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STRUCTURE OF SOME $\mathbb{Z}$-GRADED LIE SUPERALGEBRAS OF VECTOR FIELDS

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Dedicated to the memory of Claude Chevalley

Abstract. In this paper we classify $\mathbb{Z}$-graded transitive Lie superalgebras with prescribed nonpositive parts listed in [K2]. The classification of infinite-dimensional simple linearly compact Lie superalgebras given in [K2] is based on this result. We also study the structure of the exceptional $\mathbb{Z}$-graded transitive Lie superalgebras and give their geometric realization.

0. Introduction

Let L be a linearly compact Lie superalgebra, that is, a complete topological Lie superalgebra which admits a fundamental system of neighborhoods of 0 consisting of subspaces of finite codimension. (The formal completion of a Lie superalgebra of vector fields on a finite-dimensional supermanifold X at a neighborhood of a point of X is of this type.) Suppose that L is simple (i.e., has no nontrivial closed ideals). Then we may construct a Weisfeiler filtration of L by open (and hence closed of finite codimension) subspaces

$$L = L_{-h} \supset L_{-h+1} \supset \dots \supset L_0 \supset L_1 \supset \dots,$$

such that the associated graded Lie superalgebra $\mathrm{Gr}L = \oplus_{j=-h}^{\infty}\mathfrak{g}_j$, $\mathfrak{g}_j = L_j/L_{j+1}$, of depth h has the properties [W]:

(G0) dim $\mathfrak{g}_j < \infty$,

(G1) $\mathfrak{g}_{-j} = \mathfrak{g}_{-1}^{j}$, for $j \geq 1$,

(G2) if $a \in \mathfrak{g}_j$, $j \geq 0$, then $[a, \mathfrak{g}_{-1}] = 0$ implies that $a = 0$,

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(G3) the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is irreducible.

A graded Lie superalgebra of finite depth satisfying (G1) and (G2) (respectively (G1) and (G3)) is called *transitive* (respectively *irreducible*). It is shown in [K2] that an infinite-dimensional simple linearly compact Lie superalgebra L admits a subalgebra L_0 such that (L, L_0) is "even primitive", which implies that in addition to (G0)–(G3) above the associated graded Lie superalgebra $\mathfrak{g}$ satisfies

(G4) $\mathfrak{g}_{-1}$ is a strongly transitive $\mathfrak{g}_0$-module.

Recall that a faithful irreducible finite-dimensional module over a Lie superalgebra $\mathfrak{p}$ is called *strongly transitive*, if for every nonzero even element $v \in V_{\bar 0}$, we have $V = \mathfrak{p} \cdot v$. (Note that $V = V_{\bar 1}$ is strongly transitive.)

In [K2] all strongly transitive modules V with $V_{\bar 0} \neq 0$ over a finite-dimensional Lie superalgebra were classified and from this all possible choices for the nonpositive part $\mathfrak{g}_{\leq 0} = \oplus_{j=-h}^{0} \mathfrak{g}_j$ of the associated graded Lie superalgebra of L were derived. The list is as follows (the subalgebra $\oplus_{j=-h}^{0} \mathfrak{g}_j$ is written below as the $h+1$-tuple $(\mathfrak{g}_{-h}, \mathfrak{g}_{-h+1}, \dots, \mathfrak{g}_{-1}, \mathfrak{g}_0)$):

Inconsistent gradation and depth 1*:*

(I1) $(\mathbb{C}^{m|n}, gl(m,n))$
(I2) $(\mathbb{C}^{m|n}, sl(m,n))$
(I3) $(\mathbb{C}^{m|n}, spo(m,n))$
(I4) $(\mathbb{C}^{m|n}, cspo(m,n))$
(I5) $(\mathbb{C}^{n|0} \otimes \Lambda(1), sl(n) \otimes \Lambda(1) + \mathfrak{a})$
(I6) $(\mathbb{C}^{2n|0} \otimes \Lambda(1), sp(2n) \otimes \Lambda(1) + \mathfrak{a})$
(I7) $(\mathbb{C}^{n|n}, \tilde{P}(n))$
(I8) $(\mathbb{C}^{n|n}, c\tilde{P}(n))$
(I9) $(\mathbb{C}^{n|n}, P(n))$
(I10) $(\mathbb{C}^{n|n}, cP(n))$
(I11) $(\mathbb{C}^{4|4}, \widehat{P}(4))$
(I12) $(\mathbb{C}^{2|2}, spin_4^0 + \mathfrak{a})$
(I13) $(\mathbb{C}^{m|m}, Q(m))$
(I14) $(\mathbb{C}^{m|m}, cQ(m))$
(I15) $(\Pi(\Lambda(2)^{\lambda}, W(0,2)), \lambda \neq 0, 1$
(I16) $(\Pi(\Lambda(2)^{\lambda}, cW(0,2)), \lambda \neq 0, 1$
(I17) $(\Pi(\Lambda(2)), W(0,2) + \Lambda(2))$
(I18) $(\Pi(\Lambda(2)), S(0,2))$
(I19) $(\Pi(\mathbb{C}1 + \mathbb{C}\xi_1 + \mathbb{C}\xi_2), S(0,2))$

Inconsistent gradation and depth 2*:*

(J1) $(\mathbb{C}^{1|0}, \mathbb{C}^{m|n}, spo(m,n))$
(J2) $(\mathbb{C}^{1|0}, \mathbb{C}^{m|n}, cspo(m,n))$
(J3) $(\mathbb{C}^{0|1}, \mathbb{C}^{n|n}, \tilde{P}(n))$
(J4) $(\mathbb{C}^{0|1}, \mathbb{C}^{n|n}, c\tilde{P}(n))$
(J5) $(\mathbb{C}^{0|1}, \mathbb{C}^{n|n}, P(n))$
(J6) $(\mathbb{C}^{0|1}, \mathbb{C}^{n|n}, cP(n))$
(J7) $(\mathbb{C}^{0|1}, \mathbb{C}^{n|n}, P(n) + \mathbb{C}(I + \beta\Phi))$
(J8) $(\mathbb{C}^{1|0} \otimes \xi_1, \mathbb{C}^{2n|0} \otimes \Lambda(1),$
$sp(2n) \otimes \Lambda(1) + \mathbb{C}\xi_1 \frac{\partial}{\partial \xi_1} + \mathbb{C}\frac{\partial}{\partial \xi_1})$
(J9) $(\mathbb{C}^{1|0} \otimes \Lambda(1), \mathbb{C}^{2n|0} \otimes \Lambda(1),$
$csp(2n) \otimes \Lambda(1) + \mathfrak{a})$

Recall that a gradation of $\mathfrak{g}$ is called *inconsistent* if $\mathfrak{g}_{-1}$ is not purely odd; otherwise it is called *consistent*. In the latter case $\mathfrak{g}_j$ with j odd (respectively j even) is purely odd (respectively even); in particular, $\mathfrak{g}_0$ is a Lie algebra. Using some growth considerations, the following list of $\mathfrak{g}_{\leq 0}$ for consistent $\mathbb{Z}$-gradations was obtained in [K2]:

Consistent gradation:

(C1) $(\mathbb{C}, \mathbb{C}^n, cso(n))$, $n \geq 1$ and $n \neq 2$
(C2) $(\mathbb{C}^{5*}, \Lambda^2(\mathbb{C}^5), sl(5))$
(C3) $(\mathbb{C}^{5*}, \Lambda^2(\mathbb{C}^5), gl(5))$
(C4) $(\mathbb{C}^{3*}, \mathbb{C}^3 \otimes \mathbb{C}^2, gl(3) \oplus sl(2))$
(C5) $(\mathbb{C}^{3*}, \mathbb{C}^3 \otimes \mathbb{C}^2, sl(3) \oplus sl(2))$
(C6) $(\mathbb{C}^2, \mathbb{C}^{3*}, \mathbb{C}^3 \otimes \mathbb{C}^2, gl(3) \oplus sl(2))$
(C7) $(\mathbb{C}^2, \mathbb{C}^{3*}, \mathbb{C}^3 \otimes \mathbb{C}^2, sl(3) \oplus sl(2))$

Below we explain above notation.

First, $m \geq 1$ unless otherwise indicated. The prefix c denotes the addition of a 1-dimensional center, e.g., $csl(m) \cong gl(m)$. Next, whenever we have a nontrivial center in $\mathfrak{g}_0$, it is assumed that it contains an element, called the *grading operator*, that acts as the scalar j on $\mathfrak{g}_j$ for all j.

The Lie superalgebra $\mathfrak{a}$ in (I5), (I6) and (J9) stands for a subalgebra of the Lie superalgebra $\mathbb{C} + \mathbb{C}\xi_1 + \mathbb{C}\xi_1\frac{\partial}{\partial\xi_1} + \mathbb{C}\frac{\partial}{\partial\xi_1} \cong gl(1,1)$ such that its projection onto $\mathbb{C}\frac{\partial}{\partial\xi_1}$ is nontrivial.

The Lie superalgebra $\tilde{P}(m)$ is the finite-dimensional Lie superalgebra preserving an odd supersymmetric nondegenerate bilinear form [K1]. It has a $\mathbb{Z}$-gradation of the form $\tilde{P}(m) = \tilde{P}(m)_{-1} \oplus \tilde{P}(m)_0 \oplus \tilde{P}(m)_1$ (see Section 1). $\tilde{P}(m)_0 \cong gl(m) = sl(m) \oplus \mathbb{C}\Phi$, where Φ acts as the grading operator of this gradation. $P(m) = [\tilde{P}(m), \tilde{P}(m)]$ and $c\tilde{P}(m) = \tilde{P}(m) \oplus \mathbb{C}I$. $\widehat{P}(4)$ is the unique nontrivial central extension of the Lie superalgebra $P(4)$ discovered by Sergeev (see [S]).

$W(0,2)$ is the Lie superalgebra of polynomial vector fields in two odd indeterminates, while $S(0,2)$ is the subalgebra of divergence zero vector fields. They act on the space on polynomial functions $\Lambda(2)$. For $\lambda \in \mathbb{C}$, we may define on the space $\Lambda(2)$ an action of $W(0,2)$, given by $D \in W(0,2) \to D + \lambda\, \mathrm{div}D$, where $\mathrm{div}D$ is the divergence of the vector field D. To distinguish this action from the usual action of $W(0,2)$, we denote this representation space by $\Lambda(2)^\lambda$. Of course, this construction generalizes to vector fields on the superspace $\mathbb{C}^{m|n}$.

In general given a representation V of a Lie superalgebra, we use $\Pi(V)$ to denote the same representation, but with reversed parity.

Let $\mathfrak{h}_n$ be the odd Heisenberg superalgebra of rank n, that is, $\mathfrak{h}_n$ is spanned over $\mathbb{C}$ by n odd vectors $q_1, \ldots, q_n$ and an even central element z with relations $[q_i, q_j] = \delta_{i,j}z$. It is well-known that $\mathfrak{h}_n$ has a unique finite-dimensional irreducible module on which z acts as a fixed nonzero scalar. We denote this module by $\mathbb{C}^{[\frac{n-1}{2}] \,|\, [\frac{n-1}{2}]}$, in accordance with its super-dimension. On this space we have an action of the universal enveloping algebra of $\mathfrak{h}_n$, which is the Clifford superalgebra in n indeterminates. Since the quadratic terms of the Clifford superalgebra form a Lie algebra isomorphic to $so(n)$, we obtain a natural action of $\mathfrak{h}_n + so(n)$ on the Fock space $\mathbb{C}^{[\frac{n-1}{2}] \,|\, [\frac{n-1}{2}]}$. Now let $n = 4$. We have $so(4) \cong \mathfrak{a}_1 \oplus \mathfrak{a}_2$, where $\mathfrak{a}_i \cong sl(2)$, for $i = 1, 2$. We denote by $spin_4^0$ the subalgebra $\mathfrak{a}_1 + \mathfrak{h}_4$. This acts strongly transitively on $\mathbb{C}^{2|2}$. We let $\mathfrak{a}$ in (I12) be a subalgebra of $\mathfrak{a}_2$; then, of course, $spin_4^0 + \mathfrak{a}$ still acts strongly transitively on $\mathbb{C}^{2|2}$.

$Q(m)$ is the simple Lie superalgebra such that $Q(m)_{\bar{0}} = sl(m)$ and $Q(m)_{\bar{1}} = \operatorname{ad} sl(m)$ [K1].

The present paper is the last part of the program formulated in [K2] for classifying simple infinite-dimensional linearly compact Lie superalgebras. Its main purpose is to classify all transitive $\mathbb{Z}$-graded Lie superalgebras with given $\mathfrak{g}_{<0}$ from the lists above. This classification is used in an essential way in [K2]. Another essential part of the program, the reconstruction of the filtered Lie superalgebra from its associated graded Lie superalgebra, was worked out in [CK2] by means of Spencer cohomology (some cases are treated in [K2]). At the same time, the present paper provides proofs of some results on full prolongation of certain graded Lie superalgebras which are needed in [CK2] in order to prove the nonexistence of filtered deformations.

We now come to the organization of this paper. Section 1 establishes notation to be used throughout the paper and provides the reader with examples of transitive Lie superalgebras and some of their elementary properties that will be called upon later on. Sections 2 and 3 are devoted to classification of transitive Lie superalgebras with inconsistent $\mathbb{Z}$-gradation. More precisely, in Section 2 those of depth 1 are classified, while in Section 3, those of depth 2 are classified, which in total encompasses the cases (I1)–(I19) and (J1)–(J9). In Section 4 transitive Lie superalgebras with consistent $\mathbb{Z}$-gradation are determined, i.e., the cases (C1)–(C7) are covered. In Section 5 we study the properties of exceptional Lie superalgebras of vector fields in more detail and give a geometric construction of these algebras. It is our hope that this will shed some light on their algebraic structure.

In this paper all vector spaces, algebras and their tensor products are assumed to be over the field of complex numbers $\mathbb{C}$. By graded we will always mean $\mathbb{Z}$-graded, satisfying (G0). For a general graded Lie superalgebra $\mathfrak{g}$ of depth h, we put $\mathfrak{g}_- = \oplus_{j=-h}^{-1}\mathfrak{g}_j$ and $\mathfrak{g}_{\leq 0} = \oplus_{j=-h}^{0}\mathfrak{g}_j$.

1. Preliminaries

In this section we recall the definitions and elementary properties of some basic Lie superalgebras of vector fields that will be used in this paper. Simplicity or nonsimplicity of these Lie superalgebras is checked using results of Section 1.5.

1.1. The superalgebras $W(m,n)$, $S(m,n)$ and $S'(m,n)$

Let $\Lambda(n)$ be the Grassmann superalgebra in the n odd indeterminates $\xi_1, \xi_2, \dots, \xi_n$. Let $x_1, x_2, \dots, x_m$ be m even indeterminates. Set $\Lambda(m,n) = \mathbb{C}[x_1, \dots, x_m] \otimes \Lambda(n)$. Then $\Lambda(m,n)$ is an associative commutative superalgebra. Let $W(m,n)$ be the Lie superalgebra of derivations of $\Lambda(m,n)$. Then $W(m,n)$ consists of elements of the form [K1]: $\sum_{i=1}^m f_i \frac{\partial}{\partial x_i} + \sum_{i=1}^n g_i \frac{\partial}{\partial \xi_i}$, where $f_i, g_i \in \Lambda(m,n)$ and $\frac{\partial}{\partial x_i}$ (respectively $\frac{\partial}{\partial \xi_i}$) is the even (respectively odd) derivation uniquely determined by $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$ and $\frac{\partial}{\partial x_i}(\xi_j) = 0$ (respectively $\frac{\partial}{\partial \xi_i}(x_j)=0$ and $\frac{\partial}{\partial \xi_i}(\xi_j)=\delta_{ij}$). To each vector field $D=\sum_{i=1}^m f_i\frac{\partial}{\partial x_i} + \sum_{i=1}^n g_i\frac{\partial}{\partial \xi_i}$ we may associate its divergence by

$$\mathrm{div}D=\sum_{i=1}^m \frac{\partial f_i}{\partial x_i}+\sum_{i=1}^n(-1)^{p(g_i)}\frac{\partial g_i}{\partial \xi_i}.$$

The subspace of $W(m,n)$ consisting of vector fields D with zero divergence is a subalgebra of $W(m,n)$, which we will denote by $S'(m,n)$. The derived algebra of $S'(m,n)$, denoted by $S(m,n)$, coincides with $S'(m,n)$, provided $m \neq 1$ or $m = 0$ and $n \geq 3$. In the case when $m = 1$, we have an exact sequence $0 \to S(1,n) \to S'(1,n) \to \mathbb{C}\xi_1 \dots \xi_n \frac{\partial}{\partial x_1} \to 0$.

We may define a gradation on $W(m,n)$ by letting $\deg x_i = a_i = -\deg \frac{\partial}{\partial x_i}$ and $\deg \xi_j = b_j = -\deg \frac{\partial}{\partial \xi_j}$, where $a_i \in \mathbb{N}$ and $b_i \in \mathbb{Z}$, called the gradation of type $(a_1,\dots,a_m|b_1,\dots,b_n)$. $W(m,n)$ thus becomes a graded Lie superalgebra of finite depth, i.e., we have $W(m,n) = \oplus_{j=-h}^{\infty} W(m,n)_j$, where h is a positive integer. All these gradations induce those on $S'(m,n)$ and $S(m,n)$. The gradation of type $(1,\dots,1|1,\dots,1)$ is called the *principal* gradation of $W(m,n)$. In this gradation $W(m,n)$ is a Lie superalgebra of depth 1 such that $W(m,n)_0 \cong gl(m,n)$ and $W(m,n)_{-1} \cong \mathbb{C}^{m|n}$, the standard representation of $gl(m,n)$. This gradation induces gradations on $S(m,n)$ and $S'(m,n)$, also called the *principal* gradations, with the 0-th graded component isomorphic to $sl(m,n)$ and the -1-st graded component isomorphic to $\mathbb{C}^{m|n}$. The gradation of type $(1,\dots,1|0,\dots,0)$ is called the *subprincipal* gradation of $W(m,n)$, $S'(m,n)$ and $S(m,n)$. In this gradation $W(m,n)$ is of depth 1 with $W(m,n)_0 \cong gl(m)\otimes\Lambda(n)+W(0,n)$ and $W(m,n)_{-1} \cong \mathbb{C}^m\otimes\Lambda(n)$. We also have $S'(m,n)_0 \cong sl(m)\otimes\Lambda(n)+S(0,n)$ and $S'(m,n)_{-1} \cong \mathbb{C}^m\otimes\Lambda(n)$. Furthermore $S(1,n)_0 \cong S(0,n)$ and $S(1,n)_{-1}$ is spanned by all monomials in $\Lambda(n)\frac{\partial}{\partial x_1}$, except for $\xi_1\dots\xi_n\frac{\partial}{\partial x_1}$.

Letting $\deg x_i = \deg \xi_j = 1$ defines the principal gradation: $\Lambda(m,n) = \oplus_{j=0}^{\infty}\Lambda(m,n)_j$. From now on when we write $\Lambda(m,n)_j$, we will always mean the j-th graded component of $\Lambda(m,n)$ with respect to this gradation. Let $\Omega(m,n)$ be the superalgebra of differential forms over $\Lambda(m,n)$ [K1]. Then $W(m,n)$ acts on $\Omega(m,n)$ via Lie derivatives.

1.2. The superalgebras $H(2k,n)$ and $K(2k+1,n)$

Let $p_1,p_2,\dots,p_k,q_1,q_2,\dots,q_k$ be $2k$ even and $\xi_1,\xi_2,\dots,\xi_n$ be n odd indeterminates. Consider the differential form $\sigma = \sum_{i=1}^k dp_i dq_i + \sum_{i=1}^n d\xi_i d\xi_i \in \Omega(2k,n)$. Define the Hamiltonian superalgebra to be [K1]

$$H(2k,n) = \{D \in W(2k,n) \mid D\sigma = 0\}.$$

Let $\Lambda(2k,n)=\mathbb{C}[p_1,\dots,p_k,q_1,\dots,q_k]\otimes\Lambda(n)$. For $f,g\in\Lambda(2k,n)$, we define the Poisson bracket $[f,g]=\sum_{i=1}^k(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}-\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i})-(-1)^{p(f)}\sum_{i=1}^n\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_i}$.

$\Lambda(2k,n)$ with this Poisson bracket is a Lie superalgebra denoted by $\widehat{H}(2k,n)$. The map $f \to \sum_{i=1}^{k}(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}) - (-1)^{p(f)}\sum_{i=1}^{n}\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_i}$ defines a surjective Lie superalgebra homomorphism from $\Lambda(2k,n)$ onto $H(2k,n)$. The kernel of this map consists of constant functions so that we may (and will) identify $H(2k,n)$ with $\Lambda(2k,n)/\mathbb{C}1$. The principal gradation of $W(2k,n)$ induces a gradation on $H(2k,n)$, also called *principal*. With respect to this gradation, $H(2k,n)$ becomes a graded Lie superalgebra of depth 1.

The 0-th graded component of $H(2k,n)$ is the Lie superalgebra $spo(2k,n)$. Now $spo(2k,n)_{\bar{0}}$ has a basis consisting of vectors of the form $\{p_ip_j, p_iq_j, q_iq_j\}$ for $i,j = 1,\dots,k$ and $\{\xi_i\xi_j\}_{i\neq j}$ for $i,j=1,\dots,n$, and hence is isomorphic to the Lie algebra $sp(2k)\oplus so(n)$. $spo(2k,n)_{\bar{1}}$ has a basis consisting of vectors of the form $\{p_i\xi_j, q_i\xi_j\}$ for $i=1,\dots,k$ and $j=1,\dots,n$. Its span is isomorphic to the $sp(2k)\oplus so(n)$-module $\mathbb{C}^{2k}\otimes\mathbb{C}^n$, where $\mathbb{C}^{2k}$ and $\mathbb{C}^n$ are the respective standard representations of $sp(2k)$ and $so(n)$. $H(2k,n)_{-1}$ has a basis consisting of vectors of the form $\{p_i,q_i\}$ and $\{\xi_j\}$, $i=1,\dots,k$ and $j=1,\dots,n$. Evidently the span of $\{p_i,q_i\}$ is isomorphic to $\mathbb{C}^{2k}$, while the span of $\{\xi_j\}$ is isomorphic to $\mathbb{C}^n$. It is the standard representation of $spo(2k,n)$, denoted by $\mathbb{C}^{2k|n}$.

Let $t,p_1,p_2,\dots,p_k,q_1,q_2,\dots,q_k$ be $2k+1$ even and $\xi_1,\xi_2,\dots,\xi_n$ be n odd indeterminates. Consider the contact form $\Sigma = dt + \sum_{i=1}^{k}(p_idq_i - q_idp_i) + \sum_{i=1}^{n}\xi_id\xi_i \in \Omega(2k+1,n)$. Define the contact superalgebra to be [K1]

$$K(2k+1,n) = \{D\in W(2k+1,n) \mid D\Sigma = f_D\Sigma\},\ f_D\in\Lambda(2k+1,n).$$

We may realize $K(2k+1,n)$ as follows. Define the contact bracket on the space $\Lambda(2k+1,n)$ by $[f,g] = (2-E)f\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2-E)g - \sum_{i=1}^{k}(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}) + (-1)^{p(f)}\sum_{i=1}^{n}\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_i}$, where $E = \sum_{i=1}^{k}(p_i\frac{\partial}{\partial p_i} + q_i\frac{\partial}{\partial q_i}) + \sum_{i=1}^{n}\xi_i\frac{\partial}{\partial \xi_i}$ is the Euler operator. $\Lambda(2k+1,n)$ with this bracket becomes a Lie superalgebra and the map $\Lambda(2k+1,n)\to K(2k+1,n)$ given by $f\to(2-E)f\frac{\partial}{\partial t} + \frac{\partial f}{\partial t}E - \sum_{i=1}^{k}(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}) + (-1)^{p(f)}\sum_{i=1}^{n}\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_i}$, is an isomorphism of Lie superalgebras. Hence we may (and will) identify $K(2k+1,n)$ with $\Lambda(2k+1,n)$.

The gradation of type $(2,1,\dots,1|1,\dots,1)$ induces a gradation of depth 2, called the *principal* gradation of $K(2k+1,n)$. The 0-th graded component of $K(2k+1,n)$ is the Lie superalgebra $cspo(2k,n)$. Now $cspo(2k,n)_{\bar{0}}$ has a basis consisting of vectors of the form $\{t,p_ip_j,p_iq_j,q_iq_j\}$ for $i,j=1,\dots,k$ and $\{\xi_i\xi_j\}_{i\neq j}$ for $i,j=1,\dots,n$, and hence is isomorphic to the Lie algebra $csp(2k)\oplus so(n)$. $cspo(2k,n)_{\bar{1}}$ has a basis consisting of vectors of the form $\{p_i\xi_j,q_i\xi_j\}$ for $i=1,\dots,k$ and $j=1,\dots,n$. Its span is isomorphic to the $csp(2k)\oplus so(n)$-module $\mathbb{C}^{2k}\otimes\mathbb{C}^n$, where $\mathbb{C}^{2k}$ and $\mathbb{C}^n$ are the respective standard representations of $csp(2k)$ and $so(n)$. $K(2k+1,n)_{-1}$ has a basis consisting of vectors of the form $\{p_i,q_i\}$ and $\{\xi_j\}$, $i=1,\dots,k$ and $j=1,\dots,n$. Evidently the span of $\{p_i,q_i\}$ is isomorphic to $\mathbb{C}^{2k}$, while the span of $\{\xi_j\}$ is isomorphic to $\mathbb{C}^n$, on which t acts as the scalar -1. Finally $K(2k+1,n)_{-2} \cong \mathbb{C}$, on which t acts as the scalar -2.

When $n = 2l$, the contact form may be written as $\Sigma = dt + \sum_{i=1}^{k}(p_i dq_i - q_i dp_i) + \sum_{i=1}^{l} \xi_i d\xi_{2k-i+1} \in \Omega(2k+1, 2l)$. For $k > 0$, the gradation of type $(2, 1, \ldots, 1|2, \ldots, 2, 0, \ldots, 0)$ (l zeros) induces a gradation of $K(2k+1, n)$ of depth 2, which we will call the *subprincipal* gradation. The 0-th graded component is isomorphic to $csp(2k) \otimes \Lambda(l) + W(0, l)$. The -1-st graded component is isomorphic to $\mathbb{C}^{2k} \otimes \Lambda(l)$, while the -2-nd graded component is $\mathbb{C} \otimes \Lambda(l)$. This gradation induces a gradation on $\widehat{H}(2k, n)$, which will also be called *subprincipal*. Its 0-th graded component is isomorphic to $sp(2k) \otimes \Lambda(l) + W(0, l)$, the -1-st graded component is $\mathbb{C}^{2k} \otimes \Lambda(l)$, while the -2-nd graded component is $\mathbb{C} \otimes \Lambda(l)$.

1.3. The superalgebras $HO(n,n)$, $SHO(n,n)$ and $SHO'(n,n)$

Consider the differential form $\omega = \sum_{i=1}^{n} dx_i d\xi_i \in \Omega(n, n)$. Define the odd Hamiltonian superalgebra to be ([L], [ALS])

$$HO(n,n) := \{D \in W(n,n) | D\omega = 0\}.$$

The Lie superalgebra $HO(n,n)$ is simple if and only if $n \geq 2$.

The Lie superalgebra $HO(n,n)$ contains the subalgebra of divergence zero vector fields

$$SHO'(n,n) := \{D \in HO(n,n) | \operatorname{div} D = 0\}.$$

The derived algebra of $SHO'(n,n)$ is an ideal of codimension 1, denoted by $SHO(n,n)$, provided that $n \geq 2$. $SHO(n,n)$ is simple if and only if $n \geq 3$.

Another realization of $HO(n,n)$ and its subalgebras may be obtained as follows. In $\Lambda(n,n)$ we can define the Buttin bracket by

$$[f, g] := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i}\right),$$

which makes $\Lambda(n,n)$, with reversed parity, into a Lie superalgebra. It contains a one-dimensional odd center consisting of constant functions. The map $\Lambda(n,n) \to HO(n,n)$ given by $f \to \sum_{i=1}^{n}(\frac{\partial f}{\partial x_i} \frac{\partial}{d\xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x_i})$, is a surjective homomorphism of Lie superalgebras whose kernel consists of constant functions. Hence we may (and will) identify $HO(n,n)$ with $\Lambda(n,n)/\mathbb{C}1$ with reversed parity. In this identification we have $SHO'(n,n) = \{f \in \Lambda(n,n)/\mathbb{C}1 | \Delta(f) = 0\}$, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial \xi_i}$ is the odd Laplace operator, and $SHO(n,n)$ is identified with the subspace consisting of elements not containing the monomial $\xi_1 \xi_2 \ldots \xi_n$.

The principal gradation of $W(n,n)$ induces gradations, which we also call *principal*, on $HO(n,n)$ and its subalgebras. The Lie superalgebras

$HO(n,n)$, $SHO'(n,n)$ and $SHO(n,n)$ become graded Lie superalgebras of depth 1. Since $[x_i,\xi_j]=\delta_{ij}1$ in the odd Poisson bracket we obtain, by adding $\mathbb{C}1$ to degree -2, nontrivial central extensions of $HO(n,n)$, $SHO'(n,n)$ and $SHO(n,n)$, denoted by $\widehat{HO}(n,n)$, $\widehat{SHO}'(n,n)$ and $\widehat{SHO}(n,n)$, respectively.

The 0-th graded components of $HO(n,n)$ and $\widehat{HO}(n,n)$ have a basis consisting of vectors of the form $\{x_ix_j\}$, $\{x_i\xi_j\}$ and $\{\xi_i\xi_j\}_{i\neq j}$ for $i,j=1,2,\dots,n$. This is the graded finite-dimensional Lie superalgebra $\tilde{P}(n)=\tilde{P}(n)_{-1}+\tilde{P}(n)_0+\tilde{P}(n)_1$ (cf. [K1]), where $\tilde{P}(n)_0\cong gl_n$, $\tilde{P}(n)_{-1}\cong\Lambda^2(\mathbb{C}^{n*})$ and $\tilde{P}(n)_1\cong S^2(\mathbb{C}^n)$, where $\mathbb{C}^n$ stands for the standard representation of gl_n. Their -1-st graded components have a basis consisting of $\{x_i\}$ and $\{\xi_i\}$, $i=1,2,\dots,n$. Evidently the span of $\{x_i\}$ as a gl_n-module is isomorphic to $\mathbb{C}^n$, while the span of $\{\xi_i\}$ is isomorphic to $\mathbb{C}^{n*}$.

