TWISTED GRAVITON SPECTRA OF $AdS_4 \times S^7$ AND $AdS_7 \times S^4$

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Abstract.

Contents

Having described our eleven-dimensional model on flat spacetimes, we now pursue descriptions on maximally symmetric spacetimes. We begin by describing twisted versions of the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds. In eleven-dimensional supergravity, these backgrounds arise as near-horizon limits of the backgrounds sourced by some number of M2 and M5 branes in flat space respectively. In the first section of this chapter 3, we describe an analogous procedure natively in our twisted context.

To do so, we motivate ansatzae for the leading order couplings of our eleven-dimensional model to M2 and M5 branes. These couplings determine certain curved deformations of the L_∞-algebra underlying our model - we conjecture that deforming the theory in the complement of the brane by a solution to the resulting curved Maurer-Cartan equation is perturbatively equivalent to the twist of the theory on $AdS_4 \times S^7$ and $AdS_7 \times S^4$.

The next two sections provide evidence for this conjecture. We begin in section 2 with numerical checks. We give definitions of supergravity states in our twisted $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds, which can be thought of as particular field configurations that are localized at points on the conformal boundary of AdS. We compute characters of the proposed state spaces and find exact matches with counts of gravitons on $AdS_4 \times S^7$ and $AdS_7 \times S^4$ respectively.

The next strand of evidence we pursue is by matching symmetries. In the physical theory, the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds carry actions of the 3d $\mathcal{N} = 8$ and 6d $\mathcal{N} = (2, 0)$ superconformal algebras. We show that our conjectural descriptions of twists of these backgrounds carry actions of the minimal twists of the corresponding superconformal algebras.

With these pieces of evidence in hand, in sections 4, 5, we then turn to study some representation theoretic aspects of the state spaces constructed in section 2. We identify certain $\mathbb{C}^\times$ actions on our eleven-dimensional model that combine rescalings in directions normal to branes with a certain rescaling of the space of fields - this induces a decomposition of the space of fields that we dub the *graviton decomposition*. The weight 0 parts of these decompositions are certain local L_∞-algebras whose costalks recover the linearly compact super-Lie algebras $E(1|6)$ and $E(3|6)$. We thus see that these linearly compact super-Lie algebras act on the spaces of supergravity states constructed in section 2. We explicate their action on nonzero weight spaces of the graviton decomposition.

In the final section of the chapter, we motivate some current work in progress that leverages the uncovered appearance of exceptional linearly compact super-Lie algebras for holographic means. Eleven-dimensional supergravity on $AdS_4 \times S^7$ and $AdS_7 \times S^4$ is expected to be equivalent to the large N limit of the worldvolume theories of N M2 branes and N M5 branes respectively.

1. Twisted Backreactions

As remarked above, in eleven-dimensional supergravity, the $AdS_7 \times S^4$ and $AdS_4 \times S^7$ backgrounds are obtained by backreacting a number of M5 branes and M2 branes in flat space [Mal98, Wit98]. In this section, we wish to give an account of this procedure at the level of our twisted theory in eleven-dimensions. Before describing the specific examples of interest, we begin with some generalities.

Suppose we have a theory of gravity on the total space of a vector bundle. In this thesis, we are interested in holomorphic-topological field theories, and in this context, the bundle projection is a map of THF manifolds, and the gravitational theory is a local moduli problem that describes, in part, deformations of the THF structure on the total space. Operationally, producing the theory in the backreacted geometry is the output of the following two-step procedure.

- Place the theory on the complement of the zero section.
- Deform the theory on the complement of the zero section by a certain Maurer–Cartan element, thought of as the flux sourced by branes wrapping the zero section. More rigorously, the zero section determines a certain curved Maurer-Cartan equation, and the desired Maurer-Cartan element is a solution to this equation.

This procedure is implemented at the level of the Ω-deformed nonminimal twist on flat space in the appendix of [Cos16], and in [RW22] the procedure is carried about for M5 branes in our eleven-dimensional model in some global generality. For the purposes of this thesis however, we will content ourselves with examples on flat space.

1.1. **The $\mathrm{A}dS_4 \times S^7$ background.** In this section we introduce the analog of the $\mathrm{A}dS_4 \times S^7$ background in our conjectural description of the minimal twist of eleven-dimensional supergravity.

1.1.1. We begin by viewing the eleven-dimensional manifold $\mathbb{R} \times \mathbb{C}^5$ as

$$\mathrm{Tot}(K_{\mathbb{C}}^{1/4} \otimes \mathbb{C}^4 \to \mathbb{R} \times \mathbb{C}_z)$$

where we have abusively used $K_{\mathbb{C}}^{1/4} \otimes \mathbb{C}^4$ to denote its pullback along the natural projection $\mathbb{R} \times \mathbb{C} \to \mathbb{C}$. Thinking of flat space in this way is simply a way to record weights under natural scaling actions. We will use w_a to denote holomorphic fiber coordinates on $K_{\mathbb{C}}^{1/4} \otimes \mathbb{C}^4$.

We carry out the above procedure. Consider a stack of N M2 branes wrapping the zero section $\mathbb{R} \times \mathbb{C}_z$. A natural interaction to consider is

$$I_{M2}(\gamma) = N \int_{\mathbb{C}_z} \gamma + \cdots$$

which is nonzero only on the component of γ in $\Omega^1(\mathbb{R}) \otimes \Omega^{1,1}(\mathbb{C}^5)$. We have only indicated the lowest order coupling, the $\cdots$ indicate higher-order couplings which will be higher order in the fields of the eleven-dimensional theory and explicitly involve the fields in the worldvolume theory.

This coupling is justified by comparison with the physical theory and by dimensional reduction. Indeed, as discussed in §**??**, the component of γ which participates in the above coupling is a component of the C-field of eleven dimensional supergravity. Thus, the proposal mirrors electric couplings of M2 branes in the physical theory, which simply involves integrating the C-field over the worldvolume of the brane.

Moreover, reducing on a circle transverse to the M2 brane yields the $SU(4)$ twist of type IIA supergravity on $\mathbb{R}^2 \times \mathbb{C}_z \times \mathbb{C}^3$ with N $D2$ branes wrapping $\mathbb{R} \times \mathbb{C}_z$. As is shown in [**?**], an electric coupling of D2 branes to the $SU(4)$ twist of type IIA supergravity is given by

$$I_{D2}(\gamma) = N \int_{\mathbb{R} \times \mathbb{C}_z} \gamma + \cdots$$

where γ now denotes the 1-form field of the $SU(4)$ twist of type IIA supergravity. It is immediate that the pullback of I_{M2} along the map in the proof of proposition **??** recovers I_{D2}.

1.1.2. The backreacted geometry will be given by a solution to the equations of motion upon deforming the eleven-dimensional action by the interaction $I_{M2}(\gamma)$. Varying the deformed action with respect to γ, we obtain the equation of motion

$$\bar{\partial}\mu + \frac{1}{2}[\mu, \mu] + \partial\gamma\partial\gamma = N\Omega^{-1}\delta_{w=0}. \tag{1}$$

Here $[-,-]$ is the Schouten bracket. Varying β, we obtain the equation of motion

$$\partial_\Omega \mu = 0. \tag{2}$$

Lemma 1.1. Let

$$F_{M2} = \frac{6}{(2\pi i)^4} \frac{\sum_{a=1}^4 \bar{w}_a \mathrm{d}\bar{w}_1 \cdots \widehat{\mathrm{d}\bar{w}_a} \cdots \mathrm{d}\bar{w}_4}{\|w\|^8} \partial_z .$$

Then the background where $\mu = NF_{M2}$ and $\gamma = 0$ satisfies the above equations of motion in the presence of a stack of N M2 branes:

$$\bar{\partial}(NF_{M2}) + \frac{1}{2}[NF_{M2}, NF_{M2}] = N\Omega^{-1}\delta_{w=0}$$
$$\partial_\Omega(NF_{M2}) = 0.$$

Here we set all components of the field γ equal to zero (as well as the fields ν, β).

Proof. Upon specializing $\gamma = 0$, the last term in the first equation above vanishes. The equation $\bar{\partial}F_{M2} = \Omega^{-1}\delta_{w=0}$ characterizes the Bochner–Martinelli kernel representing the residue class on $\mathbb{C}^4 \setminus 0$. It is clear that $\partial_\Omega F_{M2} = 0$ and

$$[F_{M2}, F_{M2}] = 0$$

by simple type reasons. $\square$

We summarize the output of our computation with a definition.

Definition 1.2. Let $\mathcal{E}^N_{AdS_4 \times S^7}$ denote the classical BV theory on $\mathrm{Tot}(K_{\mathbb{C}}^{1/4} \otimes \mathbb{C}^4 \to \mathbb{R} \times \mathbb{C}_z) \setminus 0(\mathbb{R} \times \mathbb{C})$ given by the sheaf of cochain complexes $\mathcal{E}|_{(\mathbb{R}\times\mathbb{C})\times(\mathbb{C}^4 \setminus \{0\})}$, with BV pairing induced from $\mathcal{E}$, deformed by the interaction

$$S_{BF,\infty}(\mu + NF_{M2}, \nu, \beta, \gamma) + J(\gamma).$$

Remark 1.3. Note that upon expanding the interaction around NF_{M2}, the cubic term in $S_{BF,\infty}$ will contribute a differential which acts on γ and μ by bracketing with NF_{M2}. We accordingly denote this differential $[NF_{M2}, -]$, and we see that the sheaf of cochain complexes underlying $\mathcal{E}_{AdS_4 \times S^7}$ is in fact

$$\left(\mathcal{E}|_{(\mathbb{R}\times\mathbb{C})\times(\mathbb{C}^4\setminus\{0\})}, \delta^{(1)} + [NF_{M2}, -]\right)$$

where $\delta^{(1)}$ denotes the original linearized BRST differential.

Conjecture 1.4. The minimal twist of eleven-dimensional supergravity on the $AdS_4 \times S^7$ background with N units of M2 brane flux supported on S^7 is perturbatively equivalent to $\mathcal{E}^N_{AdS_4 \times S^7}$.

To verify this conjecture, we should directly twist eleven-dimensional supergravity on the $AdS_4 \times S^7$ spacetime. Doing so seems difficult - while it is likely not hard to identify the covariantly constant nilpotent spinors which define the twist, it seems more difficult to establish a perturbative equivalence with our description above. A modification of the pure spinor superfield formalism to symmetric spaces such as cosets for the superconformal group might make such checks more feasible.

In lieu of such, we will instead pursue other consistency checks in the following two sections.

1.2. **The $\mathrm{A}dS_7 \times S^4$ background.** We similarly introduce an analog of the $\mathrm{A}dS_7 \times S^4$ background in our description of the minimal twist of eleven-dimensional supergravity. As before, we begin by viewing our eleven-dimensional manifold $\mathbb{R} \times \mathbb{C}^5$ as

$$\mathrm{Tot}(\mathbb{R} \oplus K_{\mathbb{C}^3}^{1/2} \otimes \mathbb{C}^2 \to \mathbb{C}^3_z)$$

to record weights under natural scaling actions. We once again will use w_a to denote holomorphic fiber coordinates on $K_{\mathbb{C}^3}^{1/2} \otimes \mathbb{C}^2$, and we use t to denote a fiber coordinate on the trivial bundle $\mathbb{R} \to \mathbb{C}^3_z$.

1.2.1. To repeat the procedure in the previous subsection, we begin by discussing how the eleven-dimensional theory couples to M5 branes. Consider a stack of N M5 branes wrapping the zero section $\mathbb{C}^3_z$.

It is natural to consider the nonlocal interaction

$$I_{M5} = N \int_{\mathbb{C}^3_z} \partial_\Omega^{-1} \mu \vee \Omega + \cdots$$

Note that this expression is only nonzero on the component of μ in $\mathrm{PV}^{1,3}$. We argue that this coupling is consistent with expectations from the physical theory and from dimensional reduction.

The twisted field $\mu^{1,3}$ is a component of the Hodge star of the G-flux in the physical theory (§**??**). In the physical theory, M5 branes magnetically couple to the C-field; the coupling involves choosing a primitive for the Hodge star of the G-flux and integrating it over the M5 worldvolume. Our twist contains no fields corresponding to components of such a primitive; hence such a magnetic coupling is reflected in the appearance of ∂_Ω^{-1}.

We may once again justify this coupling by dimensional reduction to IIA supergravity. Reducing on the circle along the directions the M5 branes wrap yields the $SU(4)$ invariant twist of type IIA supergravity on $\mathbb{C}^4 \times \mathbb{R}^2$ with N $D4$ branes wrapping $\mathbb{C}^2 \times \mathbb{R}$.

In [**?**], it is shown that the magnetic coupling of $D4$ branes to the $SU(4)$ twist of IIA is of the form

$$N \int_{\mathbb{C}^2 \times \mathbb{R}} \partial_\Omega^{-1} \mu \vee \Omega_{\mathbb{C}^4} + \cdots .$$

Again, we have only explicitly indicated the first-order piece of the coupling.

1.2.2. The backreacted geometry will be given by a solution to the equations of motion upon deforming the eleven-dimensional action by the interaction $I_{M5}(\mu)$.

Varying the potential $\partial_\Omega^{-1}\mu$, we obtain the following equation of motion involving the field γ:

$$\bar{\partial}\partial\gamma + \partial_\Omega \left(\frac{1}{1-\nu} \mu \right) \wedge \partial\gamma = N \delta_{w_1 = w_2 = t = 0}. \tag{3}$$

Notice that there is an extra derivative compared to the equation of motion arising from varying the field μ. This equation only depends on γ through its field strength $\partial\gamma$.

Varying γ we obtain the equation of motion

$$(\bar{\partial} + \mathrm{d}_{\mathbb{R}})\mu + \partial\gamma\partial\gamma = 0. \tag{4}$$

Again, this only depends on γ through its field strength $\partial\gamma$.

Lemma 1.5. Let

$$F_{M5} = \frac{1}{(2\pi i)^3} \frac{\bar{w}_1 \mathrm{d}\bar{w}_2 \wedge \mathrm{d}t - \bar{w}_2 \mathrm{d}\bar{w}_1 \wedge \mathrm{d}t + t\mathrm{d}\bar{w}_1 \wedge \mathrm{d}\bar{w}_2}{(\|w\|^2 + t^2)^{5/2}} \wedge \mathrm{d}w_1 \wedge \mathrm{d}w_2$$

Then, $\partial\gamma = NF_{M5}$, $\mu = 0$, and $\nu = 0$ satisfies the equations of motion in the presence of a stack of N M5 branes sourced by the term $N\delta_{w_1=w_2=t=0}$:

$$\begin{aligned} \bar{\partial}(NF_{M5}) + \mathrm{d}_{\mathbb{R}}(NF_{M5}) &= N\delta_{w_1=w_2=t=0} \\ (NF_{M5}) \wedge (NF_{M5}) &= 0. \end{aligned}$$

Here, we set all components of the field μ equal to zero (as well as the fields ν, β).

Proof. The first equation,

$$\bar{\partial}F + \mathrm{d}_{\mathbb{R}}F = N\delta_{w_1=w_2=t=0},$$

characterizes the kernel representing N times the residue class for a four-sphere in

$$(\mathbb{C}^2 \times \mathbb{R}) \setminus 0 \simeq S^4 \times \mathbb{R}.$$

That is

$$\oint_{S^4} NF = N$$

for any four-sphere centered at $0 \in \mathbb{C}^2 \times \mathbb{R}$.

The second equation $F \wedge F = 0$ follows by simple type reasons. □

Once again, we summarize our findings in a definition.

Definition 1.6. Let $\mathcal{E}^N_{AdS_7 \times S^4}$ denote the classical BV theory on $\mathrm{Tot}(\mathbb{R} \oplus K^{1/2}_{\mathbb{C}^3} \otimes \mathbb{C}^2 \to \mathbb{C}^3_z) \setminus 0(\mathbb{C}^3_z)$ given by the sheaf of cochain complexes $\mathcal{E}|_{\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2 \setminus \{0\})}$, with BV pairing induced from that on $\mathcal{E}$, deformed by the interaction

$$S_{BF,\infty}(\mu, \nu, \beta, \gamma + N\partial^{-1}F_{M5}) + J(\gamma + N\partial^{-1}F_{M5}).$$

Remark 1.7. Note that both terms in the action only depend on γ through its holomorphic derivatives so the above expression for the action is indeed well-defined.

As before, upon expanding the interactions around NF_{M5}, the cubic terms in both $S_{BF,\infty}$ and J will contribute differentials. From $S_{BF,\infty}$, we get a differential which takes a μ type field to the Schouten bracket $N[F_{M5}, \mu]$ and from J, we get a differential which acts as $\gamma \mapsto NF_{M5} \wedge \partial\gamma$. We accordingly denote this differential $[NF_{M5}, -]$, and the sheaf of cochain complexes underlying $\mathcal{E}_{AdS_7 \times S^4}$ is in fact

$$\left(\mathcal{E}|_{\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2 \setminus \{0\})}, \delta^{(1)} + [NF_{M5}, -]\right)$$

where $\delta^{(1)}$ denotes the original linearized BRST differential.

Conjecture 1.8. The minimal twist of eleven-dimensional supergravity on the $AdS_7 \times S^4$ background with N units of M5 brane flux supported on S^4 is perturbatively equivalent to $\mathcal{E}_{AdS_7 \times S^4}$.

2. Twisted supergravity states

In this section, we pursue a check of conjectures 1.4 and 1.8. We enumerate supergravity states in the twists of the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds and compare them with expressions in the literature enumerating gravitons in these geometries.

We begin by giving a definition of supergravity states suited to our context. The definition is meant to codify the following situation. Suppose we have a theory of gravity defined on a background of the form $AdS_{d+1} \times S^{d'}$ where the conformal boundary of AdS_{d+1} is a d manifold M. We can compactify the theory, retaining all the Kaluza-Klein harmonics, to get a theory on AdS_{d+1}. A supergravity state is traditionally defined to be a solution to linearized equations of motion in this compactified theory with a given boundary value on M^d [Wit98]. Often, this definition is made in situations where the relevant boundary value problem has a unique solution, in which case one may label states by the corresponding boundary values. Moreover, one may think of such boundary values as arising from modifications of a vacuum boundary condition at a point.

2.0.1. For twists of supergravity, we procedurally implement this as follows. First, we wish to describe a model for sphere compactification on twisted AdS backgrounds. As we saw in the last section, our proposal for twists of backgrounds of the form $AdS_{d+1} \times S^{d'}$ involve placing certain local L_∞-algebras on certain manifolds of the form $X = \mathrm{Tot}(V \to Z) \setminus 0(Z)$. Note that Z can be written as a sphere bundle over $\mathbb{R}_{>0} \times Z$ - the sphere compactification of our theory on X is given by the pushforward to $\mathbb{R}_{>0} \times Z$. Such sphere compactifications can be described using a method for computing pushforwards of modules for Lie algebroids associated to foliations [Kor14, Sec. 4.2], [KT75].

The compactified theory will admit a natural boundary condition at $\{\infty\} \times Z \subset \mathbb{R}_{>0} \times Z$, given by certain local L_∞-algebras $\mathcal{L}$. In fact, the presence of extra differentials involving bracketing with the flux sourced by branes wrapping Z will induce an interesting shifted-Poisosn structure on $\mathcal{L}$. Given a local L_∞-algebra $\mathcal{L}$ underlying a perturbative classical field theory, the Chevalley-Eilenberg cochains $\mathrm{C}^\bullet(\mathcal{L})$ carry the structure of a $\mathbb{P}_0$-factorization algebra [CG17], [CG21b]. Associated to a factorization algebra, we can define a space of local operators, which is the costalk at a point of the underlying cosheaf. The sought-after spaces of supergravity states will be a space of local operators associated to $\mathrm{C}^\bullet(\mathcal{L})$.

Remark 2.1. Crucially, we will only implement this procedure at the level of the free limits of the theories defined in 1.2, 1.6 - this will suffice for the purposes of extracting spaces of states and will allow us to forgo discussion involving homotopy transfer of L_∞ structure. As such, the boundary conditions we specify at $\{\infty\} \times Z$ will in fact be abelian local Lie algebras. On the other hand, in subsection 2.3 we will identify boundary conditions for the pushforward of the *interacting* theories that exist after turning off the flux, which yield the same spaces of supergravity states.

Let us now carry out this construction for the theories defined in 1.2, 1.6.

2.1. **States on twisted $AdS_4 \times S^7$.** Following the above prescription, we consider the S^7 bundle

$$\begin{array}{ccc} S^7 & \longrightarrow & \mathrm{Tot}(K_{\mathbb{C}}^{1/4} \otimes \mathbb{C}^4 \to \mathbb{R} \times \mathbb{C}) \setminus 0(\mathbb{R} \times \mathbb{C}) \\ & & \big\downarrow p \\ & & \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{C} \end{array}$$

We wish to describe the free limit of the pushforward $p_* \mathcal{E}^N_{AdS_4 \times S^7}$ as a sheaf of cochain complexes on $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{C}$.

Proposition 2.2. The pushforward is given by the sheaf of complexes $\Omega^\bullet_{\mathbb{R}_{>0} \times \mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{V}^N_{\mathbb{C}})$ on $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{C}$, where $\mathcal{V}^N_{\mathbb{C}}$ is the following dg-vector bundle on $\mathbb{C}$:

$$\begin{array}{ccc} \underline{-} & & \underline{+} \\ H^\bullet(\mathbb{C}^4 \setminus 0, \mathrm{T}) \otimes \mathcal{O} & \xrightarrow{\partial_\Omega^W} & H^\bullet(\mathbb{C}^4 \setminus 0) \otimes \mathcal{O} \\ & \nearrow_{\partial_\Omega^Z} & \\ H^\bullet(\mathbb{C}^4 \setminus 0) \otimes \mathrm{T} & & \\ & & \\ H^\bullet(\mathbb{C}^4 \setminus 0) \otimes \mathcal{O} & \xrightarrow{\partial_Z} & H^\bullet(\mathbb{C}^4 \setminus 0) \otimes \Omega^1 \\ & \searrow^{\partial_W} & \\ & & H^\bullet(\mathbb{C}^4 \setminus 0, \Omega^1) \otimes \mathcal{O} \end{array} \tag{5}$$

where the differentials are as follows:

- The differentials ∂_Ω^Z and ∂_Ω^W are the divergence operators along the base and fiber respectively.
- The differentials ∂_Z and ∂_W are components of the holomorphic deRham differentials along the base and fiber respectively.
- Internal to each summand is a differential given by bracketing with the flux NF_{M2}.

