we have: $D^\sigma[\partial_3^q x_1^p] = \bar{D}^\sigma[\partial_3^{q-\sigma_2} x_1^{p-\sigma_1}]\bar{s}$, where $\bar{s} = p^{[\sigma_1]}(p+q+1)^{[\sigma_1]}q^{[\sigma_2]}\,s$.

Without loss of generality we can assume that $\bar{s} = 1$. Let $t_\sigma = [\partial_3^{q-\sigma_2} x_1^{p-\sigma_1}]$. Using relations (4.2), we compute:

$$\begin{aligned} e_0' \cdot w_n = e_0'\bar{D}^\sigma t_\sigma = -\sigma_1 d_3^- \bar{D}_1^{\sigma_1 - 1}\bar{D}_2^{\sigma_2}\bar{D}_3^{\sigma_3}[x_3]t_\sigma \\ + \sigma_2 d_3^- \bar{D}_1^{\sigma_1}\bar{D}_2^{\sigma_2-1}\bar{D}_3^{\sigma_3}[\partial_2]t_\sigma - \sigma_3 d_3^- \bar{D}_1^{\sigma_1}\bar{D}_2^{\sigma_2}\bar{D}_3^{\sigma_3-1}t_\sigma. \end{aligned} \tag{4.5}$$

Let $P_m = \sum_{i,j} P_{(m;i,j)}$. It follows from (4.5) that $e_0' \cdot w_k = P_{n-1}e_0' \cdot w_n$. We will use this formula in the following way. As (4.1) shows, $P_{n-1}e_0'w = P_{n-1}e_0'w_n + P_{n-1}e_0'w_{n-1}$ and we have $e_0'w = 0$; hence

$$P_{n-1}e_0'w_n = -P_{n-1}e_0'w_{n-1}. \tag{$*$}$$

We already have quite an explicit expression for the left-hand side. We will write a similar expression for the right-hand side and study the restrictions imposed by the equality $(*)$. We will see that there are very few solutions for these equations in our context, and that in the end no one of them makes it to the singular highest weight vector.

We know that

$$w_{n-1} = w_{n-1;1,1} = \textstyle\sum_{|\beta|=n-1} D^\beta T_\beta \in \mathrm{S}\Lambda_1^-\Lambda_1^+ V, \tag{4.6}$$

where T_β are the $s\ell(3)$ highest weight vectors in $\Lambda_1^-\Lambda_1^+V$.

Lemma 4.6. *Let $|\beta| = n-1$ and $T_\beta \neq 0$. There are at most six choices for $\sigma - \beta$: $(-1,1,1)$, $(0,0,1)$, $(0,1,0)$, $(1,-1,1)$, $(1,0,0)$, $(1,1,-1)$.*

Proof. It is clear that $\mathrm{wt}\,_3 w = \mathrm{wt}\,_3 w_{n-1} = (-\beta_1 + \beta_2, -\beta_2 + \beta_3) + (\lambda_1, \lambda_2) + (p,q)$, where $\lambda = (\lambda_1, \lambda_2)$ is a weight of $\Lambda_1^-\Lambda_1^+$ and there are six of these weights: $(2,0)$, $(0,1)$, $(1,-1)$, $(-2,2)$, $(-1,0)$, $(0,-2)$. But, by (4.4), $\mathrm{wt}\,_3 w = (-\sigma_1 + \sigma_2, -\sigma_2 + \sigma_3) + (p,q)$ as well, so given λ we have two linear equations on β. The fact that $|\beta| = n-1$ provides the third equation and thus the difference $\sigma - \beta$ is determined. We get the six values for $\sigma - \beta$ that correspond to the above six choices for λ. $\square$

Lemma 4.7. *There are the following possibilities for T_β (where $t_i \in \mathbb{C}$):*

(1) $\beta^{(1)} = \sigma - (-1,1,1)$, $\mathrm{wt}\,_3 T_{\beta^{(1)}} = (p,q) + (2,0)$, *and* $T_{\beta^{(1)}} = d_1^- d_1^+[\partial_3^q x_1^p]t_1$.