The 0-th graded components of $SHO(n,n)$, $SHO'(n,n)$, $\widehat{SHO}(n,n)$ and $\widehat{SHO}'(n,n)$ have a basis consisting of vectors of the form $\{x_ix_j\}$, $\{x_i\xi_j\}_{i\neq j}$, $\{x_i\xi_i-x_{i+1}\xi_{i+1}\}_{i<n}$ and $\{\xi_i\xi_j\}_{i\neq j}$ for $i,j=1,2,\dots,n$. This is the graded subalgebra $P(n)$ of $\tilde{P}(n)$ (cf. [K1]), where $P(n)_0\cong sl_n$ and $P(n)_{-1}\cong\Lambda^2(\mathbb{C}^{n*})$ and $P(n)_1\cong S^2(\mathbb{C}^n)$, where $\mathbb{C}^n$ stands for the standard representation of sl_n. Similarly, their -1-st graded components have a basis consisting of $\{x_i\}$ and $\{\xi_i\}$, $i=1,2,\dots,n$ with the span of $\{x_i\}$ isomorphic to $\mathbb{C}^n$ and the span of $\{\xi_i\}$ isomorphic to $\mathbb{C}^{n*}$.

The *subprincipal* gradation of $W(n,n)$ induces gradations, also called *subprincipal*, on $HO(n,n)$ and its subalgebras. $HO(n,n)$ (respectively $\widehat{HO}(n,n)$) in this gradation is of depth 1 with the 0-th graded component isomorphic to $W(0,n)$ and the -1-st component isomorphic to the $W(0,n)$-module $\Pi(\Lambda(n)/\mathbb{C}1)$ (respectively $\Pi(\Lambda(n))$). In this gradation $SHO'(n,n)_0=\widehat{SHO}'(n,n)_0\cong S(0,n)$, while $SHO'(n,n)_{-1}\cong\Pi(\Lambda(n)/\mathbb{C}1)$ and $\widehat{SHO}'(n,n)_{-1}\cong\Pi(\Lambda(n))$. $SHO(n,n)_0=\widehat{SHO}(n,n)_0\cong S(0,n)$, while $SHO(n,n)_{-1}$ is the span of monomials in $\Lambda(n)/\mathbb{C}1$ not containing $\mathbb{C}\xi_1\dots\xi_n$ and $\widehat{SHO}'(n,n)_{-1}$ is the span of all monomials in $\Lambda(n)$ except for $\xi_1\dots\xi_n$.

1.4. The superalgebras $KO(n,n+1)$, $SKO(n,n+1;\beta)$ and $SKO'(n,n+1;\beta)$

Let $x_1,x_2,\dots,x_n$ be n even indeterminates and $\xi_1,\xi_2,\dots,\xi_n,\xi_{n+1}=\tau$ be $n+1$ odd indeterminates. Define the odd contact form to be $\Omega=d\tau+\sum_{i=1}^n(\xi_idx_i+x_id\xi_i)\in\Omega(n,n+1)$. The odd contact superalgebra $KO(n,n+1)$ is the following subalgebra of $W(n,n+1)$ [ALS]:

$$KO(n,n+1)=\{D\in W(n,n+1)\mid D\Omega=f_D\Omega\},$$

for some $f_D\in\mathbb{C}[x_1,\dots,x_n,\xi_1,\dots,\xi_n,\tau]$. The Lie superalgebra $KO(n,n+1)$ can be realized as follows. We may define the odd contact bracket on the space $\Lambda(n,n+1)$ by

$$[f,g]=(2-E)f\frac{\partial g}{\partial\tau}+(-1)^{p(f)}\frac{\partial f}{\partial\tau}(2-E)g-\sum_{i=1}^n\Big(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial\xi_i}+(-1)^{p(f)}\frac{\partial f}{\partial\xi_i}\frac{\partial g}{\partial x_i}\Big),$$

where $E = \sum_{i=1}^n (x_i \frac{\partial}{\partial x_i} + \xi_i \frac{\partial}{\partial \xi_i})$ is the Euler operator. Reversing parity, $\Lambda(n, n+1)$ with this bracket becomes a Lie superalgebra and the map $\Lambda(n, n+1) \to KO(n, n+1)$, given by $f \to (2-E)f\frac{\partial}{\partial \tau} - (-1)^{p(f)} \frac{\partial f}{\partial \tau} E - \sum_{i=1}^n (\frac{\partial f}{\partial x_i} \frac{\partial}{d\xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x_i})$, is an isomorphism of Lie superalgebras. Hence we may (and will) identify the Lie superalgebra $KO(n, n+1)$ with $\Lambda(n, n+1)$ with reversed parity.

For $\beta \in \mathbb{C}$, we let $\mathrm{div}_\beta = \Delta + (E - n\beta)\frac{\partial}{\partial \tau}$, where Δ is the odd Laplace operator. We set [Ko]

$$SKO'(n, n+1; \beta) = \{f \in \Lambda(n, n+1) \mid \mathrm{div}_\beta f = 0\}.$$

Let $SKO(n, n+1; \beta)$ denote the derived algebra of $SKO'(n, n+1; \beta)$. Then the algebra $SKO(n, n+1; \beta)$ is simple for $n \geq 2$ and coincides with $SKO'(n, n+1; \beta)$ unless $\beta = 1$ or $\beta = \frac{n-2}{n}$. The Lie superalgebra $SKO(n, n+1; 1)$ (respectively $SKO(n, n+1; \frac{n-2}{n})$) consists of elements of $SKO'(n, n+1; 1)$ (respectively $SKO'(n, n+1; \frac{n-2}{n})$) not containing the monomial $\tau\xi_1\xi_2 \ldots \xi_n$ (respectively $\xi_1\xi_2 \ldots \xi_n$). Note that $SKO(n, n+1; \frac{1}{n})$ is the subalgebra of $KO(n, n+1)$ consisting of divergence zero vector fields.

The gradation of type $(1, \ldots, 1|1, \ldots, 1, 2)$ induces one of depth 2 on $KO(n, n+1)$ and hence on $SKO(n, n+1; \beta)$ and $SKO'(n, n+1; \beta)$ (since div_β is homogeneous with respect to this gradation), called their *principal* gradations. The 0-th graded component of $KO(n, n+1)$ in its principal gradation is spanned by vectors of the form $\{x_i x_j\}$, $\{x_i \xi_j\}$, $\{\xi_i \xi_j\}_{i \neq j}$ and τ. This is the Lie superalgebra $c\tilde{P}(n)$. $KO(n, n+1)_{-1}$ is spanned by $\{x_i\}$ and $\{\xi_i\}$, $i = 1, \ldots, n$, on which $\tilde{P}(n)$ acts as the standard representation. $KO(n, n+1)_{-2}$ is spanned by 1. The 0-th graded components of $SKO(n, n+1; \beta)$ and $SKO'(n, n+1; \beta)$ are spanned by the vectors $\{x_i x_j\}$, $\{x_i \xi_j\}_{i \neq j}$, $\{x_i \xi_i - x_{i+1}\xi_{i+1}\}_{i<n}$, $\{\xi_i \xi_j\}_{i \neq j}$ and $\tau + \beta\Phi$, where $i, j = 1, 2, \ldots, n$ and $\Phi = \sum_{i=1}^n x_i \xi_i$. This is the Lie superalgebra $\tilde{P}(n) = P(n) + \mathbb{C}(\tau + \beta\Phi)$. Their -2-nd graded components are spanned by $\mathbb{C}1$, on which $\tau + \beta\Phi$ acts as the scalar -2. Their -1-st graded components are spanned by the vectors $\{x_i\}$ and $\{\xi_i\}$ for $i = 1, \ldots, n$. With respect to $P(n)$ this is the standard representation, and $\tau + \beta\Phi$ acts on $\sum_{i=1}^n \mathbb{C}x_i$ (respectively $\sum_{i=1}^n \mathbb{C}\xi_i$) as the scalar $-1 + \beta$ (respectively $-1 - \beta$).

The *subprincipal* gradation of $KO(n, n+1)$ is that induced by the gradation of type $(1, \ldots, 1|0, \ldots, 0, 1)$. It is of depth 1 with the 0-th graded component isomorphic to $W(0, n) + \Lambda(n)$. Its -1-st component is $\Pi(\Lambda(n))$, on which $W(0, n) + \Lambda(n)$ acts in a natural way. This gradation induces the *subprincipal* gradations on its subalgebras. For $\beta \neq 1, \frac{n-2}{n}$ in this gradation $SKO'(n, n+1; \beta)_0 = SKO(n, n+1; \beta)_0 \cong W(0, n)$ and $SKO'(n, n+1; \beta)_{-1} = SKO(n, n+1; \beta)_{-1} \cong \Pi(\Lambda(n)^{\frac{2}{n(1-\beta)}})$, the representation of $W(0, n)$ on volume forms of density $\frac{2}{n(1-\beta)}$ (with reversed parity). For $\beta = \frac{n-2}{n}$ we have $SKO'(n, n+1; \frac{n-2}{n})_0 = SKO(n, n+1; \frac{n-2}{n})_0 \cong W(0, n)$ with $SKO'(n, n+1; \frac{n-2}{n})_{-1} \cong \Pi(\Lambda(n)^1)$. Note that $\Lambda(n)^1$ is not irreducible.

It has an irreducible submodule spanned by monomials in $\Lambda(n)^1$ except for $\xi_1 \dots \xi_n$. This submodule is isomorphic to $SKO(n, n+1; \frac{n-2}{n})_{-1}$. For $\beta = 1$ we have $SKO'(n, n+1; 1)_0 \cong S(0, n) + \Lambda(n)$ and $SKO(n, n+1; 1)_0$ is a semidirect sum of $S(0, n)$ and the subspace of the $\Lambda(n)$ spanned by monomials not including $\xi_1 \dots \xi_n$. $SKO'(n, n+1; 1)_{-1} = SKO(n, n+1; 1)_{-1} \cong \Pi(\Lambda(n))$ with obvious actions of the 0-th degree components.

1.5. Bitransitivity

Recall that the local part of a graded Lie superalgebra $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ is the subspace $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with the brackets defined between $a \in \mathfrak{g}_i$ and $b \in \mathfrak{g}_j$ such that $|i|, |j|, |i+j| \leq 1$. Recall that a graded Lie superalgebra is called *bitransitive* if it is transitive, and, in addition, it satisfies the following two properties:

(G5) $\mathfrak{g}_j = \mathfrak{g}_1^j$, for $j \geq 1$,
(G6) if $a \in \mathfrak{g}_j$ with $j \leq 0$, then $[a, \mathfrak{g}_1] = 0$ implies that $a = 0$.

By [K1] two bitransitive graded Lie superalgebras are isomorphic if and only if their local parts are isomorphic. The following simple proposition allows one to easily check simplicity.

Proposition 1.5.1. ([K1]) *A bitransitive irreducible graded Lie superalgebra $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ is simple if and only if the following two properties are satisfied:* $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$ and $[\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1$.

It is straightforward to check by a direct inspection the following proposition.

Proposition 1.5.2. *All graded Lie superalgebras listed in Sections 1.1–1.4 are bitransitive except for $W(1,0)$, $K(1,0)$ and $K(1,1)$ in their principal gradations.*

Remark 1.5.1. One often needs to find the members $\mathfrak{g}_j$ with $|j| \geq 2$ of a bitransitive irreducible graded Lie superalgebra $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ with given local part. If the gradation is consistent and $\mathfrak{g}_0$ is reductive, this is done as follows. Let M_i, $i = 1, 2, \dots$ (respectively Λ_i, $i = 1, 2, \dots$) be the highest (respectively lowest) vectors of a decomposition into irreducibles of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ (respectively $\mathfrak{g}_{-1}$). Let Λ' be a lowest weight vector of the $\mathfrak{g}_0$-module $S^2(\mathfrak{g}_{-1}) (\supseteq \mathfrak{g}_{-2})$. Then it does not occur in $\mathfrak{g}_{-2}$ if and only if $[\Lambda', M_i] = 0$ for all i. Having found $\mathfrak{g}_{-2}$, we look at the lowest weight vectors Λ'' of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2} (\supseteq \mathfrak{g}_{-3})$. Then Λ'' does not occur in $\mathfrak{g}_{-3}$ if and only if $[\Lambda'', M_i] = 0$ for all i, etc. Similarly, we calculate inductively $\mathfrak{g}_j$ with $j \geq 2$.

1.6. Full prolongation

Proposition 1.6.1. *Let $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ be a transitive graded Lie superalgebra for which* $\dim \mathfrak{g}_- = (m, n)$. *Suppose that there exists an embedding of $\phi_{\leq 0}$: $\mathfrak{g}_{\leq 0} \to W(m,n)_{\leq 0}$, where $W(m,n) = \oplus_j W(m,n)_j$ is a gradation of type*

$(a_1, \ldots, a_m | b_1, \ldots, b_n)$ *with* $a_i, b_j \in \mathbb{N}$. *Then the embedding* $\phi_{\leq 0}$ *extends to a graded Lie superalgebra embedding.*

Proof. We may assume without loss of generality that $\mathfrak{g}$ contains a grading operator z (by adding it if necessary). Let $\bar{\mathfrak{g}}$ and $\bar{W}(m,n)$ denote the completed filtered Lie superalgebras. By the realization theorem ([GS], [B]; the proof of [B] generalizes easily to the super case), there exists an embedding of filtered Lie superalgebras $\psi : \bar{\mathfrak{g}} \to \bar{W}(m,n)$ and any two such embeddings are equivalent, i.e., can be transformed into each other by an automorphism of $\bar{W}(m,n)$. Let $\psi_{\leq 0}$ denote the restriction of ψ to the subalgebra $\mathfrak{g}_{\leq 0}$. Using the equivalence of $\psi_{\leq 0}$ and $\phi_{\leq 0}$, we may assume that $\psi_{\leq 0} = \phi_{\leq 0}$. Since all a_i and b_j are positive and $\dim \mathfrak{g}_- = (m,n)$, $\phi_{\leq 0}(\mathfrak{g}_-)$ evaluated at 0 contains are partial derivatives. It follows that $\psi_{\leq 0}(z) = E + D$, where $E = \sum_{i=1}^m a_i x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^n b_j \xi_j \frac{\partial}{\partial \xi_j}$ and D has principal degree ≥ 1. But $[D, \mathfrak{g}_j] = 0$ for $j \leq 0$, since both $\mathrm{ad}\,(\psi_{\leq 0}(z))$ and $\mathrm{ad}\, E$ act on $\mathfrak{g}_j$, $j \leq 0$, as jI. Since $\dim \mathfrak{g}_- = (m,n)$, it follows that $D = 0$. Hence $\psi(z) = E$ and $\psi(\mathfrak{g}_j) \subseteq W(m,n)_j$. □

Under the hypothesis of Proposition 1.6.1 set $\mathfrak{F}_- = \mathfrak{g}_-$, $\mathfrak{F}_0 = \mathfrak{g}_0$ and define inductively for $j \geq 1$: $\mathfrak{F}_j = \{D \in W(m,n)_j | [D, \mathfrak{g}_-] \subseteq \oplus_{k<j} \mathfrak{F}_k\}$. Then $\mathfrak{F}$ is a graded transitive subalgebra of $W(m,n)$ such that $\mathfrak{F}_{\leq 0} = \mathfrak{g}_{\leq 0}$, called the *full prolongation* of the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$. Obviously $\mathfrak{g}$ is a subalgebra of $\mathfrak{F}$. Alternatively full prolongation may be defined as follows: Given a graded Lie superalgebra $\mathfrak{g}$ as before, we may define derivations of $\mathfrak{g}_-$ into $\mathfrak{g}$. The space of such derivations inherits a $\mathbb{Z}$-gradation from $\mathfrak{g}$. Let us denote the subspace of derivations of positive degrees by $\mathrm{der}(\mathfrak{g}_-, \mathfrak{g})_+$. One may show that on the space $\mathfrak{C} = \mathfrak{g}_- + \mathfrak{g}_0 + \mathrm{der}(\mathfrak{g}_-, \mathfrak{g})_+$ one may define a bracket making it into a graded transitive Lie superalgebra (see Remark 1.6.1 below). Evidently one has $\mathfrak{F} \subseteq \mathfrak{C}$. Conversely since $\mathfrak{C}_0 = \mathfrak{g}_0$ and $\mathfrak{C}_- = \mathfrak{g}_-$, we may embed $\mathfrak{C}$ into $W(m,n)$ so that $\mathfrak{C} \subseteq \mathfrak{F}$. Thus a graded transitive Lie superalgebra of finite depth is a full prolongation if and only if $\mathfrak{g}$ contains all derivations of $\mathfrak{g}_-$ into $\mathfrak{g}$ of positive degrees, which is equivalent to saying that $H^1(\mathfrak{g}_-, \mathfrak{g})_j = 0$, for $j \geq 1$, which is the definition we have used in [CK2].

Remark 1.6.1. For an abstract pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ one may define its full prolongation to be the Lie superalgebra $\mathfrak{g}_- + \mathfrak{g}_0 + \mathrm{der}(\mathfrak{g}_-, \mathfrak{g})_+$. The bracket between two derivations D_1, D_2 of positive degrees is defined to be $D_1 D_2 - (-1)^{p(D_1)p(D_2)} D_2 D_1$. To make sense of this operator one needs to extend the domains of D_1 and D_2 to $\oplus_{j \leq \deg D_2 - 1} \mathfrak{F}_j$ and $\oplus_{j \leq \deg D_1 - 1} \mathfrak{F}_j$, respectively. This, however, can be done in a unique way due to transitivity. So full prolongation is a well-defined notion and does not require an embedding of $(\mathfrak{g}_-, \mathfrak{g}_0)$ into $W(m,n)_{\leq 0}$.

A transitive infinite-dimensional graded Lie superalgebra $\mathfrak{g} = \oplus_{j=-h}^{\infty} \mathfrak{g}_j$ is called a prolongation of $\mathfrak{g}_{\leq 0}$. If $\mathfrak{g}_{\leq 0}$ is of type (XN), we will say that $\mathfrak{g}$ is a *prolongation of type* (XN), and if $\mathfrak{g}$ is a full prolongation of $\mathfrak{g}_{\leq 0}$, we will say that $\mathfrak{g}$ is a full prolongation of type (XN).

2. Subalgebras of full prolongation: the case of inconsistent gradation of depth 1

In this section we will construct all prolongations of types (I1)–(I19).

2.1. The cases (I1) and (I2)

Consider $W(m,n)$ in its principal gradation. It contains the subalgebra $S(m,n)$ in its principal gradation. We have $S(m,n)_0 \cong sl(m,n)$. It is straightforward to verify the following two lemmas.

Lemma 2.1.1. *$S(m,n)_j \subseteq W(m,n)_j$ is an irreducible $sl(m,n)$-module for every $j \geq -1$.*

Lemma 2.1.2. *For $D_1, D_2 \in W(m,n)$ one has* $\mathrm{div}[D_1,D_2] = D_1(\mathrm{div} D_2) - (-1)^{p(D_1)p(D_2)} D_2(\mathrm{div} D_1)$.

It follows from Lemma 2.1.2 that the map $\mathrm{div} : W(m,n) \to \Lambda(m,n)$ is an epimorphism of $S(m,n)$-modules with kernel $S'(m,n)$.

It is more convenient to dualize and assume that -1-st graded components in both cases (I1) and (I2) are $\mathbb{C}^{m|n*}$.

Lemma 2.1.3. *$W(m,n)_1$ as a module over $sl(m,n)$ decomposes as follows:*

(a) *If $n = m+1$, $W(m,n)_1$ is not completely reducible. We have the following unique composition series:*

$$0 \subsetneq S(m,n)_1 \subsetneq W(m,n)_1 \text{ with } W(m,n)_1/S(m,n)_1 \cong \mathbb{C}^{m|n}.$$

(b) *If $n \neq m+1$, $W(m,n)_1$ is a direct sum of $S(m,n)_1$ and $\mathbb{C}^{m|n}$.*

Proof. A direct calculation of $W(m,n)_1$ shows that the vector $x_1^2 \frac{\partial}{\partial \xi_n}$ is a highest weight vector for the submodule $S(m,n)_1$. Also it is not hard to check that a highest weight vector for $W(m,n)_1/S(m,n)_1$ is necessarily a nonzero vector of the form $\lambda_1 x_1 (\sum_{i=1}^m x_i \frac{\partial}{\partial x_i}) + \lambda_2 x_1 (\sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i})$, where $\lambda_1, \lambda_2 \in \mathbb{C}$. Evidently both vectors are eigenvectors of the Borel subalgebra of $sl(m) \oplus sl(n)$. We compute $[x_k \frac{\partial}{\partial \xi_l}, \lambda_1 x_1 (\sum_{i=1}^m x_i \frac{\partial}{\partial x_i}) + \lambda_2 x_1 (\sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i})] = (\lambda_2 - \lambda_1) x_1 x_k \frac{\partial}{\partial \xi_l}$, for all $l = 1, \dots, n$ and $k = 1, \dots, m$. Hence such a vector is singular if and only if $\lambda_1 = \lambda_2 \neq 0$. But in this case this vector has divergence $\lambda_1(m+1-n)$, and hence is not contained in $S(m,n)_1$ if and only if $m+1 \neq n$. $\square$

Let $E = \sum_{i=1}^m x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^n \xi_j \frac{\partial}{\partial \xi_j}$ be the Euler operator.

Lemma 2.1.4. *Let $D \in W(m,n)_j$ and suppose that $[\frac{\partial}{\partial x_i}, D]$ and $[\frac{\partial}{\partial \xi_j}, D]$ lie $S'(m,n)$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. Then $D \in S'(m,n) + \mathbb{C}E$. Hence the full prolongation of type* (I2) *is $S'(m,n)$.*

Proof. Let $D = \sum_{i=1}^m f_i \frac{\partial}{\partial x_i} + \sum_{j=1}^n g_j \frac{\partial}{\partial \xi_j}$. Then $[\frac{\partial}{\partial x_k}, D] = \sum_{i=1}^m \frac{\partial f_i}{\partial x_k}\frac{\partial}{\partial x_i} + \sum_{j=1}^n \frac{\partial g_j}{\partial x_k}\frac{\partial}{\partial \xi_j}$ and hence $\mathrm{div}[\frac{\partial}{\partial x_k}, D] = \frac{\partial}{\partial x_k}(\mathrm{div} D)$. Similarly for $[\frac{\partial}{\partial \xi_k}, D]$. By assumption $\frac{\partial}{\partial x_k}(\mathrm{div} D) = 0$ and $\frac{\partial}{\partial \xi_k}(\mathrm{div} D) = 0$. Thus $\mathrm{div} D$ is a constant. But then $D \in S'(m,n) + \mathbb{C}E$. □

Corollary 2.1.1. *Let $\mathfrak{g}$ be an infinite-dimensional graded subalgebra of $W(m,n)$ in its principal gradation. Suppose that $\mathfrak{g}_{\leq 0}$ contains $(\mathbb{C}^{m|n*}, sl(m,n))$. Then $\mathfrak{g}$ is one of the following: $S(m,n)$, $S'(m,n)$, $S(m,n)+\mathbb{C}E$, $S'(m,n)+\mathbb{C}E$ or $W(m,n)$.*

Proof. First assume that $n = m+1$. Since $\mathfrak{g}_1 \neq 0$, $\mathfrak{g}_1 \supseteq S(m,n)_1$ due to Lemma 2.1.3. By Lemma 2.1.1 $\mathfrak{g}$ contains $S(m,n)$. Thus $\mathfrak{g}$ is an $S(m,n)$-submodule of $W(m,n)$ containing $S(m,n)$. Thus, we are to consider $S(m,n)$-submodules of $W(m,n)/S(m,n)$. Now $W(m,n)/S(m,n)$ is isomorphic to $\Lambda(m,n)$ in the case $m \neq 1$ and $\Lambda(m,n) + \mathbb{C}\xi_1 \dots \xi_n \frac{\partial}{\partial x_1}$ in the case $m = 1$. In either case all $S(m,n)$-submodules are easily determined. Namely, if $\mathfrak{g}$ properly contains $S(m,n)$ and $\mathfrak{g}_0 = sl(m,n)$, then $\mathfrak{g} = S'(m,n)$ by Lemma 2.1.4. If $\mathfrak{g}_0 = gl(m,n)$, then $\mathfrak{g} \supseteq S(m,n) + \mathbb{C}E$ (in the case when $m = 1$, we have the additional choices of $\mathfrak{g} \supseteq S(1,n) + \mathbb{C}\xi_1 \dots \xi_n \frac{\partial}{\partial x_1}$ and $\mathfrak{g} \supseteq S(1,n) + \mathbb{C}E + \mathbb{C}\xi_1 \dots \xi_n \frac{\partial}{\partial x_1}$). Since $W(m,n)/(S'(m,n)+\mathbb{C}E) \cong (\Lambda(m,n)/\mathbb{C}1)$ is irreducible, if $\mathfrak{g}$ contains $S'(m,n)+\mathbb{C}E$ properly, then $\mathfrak{g} = W(m,n)$.

Now consider the case when $n \neq m+1$. It is readily checked that in order for $\mathfrak{g}$ to be infinite-dimensional, $\mathfrak{g}_1$ must contain $S(m,n)_1$. (In fact if $\mathfrak{g}_1 \cong \mathbb{C}^{m|n}$, then $\mathfrak{g}_2 = 0$.) From here on, we may apply the same argument as in the previous case. □

From Corollary 2.1.1 the following is immediate.

Proposition 2.1.1. (i) *Any prolongation of type* (I1) *is isomorphic to either $S(m,n)+\mathbb{C}E$, $S'(m,n)+\mathbb{C}E$ or $W(m,n)$ in their principal gradation.*

(ii) *Any prolongation of type* (I2) *is isomorphic to either $S(m,n)$ or $S'(m,n)$ in their principal gradation.*

2.2. The cases (I3) and (I4)

Lemma 2.2.1. *The $spo(m,n)$-module $H(m,n)_j$ is irreducible for all j.*

Proof. This is a straightforward verification using the $(sp(m) \oplus so(n))$-module structure of $H(m,n)_j$, which can be found for example in [CK2]. □

Lemma 2.2.2. *Let $\sigma = \sum_{i=1}^k dp_i dq_i + \sum_{j=1}^n d\xi_j d\xi_j$, where $m = 2k$, be the standard even symplectic form on the superspace $\mathbb{C}^{m|n}$. Let $D \in W(m,n)$ be such that $\frac{\partial}{\partial p_i}(D\sigma) = \frac{\partial}{\partial q_i}(D\sigma) = \frac{\partial}{\partial \xi_j}(D\sigma) = 0$ for all i and j. Then $D \in H(m,n) + W(m,n)_0$.*

Proof. We may assume that D is a homogeneous element of $W(m,n)$ in its principal gradation. Let

$$D=\sum_{i=1}^{k} f_i\frac{\partial}{\partial p_i}+\sum_{i=1}^{k} g_i\frac{\partial}{\partial q_i}+\sum_{i=1}^{n} h_j\frac{\partial}{\partial \xi_j}, \tag{2.2.1}$$

where $f_i, g_i, h_j \in \Lambda(m,n)$, so that

$$D\sigma=\sum_{i=1}^{k}(-1)^{p(D)}df_i dq_i+\sum_{i=1}^{k} dp_i dg_i+2(-1)^{p(D)}\sum_{i=1}^{n} dh_j d\xi_j. \tag{2.2.2}$$

Now $D\sigma$ is a 2-form, which is annihilated by all partial derivatives on $\mathbb{C}^{m|n}$. Thus it must be a 2-form with constant coefficients. Therefore if $D\sigma \neq 0$, then $D \in W(m,n)_0$. □

Proposition 2.2.1. *$H(m,n)$ is the full prolongations of $(\mathfrak{g}_{-1},\mathfrak{g}_0)$ of type* (I3). *Furthermore if $\mathfrak{g}$ is a prolongation of type* (I3)*, then $\mathfrak{g} \cong H(m,n)$ in its principal gradation.*

Proof. Let $j \geq 1$ and suppose $D \in W(m,n)_j$ is contained in the full prolongation of $(\mathfrak{g}_{-1},\mathfrak{g}_0)$ of either type (I3). Then $0=[\mathfrak{g}_{-1},D]\sigma=\mathfrak{g}_{-1}(D\sigma)$. Hence by Lemma 2.2.2 $D \in H(m,n)$. The second statement follows from Lemma 2.2.1. □

To prove that any prolongation of type (I4) is $H(m,n)+\mathbb{C}E$, we need a stronger version of Lemma 2.2.2.

Lemma 2.2.3. *Let σ be the even standard symplectic form on $\mathbb{C}^{m|n}$ given as in Lemma* 2.2.2. *Suppose that $D \in W(m,n)$ is such that $D\sigma = f\sigma$, where $f \in \Lambda(m,n)$. Then f is a scalar.*

Proof. Let D be as in (2.2.1) so that $D\sigma$ is as in (2.2.2). The condition $D\sigma = f\sigma$ gives the following set of equations for $i \neq j$:

$$(-1)^{p(D)}\frac{\partial f_i}{\partial p_j}+\frac{\partial g_j}{\partial q_i}=0,\ \ \frac{\partial g_i}{\partial p_j}-\frac{\partial g_j}{\partial p_i}=0,\ \ (-1)^{p(D)}(\frac{\partial f_j}{\partial q_i}-\frac{\partial f_i}{\partial q_j})=0, \tag{2.2.3}$$

$$(-1)^{p(D)}\frac{\partial f_i}{\partial p_i}+\frac{\partial g_i}{\partial q_i}=f,\ \ 2(-1)^{p(D)}\frac{\partial h_j}{\partial \xi_j}=f. \tag{2.2.4}$$

Using the second expression for f in (2.2.4), we see that the partial derivative of f by ξ_j is 0. Equating expressions for f given by the first expression in (2.2.4) for $i \neq j$ and differentiating by p_i, we conclude, using the first two identities in (2.2.3), that the partial derivative of f by p_i is 0. Similarly the partial derivative of f by q_i is 0, using the last two identities in (2.2.3). Thus f is a constant. □

Proposition 2.2.2. *Any prolongation of type* (I4) *is isomorphic to* $H(m,n) + \mathbb{C}E$ *in its principal gradation.*

Proof. Let $D \in W(m,n)_1$ such that $[\mathfrak{g}_{-1}, D] \in H(m,n)_0 + \mathbb{C}E$. Thus $\mathfrak{g}_{-1}(D\sigma)$ is a scalar multiple of σ. But then $D\sigma = f\sigma$, where $f \in \Lambda(m,n)$. By Lemma 2.2.3 f is a scalar, hence $f = 0$. Thus $D\sigma = 0$, so that $D \in H(m,n)$. Now we may complete the proof by induction. □

2.3. The cases (I7)–(I10)

We will identify $HO(n,n)$ with $\Lambda(n,n)/\mathbb{C}1$ with reversed parity. Note that, contrary to the case of the even Hamiltonian superalgebra $H(m,n)$, the divergence operator div : $HO(n,n) \to \Lambda(n,n)$ is nontrivial. Translating the divergence operator to $\Lambda(n,n)/\mathbb{C}1$, we obtain a multiple of the odd Laplace operator on $\Lambda(n,n)/\mathbb{C}1$ ([S]). By Lemma 2.1.2, Δ is a homomorphism of $SHO(n,n)$-modules. The image of Δ is spanned by all monomials in $\Lambda(n,n)$ except for $\xi_1 \dots \xi_n$, and thus, as a module over $SHO(n,n)$, can be identified with $SHO(n,n) + \mathbb{C}1$.