Before proceeding with the proof, we explicate the internal differential in the third item above. Recall that for $\mathcal{F} = \mathcal{O}, \mathrm{T}$, or Ω^1, the cohomology $H^\bullet(\mathbb{C}^4 \setminus 0, \mathcal{F})$ is concentrated in degrees 0 and 3. We will make use of the following dense embeddings.

$$\begin{aligned} \mathbb{C}[w_1, \ldots, w_4] &\hookrightarrow H^0(\mathbb{C}^4 \setminus 0, \mathcal{O}) \\ \mathbb{C}[w_1, \ldots, w_4]\{\partial_{w_a}\} &\hookrightarrow H^0(\mathbb{C}^4 \setminus 0, \mathrm{T}) \\ \mathbb{C}[w_1, \ldots, w_4]\{\mathrm{d}w_a\} &\hookrightarrow H^0(\mathbb{C}^4 \setminus 0, \Omega^1) \end{aligned}$$

and

$$\begin{aligned} (w_1 \cdots w_4)^{-1} \mathbb{C}[w_1^{-1}, \ldots, w_4^{-1}] &\hookrightarrow H^3(\mathbb{C}^4 \setminus 0, \mathcal{O}) \\ (w_1 \cdots w_4)^{-1} \mathbb{C}[w_1^{-1}, \ldots, w_4^{-1}]\{\partial_{w_a}\} &\hookrightarrow H^3(\mathbb{C}^4 \setminus 0, \mathrm{T}) \\ (w_1 \cdots w_4)^{-1} \mathbb{C}[w_1^{-1}, \ldots, w_4^{-1}]\{\mathrm{d}w_a\} &\hookrightarrow H^3(\mathbb{C}^4 \setminus 0, \Omega^1). \end{aligned}$$

The flux NF_{M2} of lemma 1.1 is represented by the section $N(w_1 \cdots w_4)^{-1} \partial_z \in H^3(\mathbb{C}^4 \setminus 0) \otimes \mathrm{T}$ and acts on each summand by Lie derivative along z and multiplying by $(w_1 \cdots w_4)^{-1}$

Proof. To compute the pushforward of the kind of local L_∞-algebra associated to a holomorphic-topological field theory along a map of THF manifolds, we can use a result of [Kor14, Sec. 4.2], [KT75] for describing direct images of Lie algebroid modules along maps of Lie algebroids. Schematically, if we have a proper submersion $p : X \to Z$ of THF manifolds, and a sheaf of complexes $\mathcal{E}$ on X which resolves sections of some bundle flat along the leaves of the foliation on X, then the pushforward $p_*\mathcal{E}$ has a model as a partially flat bundle on Z. The fiber of this partially flat bundle on Z is the THF cohomology of the fiber of p, with respect to the induced foliation on the fiber, with coefficients in the pullback of the bundle to the fiber. In the case of a holomorphic submersion, this recovers the usual construction of the Gauss-Manin connection for instance.

In our case, the pushforward $p_*\mathcal{E}_{AdS_4\times S^7}$ is a complex of $\Omega^\bullet_{\mathbb{R}_{>0}} \otimes \Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}$-modules given by

$$
\begin{tikzcd}
\underline{-} & \underline{\pm} \\
H^\bullet_{\mathrm{THF}}(S^7, \mathrm{T}) \otimes \mathcal{O} \arrow[r, "\partial^W_\Omega"] & H^\bullet_{\mathrm{THF}}(S^7) \otimes \mathcal{O} \\
H^\bullet_{\mathrm{THF}}(S^7) \otimes \mathrm{T} \arrow[ur, "\partial^Z_\Omega"'] & \\
H^\bullet_{\mathrm{THF}}(S^7) \otimes \mathcal{O} \arrow[r, "\partial_Z"] \arrow[dr, "\partial_W"'] & H^\bullet_{\mathrm{THF}}(S^7) \otimes \Omega^1 \\
& H^\bullet_{\mathrm{THF}}(S^7, \Omega^1) \otimes \mathcal{O}
\end{tikzcd} \tag{6}
$$

Then the proposition follows from the fact that the map $S^7 \to \mathbb{C}^4 \setminus \{0\}$ induces an isomorphism in THF cohomology. $\square$

2.1.1. Continuing, we wish to understand a natural boundary condition we can place on the fields at $\{\infty\} \times \mathbb{R} \times \mathbb{C} \subset \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{C}$. Such a boundary condition is specified by a Lagrangian in the phase space. The restriction $(p_*\mathcal{E}^N_{AdS_4\times S^7})|_{\{\infty\}\times\mathbb{C}}$ describes the phase space and is easily seen to be $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{V}^N_{\mathbb{C}})$. We wish to describe a shifted Lagrangian therein.

We begin by rewriting the phase space in the following form. Note that there is a higher residue pairing

$$H^0(\mathbb{C}^4 \setminus 0) \otimes H^3(\mathbb{C}^4 \setminus 0) \to \mathbb{C};$$

together with the natural pairings between T, Ω^1 and the integration pairing along $\mathbb{R} \times \mathbb{C}_z$, this equips $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{V}^N_{\mathbb{C}})$ with a local even-shifted symplectic structure.

Together with this shifted symplectic structure, we can identify the phase space with a twisted cotangent bundle

$$
\Omega^\bullet_{\mathbb{R}}\otimes\Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{V}^N_{\mathbb{C}}) = T^*_\Pi \left(
\begin{array}{ccc}
- & & \pm \\
\mathbb{C}[w_1,\ldots,w_4]\{\partial_{w_a}\}\otimes\Omega^\bullet_{\mathbb{R}}\otimes\Omega^{0,\bullet}_{\mathbb{C}} & \xrightarrow{\partial^W_\Omega} & \mathbb{C}[w_1,\ldots,w_4]\otimes\Omega^\bullet_{\mathbb{R}}\otimes\Omega^{0,\bullet}_{\mathbb{C}} \\
\mathbb{C}[w_1,\ldots,w_4]\otimes\Omega^\bullet_{\mathbb{R}}\otimes\Omega^{0,\bullet}_{\mathbb{C}}(\mathrm{T}) & \xrightarrow{\partial^Z_\Omega} & \\
\mathbb{C}[w_1,\ldots,w_4]\otimes\Omega^\bullet_{\mathbb{R}}\otimes\Omega^{0,\bullet}_{\mathbb{C}} & \xrightarrow{\partial_Z} & \mathbb{C}[w_1,\ldots,w_4]\otimes\Omega^\bullet_{\mathbb{R}}\otimes\Omega^{0,\bullet}_{\mathbb{C}}(\Omega^1) \\
 & \xrightarrow{\partial_W} & \mathbb{C}[w_1,\ldots,w_4]\{\mathrm{d}w_a\}\otimes\Omega^\bullet_{\mathbb{R}}\otimes\Omega^{0,\bullet}_{\mathbb{C}}
\end{array}
\right)
$$

For now, the subscript of Π is just meant to indicate that the extra differential given by bracketing with the flux constitutes a deformation of the cotangent bundle.

Thus we see that a natural Lagrangian in the phase space is given by the sheaf of cochain complexes $\Omega^\bullet_{\mathbb{R}}\otimes\Omega^\bullet_{\mathbb{C}}(\mathcal{L}^N_{AdS_4\times S^7})$ where $\mathcal{L}^N_{AdS_4\times S^7}$ is the following dg-vector bundle

(7)

$$
\begin{array}{ccc}
\pm & & - \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\ldots,w_4^{-1}]\{\partial_{w_a}\}\otimes\mathcal{O} & \xrightarrow{\partial^W_\Omega} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\ldots,w_4^{-1}]\otimes\mathcal{O} \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\ldots,w_4^{-1}]\otimes\mathrm{T} & \xrightarrow{\partial^Z_\Omega} & \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\ldots,w_4^{-1}]\otimes\mathcal{O} & \xrightarrow{\partial_Z} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\ldots,w_4^{-1}]\otimes\Omega^1 \\
 & \xrightarrow{\partial_W} & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\ldots,w_4^{-1}]\{\mathrm{d}w_a\}\otimes\mathcal{O}
\end{array}
$$

Remark 2.3. The extra term in the differential in fact equips $\Omega^\bullet_{\mathbb{R}}\otimes\Omega^\bullet_{\mathbb{C}}(\mathcal{L}^N_{AdS_4\times S^7})$ with an interesting N dependent odd-shifted Poisson structure. This will not play a significant role in the narrative of this thesis, but we flag it for later commentary nonetheless.

Definition 2.4. The *space of supergravity states* on twisted $AdS_4\times S^7$ is given by the costalk at zero of the factorization algebra on $\mathbb{R}\times\mathbb{C}$ given by $\mathrm{C}^\bullet\left(\Pi\Omega^\bullet_{\mathbb{R}}\otimes\Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^N_{AdS_4\times S^7})\right)$

2.1.2. Let us explicate definition 2.4 a bit. We wish to first understand the space of supergravity states as a cochain complex.

Lemma 2.5. The space of supergravity states on twisted $AdS_4 \times S^7$ is the symmetric algebra $\mathrm{Sym}(\mathcal{H}_{AdS_4\times S^7})$ where $\mathcal{H}_{AdS_4\times S^7}$ is the cochain complex

$$
\begin{tikzcd}[row sep=small]
\underline{-} & \underline{+} \\
\mathbb{C}[w_1,\dots,w_4]\{\partial_{w_a}\}\otimes\mathbb{C}[\partial_z]\delta_{z=0} \arrow[r, "\partial_\Omega^W"] & \mathbb{C}[w_1,\dots,w_4]\otimes\mathbb{C}[\partial_z]\delta_{z=0} \\
\mathbb{C}[w_1,\dots,w_4]\partial_z\otimes\mathbb{C}[\partial_z]\delta_{z=0} \arrow[ur, "\partial_\Omega^Z"'] & \\
\mathbb{C}[w_1,\dots,w_4]\otimes\mathbb{C}[\partial_z]\delta_{z=0} \arrow[r, "\partial_Z"] \arrow[dr, "\partial_W"'] & \mathbb{C}[w_1,\dots,w_4]\mathrm{d}z\otimes\mathbb{C}[\partial_z]\delta_{z=0} \\
& \mathbb{C}[w_1,\dots,w_4]\{\mathrm{d}w_a\}\otimes\mathbb{C}[\partial_z]\delta_{z=0}
\end{tikzcd}
\tag{8}
$$

Proof. To compute the costalk at 0 of a factorization algebra, we consider a nested sequence of open sets containing the origin and compute the limit of the value of the factorization algebra over this sequence. Consider open sets in $\mathbb{R}\times\mathbb{C}$ of the form $I \times D$ where $I \subset \mathbb{R}$ is an interval and $D \subset \mathbb{C}$ is a disc. The sections of the sheaf of cochain complexes $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^N_{AdS_4\times S^7})$ over this open set is given by

$$
\begin{tikzcd}[row sep=small]
\underline{+} & \underline{-} \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\{\partial_{w_a}\}\otimes\mathcal{O}(D) \arrow[r, "\partial_\Omega^W"] & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\otimes\mathcal{O}(D) \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\otimes\Gamma(D,\mathrm{T}) \arrow[ur, "\partial_\Omega^Z"'] & \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\otimes\mathcal{O}(D) \arrow[r, "\partial_Z"] \arrow[dr, "\partial_W"'] & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\otimes\Gamma(D,\Omega^1) \\
& (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\{\mathrm{d}w_a\}\otimes\mathcal{O}(D)
\end{tikzcd}
\tag{9}
$$

Now note that there is a canonical map $\mathcal{O}(D) \to \mathbb{C}[\![z]\!]$ given by taking the Taylor expansion at the origin. Given a functional on the fields that only depends on the value of their derivatives at the origin, then the functional must factor through the Taylor expansion. Therefore, we have that the costalk of our factorization algebra is given by

$$
\mathrm{C}^\bullet\left(
\begin{tikzcd}[row sep=small]
\underline{-} & \underline{+} \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\{\partial_{w_a}\}\otimes\mathbb{C}[\![z]\!] \arrow[r, "\partial_\Omega^W"] & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\otimes\mathbb{C}[\![z]\!] \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\otimes\mathbb{C}[\![z]\!]\partial_z \arrow[ur, "\partial_\Omega^Z"'] & \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\otimes\mathbb{C}[\![z]\!] \arrow[r, "\partial_Z"] \arrow[dr, "\partial_W"'] & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\otimes\mathbb{C}[\![z]\!]\mathrm{d}z \\
& (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]\{\mathrm{d}w_a\}\otimes\mathbb{C}[\![z]\!]
\end{tikzcd}
\right)
\tag{10}
$$

The definition of the Chevalley-Eilenberg complex above involves the continuous linear dual of a chain complex of topological vector spaces. The duals of each of the tensor factors are as follows:

	z	w_1	w_2	w_3	w_4
t_1	0	1	0	0	-1
t_2	0	0	1	0	-1
t_3	0	0	0	1	-1
q	-1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

TABLE 1. Fugacities for the fields of the holomorphic twist of eleven-dimensional supergravity for the geometry $\mathbb{R}\times\mathbb{C}^5\setminus(\mathbb{R}\times\mathbb{C}^2)$.

- There is an isomorphism between the continuous linear dual of $\mathbb{C}[\![z]\!]$ and $\mathbb{C}[\partial_z]\delta_{z=0}$: every continuous linear functional on $\mathbb{C}[\![z]\!]$ is given by a derivative of the δ-function at zero.
- The higher residue pairing lets us identify the continuous linear dual of $(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\dots,w_4^{-1}]$ with $\mathbb{C}[w_1,\cdots,w_4]$.
- The tensor factors involving one-forms and vector fields are dual to each other in the obvious way.

Thus, we see that $\mathcal{H}_{AdS_4\times S^7}$ is indeed as claimed. $\square$

2.1.3. We proceed to computing a local character for the factorization algebra defined in 2.4; thanks to lemma 2.5 we can compute this as a character of $\mathrm{Sym}(\mathcal{H}_{AdS_4\times S^7})$.

We first observe the following action of $\mathfrak{sl}(4)\oplus\mathfrak{sl}(2)$ on $\mathcal{H}_{AdS_4\times S^7}$. The $\mathfrak{sl}(4)$ summand acts on the tensor factor $\mathbb{C}[w_1,w_2,w_3,w_4]$ in the obvious way - it's the symmetric algebra on the fundamental representation. The $\mathfrak{sl}(2)$ summand acts by bracketing with the vector fields

$$\frac{\partial}{\partial z},\quad z\frac{\partial}{\partial z}-\frac{1}{4}\sum_{a=1}^{4}w_a\frac{\partial}{\partial w_a},\quad z\left(z\frac{\partial}{\partial z}-\frac{1}{2}\sum_{a=1}^{4}w_a\frac{\partial}{\partial w_a}\right).$$

We choose the following explicit generators for a choice of Cartan as follows:

- t_1,t_2,t_3 denote generators for the Cartan of $\mathfrak{sl}_4$ which is spanned by the vector fields

$$h_1=w_1\frac{\partial}{\partial w_1}-w_4\frac{\partial}{\partial w_4},\qquad h_2=w_2\frac{\partial}{\partial w_2}-w_4\frac{\partial}{\partial w_4},\qquad h_3=w_3\frac{\partial}{\partial w_3}-w_4\frac{\partial}{\partial w_4}$$

- q denotes a generator for the Cartan of $\mathfrak{sl}_2$ which is spanned by the vector field

$$\Delta=\frac{1}{4}\sum_{a=1}^{4}w_a\frac{\partial}{\partial w_a}-z\frac{\partial}{\partial z}.$$

The weights of $\mathcal{H}_{AdS_4\times S^7}$ with respect to the generators of this Cartan subalgebra are entirely determined by the weights of the holomorphic coordinates $z,w_a,a=1,\cdots,4$, which we summarize in table 1

With this in hand, we wish to compute the character of the space of supergravity states $\mathrm{Sym}(\mathcal{H}_{AdS_4\times S^7})$. Note that the space of supergravity states was defined to be a symmetric algebra - therefore its character can be computed using plethystic exponentiation of the character of $\mathcal{H}_{AdS_4\times S^7}$ - the latter may be referred to as a *single particle index* and is defined by

$$f_{AdS_4\times S^7}(t_1,t_2,t_3,q)=\mathrm{Tr}_{\mathcal{H}_{AdS_4\times S^7}}(-1)^F q^{\Delta}t_1^{h_1}t_2^{h_2}t_3^{h_3}.$$

Proposition 2.6. The single particle index of the space of supergravity states $\mathcal{H}_{AdS_4 \times S^7}$ is given by

$$f_{AdS_4 \times S^7}(t_1, t_2, t_3, q) = \frac{q \begin{pmatrix} q^{1/4}(t_1 + t_2 + t_3 + t_1^{-1}t_2^{-1}t_3^{-1}) + q^{-1} \\ -q^{-1/4}(t_1^{-1} + t_2^{-1} + t_3^{-1} + t_1t_2t_3) - q \end{pmatrix}}{(1-q)(1-q^{1/4}t_1)(1-q^{1/4}t_2)(1-q^{1/4}t_3)(1-q^{1/4}t_1^{-1}t_2^{-1}t_3^{-1})}$$

Proof. The two summands not involving holomorphic vector fields or forms appear with opposite parity, so their contributions to the character cancel. For the remaining summands, It is straightforward to compute the character of each tensor factor:

- The factor $\mathbb{C}[\partial_z]\delta_{z=0}$ contributes a factor of
$$\frac{q}{1-q}.$$
- The tensor factor $\mathbb{C}[w_1, \cdots, w_4]$ contributes a factor of
$$\frac{1}{(1-q^{1/4}t_1)(1-q^{1/4}t_2)(1-q^{1/4}t_3)(1-q^{1/4}t_1^{-1}t_2^{-1}t_3^{-1})}.$$
- The tensor factors involving vector fields and forms contribute a factor of

$$-q^{-1/4}(t_1^{-1} + t_2^{-1} + t_3^{-1} + t_1t_2t_3) + q^{1/4}(t_1 + t_2 + t_3 + t_1^{-1}t_2^{-1}t_3^{-1}) - q + q^{-1}.$$

□

2.1.4. Upon subtracting one and making the substitution

$$q = x^2, \qquad t_1 = (y_2y_3)^{1/2}/y_1^{1/2}, \qquad t_2 = (y_1y_3)^{1/2}/y_2^{1/2}, \qquad t_3 = (y_1y_2)^{1/2}/y_3^{1/2}$$

this character matches the expression in [BBMR08, Eq. 2.17]. The discrepancy of one is accounted for by a zero mode that we have introduced in writing our theory in such a way that M2 branes couple electrically. Indeed, this is an avatar of the central element in the central extension $\widehat{E(5|10)}$ of section **??**.

2.2. **States on twisted $AdS_7 \times S^4$.** Next, we consider the sphere reduction of $\mathcal{E}^N_{AdS_7 \times S^4}$ As before, we consider the S^4 bundle

$$\begin{array}{ccc} S^4 & \longrightarrow & \mathrm{Tot}(\mathbb{R} \oplus K^{1/2} \otimes \mathbb{C}^2 \to \mathbb{C}^3) \setminus 0(\mathbb{C}^3) \\ & & \big\downarrow p \\ & & \mathbb{R}_{>0} \times \mathbb{C}^3 \end{array}$$

We wish to describe the free limit of the pushforward $p_*\mathcal{E}_{AdS_7 \times S^4}$ as a sheaf of cochain complexes on $\mathbb{R}_{>0} \times \mathbb{C}^3$.

Proposition 2.7. The pushforward $p_*\mathcal{E}^N_{AdS_7 \times S^4}$ is given by the sheaf of cochain complexes $\Omega^\bullet_{\mathbb{R}_{>0}} \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{V}^N_{\mathbb{C}^3})$ where $\mathcal{V}^N_{\mathbb{C}^3}$ is the following dg-vector bundle on $\mathbb{C}^3$:

$$
\begin{tikzcd}[column sep=small]
- & \pm \\
H^\bullet_{\mathrm{THF}}\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0, \mathrm{T}\right)\otimes\mathcal{O} \arrow[r, "\partial_\Omega^W"] \arrow[ddr, dotted] & H^\bullet_{\mathrm{THF}}\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)\otimes\mathcal{O} \\
H^\bullet_{\mathrm{THF}}\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)\otimes\mathrm{T} \arrow[ur, "\partial_\Omega^Z"] & \\
H^\bullet_{\mathrm{THF}}\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)\otimes\mathcal{O} \arrow[r, "\partial_Z"] \arrow[dr, "\partial_W"'] & H^\bullet_{\mathrm{THF}}\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)\otimes\Omega^1 \\
& H^\bullet_{\mathrm{THF}}\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0, \Omega^1\right)\otimes\mathcal{O} \arrow[uul, dotted]
\end{tikzcd}
\tag{11}
$$

where the differentials are as follows:

- The differentials ∂_Ω^Z and ∂_Ω^W are the divergence operators along the base and fiber respectively.
- The differentials ∂_Z and ∂_W are components of the holomorphic deRham differentials along the base and fiber respectively.
- The dotted arrows are N dependent differentials roughly given by bracketing with the flux, and are explicated below.

Before proceeding with the proof, it will again be useful to explicate the internal differentials above. The THF cohomology of $(\mathbb{R}\times\mathbb{C}^2)\setminus 0$ possibly with coefficients in a sheaf $\mathcal{F}$ equipped with a partial flat connection along the leaves of the THF can be described as the cohomology of the following quotient of the deRham complex

$$\Omega^\bullet\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)/(\mathrm{d}w_1, \mathrm{d}w_2).$$

The cohomology is accordingly concentrated in degrees zero and two. We will make use of the dense embeddings

$$
\begin{aligned}
\mathbb{C}[w_1,w_2] &\hookrightarrow H^0\left(\Omega^\bullet\left(\mathbb{C}^2_w\times\mathbb{R}\setminus 0\right)/(\mathrm{d}w_1,\mathrm{d}w_2)\right)\\
w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}] &\hookrightarrow H^2\left(\Omega^\bullet\left(\mathbb{C}^2_w\times\mathbb{R}\setminus 0\right)/(\mathrm{d}w_1,\mathrm{d}w_2)\right)
\end{aligned}
$$

along with the analogous versions with coefficients in the sheaf $\mathcal{F}=\Omega^1, T$.

The flux NF_{M5} is then represented by a class of the form $N(w_1w_2)^{-1}\mathrm{d}w_1\mathrm{d}w_2\in(w_1w_2)^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}]\otimes\Omega^2$. The dotted differentials in equation 11 are then explicitly given by maps

$$
\begin{gathered}
\mathbb{C}[w_1,w_2]\{\partial_{w_a}\}\otimes\mathcal{O}\to(w_1w_2)^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}]\{\mathrm{d}w_a\}\otimes\mathcal{O}\\
\mathbb{C}[w_1,w_2]\otimes\Omega^1\to(w_1w_2)^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}]\otimes\mathrm{T}
\end{gathered}
$$

where the first map is given by contracting with $\mathrm{d}w_1\mathrm{d}w_2$ and multiplying by $(w_1w_2)^{-1}$, while the second map is given by applying ∂_Z, wedging with the $\mathrm{d}w_1\mathrm{d}w_2$ and contracting with the inverse of the holomorphic volume form on $\mathbb{C}^5$ to get a vector field along $\mathbb{C}^3$.