(2) $\beta^{(2)} = \sigma - (0,0,1)$, $\mathrm{wt}\,_3 T_{\beta^{(2)}} = (p,q) + (0,1)$, *and*
$T_{\beta^{(2)}} = d_1^-(d_1^+[x_2] - d_3^+[x_1])[\partial_3^q x_1^p]t_2' + (d_1^- d_2^+ - d_2^- d_1^+)[\partial_3^q x_1^p]t_2''$.

(3) $\beta^{(3)} = \sigma - (0,1,0)$, $\mathrm{wt}\,{}_3T_{\beta^{(3)}} = (p,q) + (1,-1)$, *and*
$T_{\beta^{(3)}} = d_1^- \Delta^+[\partial_3^{q-1} x_1^p]t_3' + \Delta^- d_1^+[\partial_3^{q-1} x_1^p]t_3''$.

(4) $\beta^{(4)} = \sigma - (1,-1,1)$, $\mathrm{wt}\,{}_3T_{\beta^{(4)}} = (p,q) + (-2,2)$, *and*
$T_{\beta^{(4)}} = (d_1^-[x_2] - d_2^-[x_1])(d_1^+[x_2] - d_2^-[x_1])[\partial_3^q x_1^{p-2}]t_4$.

(5) $\beta^{(5)} = \sigma - (1,0,0)$, $\mathrm{wt}\,{}_3T_{\beta^{(5)}} = (p,q) + (-1,0)$, *and*

$$T_{\beta^{(5)}} = \big(d_1^-(d_2^+[x_3] - d_3^+[x_2]) + d_2^-(d_3^+[x_1] - d_1^+[x_3]) \\ + d_3^-(d_1^+[x_2] - d_2^+[x_1])\big)[\partial_3^q x_1^{p-1}]t_5' \\ + (d_1^-[x_2] - d_2^-[x_1])\,\Delta^+[\partial_3^{q-1} x_1^{p-1}]t_5''.$$

(6) $\beta^{(6)} = \sigma - (1,1,-1)$, $\mathrm{wt}\,{}_3T_{\beta^{(6)}} = (p,q) + (0,-2)$, *and* $T_{\beta^{(6)}} = \Delta^-\Delta^+[\partial_3^{q-2} x_1^p]t_6$.

Proof. The fact that T_β is the highest weight vector in $\Lambda_1^-\Lambda_1^+V$ and Lemma 3.12 permit us to write T_β explicitly as soon as its weight is known. This directly leads us to the above expressions. $\square$

Our next step is to look at the lexicographically highest terms on the left and right hand sides of $(*)$.

Lemma 4.8. *$\ell ht P_{n-1} e_0'(D^\beta T_\beta) = \hat\partial^\beta e_0' h^{[\beta_1]} h_1^{[\beta_1]} h_2^{[\beta_2]} T_\beta$.*

Proof. This follows from $|\beta| = n-1$ and Proposition 3.3. $\square$

We can rewrite the lemma as

$$\ell ht P_{n-1} e_0'(D^\beta T_\beta) \sim \hat\partial^\beta e_0' T_\beta\,,$$

where $\sim$ means equality up to a nonzero constant multiple, because as we know from (3.3)–(3.5), $h^{[\beta_1]}$, $h_1^{[\beta_1]}$, $h_2^{[\beta_2]}$ multiplies T_β by a nonzero constant as long as $T_\beta \neq 0$.

Thus we see that if $t_1 \neq 0$, then the ℓht of the right-hand side of $(*)$ comes from $T_{\beta^{(1)}}$ and is proportional to $\hat\partial^{\beta^{(1)}} d_1^-(-q[\partial_2\partial_3^{q-1}x_1^p]t_1)$ (cf. proof of Lemma 3.13). But the ℓht of the left-hand side of $(*)$ is smaller, as one concludes immediately from (4.5) and (3.6). This implies that $t_1 = 0$, and then the ℓht on the right side of $(*)$ comes from $T_{\beta^{(2)}}$ (if it is nonzero), and the ℓht in (4.5) are clearly the terms with $\hat\partial_1^{\sigma_1}\hat\partial_2^{\sigma_2}\hat\partial_3^{\sigma_3-1}$. Comparing the coefficients of this monomial in (4.5) we get