Lemma 2.3.1. *As an* $SHO(n,n)_0$*-module* $SHO(n,n)_j$ *is irreducible, for every* j.

Proof. One can verify this using the $sl(n)$-module structure of $SHO(n,n)$, which can be found for example in [CK2]. □

Lemma 2.3.2. $HO(n,n)$ *is the full prolongation of* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ *of type* (I7). $HO(n,n) + \mathbb{C}E$ *is the full prolongation of* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ *of type* (I8).

Proof. The argument is analogous to the one we have used earlier to prove that $H(m,n)$ and $H(m,n) + \mathbb{C}E$ are full prolongations for $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ of types (I3) and (I4), respectively. Instead of using the differential form σ we use the differential form $\omega = \sum_{i=1}^n dx_i d\xi_i$. □

Lemma 2.3.3. $SHO'(n,n)$ *and* $SHO'(n,n) + \mathbb{C}E$ *in their principal gradations are the full prolongations of* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ *of types* (I9) *and* (I10), *respectively.*

Proof. We may assume that their full prolongations are subalgebras of $HO(n,n)$ and $HO(n,n) + \mathbb{C}E$, respectively, due to Lemma 2.3.2. Now observe that if D is a vector field in $W(n,n)$ such that $\mathrm{div}[\frac{\partial}{\partial x_i}, D] = \mathrm{div}[\frac{\partial}{\partial \xi_j}, D] = 0$ for every i, j, then $\mathrm{div}D$ is a constant (Lemma 2.1.4). Thus for a homogeneous vector field D of degree ≥ 1 to lie in the full prolongation of (I9) or (I10), we must have $\mathrm{div}D = 0$. Thus $D \in SHO'(n,n)_j$, for $j \geq 1$. □

Let $\phi = \sum_{i=1}^n x_i \xi_i$ (with $\sum_{i=1}^n (-x_i \frac{\partial}{\partial x_i} + \xi_i \frac{\partial}{\partial \xi_i})$ the corresponding vector field).

Proposition 2.3.1. (i) *Any prolongation of type* (I7) *is isomorphic to either* $SHO(n,n) + \mathbb{C}\Phi$, $SHO'(n,n) + \mathbb{C}\Phi$ *or* $HO(n,n)$ *in their principal gradation.*

(ii) *Any prolongation of type* (I8) *is isomorphic to either* $SHO+\mathbb{C}\Phi+\mathbb{C}E$, $SHO'+\mathbb{C}\Phi+\mathbb{C}E$ *or* $HO(n,n)+\mathbb{C}E$ *in their principal gradation.*

(iii) *Any prolongation of type* (I9) *is isomorphic to either* $SHO(n,n)$ *or* $SHO'(n,n)$ *in their principal gradation.*

(iv) *Any prolongation of type* (I10) *is isomorphic to either* $SHO(n,n)+\mathbb{C}E$ *or* $SHO'(n,n)+\mathbb{C}E$ *in their principal gradation.*

Proof. From the $sl(n)$-module structure of $HO(n,n)_k$ given in [CK2] it follows that for $k \geq 1$, $SHO(n,n)_k$ is the unique irreducible $P(n)$-submodule in $HO(n,n)_k$ generated by the highest weight vector x_1^{k+2}. Since $\mathfrak{g}_1 \neq 0$, $\mathfrak{g}_1$ contains $SHO(n,n)_1$, and hence by Lemma 2.3.1 $\mathfrak{g}$ contains $SHO(n,n)$. Thus we are led to find all $SHO(n,n)$-submodules of $HO(n,n)/SHO(n,n)$. By our discussion of the map Δ in this section we know that $HO(n,n)/SHO(n,n)$ as an $SHO(n,n)$-module contains two trivial submodules: one is degree 0 generated by the vector $\Phi+SHO(n,n)$, and one in degree $n-2$ generated by the vector $\xi_1\dots\xi_n+SHO(n,n)$. The quotient by the direct sum of these two trivial modules is isomorphic to $SHO(n,n)$. So any subalgebra $\mathfrak{g}$ of $HO(n,n)$ for which $\mathfrak{g}_{<0}=(\mathbb{C}^{n|n},P(n))$ must contain $SHO(n,n)$ and is either $SHO(n,n)$ or $SHO'(n,n)$ by Lemma 2.3.3. This gives (iii). It also follows from this that any prolongation $\mathfrak{g}$ of type $\mathfrak{g}_{<0}=(\mathbb{C}^{n|n},cP(n))$ must contain $SHO(n,n)+\mathbb{C}E$ and is either $SHO(n,n)+\mathbb{C}E$, $SHO'(n,n)+\mathbb{C}E$ by Lemma 2.3.3, which gives (iv). (i) and (ii) are completely analogous using Lemma 2.3.2 instead of Lemma 2.3.3. □

2.4. The cases (I5) and (I6)

In this section and further on we will denote by π_i, $i=1,2,\dots$, the fundamental weights of $sl(n)$ or $sp(n)$ arranged in the usual order, and let $R(\sum_i k_i\pi_i)$ stand for the finite-dimensional irreducible representation with highest weight $\sum_i k_i\pi_i$, where $k_i \in \mathbb{Z}_+$. We will need the following proposition.

Proposition 2.4.1. *$W(n,1)$ in its subprincipal gradation is the full prolongation of the pair* $(\mathbb{C}^{n*}\otimes\Lambda(1), gl(n)\otimes\Lambda(1)+W(0,1))$.

Proof. We embed $(\mathbb{C}^{n*}\otimes\Lambda(1), gl(n)\otimes\Lambda(1)+W(0,1))$ into $(W(n,n)_{-1}, W(n,n)_0)$ in its principal gradation. (The explicit embedding is given below as $(\mathfrak{F}_{-1},\mathfrak{F}_0)$.) Let x_i,ξ, $i=1,\dots,n$ be the coordinates for $\mathbb{C}^{n|1}$ and let y_i,θ_j, $i,j=1,\dots,n$, be the coordinates for $\mathbb{C}^{n|n}$. Let $\mathfrak{F}$ denote the full prolongation of $(\mathbb{C}^{n*}\otimes\Lambda(1), gl(n)\otimes\Lambda(1)+W(0,1))$ in $W(n,n)$ (see Proposition 1.6.1). Now $\{x_i\frac{\partial}{\partial x_j}+\theta_i\frac{\partial}{\partial\theta_j} \mid i,j=1,\dots,n\} \cong gl(n) \cong sl(n)\oplus\mathbb{C}I \subseteq (W(n,n)_0)_{\bar{0}}$. Then we have with respect to this $sl(n)$ the following decomposition of $\mathfrak{F}$ as an $sl(n)$-module (where we include a highest weight vector in each component and $E_1=\sum_{i=1}^n\theta_i\frac{\partial}{\partial\theta_i}$):

$$\mathfrak{F}_{-1} \quad \{R(\pi_{n-1}),\frac{\partial}{\partial y_n}\},\{R(\pi_{n-1}),\frac{\partial}{\partial\theta_{n-1}}\}.$$

$$\mathfrak{F}_0 \quad \{R(\pi_1+\pi_{n-1}), y_1\frac{\partial}{\partial y_n}+\theta_1\frac{\partial}{\partial \theta_n}\}, \{R(\pi_1+\pi_{n-1}), y_1\frac{\partial}{\partial \theta_n}\},$$
$$\{R(0), \sum_{i=1}^n y_i\frac{\partial}{\partial \theta_i}\}, \{R(0), E_1\}, \{R(0), \sum_{i=1}^n y_i\frac{\partial}{\partial y_i}\}, \{R(0), \sum_{i=1}^n \theta_i\frac{\partial}{\partial y_i}\}.$$
$$\mathfrak{F}_1 \quad \{R(2\pi_1+\pi_{n-1}), y_1^2\frac{\partial}{\partial y_n}+2y_1\theta_1\frac{\partial}{\partial \theta_n}\}, \{R(2\pi_1+\pi_{n-1}), y_1^2\frac{\partial}{\partial \theta_n}\},$$
$$\{R(\pi_1), y_1(\sum_{i=1}^n y_i\frac{\partial}{\partial y_i})+\theta_1(\sum_{i=1}^n y_i\frac{\partial}{\partial \theta_i})\}, \{R(\pi_1), y_1(\sum_{i=1}^n \theta_i\frac{\partial}{\partial y_i})-\theta_1 E_1\},$$
$$\{R(\pi_1), y_1E_1\}, \{R(\pi_1), y_1(\sum_{i=1}^n y_i\frac{\partial}{\partial \theta_i})\}.$$

Note that $W(n,1) \subseteq \mathfrak{F}$. The Euler operator $E \in W(n,1)$ in $W(n,n)$ takes the form $z = \sum_{i=1}^n y_i\frac{\partial}{\partial y_i}$.

Now z acts semisimply on $W(n,n)$ with integer eigenvalues ≥ -1. Also, since $z \in \mathfrak{F}$, z acts semisimply on $\mathfrak{F}$ with integer eigenvalues ≥ -1. Let us use superscript to denote the eigenspaces of z, e.g., $\mathfrak{F}^{-1}$ and $\mathfrak{F}^0$ denote the -1-st and 0-th eigenspaces of z in $\mathfrak{F}$, respectively. To show that $W(n,1) = \mathfrak{F}$ it is enough to show that $\mathfrak{F}^{-1} = W(n,1)^{-1}$ and $\mathfrak{F}^0 = W(n,1)^0$. This is because we know that $W(n,1)$ is the full prolongation of $(W(n,1)^{-1}, W(n,1)^0)$. Hence if $D \in \mathfrak{F}^i$, then $[D, \mathfrak{F}^{-1}] \subseteq \mathfrak{F}^{i-1}$, so that $\mathfrak{F}^i \subseteq W(n,1)^i$ by induction on i.

Now it is easy to check that $W(n,n)^{-1}$ is the linear span of $\Lambda(n)\frac{\partial}{\partial y_i}$, $i=1,\ldots,n$. Since $\mathfrak{F}_1$ does not contain any subspace spanned by $\{\theta_i\theta_j\frac{\partial}{\partial y_k}\}$, $i,j,k=1,\ldots,n$, $\mathfrak{F}_k$ cannot contain any subspace spanned by $\{\theta_{i_1}\ldots\theta_{i_{k+1}}\frac{\partial}{\partial y_j}\}$ for $k \geq 2$ (since all $\frac{\partial}{\partial \theta_i}$ are contained in $\mathfrak{F}_{-1}$). Thus $\mathfrak{F}_k^{-1} = 0$, for $k \geq 1$. Now $\mathfrak{F}_0^{-1}$ is the linear span of the vector $\sum_{i=1}^n \theta_i\frac{\partial}{\partial y_i}$ and $\mathfrak{F}_{-1}^{-1}$ is the linear span of $\{\frac{\partial}{\partial y_i} \mid i=1,\ldots,n\}$. Thus $\mathfrak{F}^{-1}$ is $n+1$-dimensional and hence is $W(n,1)^{-1}$.

We claim that $\mathfrak{F}^0$ is contained in $\mathfrak{F}_{-1}\oplus\mathfrak{F}_0\oplus\mathfrak{F}_1$. From this we conclude by inspection of our table above that $\dim_{\mathbb{C}}\mathfrak{F}^0 = \dim_{\mathbb{C}} W(n,1)^0$ so that $\mathfrak{F}^0 = W(n,1)^0$. Now we may check directly that $\mathfrak{F}_2^0 = 0$. Therefore if $D \in \mathfrak{F}_3^0$, then $[\frac{\partial}{\partial y_i}, D] \in \mathfrak{F}_2^{-1} = 0$ (from above) and $[\frac{\partial}{\partial \theta_i}, D] \in \mathfrak{F}_2^0 = 0$. Hence by transitivity $\mathfrak{F}_3^0 = 0$. By induction $\mathfrak{F}_k^0 = 0$ for $k \geq 3$, which proves our claim. $\square$

Now suppose that $\mathfrak{g}_{\leq 0}$ is of type (I5), i.e., $\mathfrak{g}_0 \cong sl(n)\otimes\Lambda(1)+\mathfrak{a}$ and $\mathfrak{g}_{-1} \cong \mathbb{C}^{n*}\otimes\Lambda(1)$, where $\mathfrak{a}$ is one of the following subalgebras of $\mathbb{C}I+\mathbb{C}\xi+\mathbb{C}\frac{\partial}{\partial\xi}+\mathbb{C}\xi\frac{\partial}{\partial\xi}$ [K2]:

(a) $\mathbb{C}\frac{\partial}{\partial\xi}+\mathbb{C}(\alpha I+\beta\xi\frac{\partial}{\partial\xi})$, where $\alpha,\beta\in\mathbb{C}$,

(b) $\mathbb{C}\frac{\partial}{\partial\xi}+\mathbb{C}I+\mathbb{C}\xi\frac{\partial}{\partial\xi}$,

(c) $\mathbb{C}\frac{\partial}{\partial\xi} + \mathbb{C}I + \mathbb{C}\xi$,
(d) $\mathbb{C}I + \mathbb{C}\xi + \mathbb{C}\frac{\partial}{\partial\xi} + \mathbb{C}\xi\frac{\partial}{\partial\xi}$,
(e) $\mathbb{C}(\frac{\partial}{\partial\xi} + \alpha\xi) + \mathbb{C}I$, where $\alpha \in \mathbb{C}$.

Due to Proposition 2.4.1 we may embed $\mathfrak{g}_{\leq 0}$ of type (I5) into $W(n,1)$ with subprincipal gradation. As usual let us denote the coordinates of $\mathbb{C}^{n|1}$ by x_i, $i = 1, \dots, n$ and ξ. We have $sl(n) \oplus \mathbb{C}I \cong gl(n) \cong \{x_i\frac{\partial}{\partial x_j} \mid i,j = 1,\dots,n\} \subseteq (W(n,1)_0)_{\bar{0}}$. With respect to this $sl(n)$ the module $W(n,1)$ decomposes as follows (as usual a highest weight vector is included and $E_0 = \sum_{i=1}^n x_i\frac{\partial}{\partial x_i}$):

$$
\begin{aligned}
W(n,1)_{-1}\ & \{R(\pi_{n-1}), \frac{\partial}{\partial x_n}\}, \{R(\pi_{n-1}), \xi\frac{\partial}{\partial x_n}\}.\\
W(n,1)_0\ \ & \{R(\pi_1+\pi_{n-1}), x_1\frac{\partial}{\partial x_n}\}, \{R(\pi_1+\pi_{n-1}), x_1\xi\frac{\partial}{\partial x_n}\}, \{R(0), E_0\},\\
& \{R(0), \xi E_0\}, \{R(0), \frac{\partial}{\partial\xi}\}, \{R(0), \xi\frac{\partial}{\partial\xi}\}.\\
W(n,1)_1\ \ & \{R(2\pi_1+\pi_{n-1}), x_1^2\frac{\partial}{\partial x_n}\}, \{R(2\pi_1+\pi_{n-1}), \xi x_1^2\frac{\partial}{\partial x_n}\},\\
& \{R(\pi_1), x_1E_0\}, \{R(\pi_1), \xi x_1 E_0\}, \{R(\pi_1), x_1\frac{\partial}{\partial\xi}\}, \{R(\pi_1), \xi x_1\frac{\partial}{\partial\xi}\}.\\
W(n,1)_2\ \ & \{R(3\pi_1+\pi_{n-1}), x_1^3\frac{\partial}{\partial x_n}\}, \{R(3\pi_1+\pi_{n-1}), \xi x_1^3\frac{\partial}{\partial x_n}\},\\
& \{R(2\pi_1), x_1^2E_0\}, \{R(2\pi_1), \xi x_1^2E_0\},\\
& \{R(2\pi_1), x_1^2\frac{\partial}{\partial\xi}\}, \{R(2\pi_1), \xi x_1^2\frac{\partial}{\partial\xi}\}.
\end{aligned}
$$

Lemma 2.4.1. *Let* $\mathfrak{g}$ *be a prolongation of type* (I5). *Then* $\mathfrak{g}$ *contains* $S(n,0)\otimes\Lambda(1)+\mathfrak{a}$.

Proof. Even though in general $sl(n)\otimes\Lambda(1)+\mathfrak{a}$ is not semisimple one may use the same method as in [C2] to show that the finite-dimensional irreducible representations of $sl(n)\otimes\Lambda(1)+\mathfrak{a}$, on which $sl(n)$ acts nontrivially, is necessarily of the form $R(\sum_{i=1}^n k_i\pi_i)\otimes\Lambda(1)$. Hence if $\mathfrak{g}_1$ does not contain one of the components isomorphic to $R(2\pi_1+\pi_{n-1})$, then it contains none of the components. Now if $\mathfrak{g}_1$ doesn't contain the $R(2\pi_1+\pi_{n-1})$-components, then $\mathfrak{g}_2$ cannot contain any of the $R(3\pi_1+\pi_{n-1})$-components either. Hence if $\mathfrak{g}_2$ is nonzero, then it must contain a vector of the form $\alpha x_1^2\frac{\partial}{\partial\xi} + \beta x_1^2\xi(\sum_{i=1}^n x_i\frac{\partial}{\partial x_i})$, $\alpha,\beta\in\mathbb{C}$. But if $\beta\neq 0$, then $[\frac{\partial}{\partial x_1}, \alpha x_1^2\frac{\partial}{\partial\xi} + \beta x_1^2\xi(\sum_{i=1}^n x_i\frac{\partial}{\partial x_i})]$ projects nontrivially onto the component $R(2\pi_1+\pi_{n-1})\otimes\Lambda(1)$ in $\mathfrak{g}_1$. Hence $\beta = 0$. But $[\xi\frac{\partial}{\partial x_1}, x_1^2\frac{\partial}{\partial\xi}]$ projects nontrivially onto $R(2\pi_1+\pi_{n-1})\otimes\Lambda(1)$ in $\mathfrak{g}_1$. Hence $\alpha = 0$ so that $\mathfrak{g}_2 = 0$. Hence if $\mathfrak{g}_1$ does not contain the representation $R(2\pi_1+\pi_{n-1})\otimes\Lambda(1)$, then $\mathfrak{g}$ is finite-dimensional. Now it is easy to see that the component $R(2\pi_1+\pi_{n-1})\otimes\Lambda(1)$ generates the graded components of degree ≥ 1 of $S(n,0)\otimes\Lambda(1)$. Thus $\mathfrak{g}$ contains $S(n,0)\otimes\Lambda(1)$. $\square$

Thus to find transitive Lie superalgebras with $\mathfrak{g}_{<0}$ of type (I5), we are led to find subalgebras of $W(n,1)$ containing $S(n,0)\otimes\Lambda(1)$.

Consider the map $\mathrm{ev}_\xi : W(n,1)\to\Lambda(n,1)$, given by $\mathrm{ev}_\xi(D)=D(\xi)$.

Lemma 2.4.2. ev_ξ *is a homomorphism of* $W(n,0)\otimes\Lambda(1)+\mathbb{C}\frac{\partial}{\partial\xi}$*-modules.*

Proof. It is readily checked that $\ker(\mathrm{ev}_\xi)=W(n,0)\otimes\Lambda(1)$. Thus for $D\in W(n,0)\otimes\Lambda(1)$ and $D'\in W(n,1)$ we have $\mathrm{ev}_\xi([D,D'])=[D,D'](\xi)=DD'(\xi)=D(\mathrm{ev}_\xi(D'))$. Now since $D'(\mathrm{ev}_\xi(\frac{\partial}{\partial\xi}))=0$, ev_ξ also commutes with the action of $\frac{\partial}{\partial\xi}$. $\square$

Recall that $\mathrm{div}: W(n,1)\to\Lambda(n,1)$ is the map that assigns to each vector field its divergence, so that its kernel is $S(n,1)$. We have seen that div is a homomorphism of $S(n,1)$-modules. Now $\Lambda(n,1)$ as an $S(n,0)\otimes\Lambda(1)+\mathbb{C}\frac{\partial}{\partial\xi}$-module has three composition factors isomorphic to $(\mathbb{C}[x_1,\dots,x_n]/\mathbb{C}1)\otimes\Lambda(1)$ and two copies of the trivial module of opposite parity. Restricting the map ev_ξ to $S(n,1)$, we have an epimorphism of $S(n,0)\otimes\mathbb{C}\frac{\partial}{\partial\xi}$-modules $\mathrm{ev}_\xi: S(n,1)\to\Lambda(n,1)$ with kernel $S(n,0)\otimes\Lambda(1)$. This proves

Proposition 2.4.2. $W(n,1)$, *as an* $S(n,0)\otimes\Lambda(1)+\mathbb{C}\frac{\partial}{\partial\xi}$*-module, has the following composition factors: two copies of opposite parity isomorphic to* $(\mathbb{C}[x_1,\dots,x_n]/\mathbb{C}1)\otimes\Lambda(1)$, *two copies of the even trivial module, two copies of the odd trivial module and one copy of* $S(n,0)\otimes\Lambda(1)$.

Proposition 2.4.3. *A prolongation of type* (I5) *is isomorphic to either* $S(n,0)\otimes\Lambda(1)+\mathfrak{a}$, $W(n,0)\otimes\Lambda(1)+\mathfrak{a}$, *or* $S(n,1)+\mathfrak{a}$, $W(n,1)$ *in their subprincipal gradation.*

Proof. Let us first consider the cases (a)–(d). In these cases $\mathbb{C}\frac{\partial}{\partial\xi}\subseteq\mathfrak{a}$. Hence by Lemma 2.4.1 we need to find all infinite-dimensional $S(n,0)\otimes\Lambda(1)+\mathfrak{a}$-submodules of $W(n,1)$ containing $S(n,0)\otimes\Lambda(1)+\mathfrak{a}$ with the 0-th graded component equal to $sl(n)\otimes\Lambda(1)+\mathfrak{a}$.

Let $\mathfrak{g}$ denote such a module. Dividing by the module $S(n,0)\otimes\Lambda(1)+\mathfrak{a}$ we are left at most with the factors $(\mathbb{C}[x_1,\dots,x_n]/\mathbb{C}1)\otimes\Lambda(1)$ and $\Pi((\mathbb{C}[x_1,\dots$ $\dots,x_n]/\mathbb{C}1)\otimes\Lambda(1))$, the same module but with reversed parity and trivial modules that are homogeneous of degree 0. But if $\mathfrak{g}/(S(n,0)\otimes\Lambda(1)+\mathfrak{a})$ contains a trivial factor, then it would mean that $\mathfrak{g}_0$ contains $sl(n)\otimes\Lambda(1)+\mathfrak{a}$ properly. Thus the composition factors of $\mathfrak{g}/(S(n,0)\otimes\Lambda(1)+\mathfrak{a})$ is at most the two nontrivial infinite-dimensional modules.

Now if $S(n,0)\otimes\Lambda(1)+\mathfrak{a}\subsetneq\mathfrak{g}\subsetneq W(n,1)$, then either

(i) $\mathfrak{g}/(S(n,0)\otimes\Lambda(1)+\mathfrak{a})\cong(\mathbb{C}[x_1,\dots,x_n]/\mathbb{C}1)\otimes\Lambda(1)$ or
(ii) $\mathfrak{g}/(S(n,0)\otimes\Lambda(1)+\mathfrak{a})\cong\Pi((\mathbb{C}[x_1,\dots,x_n]/\mathbb{C}1)\otimes\Lambda(1))$.

Now in $W(n,1)_1$ the only composition series of $sl(n)\otimes\Lambda(1)+\mathbb{C}\frac{\partial}{\partial\xi}$-modules are either

(i′) $(S(n,0)\otimes\Lambda(1))_1 \subsetneq S(n,1)_1 \subsetneq W(n,1)_1$ or
(ii′) $(S(n,0)\otimes\Lambda(1))_1 \subsetneq (W(n,0)\otimes\Lambda(1))_1 \subsetneq W(n,1)_1$.

Hence $\mathfrak{g}_1$ is either $S(n,1)_1$ or $(W(n,0)\otimes\Lambda(1))_1$. But $S(n,1)_k = S(n,1)_1^k$ and $(W(n,0)\otimes\Lambda(1))_k = (W(n,0)\otimes\Lambda(1))_1^k$, for $k\geq 1$. Thus in (i), $\mathfrak{g}\cong S(n,1)+\mathfrak{a}$, and in (ii), $\mathfrak{g}\cong W(n,0)\otimes\Lambda(1)+\mathfrak{a}$.

Consider now the case (e). Note that $\mathrm{ev}_\xi : W(n,1)\to\Lambda(n,1)$ is a homomorphism of $S(n,0)\otimes\Lambda(1)+\mathfrak{a}$-modules with kernel $W(n,0)\otimes\Lambda(1)$. Now we restrict the map div to the kernel of ev_ξ and obtain a homomorphism of $S(n,0)\otimes\Lambda(1)+\mathfrak{a}$-modules with kernel $S(n,0)\otimes\Lambda(1)$. Thus the composition factors of $W(n,1)$ as an $S(n,0)\otimes\Lambda(1)+\mathfrak{a}$-modules is the same as the factors of $W(n,1)$ as $S(n,0)\otimes\Lambda(1)+\mathbb{C}\frac{\partial}{\partial\xi}$-module in the cases (a)–(d), which allows us to employ the same argument as before. □

We now turn our attention to $\mathfrak{g}_{\leq 0}$ of type (I6). As in the case of (I5) we may embed its full prolongation into $W(2n,1)$ with the subprincipal gradation due to Proposition 2.4.1. In this embedding $sp(2n)\otimes\Lambda(1)\subseteq W(2n,1)_0$ takes the following form: it is the linear span of vectors of the form $p_i\frac{\partial}{\partial p_j}-q_j\frac{\partial}{\partial q_i}$, $p_i\frac{\partial}{\partial q_j}+p_j\frac{\partial}{\partial q_i}$, $q_i\frac{\partial}{\partial p_j}+q_j\frac{\partial}{\partial p_i}$, $\xi(p_i\frac{\partial}{\partial p_j}-q_j\frac{\partial}{\partial q_i})$, $\xi(p_i\frac{\partial}{\partial q_j}+p_j\frac{\partial}{\partial q_i})$, $\xi(q_i\frac{\partial}{\partial p_j}+q_j\frac{\partial}{\partial p_i})$, where $i,j=1,\dots,n$.

We decompose $W(2n,1)$ with respect to $sp(2n)$ and obtain ($E_0=\sum_{i=1}^n(p_i\frac{\partial}{\partial p_i}+q_i\frac{\partial}{\partial q_i})$):

$$
\begin{array}{ll}
W(2n,1)_{-1} & \{R(\pi_1),\frac{\partial}{\partial q_1}\},\{R(\pi_1),\xi\frac{\partial}{\partial q_1}\}.\\
W(2n,1)_0 & \{R(2\pi_1),p_1\frac{\partial}{\partial q_1}\},\{R(\pi_2),(p_1\frac{\partial}{\partial q_2}-p_2\frac{\partial}{\partial q_1})\},\{R(0),E_0\},\\
& \{R(2\pi_1),\xi p_1\frac{\partial}{\partial q_1}\},\{R(\pi_2),\xi(p_1\frac{\partial}{\partial q_2}-p_2\frac{\partial}{\partial q_1})\},\{R(0),\xi E_0\},\\
& \{R(0),\frac{\partial}{\partial\xi}\},\{R(0),\xi\frac{\partial}{\partial\xi}\}.\\
W(2n,1)_1 & \{R(3\pi_1),p_1^2\frac{\partial}{\partial q_1}\},\{R(\pi_1+\pi_2),p_1(p_1\frac{\partial}{\partial q_2}-p_2\frac{\partial}{\partial q_1})\},\\
& \{R(\pi_1),p_1E_0\},\{R(3\pi_1),\xi p_1^2\frac{\partial}{\partial q_1}\},\\
& \{R(\pi_1+\pi_2),p_1\xi(p_1\frac{\partial}{\partial q_2}-p_2\frac{\partial}{\partial q_1})\},\{R(\pi_1),p_1\xi E_0\}.\\
& \qquad\vdots\quad\vdots\\
W(2n,1)_k & \{R((k+2)\pi_1),p_1^{k+1}\frac{\partial}{\partial q_1}\},\{R(k\pi_1+\pi_2),p_1^k(p_1\frac{\partial}{\partial q_2}-p_2\frac{\partial}{\partial q_1})\},\\
& \{R(k\pi_1),p_1^kE_0\},\{R((k+2)\pi_1),\xi p_1^{k+1}\frac{\partial}{\partial q_1}\},
\end{array}
$$

$$\{R(k\pi_1 + \pi_2), p_1^k\xi(p_1\frac{\partial}{\partial q_2} - p_2\frac{\partial}{\partial q_1})\}, \{R(k\pi_1), p_1^k\xi E_0\}.$$

Using again the same method as in [C2], one can show that all finite-dimensional irreducible representations of $sp(2n)\otimes\Lambda(1)+\mathfrak{a}$, on which $sp(2n)$ acts nontrivially, are of the form $R(\sum_{i=1}^n k_i\pi_i)\otimes\Lambda(1)$.