Proof. The proof is exactly analogous to that of proposition 2.2 . Using results from [Kor14, Sec. 4.2], [KT75] to compute the pushforward, we find a sheaf of $\Omega^\bullet_{\mathbb{R}_{>0}}\otimes\Omega^{0,\bullet}_{\mathbb{C}^3}$-modules whose sections have a tensor factor given by the THF cohomology of S^4. Next, we use the isomorphism in THF cohomology afforded by the deformation retraction of $(\mathbb{R}\times\mathbb{C}^2)\setminus 0$ onto S^4. $\square$

2.2.1. As in the previous subsection, we ask for a boundary condition we can place on the fields at $\{\infty\}\times\mathbb{C}^3 \subset \mathbb{R}_{>0}\times\mathbb{C}^3$. The phase space, given by $(p_*\mathcal{E}^N_{AdS_7\times S^4})|_{\{\infty\}\times\mathbb{C}^3}$, is seen to be $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{V}_{\mathbb{C}^3})$, and we search for a shifted Lagrangian therein.

Completely analogously to before, we may rewrite the phase space as a shifted cotangent bundle. The higher residue pairing, the natural pairing between T, Ω^1, and the integration pairing along $\mathbb{C}^3$, all conspire to give the phase space an even-shifted symplectic structure. Together with this, we may ionce again identify the phase space with a twisted cotangent bundle

$$\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{V}_{\mathbb{C}^3}) = T^*_\Pi \left(\begin{tikzcd} \underline{-} & \pm \\ \mathbb{C}[w_1,w_2]\{\partial_{w_a}\}\otimes\Omega^{0,\bullet}_{\mathbb{C}^3} \arrow[r,"\partial^W_\Omega"] & \mathbb{C}[w_1,w_2]\otimes\Omega^{0,\bullet}_{\mathbb{C}^3} \\ \mathbb{C}[w_1,w_2]\otimes\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathrm{T}) \arrow[ur,"\partial^Z_\Omega"'] & \\ \mathbb{C}[w_1,w_2]\otimes\Omega^{0,\bullet}_{\mathbb{C}^3} \arrow[r,"\partial_Z"] \arrow[dr,"\partial_W"'] & \mathbb{C}[w_1,w_2]\otimes\Omega^{0,\bullet}_{\mathbb{C}^3}(\Omega^1) \\ & \mathbb{C}[w_1,w_2]\{\mathrm{d}w_a\}\otimes\Omega^{0,\bullet}_{\mathbb{C}^3} \end{tikzcd} \right)$$

Exactly analogously to the case of $AdS_4 \times S^7$, Π here denotes the extra N-dependent differential induced by bracketing with the flux, which deforms the cotangent bundle.

A natural Lagrangian in the phase space is thus given by $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}_{AdS_7\times S^4})$ where $\mathcal{L}_{AdS_7\times S^4}$ is the dg-vector bundle given by

$$\begin{tikzcd} \underline{-} & \pm \\ (w_1w_2)^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}]\{\partial_{w_a}\}\otimes\mathcal{O} \arrow[r,"\partial^W_\Omega"] & (w_1w_2)^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}]\otimes\mathcal{O} \\ (w_1w_2)^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}]\otimes\mathrm{T} \arrow[ur,"\partial^Z_\Omega"'] & \\ (w_1w_2)^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}]\otimes\mathcal{O} \arrow[r,"\partial_Z"] \arrow[dr,"\partial_W"'] & (w_1w_2)^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}]\otimes\Omega^1 \\ & (w_1w_2)^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}]\{\mathrm{d}w_a\}\otimes\mathcal{O} \end{tikzcd} \tag{12}$$

Definition 2.8. The space of supergravity states on twisted $AdS_7\times S^4$ is given by the costalk at zero of the factorization algebra on $\mathbb{C}^3$ given by $\mathrm{C}^\bullet\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^N_{AdS_7\times S^4})\right)$.

2.2.2. An argument exactly analogous to the proof of lemma 2.5 gives us the following

Lemma 2.9. The space of supergravity states on twisted $AdS_7\times S^4$ is the symmetric algebra $\mathrm{Sym}(\mathcal{H}_{AdS_7\times S^4})$ where $\mathcal{H}_{AdS_7\times S^4}$ is given by the cochain complex

$$
\begin{array}{ccc}
+ & & - \\
\hline
\mathbb{C}[w_1, w_2]\{\partial_{w_a}\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0} & \xrightarrow{\partial_\Omega^W} & \mathbb{C}[w_1, w_2] \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0} \\
 & \nearrow{\scriptstyle \partial_\Omega^Z} & \\
\mathbb{C}[w_1, w_2]\{\partial_{z_i}\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0} & & \\
 & & \\
\mathbb{C}[w_1, w_2] \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0} & \xrightarrow{\partial_Z} & \mathbb{C}[w_1, w_2]\{\mathrm{d}z_a\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0} \\
 & \searrow{\scriptstyle \partial_W} & \\
 & & \mathbb{C}[w_1, w_2]\{\mathrm{d}w_a\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0}
\end{array}
\tag{13}
$$

2.2.3. As before, we may compute the local character of the factorization algebra defined in 2.8 as a character of $\mathrm{Sym}(\mathcal{H}_{AdS_7\times S^4})$.

We will use an action of $\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathfrak{gl}(1)$ on $\mathcal{H}_{AdS_7\times S^4}$ which we may explicitly realize as follows:

- The subalgebra $\mathfrak{sl}(3)$ acts as vector fields rotating the plane $\mathbb{C}^3_z$

$$\sum_{ij} A_{ij} z_i \frac{\partial}{\partial z_j} \quad (A_{ij}) \in \mathfrak{sl}(3). \tag{14}$$

- The subalgebra $\mathfrak{sl}(2)$ acts by the triple of vector fields

$$w_1 \frac{\partial}{\partial w_2}, \quad w_2 \frac{\partial}{\partial w_1}, \quad \frac{1}{2}\left(w_1 \frac{\partial}{\partial w_1} - w_2 \frac{\partial}{\partial w_2}\right). \tag{15}$$

- The subalgebra $\mathfrak{gl}(1)$ acts as the vector field

$$\Delta = \sum_{i=1}^{3} z_i \frac{\partial}{\partial z_i} - \frac{3}{2}\sum_{a=1}^{2} w_a \frac{\partial}{\partial w_a}. \tag{16}$$

The character will be a function on a Cartan in $\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathfrak{gl}(1)$; we choose one whose generators are given as follows.

- t_1, t_2 denote generators for the Cartan of $\mathfrak{sl}(3)$ which is spanned by the vector fields

$$h_1 = z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}, \quad h_2 = z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3}. \tag{17}$$

- r denotes a generator for the Cartan of a $\mathfrak{sl}(2)$ which is generated by the element

$$h = \frac{1}{2}\left(w_1 \frac{\partial}{\partial w_1} - w_2 \frac{\partial}{\partial w_2}\right). \tag{18}$$

- q denotes a generator for the Cartan of the $\mathfrak{gl}(1)$ which is generated by the element Δ from equation (16).

The weights of twisted supergravity states with respect to the generators of the Cartan subalgebra above are completely determined by the weights of the holomorphic coordinates w_1, w_2, z_1, z_2, z_3, which we summarize in table 2.

We enumerate single particle supergravity states via computing the super trace of the operator $q^\Delta t_1^{h_1} t_2^{h_2} r^h$ acting on $\mathcal{H}_{AdS_7\times S^4}$:

$$f_{AdS_7\times S^4}(t_1, t_2, r, q) = \mathrm{Tr}_{\mathcal{H}_{AdS_7\times S^4}} (-1)^F q^\Delta t_1^{h_1} t_2^{h_2} r^h. \tag{19}$$

	z_1	z_2	z_3	w_1	w_2
t_1	1	0	-1	0	0
t_2	0	1	-1	0	0
r	0	0	0	1	-1
q	-1	-1	-1	$\frac{3}{2}$	$\frac{3}{2}$

TABLE 2. Fugacities for the fields of the holomorphic twist of eleven-dimensional supergravity for the geometry $\mathbb{R} \times \mathbb{C}^5 \setminus \mathbb{C}^3$.

Proposition 2.10. The single particle index of the space of twisted supergravity states $\mathcal{H}_{AdS_7 \times S^4}$ is given by the following expression

$$f_{AdS_7 \times S^4}(t_1, t_2, r, q) = \frac{q^4(t_1^{-1} + t_1 t_2^{-1} + t_2) - q^2(t_1 + t_1^{-1} t_2 + t_2^{-1}) + (q^{3/2} - q^{9/2})(r + r^{-1})}{(1 - t_1^{-1} q)(1 - t_2 q)(1 - t_1 t_2^{-1} q)(1 - r q^{3/2})(1 - r^{-1} q^{3/2})}. \tag{20}$$

The full (multiparticle) index is defined to be the plethystic exponential

$$\mathrm{PExp}\left[f_{AdS_7 \times S^4}(t_1, t_2, r, q)\right]. \tag{21}$$

Proof. As before, the two summands not involving holomorphic vector fields or forms appear with opposite parity, so their contributions to the character will cancel. For the remaining summands, it is again straightforward to compute the character of each tensor factor.

- The factor $\mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0}$ contributes a factor of
$$\frac{q^3}{(1 - t_1^{-1} q)(1 - t_2 q)(1 - t_1 t_2^{-1} q)}.$$
- The factor of $\mathbb{C}[w_1, w_2]$ contributes a factor of
$$\frac{1}{(1 - r q^{3/2})(1 - r^{-1} q^{3/2})}.$$
- The tensor factors involving vector fields and forms contribute a factor of
$$q(t_1^{-1} + t_1 t_2^{-1} + t_2) - q^{-1}(t_1 + t_1^{-1} t_2 + t_2^{-1}) + (q^{-3/2} - q^{3/2})(r + r^{-1})$$

□

2.2.4. To simplify the form of this index we can introduce a different parametrization of the Cartan of $\mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1)$. First, we can parameterize the Cartan of $\mathfrak{sl}(3)$ by the vector fields

$$-(\log y_1) z_1 \frac{\partial}{\partial z_1} - (\log y_2) z_2 \frac{\partial}{\partial z_2} - (\log y_3) z_3 \frac{\partial}{\partial z_3}. \tag{22}$$

where y_1, y_2, y_3 are parameters which satisfy the single constraint

$$y_1 y_2 y_3 = 1. \tag{23}$$

In terms of the variables t_1, t_2 used above we have

$$y_1 = t_1^{-1}, \quad y_2 = t_1 t_2^{-1}, \quad y_3 = t_2. \tag{24}$$

Second, we can parametrize the Cartan of the remaining subalgebra $\mathfrak{sl}(2) \oplus \mathfrak{gl}(1)$ by the two vector fields

$$\tilde{h} = h + \frac{1}{2}\Delta \quad \text{and} \quad \Delta \tag{25}$$

where Δ is as in equation (16) and h is as in (18). We denote by y the generator of the Cartan corresponding to the vector field $\tilde{h}$ and by q (as above) the generator corresponding to Δ. In terms of the variable r used above we have

$$y = q^{1/2} r. \tag{26}$$

Using the parametrization of the Cartan given by the variables y_i, y, Δ we obtain the equivalent expression for the index (20) as

$$f_{AdS_7 \times S^4}(y_i, y, q) = \frac{q^4(y_1+y_2+y_3) - q^2(y_1^{-1}+y_2^{-1}+y_3^{-1}) + (1-q^3)(yq+y^{-1}q^2)}{(1-y_1q)(1-y_2q)(1-y_3q)(1-yq)(1-y^{-1}q^2)}, \tag{27}$$

We note that this matches exactly with the index computed in [KKKL13, Eq. (3.23)] with the change of variables.

Our formula (20) also matches with [BBMR08, Eq. (3.24)] where we use the change of variables

$$q = x^4, \quad t_1 = y_2, \quad t_2 = y_1, \quad r^2 = z. \tag{28}$$

(Notice the variables y_1, y_2 used in [BBMR08] differ from the variables we introduced in (22).)

2.2.5. We consider the specialization of this index

$$q = r^2, t_2 = 1 \tag{29}$$

which is known as the Schur limit. Applying this limit to (20) yields the plethystic exponential of the following single particle index

$$f_{AdS_7 \times S^4}(q, t_1, t_2 = 1, r = q^{1/2}) = \frac{q}{(1-q)^2}$$

This plethystic exponential yields the MacMahon function, which is the character of the vacuum module of the $W_{1+\infty}$-algebra. We will revisit this observation in section 6.

2.3. **Transverse boundary conditions.** In the previous subsections, we discussed boundary conditions in the phase spaces of the sphere compactifications $p_*\mathcal{E}^N_{AdS_4 \times S^7}$ and $p_*\mathcal{E}^N_{AdS_7 \times S^4}$ viewed as free theories, that exist for generic values of N. However, there are distinguished boundary conditions that exist for $N = 0$ that we will use in the sequel. Moreover, these distinguished boundary conditions are in fact boundary conditions for the interacting theory.

Indeed, recall that in equations (6.4.4), (6.4.4) we wrote the phase spaces as twisted cotangent bundles, where the nontrivial Poisson tensor was induced by the terms in the differential coming from bracketing with the flux. When we specialize $N = 0$, this Poisson tensor vanishes, and there is an additional Lagrangian given by the zero section.

Explicitly these Lagrangians in the phase space are described as follows.

- Let $\mathcal{L}^{r=0}_{AdS_4\times S^7}$ denote the following dg-vector bundle on $\mathbb{C}$

(30)
$$\begin{tikzcd}
- & \pm \\
\mathbb{C}[w_1,\dots,w_4]\{\partial_{w_a}\}\otimes\mathcal{O} \arrow[r, "\partial^W_\Omega"] & \mathbb{C}[w_1,\dots,w_4]\otimes\mathcal{O} \\
\mathbb{C}[w_1,\dots,w_4]\otimes\mathrm{T} \arrow[ur, "\partial^Z_\Omega"'] & \\
\mathbb{C}[w_1,\dots,w_4]\otimes\mathcal{O} \arrow[r, "\partial_Z"] \arrow[dr, "\partial_W"'] & \mathbb{C}[w_1,\dots,w_4]\otimes\Omega^1 \\
& \mathbb{C}[w_1,\dots,w_4]\{\mathrm{d}w_a\}\otimes\mathcal{O}
\end{tikzcd}$$

The desired Lagrangian in $(p_*\mathcal{E}^0_{AdS_4\times S^7})|_{\{\infty\}\times\mathbb{R}\times\mathbb{C}}$ is given by $\Omega^\bullet_{\mathbb{R}}\otimes\Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4\times S^7})$.

- Let $\mathcal{L}^{r=0}_{AdS_7\times S^4}$ denote the following dg-vector bundle on $\mathbb{C}^3$

(31)
$$\begin{tikzcd}
\pm & - \\
\mathbb{C}[w_1,w_2]\{\partial_{w_a}\}\otimes\mathcal{O} \arrow[r, "\partial^W_\Omega"] & \mathbb{C}[w_1,w_2]\otimes\mathcal{O} \\
\mathbb{C}[w_1,w_2]\otimes\mathrm{T} \arrow[ur, "\partial^Z_\Omega"'] & \\
\mathbb{C}[w_1,w_2]\otimes\mathcal{O} \arrow[r, "\partial_Z"] \arrow[dr, "\partial_W"'] & \mathbb{C}[w_1,w_2]\otimes\Omega^1 \\
& \mathbb{C}[w_1,w_2]\{\mathrm{d}w_a\}\otimes\mathcal{O}
\end{tikzcd}$$

The desired Lagrangian in $(p_*\mathcal{E}^0_{AdS_7\times S^4})|_{\{\infty\}\times\mathbb{C}^3}$ is given by $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$.

Note that the same formulae defining the L_∞ structure on the parity shift of the eleven-dimensional theory $\mathcal{E}$ **??** equip these Lagrangians in the $N=0$ phase spaces with L_∞ structures. Moreover, the canonical maps

$$\Omega^\bullet_{\mathbb{R}}\otimes\Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4\times S^7}) \to (p_*\mathcal{E}^0_{AdS_4\times S^7})|_{\{\infty\}\times\mathbb{R}\times\mathbb{C}}, \quad \Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4}) \to (p_*\mathcal{E}^0_{AdS_7\times S^4})|_{\{\infty\}\times\mathbb{C}^3}$$

preserve the L_∞ brackets even taking into account the potential for additional higher brackets on the target coming from homotopy transfer. Indeed, such brackets must necessarily involve classes in $H^2_{\mathrm{THF}}\left((\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)$

Remark 2.11. These boundary conditions have a very natural physical interpretation. Recall that we constructed our avatars of the $AdS_4\times S^7$ and $AdS_7\times S^4$ backgrounds by codifying their appearance as backreactions of M2 and M5 branes respectively. In the absence of the fluxes sourced by these branes, we may ask that the supergravity fields extend over the former locations of these branes. As such, we can think of the boundary conditions defined above as finite-type models for the restriction of the fields of the eleven-dimensional theory to the location of branes.

2.3.1. The following lemma illustrates that the state space of definitions 2.4 2.8 can also be computed using these alternate boundary conditions.

These Lagrangians afford an alternative description of twisted supergravity states, which will be used to investigate their representation theoretic properties.

Proposition 2.12. There are isomorphisms

$$\mathcal{U}\left(\Omega^\bullet_{\mathbb{R}}\otimes\Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4\times S^7})\right)(0)\cong\mathrm{Sym}(\mathcal{H}_{AdS_4\times S^7})$$
$$\mathcal{U}\left(\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^S})\right)(0)\cong\mathrm{Sym}(\mathcal{H}_{AdS_7\times S^4})$$

Proof. For a local L_∞ algebra $\mathcal{L}$, its factorization envelope $\mathcal{U}(L)$ is defined to be $\mathrm{C}_\bullet(\mathcal{L}_c)$ the Lie algebra chains on the cosheaf of compactly supported sections. Therefore, it suffices to show that in each case, the costalk of the cosheaf of comapctly supported sections is quasi-isomorphic to $\mathcal{H}_{AdS_4\times S^7}$ and $\mathcal{H}_{AdS_7\times S^4}$ respectively. This is a consequence of the following observations.

Note that by ellipticity, there are quasi-isomorphisms

$$\overline{\Omega}^{0,\bullet}_{\mathbb{C},c}(D) \to \Omega^{0,\bullet}_{\mathbb{C},c}(D), \quad \overline{\Omega}^{0,\bullet}_{\mathbb{C}^3,c} \to \Omega^{0,\bullet}_{\mathbb{C}^3,c}(D^3).$$

coming from the inclusion of compactly supported distributional sections into compactly supported smooth sections. Now contracting the Dolbeault resolution, there are quasi-isomorphisms

$$\mathbb{C}[\partial_z]\delta_{z=0} \to \overline{\Omega}^{0,\bullet}_{\mathbb{C},c}(D), \quad \mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z=0} \to \overline{\Omega}^{0,\bullet}_{\mathbb{C}^3,c}(D^3).$$

These results apply equally as well for sections of holomorphic bundles. Now, computing the limit in the definition of the costalk over a collection of open sets containing the origin gives the result. □

3. Twisted global symmetries

As we indicated in the beginning of this chapter, a feature of the physical $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds is that they have as isometries, the 3d $\mathcal{N} = 8$ and 6d $\mathcal{N} = (2,0)$ superconformal algebras respectively. In fact, the complex forms of these two super-Lie algebras are the same.

In this section we provide evidence for conjectures 1.41.8 by arguing that the global sections of the local moduli problems $p_*\mathcal{E}_{AdS_4\times S^7}$ and $p_*\mathcal{E}_{AdS_7\times S^4}$ carry actions by the minimal twists of the relevant superconformal algebras. We will find that the twist of the superconformal algebra is the same in each case, but the actions are slightly different.

3.1. Superconformal algebras.

The complex form of the algebra of isometries for supergravity in both the $\mathrm{A}dS_4$ and $\mathrm{A}dS_7$ backgrounds is $\mathfrak{osp}(8|4)$ (though, their real forms differ). This agrees with the complex form of the 6d $\mathcal{N} = (2,0)$ superconformal algebra and the 3d $\mathcal{N} = 8$ superconformal algebra. The bosonic part of this algebra is isomorphic to $\mathfrak{so}(8) \oplus \mathfrak{sp}(2) \cong \mathfrak{so}(8) \oplus \mathfrak{so}(5)$.

The following is a mild rephrasing of a result in [SW22] where twisted superconformal symmetry in six dimensions is studied in some detail.

Theorem 3.1 (Saberi-Williams)**.** There is a map of super-Lie algebras $\phi : \mathfrak{siso}_{11d} \to \mathfrak{osp}(8|4)$. Letting $Q \in \mathfrak{siso}_{11d}$ be the odd square-zero element used to define the minimal twist of eleven-dimensional supergravity, there is an equivalence of dg super-Lie algebras

$$(\mathfrak{osp}(8|4), [\phi(Q), -]) \cong \mathfrak{osp}(6|2).$$

The super-Lie algebra $\mathfrak{osp}(6|2)$ will therefore play the role of the residual isometries of the twisted AdS background. The bosonic part of $\mathfrak{osp}(6|2)$ is the direct sum Lie algebra $\mathfrak{sl}(4) \oplus \mathfrak{sl}(2)$. The odd part of the algebra $\mathfrak{osp}(6|2)$ is $\wedge^4 W \otimes R$ where W is the fundamental $\mathfrak{sl}(4)$ representation and R is the fundamental $\mathfrak{sl}(2)$ representation.