$$\sigma_3 d_3^-[\partial_3^q x_1^p] \sim e_0'(d_1^-(d_1^+[\partial_3^q x_1^{p-1}x_2]t_2' - d_2^+[\partial_3^q x_1^p]t_2') + (d_1^- d_2^+ - d_2^- d_1^+)[\partial_3^q x_1^p]t_2''),$$

or

$$\sigma_3 d_3^-[\partial_3^q x_1^p] \sim -d_1^-(f_2[\partial_3^q x_1^{p-1}x_2]t_2' + f_{12}[\partial_3^q x_1^p]t_2') + (d_1^- f_{12} + d_2^- f_2)[\partial_3^q x_1^p]t_2''.$$

This clearly implies that $\sigma_3 = 0$, $t_2' = t_2'' = 0$.

Taking this into account, we can rewrite $(*)$:

$$\sigma_2 d_3^- \bar D_2^{\sigma_2-1}[\partial_2\partial_3^{q-\sigma_2}x_1^{p-\sigma_1}] - \sigma_1 d_3^- \bar D_1^{\sigma_1-1}\bar D_2^{\sigma_2}[\partial_3^{q-\sigma_2}x_1^{p-\sigma_1}x_3] \\ = -P_{n-1}e_0'\left(D_1^{\sigma_1}D_2^{\sigma_2-1}T_{\beta^{(3)}} + D_1^{\sigma_1-1}D_2^{\sigma_2}T_{\beta^{(5)}} + D_1^{\sigma_1-1}D_2^{\sigma_2-1}D_3T_{\beta^{(6)}}\right), \quad (4.7)$$

where $T_{\beta^{(4)}}$ is absent because there are no such terms when $\sigma_3 = 0$, as the components of $\beta^{(4)}$ are nonnegative.

Let us be more careful with the constant factors here. In computing the ℓht on the right side we apply Lemma 4.8 to $\beta^{(3)}$ and we get $\ell ht P_{n-1} e_0'(\mathcal{D}_1^{\sigma_1}\mathcal{D}_2^{\sigma_2-1}T_{\beta^{(3)}}) = \hat{\partial}_1^{\sigma_1}\hat{\partial}_2^{\sigma_2-1}e_0' h^{[\sigma_1]}h_1^{[\sigma_1]}h_2^{[\sigma_2-1]}T_{\beta^{(3)}} = \hat{\partial}_1^{\sigma_1}\hat{\partial}_2^{\sigma_2-1}e_0' T_{\beta^{(3)}}(p+1)^{[\sigma_1]}(p+q+1)^{[\sigma_1]}(q-1)^{[\sigma_2-1]}$ because $\mathrm{wt}\,_3 T_{\beta^{(3)}} = (p+1, q-1)$ (and of course $(q-1)^{[\sigma_2-1]} \neq 0$ as $\sigma_2 \le q$)). Letting $b = ((p+1)^{[\sigma_1]}(p+q+1)^{[\sigma_1]}(q-1)^{[\sigma_2-1]})^{-1}$, we arrive at the following equation:

$$\begin{aligned}\sigma_2 q d_3^-[\partial_2\partial_3^{q-1}x_1^p]b &= d_1^- e_0'\left(d_1^+[\partial_1\partial_3^{q-1}x_1^p]t_3' + d_2^+[\partial_2\partial_3^{q-1}x_1^p]t_2' + d_3^+[\partial_3^q x_1^p]t_3'\right)\\ &\quad + d_1^- e_0' d_1^+[\partial_1\partial_3^{q-1}x_1^p]t_3'' + d_2^- e_0' d_1^+[\partial_2\partial_3^{q-1}x_1^p]t_3'' + d_3^- e_0' d_1^+[\partial_3^q x_1^p]t_3''\\ &= d_1^-\left(f_2[\partial_1\partial_3^{q-1}x_1^p]t_3' - f_{12}[\partial_2\partial_3^{q-1}x_1^p]t_3' + 0 + f_2[\partial_1\partial_3^{q-1}x_1^p]t_3''\right)\\ &\quad + d_2^- f_2[\partial_2\partial_3^{q-1}x_1^p]t_3'' + d_3^- f_2[\partial_3^q x_1^p]t_3''\,.\end{aligned}$$