Proposition 2.4.4. *Let $\mathfrak{F}$ be the full prolongation of $\mathfrak{g}_{\leq 0}$ of type* (I6). *Then $\mathfrak{F}\cong H(2n)\otimes\Lambda(1)+\mathfrak{a}$.*

Proof. We will only give a sketch of the proof. First one shows that $\mathfrak{F}_1$ is the component $R(3\pi_1)\otimes\Lambda(1)$. Of course, we know that $R(3\pi_1)\otimes\Lambda(1)\subseteq\mathfrak{F}_1$. So it amounts to showing that none of the other components listed in the table above can lie in $\mathfrak{F}_1$, which in turn can be shown by computing the bracket between $\mathfrak{F}_{-1}$ and the highest weight vectors of each of the components. (In fact a direct computation shows that the bracket between $\mathfrak{F}_{-1}$ and any highest weight vector of the other components has nontrivial projection onto the $R(\pi_2)\otimes\Lambda(1)$-component of $W(2n,1)_0$, which does not lie in $\mathfrak{g}_0$.) Analogously one shows that $\mathfrak{F}_k\cong R((k+2)\pi_1)\otimes\Lambda(1)$ using the fact that $\mathfrak{F}_{k-1}\cong R((k+1)\pi_1)\otimes\Lambda(1)$, for $k\geq 2$. □

Corollary 2.4.1. *Any prolongation of type* (I6) *is isomorphic to $H(2n)\otimes\Lambda(1)+\mathfrak{a}$.*

Proof. Since the irreducible (with respect to $\mathfrak{g}_0$) component $R(3\pi_1)\otimes\Lambda(1)$ generates the graded components of $H(2n)\otimes\Lambda(1)$ of degree ≥ 1, it follows that $\mathfrak{g}\cong H(2n)\otimes\Lambda(1)+\mathfrak{a}$. □

2.5. The case (I12)

We first classify finite-dimensional irreducible representations of the Lie superalgebra $spin_4^0$. The proof we will give can be applied to classifying representations of $\mathfrak{h}_n+\mathfrak{b}$, where $\mathfrak{b}$ is a semisimple subalgebra of $so(n)$ in general, but since we will not need a result in such a generality here, we will concentrate on the case $n=4$ and $\mathfrak{b}\cong sl(2)$.

Let $\mathbb{C}^{2|2}$ denote the Fock space of $\mathfrak{h}_4$. Recall that $so(4)\cong\mathfrak{a}_1\oplus\mathfrak{a}_2$, where $\mathfrak{a}_i\cong sl(2)$, and by definition $spin_4^0=\mathfrak{a}_1+\mathfrak{h}_4$. Now since $so(4)$ acts on $\mathbb{C}^{2|2}$, $\mathfrak{a}_1$ acts on $\mathbb{C}^{2|2}$. Thus $\mathbb{C}^{2|2}$ is an irreducible representation of $spin_0^4$. Now let $R(m)$ denote the irreducible representation of $\mathfrak{a}_1(\cong sl(2))$ of highest weight m. Since $\mathfrak{h}_4$ is an ideal, we may extend $R(m)$ to an irreducible representation of $spin_4^0$ in a trivial way. Now we have a natural triangular decomposition of $spin_4^0$ and hence the notion of a highest weight makes sense for a finite-dimensional irreducible representation of $spin_4^0$. To be more concrete, let z denote the center of $\mathfrak{h}_4$ and let E,H,F denote the Chevalley generators of $\mathfrak{a}_1$. The Cartan subalgebra of $spin_4^0$ is then $\widehat{\mathfrak{h}}=\mathbb{C}H+\mathbb{C}z$. Let $\Lambda\in\widehat{\mathfrak{h}}^*$. We will denote the unique irreducible highest weight representation of $spin_0^4$ of highest weight Λ by $L(\Lambda)$ or by $L(m,\lambda)$, where $\Lambda(H)=m$ and $\Lambda(z)=\lambda$. The representation theory of $spin_4^0$ is similar to that of the algebras considered in [C1] (cf. [KT]). To be more precise, one can use the same argument as in Proposition 4.1 of [C1] to prove that for $\lambda \neq 0$, $L(m,\lambda)$ is finite-dimensional if and only if $m \in \mathbb{N}$. $\mathbb{C}^{2|2}$ is then the "minimal" finite-dimensional irreducible representation of $spin_4^0$ for a given nonzero scalar λ. All other finite-dimensional irreducible representations are obtained by taking the tensor product of this minimal representation and an irreducible representation of $\mathfrak{a}_1$, which is regarded as a representation of $spin_4^0$, on which the center acts trivially. (The fact that such modules are in fact irreducible follows from the proof Proposition 5.1 in [C1]). Hence we see that as an $\mathfrak{a}_1$-module we have $L(m,\lambda) \cong L(m,\mu)$, for any nonzero complex numbers λ and μ. Since we will only need to know the structures of $L(m,\lambda)$ as an $\mathfrak{a}_1$-module, we will denote all $L(m,\lambda)$ simply by $L(m)$, when the value of λ is obvious. We summarize our above discussion in the following proposition.

Proposition 2.5.1. *The $spin_4^0$-module $L(m)$ is finite-dimensional if and only if $m \in \mathbb{N}$. Furthermore $L(m)$ is isomorphic to $\mathbb{C}^{2|2} \otimes R(m-1)$.*

It is easy to verify that with respect to $\mathfrak{a}_1$, $\mathbb{C}^{2|2}$ decomposes into one copy of the standard representation and two copies of the trivial representation. From this we obtain the decomposition of $L(m)$ as an $\mathfrak{a}_1$-module.

Corollary 2.5.1. *As an $\mathfrak{a}_1 = sl(2)$-module, $L(m)$ decomposes as follows:*

$$L(m) \cong R(m) \oplus R(m-1) \oplus R(m-1) \oplus R(m-2).$$

We embed $(\mathbb{C}^{2|2}, \mathfrak{h}_4 + so(4))$ into $W(2,2)_{\leq 0}$ in its principal gradation as follows: $\mathbb{C}^{2|2}$ is, of course, mapped to $\mathbb{C}\frac{\partial}{\partial x_1} + \mathbb{C}\frac{\partial}{\partial x_2} + \mathbb{C}\frac{\partial}{\partial \xi_1} + \mathbb{C}\frac{\partial}{\partial \xi_2}$, while $spin_4^0$ is the span of the vectors $x_1\frac{\partial}{\partial x_2}, x_2\frac{\partial}{\partial x_1}$ and $x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}$, which span $\mathfrak{a}_1$, and the span of the vectors $\sum_{i=1}^2(x_i\frac{\partial}{\partial x_i} + \xi_i\frac{\partial}{\partial \xi_i})$, $\xi_1\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial \xi_2}$, $\xi_2\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial \xi_1}$, $\xi_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial \xi_2}$, $x_1\frac{\partial}{\partial \xi_1} - \xi_2\frac{\partial}{\partial x_2}$, which form the copy of $\mathfrak{h}_4$. To complete $so(4) + \mathfrak{h}_4$, we need $\mathfrak{a}_2 \cong sl(2)$, which is spanned by the vectors $\xi_1\frac{\partial}{\partial \xi_2}, \xi_2\frac{\partial}{\partial \xi_1}$ and $\xi_1\frac{\partial}{\partial \xi_1} - \xi_2\frac{\partial}{\partial \xi_2}$.

Let $\mathfrak{F}$ denote the full prolongation of $(\mathbb{C}^{2|2}, spin_4^0 + \mathfrak{a})$, where $\mathfrak{a}$ is a subalgebra of $\mathfrak{a}_2$. It is a straightforward verification that $\mathfrak{F}_1 \cong L(3)$ with highest weight vector $x_1^2\frac{\partial}{\partial x_2}$. Furthermore $[\mathfrak{F}_{-1}, \mathfrak{F}_1] = spin_4^0$. Now decompose $W(2,2)_k$, for $k \geq 2$, with respect to the copy of $sl(2)$ inside $spin_4^0$. We have (where $j,l = 1,2$ and $E_0 = \sum_{i=1}^2 x_i\frac{\partial}{\partial x_i}$)

$$\begin{aligned} W(2,2)_2 \quad & \{R(4), x_1^3\frac{\partial}{\partial x_2}\}, \{R(2), x_1^2E_0\}, \{2R(3), x_1^2\xi_j\frac{\partial}{\partial x_2}\}, \\ & \{2R(1), x_1\xi_jE_0\}, \{2R(3), x_1^3\frac{\partial}{\partial \xi_j}\}, \{4R(2), x_1^2\xi_l\frac{\partial}{\partial \xi_j}\}, \\ & \{2R(1), x_1\xi_1\xi_2\frac{\partial}{\partial \xi_j}\}, \{R(2), \xi_1\xi_2x_1\frac{\partial}{\partial x_2}\}, \{R(0), \xi_1\xi_2E_0\}. \end{aligned}$$

$$\vdots \quad \vdots$$

$$\begin{aligned} W(2,2)_k \quad & \{R(k+2), x_1^{k+1}\frac{\partial}{\partial x_2}\}, \{R(k), x_1^k E_0\}, \{2R(k+1), x_1^k \xi_j \frac{\partial}{\partial x_2}\}, \\ & \{2R(k-1), x_1^{k-1}\xi_j E_0\}, \{2R(k+1), x_1^{k+1}\frac{\partial}{\partial \xi_j}\}, \\ & \{4R(k), x_1^k \xi_l \frac{\partial}{\partial \xi_j}\}, \{2R(k-1), x_1^{k-1}\xi_1\xi_2\frac{\partial}{\partial \xi_j}\}, \\ & \{R(k), \xi_1\xi_2 x_1^{k-1}\frac{\partial}{\partial x_2}\}, \{R(k-2), x_1^{k-2}\xi_1\xi_2 E_0\}. \end{aligned}$$

So each $W(2,2)_k$ as an $sl(2)$-module decomposes into

$$W(2,2)_k \cong R(k+2) \oplus 4R(k+1) \oplus 6R(k) \oplus 4R(k-1) \oplus R(k-2). \quad (2.5.1)$$

Proposition 2.5.2. *As a module over $spin_4^0$, $\mathfrak{F}_k \cong L(k+2)$.*

Proof. By Corollary 2.5.1 and (2.5.1), as a $spin_4^0$-module, $W(2,2)_k$ has the following composition factors: $L(k+2)$, two copies of $L(k+1)$ and $L(k)$. We claim that only the submodule $L(k+2)$ can be in $\mathfrak{F}_k$. To see this we will argue by induction on k. The claim is true for $k=1$. Suppose that the claim is true for $k-1$, $k \geq 2$. Now with respect to $sl(2)$ the composition factors of $W(2,2)_k$ as a $spin_4^0$-module decompose as follows:

$$\begin{aligned} L(k+2) &\cong R(k+2) \oplus R(k+1) \oplus R(k+1) \oplus R(k) \\ L(k+1) &\cong R(k+1) \oplus R(k) \oplus R(k) \oplus R(k-1) \\ L(k+1) &\cong R(k+1) \oplus R(k) \oplus R(k) \oplus R(k-1) \\ L(k) &\cong R(k) \oplus R(k-1) \oplus R(k-1) \oplus R(k-2) \end{aligned}$$

Our claim is that only $L(k+2)$ lie in $\mathfrak{F}_k$. Obviously $L(k+2)$ lies in $\mathfrak{F}$, since its highest weight vector is $x_1^{k+1}\frac{\partial}{\partial x_2}$, and $[\frac{\partial}{\partial x_i}, x_1^{k+1}\frac{\partial}{\partial x_2}]$ lies in the component $L(k+1) \subseteq \mathfrak{F}_{k-1}$ and $[\frac{\partial}{\partial \xi_i}, x_1^{k+1}\frac{\partial}{\partial x_2}] = 0$, for $i=1,2$. Now to see that this is the only component lying in $\mathfrak{F}_k$, it is enough to show that none of the $sl(2)$-components $R(k-1)$ lies in $\mathfrak{F}_k$. Now the $sl(2)$-highest weight vectors of the $R(k-1)$-components in $\mathfrak{F}_k$ are by inspection from the table above $x_1^{k-1}\xi_j E_0$ and $x_1^{k-1}\xi_1\xi_2\frac{\partial}{\partial \xi_j}$, for $j=1,2$. Let $v = \sum_{j=1}^2 \alpha_j x_1^{k-1}\xi_j E_0 + \sum_{j=1}^2 \beta_j x_1^{k-1}\xi_1\xi_2\frac{\partial}{\partial \xi_j}$. We have

$$[\frac{\partial}{\partial x_1}, v] = (k-1)\sum_{j=1}^{2}\alpha_j x_1^{k-2}\xi_j E_0 + (k-1)\sum_{j=1}^{2}\beta_j x_1^{k-2}\xi_1\xi_2\frac{\partial}{\partial \xi_j} + \alpha_1 x_1^{k-1}\frac{\partial}{\partial x_1},$$

which, for some nonzero α_i or β_i, $i=1,2$, would project nontrivially onto a $R(k-2)$-component of $W(2,2)_{k-1}$. By induction, however, no such component can lie in $\mathfrak{F}_{k-1}$. Thus $\alpha_i = \beta_i = 0$, for $i=1,2$, and hence $\mathfrak{F}_k = L(k+2)$. □

It is straightforward to verify that $SKO(2,3;1)$ with the subprincipal gradation is of type (I12) with $\mathfrak{a} = 0$. Hence from the above discusssion we obtain the main result of this subsection.

Theorem 2.5.1. *Any prolongation of type* (I12) *is* $SKO(2,3;1)+\mathfrak{a}$, *where* $SKO(2,3;1)$ *is considered in its subprincipal gradation and* $\mathfrak{a}$ *is a subalgebra of* $sl(2)$ *of outer derivations of* $SKO(2,3;1)$ *put in* 0 *degree* (*cf. Proposition 5.3.4*).

2.6. The cases (I15)–(I19)

We let $\Lambda(2)$ denote the Grassmann superalgebra in the two odd indeterminates θ_1, θ_2 and let $W(0,2)$ be its derivation superalgebra. On $\Lambda(2)$ we have a natural action of $W(0,2)+\Lambda(2)$. From Section 1.3 we know that the simple Lie superalgebra $KO(2,3)$ in its subprincipal gradation satisfies $KO(2,3)_{\leq 0} = (\Pi(\Lambda(2)), W(0,2)+\Lambda(2))$. The following proposition is true for $KO(n,n+1)$ and can be proved analogously. However, to keep notation simple, we will assume that $n=2$, the only case we will use in this paper.

Proposition 2.6.1. $KO(2,3)$ *in its subprincipal gradation is the full prolongation of* $(\Pi(\Lambda(2)), W(0,2)+\Lambda(2))$.

Proof. We embed the full prolongation $\mathfrak{F}$ of $(\Pi(\Lambda(2)), W(0,2)+\Lambda(2))$ into $W(2,2)$ in its principal gradation. Denoting the coordinates of the superspace $\mathbb{C}^{2|2}$ by x_1, x_2 and ξ_1, ξ_2, it is straightforward to verify that the following gives an embedding of $(\Pi(\Lambda(2)), W(0,2)+\Lambda(2))$ into $W(2,2)_{\leq 0}$:

$$
\begin{aligned}
\mathfrak{g}_{-1}: \quad & \theta_1 \to \frac{\partial}{\partial x_1}, \quad \theta_2 \to \frac{\partial}{\partial x_2}, \quad 1 \to \frac{\partial}{\partial \xi_1}, \quad \theta_1\theta_2 \to \frac{\partial}{\partial \xi_2}. \\
\mathfrak{g}_0: \quad & \frac{\partial}{\partial \theta_1} \to -x_1\frac{\partial}{\partial \xi_1} + \xi_2\frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial \theta_2} \to -x_2\frac{\partial}{\partial \xi_1} - \xi_2\frac{\partial}{\partial x_1}, \\
& \theta_1\frac{\partial}{\partial \theta_2} \to -x_2\frac{\partial}{\partial x_1}\theta_2\frac{\partial}{\partial \theta_1} \to -x_1\frac{\partial}{\partial x_2}, \\
& \theta_1\frac{\partial}{\partial \theta_1} - \theta_2\frac{\partial}{\partial \theta_2} \to -x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}, \\
& \theta_1\frac{\partial}{\partial \theta_1} + \theta_2\frac{\partial}{\partial \theta_2} \to -x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} - 2\xi_2\frac{\partial}{\partial \xi_2} \\
& \theta_1\theta_2\frac{\partial}{\partial \theta_1} \to -x_1\frac{\partial}{\partial \xi_2}, \quad \theta_1\theta_2\frac{\partial}{\partial \theta_2} \to -x_2\frac{\partial}{\partial \xi_2}, \\
& 1 \to -\sum_{i=1}^{2}(x_i\frac{\partial}{\partial x_i} + \xi_i\frac{\partial}{\partial \xi_i}), \quad \theta_1\theta_2 \to \xi_1\frac{\partial}{\partial \xi_2}, \\
& \theta_1 \to -x_2\frac{\partial}{\partial \xi_2} - \xi_1\frac{\partial}{\partial x_1}, \quad \theta_2 \to -x_1\frac{\partial}{\partial \xi_2} + \xi_1\frac{\partial}{\partial x_2}.
\end{aligned}
$$

Note that the grading operator E of $KO(2,3)$ in its standard gradation via this embedding is sent to $E \to -x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} - 2\xi_1\frac{\partial}{\partial \xi_1}$.

The idea of the proof is the same as the one given in the proof of Proposition 2.4.1. The element τ acts on $W(2,2)$ semisimply with integer eigenvalues ≥ -2, hence it does the same on the invariant subspace $\mathfrak{F}$. On an invariant subspace V of $W(2,2)$, let us denote the j-th eigenspace by V^j, for $j \geq -2$. By Lemma 3.2.2 we know that $KO(n,n+1)$ in its principal gradation is the full prolongation of its nonpositive part. Using this, in order to complete the proof, it suffices to show that $\mathfrak{F}^{-2} = KO(2,3)_{-2}$, $\mathfrak{F}^{-1} = KO(2,3)_{-1}$ and $\mathfrak{F}^0 = KO(2,3)_0$, where the gradation of $KO(2,3)$ here, and in the remainder of the proof, is assumed to be principal.

It is easy to see that the span of the following vectors is $W(2,2)^j$, for $j = -2,-1,0$ ($i,k = 1,2$):

$$
\begin{array}{ll}
W(2,2)^{-2} & \dfrac{\partial}{\partial \xi_1},\ \xi_2\dfrac{\partial}{\partial \xi_1}. \\
W(2,2)^{-1} & \dfrac{\partial}{\partial x_i},\ \xi_2\dfrac{\partial}{\partial x_i},\ x_i\dfrac{\partial}{\partial \xi_1},\ x_i\xi_2\dfrac{\partial}{\partial \xi_1}. \\
W(2,2)^{0} & \dfrac{\partial}{\partial \xi_2},\ \xi_2\dfrac{\partial}{\partial \xi_2},\ x_i\dfrac{\partial}{\partial x_k},\ \xi_2 x_i\dfrac{\partial}{\partial x_k}, \\
 & x_i x_k\dfrac{\partial}{\partial \xi_1},\ \xi_2 x_i x_k\dfrac{\partial}{\partial \xi_1},\ \xi_1\dfrac{\partial}{\partial \xi_1},\ \xi_1\xi_2\dfrac{\partial}{\partial \xi_1}.
\end{array}
$$

Since $\xi_2\frac{\partial}{\partial \xi_1} \notin \mathfrak{F}$ by inspection of our above embedding of $(\Lambda(2), W(0,2) + \Lambda(2))$ into $W(2,2)_{\leq 0}$, we have that $\mathfrak{F}^{-2}$ is one-dimensional, and hence $\mathfrak{F}^{-2} = KO(2,3)_{-2}$.

Now of $W(2,2)_0^{-1}$ only the four-dimensional subspace spanned by $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$, $-x_1\frac{\partial}{\partial \xi_1}+\xi_2\frac{\partial}{\partial x_2}$, $-x_2\frac{\partial}{\partial \xi_1}-\xi_2\frac{\partial}{\partial x_1}$ lies in $\mathfrak{F}$ by inspection of the embedding above. But $[\frac{\partial}{\partial x_i}, x_i\xi_2\frac{\partial}{\partial \xi_1}] = \xi_2\frac{\partial}{\partial \xi_1} \notin \mathfrak{F}$. Thus $\mathfrak{F}^{-1}$ is four-dimensional and so is equal to $KO(2,3)_{-1}$.

Finally, it follows again by inspection of our embedding and a simple computation that $\mathfrak{F}^0$ is as follows: it contains a five-dimensional subspace spanned by $\xi_i\frac{\partial}{\partial \xi_i}$ and $x_i\frac{\partial}{\partial x_k}$, where $i,k = 1,2$. Furthermore it contains $\mathbb{C}\frac{\partial}{\partial \xi_2}$ and the $sl(2)$-module generated by the vector $2x_1\xi_1\frac{\partial}{\partial x_2} + x_1^2\frac{\partial}{\partial \xi_2}$. Thus it is nine-dimensional and hence is $KO(2,3)_0$. □

Proposition 2.6.2. *Let* $\mathfrak{g}$ *be a prolongation of type* (I15)–(I19). *Then* $\mathfrak{g}$ *contains the Lie superalgebra* $\widehat{SHO}(2,2)$ *in its subprincipal gradation.*

Proof. We may assume by Proposition 2.6.1 that $\mathfrak{g}$ is a subalgebra of $KO(2,3)$ in its subprincipal gradation. Note also that, in all cases under consideration, the vectors $1, \xi_1, \xi_2$ and $x_1\xi_2, x_2\xi_1, x_1\xi_1 - x_2\xi_2, x_1, x_2$ are contained in $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$, respectively.

Consider $KO(2,3)_1$ in its subprincipal gradation as a module over the Lie superalgebra $S(0,2)$, generated by the vectors $x_1\xi_2, x_2\xi_1, x_1\xi_1 - x_2\xi_2, x_1, x_2$. As an $sl(2)$-module it is a direct sum of two (nonirreducible) modules, namely $\sum_{i,j}\mathbb{C}x_ix_j \otimes \Lambda(2)$ and $\sum_i \mathbb{C}x_i\tau \otimes \Lambda(2)$. It is easy to see that any nonzero vector in $KO(2,3)_1$, not lying in the component $\sum_{i,j}\mathbb{C}x_ix_j$, is transitive with respect to the action of x_1, x_2. Hence any nonzero $S(0,2)$-submodule of $KO(2,3)_1$ contains the minimal submodule $\sum_{i,j}\mathbb{C}x_ix_j$. Similarly, any submodule of $KO(2,3)_2$ contains the minimal submodule $\sum_{i,j,k}\mathbb{C}x_ix_jx_k$. Hence $\mathfrak{g}$ contains $\widehat{SHO}(2,2)_{-2}$, $\widehat{SHO}(2,2)_0$ and a nonzero vector of both $\widehat{SHO}(2,2)_{\pm 1}$ in the principal gradation. But then it follows from Lemma 2.3.1 that $\widehat{SHO}(2,2)$ is contained in $\mathfrak{g}$. □

Thus we are led to study $\widehat{SHO}(2,2)$-submodules of $KO(2,3)$ containing $\widehat{SHO}(2,2)$. This will be done for general $KO(n,n+1)$ in Section 3.2. From this we obtain that the only $\widehat{SHO}(2,2)$-submodules of $KO(2,3)$ are as follows: $\widehat{SHO}(2,2)$, $\widehat{SHO}'(2,2)$, $\widehat{SHO}(2,2)+\mathbb{C}\Phi$, $\widehat{SHO}'(2,2)+\mathbb{C}\Phi$, $\widehat{SHO}(2,2)+\mathbb{C}E+\mathbb{C}\Phi$, $\widehat{SHO}'(2,2)+\mathbb{C}E+\mathbb{C}\Phi$, $\widehat{HO}(2,2)$, $\widehat{HO}(2,2)+\mathbb{C}E$ $SKO(2,3;\beta)$, $SKO'(2,3;\beta)$, $SKO(2,3;\beta)+\mathbb{C}E$, $SKO'(2,3;\beta)+\mathbb{C}E$ and $KO(2,3)$. From this list we obtain (we include the case $\lambda=0,1$ as well for the sake of completeness):

Theorem 2.6.1. *The prolongations of types* (I15)–(I19) *are the following superalgebras in their subprincipal gradation*

$$\begin{aligned}
&(\mathrm{I15}) && SKO(2,3;1-\tfrac{1}{\lambda}), \quad \lambda\neq 0,1,\\
& && \widehat{HO}(2,2), \quad \lambda=0,\\
& && SKO'(2,3;0), \quad \lambda=1,\\
&(\mathrm{I16}) && SKO(2,3;1-\tfrac{1}{\lambda})+\mathbb{C}E, \quad \lambda\neq 0,1,\\
& && \widehat{HO}(2,2)+\mathbb{C}E, \quad \lambda=0,\\
& && SKO'(2,3;0)+\mathbb{C}E, \quad \lambda=1,\\
&(\mathrm{I17}) && KO(2,3),\\
&(\mathrm{I18}) && \widehat{SHO}'(2,2),\\
&(\mathrm{I19}) && \widehat{SHO}(2,2).
\end{aligned}$$

Proof. The proof follows from the list of possible $\widehat{SHO}(2,2)$-submodules of $KO(2,3)$ containing $\widehat{SHO}(2,2)$. We only need to check $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$ for all these submodules and match them with (I15)–(I19). □

2.7. The cases (I13) and (I14)

It was proved in [Wi] that for a pair $(\mathfrak{g}_{-1},\mathfrak{g}_0)$ to have an infinite prolongation, it is necessary and sufficient that there exists an element in $x\in\mathfrak{g}_0$ such that the matrix $\mathrm{ad}\,x:\mathfrak{g}_{-1}\to\mathfrak{g}_{-1}$ has rank 1. The argument given there is also applicable to Lie superalgebras of depth 1. Now in the cases (I13) and (I14) no such x exists. Thus these cases give finite prolongations. In fact, it is easy to check directly in this case that any prolongation is zero beyond $\mathfrak{g}_1$.

2.8. The case (I11)

It was shown in [S] that the full prolongation of type $(\mathbb{C}^{4|4},\widehat{P}(4))$ is a simple Lie superalgebra, denoted by $E(4,4)$ in [K2]. Furthermore $E(4,4)_j$ is an irreducible $\widehat{P}(4)$-module and $[E(4,4)_1, E(4,4)_j] \neq 0$, for $j \geq 1$. Hence the exceptional Lie superalgebra $E(4,4)$ is the unique prolongation of type (I11).

3. Subalgebras of full prolongation: the case of inconsistent gradation of depth 2

3.1. The cases (J1) and (J2)

Let $\mathfrak{a} = \oplus_{j=-d}^{-1}\mathfrak{a}_j$ be a finite-dimensional graded commutative Lie superalgebra. Then $\mathfrak{a}$ acts naturally as derivations on its graded algebra of polynomial functions $S^\bullet(\mathfrak{a}^*)$. We start with the following lemma, which is useful for the study of graded transitive Lie superalgebras of depth ≥ 2.

Lemma 3.1.1. *Let $\mathfrak{g} = \oplus_{j=-h}^{\infty}\mathfrak{g}_j$ be a transitive Lie superalgebra of finite depth. Let $\mathfrak{a}$ be a graded $\mathfrak{g}_0$-invariant commutative subalgebra of $\mathfrak{g}_- = \oplus_{j=-h}^{-1}\mathfrak{g}_j$ and let $C(\mathfrak{a})$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Then $\mathfrak{g}$, as a $(\mathfrak{a}+\mathfrak{g}_0)$-module, is a submodule of $S^\bullet(\mathfrak{a}^*) \otimes C(\mathfrak{a})$.*

Proof. Let $x \in \mathfrak{g}$ be a homogeneous element in both its $\mathbb{Z}_2$- and $\mathbb{Z}$-gradation. If $[x,\mathfrak{a}] = 0$, then $x \in C(\mathfrak{a})$. Suppose that $[x,\mathfrak{a}] \neq 0$. Then there exists a unique nonzero integer m such that $[\dots[x,\underbrace{\mathfrak{a}],\mathfrak{a}]\dots],\mathfrak{a}]}_{m \text{ times}} \neq 0$, and $[\dots[x,\underbrace{\mathfrak{a}],\mathfrak{a}]\dots],\mathfrak{a}]}_{m+1 \text{ times}} = 0$. Hence $[\dots[x,\underbrace{\mathfrak{a}],\mathfrak{a}]\dots],\mathfrak{a}]}_{m \text{ times}}$ is a nonzero element in $C(\mathfrak{a})$ and therefore, due to commutativity of $\mathfrak{a}$, x gets mapped into

$$\mathrm{Hom}(S^m(\mathfrak{a}), C(\mathfrak{a})) = S^m(\mathfrak{a}^*) \otimes C(\mathfrak{a}).$$

This map is a homomorphism of $(\mathfrak{a}+\mathfrak{g}_0)$-modules and it is clearly injective. □

Let m be an odd integer and write $m = 2k+1$. Recall that the contact superalgebra $K(m,n)$ in its principal gradation is a Lie superalgebra of depth 2 with $K(m,n)_0 \cong cspo(2k,n)$, where the center in $cspo(2k,n)$ acts as the grading operator. As $spo(2k,n)$-modules, one has $K(m,n)_{-2} \cong \mathbb{C}$ and $K(m,n)_{-1} \cong \mathbb{C}^{2k|n}$, the standard representation. Furthermore we may identify $K(m,n)$ with the polynomial superalgebra $\Lambda(m,n)$ with the contact bracket. Choosing even indeterminates $t, p_1, \dots, p_k, q_1, \dots, q_k$ and odd indeterminates $\xi_1, \dots, \xi_n$ the subalgebra $\widehat{H}(2k,n)$ in its principal gradation is realized naturally as the polynomial subalgebra in the indeterminates p_i, q_i and ξ_j, $i = 1, \dots, k$ and $j = 1, \dots, n$. We have $\widehat{H}(2k,n)_{-2} = \mathbb{C}$, $\widehat{H}(2k,n)_{-1} = \mathbb{C}^{2k|n}$ and $\widehat{H}(2k,n)_0 = spo(2k,n)$.

Lemma 3.1.2. *$\widehat{H}(2k,n)$ is the full prolongation of $(\mathbb{C}, \mathbb{C}^{2k|n}, spo(2k,n))$.*

Proof. Let $\mathfrak{F}$ be the full prolongation of the triple above and set $\mathfrak{F}_- = \mathfrak{F}_{-2} \oplus \mathfrak{F}_{-1}$. We claim that if $D : \mathfrak{F}_- \to \mathfrak{F}_{\leq 0}$ is a derivation of degree 1, then $D \in \widehat{H}(2k,n)_1$. For if $D : \mathfrak{F}_- \to \mathfrak{F}_{\leq 0}$ is a derivation, then $0 = D([1, \mathfrak{F}_{-1}]) =$

$[D(1), \mathfrak{F}_{-1}] + [1, D(\mathfrak{F}_{-1})] = [D(1), \mathfrak{F}_{-1}]$. Hence, since $D(1) \in \mathfrak{F}_{-1}$, we get $D(1) = 0$. But then D is a derivation of $\widehat{H}(2k,n)/\mathbb{C}1 \cong H(2k,n)$, which by Proposition 2.2.1 contains all its derivations of positive degrees. Hence $D \in H(2k,n)$ and so $D \in \widehat{H}(2k,n)$. Using transitivity one similarly proves that if D is of degree 2, then $D \in \widehat{H}(2k,n)$, etc. $\square$

Corollary 3.1.1. *Let* $\mathfrak{g}$ *be a prolongation of type* (J1). *Then* $\mathfrak{g} \cong \widehat{H}(2k,n)$, $k \geq 1$.