3.2. **Global symmetries of twisted $AdS_4 \times S^7$.** To provide further evidence for conjecture 1.4 we wish to articulate a sense in which $\mathfrak{osp}(6|2)$ is witnessed as a symmetry of $\mathcal{E}^N_{AdS_4\times S^7}$. To this end, we will provide evidence for the claim that there is a Lie map

$$\mathfrak{osp}(6|2) \to H^\bullet\left(\Pi\mathcal{E}^N_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4\setminus\{0\}\right)\right).$$

We first focus on the case where the flux $N=0$. In this case, we will show that the embedding factors through the natural restriction map from the theory on flat space

$$\begin{tikzcd} & H^\bullet\left(\Pi\mathcal{E}(R\times\mathbb{C}^5)\right)\cong\widehat{E(5|10)} \arrow[d, "\mathrm{res}"] \\ \mathfrak{osp}(6|2) \arrow[ur, "i_{M2}"] \arrow[r] & H^\bullet\left(\Pi\mathcal{E}^0_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4\setminus\{0\}\right)\right) \end{tikzcd}$$

3.2.1. We begin by describing the map i_{M2}. As recalled above, the bosonic part of $\mathfrak{osp}(6|2)$ is $\mathfrak{sl}(4)\oplus\mathfrak{sl}(2)$. In its incarnation as a twist of 3d $\mathcal{N}=8$ superconformal symmetry, it is useful to think of the $\mathfrak{sl}(2)$-summand as describing conformal transformations on $\mathbb{C}_z$, while the Lie algebra $\mathfrak{sl}(4)$ is a residual R-symmetry describing rotations on $\mathbb{C}^4_w$.

The restriction of i_{M2} to the bosonic summand will actually realize $\mathfrak{sl}(4)\oplus\mathfrak{sl}(2)$ as the global symmetries corresponding to the vector fields in equation 6.4.4.

- The image of the bosonic summand $\mathfrak{sl}(2)$ under i_{M2} is spanned by the vector fields

$$\frac{\partial}{\partial z},\quad z\frac{\partial}{\partial z}-\frac{1}{4}\sum_{a=1}^4 w_a\frac{\partial}{\partial w_a},\quad z\left(z\frac{\partial}{\partial z}-\frac{1}{2}\sum_{a=1}^4 w_a\frac{\partial}{\partial w_a}\right)\in\mathrm{PV}^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R}).$$

  These vector fields are divergence free and reduce to the usual holomorphic conformal transformations along $w=0$.
- The image of $B_{ab}\in\mathfrak{sl}(4)$ under i_{M2} is given by the vector field

$$B_{ab}w_a\frac{\partial}{\partial w_b}\in\mathrm{PV}^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R}).$$

To describe the image of the fermionic part of $\mathfrak{osp}(6|2)$ under the map i_{M2} It is natural to split $R=\mathbb{C}_{+1}\oplus\mathbb{C}_{-1}$, so that the odd part decomposes as

$$(\wedge^2\mathbb{C}^4)_{+1}\oplus(\wedge^2\mathbb{C}^4)_{-1}.$$

In terms of residual 3d $\mathcal{N}=8$ superconformal symmetries, the fermionic summand $(\wedge^2\mathbb{C}^4)_{+1}$ consists of residual supertranslations, while the fermionic summand $(\wedge^2\mathbb{C}^4)_{-1}$ consists of the remaining superconformal transformations.

- For $e_a\wedge e_b\in(\wedge^2\mathbb{C}^4)_{+1}$ we have that

$$i_{M2}(e_a\wedge e_b)=\frac{1}{2}(w_a\mathrm{d}w_b-w_b\mathrm{d}w_a)\in\Omega^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R}).$$

- For $e_a\wedge e_b\in(\wedge^2\mathbb{C}^4)_{-1}$ we have that

$$i_{M2}(e_a\wedge e_b)=\frac{1}{2}z(w_a\mathrm{d}w_b-w_b\mathrm{d}w_a)\in\Omega^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R}).$$

The following is a straightforward check.

Lemma 3.2. The map i_{M2} is a Lie map.

It is clear that the image of the chain-level map i_{M2} defined above is closed for the linearized BRST differential $\delta^{(1)}$ so descends to a map $i_{M2} : \mathfrak{osp}(6|2) \to H^\bullet\left(\Pi\mathcal{E}(R \times \mathbb{C}^5)\right) \cong \widehat{E(5|10)}$ as claimed. As such, the composition $\mathrm{res} \circ i_{M2}$ defines an inner action of $\mathfrak{osp}(6|2)$ on the cohomology of global sections $H^\bullet\left(\Pi\mathcal{E}^0_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4 \setminus \{0\}\right)\right)$.

3.2.2. Next, we turn on $N \neq 0$ units of nontrivial flux. Note that not all fields in the image of the map $\mathrm{res} \circ i_{M2}$ commute with bracketing with the flux NF_{M2}, and as such are not compatible with the total differential $\delta^{(1)} + [NF_{M2}, -]$ on $\Pi\mathcal{E}^N_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4 \setminus \{0\}\right)$. Nevertheless, we have the following:

Proposition 3.3. There exist N-dependent corrections to the fields defining the embedding of $\mathfrak{osp}(6|2)$ summarized above which are closed for the modified BRST differential $\delta^{(1)} + [NF_{M2}, -]$. Furthermore, these order N corrections define an embedding of

$$\mathfrak{osp}(6|2) \to H^\bullet\left(\Pi\mathcal{E}^N_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4 \setminus \{0\}\right)\right).$$

Proof. For notational convenience, we will let $\mathcal{L}$ denote the local L_∞ algebra on $(\mathbb{R}\times\mathbb{C})\times(\mathbb{C}^4\setminus\{0\})$ given by $\Pi\mathcal{E}^N_{AdS_4\times S^7}$ and set $F = F_{M2}$. We will show that the image of the map $\mathrm{res}\circ i_{M2}$ survives to the last page of a spectral sequence that abuts to the target of the above map. The spectral sequence is the one associated to the bicomplex whose differentials are the linearized BRST differential $\delta^{(1)}$ and the operator $[F, -]$ given by bracketing with the flux. Recall from 6.4.4 that F is an element of $\mathrm{PV}^{1,3}(\mathbb{C}^4_w \setminus 0)\otimes\Omega^{0,0}(\mathbb{C}_z)\otimes\Omega^0(\mathbb{R})$ and $[F, -]$ acts on the fields according to two types of maps:

$$[F,-] : \mathrm{PV}^{i,\bullet}(\mathbb{C}^4_w\setminus 0)\otimes\mathrm{PV}^{j,\bullet}(\mathbb{C}_z)\otimes\Omega^\bullet(\mathbb{R}) \to \mathrm{PV}^{i,\bullet+3}(\mathbb{C}^4_w\setminus 0)\otimes\mathrm{PV}^{j,\bullet}(\mathbb{C}_z)\otimes\Omega^\bullet(\mathbb{R})$$
$$[F,-] : \Omega^{i,\bullet}(\mathbb{C}^4_w\setminus 0)\otimes\Omega^{j,\bullet}(\mathbb{C}_z)\otimes\Omega^\bullet(\mathbb{R}) \to \Omega^{i,\bullet+3}(\mathbb{C}^4_w\setminus 0)\otimes\Omega^{j,\bullet}(\mathbb{C}_z)\otimes\Omega^\bullet(\mathbb{R}).$$

The first page of the spectral sequence is the cohomology with respect to the original linearized BRST differential $\delta^{(1)}$; this is exactly $H^\bullet\left(\Pi\mathcal{E}^0_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4 \setminus \{0\}\right)\right)$. It will be useful to compute this page explicitly.

Recall that the linearized BRST differential decomposes as

$$\delta^{(1)} = \bar\partial + \mathrm{d}_{\mathbb{R}} + \partial_\Omega|_{\mu\to\nu} + \partial|_{\beta\to\gamma}.$$

To compute this page, we use an auxiliary spectral sequence which simply filters by the holomorphic form and polyvector field type. This first page of this auxiliary spectral sequence is simply given by the cohomology with respect to $\bar\partial + \mathrm{d}_{\mathbb{R}}$. This cohomology is given by

$$\begin{array}{ccc}
 & \pm & - \\
 & H^\bullet(\mathbb{C}^4\setminus 0, \mathrm{T})\otimes H^\bullet(\mathbb{C},\mathcal{O}) & H^\bullet(\mathbb{C}^4\setminus 0, \mathcal{O})\otimes H^\bullet(\mathbb{C},\mathcal{O}) \\
(32) & H^\bullet(\mathbb{C}^4\setminus 0, \mathcal{O})\otimes H^\bullet(\mathbb{C},\mathrm{T}) & \\
 & H^\bullet(\mathbb{C}^4\setminus 0, \mathcal{O})\otimes H^\bullet(\mathbb{C},\mathcal{O}) & H^\bullet(\mathbb{C}^4\setminus 0, \mathcal{O})\otimes H^\bullet(\mathbb{C},\Omega^1) \\
 & & H^\bullet(\mathbb{C}^4\setminus 0, \Omega^1)\otimes H^\bullet(\mathbb{C},\mathcal{O})
\end{array}$$

The cohomology of $\mathbb{C}$ is of course concentrated in degree zero and there is a dense embedding $\mathbb{C}[z] \hookrightarrow H^\bullet(\mathbb{C}, \mathcal{F})$ for $\mathcal{F} = \mathcal{O}, \mathrm{T}$, or Ω^1. It follows that up to completion, the cohomology $H^\bullet\left(\mathcal{L}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4 \setminus \{0\}\right); d+\bar\partial\right)$ is given by the direct sum of $H^\bullet\left(\Pi\mathcal{E}(\mathbb{R}\times\mathbb{C}^5); d+\bar\partial\right)$ with

(33)

$$\begin{tikzcd}
- & & \pm \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\cdots,w_4^{-1}][z]\{\partial_{w_i}\} \arrow[rr, dotted, "\partial_\Omega^W"] & & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\cdots,w_4^{-1}][z] \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\cdots,w_4^{-1}][z]\partial_z \arrow[urr, dotted, "\partial_\Omega^Z"'] & & \\
(w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\cdots,w_4^{-1}][z] \arrow[rr, dotted, "\partial_Z"] \arrow[drr, dotted, "\partial_W"'] & & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\cdots,w_4^{-1}][z]\mathrm{d}z \\
& & (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\cdots,w_4^{-1}][z]\{\mathrm{d}w_i\}.
\end{tikzcd}$$

The remaining piece of the original BRST operator is drawn in dotted lines. The first page of the spectral sequence converging to the cohomology with respect to $\delta^{(1)} + [NF, -]$ is thus given by the cohomology of the global symmetry algebra on $\mathbb{C}^5 \times \mathbb{R}$, which we computed in §**??**, plus the cohomology of the above complex with respect to the dotted-line operators. Indeed, this is exactly $H^\bullet\left(\Pi\mathcal{E}^0_{AdS_4\times S^7}\left((\mathbb{R}\times\mathbb{C})\times\mathbb{C}^4\setminus\{0\}\right)\right)$

Recall that the image of the flux F at this page in the spectral sequence corresponds to the class

$$[F] = (w_1\cdots w_4)^{-1}\partial_z \in (w_1\cdots w_4)^{-1}\mathbb{C}[w_1^{-1},\cdots,w_4^{-1}][z]\partial_z$$

The next page of the spectral sequence is given by computing the cohomology with respect to the operator $[NF, -]$. As observed above, this operator maps Dolbeault degree zero elements to Dolbeault degree three elements. For degree reasons, there are no further differentials and the spectral sequence collapses after the second page.

We now wish to argue that the image of the map $res \circ i_{M2}$ is annihilated by $\big[N[F], -\big]$. This is a direct calculation. For instance, recall that an element in the image of the odd summand $(\wedge^2\mathbb{C}^2)_{-1}$ (which corresponds to a superconformal transformation) is of the form $zw_a \wedge \mathrm{d}w_b = z(w_a\mathrm{d}w_b - w_b\mathrm{d}w_a)$. We have

$$\big[[F], z(w_a\mathrm{d}w_b - w_b\mathrm{d}w_a)\big] = (w_1\cdots w_4)^{-1}(w_a\mathrm{d}w_b - w_b\mathrm{d}w_a) = 0$$

since the class $(w_1\cdots w_4)^{-1}$ is in the kernel of the operator given by multiplication by w_a for any $a = 1,\ldots 4$. □

Remark 3.4. We comment on an alternate method to compute the first page of the auxiliary spectral sequence we used to compute the first page of the spectral sequence converging to the cohomology with respect to $\delta^{(1)} + [NF, -]$. We could have used a Serre-type spectral sequence for certain kinds of sheaves on THF manifolds [KT75], [Kor14], applied to the pushforward $p_*\Pi\mathcal{E}^N_{AdS_4\times S^7}$ from section 2.7. In this case, this Serre-type spectral sequence degenerates at the E_2-page.

3.3. **Global symmetries of twisted $AdS_7 \times S^4$.** We now wish to repeat the analysis of the previous section for the twisted $AdS_7 \times S^4$ background so as to provide evidence for conjecture 1.8. As before, we wish to provide evidence for the claim that there is a Lie map

$$\mathfrak{osp}(6|2) \to H^\bullet\left(\Pi\mathcal{E}^N_{AdS_7\times S^4}\left(\mathbb{C}^3\times(\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)\right).$$

We first focus on the case $N=0$ where once again the embedding factors through the natural restriction map from the theory on flat space

$$\begin{tikzcd} & H^\bullet\left(\Pi\mathcal{E}(R\times\mathbb{C}^5)\right)\cong\widehat{E(5|10)} \arrow[d, "\mathrm{res}"] \\ \mathfrak{osp}(6|2) \arrow[ur, "i_{M5}"] \arrow[r] & H^\bullet\left(\Pi\mathcal{E}^0_{AdS_7\times S^4}\left(\mathbb{C}^3\times(\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)\right) \end{tikzcd}$$

3.3.1. We begin by describing the map i_{M5}. Recall that the bosonic part of $\mathfrak{osp}(6|2)$ is the direct sum Lie algebra $\mathfrak{sl}(4)\oplus\mathfrak{sl}(2)$. In its incarnation as the minimal twist of the 6d $\mathcal{N}=(2,0)$ superconformal algebra, the roles of the $\mathfrak{sl}(4)$ and $\mathfrak{sl}(2)$ summands are interchanged compared to the case of the M2 brane. Indeed, the Lie algebra $\mathfrak{sl}(4)$ represents conformal transformations along $\mathbb{C}^3_z$, while $\mathfrak{sl}(2)$ is a residual R-symmetry describing rotations on $\mathbb{C}^2_w$.

Moreover, the restriction of i_{M5} to a copy of $\mathfrak{sl}(3)\oplus\mathfrak{gl}(1)\oplus\mathfrak{sl}(2)\subset\mathfrak{sl}(4)\oplus\mathfrak{sl}(2)$ will realize this subalgebra as the global symmetries corresponding to the vector fields in equation 2.2.3.

- The bosonic abelian subalgebra $\mathbb{C}^3\subset\mathfrak{sl}(4)$ of translations is mapped to the obvious vector fields
$$\frac{\partial}{\partial z_i}\in \mathrm{PV}^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R}),\quad i=1,2,3.$$
- The image of $A_{ij}\in\mathfrak{sl}(3)\subset\mathfrak{sl}(4)$ under i_{M5} is given by the vector field
$$A_{ij}z_i\frac{\partial}{\partial z_j}\in \mathrm{PV}^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R}),\quad (A_{ij})\in\mathfrak{sl}(3).$$
- The image of $\mathfrak{gl}(1)\subset\mathfrak{sl}(4)$ corresponding to rescaling $\mathbb{C}^3$ under i_{M5} is the element
$$\sum_{i=1}^3 z_i\frac{\partial}{\partial z_i}-\frac{3}{2}\sum_{a=1}^2 w_a\frac{\partial}{\partial w_a}\in \mathrm{PV}^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R}).$$
- The image of the remaining subalgebra of $\mathfrak{sl}(4)$, which describes special conformal transformations on $\mathbb{C}^3$, is spanned by the elements
$$z_j\left(\sum_{i=1}^3 z_i\frac{\partial}{\partial z_i}-2\sum_{a=1}^2 w_a\frac{\partial}{\partial w_a}\right)\in \mathrm{PV}^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R}).$$
Notice that these vector fields are divergence-free and restrict to the ordinary special conformal transformations along $w=0$.
- The image of the bosonic summand $\mathfrak{sl}(2)$ corresponding to residual R-symmetry is spanned by the vector fields
$$w_1\frac{\partial}{\partial w_2},w_2\frac{\partial}{\partial w_1},\frac{1}{2}\left(w_1\frac{\partial}{\partial w_1}-w_2\frac{\partial}{\partial w_2}\right)\in \mathrm{PV}^{1,0}(\mathbb{C}^5)\otimes\Omega^0(\mathbb{R}).$$

To describe the image of the fermionic part of $\mathfrak{osp}(6|2)$, which is given by $\wedge^2 W \oplus R$ with W the fundamental $\mathfrak{sl}(4)$ representation and R the fundamental $\mathfrak{sl}(2)$ representation, it is natural to split $W = L \oplus \mathbb{C}$ with $L = \mathbb{C}^3$ the fundamental $\mathfrak{sl}(3) \subset \mathfrak{sl}(4)$ representation. The odd part then decomposes as

$$L \otimes R \oplus \wedge^2 L \otimes R \cong \mathbb{C}^3 \otimes \mathbb{C}^2 \oplus \wedge^2 \mathbb{C}^3 \otimes \mathbb{C}.$$

- The summand $L \otimes R$ consists of the remaining 6d supertranslations. Its image under i_{M5} is spanned by the fields
$$z_i \mathrm{d}w_a \in \Omega^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}), \quad a = 1,2, \quad i = 1,2,3.$$
- The summand $\wedge^2 L \otimes R$ consists of the remaining 6d superconformal transformations. Its image under i_{M5} is spanned by the fields
$$\frac{1}{2} w_a (z_i \mathrm{d}z_j - z_j \mathrm{d}z_i) \in \Omega^{1,0}(\mathbb{C}^5) \otimes \Omega^0(\mathbb{R}), \quad a = 1,2, \quad k = 1,2,3.$$

The following is a straightforward check.

Lemma 3.5. The map i_{M5} is a Lie map.

It is again clear that the image of the chain-level map i_{M5} defined above is closed for the linearized BRST differential $\delta^{(1)}$ on $\Pi\mathcal{E}$ so descends to a map i_{M5} : $\mathfrak{osp}(6|2) \to H^\bullet(\Pi\mathcal{E}(\mathbb{R} \times \mathbb{C}^5)) \cong \widehat{E(5|10)}$ as claimed. As such, the composition $\mathrm{res} \circ i_{M5}$ will define an inner action of $\mathfrak{osp}(6|2)$ on the cohomology of the global sections $H^\bullet \left(\Pi\mathcal{E}^0_{AdS_7 \times S^4} \left(\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2) \setminus 0 \right) \right)$.

3.3.2. Next, we turn on $N \neq 0$ units of nontrivial flux. Again, not all fields in the image of the map $\mathrm{res} \circ i_{M5}$ are compatible with the total differential $\delta^{(1)} + [NF, -]$ on $\Pi\mathcal{E}^N_{AdS_7 \times S^4} \left(\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2) \setminus 0 \right)$. Nevertheless, we have the following version of proposition 3.3

Proposition 3.6. There exist N-dependent corrections to the fields defining the embedding of $\mathfrak{osp}(6|2)$ summarized above which are closed for the modified BRST differential $\delta^{(1)} + [NF_{M5}, -]$. Furthermore, these N-dependent corrections define an embedding

$$H^\bullet \left(\Pi\mathcal{E}^N_{AdS_7 \times S^4} \left(\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2) \setminus 0 \right) \right).$$

Proof. We proceed exactly analogously to the proof of proposition 3.3. For notational convenience, we will let $\mathcal{L}$ denote the local L_∞ algebra on $\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2) \setminus 0$ given by $\Pi\mathcal{E}^N_{AdS_7 \times S^4}$ and set $F = F_{M5}$. We will show that the image of the map $\mathrm{res} \circ i_{M5}$ survives to the last page of a spectral sequence that abuts to the target of the above map. The spectral sequence is the one associated to the bicomplex whose differentials are the linearized BRST differential $\delta^{(1)}$ and the operator $[F, -]$ given by bracketing with the flux.

The first page of this spectral sequence is the cohomology with respect to the original linearized BRST differential $\delta^{(1)}$; this is exactly $H^\bullet \left(\Pi\mathcal{E}^0_{AdS_7 \times S^4} \left(\mathbb{C}^3 \times (\mathbb{R} \times \mathbb{C}^2) \setminus 0 \right) \right)$. It will be useful to compute this page explicitly.

We once again do so by way of an auxiliary spectral sequence which simply filters by the holomorphic form and polyvector field type. This first page of this auxiliary spectral sequence is simply given by the cohomology with respect to $\mathrm{d}_\mathbb{R} + \bar{\partial}$.

It follows that up to completions, the cohomology $H^\bullet\left(\mathcal{L}\left(\right);\mathrm{d}_{\mathbb{R}}+\bar\partial\right)$ is the direct sum of the cohomology on flat space $H^\bullet(\Pi\mathcal{E}(\mathbb{C}^5\times\mathbb{R}),\mathrm{d}_{\mathbb{R}}+\bar\partial)$ with

(34)

$$
\begin{tikzcd}
\pm & & - \\
w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}][z_1,z_2,z_3]\{\partial_{w_i}\} \arrow[rr, dotted, "\partial_\Omega^W"] & & w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}][z_1,z_2,z_3] \\
w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}][z_1,z_2,z_3]\{\partial_{z_i}\} \arrow[urr, dotted, "\partial_\Omega^Z"'] & & \\
w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}][z_1,z_2,z_3] \arrow[rr, dotted, "\partial_W"] \arrow[drr, dotted, "\partial_Z"'] & & w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}][z_1,z_2,z_3]\{\mathrm{d}z_i\} \\
& & w_1^{-1}w_2^{-1}\mathbb{C}[w_1^{-1},w_2^{-1}][z_1,z_2,z_3]\{\mathrm{d}w_i\}.
\end{tikzcd}
$$

The first page of the spectral sequence converging to the cohomology with respect to $\delta^{(1)}+[NF,-]$ is given by the cohomology of the global symmetry algebra on $\mathbb{C}^5\times\mathbb{R}$, which we computed in §**??**, plus the cohomology with respect to the dotted-line operators in (34). This is indeed the cohomology of global sections $H^\bullet\left(\Pi\mathcal{E}^0_{AdS_7\times S^4}\left(\mathbb{C}^3\times(\mathbb{R}\times\mathbb{C}^2)\setminus 0\right)\right)$.

Recall that the flux F was defined as the image under ∂ of some γ-type field. Therefore, the class $[F]$ does not live inside this page of the spectral sequence, but the operator $[[F],-]$ does act on this page nevertheless. For instance, if $f^i(z,w)\mathrm{d}z_i$ is a one-form living in $H^0(\mathbb{C}^5,\Omega^1)\otimes H^0(\mathbb{R})$, then

$$[[F],f^i(z,w)\mathrm{d}z_i]=\epsilon_{ijk}w_1^{-1}w_2^{-1}\partial_{z_j}f^i(z,w)\partial_{z_k}$$

which is an element in

$$\mathbb{C}[w_1^{-1},w_2^{-1}][z_1,z_2,z_3]\{\partial_{z_i}\}\subset H^0(\mathbb{C}^3,\mathrm{T})\otimes H^2\big(\Omega^\bullet(\mathbb{C}^2\times\mathbb{R}\setminus 0)/(\mathrm{d}w_1,\mathrm{d}w_2)\big).$$

The next page of the spectral sequence is given by computing the cohomology with respect to the operator $[NF,-]$. This operator maps Dolbeault-de Rham degree zero elements to Dolbeault-de Rham degree two elements. For degree reasons, there are no further differentials and the spectral sequence collapses after the second page.