Looking at the coefficients of d_1^-, we conclude that $pt_3' = 0$ and since $p \neq 0$, $t_3' = 0$. From the coefficients of d_2^- we see that $(q-1)t_3'' = 0$, and from the coefficients of d_3^- we conclude that $q\sigma_2 b = -qt_3''$. Thus, either $\sigma_2 = t_3' = t_3'' = 0$ or $\sigma_2 > 0$, $q = 1$, $t_3' = 0$, $t_3'' = -\sigma_2 b$. Since $\sigma_2 \le q$, in the latter case we have $\sigma_2 = 1$, $t_3'' = -b$.

If $\sigma_2 = 0$, then (4.7) reduces to

$$\sigma_1 d_3^- \bar{D}_1^{\sigma_1-1}[\partial_3^q x_1^{p-\sigma_1}x_3] \sim P_{n-1}e_0' D^{\sigma_1-1}T_{\beta^{(5)}}. \tag{4.8}$$

Now we look at the coefficients of $\sigma_1^{\sigma_1-1}$ in the equation. In the left-hand side we get

$$\sigma_1 d_3^-[\partial_3^q x_1^{p-1}x_3].$$

Furthermore, since $\sigma_1 = n$, we can use Lemma 4.8 on the right of (4.8), hence the coefficient of $\hat{\partial}_1^{\sigma_1-1}$ on the right of (4.8) is equal to $e_0' T_{\beta^{(5)}}$, which we now compute:

$$\begin{aligned}e_0' T_{\beta^{(5)}} &= -d_1^- e_0'(d_2^+[\partial_3^q x_1^{p-1}x_3] - d_3^+[\partial_3^q x_1^{p-1}x_2])t_5'\\ &\quad - d_2^- e_0'(d_3^+[\partial_3^q x_1^p] - d_1^+[\partial_3^q x_1^{p-1}x_3])t_5'\\ &\quad - d_3^- e_0'(d_1^+[\partial_3^q x_1^{p-1}x_2] - d_2^+[\partial_3^q x_1^p])t_5'\\ &\quad - d_1^- e_0'\Delta^+[\partial_3^q x_1^{p-1}x_2]t_5'' - d_2^- e_0'\Delta^+[\partial_3^{q-1}x_1^p]t_5''\\ &= d_1^- f_{12}[\partial_3^q x_1^{p-1}x_3]t_5' + d_2^- f_2[\partial_3^q x_1^{p-1}x_3]t_5'\\ &\quad - d_3^-(f_2[\partial_3^q x_1^{p-1}x_2]t_5' + f_{12}[\partial_3^q x_1^p]t_5')\\ &\quad - d_1^-(f_2[\partial_1\partial_3^{q-1}x_1^{p-1}x_2]t_5'' - f_{12}[\partial_2\partial_3^{q-1}x_1^{p-1}x_2]t_5'')\\ &\quad - d_2^-(f_2[\partial_1\partial_3^{q-1}x_1^p]t_5'' - f_{12}[\partial_2\partial_3^{q-1}x_1^p]t_5'')\,.\end{aligned}$$

At the end we get for $e_0' T_{\beta^{(5)}}$:

$$\begin{aligned}e_0' T_{\beta^{(5)}} &= d_1^-\left(\,(-qt_5' - t_5'')[\partial_1 x_1] + (p-1)t_5''[\partial_2 x_2] + (p-1)t_5'[\partial_3 x_3]\,\right)[\partial_3^{q-1}x_1^{p-2}x_3]\\ &\quad + d_2^-(-qt_5' - pt_5'')[\partial_2\partial_3^{q-1}x_1^{p-1}x_3] + d_3^-((p+q+1)t_5')[\partial_3^q x_1^{p-1}x_3]\,.\end{aligned} \tag{4.9}$$

We conclude that the terms with d_1^- and d_2^- disappear iff either $p=1, qt_5'+t_5''=0$ or when $T_{\beta^{(5)}}=0$. Now for the case when $\sigma_3=\sigma_2=0$ and $T_{\beta^{(5)}}=0$ we get $\sigma_1=0$ and this is a contradiction.