Proof. By Lemma 3.1.2, $\mathfrak{g}$ is a subalgebra of $\widehat{H}(2k,n)$. Since $\mathfrak{g}$ is infinite-dimensional $\mathfrak{g}_1 \neq 0$. However by Lemma 2.2.1, $\widehat{H}(2k,n)_1$ is an irreducible $spo(2k,n)$-module generating $\widehat{H}(2k,n)_j$ for $j \geq 1$. Thus $\mathfrak{g} = \widehat{H}(2k,n)$. $\square$

Lemma 3.1.3. *$K(m,n)$ in its principal gradation is the full prolongation of type* (J2).

Proof. Let $\mathfrak{F}$ be the full prolongation of $(\mathbb{C}, \mathbb{C}^{2k|n}, cspo(2k,n))$. Let $\mathfrak{a} = \mathbb{C}$ be the subalgebra of degree -2. We have by Lemma 3.1.1 that $\mathfrak{F} \subseteq S^{\bullet}(\mathbb{C}^*) \otimes C(\mathfrak{a})$. We want to know $C(\mathbb{C})$, i.e., the centralizer of $\mathbb{C}$ in the full prolongation $\mathfrak{F}$. But by definition, $\oplus_{j \leq 0} \mathfrak{F}_j = \oplus_{j \leq 0} K(m,n)_j$. Thus we know that $C(\mathbb{C})_{-2} = \mathbb{C}$, $C(\mathbb{C})_{-1} = H(2k,n)_{-1}$ and $C(\mathbb{C}) = H(2k,n)_0$. Hence $C(\mathbb{C})$ is a transitive Lie superalgebra with $C(\mathbb{C})_{\leq 0}$ of the type in Lemma 3.1.2. Hence $C(\mathbb{C})$ is contained in the full prolongation of $(\mathbb{C}, \mathbb{C}^{2k|m}, spo(2k,m))$, i.e. $\widehat{H}(2k,n)$. But $\widehat{H}(2k,n)$ is the centralizer of $\mathbb{C}$ in $K(m,n)$, which is contained in $\mathfrak{F}$. Therefore $C(\mathbb{C}) = \widehat{H}(2k,n)$ and we have $\mathfrak{F} \subseteq S^{\bullet}(\mathbb{C}^*) \otimes \Lambda(2k,n)$. Since $K(m,n) = S^{\bullet}(\mathbb{C}^*) \otimes \Lambda(2k,n)$, we have $\mathfrak{F} = K(m,n)$. $\square$

Proposition 3.1.2. *Suppose that* $\mathfrak{g}$ *is a prolongation of type* (J2) (*where* m *is replaced by* $2k$) *with* $m = 2k + 1 \geq 3$. *Then* $\mathfrak{g}$ *is either* $\widehat{H}(2k,n) + \mathbb{C}t$ *or* $K(m,n)$ *in their principal gradation.*

Proof. By Lemma 3.1.3 we may assume that $\mathfrak{g}$ is a subalgebra of $K(m,n)$ in its principal gradation with $\mathfrak{g}_{<0} \cong K(m,n)_{<0}$. Consider $K(m,n)_1$ as a module over $K(m,n)_0 \cong cspo(2k,n)$. It is easy to check that $K(m,n)_1$ is a direct sum of two irreducible submodules, namely $H(2k,n)_1$ and $\mathbb{C}^{2k|n}$, which is spanned by the vectors $p_i t$, $q_i t$ and $\xi_j t$, where $i = 1, \ldots, k$ and $j = 1, \ldots, n$. If $\mathfrak{g}_1 \cong \mathbb{C}^{2k|n}$, it is easy to verify that $\mathfrak{g}_2$ is at most the one-dimensional space spanned by t^2. From this it follows that $\mathfrak{g}_3 = 0$, and hence $\mathfrak{g}$ is finite-dimensional. Hence $\mathfrak{g}_1$ contains at least $H(2k,n)$. Suppose first that $\mathfrak{g}_1 = H(2k,n)$. Then by Lemma 2.2.1, $\widehat{H}(2k,n) + \mathbb{C}t \subseteq \mathfrak{g}$. Now $K(m,n)_2$ consists of three irreducible $cspo(2k,n)$-submodules, namely $H(2k,n)_2$, $\operatorname{ad} spo(2k,n)$, spanned by $t(\Lambda(2k,n)_2)$, and the trivial module spanned by the vector t^2. Since $[p_i, t^2]$ and $[q_i, p_1^2 t]$ project nontrivially onto the component $\mathbb{C}^{m|n}$ in $K(m,n)_1$, neither the trivial nor the adjoint components of $K(m,n)_2$ can lie in $\mathfrak{g}$. Thus $\mathfrak{g}_2 = H(2k,n)_2$. We claim that in general $\mathfrak{g}_j = H(2k,n)_j$, for $j \geq 1$, so that $\mathfrak{g} = \widehat{H}(2k,n) + \mathbb{C}t$. We will show this by induction on j. We know that the claim is true for $j = 1, 2$. Suppose now that $x \in K(m,n)_j$, $j \geq 3$, of the form $x = \sum_{i\geq 1}^N f_i t^i$, $f_i \in \Lambda(2k,n)$. We want to show if x lies in $\mathfrak{g}_j$, then $x = 0$. We have $[1, x] = 2\sum_{i=1}^N t^{i-1} f_i \in H(2k,n)_{j-2}$, $j \geq 3$. Thus by induction $x = f_1 t$. Now let $y \in \mathfrak{g}_{-1}$ be either p_i, q_i or ξ_j, where $i = 1, \dots, k$ and $j = 1, \dots, n$. Then we have $[y, x] = y f_1 \pm t\frac{\partial f_1}{\partial \bar{y}}$, where $\bar{p_i} = q_i$, $\bar{q_i} = p_i$ and $\bar{\xi_i} = \xi_i$. Since this expression is supposed to lie in $H(2k,n)$, we have $\frac{\partial f_1}{\partial y} = 0$ for all $y \in \mathfrak{g}_{-1}$. Thus f_1 is a constant, and so x is a scalar multiple of t. But if $x \neq 0$, then $x \in K(m,n)_1$, which is a contradiction. Hence $x = 0$. Thus we have proved that if $\mathfrak{g}_1 = H(2k,n)_1$, then $\mathfrak{g} = \widehat{H}(2k,n) + \mathbb{C}t$.

Now it follows from Proposition 1.5.2 that if $\mathfrak{g}_1 = K(m,n)_1$, then $\mathfrak{g} = K(m,n)$. □

3.2. The cases (J3)–(J7)

Recall that the odd contact Lie superalgebra $KO(n, n+1)$ is isomorphic to $\Lambda(n, n+1)$ with reversed parity equipped with the odd contact bracket. Choosing even indeterminates $x_1, \dots, x_n$ and odd indeterminates $\xi_1, \dots, \xi_n$, τ, we may identify the odd Poisson subalgebra $\widehat{HO}(n,n)$ inside $KO(n, n+1)$ with polynomials in the indeterminates $x_1, \dots, x_n$ and $\xi_1, \dots, \xi_n$ with reversed parity. In its principal gradation $KO(n, n+1)$ is a graded Lie superalgebra of depth 2 with $KO(n,n+1)_{-2} = \mathbb{C}^{0|1}$, $KO(n,n+1)_{-1} = \mathbb{C}^{n|n}$ and $KO(n, n+1) = c\tilde{P}(n)$. The subalgebra $\widehat{HO}(n,n)$ is of depth 2 with $\widehat{HO}(n,n)_{-2} = \mathbb{C}^{0|1}$, $\widehat{HO}(n,n)_{-1} = \mathbb{C}^{n|n}$ and $\widehat{HO}(n,n) = \tilde{P}(n)$. Note that $\tilde{P}(n) = P(n) + \mathbb{C}\Phi$, where $\Phi = \sum_{i=1}^n x_i \xi_i$.

Exactly the same proof as the one given in Lemma 3.1.2, with $HO(n,n)$ and Lemma 2.3.2 (respectively $SHO'(n,n)$ and Lemma 2.3.3) replacing $H(2k,n)$ and Proposition 2.2.1 gives

Lemma 3.2.1. *The Lie superalgebra $\widehat{HO}(n,n)$ is the full prolongation of the triple $(\mathbb{C}^{0|1}, \mathbb{C}^{n|n}, \tilde{P}(n))$, while $\widehat{SHO}'(n,n)$ is the full prolongation of the triple $(\mathbb{C}^{0|1}, \mathbb{C}^{n|n}, P(n))$.*

Now the same argument as in the proof of Lemma 3.1.3, with Lemma 3.2.1 replacing Lemma 3.1.2 and $\widehat{HO}(n,n)$ taking the role of $\widehat{H}(2k,n)$, allows us to prove

Lemma 3.2.2. *$KO(n, n+1)$ in its principal gradation is the full prolongation of the triple $(\mathbb{C}^{0|1}, \mathbb{C}^{n|n}, c\tilde{P}(n))$.*

In $KO(n, n+1) = \Lambda(n, n+1)$ we have the subalgebra $SKO'(n, n+1; \beta) = \{f \in \Lambda(n, n+1) |\ \mathrm{div}_\beta f = 0\}$ with $\mathrm{div}_\beta = 2(-1)^{p(f)}(\Delta + (E - n\beta)\frac{\partial}{\partial \tau})$, where Δ is the odd Laplace operator, $E = \sum_{i=1}^n (x_i \frac{\partial}{\partial x_i} + \xi_i \frac{\partial}{\partial \xi_i})$ and $\beta \in \mathbb{C}$. Recall that $SKO(n, n+1; \beta)_{\leq 0}$ is $(\mathbb{C}^{0|1}, \mathbb{C}^{n|n}, P(n) + \mathbb{C}(\tau + \beta\Phi))$.

Lemma 3.2.3. *$SKO'(n,n+1;\beta)$ is the full prolongation of the triple ($\mathbb{C}^{0|1}$, $\mathbb{C}^{n|n}, P(n)+\mathbb{C}(\tau+\beta\Phi)$).*

Proof. We may assume by Lemma 3.2.2 that the full prolongation $\mathfrak{F}$ of the above triple is a subalgebra of $KO(n,n+1)$. Let $f\in\mathfrak{F}$ be such that $[1,f]=2\frac{\partial f}{\partial\tau}$, $[x_i,f]=x_i\frac{\partial f}{\partial\tau}-\frac{\partial f}{\partial\xi_i}$ and $[\xi_i,f]=\xi_i\frac{\partial f}{\partial\tau}+\frac{\partial f}{\partial x_i}$ are all annihilated by the operator div_β. We want to show that $\mathrm{div}_\beta f=0$. We may assume that $f\in KO(n,n+1)_j$ for $j\geq 1$.

First note that $0=\mathrm{div}_\beta\frac{\partial f}{\partial\tau}=2(-1)^{p(f)+1}(\Delta\frac{\partial f}{\partial\tau})=2(-1)^{p(f)}(\frac{\partial}{\partial\tau}\Delta(f))=\frac{\partial}{\partial\tau}\mathrm{div}_\beta(f)$. Now

$$\begin{aligned}
0&=\mathrm{div}_\beta(x_i\frac{\partial f}{\partial\tau}-\frac{\partial f}{\partial\xi_i})=\mathrm{div}_\beta(x_i\frac{\partial f}{\partial\tau})-\mathrm{div}_\beta(\frac{\partial f}{\partial\xi_i})\\
&=2(-1)^{p(f)+1}(\Delta(x_i\frac{\partial f}{\partial\tau})+(E-n\beta)(x_i\frac{\partial^2 f}{\partial\tau^2})-\Delta(\frac{\partial f}{\partial\xi_i})-(E-n\beta)(\frac{\partial^2 f}{\partial\tau\partial\xi_i}))\\
&=2(-1)^{p(f)+1}(\frac{\partial^2 f}{\partial\xi_i\partial\tau}+x_i\underbrace{\Delta(\frac{\partial f}{\partial\tau})}_{=0}-\Delta(\frac{\partial f}{\partial\xi_i})-(E-n\beta)(\frac{\partial^2 f}{\partial\tau\partial\xi_i}))\\
&=2(-1)^{p(f)+1}(\frac{\partial}{\partial\xi_i}(\frac{\partial f}{\partial\tau}+\Delta(f)+(E-n\beta)(\frac{\partial f}{\partial\tau})-\frac{\partial f}{\partial\tau}))=-\frac{\partial}{\partial\xi_i}(\mathrm{div}_\beta f).
\end{aligned}$$

A similar calculation shows that $\frac{\partial}{\partial x_i}(\mathrm{div}_\beta f)=0$. But then $\mathrm{div}_\beta f$ is a constant. Since $f\in KO(n,n+1)_j$ with $j\geq 1$, we conclude that $\mathrm{div}_\beta=0$. □

The following is the key lemma of this section.

Lemma 3.2.4. *For $n\geq 2$, all nonzero proper $c\tilde{P}(n)$-submodules of $KO(n,n+1)_1$ are $HO(n,n)_1$, $SHO(n,n)_1$, $SKO(n,n+1;\beta)_1$ and $\Lambda(n,n)_1\tau$. In the case when $n=3$ we have additional submodules $SHO'(n,n)_1$ and $SKO'(n,n+1;\frac{1}{3})_1$.*

Proof. Let V be a nonzero proper $P(n)$-submodule of $KO(n,n+1)_1$. Let $0\neq x\in V\subseteq KO(n,n+1)_1$. We may write $x=f+g\tau$, $f\in\Lambda(n,n)_3$ and $g\in\Lambda(n,n)_1$. Successive applications of elements in $P(n)$ to x gives a nonzero element $x=f'+g'\tau$, such that either

(i) $f'\in SHO'(n,n)_1$ with $f'\neq 0$ and $g'=0$,
(ii) $f'\in SHO'(n,n)_1$ with $f'\neq 0$ and $g'\neq 0$, or
(iii) $f'=0$ and $g'\neq 0$.

In case (i) we conclude that $SHO(n,n)_1$ is contained in V. We distinguish between two cases, namely, $SHO'(n,n)_1\not\subseteq V$ and $SHO'(n,n)_1\subseteq V$. (We need to make this distinction between $SHO'(n,n)_1$ and $SHO(n,n)_1$ only in the case when $n=3$.) Consider first the case $SHO'(n,n)_1\subseteq V$. Then the quotient module $V/SHO'(n,n)_1$ is a submodule of $KO(n,n+1)_1/SHO'(n,n)_1$, which is a direct sum of two copies of the standard representation of $P(n)$. Thus if $SHO'(n,n)_1\subsetneq V$, then $V=SKO(n,n+1;\beta)_1$, for some $\beta\in\mathbb{C}$, or $V=HO(n,n)_1$.

Next suppose that $SHO'(n,n)_1$ is not contained in V. Now any nonzero vector in $KO(n,n+1)_1/SHO(n,n)_1$ can be brought via successive applications of elements in $P(n)$ to a nonzero vector of the form $(\alpha\xi_1\xi_2\dots\xi_n + \tau f) + SHO(n,n)_1$, where $f \in \Lambda(n,n)_1$ and $\alpha \in \mathbb{C}$. Thus if V properly contains $SHO(n,n)_1$, then we may assume that we have such an element in $V/SHO(n,n)_1$. Since $SHO'(n,n)_1 \not\subseteq V$, $f \neq 0$. But if $\alpha \neq 0$, then applying any element of $P(n)$ that does not annihilate $f\tau$ implies that $\Lambda(n,n)_1\tau \subseteq V$, which in turn implies that $\xi_1 \dots \xi_n \in V$. But then $SHO'(n,n) \subseteq V$. Hence $\alpha = 0$ so that V contains $SHO(n,n)_1 \oplus \Lambda(n,n)_1\tau$. Since the quotient of $KO(n,n+1)_1$ by $SHO(n,n)_1 \oplus \Lambda(n,n)_1\tau$ has a unique minimal submodule generated by the image of $\mathbb{C}\xi_1 \dots \xi_n$ under the natural projection, we conclude that $V = SHO(n,n)_1 \oplus \Lambda(n,n)_1\tau$.

In case (ii), since $SHO(n,n)_1 \not\cong \mathbb{C}^{n|n}$, we conclude that V contains both $SHO(n,n)_1$ and $\Lambda(n,n)_1\tau$. Now the quotient of $KO(n,n+1)_1$ by $SHO(n,n)_1 \oplus \Lambda(n,n)_1\tau$ is irreducible in the case when $SHO'(n,n)_1 = SHO(n,n)_1$. In the case when $SHO(n,n)_1 \subsetneq SHO'(n,n)_1$, the quotient has a unique submodule generated by the vector $\xi_1\xi_2\dots\xi_n + SHO(n,n)_1$ such that the quotient by it is irreducible. Thus we conclude that V is $SHO(n,n) \oplus \Lambda(n,n)_1\tau$ in the first case and $SHO'(n,n) \oplus \Lambda(n,n)_1\tau$ in the second case.

In case (iii) $\Lambda(n,n)_1\tau$ is a submodule of V. Dividing by this submodule, we obtain $HO(n,n)_1$. Here we know that the only nonzero proper submodules are either $SHO(n,n)_1$ or $SHO'(n,n)_1$ so that V is either $\Lambda(n,n)_1\tau$, $\Lambda(n,n)_1\tau \oplus SHO(n,n)_1$ or $\Lambda(n,n)_1\tau \oplus SHO'(n,n)_1$. □

Lemma 3.2.5. *Let $\mathfrak{g}$ be a graded subalgebra of a graded Lie superalgebra $\mathfrak{h}$ such that $\mathfrak{g}_- = \mathfrak{h}_-$ and $\mathfrak{h}_{-1}^j = \mathfrak{h}_{-j}$. Suppose that $\mathfrak{g}$ is the full prolongation of $(\mathfrak{g}_-, \mathfrak{g}_0)$. Let $\mathfrak{a} \subseteq \mathfrak{h}_0$ be a subalgebra such that $[\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{g}$. Then any graded subalgebra $\mathfrak{n}$ of $\mathfrak{h}$ such that $\mathfrak{n}_- = \mathfrak{g}_-$, $\mathfrak{n}_0 = \mathfrak{g}_0 + \mathfrak{a}$ and $\mathfrak{n}_1 = \mathfrak{g}_1$ is a subalgebra of $\mathfrak{g} + \mathfrak{a}$.*

Proof. Let $x \in \mathfrak{n}_2$. We have $[\mathfrak{g}_{-1}, x] = [\mathfrak{n}_{-1}, x] \subseteq \mathfrak{n}_1 = \mathfrak{g}_1$. Since $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$ it follows that $[\mathfrak{g}_-, x] \subseteq \mathfrak{g}$. Since $\mathfrak{g}$ is a full prolongation, this implies that $x \in \mathfrak{g}_2$. Hence $\mathfrak{n}_2 \subseteq \mathfrak{g}_2$. By induction it follows that $\mathfrak{n}_j \subseteq \mathfrak{g}_j$ for $j \geq 2$. □

We are now in position to prove the main theorem of this section.

Theorem 3.2.1. *Let $\mathfrak{g}$ be an infinite-dimensional graded transitive Lie superalgebra.*

(i) *If $\mathfrak{g}_{\leq 0}$ is of type* (J3), *then $\mathfrak{g}$ is either $\widehat{HO}(n,n)$, $\widehat{SHO}(n,n) + \mathbb{C}\Phi$ or $\widehat{SHO}'(n,n) + \mathbb{C}\Phi$.*

(ii) *If $\mathfrak{g}_{\leq 0}$ is of type* (J4), *then $\mathfrak{g}$ is either $KO(n,n+1)$, $SKO(n,n+1;\beta) + \mathbb{C}\Phi$, $SKO'(n,n+1;\beta) + \mathbb{C}\Phi$, $\widehat{HO}(n,n) + \mathbb{C}\tau$, $\widehat{SHO}(n,n) + \mathbb{C}\Phi + \mathbb{C}\tau$, or $\widehat{SHO}'(n,n) + \mathbb{C}\Phi + \mathbb{C}\tau$.*

(iii) *If $\mathfrak{g}_{\leq 0}$ is of type* (J5), *then $\mathfrak{g}$ is either $\widehat{SHO}(n,n)$ or $\widehat{SHO}'(n,n)$.*

(iv) *If* $\mathfrak{g}_{\leq 0}$ *is of type* (J6), *then* $\mathfrak{g}$ *is either* $SKO(n, n+1; 0)$, $SKO'(n, n+1; 0)$, $\widehat{SHO}(n,n) + \mathbb{C}\tau$ *or* $\widehat{SHO}'(n,n) + \mathbb{C}\tau$.

(v) *If* $\mathfrak{g}_{\leq 0}$ *is of type* (J7), *then* $\mathfrak{g}$ *is either* $SKO(n, n+1; \beta)$, $SKO'(n, n+1; \beta)$, $\widehat{SHO}(n,n) + \mathbb{C}(\tau + \beta\Phi)$ *or* $\widehat{SHO}'(n,n) + \mathbb{C}(\tau + \beta\Phi)$.

In all cases the algebras are considered in their principal gradation.

Proof. We will only show how to prove this theorem in the most complicated case (ii), as all other cases are analogous.

First we may assume that $\mathfrak{g}$ is embedded inside $KO(n, n+1)$ due to Lemma 3.2.2. Now $\mathfrak{g}_{\leq 0}$ is given. So we consider the possibility for $\mathfrak{g}_1$. We know that it must be a $c\tilde{P}(n)$-submodule of $KO(n, n+1)_1$ and we have studied all possible submodules in Lemma 3.2.4. They are $KO(n, n+1)_1$, $HO(n,n)_1$, $SHO(n,n)_1$, $SKO(n, n+1; \beta)_1$ and $\Lambda(n,n)_1\tau$ and when $n = 3$ we have the additional choices of $SHO'(n,n)_1$ and $SKO'(n, n+1; \frac{1}{3})_1$. Now we go through all cases one by one.

It is easy to verify that $KO(n, n+1)_1^j = KO(n, n+1)_j$, for $j \geq 1$, and hence in the very first case $\mathfrak{g} = KO(n, n+1)$.

In the case when $\mathfrak{g}_1 = \widehat{HO}(n,n)_1$ we know by Lemmas 3.2.1 and 3.2.5 that $\mathfrak{g}$ must be a subalgebra of $\widehat{HO}(n,n) + \mathbb{C}\tau$. But since $\widehat{HO}(n,n)_1^j = \widehat{HO}(n,n)_j$, for $j \geq 1$, we must have $\mathfrak{g} = \widehat{HO}(n,n) + \mathbb{C}\tau$.

In the case when $\mathfrak{g}_1 = SHO(n,n)_1$, we know by Lemmas 3.2.1 and 3.2.5 that $\mathfrak{g}$ is contained in $SHO'(n,n) + \mathbb{C}\Phi + \mathbb{C}\tau$. But since $\widehat{SHO}(n,n)_1^j = \widehat{SHO}(n,n)_j$, for $j \geq 1$, we have $\mathfrak{g}$ is either $\widehat{SHO}(n,n) + \mathbb{C}\Phi + \mathbb{C}\tau$ or $\widehat{SHO}'(n,n) + \mathbb{C}\Phi + \mathbb{C}\tau$.

The case when $\mathfrak{g}_1 = \Lambda(n,n)\tau$ is special in the sense that one can show, using the similar argument as in the proof of Proposition 3.1.2, that $\mathfrak{g}_2 = 0$. Hence $\mathfrak{g}$ must be finite-dimensional.

Similarly, we may go through all the remaining cases. □

3.3. The cases (J8) and (J9)

We will embed the full prolongations of Lie superalgebras of these types inside $K(2n+1, 2)$ in its subprincipal gradation. To carry out this task, we need first to study $K(2n+1, 2)$ in its subprincipal gradation. Recall that giving $K(2n+1, 2)$ the subprincipal gradation makes it a Lie superalgebra of depth 2. We have $K(2n+1, 2)_0 \cong csp(2n) \otimes \Lambda(1) + \mathbb{C}\frac{\partial}{\partial\xi} + \mathbb{C}\xi\frac{\partial}{\partial\xi}$, where the generator of $\Lambda(1)$ is ξ. As $K(2n+1,2)_0$-modules $K(2n+1,2)_{-2} \cong \mathbb{C} \otimes \Lambda(1)$ and $K(2n+1, 2)_{-1} \cong \mathbb{C}^{2n} \otimes \Lambda(1)$, where $\mathbb{C}$ and $\mathbb{C}^{2n}$ are the trivial and the standard representations of $sp(2n)$, respectively. Of course, the subalgebra $\widehat{H}(2n, 2)$ inherits a gradation, which we also call subprincipal. We have $\widehat{H}(2n, 2)_{\leq 0}$ is isomorphic to the triple $(\mathbb{C} \otimes \Lambda(1), \mathbb{C}^{2n} \otimes \Lambda(1), sp(2n) \otimes \Lambda(1) + \mathbb{C}\frac{\partial}{\partial\xi} + \mathbb{C}\xi\frac{\partial}{\partial\xi})$.

Proposition 3.3.1. $K(2n+1, 2)$ *in its subprincipal gradation is the full prolongation of the triple* $(\mathbb{C} \otimes \Lambda(1), \mathbb{C}^{2n} \otimes \Lambda(1), csp(2n) \otimes \Lambda(1) + \mathbb{C}\frac{\partial}{\partial\xi} + \mathbb{C}\xi\frac{\partial}{\partial\xi})$.

Proof. Let $t, p_1, \ldots, p_n, q_1, \ldots, q_n$ be $2n+1$ even indeterminates and τ an odd indeterminate. Put $\deg t = \deg \tau = 2$ and $\deg p_i = \deg q_i = 1$. Then this induces a gradation on $W(2n+1,1)$ making it a Lie superalgebra of depth 2. The following is a basis of a copy of the Lie superalgebra $\mathfrak{g}_{\leq 0} = \mathbb{C}\otimes\Lambda(1)+\mathbb{C}^{2n}\otimes\Lambda(1)+(csp(2n)\otimes\Lambda(1)+\mathbb{C}\frac{\partial}{\partial\xi}+\mathbb{C}\xi\frac{\partial}{\partial\xi})$ inside $W(2n+1,1)_{\leq 0}$:

$$\begin{array}{ll}
\mathfrak{g}_{-2} & \frac{\partial}{\partial t}, \frac{\partial}{\partial \tau}, \\
\mathfrak{g}_{-1} & \{p_i\frac{\partial}{\partial t}+\frac{\partial}{\partial q_i}, q_i\frac{\partial}{\partial t}-\frac{\partial}{\partial p_i} | i=1,\ldots,n\} \cong \mathbb{C}^{2n}, \\
 & \{q_i\frac{\partial}{\partial \tau}, p_i\frac{\partial}{\partial \tau} | i=1,\ldots,n\} \cong \mathbb{C}^{2n}\otimes\xi, \\
\mathfrak{g}_0 & \{p_i\frac{\partial}{\partial p_j}-q_j\frac{\partial}{\partial q_i}, p_i\frac{\partial}{\partial q_j}+p_j\frac{\partial}{\partial q_i}, q_i\frac{\partial}{\partial p_j}+q_j\frac{\partial}{\partial p_i} | i,j=1,\ldots,n\}\cong sp(2n), \\
 & \{p_iq_j\frac{\partial}{\partial \tau}, p_ip_j\frac{\partial}{\partial \tau}, q_iq_j\frac{\partial}{\partial \tau} | i,j=1,\ldots,n\} \cong sp(2n)\otimes\xi, \\
 & \{z, \tau\frac{\partial}{\partial \tau}, \tau\frac{\partial}{\partial t}, t\frac{\partial}{\partial t}\} \cong \Lambda(1)+W(0,1),
\end{array}$$

where $z = 2t\frac{\partial}{\partial t}+\sum_{i=1}^{n}(p_i\frac{\partial}{\partial p_i}+q_i\frac{\partial}{\partial q_i})+\tau\frac{\partial}{\partial \tau}$. Then $\operatorname{ad} z$ acts semisimply on $W(2n+1,1)$ with integer eigenvalues ≥ -2, and hence does the same on the full prolongation of the triple $(\mathbb{C}\otimes\Lambda(1), \mathbb{C}^{2n}\otimes\Lambda(1), csp(2n)\otimes\Lambda(1)+\mathbb{C}\frac{\partial}{\partial\xi}+\mathbb{C}\xi\frac{\partial}{\partial\xi})$. Let V^j denote the j-eigenspace of z in an invariant subspace $V\subseteq W(2n+1,1)$. The idea of the proof is the same as the one in the proof of Proposition 2.4.1: Let $\mathfrak{F}$ denote the full prolongation. Since, by Lemma 3.1.1, $K(2n+1,2)$ in its principal gradation is the full prolongation of $K(2n+1,2)_{\leq 0}$, it suffices to show that $\mathfrak{F}^j = K(2n+1,2)^j$, for $j=-2,-1,0$.

Bases for the eigenspaces of z in $W(2n+1,1)$ of nonpositive eigenvalues are as follows ($i,j=1,\ldots,n$):

$$\begin{array}{ll}
W(2n+1,1)^{-2} & \frac{\partial}{\partial t}, \\
W(2n+1,1)^{-1} & \frac{\partial}{\partial \tau}, \frac{\partial}{\partial p_i}, \frac{\partial}{\partial q_i}, p_i\frac{\partial}{\partial t}, q_i\frac{\partial}{\partial t}, \tau\frac{\partial}{\partial t}, \\
W(2n+1,1)^{0} & p_iq_j\frac{\partial}{\partial t}, p_ip_j\frac{\partial}{\partial t}, q_iq_j\frac{\partial}{\partial t}, p_i\tau\frac{\partial}{\partial t}, q_i\tau\frac{\partial}{\partial t}, p_i\frac{\partial}{\partial p_j}, \\
 & q_i\frac{\partial}{\partial p_j}, p_i\frac{\partial}{\partial q_j}, q_i\frac{\partial}{\partial q_j}, p_i\frac{\partial}{\partial \tau}, q_i\frac{\partial}{\partial \tau}, \tau\frac{\partial}{\partial \tau}, t\frac{\partial}{\partial t}.
\end{array}$$

Among these vectors we want to determine those that are in $\mathfrak{F}$. Note that all the above vectors, with the exception of $p_i\tau\frac{\partial}{\partial t}$ and $q_i\tau\frac{\partial}{\partial t}$, lie in $W(2n+1,1)_{\leq 0}$. Hence we only need to show that the linear span of $p_i\tau\frac{\partial}{\partial t}$ and $q_i\tau\frac{\partial}{\partial t}$ do not lie in $\mathfrak{F}$. But this is obvious, since $p_i\frac{\partial}{\partial t}$ and $q_i\frac{\partial}{\partial t}$ do not lie in the $\mathfrak{g}_{\leq 0}$. $\square$

Remark 3.3.1. Set $\deg \xi = 0$, $\deg t = 2$ and $\deg p_i = \deg q_i = 1$. Then $\widehat{H}(2n,0) \otimes \Lambda(1) \subseteq K(2n+1,0) \otimes \Lambda(1)$ are both graded Lie superalgebras with respect to this gradation.