We now wish to argue that the image of the map $\mathrm{res}\circ i_{M5}$ is annihilated by $\big[N[F],-\big]$. This is a direct calculation. For instance, recall that an element in the image of the odd summand $\wedge^2L\otimes R=\wedge^2\mathbb{C}^3\otimes\mathbb{C}^2$ (which corresponds to a superconformal transformation) is of the form $w_a(z_i\mathrm{d}z_j-z_j\mathrm{d}z_i)$, $a=1,2,i,j=1,2,3$. We have

$$\big[[F],w_a(z_i\mathrm{d}z_j-z_j\mathrm{d}z_i)\big]=2\epsilon_{ijk}(w_1^{-1}w_2^{-1})\cdot w_a\partial_{z_k}=0$$

since the class $w_1^{-1}w_2^{-1}$ is in the kernel of the operator given by multiplication by w_a for $a=1,2$. Verifying that the remaining elements in the image of i_{M5} are in the kernel of $\big[[F],-\big]$ is similar. This completes the proof. □

Remark 3.7. As in remark 3.4 comment on an alternate method to compute the first page of the auxiliary spectral sequence we used to compute the first page of the spectral sequence converging to the cohomology with respect to $\delta^{(1)}+[NF,-]$. We could have used a Serre-type spectral sequence for certain kinds of sheaves on THF manifolds [KT75], [Kor14], applied to the pushforward $p_*\Pi\mathcal{E}^N_{AdS_7\times S^4}$ from section 2.7. In this case, this Serre-type spectral sequence degenerates at the E_2-page. We will return to similarly flavored constructions in later work.

4. $E(1|6)$ MODULES FROM GRAVITONS ON $AdS_4 \times S^7$

Having justified that the spaces of supergravity states constructed in the previous subsection are in fact counting gravitons on $AdS_4 \times S^7$ and $AdS_7 \times S^4$ respectively, we turn to studying representation theoretic properties of these state spaces. In this section, we focus on the case of gravitons on $AdS_4 \times S^7$, using the description of the state space afforded by proposition 2.12 which describes it as the costalk of a factorization envelopes of the boundary conditions $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$.

We construct a certain $\mathbb{C}^\times$-action on the boundary fields $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$ equipped with the L_∞ structure from remark with the feature that the zeroth weight spaces are a local version of another exceptional linearly compact super-Lie algebras, $E(1|6)$. This in particular readily gives a decomposition of the state space $\mathcal{H}_{AdS_4 \times S^7}$ into $E(1|6)$-modules. We explicitly characterize the summands of this decomposition with their module structures and give closed form expressions for their characters.

4.1. The graviton decomposition of twisted $AdS_4 \times S^7$.

4.1.1. We consider a particular decomposition of the space of states $\mathcal{H}_{AdS_4 \times S^7}$. It is induced by a decomposition of the boundary fields $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$ introduced in section 2.3 The decomposition is induced by a $\mathbb{C}^\times$ action on the boundary fieldsboundary fields $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$ that mixes fiberwise rescalings on spacetime with a fiberwise rescaling of the space of fields.

Explicitly, the action is given as follows

- On the fields

$$\mu(t; w_a, z) \in \mathbb{C}[w_1, \cdots, w_4]\{\partial_{w_a}\} \otimes \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U) \oplus \mathbb{C}[w_1, \cdots, w_4] \otimes \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U, T)$$

  the action is
$$\lambda \cdot \mu(t; w_a, z) = \mu(t; \lambda w_a, z).$$
- On the fields $\nu(t; w_a, z) \in \mathbb{C}[w_1, \cdots, w_4] \otimes \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U)$ the action is
$$\lambda \cdot \nu(t; w_a, z) = \nu(t; \lambda w, z).$$
- On the fields $\beta(t; w_a, z) \in \mathbb{C}[w_1, \cdots, w_4] \otimes \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U)$ the action is
$$\lambda \cdot \beta(t; w_a, z) = \lambda^{-2} \beta(t; \lambda w_a, z).$$
- On the fields

$$\gamma(t; w_a, z) \in \mathbb{C}[w_1, \cdots, w_4]\{\mathrm{d}w_a\} \otimes \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U) \oplus \mathbb{C}[w_1, \cdots, w_4] \otimes \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U, \Omega^1)$$

  the action is
$$\lambda \cdot \gamma(t; w_a, z) = \lambda^{-2} \gamma(t; \lambda w_a, z).$$

The following result is a straightforward if lengthy computation. We state it without proof.

Proposition 4.1. The L_∞ structure on $\Pi \Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$ identified in section 2.3 is equivariant for this $\mathbb{C}^\times$ action.

This result induces a product decomposition
$$\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7}) = \prod_{n \geq 2} \mathcal{F}^{(n)}_{\mathbb{R} \times \mathbb{C}}$$

where for each open set $I \times U \subset \mathbb{R} \times \mathbb{C}$, we have that

$$\mathcal{F}^{(n)}_{\mathbb{R}\times\mathbb{C}}(I \times U) \subset \Omega^{\bullet}_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(U, \mathcal{L}^{r=0}_{AdS_4\times S^7})$$

is the weight n eigenspace with respect to the above $\mathbb{C}^\times$ action. In particular, we see that $\mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C}}$ is itself a local dg-Lie algebra, for which every $\mathcal{F}^{(n)}_{\mathbb{R}\times\mathbb{C}}$ is a module.

4.2. The lowest piece: the holomorphic-topological twist of the 3d $\mathcal{N} = 8$ BLG theory.

4.2.1. The first nontrivial case is the weight (-2) piece. We have the following

Lemma 4.2. There is a quasi-isomorphism

$$\mathcal{F}^{(-2)}_{\mathbb{R}\times\mathbb{C}} \cong \Omega^{\bullet}_{\mathbb{R}\times\mathbb{C}}.$$

Proof. The only sections which contribute are those of type β or γ with no form components along the fiber directions. Therefore, we see directly that

$$\mathcal{F}^{(-2)}_{\mathbb{R}\times\mathbb{C}} \cong \Omega^{\bullet}_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{O} \xrightarrow{\partial} \Omega^1).$$

□

4.2.2. The next nontrivial case is the weight (-1) piece.

Lemma 4.3. There is a quasi-isomorphism

$$\mathcal{F}^{(-1)}_{\mathbb{R}\times\mathbb{C}} \cong \Omega^{\bullet}_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}} \left(K^{1/4} \otimes (\mathbb{C}^4)^* \oplus \Pi K^{3/4} \otimes \mathbb{C}^4\right)$$

Proof. On an open set of the form $I \times U$, the sections of the specified weight are:

- fields of type μ of the form $\mu_a(t; z)\partial_{w_a}$. As the w_a are fiber coordinates on $K^{1/4}_{\mathbb{C}}$, these fields transform as sections of $K^{1/4}_{\mathbb{C}}$.
- fields of type γ of the form $\gamma_a(t; z)\mathrm{d}w_a$. These fields transform as sections of $K^{3/4}_{\mathbb{C}}$.

□

4.2.3. We wish to flag an appearance of $\mathcal{F}^{(-1)}_{\mathbb{R}\times\mathbb{C}}$ in supersymmetric physics in three-dimensions. There is a highly supersymmetric Chern-Simons-matter theory discovered independently by Bagger-Lambert [BL07], [BL08] and Gustavsson [Gus09]. The aptly named BLG theory has $\mathcal{N} = 8$ superconformal symmetry, and admits a holomorphic-topological twist that was computed by Garner in [?].

The sheaf of complexes $\mathcal{F}^{(-1)}_{\mathbb{R}\times\mathbb{C}}$ matches the field contents of the holmorphic-topological twist of the BLG theory, and as such, it can be equipped with an L_∞ structure under which it is perturbatively equivalent to the twisted BLG theory. In work-in-progress with Garner and Williams, we show that the action of $\mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C}}$ on $\mathcal{F}^{(-1)}_{\mathbb{R}\times\mathbb{C}}$ in fact preserves this L_∞-structure.

4.3. The zero-th piece: A local version of $E(1|6)$.

4.3.1. The next nontrivial case is the weight (0) piece. This factor is special because it carries the induced structure of a local L_∞ algebra on $\mathbb{R} \times \mathbb{C}$. We will prove that it is equivalent to a local Lie algebra version of the exceptional super-Lie algebra $E(1|6)$.

We first recall the definition of this super-Lie algebra [Kac98]

Definition 4.4. Let $E(1|6)$ be the following super-Lie algebra.

- The even part of $E(1|6)_0$ given by the semidirect product Lie algebra $\Gamma(\widehat{D}, T) \ltimes \left(\Gamma(\widehat{D}, \mathcal{O}) \otimes \mathfrak{sl}(4)\right)$
- The odd part $E(1|6)_1$ is given by the (unique) nontrivial extension of $E(1|6)_0$-modules

$$0 \to \mathrm{Sym}^2(\mathbb{C}^4) \otimes K_{\mathbb{C}}^{1/2} \to E(1|6)_1 \to \wedge^2(\mathbb{C}^4) \otimes K_{\mathbb{C}}^{-1/2} \to 0.$$

The only remaining bracket to be specified, the odd bracket, is given as follows.

- Given sections $A \otimes f \mathrm{d}z^{1/2} \in \mathrm{Sym}^2(\mathbb{C}^4) \otimes K_{\mathbb{C}}^{1/2}$ and $B \otimes g\partial_z^{1/2} \in \wedge^2(\mathbb{C}^4) \otimes K_{\mathbb{C}}^{-1/2}$, we have that

$$[A \otimes f \mathrm{d}z^{1/2}, B \otimes g\partial_z^{1/2}] = A * B \otimes fg \in \mathfrak{sl}(4) \otimes \mathcal{O}.$$

  Here, $*$ refers to the hodge star of B and we are viewing A and $*B$ as symmetric and skew-symmetric 4×4 matrices respectively; their product is traceless.
- Given sections $A \otimes f\partial_z^{1/2}, B \otimes g\partial_z^{1/2} \in \Gamma(\widehat{D}, \wedge^2(\mathbb{C}^4) \otimes K_{\mathbb{C}}^{1/2})$, we have that

$$[A \otimes f \mathrm{d}z^{-1/2}, B \otimes g \mathrm{d}z^{-1/2}] = \mathrm{Tr}(A * B) \otimes fg\partial_z + \frac{1}{2}(A * B)_0 \otimes \left(\partial(f \mathrm{d}z^{-1/2}) g \mathrm{d}z^{-1/2} + f \mathrm{d}z^{-1/2} \partial(g \mathrm{d}z^{-1/2})\right)$$
$$\in \Gamma(\widehat{D}, T) \ltimes \left(\mathfrak{sl}(4) \otimes \Gamma(\widehat{D}, \mathcal{O})\right).$$

  where again $*$ denotes the Hodge star and the subscript of zero denotes projection to the traceless part.

The relationship between this super-Lie algebra and our decomposition is established through the following result.

Proposition 4.5. There is an equivalence of super-Lie algebras

$$\mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C},c}(0) \cong E(1|6).$$

Proof. We will begin by trying to characterize the local L_∞-algebra $\mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C}}$. We claim that it is quasi-isomorphic to a local version of $E(1|6)$.

Indeed, it is easy to see that the weight zero sections consists of the following cochain complex

$$
\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}} \left(\begin{tikzcd} \underline{even} & & \underline{odd} \\ \mathbb{C}[w_a \partial_{w_b}] \otimes \mathcal{O} \arrow[rr, "\partial^W_\Omega"] & & \mathcal{O} \\ \mathrm{T} \arrow[urr] & & \\ \mathrm{Sym}^2(\mathbb{C}^4) \arrow[r, "\partial_W"] \arrow[dr] & \mathbb{C}[w_a dw_b] \otimes K^{-1/2} & \\ & \mathrm{Sym}^2(\mathbb{C}^4) \otimes \Omega^1 \otimes K^{-1/2} & \end{tikzcd} \right) \tag{35}
$$

Of course, the differentials are just appropriate components of the divergence operator and holomorphic deRham operator. We can compute cohomology by way of a spectral sequence whose first page is the cohomology with respect to $\partial^W_\Omega + \partial_W$. We see that the differential ∂^W maps surjectively onto functions and its kernel is isomorphic to $\mathfrak{sl}(4) \otimes \mathcal{O}$. Likewise, the differential ∂_W is the canonical inclusion of $\mathfrak{sl}(4)$ representations $\mathrm{Sym}^2(\mathbb{C}^4) \to \mathbb{C}^4 \otimes \mathbb{C}^4$. Its cokernel is a copy of $\wedge^2 \mathbb{C}^4$.

Thus, we see that this page of the spectral sequence is given by

$$
\mathcal{E}(1|6) \overset{def}{=} \Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}} \left(\begin{array}{cc} \underline{even} & \underline{odd} \\ \\ \mathrm{T} & \wedge^2(\mathbb{C}^4) \otimes K^{-1/2} \\ \\ \mathfrak{sl}(4) \otimes \mathcal{O} & \mathrm{Sym}^2(\mathbb{C}^4) \otimes K^{1/2} \end{array} \right) \tag{36}
$$

and there are no non-zero differentials so the spectral sequence degenerates.

To see that the Lie structure induced from the L_∞-structure on $\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4 \times S^7})$ is in fact given by the same formulae as the brackets on $E(1|6)$ in equation 4.4, it will be useful to provide an explicit quasi-isomorphism $\Psi^{(0)} : \mathcal{E}(1|6) \to \mathcal{F}^{(0)}_{\mathbb{R} \times \mathbb{C}}$. On an open set $I \times D \subset \mathbb{R} \times \mathbb{C}$, this is defined as follows

- Given a section $g(t;z)\partial_z \in \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(D, T)$ where $g(t;z)$ is a mixed deRham-Dolbeault form on $I \times D$, we define
$$
\begin{aligned} \Psi^{(0)}(g(t;z)\partial_z) &= g(t;z)\partial_z - \frac{1}{4}(\partial_z g(t;z)) w_a \partial_{w_a} \\ &\in \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(D, T \oplus \mathbb{C}\{w_a \partial_{w_b}\}) \end{aligned}
$$
- Given a section $A_{ab} \otimes g(t;z) \in \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(D, \mathfrak{sl}(4) \otimes \mathcal{O})$ where $g(t;z)$ is a mixed deRham-Dolbeault form on $I \times D$ and $A_{ab} \in \mathfrak{sl}(4)$
$$
\begin{aligned} \Psi^{(0)}(A_{ab} \otimes g(t;z)) &= g(t;z) A_{ab} w_a \partial_{w_b} \\ &\in \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}(D, \mathbb{C}\{w_a \partial_{w_b}\}) \end{aligned}
$$

- Given a section $A_{ab} \otimes g(t;z)\mathrm{d}z^{-1/2} \in \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}\left(D, \wedge^2(\mathbb{C}^4) \otimes K^{-1/2}\right)$ where $g(t;z)$ is a mixed deRham-Dolbeault form on $I \times D$ and $A_{ab} \in \wedge^2(\mathbb{C}^4)$ we define

$$\begin{aligned}\Psi^{(0)}(A_{ab} \otimes g(t;z)\mathrm{d}z^{-1/2}) &= g(t;z)A_{ab}w_a\partial_{w_b}\\ &\in \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}\Big(D, \mathbb{C}\{w_a\partial_{w_b}\} \otimes K^{-1/2}\Big)\end{aligned}$$

- Given a section $A_{ab} \otimes g(t;z)\mathrm{d}z^{1/2} \in \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}\left(D, \mathrm{Sym}^2(\mathbb{C}^4) \otimes K^{1/2}\right)$ where $g(t;z)$ is a mixed deRham-Dolbeault form on $I \times D$ and $A_{ab} \in \mathrm{Sym}^2(\mathbb{C}^4)$ we define

$$\begin{aligned}\Psi^{(0)}(A_{ab} \otimes g(t;z)\mathrm{d}z^{1/2}) &= g(t;z)A_{ab}w_aw_b\mathrm{d}z\\ &\in \Omega^\bullet_{\mathbb{R}}(I) \otimes \Omega^{0,\bullet}_{\mathbb{C}}\Big(D, \mathrm{Sym}^2(\mathbb{C}^4) \otimes \Omega^1 \otimes K^{-1/2}\Big)\end{aligned}$$

It is easy to see that $\Psi^{(0)}$ is a quasi-isomorphism and a straightforward if lengthy check confirms that it preserves Lie brackets. The result then follows from computing the limit of $\mathcal{E}(1|6)_c(I \times D)$ over open sets containing the origin. $\square$

Remark 4.6. We note that the map i_{M2} from lemma 3.2 in fact defines a Lie map from $\mathfrak{osp}(6|2)$ to the sections of the boundary condition $\Pi\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}(\mathcal{L}^{r=0}_{AdS_4\times S^7})$ over every open set containing the origin. The image of the map lands exactly in the step $\mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C}}$ of the decomposition from proposition 4.1. Therefore we see that $E(1|6)$ contains $\mathfrak{osp}(6|2)$ as a finite dimensional subalgebra.

4.4. **General summands and $E(1|6)$-modules.** We now move on to giving an explicit description of the general summand $\mathcal{F}^{(j)}$ for $j \geq 1$.

We first fix some notation for irreducible highest weight representations of $\mathfrak{sl}(4)$. Let $\mathfrak{h} \subset \mathfrak{sl}$ be the Cartan given by diagonal matrices and let $L_i \in \mathfrak{h}^*$ be the linear functional that picks out the i-th diagonal entry. We may accordingly write $\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, L_3, L_4\}/(L_1 + \cdots + L_4)$. We will write Γ_{a_1,a_2,a_3} for the irreducible representation of $\mathfrak{sl}(4)$ of highest weight $(a_1 + a_2 + a_3)L_1 + (a_2 + a_3)L_2 + a_3L_3$.

Proposition 4.7. Let $j \geq 1$. The complex of vector bundles $\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C}}$ is quasi-isomorphic to

(37)
$$\Omega^\bullet_{\mathbb{R}} \otimes \Omega^{0,\bullet}_{\mathbb{C}}\left(\begin{array}{cc}\underline{\textit{even}} & \underline{\textit{odd}}\\ & \\ \Gamma_{j,1,0} \otimes K^{-j/4} & \mathrm{Sym}^{j+2}(\mathbb{C}^4) \otimes \Omega^1 \otimes K^{-(j+2)/4}\\ & \\ \mathrm{Sym}^j(\mathbb{C}^4) \otimes \mathrm{T} \otimes K^{-j/4} & \Gamma_{j+1,0,1} \otimes K^{-(j+2)/4}\end{array}\right)$$

Proof. We begin by noting that we can explicitly describe $\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C}}$ as $\Omega^\bullet_{\mathbb{R}}\otimes\Omega^{0,\bullet}_{\mathbb{C}}(F^{(j)})$ where $F^{(j)}$ denotes the following dg-vector bundle:

$$
\begin{tikzcd}[column sep=small]
\underline{\textit{even}} & \underline{\textit{odd}} \\
\mathrm{Sym}^{j+1}(\mathbb{C}^4)\otimes(\mathbb{C}^4)^*\otimes K^{-j/4} \arrow[dr, "\partial_\Omega^W"] & \\
& \mathrm{Sym}^j(\mathbb{C}^4)\otimes K^{-j/4} \\
\mathrm{Sym}^j(\mathbb{C}^4)\otimes \mathrm{T}\otimes K^{-j/4} \arrow[ur, "\partial_\Omega^V"'] & \\
& \mathrm{Sym}^{j+2}(\mathbb{C}^4)\otimes \mathrm{T}^*\otimes K^{-(j+2)/4} \\
\mathrm{Sym}^{j+2}(\mathbb{C}^4)\otimes K^{-(j+2)/4} \arrow[ur, "\partial_V"] \arrow[dr, "\partial_W"'] & \\
& K^{-(j+2)/4}\otimes \mathrm{Sym}^{j+1}(\mathbb{C}^4)\otimes\mathbb{C}^4.
\end{tikzcd}
\tag{38}
$$

Note that the differentials here are all $\mathfrak{sl}(4)$ equivariant maps, tensored with a differential operator acting on sections of a bundle on $\mathbb{C}$. In particular

- The differential ∂_Ω^W involves the canonical projection
$$\mathrm{Sym}^{j+1}(\mathbb{C}^4)\otimes(\mathbb{C}^4)^*\to\mathrm{Sym}^j(\mathbb{C}^4).$$
Its kernel is precisely the irreducible highest weight representation $\Gamma_{j+1,0,1}$.
- The differential ∂_W is the canonical inclusion
$$\mathrm{Sym}^{j+2}(\mathbb{C}^4)\to\mathrm{Sym}^{j+1}(\mathbb{C}^4)\otimes\mathbb{C}^4.$$
Its cokernel is the irreducible highest weight representation $\Gamma_{j,1,0}$.

We can compute the cohomology using a spectral sequence whose first page is given by the cohomology with respect to $\partial_\Omega^W+\partial_W$. There are no further differentials on this page so the result follows. □

4.5. **Characters of $E(1|6)$-modules.** Note that the decomposition of the state space $\mathrm{Sym}(\mathcal{H}_{AdS_4\times S^7})=\prod_{j\geq -2}\mathcal{U}(\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C}})(0)$ gives a product formula for the characters computed in proposition 6.4.4

$$\chi\left(\mathrm{Sym}(\mathcal{H}_{AdS_4\times S^7})\right)=\prod_{j\geq-2}\chi\left(\mathcal{U}(\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C}})(0)\right).$$

We end this section by computing each of the characters $\chi\left(\mathcal{U}(\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C}})(0)\right)$. We will express our characters in terms of characters of highest weight respresentations of $\mathfrak{sl}(4)$ which we denote $\chi^{\mathfrak{sl}(4)}(\Gamma_{a_1,a_2,a_3})$.

4.5.1. From the characterization in 6.4.4, the lowest step of the decomposition $\mathcal{F}^{(-2)}$ is just given by the deRham complex on $\mathbb{R}\times\mathbb{C}$, and accordingly the character of $\mathcal{F}_c^{(-2)}(0)$ is the constant function 1.