If $\sigma_2=\sigma_3=0$ and $p=1$, then $\sigma_1=1$ because $\sigma_1\le p$, and it could not be 0 as this gives $|\sigma|=0$. So it becomes $|\sigma|=n=1$ and $w=w_n+w_{n-1}=w_1+w_0=D_1[\partial_3^q]+T_{\beta^{(5)}}$ ($T_{\beta^{(6)}}$ disappears for $\sigma_2=0$). We can check $e_3\cdot w$ now: $e_3\cdot w=0+e_3T_{\beta^{(5)}}=2(d_{12}^+[x_3]+d_{23}^+[x_1]+d_{31}^+[x_2])[\partial_3^q]t_5'+(d_1^+[x_2]-d_2^+[x_1])\Delta^+[\partial_3^{q-1}]t_5''$, therefore $e_3\cdot w=0$ implies $t_5'=t_5''=0$ and we arrive at a contradiction.

If $\sigma_2=1$, we are left with the situation when $q=1,\sigma_2=1,t_3'=0,t_3''=-b$. Then (4.7) reduces to

$$d_3^-\bar{D}_1^{\sigma_1}[\partial_2x_1^{p-\sigma_1}]-\sigma_1d_3^-\bar{D}_1^{\sigma_1-1}D_2[x_1^{p-\sigma_1}x_3]=-P_{n-1}e_0'(D_1^{\sigma_1}T_{\beta^{(3)}}+D^{\sigma_1-1}D_2T_{\beta^{(5)}}). \quad (4.10)$$

Note that $\sigma_1\neq 0$ because $\sigma_1+1=n$, and, if $\sigma_1=0$, then $n=1$, the term with $T_{\beta^{(5)}}$ disappears and $w=w_n+w_{n-1}$, hence we have $w=\bar{D}_2t_\sigma-D_1\Delta^-d_1^+[x_1^p]b$. Since $e_3w=0$ and e_3 annihilates the first summands but does not annihilate the second one, we arrive at a contradiction. Therefore $\sigma_1\neq 0$.

We already know that the ℓht in both sides of (4.10) are equal, so let us look at the next terms, i.e., the coefficients of $\hat{\partial}_1^{\sigma_1-1}\hat{\partial}_2$. In the left-hand side we get

$$\sigma_1d_3^-[x_1^{p-1}]([\partial_2x_2]-[\partial_3x_3])\,. \quad (4.11)$$

In order to do the same for the right-hand side, we need the second lexicographically ordered term of $D_1^{\sigma_1}$. Using Lemma 3.8 twice, we have

$$\begin{aligned}D_1^{\sigma_1}&=A^{\sigma_1}h^{[\sigma_1]}+\ldots=\hat{\partial}_1^{\sigma_1}h_1^{[\sigma_1]}h^{[\sigma_1]}+\sigma_1\hat{\partial}_2\hat{\partial}_1^{\sigma_1-1}f_1(h_1-1)^{[\sigma_1-1]}h^{[\sigma_1]}+\ldots\\&=\hat{\partial}_1^{\sigma_1}h^{[\sigma_1]}+\sigma_1\hat{\partial}_1^{\sigma_1-1}\hat{\partial}_2(h_1+1)^{[\sigma_1-1]}(h+1)^{[\sigma_1]}f_1+\ldots\,.\end{aligned}$$

Hence the coefficient of $\hat{\partial}_1^{\sigma_1-1}\hat{\partial}_2$ on the right-hand side of (4.10) is

$$-e_0'\left((h_1+1)^{[\sigma_1-1]}(h+1)^{[\sigma_1]}f_1T_{\beta^{(3)}}+h^{[\sigma_1-1]}h_1^{[\sigma_1-1]}h_2T_{\beta^{(5)}}\right).$$