Lemma 3.3.1. *$\widehat{H}(2n,0) \otimes \Lambda(1)$ and $K(2n+1,0) \otimes \Lambda(1)$ are both full prolongations.*

Proof. We may embed the nonpositive graded components into $K(2n+1,2)_{\leq 0}$ in its subprincipal gradation. Using Proposition 3.3.1, it is enough to compute the full prolongations in $K(2n+1,2)$. This is straightforward, and we will omit the details. □

Lemma 3.3.2. *$\widehat{H}(2n,2)$ in its subprincipal gradation is the full prolongation of the triple $(\mathbb{C} \otimes \Lambda(1), \mathbb{C}^{2n} \otimes \Lambda(1), sp(2n) \otimes \Lambda(1) + \mathbb{C}\frac{\partial}{\partial \xi} + \mathbb{C}\xi\frac{\partial}{\partial \xi})$.*

Proof. We embed the triple inside $K(2n+1,2)_{\leq 0}$ in its subprincipal gradation. Let $\mathfrak{F}$ denote its full prolongation inside $K(2n+1,2)$. It is then not hard to verify that $\widehat{H}(2n,2)_j$, $j = -2,-1,0$ in its principal gradation lie in $\mathfrak{F}$. From this the lemma follows from the fact that $\widehat{H}(2n,2)$ in its principal gradation is the full prolongation of its nonpositive parts (Lemma 3.1.2). □

Theorem 3.3.1. *Any prolongation of type* (J9) *is isomorphic to either $\widehat{H}(2n,0) \otimes \Lambda(1) + \mathfrak{a}$, $K(2n+1,0) \otimes \Lambda(1) + \mathfrak{a}$, or $\widehat{H}(2n,2) + \mathfrak{a}$, $K(2n+1,2)$ in their subprincipal gradation.*

Proof. From Proposition 3.3.1 we may embed the full prolongations of Lie superalgebras of type (J9) inside $K(2n+1,2)$ with the subprincipal gradation. We let t, p_i, q_i, $i = 1, \dots, n$ be the even indeterminates and θ_1, θ_2 be the odd indeterminates with degrees assigned as earlier in this section. Hence to study subalgebras of full prolongations of type (J9) we need to study the module structure of $K(2n+1,2)_1$. Now as a module over the 0-th graded component $K(2n+1,2)_1$ is generated by the highest weight vectors $p_1^3\theta_2$, $p_1 t\theta_2$ and $p_1\theta_1\theta_2$. It is easy to check that $K(2n+1,2)_1$ contains the unique irreducible submodule generated by $p_1^3\theta_2$. Dividing by this submodule $K(2n+1,2)_1$ decomposes into a direct sum of two nonisomorphic irreducible modules, generated by the remaining highest weight vectors given above. Hence all possible proper submodules of $K(2n+1,2)_1$ are as follows: It is generated by $p_1^3\theta_2$, which is isomorphic to $(\widehat{H}(2n,0) \otimes \Lambda(1))_1$, it is generated by $p_1^3\theta_2$ and $p_1 t\theta_2$, which is isomorphic to $(K(2n+1,0) \otimes \Lambda(1))_1$ or it is generated by $p_1^3\theta_2$ and $p_1\theta_1\theta_2$, which is isomorphic to $\widehat{H}(2n,2)_1$, all in the gradation described in Lemmas 3.3.1 and 3.3.2. Since the positive degree components of $\widehat{H}(2n,0) \otimes \Lambda(1)$, $K(2n+1,0) \otimes \Lambda(1)$ and $\widehat{H}(2n,2)$ in these gradations are all generated by its degree 1 components, it follows by Lemma 3.3.1 and 3.3.2 that the corresponding Lie superalgebras of type (J9) are either $\widehat{H}(2n,0) \otimes \Lambda(1) + \mathfrak{a}$, $K(2n+1,0) \otimes \Lambda(1) + \mathfrak{a}$, $\widehat{H}(2n,2) + \mathfrak{a}$ or $K(2n+1,2)$. □

Since dividing a full prolongation of depth 2 by a central element lying in degree -2 remains a full prolongation, the proof of the Theorem 3.3.1 may be adapted to prove

Theorem 3.3.2. *Any prolongation of type* (J8) *is isomorphic to either* $H(2n,2)$ *or* $(\widehat{H}(2n,0) \otimes \Lambda(1) + \mathbb{C}\frac{\partial}{\partial \xi} + \mathbb{C}\xi\frac{\partial}{\partial \xi})/\mathbb{C}1$ *in the gradation of Remark 3.3.1.*

4. Subalgebras of full prolongation: the case of consistent gradation

4.1. The case (C1), $n \neq 6$

It follows from Lemma 3.1.3 that any prolongation of type (C1) is necessarily a subalgebra of $K(1,n)$ in its principal gradation. Thus to find all such subalgebras, we will need to study the $K(1,n)_0$-module structure of $K(1,n)_1$. Recall that $K(1,n)_0 \cong cso(n)$. Note also that $K(1,2)$ is not irreducible, hence we will assume that $n \neq 2$ in the rest of this section.

We let t be the even indeterminate and denote the n odd indeterminates by $\xi_1, \dots, \xi_n$. Identifying $K(1,n)$ with $\Lambda(1,n)$ as usual, $K(1,n)_0$ is spanned by $\xi_i\xi_j$, which is isomorphic to $so(n)$ and the grading operator t, while $K(1,n)_1$ is spanned by vectors of the form $\xi_i\xi_j\xi_k$ and $t\xi_i$, where $i,j,k = 1,\dots,n$ with i, j and k all distinct. Thus, the $so(n)$-module $K(1,n)_1$ is isomorphic to $\Lambda^3(\mathbb{C}^n) \oplus \mathbb{C}^n$. The following lemma is obvious.

Lemma 4.1.1. *The $so(n)$-module spanned by $\xi_i\xi_j\xi_k$ is irreducible for $n \neq 6$. The $so(n)$-module spanned by $\xi_i t$ is irreducible for all n. Furthermore, these two modules are isomorphic if and only if $n = 4$.*

Lemma 4.1.2. *Let $\mathfrak{g}$ be a prolongation of type* (C1). *Suppose that $\mathfrak{g}_1$ is spanned by either $\xi_i\xi_j\xi_k$ or $\xi_i t$, but not both, for $i,j,k = 1,\dots,n$ and all distinct. Then $\mathfrak{g}$ is finite-dimensional.*

Proof. In the case when $\mathfrak{g}_1$ is spanned by $\xi_i t$, we have seen already in Section 3.1 that $\mathfrak{g}_2$ is at most $\mathbb{C}t^2$ and $\mathfrak{g}_j = 0$, for $j \geq 3$. In the case when $\mathfrak{g}_1$ is spanned by $\xi_i\xi_j\xi_k$, it follows from Lemma 3.2.5 and the fact that $\widehat{H}(0,n)$ is a full prolongation that $\mathfrak{g}$ is a subalgebra of $\widehat{H}(0,n) + \mathbb{C}t$. Thus it is also finite-dimensional. □

Proposition 4.1.1. *Let $\mathfrak{g}$ be a prolongation of type* (C1) *for $n \neq 2,4,6$. Then $\mathfrak{g} \cong K(1,n)$.*

Proof. By Lemmas 4.1.1 and 4.1.2 we know that $\mathfrak{g}_1$ must be all of $K(1,n)_1$. But then Lemmas 3.1.3 and 3.1.4 tell us that $\mathfrak{g}$ must be all of $K(1,n)$. □

Remark 4.1.1. The Lie superalgebra $S(1,2)$ has a gradation given as follows: We put the degree of the even indeterminate to 2, and the degrees of the odd indeterminates to 1. This produces a Lie superalgebra of depth 2 with the 0-th graded component isomorphic to gl_2. The center of $gl(2)$

acts as the grading operator and furthermore the -2-nd graded component is a 1-dimensional representation, while the -1-st graded component is isomorphic to two copies of the standard representation. This Lie superalgebra is shown in [KL] to be isomorphic to the "annihilation part" of the $SU(2)$-superconformal algebra, also called the "small" $N = 4$ superconformal algebra [A]. In [P] it was shown that the Lie superalgebra of outer derivations of $S(1,2)$ is a Lie algebra isomorphic to $sl(2)$. Hence for the remainder of this section when we write $sl(2)$, we will mean this algebra of outer derivations.

Proposition 4.1.2. *Let* $\mathfrak{g}$ *be a prolongation of type* (C1) *for* $n = 4$. *Then* $\mathfrak{g}$ *is either* $K(1,4)$ *or* $S(1,2)+sl(2)$, *in the gradation given in Remark* 4.1.1.

Proof. If $\mathfrak{g}_1 = K(1,4)_1$, then $\mathfrak{g}$ is $K(1,4)$ by Lemma 3.1.3 and 3.1.4.

Now $K(1,4)_1$ is a direct sum of two isomorphic irreducible representations of $so(4)$. An explicit isomorphism is established by the map $\xi_i t \to \xi_i^*$, where ξ_i^* denotes the Hodge dual of ξ_i, for $i = 1,\ldots,4$. Thus in addition to the two submodules of $K(1,4)_1$ discussed in Lemma 4.1.2, we have also $so(4)$-submodules V_λ, spanned by vectors of the form $\xi_i t + \lambda\xi_i^*$, where $\lambda \in \mathbb{C}^*$. Now a direct calculation of the Lie bracket of arbitrary two such vectors shows that they generate a proper submodule of $K(1,4)_2$ if and only if $\lambda = \pm 1$. The Lie superalgebra thus generated is isomorphic to the positive part of the "small" $N = 4$ superconformal algebra. Thus $\mathfrak{g}$ at least contains $S(1,2) + sl(2)$ with the above gradation.

Now from the explicit realization as fields of the "small" $N = 4$ in [CK1], it follows that the j-th graded component of the "small" $N = 4$ does not contain a (nonzero) vector of the form $\xi_{i_1}\xi_{i_2}\ldots\xi_{i_l}t^k$ with $l+2k-2 = j \geq 1$. It also follows from the formulas in [CK1] that if $\mathfrak{g}$ contains $S(1,2) + sl(2)$ properly, then $\mathfrak{g}_j$, for some $j \geq 2$, must contain all vectors of such a form for some fixed k and some fixed l with $l \geq 2$. Choosing j to be minimal with respect to the property that $(S(1,2) + sl(2))_j \subsetneq \mathfrak{g}_j$ and applying ξ_{i_1} to any such a vector, we obtain a contradiction. Thus $\mathfrak{g}$ must be $S(1,2) + sl(2)$. □

4.2. The case (C1), $n = 6$

Again we study the $so(6)$-module structure of $K(1,6)_1$. In this case the submodule generated by $\xi_i\xi_j\xi_k$, for $i,j,k = 1,\ldots,6$ distinct, decomposes into a direct sum of two mutually contragredient irreducible 10-dimensional submodules, say V_+ and V_-. Thus $K(1,6)_1$ is a direct sum of V_+, V_- and the submodule spanned by the vectors $\xi_i t$, which we will denote by M. All three submodules are nonisomorphic so that it is easy to find all submodules of $K(1,6)_1$. By Lemma 4.1.2 among those submodules only $V_+ \oplus M$ and $V_- \oplus M$ and $K(1,6)_1$ may give infinite dimensional subalgebras of $K(1,6)$. Of course, the choice of $K(1,6)_1$ gives $K(1,6)$.

Let $E(1,6)_j = K(1,6)_j$, for $j \leq 0$, and $E(1,6)_j = (V_+ + M)^j$. Then, using Proposition 1.5.1, we see that the graded Lie superalgebra $E(1,6) =$

$\oplus_{j\geq -2}E(1,6)_j$ is simple. It is clear that, replacing $+$ by $-$ we get an isomorphic Lie superalgebra. This superalgebra was found in [CK1] as the "annihilation part" of a new $N=6$ superconformal algebra CK_6. Independently it was also found in [S]. From the explicit construction in [CK1] it is clear that $E(1,6)$ is infinite-dimensional.

Proposition 4.2.1. *Let $\mathfrak{g}$ be a prolongation of type* (C1) *for $n=6$. Then $\mathfrak{g}$ is isomorphic to either $K(1,6)$ or $E(1,6)$.*

Proof. Let $\mathfrak{g}$ be a graded proper subalgebra of $K(1,6)$ containing $E(1,6)$. From the explicit field realizations of CK_6 in [CK1] one sees that $E(1,6)_j$ cannot contain a (nonzero) vector of the form $\xi_{i_1}\xi_{i_2}\dots\xi_{i_l}t^k$ with $l+2k-2=j\geq 1$, except for t^2 and $\xi_i t$. It also follows from the formulas in [CK1] that, since $\mathfrak{g}$ contains $E(1,6)$ properly, $\mathfrak{g}_j$ must contain all vectors of such a form for some fixed k and some fixed l with $l\geq 3$. Let $j\geq 1$ be the smallest integer such that $E(1,6)_j \subsetneq \mathfrak{g}_j$. Then, of course, $j\geq 2$. Thus $\mathfrak{g}_j$ contains all vectors of the form $\xi_{i_1}\xi_{i_2}\dots\xi_{i_l}t^k$ with $l\geq 3$. Take any such a vector in $\mathfrak{g}_j$. We apply ξ_{i_1} to this vector and we obtain a vector $\xi_{i_2}\dots\xi_{i_l}t^k\in\mathfrak{g}_{j-1}=E(1,6)_{j-1}$. But we know that this can only happen when $l=1$ and $k=1$ or else $l=0$ and $k=2$. But this is a contradiction, since we have assumed that $l\geq 3$. □

4.3. The case of (C2) and (C3)

We denote by $E(5,10)$ and $E'(5,10)$ the full prolongations of the type (C2) and (C3), respectively: $E(5,10)=\oplus_{j\geq -2}\mathfrak{F}_j\subseteq E'(5,10)=\oplus_{j\geq -2}\mathfrak{F}'_j$. Let $C=\oplus_{j\geq -2}C_j$ be the centralizer of $\mathfrak{F}_{-2}=\mathfrak{F}'_{-2}=\mathbb{C}^{5*}$ in $E'(5,10)$. Since $C_0=0$, we conclude from transitivity that $C_j=0$ for $j\geq 0$. Applying Lemma 3.1.1 we conclude that $E'(5,10)\subseteq S^{\bullet}(\mathbb{C}^5)\otimes(\mathbb{C}^{5*}+\Lambda^2(\mathbb{C}^5))$. In particular, $\mathfrak{F}'_1\subseteq\mathbb{C}^5\otimes\Lambda^2(\mathbb{C}^5)\cong R(\pi_1+\pi_2)\oplus\Lambda^2(\mathbb{C}^{5*})$ as $sl(5)$-modules. However, $\mathfrak{F}'_1$ cannot contain the component $\Lambda^2(\mathbb{C}^{5*})$ due to Theorem 5.2 from [K2]. Hence $\mathfrak{F}'_1\cong R(\pi_1+\pi_2)$ or 0 as $sl(5)$-modules. But then $[\mathfrak{F}'_{-1},\mathfrak{F}_1]\subseteq sl(5)=\mathfrak{F}_0$, hence $\mathfrak{F}'_j=\mathfrak{F}_j$ for all $j\geq 1$ and thus $E'(5,10)=E(5,10)+\mathbb{C}z$, where z is the grading operator. One can show (cf. [K2]) that the bitransitive Lie superalgebra with local part $\mathfrak{F}_{-1}\oplus\mathfrak{F}_0\oplus R(\pi_1+\pi_2)$ is a prolongation of type (C2). (It is infinite-dimensional in view of the classification of finite-dimensional Lie superalgebras [K1]). Hence $\mathfrak{F}_1\cong R(\pi_1+\pi_2)$ as $sl(5)$-modules. Alternatively, one can use the explicit construction presented in Section 5.

Consider the Lie algebra $E(5,10)_{\bar 0}$, the even part of $E(5,10)$. Since $[x,\mathfrak{F}_{-2}]=0$ implies that $x\in C$, we conclude that $E(5,10)_{\bar 0}$ is an infinite-dimensional graded transitive Lie algebra with $(E(5,10)_{\bar 0})_{\leq 0}$ equal to $(\mathbb{C}^{5*}, sl(5))$. But the only such Lie algebra is $S(5,0)$, and so the even part of $E(5,10)$ is isomorphic to $S(5,0)$ as a graded Lie algebra.

Next we consider $E(5,10)_{\bar 1}$ as a module over $E(5,10)_{\bar 0}\cong S(5,0)$. We have $\mathfrak{F}_{-1}\cong R(\pi_2)$ as $sl(5)$-modules and $[\mathfrak{F}_{-2},\mathfrak{F}_{-1}]=0$. Thus we have a homomorphism of $S(5,0)$-modules from $E(5,10)_{\bar 1}$ into the module

$\mathrm{Hom}_{U(S(5,0)_+)}(U(S(5,0)), R(\pi_2))$, which is isomorphic to $\Omega^2(5,0)$, the module of differential 2-forms with polynomial coefficients. (Here $S(5,0)_+$ is, of course, the subalgebra of $S(5,0)$ consisting of vector fields vanishing at the origin. The action of $S(5,0)_+$ on an $sl(5)$-module is the obvious one). By transitivity of $E(5,10)_{\bar{1}}$ with respect to $S(5,0)$ (see Proposition 5.2.2) this map is injective. Hence by [R], $E(5,10)_{\bar{1}}$ is either $\Omega^2(5,0)$ or it is the submodule of closed 2-forms, denoted by $d\Omega^1(5,0)$. Since $\mathfrak{F}_1 \cong R(\pi_1+\pi_2)$, we conclude that $E(5,10)_{\bar{1}} \cong d\Omega^1(5,0)$ as $S(5,0)$-modules. Hence all $\mathfrak{F}_i$ are irreducible $sl(5)$-modules.

We have established the following theorem.

Theorem 4.3.1. (i) *Any prolongation of type* (C2) *is isomorphic to* $E(5,10)$. *Furthermore, we have* $E(5,10)_{\bar{0}} \cong S(5,0)$ *and, as an* $S(5,0)$*-module, we have* $E(5,10)_{\bar{1}} \cong d\Omega^1(5,0)$.

(ii) *Any prolongation of type* (C3) *is isomorphic to* $E(5,10)+\mathbb{C}z$, *where* z *is the grading operator.*

4.4. The cases (C4) and (C5)

Denote by $E(3,6) = \oplus_{j\geq -2}\mathfrak{F}_2$ the full prolongation of type (C4). Recall that $\mathfrak{F}_0 = sl(3)\oplus sl(2)\oplus \mathbb{C}z$, where z is the grading operator. It follows from Lemmas 5.1, 5.2, 5.5 and 5.6 from [K2] that, as an $sl(3)\oplus sl(2)$-module, $\mathfrak{F}_1$ lies in the direct sum of the modules $\mathfrak{F}_1' \cong \mathbb{C}^{3*}\boxtimes\mathbb{C}^2$ and $\mathfrak{F}_1'' \cong S^2(\mathbb{C}^3)\boxtimes\mathbb{C}^2$, and moreover, $\mathfrak{F}_{-1}\oplus\mathfrak{F}_0\oplus\mathfrak{F}_1'$ is the local part of $spo(2,6)$. One can show (cf. [K2]) that the bitransitive Lie superalgebra with local part $\mathfrak{F}_{-1}\oplus\mathfrak{F}_0\oplus(\mathfrak{F}_1'+\mathfrak{F}_1'')$ is a prolongation of type (C4). Hence $\mathfrak{F}_1\cong\mathfrak{F}_1'\oplus\mathfrak{F}_1''$ as $sl(3)\oplus sl(2)$-modules.

Corollary 4.4.1. *The centralizer* C *of* $\mathfrak{F}_{-2}$ *in* $E(3,6)$ *is isomorphic to* $\mathfrak{F}_{-2}\oplus\mathfrak{F}_{-1}\oplus sl(2)$.

Proof. We know that $C = \oplus_{j=-2}^{\infty} C_j$ is a graded subalgebra of $E(3,6)$ with $C_{-2} = \mathfrak{F}_{-2}$, $C_{-1} = \mathfrak{F}_{-1}$ and $C_0 = sl(2)$. From our computation of $\mathfrak{F}_1$ above it follows that $C_1 = 0$. By transitivity $C_j = 0$ for all $j\geq 2$. □

Corollary 4.4.1 combined with Lemma 3.1.1 immediately puts restrictions on $\mathfrak{F}$ as an $sl(3)\oplus sl(2)$-module.

Corollary 4.4.2. *As an* $sl(3)\oplus sl(2)$*-module, we have* $E(3,6)_{\bar{0}} \subseteq W(3,0)\boxtimes R(0) + \mathbb{C}[x_1,x_2,x_3]\boxtimes R(2)$ *and* $E(3,6)_{\bar{1}} \subseteq \Omega^1(3,0)\boxtimes R(1)$.

Thus by Corollary 4.4.2, as $sl(3)\oplus sl(2)$-modules, $\mathfrak{F}_2$ consists of at most three components, namely $\mathfrak{F}_2' \cong R(2\pi_1+\pi_2)\boxtimes R(0)$, $\mathfrak{F}_2'' \cong R(\pi_1)\boxtimes R(0)$ and $\mathfrak{F}_2''' \cong R(\pi_1)\boxtimes R(2)$. Here, as before, $R(m)$ stands for the $(m+1)$-dimensional irreducible $sl(2)$-module. We will prove that $\mathfrak{F}_2$ consists of all three components.

Let E_i, H_i, F_i $(i=1,2)$ and E, H, F be the Chevalley generators of $sl(3)$ and $sl(2)$, respectively. Let $F_{12} = [F_1, F_2]$ and let $z\in\mathfrak{F}_0$ denote the grading operator. Let M be the lowest weight vector of $\mathfrak{F}_{-1}$ and let Λ_1 and Λ_2 be the highest weight vectors of $\mathfrak{F}_1'$ and $\mathfrak{F}_1''$, respectively. We have:

$$[M, \Lambda_1] = -\tfrac{2}{3}H_1 - \tfrac{4}{3}H_2 - \tfrac{2}{3}z + 2H, \quad [M, \Lambda_2] = 2E_1. \tag{4.4.1}$$

First the vector $v' = [\Lambda_1, [F, \Lambda_2]]$ is a highest weight vector, provided it is nonzero (since $[\Lambda_1, \Lambda_2] = 0$, otherwise it is a highest weight vector of $\mathfrak{F}_2$). To check that $v' \neq 0$ we compute using (4.4.1): $[M, v'] = [[M, \Lambda_1], [F, \Lambda_2]] - [\Lambda_1, [F, [M, \Lambda_2]]] = [-\frac{2}{3}H_1 - \frac{4}{3}H_2 - \frac{2}{3}z + 2H, [F, \Lambda_2]] = -[F, \Lambda_2] \neq 0$. Hence $\mathfrak{F}_2'$ lies in $\mathfrak{F}_2$.

Furthermore, note that the vector $v''' = [\Lambda_2, [F_{12}, \Lambda_1]]$ has weight $(\pi_1, 2)$. We compute

$$\begin{aligned}[M, v'''] &= [[M, \Lambda_2], [F_{12}, \Lambda_1]] - [\Lambda_2, [M, [F_{12}, \Lambda_1]]] \\ &= 2[E_1, [F_{12}, \Lambda_1]] - [\Lambda_2, [F_{12}, [M, \Lambda_1]]] = 2[F_2, \Lambda_1] - 2[\Lambda_2, F_{12}].\end{aligned} \tag{4.4.2}$$

Neither vector in the right-hand side of (4.4.2) above is zero, and they lie in different components of $\mathfrak{F}_1$. Hence v''' is nonzero. This proves that $\mathfrak{F}_2'''$ lies in $\mathfrak{F}_2$.

Next consider the vector $v'' = [[F_2, \Lambda_1], [F, \Lambda_1]]$. It has weight π_1. We compute $[M, v''] = [[M, [F_2, \Lambda_1]], [F, \Lambda_1]] - [[F_2, \Lambda_1], [M, [F, \Lambda_1]]] = [[F_2, [M, \Lambda_1]], [F, \Lambda_1]] - [[F_2, \Lambda_1], [F, [M, \Lambda - 1]]] = -2[F_2, [F, \Lambda_1]] - 4[[F_2, \Lambda_1], F] = 2[F, [F_2, \Lambda_1]] \neq 0$. Now this vector lies in $S^2(R(\pi_2) \boxtimes R(1)) \cong (R(2\pi_2) \boxtimes R(3)) \oplus \mathfrak{F}_2''$. Since the first component cannot lie in $\mathfrak{F}_2$, this vector must be in $\mathfrak{F}_2''$. Thus $\mathfrak{F}_2''$ lies in $\mathfrak{F}_2$. Summarizing, our computations prove

Proposition 4.4.1. *As an $sl(3) \oplus sl(2)$-module, $\mathfrak{F}_2 \cong \mathfrak{F}_2' \oplus \mathfrak{F}_2'' \oplus \mathfrak{F}_2'''$.*

Corollary 4.4.3. *$\mathfrak{F}_{\bar{0}}$ contains $W(3,0)$ as a subalgebra.*

Proof. This follows immediately from Corollary 4.4.1, Proposition 4.4.1 and Proposition 2.1.1. □

Proposition 4.4.2. *As a $W(3,0) \oplus sl(2)$-module, $E(3,6)_{\bar{1}} \cong (\Omega^1(3,0))^{-\frac{1}{2}} \boxtimes \mathbb{C}^2$.*

Proof. Let $W(3,0)_+$ be the subalgebra of $W(3,0)$ consisting of vector fields vanishing at the origin. Consider the $W(3,0)_+ \oplus sl(2)$-submodule $(E(3,6)_{\bar{1}})_+$ spanned by vectors in $E(3,6)_{\bar{1}}$ of positive degrees. Then $E(3,6)_{\bar{1}}/(E(3,6)_{\bar{1}})_+$ is a $W(3,0)_+ \oplus sl(2)$-module with $W(3,0)_{>0}$ (vector fields vanishing at the origin to the second order or more) acting trivially. As an $gl(3) \oplus sl(2)$-module it is isomorphic to $\mathbb{C}^3 \boxtimes \mathbb{C}^2$ with the identity acting as the scalar $-\frac{1}{2}$. Let V be the $W(3,0) \oplus sl(2)$-module obtained by producing from the $W(3,0)_+ \oplus sl(2)$-module $E(3,6)_{\bar{1}}/(E(3,6)_{\bar{1}})_+$. Then we obtain a homomorphism of $W(3,0) \oplus sl(2)$-modules $\phi : E(3,6)_{\bar{1}} \to V$. By [R], V is irreducible and hence ϕ is onto (since ϕ is nonzero). But then Corollary 4.4.2 tells us that the map is in fact an isomorphism. It remains to observe that V, as an $W(3,0) \oplus sl(2)$-module, is nothing but $(\Omega^1(3,0))^{-\frac{1}{2}} \boxtimes \mathbb{C}^2$. □

Using similar argument together with Proposition 4.4.1 one proves

Proposition 4.4.3. *As a $W(3,0)\oplus sl(2)$-module, $E(3,6)_{\bar{0}}$ is isomorphic to the direct sum of $W(3,0)\boxtimes R(0)$ and $\mathbb{C}[x_1,x_2,x_3]\boxtimes R(2)$.*

Remark 4.4.1. $SHO(3,3)$ has a consistent gradation of depth 2 of the following type. We choose $\deg x_i = 2$ and $\deg \xi_i = 1$, for $i=1,2,3$. Then it is straightforward to check that $SHO(3,3)_0 \cong sl(3)$ with $SHO(3,3)_{-2}$ (respectively $SHO(3,3)_{-1}$) isomorphic to $\mathbb{C}^{3*}$ (respectively $\mathbb{C}^3\oplus\mathbb{C}^3$). It has outer derivations of degree 0 in this gradation, which is isomorphic to $gl(2)$ [K2], so that $SHO(3,3)+gl(2)$ in degree -1 is isomorphic to the $sl(3)\oplus sl(2)$-module $\mathbb{C}^3\boxtimes\mathbb{C}^2$. Removing the grading operator we obtain the subalgebra $SHO(3,3)+sl(2)$.

Theorem 4.4.1. (i) *$E(3,6)_{\bar{0}}$ is isomorphic to the semidirect sum of $W(3,0)$ and $\mathbb{C}[x_1,x_2,x_3]\otimes sl(2)$, and the $E(3,6)_{\bar{0}}$-module $E(3,6)_{\bar{1}}$ is isomorphic to $\Omega^1(3,0)^{-\frac{1}{2}}\otimes\mathbb{C}^2$.*

(ii) *Any prolongation of type* (C4) *is either $E(3,6)$ or $SHO(3,3)+gl(2)$.*

(iii) *Any prolongation of type* (C5) *is $SHO(3,3)+sl(2)$.*

Proof. (i) follows by combining Propositions 4.4.2 and 4.4.3. We proceed now to prove (ii) and (iii).

$\mathfrak{F}_1$ consists of two components, namely, $R(2\pi_1)\boxtimes R(1)$ and $R(\pi_2)\boxtimes R(1)$. $\mathfrak{F}_2$ consists of three components, namely, $R(2\pi_1+\pi_2)\boxtimes R(0)$, $R(\pi_1)\boxtimes R(0)$ and $R(\pi_1)\boxtimes R(2)$. Since $S^2(R(2\pi_1)\boxtimes R(1))\cong(R(4\pi_1)\boxtimes R(2))\oplus(R(2\pi_1)\boxtimes R(2))\oplus(R(2\pi_1+\pi_2)\boxtimes R(0))$, it follows that we have $[R(2\pi_1)\boxtimes R(1), R(2\pi_1)\boxtimes R(1)] = R(2\pi_1+\pi_2)\boxtimes R(0)$ in $\mathfrak{F}_2$. Similarly, one shows that $[R(\pi_2)\boxtimes R(1), R(\pi_2)\boxtimes R(1)] = R(\pi_1)\boxtimes R(0)$ in $\mathfrak{F}_2$. Thus $[\mathfrak{g}_{-1}, R(\pi_1)\boxtimes R(0)] = [\mathfrak{g}_{-1},[R(\pi_2)\boxtimes R(1), R(\pi_2)\boxtimes R(1)]]\subseteq[\mathfrak{g}_0, R(\pi_2)\boxtimes R(1)]\subseteq R(\pi_2)\boxtimes R(1)$. From (4.4.2) we see that

$$[\mathfrak{g}_{-1}, R(\pi_1)\boxtimes R(2)]\not\subseteq R(2\pi_1)\boxtimes R(1). \tag{4.4.3}$$

Let $\mathfrak{g}$ be a prolongation of type (C4). Then there are three possible choices for $\mathfrak{g}_1$, namely, either $\mathfrak{g}_1 = R(2\pi_1)\boxtimes R(1)$, $\mathfrak{g}_1 = R(\pi_2)\boxtimes R(1)$ or $\mathfrak{g}_1$ consists of both components.