4.5.2. We proceed to the next step of the decomposition, using the characterization in 6.4.4.

Proposition 4.8. The character $\chi\left(\mathcal{U}(\mathcal{F}^{(-1)}_{\mathbb{R}\times\mathbb{C}})(0)\right)$ is given by the plethystic exponential of the following expression:

(39)
$$f_{-1}(t_1,t_2,t_3,q) = \frac{q\left(q^{-3/4}(t_1+t_2+t_3+t_1^{-1}t_2^{-1}t_3^{-1}) - q^{-1/4}(t_1^{-1}+t_2^{-1}+t_3^{-1}+t_1t_2t_3)\right)}{(1-q)}$$

Proof. The proof proceeds by the same trick as in the proof of proposition 2.12. To describe the costalk, we wish to compute a limit of sections of $\mathcal{F}^{(-1)}_{\mathbb{R}\times\mathbb{C},c}$ on open sets of the form $I\times D$ containing the origin in $\mathbb{R}\times\mathbb{C}$. Using ellipticity, we can describe such sections as a module over the ring generated by holomorphic derivatives of the delta function.

Accordingly, we have contributions from the following summands:

- An even copy of $\mathbb{C}^4\otimes\mathbb{C}\{\mathrm{d}z^{3/4}\}\otimes\mathbb{C}[\partial_z]\delta_{z=0}$. The character of this summand is
$$\frac{q\left(q^{-3/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{1,0,0})\right)}{(1-q)} = \frac{q\left(q^{-3/4}(t_1+t_2+t_3+t_1^{-1}t_2^{-1}t_3^{-1})\right)}{(1-q)}$$
- An odd copy of $\mathbb{C}^4\otimes\mathbb{C}\{\mathrm{d}z^{1/4}\}\otimes\mathbb{C}[\partial_z]\delta_{z=0}$. The character of this summand is
$$\frac{-q\left(q^{-1/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{0,0,1})\right)}{(1-q)} = \frac{-q\left(q^{-3/4}(t_1^{-1}+t_2^{-1}+t_3^{-1}+t_1t_2t_3)\right)}{(1-q)}$$

□

Note that under the change of fugacities in 6.4.4, this matches exactly with the single particle index for the theory on a single M2 brane [BBMR08, Eq. (2.32)].

4.5.3. We continue to the next step of the decomposition given by $\mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C}}$.

Arguing similarly as in the proof of the previous proposition, we have the following.

Proposition 4.9. The character $\chi\left(\mathcal{U}(\mathcal{F}^{(0)}_{\mathbb{R}\times\mathbb{C}})(0)\right)$ is given by the plethystic exponential of the following expression:

(40)
$$f_0(t_1,t_2,t_3,q) = \frac{q}{(1-q)}\left(q^{1/2}\chi^{\mathfrak{sl}(4)}(\Gamma_{0,1,0}) + q^{-1/2}\chi^{\mathfrak{sl}(4)}(\Gamma_{2,0,0}) - q - \chi^{\mathfrak{sl}(4)}(\Gamma_{1,0,1})\right)$$

4.5.4. Finally, we continue to the general step of the decomposition.

Proposition 4.10. Let $j\geq 1$. The character $\chi\left(\mathcal{U}(\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C}})(0)\right)$ is the plethystic exponential of the following expression:

$$f_j(t_1,t_2,t_3,q) = \frac{q}{(1-q)}\begin{pmatrix} q^{(j-2)/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j+2,0,0}) + q^{(j+2)/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j+1,0,1}) \\ -q^{j/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j,1,0}) - q^{(j+1)/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j,0,0}) \end{pmatrix} \tag{41}$$

4.5.5. As a consequence, of the above we have that $f_{AdS_4\times S^7}(t_i,q)=\sum_{j\geq -2} f_j(t_i,q)$, or explicitly:

$$\frac{q\begin{pmatrix} q^{1/4}(t_1+t_2+t_3+t_1^{-1}t_2^{-1}t_3^{-1})+q^{-1} \\ -q^{-1/4}(t_1^{-1}+t_2^{-1}+t_3^{-1}+t_1t_2t_3)-q \end{pmatrix}}{(1-q)(1-q^{1/4}t_1)(1-q^{1/4}t_2)(1-q^{1/4}t_3)(1-q^{1/4}t_1^{-1}t_2^{-1}t_3^{-1})}$$
$$=1+\frac{q}{1-q}\begin{pmatrix} q^{-3/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{1,0,0})-q^{-1/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{0,0,1}) \\ +q^{1/2}\chi^{\mathfrak{sl}(4)}(\Gamma_{0,1,0})+q^{-1/2}\chi^{\mathfrak{sl}(4)}(\Gamma_{2,0,0})-q-\chi^{\mathfrak{sl}(4)}(\Gamma_{1,0,1}) \end{pmatrix}$$
$$+\frac{q}{1-q}\sum_{j\geq 1}\begin{pmatrix} q^{(j-2)/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j+2,0,0})+q^{(j+2)/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j+1,0,1}) \\ -q^{j/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j,1,0})-q^{(j+1)/4}\chi^{\mathfrak{sl}(4)}(\Gamma_{j,0,0}) \end{pmatrix}$$

In [BBMR08, Eq. (2.15, 2.16)], the index counting gravitons on $f_{AdS_4\times S^7}$ is expressed as a sum of characters of irreducible representations of the 3d $\mathcal{N}=8$ superconformal algebra that the authors call *graviton representations*. Comparison with the above expansion suggests the following conjecture

Conjecture 4.11. For $j\geq -1$, the minimal twist of the $j+2$nd graviton representation in [BBMR08, Eq. (2.15, 2.16)] is exactly $\mathcal{F}^{(j)}_{\mathbb{R}\times\mathbb{C},c}(0)$.

Remark 4.12. This conjecture implies that the minimal twist of these graviton representations, which is a priori a module for the minimally twisted 3d $\mathcal{N}=8$ superconformal algebra $\mathfrak{osp}(6|2)$, is in fact a module for the larger infinite dimensional super-Lie algebra $E(1|6)$. This can be thought of as analogous to the enhancement of conformal symmetries to the action of the Witt algebra of vector fields in 2d chiral conformal field theory. Such symmetry enhancements in 3 dimensions is the topic of joint work in progress with Garner and Williams.

5. $E(3|6)$-modules from gravitons on $AdS_7\times S^4$

We now repeat the analysis of the previous subsection for gravitons on $AdS_7\times S^4$ respectively, making use of the description of supergravity states on $AdS_7\times S^4$ as the costalk of the factorization envelopes of the boundary condition and $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$

As before, we construct certain $\mathbb{C}^\times$ actions on the boundary fields $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$; we find that the zeroth weight space is a local version of another exceptional linearly compact super-Lie algebra $E(3|6)$. The decomposition of $\mathcal{H}_{AdS_7\times S^4}$ as a direct sum of $E(3|6)$ modules incidentally turns out to be very closely related to a decomposition of $E(5|10)$ into $E(3|6)$ modules studied by Cheng-Kac [].

5.1. The graviton decomposition of twisted $AdS_7\times S^4$.

We wish to consider a particular decomposition of the space of states $\mathcal{H}_{AdS_7\times S^4}$. It is induced by a decomposition of the boundary fields $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ introduced in section 2.3.

Let $U\subset\mathbb{C}^3$ be an open; explicitly, the $\mathbb{C}^\times$ action on $\Omega^{0,\bullet}_{\mathbb{C}^3}(U,\mathcal{L}^{r=0}_{AdS_7\times S^4})$ is given as follows.

- On the fields $\mu(w_a,z_i)\in\mathbb{C}[w_1,w_2]\{\partial_{w_a}\}\otimes\Omega^{0,\bullet}_{\mathbb{C}^3}(U)\oplus\mathbb{C}[w_1,w_2]\otimes\Omega^{0,\bullet}_{\mathbb{C}^3}(U,T)$ the action is
$$\lambda\cdot\mu(w_a,z_i)=\mu(\lambda w_a,z_i).$$
- On the fields $\nu(w_a,z_i)\in\mathbb{C}[w_1,w_2]\otimes\Omega^{0,\bullet}_{\mathbb{C}^3}(U)$ the action is
$$\lambda\cdot\nu(w_a,z_i)=\nu(\lambda w,z).$$

- On the fields $\beta(w_a, z_i) \in \mathbb{C}[w_1, w_2] \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}(U)$ the action is
$$\lambda \cdot \beta(w_a, z_i) = \lambda^{-1}\beta(\lambda w_a, z_i).$$
- On the fields $\gamma(w_a, z_i) \in \mathbb{C}[w_1, w_2]\{\mathrm{d}w_a\} \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}(U) \oplus \mathbb{C}[w_1, w_2] \otimes \Omega^{0,\bullet}_{\mathbb{C}^3}(U, \Omega^1)$ the action is
$$\lambda \cdot \gamma(w_a, z_i) = \lambda^{-1}\gamma(\lambda w_a, z_i).$$

The following proposition is a straightforward if lengthy computation - we state it without proof.

Proposition 5.1. The L_∞ structure on $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7 \times S^4})$ identified in remark 6.4.4 is equivariant for this $\mathbb{C}^\times$ action.

This result induces a product decomposition

$$\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7 \times S^4}) = \prod_{n \geq -1} \mathcal{G}^{(n)}_{\mathbb{C}^3} \tag{42}$$

where for each open set $U \subset \mathbb{C}^3$

$$\mathcal{G}^3_{\mathbb{C}}(U)^{(n)} \subset \Omega^{0,\bullet}_{\mathbb{C}^3}(U, \mathcal{L}^{r=0}_{AdS_7 \times S^4})$$

is the weight n eigenspace with respect to the above $\mathbb{C}^\times$ action. In particular, we see that $\mathcal{G}^{(0)}_{\mathbb{C}^3}$ is itself a local dg-Lie algebra (that we will soon describe). Moreover, every $\mathcal{G}^{(n)}_{\mathbb{C}^3}$, $n \geq -1$ is a (local) module for this local dg-Lie algebra.

5.2. **The lowest piece: the holomorphic twist of the abelian 6d $\mathcal{N} = (2, 0)$ tensor multiplet.** The first non trivial case is the weight (-1) piece.

Lemma 5.2. There is an equivalence of abelian local Lie algebras

$$\mathcal{G}^{(-1)}_{\mathbb{C}^3} \cong \Omega^{0,\bullet}_{\mathbb{C}^3} \begin{pmatrix} \underline{+} & & \underline{-} \\ & & \\ \mathbb{C}^2 \otimes K^{1/2}_{\mathbb{C}^3} & & \\ & & \\ \mathcal{O}_{\mathbb{C}^3} & \xrightarrow{\partial} & \Omega^1_{\mathbb{C}^3} \end{pmatrix}$$

Proof. We readily see that the fields of weight -1 include

- fields of type μ of the form $\mu_a(z_i)\partial_{w_a}$. As w_a are fiber coordinates on $K^{-1/2}_{\mathbb{C}^3}$, these fields transform as sections of $K^{-1/2}_{\mathbb{C}^3}$.
- fields of type β with no w_a-dependence. These fields constitute a copy of $\mathcal{O}_{\mathbb{C}^3}$.
- fields of type γ of the form $\gamma_i(z_i)\mathrm{d}z_i$. These fields constitute a copy of $\Pi\Omega^1_{\mathbb{C}^3}$.

Since ∂ is weight zero for this $\mathbb{C}^\times$ action, the fields of the last two type combine to give the complex of sheaves

$$\mathcal{O}_{\mathbb{C}^3} \xrightarrow{\partial} \Pi\Omega^1_{\mathbb{C}^3}.$$

□

5.2.1. In [SW20] Saberi and Williams, the authors studied the minimal twist of the 6d $\mathcal{N} = (2, 0)$ abelian tensor multiplet. The twist is a free theory and can be defined on any complex three-fold admitting a square root of its canonical bundle. On $\mathbb{C}^3$, the $\mathbb{Z} \times \mathbb{Z}/2$ graded sheaf of complexes $\mathcal{E}_{tens}$ encoding its field content is given by

$$
\begin{array}{ccc}
\underline{-1} & & \underline{0} \\
\Pi\mathbb{C}^2 \otimes \Omega^{0,\bullet}_{\mathbb{C}^3} \otimes K^{1/2}_{\mathbb{C}^3} & & \\
\Omega^{2,\bullet}_{\mathbb{C}^3} & \xrightarrow{\quad\partial\quad} & \Omega^{3,\bullet}_{\mathbb{C}^3}
\end{array} \tag{43}
$$

Here we recall in the $\mathbb{Z} \times \mathbb{Z}/2$ bigrading the differential has bidegree $(1, 0)$.

We observe the following:

Proposition 5.3. There is a quasi-isomorphism of factorization algebras valued in $\mathbb{Z}/2$ graded commutative dg algebras on $\mathbb{C}^3$

$$\mathcal{U}(\mathcal{G}^{(-1)}_{\mathbb{C}^3}) \xrightarrow{\simeq} \mathrm{C}^\bullet(\Pi\mathcal{E}_{tens})$$

Proof. Recall that the factorization algebra $\mathcal{U}(\mathcal{G}^{(-1)}_{\mathbb{C}^3})$ assigns to an open set $U \subset \mathbb{C}^3$ the graded symmetric algebra on the complex

$$
\begin{array}{ccc}
\underline{-} & & \underline{+} \\
\Omega^{0,\bullet}_{\mathbb{C}^3,c}(U) & \xrightarrow{\quad\partial\quad} & \Omega^{1,\bullet}_{\mathbb{C}^3,c}(U) \\
\Omega^{0,\bullet}_{\mathbb{C}^3,c}(U, \mathbb{C}^2 \otimes K^{1/2}) & &
\end{array} \tag{44}
$$

On the other hand, if we totalize the $\mathbb{Z} \times \mathbb{Z}/2$-grading on $\mathcal{E}_{tens}$ to a $\mathbb{Z}/2$-grading, the factorization algebra $\mathrm{C}^\bullet(\Pi\mathcal{E}_{tens})$ assigns to an open set $U \subset \mathbb{C}^3$ the symmetric algebra on the complex

$$
\begin{array}{ccc}
\underline{-} & & \underline{+} \\
\Omega^{2,\bullet}_{\mathbb{C}^3}(U)^\vee & \xrightarrow{\quad\partial\quad} & \Omega^{3,\bullet}_{\mathbb{C}^3}(U)^\vee \\
& & \Omega^{0,\bullet}_{\mathbb{C}^3}(U, \mathbb{C}^2 \otimes K^{1/2})^\vee
\end{array} \tag{45}
$$

Here the superscript refers to the topological dual, which is described in terms of compactly supported distributional sections of the Serre dual vector bundle. Thus, we see that the above complex is the same as

$$\underline{-} \qquad\qquad \underline{\pm}$$

$$\overline{\Omega}^{0,\bullet}_{\mathbb{C}^3,c}(U) \xrightarrow{\quad\partial\quad} \overline{\Omega}^{1,\bullet}_{\mathbb{C}^3,c}(U) \tag{46}$$

$$\overline{\Omega}^{0,\bullet}_{\mathbb{C}^3,c}(U, \mathbb{C}^2 \otimes K^{1/2})$$

where the degree shift is coming from Serre duality. The result then follows from the fact that by ellipticity, the natural inclusion $\Omega^{0,\bullet}_{\mathbb{C}^3,c} \to \overline{\Omega}^{0,\bullet}_{\mathbb{C}^3,c}$ is a quasi-isomorphism. □

5.3. **The zero-th piece: a local version of $E(3|6)$.** As before, the weight zero summand $\mathcal{G}^{(0)}_{\mathbb{C}^3}$ is special because it carries the induced structure of a local L_∞-algebra on $\mathbb{C}^3$ inherited from the L_∞ structure on $\Pi\Omega^{0,\bullet}(\mathcal{L}^{r=0}_{AdS_7 \times S^4})$ identified in section 2.3. We will prove that it is equivalent to a local Lie algebra version of the exceptional super-Lie algebra $E(3|6)$ [Kac98].

We first recall the definition of this super-Lie algebra.

Definition 5.4. Let $E(3|6)$ be the following super-Lie algebra.

- The even part, $E(3|6)_0$ is given by the semidirect product Lie algebra $\Gamma(\widehat{D}, T) \ltimes \left(\mathfrak{sl}(2) \otimes \Gamma(\widehat{D}, \mathcal{O})\right)$.
- The odd part, $E(3|6)_1$ is given by $\mathbb{C}^2 \otimes \Gamma(\widehat{D}, \Omega^1(K^{-1/2}))$.

The remaining brackets to be specified, are as follows:

- The action of $E(3|6)_0$ on $E(3|6)_1$ is given by the Lie derivative, along with the fundamental action of $\mathfrak{sl}(2)$.
- The bracket between two odd elements is given by

$$\begin{aligned}
&[v_1 \otimes f_i \mathrm{d}z_i \otimes (\partial_{z_1}\partial_{z_2}\partial_{z_3})^{1/2}, v_2 \otimes g_j \mathrm{d}z_j \otimes (\partial_{z_1}\partial_{z_2}\partial_{z_3})^{1/2}] \\
&\qquad = \omega(v_1, v_2)\varepsilon^{ijk} f_i g_j \partial_{z_k} \\
&\qquad + (v_1 \odot v_2)\left(\partial(f_i \mathrm{d}z_i) g_j \mathrm{d}z_j - f_i \mathrm{d}z_i \partial(g_j \mathrm{d}z_j)\right) \vee (\partial_{z_1}\partial_{z_2}\partial_{z_3})
\end{aligned}$$

  where ω denotes a symplectic form on $\mathbb{C}^2$ and $\odot : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathfrak{sl}(2)$ is the canonical $\mathfrak{sl}(2)$-equivariant projection.

The relationship between this super-Lie algebra and our decomposition is established through the following result.

Proposition 5.5. There is an equivalence of super-Lie algebras

$$\mathcal{G}^{(0)}_{\mathbb{C}^3,c}(0) \cong E(3|6)$$

Proof. We begin by characterizing the local L_∞-algebra $\mathcal{G}^{(0)}_{\mathbb{C}^3}$. We claim that it is quasi-isomorphic to a local version of $E(3|6)$.

Indeed, we readily see that the weight zero sections consist of the following cochain complex

$$\Omega^{0,\bullet}_{\mathbb{C}^3}\left(\begin{tikzcd} \underline{\textit{even}} & \underline{\textit{odd}} \\ \mathbb{C}\{w_a\partial_{w_b}\}\otimes\mathcal{O} \arrow[r, "\partial_\Omega^W"] & \mathcal{O} \\ \mathrm{T} \arrow[ur, "\partial_\Omega^Z"] & \\ \mathbb{C}^2\otimes\mathcal{O} \arrow[r, "\partial_W"] \arrow[dr, "\partial_Z"] & \mathbb{C}\{\mathrm{d}w_a\}\otimes K^{-1/2} \\ & \mathbb{C}^2\otimes\Omega^1\otimes K^{-1/2}\end{tikzcd}\right) \tag{47}$$

The differentials are again components of the divergence operator and holomorphic deRham operator. We can compute cohomology by way of a spectral sequence whose first page is the cohomology with respect to $\partial_\Omega^W + \partial_W$. We see that the differential ∂_Ω^W maps surjectively onto functions and its kernel is isomorphic to $\mathfrak{sl}(2)\otimes\mathcal{O}$. Likewise, the differential ∂_W is just the identity map between $\mathbb{C}^2$ and $\mathbb{C}\{\mathrm{d}w_a\}$.

Thus we see that this page of the spectral sequence is given by

$$\mathcal{E}(3|6) \overset{def}{=} \Omega^{0,\bullet}_{\mathbb{C}^3}\left(\begin{array}{cc} \underline{\textit{even}} & \underline{\textit{odd}} \\ \mathrm{T} & \mathbb{C}^2\otimes\Omega^1_{\mathbb{C}^3}(K^{-1/2}) \\ \mathfrak{sl}(2)\otimes\mathcal{O} & \end{array}\right) \tag{48}$$

and there are no non-zero differentials so the spectral sequence degenerates.

To see that the Lie structure induced from the L_∞-structure on $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ is in fact given by the same formulae as the brackets on $E(3|6)$ given in 5.4, it will be useful to provide an explicit quasi-isomorphism $\Phi^{(0)} : \mathcal{E}(3|6) \to \mathcal{G}^{(0)}_{\mathbb{C}^3}$. On an open set $U \subset \mathbb{C}^3$, this is defined as follows.

- Given a section $g_i(z)\partial_{z_i} \in \Omega^{0,\bullet}_{\mathbb{C}^3}(U, T)$ where $g_i(z)$ is a Dolbeault form on U, we define

$$\begin{aligned}\Phi^{(0)}(g_i(z)\partial_{z_i}) &= g_i(z)\partial_{z_i} - \frac{1}{2}(\partial_{z_i} g_i(z)) w_a \partial_{w_a} \\ &\in \Omega^{0,\bullet}_{\mathbb{C}^3}\left(U, T \oplus \mathbb{C}\{w_a\partial_{w_b}\}\otimes\mathcal{O}\right).\end{aligned}$$

- Given a section $A\otimes g(z) \in \Omega^{0,\bullet}_{\mathbb{C}^3}(U, \mathfrak{sl}(2)\otimes\mathcal{O})$ where $g(z)$ is a Dolbeault form on U, and $A_{ab}\in\mathfrak{sl}(2)$ we define

$$\begin{aligned}\Phi^{(0)}(A_{ab}\otimes g(z)) &= g(z) A_{ab} w_a \partial_{w_b} \\ &\in \Omega^{0,\bullet}_{\mathbb{C}^3}\left(\mathbb{C}\{w_a\partial_{w_b}\}\otimes\mathcal{O}\right).\end{aligned}$$

- Given a section $v \otimes g_i(z)\mathrm{d}z_i(\partial_{z_1}\partial_{z_2}\partial_{z_3}) \in \Omega^{0,\bullet}_{\mathbb{C}^3}(U, \mathbb{C}^2 \otimes \Omega^1 \otimes K^{-1/2})$ where $g_i(z)$ is a Dolbeault form on U and $v \in \mathbb{C}^2$, we define

$$\Phi^{(0)}(v \otimes g_i(z)\mathrm{d}z_i(\partial_{z_1}\partial_{z_2}\partial_{z_3})) = (w_1(v)w_1 + w_2(v)w_2) \otimes g_i(z)$$
$$\in \Omega^{0,\bullet}_{\mathbb{C}^3}\left(\mathbb{C}^2 \otimes \Omega^1 \otimes K^{-1/2}\right).$$

The result then follows from computing the limit of $\mathcal{E}(3|6)_c(D^3)$ over open sets containing the origin. □

Remark 5.6. We note that the map i_{M5} from lemma 3.5 in fact defines a Lie map from $\mathfrak{osp}(6|2)$ to the sections of the boundary condition $\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ over every open set containing the origin. The image of the map lands exactly in the step $\mathcal{G}^{(0)}_{\mathbb{C}^3}$ of the decomposition from proposition 5.1. Therefore we see that $E(3|6)$ contains $\mathfrak{osp}(6|2)$ as a finite dimensional subalgebra.