Since the weights of $f_1T_{\beta^{(3)}}$ and $T_{\beta^{(5)}}$ are both equal to $(p-1,1)$, this becomes

$$-(p(p+2)e_0'f_1T_{\beta^{(3)}}+(p-\sigma_1+1)e_0'T_{\beta^{(5)}})(p-1)^{[\sigma_1-1]}(p+1)^{[\sigma_1-1]}. \quad (4.12)$$

Comparing (4.11) and (4.12) gives

$$\begin{aligned}&\sigma_1d_3^-[x_1^{p-1}]([\partial_2x_2]-[\partial_3x_3])\\&\quad=-(p-1)^{[\sigma_1-1]}(p+1)^{[\sigma_1-1]}\left(p(p+2)e_0'f_1T_{\beta^{(3)}}+(p-\sigma_1+1)e_0'T_{\beta^{(5)}}\right). \quad (4.13)\end{aligned}$$

Since e_0' commutes with f_1, we can use our previous calculation of $e_0'T_{\beta^{(3)}}$: $e_0'T_{\beta^{(3)}}=-d_3^-[\partial_2x_1^p]b$. Hence

$$e_0'f_1T_{\beta^{(3)}}=f_1e_0'T_{\beta^{(3)}}=d_3^-[\partial_1x_1^p]b-p\,d_3^-[\partial_2x_2x_1^p]b\,. \quad (4.14)$$

Also, our previous calculation of $e'_0 T_{\beta(5)}$shows that in general terms with d_1^-, d_2^- are present in $e'_0 T_{\beta(5)}$. But (4.13) shows that these terms have to be zero because there are no such terms in the other entries in (4.13). We conclude that $p = 1$, $t'_5 + t''_5 = 0$.

So we are left with $p = 1$, $q = 1$, $\sigma_3 = 0$, $\sigma_2 = 1$ and $\sigma_1 \neq 0$. The latter condition implies $\sigma_1 = 1$ because $\sigma_1 \leq p$. Let us write again the coefficients of $\hat{\partial}_1^{\sigma_1 - 1}\hat{\partial}_2 = \hat{\partial}_2$ in (4.10), which are given by (4.12), for our specific data: $d_3^-([\partial_2 x_2] - [\partial_3 x_3]) = 2e'_0 f_1 T_{\beta(3)} - e'_0 T_{\beta(5)}$. Together with (4.14) and (4.9) we arrive at $d_3^-([\partial_2 x_2] - [\partial_3 x_3]) = d_3^-(3([\partial_1 x_1] - [\partial_2 x_2])(-1/6) - 3\,t'_5[\partial_3 x_3])$ which gives $t'_5 = 1/2$.

Let us calculate also the terms with $\hat{\partial}_3$ at both sides of (4.10). We get

$$\begin{aligned}\hat{\partial}_3 d_3^-[\partial_2 x_2]\,2 &= -P_1(e'_0(\hat{\partial}_3(f_{12}h_1 + f_2 f_1)T_{\beta(3)} + \hat{\partial}_3 f_2 T_{\beta(5)})) \\ &= -\hat{\partial}_3((f_{12}h_1 + f_2 f_1)e'_0 T_{\beta(3)} + f_2 e'_0 T_{\beta(5)}) \\ &= +\hat{\partial}_3((f_{12}h_1 + f_2 f_1)(d_3^-[\partial_2 x_1]\tfrac{1}{6}) - f_2(d_3^-[\partial_3 x_3]\tfrac{1}{2}))\,.\end{aligned}$$

Clearly $(f_{12}h_1 + f_2 f_1)d_3^-[\partial_2 x_1] = d_3^-(f_{12}h_1 + f_2 f_1)[\partial_2 x_1] = d_3^-[\partial_2 x_3]\,3$ and $f_2\, d_3^-[\partial_3 x_3] = d_3^-\, f_2[\partial_3 x_3] = -d_3^-[\partial_2 x_3]$.

Combining these equations, we get $d_3^-[\partial_3 x_3]\,2 = d_3^-[\partial_2 x_3]\frac{1}{2} + d_3^-[\partial_2 x_3]\frac{1}{2}$, a contradiction. This closes the last case and ends the proof of Theorem 4.1. □

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