In the last case we know that $\mathfrak{g}_2$ consists of all three components. Since $\mathfrak{g}_2$ contains the component $R(\pi_1)\boxtimes R(0)$, $\mathfrak{g}$ contains the subalgebra $W(3,0)$ by Proposition 2.2.1. Consider $\mathfrak{g}$ as a $W(3,0)\oplus sl(2)$-submodule of the full prolongation $\mathfrak{F}$. $\mathfrak{g}_{\bar{1}}$ must be all of $(\Omega^1(3,0))^{-\frac{1}{2}}\boxtimes\mathbb{C}^2$, since $(\Omega^1(3,0))^{-\frac{1}{2}}$ is an irreducible $W(3,0)$-module. Since $\mathfrak{g}_2$ contains the component $R(\pi_1)\boxtimes R(1)$, $\mathfrak{g}$ must contain all of $\mathbb{C}[x_1,x_2,x_3]\otimes sl(2)$, as well. (Actually this argument also implies that $\mathfrak{F}_j=\mathfrak{F}_1^j$ for $j\geq 1$.)

If $\mathfrak{g}_1$ is the component $R(\pi_2)\boxtimes R(1)$, i.e. the component contragredient to $\mathfrak{g}_{-1}$, one checks that $\mathfrak{g}_2$ is just the component $R(\pi_1)\boxtimes R(0)$ and $\mathfrak{g}_3=0$, so that we get a finite-dimensional Lie superalgebra.

If $\mathfrak{g}_1$ is the component $R(2\pi_1)\boxtimes R(1)$, we see from the discussion in the beginning of the proof that $\mathfrak{g}_2$ consists of the single component $R(2\pi_1+$

$\pi_2) \boxtimes R(0)$. Thus, by Proposition 2.1.1, $\mathfrak{g}_{2j}$ cannot contain the component $R(j\pi_1) \boxtimes R(0)$. We consider $\mathfrak{g}$ as an $S(3,0) \oplus sl(2)$-submodule of the full prolongation $\mathfrak{F}$. Since the component $R(\pi_1) \boxtimes R(2)$ is missing in degree 2 by (4.4.3), it follows that $\mathfrak{g}_{\bar{0}}$ is just $S(3,0) + sl(2)$. Since the component $R(\pi_2) \boxtimes R(1)$ is missing in degree 1, it follows that $\mathfrak{g}_{\bar{1}}$ is just $d\Omega^0(3,0)$, the space of closed 1-forms in three indeterminates. It is now easy to show that a Lie superalgebra with this structure is $SHO(3,3) + gl(2)$, with the gradation described earlier. This proves (ii).

Of course, the proof given for (ii) also shows that if $\mathfrak{g}_{\leq 0}$ is of type (C5), then $\mathfrak{g}$ is $SHO(3,3) + sl(2)$, which proves (iii). □

Remark 4.4.2. Note that $E(3,6)$ is an irreducible graded bitransitive Lie superalgebra satisfying the conditions of Proposition 1.5.1. Hence $E(3,6)$ is a simple Lie superalgebra.

4.5. The cases (C6) and (C7)

Let $\mathfrak{F} = E(3,8)$ denote the full prolongation of type (C6). Recall that $\mathfrak{F}_0 = sl(3) \oplus sl(2) \oplus \mathbb{C}z$, where z is the grading operator. Furthermore, as $sl(3) \oplus sl(2)$-modules, we have $\mathfrak{F}_{-3} \cong R(0) \boxtimes R(1)$, $\mathfrak{F}_{-2} \cong R(\pi_2) \boxtimes R(0)$ and $\mathfrak{F}_{-1} \cong R(\pi_1) \boxtimes R(1)$. From Lemmas 5.1, 5.2, 5.5 and 5.6 from [K2] it follows that $\mathfrak{F}_1$ lies in the direct sum of the $sl(3) \oplus sl(2)$-modules $\mathfrak{F}_1' \cong R(\pi_2) \boxtimes R(1)$ and $\mathfrak{F}_1'' \cong R(2\pi_1) \boxtimes R(1)$. Furthermore, $\mathfrak{F}_{-1} \oplus \mathfrak{F}_0 \oplus \mathfrak{F}_1'$ is the local part of $sl(3,2)$. One can show (cf. [K2]) that the bitransitive Lie superalgebra with local part $\mathfrak{F}_{-1} \oplus \mathfrak{F}_0 \oplus (\mathfrak{F}_1' + \mathfrak{F}_1'')$ is a prolongation of type (C4). Hence we obtain

Proposition 4.5.1. *$\mathfrak{F}_1 \cong (R(2\pi_1) \boxtimes R(1)) \oplus (R(\pi_2) \boxtimes R(1))$ as a $gl(3) \oplus sl(2)$-module.*

Lemma 4.5.1. *Let $\mathfrak{g}$ be a prolongation of type (C6) or (C7). Let $C = C_{\mathfrak{g}}(\mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2})$ denote the centralizer of $\mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2}$ in $\mathfrak{g}$. Then $C = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2}$.*

Proof. Evidently C is a graded Lie subalgebra of $\mathfrak{g}$ with $C_{-3} = \mathfrak{g}_{-3}$, $C_{-2} = \mathfrak{g}_{-2}$, $C_{-1} = 0$ and $C_0 = 0$. It follows from transitivity of $\mathfrak{g}$ that $C_i = 0$, for all $i \geq 0$. □

We may now apply Lemma 3.1.1 to obtain an "upper bound" for the full prolongation $\mathfrak{F}$ of type (C6). We thus arrive at $\mathfrak{F} \subseteq S^\bullet(\mathbb{C}^3 \oplus \mathbb{C}^2) \otimes (\mathbb{C}^{3*} \oplus \mathbb{C}^2)$, as a module over $gl(3) \oplus sl(2)$. That is, as a $gl(3) \oplus sl(2)$-module, $\mathfrak{F}$ is a submodule of $W(3,2)$, with the even and odd indeterminates having degree 2 and 3, respectively.

Choose x_1, x_2, x_3 to be our even and θ_1, θ_2 to be our odd indeterminates of $W(3,2)$. We put $\deg x_i = 2$ and $\deg \theta_j = 3$. As a $W(3,0) \oplus sl(2)$-module, $W(3,2)$ can be written as a vector space direct sum of seven, not necessarily irreducible, components. Namely,

$$
\begin{gathered}
W(3,0), W(3,0)\boxtimes(\mathbb{C}\theta_1+\mathbb{C}\theta_2), W(3,0)\boxtimes\mathbb{C}\theta_1\theta_2,\\
\mathbb{C}[x_1,x_2,x_3]\boxtimes(\mathbb{C}\tfrac{\partial}{\partial\theta_1}+\mathbb{C}\tfrac{\partial}{\partial\theta_2}), \mathbb{C}[x_1,x_2,x_3]\boxtimes\mathbb{C}(\theta_1\tfrac{\partial}{\partial\theta_1}+\theta_2\tfrac{\partial}{\partial\theta_2}),\\
\mathbb{C}[x_1,x_2,x_3]\boxtimes(\mathbb{C}(\theta_1\tfrac{\partial}{\partial\theta_2})+\mathbb{C}(\theta_2\tfrac{\partial}{\partial\theta_1})+\mathbb{C}(\theta_1\tfrac{\partial}{\partial\theta_1}-\theta_2\tfrac{\partial}{\partial\theta_2})),\\
\mathbb{C}[x_1,x_2,x_3]\boxtimes(\mathbb{C}\theta_1\theta_2\tfrac{\partial}{\partial\theta_1}+\mathbb{C}\theta_1\theta_2\tfrac{\partial}{\partial\theta_2}).
\end{gathered}
\tag{4.5.1}
$$

Our strategy for finding $\mathfrak{F}$ is as follows. We will prove that $W(3,0)\oplus sl(2)$ sits inside $\mathfrak{F}$ and compute the centralizer of $\mathfrak{g}_{-2}$ inside $\mathfrak{F}$. This will allow us to rule out three of the seven components above. The remaining four components will then be constructed by taking brackets of vectors in $\mathfrak{F}_1$.

Proposition 4.5.2. *Let $\mathfrak{g}$ be a prolongation of type* (C6). *Let C be the centralizer of $\mathfrak{g}_{-2}$ in $\mathfrak{g}$. Then, as $sl(3)\oplus sl(2)$-modules, $C_{-3}=R(0)\boxtimes R(1)$, $C_{-2}=R(\pi_2)\boxtimes R(0)$, $C_{-1}=0$, $C_0=R(0)\boxtimes R(2)$, $C_1=R(\pi_2)\boxtimes R(1)$ and $C_i=0$, for $i\geq 2$.*

Proof. It is enough to prove that $C_2=C_3=C_4=0$. Let $j\geq 2$ and $x\in C_{j+3}$. Then $[\mathfrak{g}_{-3},x]\in C_j$ with $j\geq 2$. But $C_j=0$ by induction. Thus x lies in the centralizer of $\mathfrak{g}_{-3}\oplus\mathfrak{g}_{-2}$ with $j\geq 2$. Hence by Lemma 4.5.1 $x=0$.

Using exactly the same argument as above, it follows that $C_2=0$, since $C_{-1}=0$.

By Lemma 4.5.1 $[\mathfrak{g}_{-3},C_3]\subseteq sl(2)$ (since C is a subalgebra). But then C_3 is contained in the prolongation of $(\mathbb{C}^2, sl(2))$, where $\mathbb{C}^2$ is odd. Thus C_3 is contained in $S(0,2)_1=0$. Hence $C_3=0$.

Comparing with the components of (4.5.1) in degree 4, it follows that if $C_4\neq 0$, then C_4 is the component $R(\pi_2)\boxtimes R(0)$. Now $\mathfrak{F}_2$ by (4.5.1) consists of at most three components, namely, $R(2\pi_1+\pi_2)\boxtimes R(0)$, $R(\pi_1)\boxtimes R(0)$ and $R(\pi_1)\boxtimes R(2)$. We have $C_4\subseteq\Lambda^2(\mathfrak{F}_{-1}^*)\otimes\mathfrak{F}_2$. Decomposing this tensor product we see that $[\mathfrak{F}_{-1},[\mathfrak{F}_{-1},C_4]]\subseteq R(\pi_1)\boxtimes R(2)$ in $\mathfrak{F}_2$. But then $[\mathfrak{F}_{-1},[\mathfrak{F}_{-1},[\mathfrak{F}_{-1},C_4]]]\subseteq R(2\pi_1)\boxtimes R(1)$ in $\mathfrak{F}_1$. Thus we have in $\mathfrak{F}_1$ that $[\mathfrak{F}_{-2},[\mathfrak{F}_{-1},C_4]]=[[\mathfrak{F}_{-1},\mathfrak{F}_{-1}],[\mathfrak{F}_{-1},C4]]\subseteq R(2\pi_1)\boxtimes R(1)$. On the other hand we also have in $\mathfrak{F}_1$ that $[\mathfrak{F}_{-2},[\mathfrak{F}_{-1},C_4]]=[[\mathfrak{F}_2,\mathfrak{F}_{-1}],C_4]=[\mathfrak{F}_{-3},C_4]\subseteq R(\pi_2)\boxtimes R(1)$. Thus $[\mathfrak{F}_{-3},C_4]=0$ and so, by Lemma 4.5.1, $C_4=0$. □

Now we may again apply Lemma 3.1.1 and get

Corollary 4.5.1. *As a $gl(3)\oplus sl(2)$-module we have $\mathfrak{F}\subseteq S^\bullet(\mathbb{C}^3)\otimes(\mathbb{C}^2\oplus\mathbb{C}^{3^*}\oplus sl(2)\oplus(\mathbb{C}^{3*}\boxtimes\mathbb{C}^2))$.*

Proposition 4.5.3. *$\mathfrak{F}_2$ consists of three components, namely, $R(2\pi_1+\pi_2)\boxtimes R(0)$, $R(\pi_1)\boxtimes R(0)$ and $R(\pi_1)\boxtimes R(2)$.*

Proof. By Corollary 4.5.1 this is as big as $\mathfrak{F}_2$ can be. To show that this is indeed $\mathfrak{F}_2$, we build the three highest weight vectors from $\mathfrak{F}_1$. Let Λ_1

and Λ_2 be the highest weight vectors of $\mathfrak{F}_1'$ and $\mathfrak{F}_1''$, respectively. As in Section 4.4 one may show, by taking brackets with the lowest weight vector of $\mathfrak{F}_{-1}$, that the vectors $[[F,\Lambda_2],\Lambda_1]$, $[F,[\Lambda_2,[F_{12},\Lambda_1]]]+[[F,\Lambda_2],[F_{12},\Lambda_1]]$ and $[\Lambda_2,[F_{12},\Lambda_1]]$ are nonzero and that they are the highest weight vectors of the modules $R(2\pi_1+\pi_2)\boxtimes R(0)$, $R(\pi_1)\boxtimes R(0)$ and $R(\pi_1)\boxtimes R(2)$, respectively. $\square$

Corollary 4.5.2. *$W(3,0)\oplus sl(2)$ is a subalgebra of $\mathfrak{F}$.*

Proof. Taking the prolongation of $(\mathfrak{F}_{-2}, gl(3))$ we know by Proposition 2.1.1 that either $W(3,0)$ or $S(3,0)+\mathbb{C}z$ is contained in $\mathfrak{F}$. But we have shown in Proposition 4.5.3 that $\mathfrak{F}_2$ contains the component $R(\pi_1)\otimes R(0)$. Thus $W(3,0)\subseteq\mathfrak{F}$. $\square$

Corollary 4.5.3. *As a $W(3,0)\oplus sl(2)$-module we have (cf. Section 5.1)*

$$\mathfrak{F}_{\bar 0}\cong W(3,0)\boxtimes R(0)+\mathbb{C}[x_1,x_2,x_3]\boxtimes R(2),$$
$$\mathfrak{F}_{\bar 1}\cong \Omega^3(3,0)^{-\frac{3}{2}}\boxtimes R(1)+\Omega^2(3,0)^{-\frac{1}{2}}\boxtimes R(1).$$

Proof. Consider the $W(3,0)\oplus sl(2)$-module $\mathfrak{F}_{\bar 0}/W(3,0)$. The same argument as in the proof of Proposition 4.4.1 tells us that we have a monomorphism of $W(3,0)\oplus sl(2)$-modules from $\mathfrak{F}_{\bar 0}/W(3,0)$ into $\mathbb{C}[x_1,x_2,x_3]\otimes R(2)$. Since $\mathfrak{F}_{\bar 0}/W(3,0)$ contains $sl(2)$ and contains $R(\pi_1)\boxtimes R(2)$ by Proposition 4.5.3, it follows by Corollary 4.5.1 that $\mathfrak{F}_{\bar 0}/W(3,0)\cong\mathbb{C}[x_1,x_2,x_3]\otimes R(2)$.

As for the odd part $\mathfrak{F}_{\bar 1}$ we have similarly $W(3,0)\oplus sl(2)$-monomorphisms from $\mathfrak{F}_{\bar 1}$ into both $\Omega^3(3,0)^{-\frac{3}{2}}\boxtimes\mathbb{C}^2$ and $\Omega^2(3,0)^{-\frac{1}{2}}\boxtimes\mathbb{C}^2$. Since both $\Omega^3(3,0)^{-\frac{3}{2}}\boxtimes\mathbb{C}^2$ and $\Omega^2(3,0)^{-\frac{1}{2}}\boxtimes\mathbb{C}^2$ are irreducible $W(3,0)\oplus sl(2)$-modules by [R], it follows by Proposition 4.5.1 that $\mathfrak{F}_{\bar 1}\cong\Omega^3(3,0)^{-\frac{3}{2}}\boxtimes\mathbb{C}^2+\Omega^2(3,0)^{-\frac{1}{2}}\boxtimes\mathbb{C}^2$. $\square$

Remark 4.5.1. We have $E(3,8)_{-j}=E(3,8)_{-1}^j$, for $j\geq 1$ and $[E(3,8)_{-1}, E(3,8)_1]=E(3,8)_0$. It follows from the existence of a grading operator in $E(3,8)_0$ that $E(3,8)$ is a simple Lie superalgebra. It is also easy to show (see below) that $E(3,8)$ is bitransitive.

Remark 4.5.2. Recall that we have given $SHO(3,3)$ a gradation that makes it into a depth 2 Lie superalgebra in Remark 4.4.1. We may add a central extension by a two dimensional odd center in degree -3, on which the outer derivations algebra $gl(2)$ acts as the standard module [K2]. The resulting Lie superalgebra of depth 3 will be denoted by $\mathbb{C}^2+SHO(3,3)+gl(2)$. Of course, removing the grading operator gives another Lie superalgebra of depth 3, which we will denote by $\mathbb{C}^2+SHO(3,3)+sl(2)$.

Theorem 4.5.1. (i) *Any prolongation of type* (C6) *is isomorphic to either* $E(3,8)$ *or* $\mathbb{C}^2+SHO(3,3)+gl(2)$.

(ii) *Any prolongation of type* (C7) *is isomorphic to* $\mathbb{C}^2+SHO(3,3)+sl(2)$.

Proof. Consider the case (C6). By Proposition 4.5.1 we have three choices for $\mathfrak{g}_1$, namely, $R(2\pi_1)\boxtimes R(1)$, $R(\pi_2)\boxtimes R(1)$ or their direct sum.

In the third case, $\mathfrak{g} = E(3,8)$. This is because the even part must contain $W(3,0) \oplus sl(2)$ and the odd part of $\mathfrak{F}$ is the sum of two irreducible $W(3,0) \oplus sl(2)$-modules. But $\mathfrak{F}_1$ contains nonzero vectors of either components. Since in this case $\mathfrak{F}_2 = [\mathfrak{F}_1, \mathfrak{F}_1] = \mathfrak{g}_2$, it follows similarly that $\Omega^0(3,0) \otimes sl(2)$ is also contained in $\mathfrak{g}$. (Actually this argument also proves that $\mathfrak{F}_j = \mathfrak{F}_1^j$ for $j \geq 1$.)

In the second case it is easy to verify directly that $\mathfrak{g}_2 = 0$, and hence $\mathfrak{g}$ is finite-dimensional.

Thus it remains to consider the case when $\mathfrak{g}_1 = R(2\pi_1) \boxtimes R(1)$. It is easy to check that $\mathfrak{g}_1$ generates the subalgebra $\mathbb{C}^2 + SHO(3,3) + gl(2)$. A simple computation shows that in this case $\mathfrak{g}_2$ can consist of only the component $R(2\pi_1 + \pi_2) \boxtimes R(0)$. But then $\mathfrak{g}$ can neither contain the components $\mathbb{C}[x_1, x_2, x_3] \boxtimes sl(2)$ in degree higher than 2 nor the component $\Omega^2(3,0)^{-\frac{1}{2}} \boxtimes \mathbb{C}^2$. Furthermore, the computation of $\mathfrak{g}_2$ shows that the even part of $\mathfrak{g}$ must be $S(3,0) + gl(2)$ and the odd part must be $\Omega^3(3,0)^{-\frac{3}{2}} \boxtimes \mathbb{C}^2$. But this Lie superalgebra is precisely $\mathbb{C}^2 + SHO(3,3) + gl(2)$.

The case (C7) follows from our discussion, since in this case $\mathfrak{g}_1$ must be the component $R(2\pi_1) \boxtimes R(1)$. $\square$

Remark 4.5.3. Note that we have shown along the way that all exceptional graded Lie superalgebras are bitransitive.

5. Exceptional simple Lie superalgebras of vector fields

5.1. Action of vector fields on twisted differential forms

Let $W(n,0) = W_n$ denote the Lie algebra of derivations of the polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ and consider the semidirect sum $\tilde{W}_n = W_n + \mathbb{C}[x_1, \ldots, x_n]$. For each $\lambda \in \mathbb{C}$ we have a Lie algebra homomorphism $\varphi_\lambda : W_n \to \tilde{W}_n$ defined by $\varphi_\lambda(D) = D + \lambda \, \mathrm{div} D, \quad D \in W_n$. Thus, whenever we have a W_n-module V, which extends to a $\tilde{W}_n$-module, we may twist the W_n-module structure by $\lambda \in \mathbb{C}$ by pulling back the homomorphism φ_λ. The W_n-module thus obtained from V is denoted by V^λ.

We let $\Omega^k(n,0) = \Omega^k$ denote the differential k-forms with polynomial coefficients. W_n acts naturally via Lie derivatives on Ω^k, for all $k = 0, 1, \ldots, n$ and this action extends naturally to $\tilde{W}_n$. Hence the above construction gives us a W_n-module $(\Omega^k)^\lambda$, $\lambda \in \mathbb{C}$. The W_n-module $(\Omega^n)^\lambda$ is naturally identified with the module of λ-densities $(\Omega^n)^\lambda = \{f(dx_1 \ldots dx_n)^\lambda \mid f \in \mathbb{C}[x_1, \ldots, x_n]\}$. It follows easily from the definition that $\Omega^0 \cong (\Omega^n)^{-1}$ and $(\Omega^0)^1 \cong \Omega^n$ as W_n-modules.

Let $(W_n)_+$ denote the subalgebra of W_n consisting of the vector fields vanishing at the origin. We have $(W_n)_0 \cong gl(n) \subseteq (W_n)_+$. Let $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ with $\lambda_i \geq \lambda_{i+1}$ and $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$ for all i, and let us denote the finite-dimensional irreducible representation of $gl(n)$ with this highest weight by $V(\Lambda)$. In particular the highest weight $(1, 0, \ldots, 0)$ gives the standard module, while the highest weight $(1, \ldots, 1, 0, \ldots, 0)$, with i 1's, gives the i-th exterior power of the standard module. We may extend $V(\Lambda)$ to a module of $(W_n)_+$ in a trivial way. Coinducing $V(\Lambda)$ to W_n gives a module over W_n. This module is irreducible unless Λ is one of the following weights: $(0,\dots,0),(1,\dots,0,0),\dots,(1,\dots,1,0)$ [R]. In these cases the corresponding W_n-modules are just the usual modules of differential k-forms Ω^k, for $k=0,\dots,n-1$. Furthermore, $d\Omega^{k-1}$ is the unique nontrivial W_n-submodule of Ω^k and $\Omega^k/d\Omega^{k-1}\cong d\Omega^k$. The following proposition is easy to verify.

Proposition 5.1.1. *The W_n-module $(\Omega^k)^\lambda$ is isomorphic to*

$$\mathrm{Hom}_{U((W_n)_+)}(W_n, V(\Lambda)),$$

where Λ is the weight $(\lambda+1,\dots,\lambda+1,\lambda,\dots,\lambda)$, with k $(\lambda+1)$'s.

It is known that by choosing a volume form v there exists a one-to-one correspondence ψ between vector fields and $n-1$-forms that associates to a vector field D its contraction with v. However, this map is not an isomorphism of W_n-modules. (This is clear, since the $n-1$-forms do not form an irreducible W_n-module, while W_n is a simple Lie algebra.) In fact, the following holds.

Proposition 5.1.2. *The map ψ induces an isomorphism of W_n-modules: $(\Omega^{n-1})^{-1}\cong W_n$, and an isomorphism of S_n-modules: $d\Omega^{n-2}\cong S_n$.*

The map $d:(\Omega^k)^\lambda\to(\Omega^{k+1})^\lambda$ is not W_n-equivariant unless $\lambda=0$. The obstruction is given by the following proposition.

Proposition 5.1.3. *For $D\in W_n$ and $\omega\in(\Omega^k)^\lambda$ we have*

$$[d,D]\omega=\lambda d(\mathrm{div}D)\omega.$$

The verification of Propositions 5.1.2 and 5.1.3 is straightforward.

5.2. Transitive modules over transitive Lie superalgebras

Let $\mathfrak{g}=\oplus_{j=-1}^{\infty}\mathfrak{g}_j$ be a graded transitive Lie superalgebra of depth 1. Let $V=\oplus_{i=-1}^{\infty}V_i$ be a graded $\mathfrak{g}$-module, i.e. $\mathfrak{g}_jV_i\subseteq V_{j+i}$. We will call V a *transitive* $\mathfrak{g}$-module if the condition $\mathfrak{g}_{-1}x=0$ with $x\in V_i$, for $i\geq 0$, implies that $x=0$. Obviously the adjoint module of a graded transitive Lie algebra is a transitive module over itself. In this paper we will only consider the cases when $\mathfrak{g}=W_n$ or $\mathfrak{g}=S_n$, where as before $W_n=W(n,0)$ and $S_n=S(n,0)$. Note that the modules $(\Omega^k)^\lambda$, described in Section 5.1, are examples of transitive modules over the Lie algebras S_n and W_n. Since a submodule of a transitive module is transitive, it follows that the module of closed differential k-forms is also transitive. The following proposition will prove to be useful later on.

Proposition 5.2.1. *Let $\mathfrak{g}$ be a graded transitive Lie superalgebra of depth 1 and let V^1, V^2 and V^3 be transitive $\mathfrak{g}$-modules. Let $\phi:V^1\otimes V^2\to V^3$ be* *a nonzero graded $\mathfrak{g}$-equivariant map. Then ϕ is uniquely determined by its restriction to $\phi^{-1}(V^3_{-1})$.*

Proof. Let j be the maximal integer so that $V^1_{-1}\otimes V^2_j \subseteq \phi^{-1}(V^3_{-1})$. Thus the value of ϕ, restricted to $V^1_{-1}\otimes V^2_k$ for $k \leq j$, is determined by its restriction to $\phi^{-1}(V^3_{-1})$. For $x \in V^2_{j+1}$ and $y \in V^1_{-1}$ we have $\mathfrak{g}_{-1}\phi(y\otimes x) = \phi(\mathfrak{g}_{-1}y \otimes x) \pm \phi(y\otimes \mathfrak{g}_{-1}x) = \pm\phi(y\otimes \mathfrak{g}_{-1}x)$. The right-hand side is determined by $\phi^{-1}(V^3_{-1})$, hence the left-hand side is determined by $\phi^{-1}(V^3_{-1})$. Due to the transitivity of V^3, $\phi(y\otimes x)$ is uniquely determined. The same procedure can now be repeated to prove that the value of ϕ at $V^1_{-1}\otimes V^2$ is uniquely determined by $\phi^{-1}(V^3_{-1})$. From this it follows, using the same argument, that ϕ is completely determined by $\phi^{-1}(V^3_{-1})$. □

Let $\mathfrak{g} = \oplus_{j=-1}^{\infty}\mathfrak{g}_j$ be a graded transitive Lie algebra and let $V = \oplus_{j=-1}^{\infty}V_j$ be a graded transitive $\mathfrak{g}$-module. Set $\mathfrak{g}_+ = \oplus_{j=0}^{\infty}\mathfrak{g}_j$ and $V_+ = \oplus_{j=0}^{\infty}V_j$. Since V_+ is a $\mathfrak{g}_+$-module, so is the quotient V/V_+. The next proposition describes graded transitive $\mathfrak{g}$-modules.

Proposition 5.2.2. *Let $V = \oplus_{j=-1}^{\infty}V_j$ be a transitive module over a transitive Lie algebra $\mathfrak{g} = \oplus_{j=-1}^{\infty}\mathfrak{g}_j$. Then V is a submodule of the produced module* $\mathrm{Hom}_{U(\mathfrak{g}_+)}(U(\mathfrak{g}), V/V_+)$.

Proof. We have a homomorphism of $\mathfrak{g}$-modules

$$\phi : V \to \mathrm{Hom}_{U(\mathfrak{g}_+)}(U(\mathfrak{g}), V/V_+),$$

given by $\phi(v)(u) = (uv) + V_+$ for $u \in U(\mathfrak{g})$ and $v \in V$. It is easy to see that transitivity implies that ϕ is an embedding. □

5.3. Geometric construction

The purpose of this section is to obtain a better understanding of the exceptional simple Lie superalgebras of vector fields. To simplify notation we will denote by $\Omega^k(n)$ the differential k-forms in n indeterminates with polynomial coefficients.

We begin with $E(5,10)$. We have seen in Section 4.4 that $E(5,10)_{\bar 0} \cong S_5$ and $E(5,10)_{\bar 1} \cong d\Omega^1(5)$ via the Lie derivative. Thus the bracket in $E(5,10)$ is completely determined if we know the bracket between two odd elements.

Let ω_1 and ω_2 be two closed 2-forms so that $\omega_1\wedge\omega_2$ is a closed 4-form. Choosing a volume form we may identify 4-forms with vector fields. This identification, although not an isomorphism of W_5-modules, is an isomorphism of S_5-modules by Proposition 5.1.2 (or 5.1.3). It is easy to see that under this identification closed 4-forms correspond to divergence zero vector fields. Thus we have an S_5-equivariant map from $d\Omega^1(5)\otimes d\Omega^1(5)$ to S_5.

Proposition 5.3.1. *$E(5,10)_{\bar 0} \cong S_5$, $E(5,10)_{\bar 1} \cong d\Omega^1(5)$ as S_5-modules. Furthermore, if $\omega_1, \omega_2 \in d\Omega^1(5)$, then $[\omega_1,\omega_2] = \omega_1\wedge\omega_2$, where by $\omega_1\wedge\omega_2$*

we mean the divergence zero vector field in 5 *indeterminates corresponding to the closed* 4*-form* $\omega_1 \wedge \omega_2$.