5.4. **General summands and $E(3|6)$-modules.** We move on to give the following general description of the weight j component $\mathcal{G}^{(j)}_{\mathbb{C}^3}$. Since we have already described $j = -1, 0$ we focus on $j \geq 1$.

Proposition 5.7. Let $j \geq 1$. The complex of vector bundles $\mathcal{G}^{(j)}$ is quasi-isomorphic to

$$\Omega^{0,\bullet}_{\mathbb{C}^3}\begin{pmatrix} \underline{\mathit{even}} & \underline{\mathit{odd}} \\ \mathrm{Sym}^j(\mathbb{C}^2)\otimes \mathrm{T} \otimes K^{-j/2} & \mathrm{Sym}^{j-1}(\mathbb{C}^2)\otimes K^{-(j+1)/2} \\ \mathrm{Sym}^{j+2}(\mathbb{C}^2)\otimes K^{-j/2} & \mathrm{Sym}^{j+1}(\mathbb{C}^2)\otimes \mathrm{T}^* \otimes K^{-(j+1)/2}. \end{pmatrix} \tag{49}$$

Proof. We begin by noting that we can explicitly describe the weight j component $\mathcal{G}^{(j)}_{\mathbb{C}^3}$ as $\Omega^{0,\bullet}_{\mathbb{C}^3}(G^{(j)})$ where $G^{(j)}$ is the following dg-vector bundle

$$
\begin{tikzcd}
\underline{\textit{even}} & \underline{\textit{odd}} \\
\mathrm{Sym}^{j+1}(\mathbb{C}^2)\otimes\mathbb{C}^2\otimes K^{-j/2} \arrow[dr, "\partial_\Omega^W"'] & \\
& \mathrm{Sym}^{j}(\mathbb{C}^2)\otimes K^{-j/2} \\
\mathrm{Sym}^{j}(\mathbb{C}^2)\otimes \mathrm{T}\otimes K^{-j/2} \arrow[ur, "\partial_\Omega^Z"] & \\
& \mathrm{Sym}^{j+1}(\mathbb{C}^2)\otimes \Omega^1\otimes K^{-(j+1)/2} \\
\mathrm{Sym}^{j+1}(\mathbb{C}^2)\otimes K^{-(j+1)/2} \arrow[ur, "\partial_Z"] \arrow[dr, "\partial_W"'] & \\
& \mathrm{Sym}^{j}(\mathbb{C}^2)\otimes\mathbb{C}^2\otimes K^{-(j+1)/2}
\end{tikzcd}
\tag{50}
$$

Note that the differentials here are $\mathfrak{sl}(2)$-equivariant maps, tensored with a differential operator acting on sections of a bundle on $\mathbb{C}^3$. In particular

- The differential ∂_Ω^W is the canonical projection
$$\mathrm{Sym}^{j+1}(\mathbb{C}^2)\otimes\mathbb{C}^2 \cong \mathrm{Sym}^{j+2}(\mathbb{C}^2)\oplus\mathrm{Sym}^{j}(\mathbb{C}^2)\twoheadrightarrow \mathrm{Sym}^{j}(\mathbb{C}^2)$$
tensored with the identity acting on $K^{-j/2}$.
- The differential ∂_W is the canonical inclusion
$$\mathrm{Sym}^{j+1}(\mathbb{C}^2)\hookrightarrow \mathrm{Sym}^{j-1}(\mathbb{C}^2)\oplus\mathrm{Sym}^{j+1}(\mathbb{C}^2)\cong \mathrm{Sym}^{j}(\mathbb{C}^2)\otimes\mathbb{C}^2.$$
tensored with the identity acting on $K^{-(j+1)/2}$.

There is a spectral sequence whose first term is computed by the $\partial_\Omega^W+\partial_W$-cohomology. The result is the complex of sheaves in equation 49. There are no further differentials so the spectral sequence collapses at this page and the result follows. $\square$

5.4.1. The decomposition of $\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ in equation (42) is closely related to a decomposition of the exceptional simple super Lie algebra $E(5|10)$ studied in [KR01]. In **??**, we showed that the global sections of the parity shifted fields of our eleven-dimensional theory on flat space is quasi-isomorphic to a Lie 2-extension of $E(5|10)$, which we denoted $\widehat{E(5|10)}$. More precisely, we found a Lie 2-extension of a version of $E(5|10)$ built out of polynomials rather than Taylor series. Given that the L_∞ structure on $\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ is given by the same formulas as that on $\Pi\mathcal{E}$, it is easy to see that the space of ∞-jets of $\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ at the origin, which following lemma 6.4.4 has underlying vector space $\mathcal{H}_{AdS_7\times S^4}$, is quasi-isomorphic to $\widehat{E(5|10)}$.

In [KR01] the following weight decomposition of $E(5|10)$ is constructed. Splitting $\mathbb{C}^5 = \mathbb{C}^2_{w_a} \times \mathbb{C}^3_{z_i}$ as we have been doing, we stipulate that

- the coordinate z_i has weight zero, $\widetilde{(}z_i) = 0$.
- the coordinate w_a has weight $+1$, $\widetilde{(}w_a) = +1$.
- the parity of an element carries an additional weight of -1. Thus, for example, the odd element $[\mathrm{d}w_1 \mathrm{d}z_1] \in \Omega^{2,cl}(\widehat{D}^5)$ carries weight $+1-1=0$. Viewing the odd part as the space of closed two-forms, then equivalently this grading translates to the one-form symbol $\mathrm{d}(-)$ as carrying weight $-1/2$.

It is straightforward to verify that this weight grading is compatible with the super Lie algebra structure on $E(5|10)$. Moreover, we see that similarly to the decomposition of $\mathcal{H}_{AdS_7\times S^4}$ induced by our $\mathbb{C}^\times$ action in 5.1, the weight grading is concentrated in degrees ≥ -1. In particular, there is a decomposition of super vector spaces

$$E(5|10) = \tilde{U}_{-1} \times \prod_{j\geq 0} U_j \tag{51}$$

Further, this decomposition also has the property that the 0-th piece U_0 is isomorphic to $E(3|6)$. As such, each U_j is an $E(3|6)$-module; Kac characterizes these modules explicitly and identifies them as certain irreducible $E(3|6)$-modules. In the notation of [KR01] we have that $U_{-1} = I(0,0;1;-1)^*$ and for $j \geq 1$, $U_j = I(0,0;j-1;j+1)^*$.

The decomposition of $\mathcal{H}_{AdS_7\times S^4}$ afforded by proposition 5.1 induces a weight grading of $\widehat{E(5|10)}$ which extends the one on $E(5|10)$ that we have just described by declaring that the central term have weight -1. In this way, we get a related decomposition of super L_∞ algebras

$$\widehat{E(5|10)} = \prod_{j\geq -1} U_j \tag{52}$$

were U_{-1} is a $\mathbb{C}$-extension of $\tilde{U}_{-1}$ defined in the decomposition (51) and for $j \geq 0$ the U_j's are the same as in the non centrally extended case. As a corollary, we see that each of the $E(3|6)$ modules which we have identified as the costalk at 0 $\mathcal{G}^{(j)}_{\mathbb{C}^3,c}(0)$ is in fact irreducible.

5.5. **Characters of $E(3|6)$-modules.** The decomposition of the state space $\mathrm{Sym}\,(\mathcal{H}_{AdS_7\times S^4}) = \prod_{j\geq -1} \mathcal{U}(\mathcal{G}^{(j)})(0)$ gives a product formula for the characters computed in proposition 2.10

$$\chi\left(\mathrm{Sym}\mathcal{H}_{AdS_7\times S^4}\right) = \prod_{j\geq -1} \chi\left(\mathcal{U}(\mathcal{G}^{(j)})(0)\right)$$

We end this section by computing each of the characters $\chi\left(\mathcal{U}(\mathcal{G}^{(j)})(0)\right)$. We will express our characters in terms of characters of highest weight representations of $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$, which we denote by $\chi^{\mathfrak{sl}(2)}_k$ and $\chi^{\mathfrak{sl}(3)}_{[k,l]}$.

5.5.1. We begin with the lowest step of the decomposition, using the characterization given in 5.2.

Proposition 5.8. The character $\chi\left(\mathcal{U}(\mathcal{G}^{(-1)}_{\mathbb{C}^3})(0)\right)$ is given by the plethystic exponential of the following expression:

$$g_{-1}(t_1,t_2,r,q) = \frac{q^{3/2}(r+r^{-1}) - q^2(t_1+t_1^{-1}t_2+t_2^{-1}) + q^3}{(1-t_1^{-1}q)(1-t_1t_2^{-1}q)(1-t_2q)}. \tag{53}$$

Proof. Note that in light of proposition 5.3 we can equivalently compute the character of the costalk at the origin of $\mathrm{C}^\bullet(\Pi\mathcal{E}_{tens})$. We first give a more explicit description of the costalk as a cochain complex. Proceeding exactly analogously to the proof of lemma 2.5, we see that the costalk is given by the symmetric algebra on the following cochain complex

$$\begin{array}{ccc}
\underline{-} & & \underline{+} \\
\mathbb{C}\{\mathrm{d}z_i\mathrm{d}z_j\}\otimes\mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0} & \xrightarrow{\partial} & \mathbb{C}\{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\}\otimes\mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0} \\
 & & \oplus \\
 & & \mathbb{C}^2\otimes\mathbb{C}\{\mathrm{d}z_i^{1/2}\}\otimes\mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0}
\end{array}$$

Computing summand-by-summand, we see:

- the odd summand $\mathbb{C}\{\mathrm{d}z_i\mathrm{d}z_j\}\otimes\mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0}$ contributes

$$-q^2\frac{\chi^{\mathfrak{sl}(3)}_{[0,1]}(t_1,t_2)}{(1-t_1^{-1}q)(1-t_1t_2^{-1}q)(1-t_2q)} = -q^2\frac{t_1+t_1^{-1}t_2+t_2^{-1}}{(1-t_1^{-1}q)(1-t_1t_2^{-1}q)(1-t_2q)}.$$

  where $\chi^{\mathfrak{sl}(3)}_{[1,0]}(t_1,t_2)$ is the $\mathfrak{sl}(3)$ character of highest weight $[1,0]$.
- the even summand $\mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0}$ contributes

$$q^3\frac{1}{(1-t_1^{-1}q)(1-t_1t_2^{-1}q)(1-t_2q)}.$$

- the even summand $\mathbb{C}^2\otimes\mathbb{C}\{\mathrm{d}z_i^{1/2}\}\otimes\mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0}$ contributes

$$q^{3/2}\frac{\chi^{\mathfrak{sl}(2)}_1(r)}{(1-t_1^{-1}q)(1-t_1t_2^{-1}q)(1-t_2q)} = q^{3/2}\frac{(r+r^{-1})}{(1-t_1^{-1}q)(1-t_1t_2^{-1}q)(1-t_2q)}.$$

  where $\chi^{\mathfrak{sl}(2)}_1(r)$ is the $\mathfrak{sl}(2)$ character of highest weight one.

□

In terms of the parameters y_1,y_2,y_3,y,q introduced in 6.4.4 this single particle character reads

$$g_{-1}(y_i,y,q) = \frac{qy+q^2y^{-1}-q^2(y_1^{-1}+y_2^{-1}+y_3^{-1})+q^3}{(1-y_1q)(1-y_2q)(1-y_3q)}. \tag{54}$$

The expression matches exactly with the index of the abelian six-dimensional superconformal theory. For example, compare with [KKKL13, Eq. (3.1)] or [BBMR08, Eq. (3.35)].

From now on, we will give all formulas for the index in terms of the parameters y_1,y_2,y_3,y,q.

5.5.2. We continue to the next step of the decomposition, which is given by $\mathcal{G}^{(0)}_{\mathbb{C}^3}$.

Proposition 5.9. The character $\chi\left(\mathcal{U}(\mathcal{G}^{(0)}_{\mathbb{C}^3})(0)\right)$ is the plethystic exponential of following expression:

(55)
$$g_0(y_i, y, q) = \frac{q^4(y_1+y_2+y_3) + q^2(y^2 + q + q^2y^{-2}) - q^3(y+qy^{-1})(y_1^{-1}+y_2^{-1}+y_3^{-1})}{(1-y_1q)(1-y_2q)(1-y_3q)}.$$

Proof. As usual, we wish to describe the costalk $\mathcal{G}^{(0)}_{\mathbb{C}^3,c}(0)$ more explicitly. By the same argument as in the proofs of propositions 6.4.4, 6.4.4, we may use elliptic regularity to describe the compactly supported smooth sections on a disc in terms of derivatives of the delta function at the origin in $\mathbb{C}^3$.

Accordingly, we have contributions from the following summands.

- An even copy of $\mathbb{C}\{\partial_{z_i}\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0}$. The character of this summand is
$$q^4 \frac{\chi^{\mathfrak{sl}(3)}_{[1,0]}(y_i)}{(1-y_1q)(1-y_2q)(1-y_3q)} = q^4 \frac{y_1+y_2+y_3}{(1-y_1q)(1-y_2q)(1-y_3q)}.$$
- An even copy of $\mathfrak{sl}(2) \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0}$. The character of this summand is
$$q^3 \frac{\chi^{\mathfrak{sl}(2)}_2(q^{-1/2}y)}{(1-y_1q)(1-y_2q)(1-y_3q)} = \frac{q^2y^2 + q^3 + q^4y^{-2}}{(1-y_1q)(1-y_2q)(1-y_3q)}.$$
- An odd copy of $\mathbb{C}^2 \otimes \mathbb{C}\{\mathrm{d}z_i\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0}$. The character of this summand is
$$q^{7/2} \frac{\chi^{\mathfrak{sl}(2)}_1(q^{-1/2}y)\,\chi^{\mathfrak{sl}(3)}_{[0,1]}(y_i)}{(1-y_1q)(1-y_2q)(1-y_3q)} = q^3 \frac{(y+qy^{-1})(y_1^{-1}+y_2^{-1}+y_3^{-1})}{(1-y_1q)(1-y_2q)(1-y_3q)}.$$

□

5.5.3. Finally, we continue to the general step of the decomposition.

Proposition 5.10. Let $j \geq 1$. The character $\chi\left(\mathcal{U}(\mathcal{G}^{(j)}_{\mathbb{C}^3})(0)\right)$ is the plethystic exponential of following expression:

(56)
$$g_j(y_i, y, q) = \frac{q^3\begin{pmatrix} q^{1+3j/2}\chi^{\mathfrak{sl}(2)}_j(q^{-1/2}y)\chi^{\mathfrak{sl}(3)}_{[1,0]}(y_i) + q^{3j/2}\chi^{\mathfrak{sl}(2)}_{j+2}(q^{-1/2}y) \\ -q^{3(j+1)/2}\chi^{\mathfrak{sl}(2)}_{j-1}(q^{-1/2}y) - q^{-1+3(j+1)/2}\chi^{\mathfrak{sl}(2)}_{j+1}(q^{-1/2}y)\chi^{\mathfrak{sl}(3)}_{[0,1]}(y_i)\end{pmatrix}}{(1-y_1q)(1-y_2q)(1-y_3q)}.$$

Proof. We proceed exactly analogously to all the previous cases. Using elliptic regularity on sections of $\mathcal{G}^{(j)}_{\mathbb{C}^3}$ over a disc containing the origin, we see that $\mathcal{U}(\mathcal{G}^{(j)}_{\mathbb{C}^3})(0)$ is a symmetric algebra on a cochain complex with the following summands

- An even copy of $\mathrm{Sym}^j(\mathbb{C}^2) \otimes \mathbb{C}\{\partial_{z_i}\} \otimes \mathbb{C}[\partial_{z_1}, \partial_{z_2}, \partial_{z_3}]\delta_{z_i=0} \otimes (\partial_{z_1}\partial_{z_2}\partial_{z_3})^{-j/2}$ which contributes

(57)
$$\frac{q^3\left(q^{1+3j/2}\chi^{\mathfrak{sl}(2)}_j(q^{-1/2}y)\chi^{\mathfrak{sl}(3)}_{[1,0]}(y_i)\right)}{(1-y_1q)(1-y_2q)(1-y_3q)}$$

- An even copy of $\mathrm{Sym}^{j+2}(\mathbb{C}^2)\otimes\mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0}\otimes(\partial_{z_1}\partial_{z_2}\partial_{z_3})^{-j/2}$ which contributes

$$\frac{q^3\left(q^{3j/2}\chi^{\mathfrak{sl}(2)}_{j+2}(q^{-1/2}y)\right)}{(1-y_1q)(1-y_2q)(1-y_3q)} \tag{58}$$

- An odd copy of $\mathrm{Sym}^{j-1}(\mathbb{C}^2)\otimes\mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0}\otimes(\partial_{z_1}\partial_{z_2}\partial_{z_3})^{-(j+1)/2}$ which contributes

$$\frac{-q^3\left(q^{3(j+1)/2}\chi^{\mathfrak{sl}(2)}_{j-1}(q^{-1/2}y)\right)}{(1-y_1q)(1-y_2q)(1-y_3q)} \tag{59}$$

- An odd copy of $\mathrm{Sym}^{j+1}(\mathbb{C}^2)\otimes\mathbb{C}\{\mathrm{d}z_i\}\otimes\mathbb{C}[\partial_{z_1},\partial_{z_2},\partial_{z_3}]\delta_{z_i=0}\otimes(\partial_{z_1}\partial_{z_2}\partial_{z_3})^{-(j+1)/2}$ which contributes

$$\frac{-q^3\left(q^{-1+3(j+1)/2}\chi^{\mathfrak{sl}(2)}_{j+1}(q^{-1/2}y)\chi^{\mathfrak{sl}(3)}_{[0,1]}(y_i)\right)}{(1-y_1q)(1-y_2q)(1-y_3q)} \tag{60}$$

□

5.5.4. As a consequence, we have that $f_{AdS_7\times S^4}(y_i,y,q)=\sum_{j\geq -1}g_j(y_i,y,q)$, or explicitly:

$$\begin{aligned}
&\frac{q^4(y_1+y_2+y_3)-q^2(y_1^{-1}+y_2^{-1}+y_3^{-1})+(1-q^3)(yq+y^{-1}q^2)}{(1-y_1q)(1-y_2q)(1-y_3q)(1-yq)(1-y^{-1}q^2)}\\
&=\frac{qy+q^2y^{-1}-q^2(y_1^{-1}+y_2^{-1}+y_3^{-1})+q^3}{(1-y_1q)(1-y_2q)(1-y_3q)}\\
&+\frac{q^4(y_1+y_2+y_3)+q^2(y^2+q+q^2y^{-2})-q^3(y+qy^{-1})(y_1^{-1}+y_2^{-1}+y_3^{-1})}{(1-y_1q)(1-y_2q)(1-y_3q)}\\
&+\sum_{j\geq 1}\frac{q^3\begin{pmatrix} q^{1+3j/2}\chi^{\mathfrak{sl}(2)}_{j}(q^{-1/2}y)\chi^{\mathfrak{sl}(3)}_{[1,0]}(y_i)+q^{3j/2}\chi^{\mathfrak{sl}(2)}_{j+2}(q^{-1/2}y)\\ -q^{3(j+1)/2}\chi^{\mathfrak{sl}(2)}_{j-1}(q^{-1/2}y)-q^{-1+3(j+1)/2}\chi^{\mathfrak{sl}(2)}_{j+1}(q^{-1/2}y)\chi^{\mathfrak{sl}(3)}_{[0,1]}(y_i)\end{pmatrix}}{(1-y_1q)(1-y_2q)(1-y_3q)}
\end{aligned}$$

In [BBMR08, Eq. (3.22, 3.23)], the index counting gravitons on $f_{AdS_7\times S^4}$ is expressed as a sum of characters of irreducible representations of the 6d $\mathcal{N}=(2,0)$ superconformal algebra. In [CDI16, Table 24] these representations are labeled as $\mathcal{D}_1[0,0,0]^{(0,m)}_{2m}$ where $m\geq 1$. The characters of these modules have been computed (see for example [AFI+20, Eq. (166)] and match exactly with $g_{m-2}(y_i,y,q)$ after a suitable change of variables. Thus, we conjecture the following

Conjecture 5.11. For $j\geq -1$, the minimal twist of $\mathcal{D}_1[0,0,0]^{(0,j+2)}_{2(j+2)}$ is exactly $\mathcal{G}^{(j)}_{\mathbb{C}^3,c}(0)$.

Remark 5.12. As we remarked in 4.12, this conjecture implies that the minimal twist of $\mathcal{D}_1[0,0,0]^{(0,j+2)}_{2(j+2)}$ which is a priori a module for the minimally twisted 6d $\mathcal{N}=(2,0)$ superconformal algebra $\mathfrak{osp}(6|2)$, is in fact a module for the larger infinite dimensional super-Lie algebra $E(3|6)$. This can be thought of as analogous to the enhancement of conformal symmetries to the action of the Witt algebra of vector fields in 2d chiral conformal field theory.

6. Holographic Speculations

We began this thesis with some remarks on how dualities between physical theories can often be used to uncover novel equivalences between the mathematical objects that describe them. In this final section of the thesis, we offer some speculations to this effect. We caution the reader that a large portion of this section involves recalling constructions from physics without any attention to rigor for motivational purposes.

In sections 4.2, 5.2, we commented on how minimal twists of the 3d $\mathcal{N} = 8$ BLG theory and 6d $\mathcal{N} = (2, 0)$ tensor multiplets are are visible as pieces of the graviton decompositions of twisted $AdS_4 \times S^7$ and $AdS_7 \times S^4$ respectively. Famous instances of the AdS/CFT correspondence posit equivalences between the higher rank 3d $\mathcal{N} = 8$ theories studied by ABJM and the higher rank 6d $\mathcal{N} = (2, 0)$ theories of type A_N with M-theory on $AdS_4 \times S^7$ and $AdS_7 \times S^4$ respectively. It is natural to wonder whether the twisted holograpahy proposal mentioned in the introduction can be applied to our descriptions of the twisted $AdS_4 \times S^7$ and $AdS_7 \times S^4$ backgrounds to study the minimal twists of the higher rank 3d $\mathcal{N} = 8$ and 6d $\mathcal{N} = (2, 0)$ superconformal field theories respectively.

Our goal in this section is to posit some expectations regarding the minimal twist of the 6d $\mathcal{N} = (2, 0)$ theory of type A_N. This theory is notorious for being both ubiquitous and nebulous. On the one hand, almost every superconformal field theory that has had interesting applications to geometry, topology, or representation theory occurs as one of its dimensional reductions, so it has long been expected to contain very rich mathematics. On the other hand, it does not admit a Lagrangian description. Its only free parameter is the rank of an ADE Lie algebra, and outside of the abelian case, a field realization is not even known.

We begin by recalling some features of the AdS/CFT correspondence. We will begin with a more physical language, and work towards some concrete mathematical expectations.

6.1. The AdS/CFT correspondence.

Traditional formulations of the AdS/CFT correspondence relate two theories, schematically denoted T_{CFT} and T_{grav} on manifolds M_1, M_2 respectively, together with a conformal diffeomorphism $\partial M_2 \cong M_1$. The theories have the feature that boundary values of fields of T_{grav} denoted $\phi|_\partial$, may be identified with sources for T_{CFT} denoted J. The two theories are considered to be holographically dual when their partition functions are equivalent $Z_{CFT}[J] = Z_{grav}[\phi|_\partial]$.

In examples of stringy origin, T_{CFT} describes the low energy dynamics of a stack of N branes in supergravity, in the large N limit, and T_{grav} describes gravitational dynamics in the background the branes source.

6.1.1. Let's identify some salient features of the primordial example of such a duality so as to inform our desiderata in the sequel.

Conjecture 6.1 (Maldacena [Mal98], [Wit98])**.** The following are equivalent:

- $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$. In addition to the rank of the gauge group, the theory has a parameter the Yang-Mills coupling constant g_{YM}.
- type IIB superstring theory on $AdS_5 \times S^5$ with N units of five-form flux on S^5. The theory has two free parameters, the string coupling g_s and a

parameter L/ℓ_s which describes the scale of AdS relative to the length of the string.

Under this equivalence, the parameters of the two sides are identified as follows $g_{YM}^2 = 2\pi g_s$ and $2g_{YM}^2 N = (L/\ell_s)^4$.

It is convenient to introduce a parameter $\lambda = g_{YM}^2 N$, the so-called *'t Hooft coupling*; in the perturbative regime where the number of colors is also large (a limit that we will introduce momentarily), the β-function keeps λ of the same order.

It is very difficult to perform explicit calculations of most observables associated to either theory at generic values of the parameters on either side. However, there are certain limits which afford more tractability.

- The first limit we can take involves sending the string coupling g_s to zero and keeping the parameter L/ℓ_s fixed. In this limit, contributions from higher genus worldsheets in string perturbation theory are suppressed. Under the above identification of parameters, we see that this limit should involve taking $g_{YM} \to 0$ while keeping the 't Hooft coupling finite; that is, we must take the large N limit of the gauge theory. This limit is traditionally referred to as the *'t Hooft limit*. Corrections in $\frac{1}{N}$ then correspond to turning on quantum effects in string theory.
- After taking the 't Hooft limit, we may further consider the limit where L/ℓ_s is large. In this limit, strings are small and particle like compared to the scale of AdS and the theory looks like classical type IIB supergravity on $AdS_5 \times S^5$. On the gauge theory side, this corresponds to the limit where the 't Hooft coupling is large. As such, we see that even this simplified form of the AdS/CFT correspondence is extremely powerful as it relates strongly coupled gauge theory to classical perturbative supergravity!

6.2. **BPS observables in AdS/CFT.** Many checks of the AdS/CFT correspondence involve computing quantities on either side that are independent of the coupling and comparing them. Such quantities are typically BPS, and as such can be studied at the level of twists. We introduce two such quantities which we will further expand on in our relevant example below.

6.2.1. Suppose that T_{CFT} is superconformal, such as in the above example. In such examples, one expects that the superconformal algebra in fact acts on M_2 as isometries, at least asymptotically.

Superconformal field theories admit a plethora of protected quantities that can be computed exactly at weak coupling. One such quantity is the superconformal index, which in a Hamiltonian formulation of the theory can be thought of as a Witten-index in radial quantization. Schematically, such a quantity takes the form

$$\mathrm{Tr}_{\mathcal{H}}\left((-1)^F \exp(-\beta\{Q, \overline{Q}\}) x_1^{J_1} \cdots x_n^{J_n} y_1^{H_1} \cdots y_n^{H_n}\right)$$

where $(-1)^F$ is the fermion number operator, β is an inverse temperature, Q is a supercharge and the x_i are fugacities keeping track of charges under angular momenta, and y_i are fugacities keeping track of charges under R-symmetries. The superconformal index gives a generating function for the difference between bosonic and fermionic states annihilated by a particular supercharge. Under an operator-state correspondence, the superconformal index can also be thought of as a signed count of local operators preserved under a single supercharge.

In terms of partition functions, the superconformal index is gotten by a partition function on a twisted product $M_1 = S^1 \times_\omega S^{d-1}$ where the twisting ω is determined by a background connection for the global symmetries of the problem. The expectation that the AdS/CFT correspondence can be expressed as an equality of partition functions therefore suggests a recapitulation of the superconformal index in gravitational terms. An exciting body of work aims to make this gravitiational incarnation precise, see for example [Mur20] and references therein.

Note that by definition, the superconformal index provides a lower bound on the number of fractionally BPS states of T_{CFT}. It is often the case, however, that T_{grav} includes in its spectrum, black holes, which are expected to have a thermodynamic entropy proportional to the event-horizon-area at leading order, as given by the Beckenstein-Hawking formula. As such, the growth of states in T_{CFT}, and hopefully the superconformal index, should reflect this.

6.2.2. Another such quantity is the algebra of BPS local operators in T_{CFT}. This vector space underlying this algebra is precisely a costalk of the factorization algebra of observables of a twist of T_{CFT}. In light of the aforementioned operator-state correspondence, this can be thought of as categorifying the superconformal index. Under the AdS/CFT dictionary, local operators of T_{CFT} are supposed to match with certain kinds of states in T_{grav}.

Moreover, both kinds of objects transform in representations of a superconformal algebra and the map between them preserves the actions. Local operators in the CFT are equipped with an interesting algebraic structure given by operator-product-expansion, and the AdS/CFT correspondence intertwines this algebraic structure with scattering of supergravity states. Indeed, the equality of partition functions along with the matching of sources for CFT local operators with boundary values of gravitational fields gives a prescription for computing correlation functions between CFT local operators by varying the gravitational action evaluated on field configurations subject to certain boundary values with respect to the boundary value. This recipe can be recast as a tree-level computation in the gravitational theory, involving computation of so-called Witten diagrams [**?**].

6.3. **Twisted holography.** Introduced by Costello and Li in [**?**], the twisted holography proposal posits an avatar of the AdS/CFT correspondence that holds at the level of factorization algebras associated to supersymmetric twists of T_{CFT} and T_{grav}. There is an exciting body of work being developed around this program including tests of this proposal from both the gravitational and gauge theory sides.

6.3.1. Concretely, the twisted holography proposal suggests that the type of duality between the factorization algebras associated to a gravitational theory and to the worldvolume theory of a number of branes is a general version of *Koszul duality*.

Ordinary Koszul duality for associative algebras (so quantum mechanical systems) associates to an (augmented) algebra A a dual algebra $A^!$ whose appropriate derived category of modules is the same as that of A. Following the work of [**?**, **?**] (see also the review in [PW21]) there is a simple physical interpretation of Koszul duality. If A is the algebra of operators of some bulk quantum field theory (perturbatively we can even consider a theory of gravity) then $A^!$ is the algebra of operators on the universal topological line defect. Universal here means that algebra of operators on any other line defect which couples to the bulk system admits a unique map of algebras from $A^!$.

The general theory of Koszul duality for factorization algebras has not been developed, and we do not do so here, but see [Lur17] for the case of $\mathbb{E}_n$-algebras and [GLZ22], [Tam03] for the case of particular kinds of vertex algebras. This sort of duality would allow one to make sense of universality statements as above for higher dimensional, possibly non-topological, defects in an arbitrary bulk quantum field theory. Roughly, one expects the Koszul dual of a factorization algebra to be the factorization algebra corepresenting the functor of looking at solutions to a Maurer-Cartan equation in a tensor product.

6.3.2. Let us now make a more concrete, yet slightly informal, statement of twisted holography which fits into the approach of this thesis. Let X be a smooth manifold, and let Obs_{grav} denote a factorization algebra on X that we view as the observables of a bulk gravitational theory. Suppose we have, in addition, a stack of N branes, wrapping a closed submanifold $Y \hookrightarrow X$ whose worldvolume theory has a factorization algebra of observables Obs^N_{CFT}.

Note that Obs_{grav} is a factorization algebra on X, while Obs^N_{CFT} is a factorization algebra on the closed submanifold Y so we cannot yet compare them. We can, however, attempt to restrict Obs_{grav} to a factorization algebra just on Y, which we denote by $\mathrm{Obs}_{grav}|_Y$.

Expectation 6.2 (Twisted holographic principle following [**?**])**.** There is a map of factorization algebras
$$(\mathrm{Obs}_{grav}|_Y)^! \to \mathrm{Obs}^N_{CFT}$$
that becomes an equivalence in the large N limit.

As we recalled in the previous subsection, traditional formulations of the AdS/CFT correspondence relate local operators of the gauge theory to states of the gravitational theory on AdS. Therefore, a natural desideratum in relating the above to more traditional statements is a precise relation between the source of the above map and gravitational states in AdS. Moreover, there is an operational definition of the operator-product-expansion on the costalk of a Koszul dual factorization algebra which realizes the expectation that Koszul duality corepresents the functor taking Maurer-Cartan elements in the tensor product. Another desideratum is to relate the output of this procedure with the scattering product on gravitational states computed by Witten diagrams.

Remark 6.3. In this context, the definition of Koszul duality involves another ingredient, namely the backreaction of branes wrapping Y. This is meant to capture the fact that $(\mathrm{Obs}_{grav}|_Y)$ may not be canonically augmented, but we may try to deform it in a way that kills off the obstruction to being augmented. More precisely, one expects that the correct version of Koszul duality for application in holographic contexts is a version of *curved* Koszul duality for factorization algebras.

Remark 6.4. For finite N, this map will in general be neither injective nor surjective. The kernel and cokernel of this map for finite N correspond to interesting nonperturbative effects in the gravitational theory. For instance, in gauge theories:

- This map has a kernel given by trace relations. Syzygies between trace relations are conjecturally related to the worldvolume theories of certain other branes in the gravitational theory, so-called *giant gravitons* [GL21], [CKL+23], [Ima22] [1]
- This map also has a cokernel. By fiat, these are classes that exist in the finite N cohomology of the observables of a gauge theory that are not in the image of the natural map from the large N theory. Recent developments in cohomological approaches to counting quantum microstates of $\frac{1}{16}$-BPS black holes in $AdS_5 \times S^5$ [CKL+23] [CL23] [CY13] can be cast as trying to characterize the cokernel of a specific example of this map.

6.3.3. The above expectation can be tested in instances where both sides of the duality admit explicit descriptions. This has been carried out in many examples including:

- A stack of $D3$ branes in twisted Ω-deformed type IIB supergravity on flat space. The theory on the stack of $D3$ branes is dual to the closed string B-model on the deformed conifold [CG21a]. This can be understood as a twisted Ω-deformed version of the physical AdS/CFT duality between 4d $\mathcal{N} = 4$ super Yang-Mills and type IIB string theory on $AdS_5 \times S^5$. Here the duality can be formulated in terms of vertex algebras.
- M2 branes and M5 branes in twisted Ω-deformed M-theory on Taub-NUT space [Cos16, Cos17]. In the particular Ω-background, M2 branes are localized to a topological quantum mechanical system where the duality can be phrased in terms of associative algebras and ordinary Koszul duality. The koszul dual algebra bears close relations to the spherical Cherednik algebra. The Ω-background localizes M5 to a complex plane and the observables of the localized theory are an affine W_N vertex algebra. Many celebrated features of the representation theory of these algebras and their relations with geometry have found natural explanations from the perspective of this twist of M-theory [GR20], [OZ21].

The example we consider is closely related to the second of these. Indeed, there is an odd nilpotent element in $\mathfrak{osp}(6|2)$, which we refer to as S in the sequel. Using the inner action of $\mathfrak{osp}(6|2)$ on our eleven-dimensional model on twisted $AdS_7 \times S^4$ as identified in proposition 3.6, S affords a deformation of our model. This is the deformation considered in [BRvR15], and it induces a specialization of characters called the Schur limit.

6.3.4.

6.4. **M5 branes, holomorphy, and holography.** The results in the second half of this thesis can be viewed as baby steps in investigating twisted holography for the minimal twist of the 6d $\mathcal{N} = (2,0)$ theory. Let us begin by spelling out the objects in expectation 6.2 adapted to our setting.

- The 11d spacetime manifold X is $\mathbb{R} \times \mathbb{C}^5$ and Y is a copy of $\mathbb{C}^3$.

[1] We thank Ji-Hoon Lee for conversations related to this topic

- The factorization algebra $\mathrm{Obs}_{grav}|_{\mathbb{C}^3}$ has the feature that its semiclassical free limit is the factorization algebra denoted $\mathrm{C}^\bullet\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^N_{AdS_7\times S^4})\right)$ in definition 2.8.
- The factorization algebra Obs^N_{CFT} describes local observables in the minimal twist of the theory on a stack of N M5 branes wrapping $\mathbb{C}^3$.

Our goal is to try and use this map, and expectations about its kernel and cokernel, to give a concrete description of the target. There have been various approaches to try and characterize the spectrum of $\frac{1}{16}$-BPS states in the 6d $\mathcal{N}=(2,0)$ theories of type A_{N-1}, which furnish consistency checks to test our proposal against. Some of these involve applications of instanton counting techniques in 5d $\mathcal{N}=2$ gauge theory [?] and some of them involve holographic techniques [Ima22].

As we remarked in subsection 6.2, the first ingredient is a map of representations of the superconformal algebra between local operators of the CFT and supergravity states. In order to codify such a matching in terms of the kinds of data in the statement of expectation 6.2, we require a matching between supergravity states and the costalk at the origin of the factorization algebra $(\mathrm{Obs}_{grav}|_{\mathbb{C}^3})^!$. This is precisely the content of proposition 2.12.

6.4.1. We have the following conjectural large N statements

Conjecture 6.5 (R-Saberi-Williams)**.** There is an equivalence of holomorphic $\mathbb{P}_0$-factorization algebras

$$\left(\mathrm{C}^\bullet(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^N_{AdS_7\times S^4}))\right)^! \cong \mathcal{U}_\omega\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})\right).$$

where the right hand side denotes a twisted factorization envelope of the local L_∞-algebra $\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$. Moreover, upon deforming by the Maurer-Cartan element $S\in\mathfrak{osp}(6|2)$, the factorization algebra $\mathcal{U}_\omega\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})\right)$ has no sections away from a copy of $\mathbb{C}\subset\mathbb{C}^3$, and its restriction to this copy of $\mathbb{C}$ is equivalent to a twisted factorization envelope of the local Lie algebra $\mathrm{Diff}_{\mathbb{C}}$ of holomorphic differential operators on $\mathbb{C}$.

Here, the twisting cocycle ω comes from the shifted Poisson tensor that was induced by the flux in section 3. The content in verifying this conjecture is to explicitly compute the twisting coming from the flux sourced by the brane, and check that upon deforming by the element S, it induces the correct cocycle on $\mathrm{Diff}(\mathbb{C}^\times)$

The comment in 2.2.5 constitutes a very meager consistency check for the second part of this conjecture, where we observe that the Schur limit of the character of the costalk of $\mathcal{U}_\omega\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})\right)$ recovers the vacuum character of the $W_{1+\infty}$ vertex algebra.

There is a deformation of the twisted factorization envelope of $\mathrm{Diff}_{\mathbb{C}}$ which yields the $\mathcal{W}_{1+\infty}$ vertex algebra, also referred to as the affine Yangian of $\mathfrak{gl}(1)$. In [Cos16], Costello finds this deformation from a loop level computation in his 5d noncommutative gauge theory. We also expect to be able to lift this to the minimal twist. We summarize this expectation in a conjecture.

Conjecture 6.6 (R-Saberi-Williams)**.** There is an equivalence of holomorphic factorization algebras

$$\left(\mathrm{C}^\bullet_\hbar(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^N_{AdS_7\times S^4}))\right)^! \cong \mathcal{U}_{\hbar,\omega}\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})\right).$$

where the right hand side denotes a deformation of the factorization algebra in the previous conjecture induced by loop-level effects in our eleven-dimensional model. Moreover, upon deforming by the Maurer-Cartan element $S \in \mathfrak{osp}(6|2)$, the factorization algebra $\mathcal{U}_{\hbar,\omega}\left(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})\right)$ has no sections away from a copy of $\mathbb{C} \subset \mathbb{C}^3$, and its restriction to this copy of $\mathbb{C}$ is equivalent to the $\mathcal{W}_{1+\infty}$ vertex algebra.

6.4.2. We now move on to finite N statements. For the lowest steps of the filtration, we can make some very concrete statements.

Conjecture 6.7 (R-Saberi-Williams)**.** Upon deforming by $S \in \mathfrak{osp}(6|2)$, the factorization algebra $\mathcal{U}_\omega(\mathcal{G}^{(-1)}_{\mathbb{C}^3})$ has no sections away from a copy of $\mathbb{C} \subset \mathbb{C}^3$ and its restriction to this copy of $\mathbb{C}$ is equivalent to the Heisenberg vertex algebra.

To check this conjecture, it remains to compute the pullback of the twisting cocycle ω under the inclusion of $\mathcal{G}^{(0)}$ and see that it deforms to the Heisenberg cocycle.

6.4.3. There is a distinguished Lie sub-algebra of the algebra of differential operators on $\mathbb{C}^\times$ which is given by the Witt-algebra of vector fields. The central extension of $\mathrm{Diff}(\mathbb{C}^\times)$ induced by ω above restricts to the Virasoro central extension. Similarly, in proposition 5.5 we have identified a distinguished local super-Lie algebra inside $\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ given by $\mathcal{E}(3|6)$.

Accordingly, we conjecture the following

Conjecture 6.8 (R-Saberi-Williams)**.** Upon deforming by $S \in \mathfrak{osp}(6|2)$, the factorization algebra $\mathcal{U}_\omega(\mathcal{E}(3|6))$ has no sections away from a copy of $\mathbb{C} \subset \mathbb{C}^3$ and its restriction to this copy of $\mathbb{C}$ is equivalent to the Virasoro vertex algebra.

Again, to check this conjecture it remains to compute the pullback of the twisting cocycle ω along the inclusion $\mathcal{E}(3|6) \to \Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^{r=0}_{AdS_7\times S^4})$ and compare its deformation with the cocycle giving the Virasoro central extension.

We can once again perform a consistency check at the level of characters of costalks. Indeed, we see that the plethystic exponential of the specialized character $g_0(y=1, y_3=1, q) = \frac{q^2}{1-q}$ is exactly the vacuum character of the Virasoro algebra.

Moreover, note that combining with conjecture 6.5, we expect maps

$$\mathcal{U}_\omega(\mathcal{E}(3|6)) \to \left(\mathrm{C}^\bullet(\Pi\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^N_{AdS_7\times S^4}))\right)^! \to \mathrm{Obs}^N_{CFT}$$

for every N. This map can be thought of as a Noether-type map associated to an $\mathcal{E}(3|6)$-symmetry of the minimal twist of any finite rank 6d $\mathcal{N} = (2,0)$ theory [CG21b].

6.4.4. More generally, the $\mathcal{W}_{1+\infty}$ algebra has as quotients, the $\mathcal{W}_N$ algebras when the central charge is set equal to N. Accordingly, we dream of the following:

Speculation 6.9. Under an integrality condition on the central charge, the map

$$(\mathrm{Obs}_{grav}|_{\mathbb{C}^3})^! \to \mathrm{Obs}^N_{CFT}$$

factors as

$$\begin{tikzcd}
(\mathrm{Obs}_{grav}|_{\mathbb{C}^3})^! \arrow[rr] \arrow[d] & & \mathrm{Obs}^N_{CFT} \\
\mathcal{U}_{\hbar,\omega}(\Omega^{0,\bullet}_{\mathbb{C}^3}(\mathcal{L}^N_{AdS_7\times S^4}))/\mathcal{U}_{\hbar,\omega}(\prod_{j\geq N-1}\mathcal{G}^{(j)}_{\mathbb{C}^3}) \arrow[urr] & &
\end{tikzcd}$$

We can perform a consistency check of the above speculation at the level of characters of costalks. It is expected that the superconformal deformation deforms Obs^N_{CFT} to the $\mathcal{W}_N$ vertex algebra. On the other hand, we have that

Proposition 6.10. Upon specializing $y = 1, y_3 = 1$ (so that $y_1 y_2 = 1$), one has

$$\begin{aligned}
\chi\left(\Omega^{0,\bullet}_{\mathbb{C}^3,c}(\mathcal{L}^N_{AdS_7\times S^4})(0)\Big/\left(\mathcal{G}^{(-1)}_{\mathbb{C}^3,c}\times\prod_{j\geq N-1}\mathcal{G}^{(j)}_{\mathbb{C}^3,c}\right)(0)\right) &= \sum_{j\geq 0}^{N-2} g_j(y_1,y_2,y_3=1,y=1,q) \\
&= \frac{q^2+q^3+\cdots+q^N}{1-q}
\end{aligned}$$

The plethystic exponential of the right hand side agrees with the vacuum character of the W_N vertex algebra.

Proof. By induction it suffices to show that the specialization of the single particle local character g_j of the factorization algebra $\mathcal{U}(\mathcal{G}^{(j)})$ is $q^{j+2}/(1-q)$. We have already seen this in the case $j = -1, 0$, so it suffices to show this when $k \geq 1$.

First observe that the denominator becomes

$$(1 - y_1 q)(1 - y_2 q)(1 - q). \tag{61}$$

Next, we observe that the numerator of $g_j(y_1, y_2, y_3 = 1, y = 1, q)$ can be factored as

$$\begin{aligned}
q^{3+3j/2}\left(q^{-(j+2)/2} + q^{-(j-2)/2} - q^{-j/2}(y_1+y_2)\right) &= q^{j+2}(1+q^2-q(y_1+y_2)) \\
&= q^{j+2}(1-y_1q)(1-y_2q)
\end{aligned}$$

where in the last line we have used $y_1 y_2 = 1$. The result follows. □

6.4.5. In [RW22] we try to explicitly characterize the discrepancy between the characters of $\mathcal{U}(\mathcal{G}^{(j)}_{\mathbb{C}^3})$ and expectations about the superconformal index of the finite rank 6d $\mathcal{N} = (2,0)$ theories computed via instanton counting techniques [KKKL13] and the "giant graviton expansion" [AFI+20, **?**], [Ima22]. It would be interesting to try and categorify the discrepancies and identify them in terms of modules for $E(3|6)$.

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