Proof. We let $\deg x_i = 2$ for $i = 1, \dots, 5$. This induces a gradation on W_5 and hence on S_5. Thus we have $S_5 = \oplus_{j=-1}^{\infty}(S_5)_{2j}$. Denote by $V^1 = \oplus_{j=-1}^{\infty} V^1_{2j+1}$ the S_5-module $d\Omega^1(5)$. We have $V^1_{-1} \cong R(\pi_2)$ as an $(S_5)_0 \cong sl(5)$-module. (Here, as before, π_i denotes the i-th fundamental weight and $R(\pi_i)$ is the irreducible representation of $sl(5)$ with highest weight π_i.) Now the module $R(\pi_4)$ appears in $R(\pi_2) \otimes R(\pi_2)$ with multiplicity 1. Thus there exists a unique (up to a scalar) $sl(5)$-equivariant bilinear map from $V^1_{-1} \otimes V^1_{-1}$ to $(S_5)_{-2}$. Now by Proposition 5.2.1 any S_5-equivariant map from $V^1 \otimes V^1$ to S_5, mapping $V^1_{-1} \otimes V^1_{-1}$ nontrivially onto $(S_5)_{-2}$, is uniquely determined by its restriction to $V^1_{-1} \otimes V^1_{-1}$. Since there is only one such $sl(5)$-equivariant map, it follows that there exists a unique (up to a scalar) nonzero graded S_5-equivariant map $\phi : V^1 \otimes V^1 \to S_5$ that maps $V^1_{-1} \otimes V^1_{-1}$ nontrivially onto $(S_5)_{-2}$. Since the assignment $\omega_1 \wedge \omega_2$ is certainly S_5-equivariant, we see that $[\omega_1, \omega_2] = \omega_1 \wedge \omega_2$. □

Remark 5.3.1. It is easy to verify directly that the bracket defined by Proposition 5.3.1 satisfies the Jacobi identity. Indeed, the only nontrivial identity to check is $[[\omega, \omega], \omega] = 0$ for any $\omega \in d\Omega^1(5)$. It suffices to check this for a generic ω, which, by a change of indeterminates, can be brought to a scalar multiple of $\omega = dx_1 \wedge dx_2 + \alpha dx_3 \wedge dx_4$, $\alpha \in \mathbb{C}$. In this case $[\omega, \omega]$ is a scalar multiple of $\frac{\partial}{\partial x_5}$, hence $[[\omega, \omega], \omega] = 0$. We are grateful to M. Kontsevich and P. Severa for this argument.

Next we consider $E(4,4)$. From the description of $E(4,4)$ in [S], one verifies that $E(4,4)_{\bar{0}} \cong W_4$ and $E(4,4)_{\bar{1}} \cong (\Omega^1(4))^{-\frac{1}{2}}$ as W_4-modules. We define a map $\phi : (\Omega^1(4))^{-\frac{1}{2}} \otimes (\Omega^1(4))^{-\frac{1}{2}} \to W_4$ by

$$\omega_1 \otimes \omega_2 \to d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2 \in (\Omega^3(4))^{-1} \cong W_4, \tag{5.3.1}$$

for $\omega_1, \omega_2 \in (\Omega^1(4))^{-\frac{1}{2}}$.

Proposition 5.3.2. ϕ *in* (5.3.1) *is* W_4*-equivariant.*

Proof. Let $D \in W_4$ and $\omega_1, \omega_2 \in (\Omega^1(2))^{-\frac{1}{2}}$. We have $D\phi(\omega_1 \otimes \omega_2) = D(d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2) = Dd\omega_1 \wedge \omega_2 + d\omega_1 \wedge D\omega_2 + D\omega_1 \wedge d\omega_2 + \omega_1 \wedge Dd\omega_2 = dD\omega_1 \wedge \omega_2 + \frac{1}{2}d(\operatorname{div} D)\omega_1 \wedge \omega_2 + d\omega_1 \wedge D\omega_2 + D\omega_1 \wedge d\omega_2 + \omega_1 \wedge dD\omega_2 + \frac{1}{2}\omega_1 \wedge d(\operatorname{div} D)\omega_2 = dD\omega_1 \wedge \omega_2 + d\omega_1 \wedge D\omega_2 + D\omega_1 \wedge d\omega_2 + \omega_1 \wedge dD\omega_2$. The third equality above is a consequence of Proposition 5.1.3. On the other hand, $\phi(D(\omega_1 \otimes \omega_2)) = \phi(D\omega_1 \otimes \omega_2) + \phi(\omega_1 \otimes D\omega_2) = dD\omega_1 \wedge \omega_2 + d\omega_1 \wedge D\omega_2 + D\omega_1 \wedge d\omega_2 + \omega_1 \wedge dD\omega_2$. Thus ϕ is W_4-equivariant. □

Since the $sl(4)$-module $(W_4)_{-1}$ appears in $\Omega^1(4)_{-1} \otimes \Omega^1(4)_0$ with multiplicity 1, an almost verbatim argument as in the proof of Proposition 5.3.1 gives

Proposition 5.3.3. *$E(4,4)_{\bar 0} \cong W_4$ and $E(4,4)_{\bar 1} \cong (\Omega^1(4))^{-\frac{1}{2}}$. Furthermore, for $\omega_1, \omega_2 \in E(4,4)_{\bar 1}$ we have $[\omega_1, \omega_2] = d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2$.*

Next, we discuss a construction of $SKO(2,3;1)$ which makes it obvious that $sl(2)$ is its outer derivations Lie algebra. The proof of the following proposition is straightforward using the same method as in the proof of Proposition 5.3.1.

Proposition 5.3.4. *$SKO(2,3;1)_{\bar 0} \cong W_2$ and $SKO(2,3;1)_{\bar 1} \cong (\Omega^0(2))^{-\frac{1}{2}} \oplus (\Omega^0(2))^{-\frac{1}{2}}$ as a W_2-module. $SKO(2,3;1)$ has a $\mathbb{Z}$-gradation of the form $SKO(2,3;1)^0 \cong W_2$, $SKO(2,3;1)^{\pm 1} \cong (\Omega^0(2))^{-\frac{1}{2}}$. Furthermore, for $f \in SKO(2,3;1)^1$ and $g \in SKO(2,3;1)^{-1}$ we have $[f,g] = dfg - fdg$. In particular, $sl(2)$ is a Lie algebra of outer derivations of $SKO(2,3;1)$ acting trivially on the even part.*

Now consider the case of $E(1,6)$. From the description of $E(1,6)$, which is the subalgebra of "positive modes" of the superconformal algebra CK_6 [CK1], it is straightforward to verify that $E(1,6)_{\bar 0} = W_1 + sl(4) \otimes \mathbb{C}[t]$. $E(1,6)_{\bar 1}$ is the unique nontrivial extension of $E(1,6)_{\bar 0}$-modules of the form $0 \to S^2(\mathbb{C}^4) \otimes \mathbb{C}[t]dt^{\frac{1}{2}} \to E(1,6)_{\bar 1} \to \Lambda^2(\mathbb{C}^4) \otimes \mathbb{C}[t]dt^{-\frac{1}{2}} \to 0$, where $\mathbb{C}^4$ is the standard $sl(4)$-module, [CKW]. To describe $E(1,6)$ explicitly it is more elegant to use complex 4×4 matrices as in [CK1]. Let A be a 4×4 matrix. We let $A_0 = A - \frac{1}{4}\mathrm{Tr}(A) \in sl(4)$. Furthermore, we may identify $S^2(\mathbb{C}^4)$ and $\Lambda^2(\mathbb{C}^4)$ with symmetric and skew-symmetric 4×4 matrices, respectively. For a skew-symmetric 4×4 matrix C we let $\omega(C)$ denote the Hodge dual of this matrix [CK1], that is explicitly:

$$\omega(\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}) = \begin{bmatrix} 0 & f & -e & d \\ -f & 0 & c & -b \\ e & -c & 0 & a \\ -d & b & -a & 0 \end{bmatrix}.$$

Thus we will identify an element in $sl(4) \otimes \mathbb{C}[t]$ with $A \otimes f$, where A is a traceless 4×4 matrix and $f \in \mathbb{C}[t]$. Furthermore, the spaces $S^2(\mathbb{C}^4) \otimes \mathbb{C}[t]dt^{\frac{1}{2}}$ and $\Lambda^2(\mathbb{C}^4) \otimes \mathbb{C}[t]dt^{-\frac{1}{2}}$ will be identified with the spaces of $\frac{1}{2}$-densities and $-\frac{1}{2}$-densities in t with coefficients in the symmetric and skew-symmetric 4×4 matrices, respectively. Of course, W_1 consists of polynomial vector fields in t. With these identifications we may write down the nontrivial extension as follows: $(A \otimes f)(B \otimes gdt^{\frac{1}{2}}) = (AB + BA^t) \otimes fgdt^{\frac{1}{2}}$, $(A \otimes f)(C \otimes gdt^{-\frac{1}{2}}) = (AC + CA^t) \otimes fgdt^{-\frac{1}{2}} + (AC - CA^t)dfgdt^{-\frac{1}{2}}$, where $A \in sl(4)$ and B (respectively C) is a symmetric (respectively skew-symmetric) 4×4 matrix. Furthermore, $f, g \in \mathbb{C}[t]$. Keeping this notation and denoting symmetric matrices by B_i and skew-symmetric matrices by C_i, for $i = 1,2$, we may write down the bracket of elements in $E(1,6)_{\bar 1}$ as follows: $[B_1 \otimes fdt^{\frac{1}{2}}, B_2 \otimes gdt^{\frac{1}{2}}] = 0$, $[B \otimes fdt^{\frac{1}{2}}, C \otimes gdt^{-\frac{1}{2}}] = -\frac{1}{2}B\omega(C) \otimes fdt^{\frac{1}{2}}gdt^{-\frac{1}{2}}$, $[C_1 \otimes fdt^{-\frac{1}{2}}, C_2 \otimes gdt^{-\frac{1}{2}}] = \mathrm{Tr}(C_1\omega(C_2)) \otimes fdt^{-\frac{1}{2}}gdt^{-\frac{1}{2}} + \frac{1}{2}(C_1\omega(C_2))_0 \otimes (d(fdt^{-\frac{1}{2}})gdt^{-\frac{1}{2}} -$

$fdt^{-\frac{1}{2}}d(gdt^{-\frac{1}{2}})$). Above we used the identification $W_1 \cong \mathbb{C}[t]dt^{-1}$. The proof of this realization of $E(1,6)$ (and of the remaining cases in this section) is along the same lines as the proofs we have given for other algebras considered earlier in this section, and so we will leave it to the reader.

Remark 5.3.2. At this point it might be worthwhile to present another way of defining $E(1,6)$. Recall that $K(1,n)$ is identified with $\Lambda(1,n)$ via the linear map constructed in Section 1.2. Define a linear operator $A : \Lambda(1,6) \to \Lambda(1,6)$ by

$$A(f) = (-1)^{\frac{d(d+1)}{2}} (\frac{\partial}{\partial t})^{3-d}(f^*),$$

where f is a monomial in $\Lambda(1,6)$, d is the number of odd indeterminates in f, f^* is the Hodge dual of f and the operator $(\frac{\partial}{\partial t})^{-1}$ indicates integration with respect to t. Then $E(1,6)$ is identified with the image of the operator $I - iA$.

Next consider $E(3,6)$. From Section 4.4 we have $E(3,6)_{\bar{0}} = W_3 + \Omega^0(3) \otimes sl(2)$ and $E(3,6)_{\bar{1}}$, as an $E(3,6)_{\bar{0}}$-module, can be identified with $(\Omega^1(3))^{-\frac{1}{2}} \otimes \mathbb{C}^2$. We need to describe the bracket between two odd elements. Let $\omega_1, \omega_2 \in (\Omega^1(3))^{-\frac{1}{2}}$ and $v_1, v_2 \in \mathbb{C}^2$. Then

$$[\omega_1 \otimes v_1, \omega_2 \otimes v_2] = (\omega_1 \wedge \omega_2) \otimes (v_1 \wedge v_2) + (d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2) \otimes (v_1 \bullet v_2),$$

where $v_1 \bullet v_2$ is an element in the symmetric square of $\mathbb{C}^2$, thus giving us an element in $sl(2)$, while $v_1 \wedge v_2$ is an element in the skew-symmetric square of $\mathbb{C}^2$, thus giving us a complex number. Again the identification of W_3 with $(\Omega^2(3))^{-1}$ is used. Also we have identified $\Omega^0(3)$ with $(\Omega^3(3))^{-1}$.

Finally it follows from Section 4.5 that the Lie superalgebra $E(3,8)$ has the following structure: $E(3,8)_{\bar{0}} = W_3 + \Omega^0(3) \otimes sl(2)$ (natural semidirect sum) with $E(3,8)_{\bar{1}}$, as an $E(3,8)_{\bar{0}}$-module, isomorphic to $(\Omega^0(3)^{-\frac{1}{2}} \otimes \mathbb{C}^2) + (\Omega^2(3)^{-\frac{1}{2}} \otimes \mathbb{C}^2)$. We will now describe the bracket between two odd elements. Let $\omega_1, \omega_2 \in \Omega^2(3)^{-\frac{1}{2}}$, $f_1, f_2 \in \Omega^0(3)^{-\frac{1}{2}}$ and $v_1, v_2 \in \mathbb{C}^2$. Then we have

$$[\omega_1 \otimes v_1, \omega_2 \otimes v_2] = 0, \ [f_1 \otimes v_1, f_2 \otimes v_2] = df_1 df_2 \otimes v_1 \wedge v_2,$$
$$[f_1 \otimes v_1, \omega_1 \otimes v_2] = (f_1\omega_1 \otimes v_1 \wedge v_2) + ((f_1 d\omega_1 - \omega_1 df_1) \otimes (v_1 \bullet v_2)),$$

where again the identification of W_3 with $\Omega^2(3)^{-1}$ is used, along with $\Omega^3(3)^{-1} \cong \Omega^0(3)$. As in the description of $E(3,6)$ in the previous paragraph $v_1 \bullet v_2$ is an element in the symmetric square of $\mathbb{C}^2$, which gives an element in $sl(2)$, while $v_1 \wedge v_2$, as an element in the skew-symmetric square of $\mathbb{C}^2$, gives a complex number.

5.4. $\mathbb{Z}$-gradations

This section is devoted to the classification of $\mathbb{Z}$-gradations for the simple exceptional Lie superalgebras. Note that a $\mathbb{Z}$-gradation of a superalgebra A is the same as a homomorphism of $\mathbb{C}^\times$ to the group of automorphisms of A, or, equivalently, a choice of an ad-diagonalizable element

h with integral eigenvalues in the Lie superalgebra der$A$ of derivations of A. But for all linearly compact simple Lie superalgebras L the Lie superalgebra der$L$ has a descending filtration by open subalgebras (cf. [K2], Proposition 6.1). Hence all maximal ad-diagonalizable subalgebras of der$L$ are conjugated by an automorphism of L to one of them, say $\mathfrak{h}$. Let $P^\vee = \{h \in \mathfrak{h} \mid \operatorname{ad} h$ has only integral eigenvalues$\}$. Then, up to conjugation by an automorphism of L, the $\mathbb{Z}$-gradations of L are parameterized by elements of $P^\vee$. The condition (G0), which we usually impose, is equivalent to some positivity conditions on $h \in P^\vee$. If $\mathfrak{g} = \oplus_{j \geq -h}\mathfrak{g}_j$ is $\mathbb{Z}$-graded (satisfying (G0)) such that its completion is L as above, and if two $\mathbb{Z}$-gradations of $\mathfrak{g}$ (satisfying (G0)) give conjugate gradations of L, then, clearly, they are conjugate by an automorphism of $\mathfrak{g}$. Also we may choose $\mathfrak{h}$ to be a Cartan subalgebra of a maximal reductive Lie subalgebra of $(\operatorname{der}\mathfrak{g})_0$, where $\operatorname{der}\mathfrak{g} = \oplus_j(\operatorname{der}\mathfrak{g})_j$ is the induced $\mathbb{Z}$-gradation by that of $\mathfrak{g}$.

These arguments show that the $\mathbb{Z}$-gradations constructed in this section exhaust, up to conjugation, all $\mathbb{Z}$-gradations (satisfying (G0)) of the Lie superalgebras in question (cf. [K3]).

We begin with $E(5,10)$. Recall that $E(5,10)_{\bar{0}} \cong S_5$ and $E(5,10)_{\bar{1}} \cong d\Omega^1(5)$ as an S_5-module. To define a $\mathbb{Z}$-gradation on $E(5,10)$ we let $\deg x_i = -\deg \frac{\partial}{\partial x_i} = a_i \in \mathbb{N}$. Let $\deg d = b$, where d is the exterior differential, so that we have $\deg dx_i = a_i + b$. By Proposition 5.3.1, dx_i^* (the Hodge dual of dx_i) is identified with $\frac{\partial}{\partial x_i}$ for all $i = 1, \ldots, 5$ and thus we have the relation $\sum_{i=1}^5 (a_j + b) - a_i - b = -a_i$, hence $\sum_{j=1}^5 a_i = -4b$. The degree of the element $dx_i dx_j$ is $2b + a_i + a_j$, which is thus an integer. This implies that $b \in \frac{1}{2}\mathbb{Z}$. We thus constructed for each quintuple of positive integers $(a_1, \ldots, a_5)$ with an even sum a $\mathbb{Z}$-gradation of $E(5,10)$ by letting

$$\deg x_i = -\deg \frac{\partial}{\partial x_i} = a_i, \ \deg d = -\frac{1}{4}\sum_{i=1}^{5} a_i.$$

These are all $\mathbb{Z}$-gradations, up to an automorphism. Hence $P^\vee$ is isomorphic to the root lattice of type D_5.

Our gradation of $E(5,10)$ in Section 4.3 is obtained by taking $a_i = 2$ so that $b = -\frac{5}{2}$. Another gradation of $E(5,10)$ is obtained by taking $a_i = 1$, for $i = 1, \ldots, 4$, and $a_5 = 2$. Then $b = -\frac{3}{2}$. This gradation produces a Lie superalgebra of depth 2. It is easy to check that this is the grading of $E(5,10)$ in [S]. The third irreducible gradation of $E(5,10)$ is obtained by taking $a_1 = a_2 = 3$ and $a_3 = a_4 = a_5 = 2$ (it is of depth 3) and it looks as follows: $\mathfrak{g}_0$ is a direct sum of sl_2 and $sl_3[\xi] + \mathbb{C}\frac{\partial}{\partial \xi}$, and the $\mathfrak{g}_0$-modules $\mathfrak{g}_{-j}$ for $j = 1, 2, 3$ are respectively, $\mathbb{C}^2 \boxtimes \mathbb{C}^3[\xi]$, $\mathbb{C} \boxtimes \mathbb{C}^{3*}[\xi]$ and $\mathbb{C}^2 \boxtimes \mathbb{C}$. The only other irreducible gradation corresponds to $(2,2,2,1,1)$ and has depth 2.

Next consider $E(3,6)$. We have by Section 5.3, $E(3,6)_{\bar{0}} \cong W_3 + \Omega^0(3) \otimes sl(2)$ and $E(3,6)_{\bar{1}} \cong \Omega^1(3)^{-\frac{1}{2}} \otimes \mathbb{C}^2$. Let x_i, $i = 1, 2, 3$, be the three indeterminates for W_3, let H, E, F be the standard basis for $sl(2)$ and let

e_1, e_2 be the standard basis for $\mathbb{C}^2$. One can show as above that, up to an automorphism, all $\mathbb{Z}$-gradations of $E(3,6)$ are parameterized by quadruples $(a_1, a_2, a_3, \epsilon)$, where the a_i's are positive integers, and $\epsilon \in \frac{1}{2}\mathbb{Z}$ such that $\epsilon + \frac{1}{2}\sum_{i=1}^{3} a_i \in \mathbb{Z}$, by letting

$$\deg x_i = -\deg \frac{\partial}{\partial x_i} = a_i, \ \deg d = -\frac{1}{2}\sum_{i=1}^{3} a_i,$$
$$\deg e_1 = -\deg e_2 = \epsilon, \ \deg E = -\deg F = 2\epsilon, \ \deg H = 0.$$

Hence $P^\vee$ is of type D_4.

Putting $a_i = 2$ along with $\epsilon = 0$ determines our (standard) gradation of $E(3,6)$. We may also set $a_i = 1$ for $i = 1,2,3$, $\epsilon = \frac{1}{2}$. This produces the gradation of $E(3,6)$ given in [S]. The only other irreducible gradation of $E(3,6)$ corresponds to $(2,1,1,0)$. Its depth is 2.

All $\mathbb{Z}$-gradations of $E(1,6)$ are those induced from $K(1,6)$ associated to the differential form $dt + \sum_{i=1}^{3} \xi_i d\xi_{3+i}$, obtained by letting $\deg t = -\deg \frac{\partial}{\partial t} = a \in \mathbb{N}$, $\deg \xi_i = -\deg \frac{\partial}{\partial \xi_i} = b_i \in \mathbb{Z}$ subject to the conditions $b_i + b_{3+i} = a$, $i = 1,2,3$. Hence in this case $P^\vee$ is of type B_4.

Our standard gradation is $a = 2$ and $b_i = 1$. There is only one other irreducible gradation: $a = b_1 = b_2 = b_3 = 1$ and $b_4 = b_5 = b_6 = 0$. In this gradation $\mathfrak{g}_0 = S(0,3) + \mathbb{C}E + \Lambda(3)$ and $\mathfrak{g}_{-1} = \Lambda(3)$. It is of depth 1.

Next consider $\mathbb{Z}$-gradations for $E(4,4)$. Letting x_i, $i = 1,2,3,4$, be the indeterminates for $W_4 \cong E(4,4)_{\bar{0}}$, we set $\deg x_i = a_i \in \mathbb{Z}_+$ and $\deg d = b$. The bracket between two odd elements and our identification of 3-forms with vector fields determine a relation between the a_i's and b, and it is given by the equation $b = -\frac{1}{2}(\sum_{i=1}^{4} a_i)$. Thus the $\mathbb{Z}$-gradations are parameterized by quadruples of positive integers (a_1, a_2, a_3, a_4) with an even sum. This implies that $P^\vee$ is of type D_4.

Choosing $a_i = 1$ (so that $b = -2$) gives our (standard) gradation of $E(4,4)$. This is the only irreducible gradation of $E(4,4)$.

Finally $\mathbb{Z}$-gradations of $E(3,8)$ are obtained as follows. From Section 5.3 we have $E(3,8)_{\bar{0}} \cong W_3 + \mathbb{C}[x_1,x_2,x_3] \otimes sl(2)$ and $E(3,8)_{\bar{1}} \cong (\Omega^0(3)^{-\frac{1}{2}} + \Omega^2(3)^{-\frac{1}{2}}) \otimes \mathbb{C}^2$. Let $\deg x_i = a_i$, $\deg d = b$ and $\deg e_j = \epsilon_j$, for $j = 1,2$ where $\{e_j\}$ is a basis for $\mathbb{C}^2$. Let E, F, H denote the Chevalley generators of $sl(2)$. From the description of $E(3,8)$ in Section 5.3 we obtain the following set of relations ($a = \sum_{i=1}^{3} a_i$): $a + \epsilon_1 + \epsilon_2 = 0$, $a + 2b + \epsilon_1 + \epsilon_2 = 0, a + 2b + 2\epsilon_1 - \deg E = 0$, $a + 2b + 2\epsilon_2 + \deg E = 0, \deg H = 0$, $\deg E = -\deg F$. Solving these equations we get

$$\epsilon_2 = -a - \epsilon_1, \quad \deg E = -\deg F = a + 2\epsilon_1, \quad \deg d = \deg H = 0.$$

Thus the $\mathbb{Z}$-gradations are parameterized by quadruples $(a_1, a_2, a_3, \epsilon_1)$, where $a_i \in \mathbb{N}$ and $\epsilon_1 \in \mathbb{Z}$, and so $P^\vee$ is of type B_4.

The quadruple $(2,2,2,-3)$ gives our (standard) gradation, while $(1,1,1,-2)$ gives the one in [S]. There is only one other irreducible gradation of $E(3,8)$: $a_1 = 2, a_2 = a_3 = 1$ and $\epsilon = -2$. It is of depth 2.

5.5. Embeddings

The purpose of this section is to describe several embeddings among the simple exceptional Lie superalgebras of vector fields. Our discussion is based on our realizations of these Lie superalgebras given in Section 5.3 and we will therefore adopt this same notation without further explanation.

Note that $E(m,n)$ can be embedded into $E(m',n')$ only if $m \leq m'$ (from growth considerations) and only if the maximal reductive Lie subalgebra $\mathfrak{s}$ of $E(m,n)$ can be embedded in the maximal reductive subalgebra $\mathfrak{s}'$ of $E(m',n')$ (by Levi's theorem). It follows that if $E(m,n)$ is embedded in $E(m',n')$, then any $\mathbb{Z}$-gradation of $E(m,n)$ extends to that of $E(m',n')$.

Let (a_1,a_2,a_3,a_4,a_5) denote the 5-tuple of positive integers that determine a $\mathbb{Z}$-gradation for $E(5,10)$ (see Section 5.4). Suppose that $E(3,8)$ in its standard gradation embeds into $E(5,10)$ with the above gradation. By the above remarks, we can assume that $a_1 = a_2 = a_3 = a$ and $a_4 = a_5 = b$. Thus $2\deg d = -\frac{3a}{2} - b$. We distinguish among three cases, namely, $a = b$, $a > b$ and $a < b$. The first case cannot happen, since the negative part of $E(5,10)$ in this gradation only contains two graded components. Suppose $a > b$. $(E(5,10)_-)_{\bar{0}}$ contains 3 graded components. Namely, as $sl(3)\oplus sl(2)$-modules, they are $\mathbb{C}^{3*}\boxtimes\mathbb{C}$ in degree $-a$, $\mathbb{C}\boxtimes\mathbb{C}^2$ in degree $-b$ and $\mathbb{C}^3\boxtimes\mathbb{C}^2$ in degree $-a+b$, respectively. $(E(5,10)_-)_{\bar{1}}$ contains two components. Namely, as $sl(3)\oplus sl(2)$-modules, they are $\mathbb{C}\boxtimes\mathbb{C}$ in degree $-\frac{3a}{2}+b$ and $\mathbb{C}^3\boxtimes\mathbb{C}^2$ in degree $-\frac{a}{2}$. Evidently, they cannot include $E(3,8)_-$ in its standard gradation. Thus this case cannot happen either. Finally let $a < b$. In this case $(E(5,10)_-)_{\bar{0}}$ consists of three components, namely $\mathbb{C}\boxtimes\mathbb{C}^2$ in degree $-b$, $\mathbb{C}^{3*}\boxtimes\mathbb{C}$ in degree $-a$ and $\mathbb{C}^3\boxtimes\mathbb{C}^2$ in degree $-b+a$. Furthermore, $(E(5,10)_-)_{\bar{1}}$ consists of two components, namely $\mathbb{C}^{3*}\boxtimes\mathbb{C}$ in degree $-b+\frac{a}{2}$ and $\mathbb{C}^3\boxtimes\mathbb{C}^2$ in degree $-\frac{a}{2}$. But also in this case $E(3,8)_-$ in its standard gradation cannot be embedded in $E(5,10)_-$.

We conclude that $E(3,8)$ cannot be embedded into $E(5,10)$. A similar analysis shows that $E(4,4)$ cannot be embedded in $E(5,10)$.

Hence we have only the following possibilities of embeddings: $E(1,6) \subseteq E(4,4)$ or $E(5,10)$; $E(3,6) \subseteq E(5,10)$.

The gradation of depth 1 embeds $E(1,6)$ in $E(4,4)$.

There is an embedding of $E(3,6)$ into $E(5,10)$ given as follows. Taking $D \in W_3$ and $f \in \Omega^0(3)$, $E(3,6)_{\bar{0}}$ is embedded into $E(5,10)_{\bar{0}}$ as follows: $D \to D - \frac{1}{2}(\mathrm{div}D)(x_4\frac{\partial}{\partial x_4} + x_5\frac{\partial}{\partial x_5}), f\otimes H \to f(x_4\frac{\partial}{\partial x_4} - x_5\frac{\partial}{\partial x_5})$, $f\otimes E \to fx_4\frac{\partial}{\partial x_5}$, $f\otimes F \to fx_5\frac{\partial}{\partial x_4}$. Writing (a_1,a_2) with $a_1,a_2 \in (\Omega^1(3))^{-\frac{1}{2}}$, $E(3,6)_{\bar{1}}$ is embedded into $E(5,10)_{\bar{1}}$ as follows: $(a_1,a_2) \to -d(x_4(a_1-a_2))+d(x_5(a_1+a_2))$. The verification that this is indeed is a homomorphism of Lie superalgebras is a straightforward, albeit rather tedious, calculation.

The Lie superalgebra $E(1,6)$ can be embedded in $E(5,10)$ as follows. First we embed the even part: $t^n\frac{\partial}{\partial t} \to x_5^n\frac{\partial}{\partial x_5} - \frac{n}{4}x_5^{n-1}\sum_{i=1}^4 x_i\frac{\partial}{\partial x_i}$, $A\otimes t^n \to x_5^n\sum_{i,j=1}^4 a_{ij}x_i\frac{\partial}{\partial x_j}$, where $A \in sl(4)$ is the 4×4 matrix (a_{ij}). The odd part is realized in $E(5,10)_{\bar{1}}$ as follows: $(E_{ij}+E_{ji})\otimes f \to (x_i dx_j + x_j dx_i)df$, $(E_{ij} - E_{ji}) \otimes f \to d(f(x_i dx_j - x_j dx_i))$, where $f \in \mathbb{C}[x_5]$, E_{ij} is the 4×4 matrix having 1 at the (i,j)-th entry and 0 elsewhere.

Postscript

We are taking this opportunity to make corrections to [K2]:

p. 36,	line 15:	replace $spo(2,n)$ by $spo(n,2)$,
p. 43,	line 3_-:	replace $sl_n(3,2)$ by $sl(3,2)$,
p. 43,	line 2_-:	replace $spo(3,2)$ by $spo(2,6)$,
p. 44,	line 4_-:	replace sl_5 by sl_5^*,
p. 50,	line 12:	should be $\mathrm{S}^{j+1}(sl_5)\otimes sl_5^*$.

The remaining corrections concern Propositions 4.1 and 4.2 (which are referred to for proofs in the present paper) and reflect mainly the fact (unnoticed in [K2]) that $E(2,2)$ is isomorphic to $SKO(2,3;1)$:

p. 31,	line 11:	replace $E(2,2)$ by $SKO(2,3;1)$,
p. 31,	line 19:	replace $SKO(2,3;1)$ by $\widehat{SHO}(2,2)$,
p. 31,	line 20:	replace $SKO'(2,3;1)$ by $\widehat{SHO}'(2,2)$,
p. 32,	line 12_-:	replace $E(2,2)+\mathfrak{a}$ by $SKO(2,3;1)+\mathfrak{a}$,
p. 53,	line 13_-:	remove $\bar{E}(2,2)$,
p. 54,	line 4:	replace $\bar{E}(2,2)$ by $\overline{SKO}(2,3;1)$.

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