CONSTRAINTS IN THE BV FORMALISM: SIX-DIMENSIONAL SUPERSYMMETRY AND ITS TWISTS

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ABSTRACT. We compute the holomorphic twist of the abelian $\mathcal{N} = (2, 0)$ multiplet in six dimensions, beginning with its natural formulation as a Poisson BV theory as produced by the pure spinor superfield formalism. In the holomorphic case, the result consists of symplectic-valued holomorphic bosons (from the $\mathcal{N} = (1, 0)$ hypermultiplet), together with the degenerate holomorphic theory representing the intermediate Jacobian (from the $\mathcal{N} = (1, 0)$ tensor multiplet). We check that our result matches with known ones under dimensional reduction to five and four dimensions;

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Date: Began: January 21, 2020. Last compiled: August 13, 2020.

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1. Introduction

There is a supersymmetric theory in six dimensions whose fields, in part, consist of a self-dual two-form. A precise mathematical model for the so-called "tensor multiplet" has remained elusive since the theory does not admit a Lagrangian description, meaning its equations of motion do not arise from an action functional via the usual methods of variational calculus.

Part of the desire to better understand the tensor multiplet is due to its ubiquity in the context of string theory and M-theory. The tensor multiplet with $\mathcal{N} = (2,0)$ supersymmetry famously appears as the worldvolume theory of the $M5$-brane [1]. Throughout this paper we focus entirely on the case of the "abelian tensor multiplet". BW: how to state this

Our first step is to give a mathematical formulation of the perturbative theory of the free $\mathcal{N} = (2,0)$ tensor multiplet in six dimensions. (A corresponding formulation of the $\mathcal{N} = (1,0)$ tensor multiplet follows immediately from this.) Throughout the paper, we make use of the Batalin–Vilkovisky (BV) formalism. For a modern treatment of this setup see [5], [10], and for a traditional outlook see [8]. Roughly, the data of a classical theory in the BV formalism is a graded space of fields $\mathcal{E}_{\mathrm{BV}}$ (given as the space of sections of some graded vector bundle on spacetime), together with a symplectic form ω_{BV} of cohomological degree (-1) on $\mathcal{E}_{\mathrm{BV}}$, and an action functional, whose associated Hamiltonian vector field defines a differential on $\mathcal{E}_{\mathrm{BV}}$. Under appropriate conditions, this differential provides a free resolution to the sheaf of solutions to the equations of motion of the theory.

Our formulation was motivated by the desire to understand the pure spinor superfield formalism for $\mathcal{N} = (2,0)$ supersymmetry; the relevant cohomology was first computed in [2], and was rediscovered and reinterpreted in [3]. Roughly speaking, this formalism takes as input an equivariant sheaf over the space of Maurer–Cartan elements, or nilpotence variety, of the supertranslation algebra, and produces a chain complex of locally free sheaves over the spacetime, together with a homotopy action of the corresponding supersymmetry algebra. The resulting multiplet can be interpreted as the BRST or BV formulation of the corresponding free multiplet, according to whether the action of the supersymmetry algebra closes on shell or not; the differential, which is also an output of the formalism, corresponds in the latter case to the Hamiltonian vector field mentioned above.

In the case of $\mathcal{N} = (2,0)$ supersymmetry, the action of the algebra is, as always, guaranteed on general grounds, and the differential includes the correct linearized equations of motion. One thus expects an on-shell formalism, but the interpretation of the resulting resolution as a BV theory is subtle, as there is no obvious shifted symplectic pairing. As such, developing an understanding of theories like the tensor multiplet requires a slight generalization of the standard formalism.

Forgetting about supersymmetry momentarily, we recall that the peculiarity of the six-dimensional tensor multiplet boils down to what physicists refer to as "chirality" [1], which manifests as the self-duality condition[1] on the field strength of a two-form on a Riemannian manifold M^6:

$$\alpha \in \Omega^2(M^6), \quad \star \mathrm{d}\alpha = \sqrt{-1}\, \mathrm{d}\alpha. \tag{1}$$

The putative Yang–Mills style action of a higher form gauge theory is given by the L^2-norm $\|\mathrm{d}\alpha\|_{L^2} = \int \mathrm{d}\alpha \wedge \star \mathrm{d}\alpha$. It is clear that the self-duality condition implies the norm vanishes identically, so an action functional of Yang–Mills type is not feasible. Thus, the action functional is identically zero and there is no standard BRST, let alone BV, formulation of the theory.

In general, this issue of "chirality", or self-duality, arises in real dimensions $4k+2$, where $k = 0, 1, \ldots$.[2] Here, it is the condition that a $2k$-form α satisfy $\star \mathrm{d}\alpha = \sqrt{-1}\, \mathrm{d}\alpha$ BW: fix power of **i** The simplest version of this issue of chirality actually arises in two-dimensional conformal field theory, when $k = 0$, which we briefly turn our attention to. Here, this self-duality constraint is precisely the condition of holomorphy.

The chiral boson. We take an intermezzo within this introduction to review the theory of the *chiral boson* on Riemann surfaces, as it shares many important features with our main example of the six-dimensional self-dual two-form. The theory of the chiral boson describes holomorphic $\mathrm{U}(1)$-valued maps on a Riemann surface Σ. Equivalently, this is the chiral WZW model for the Lie group $\mathrm{U}(1)$.

Working perturbatively, which we will continue to do throughout this paper, the field φ is simply a holomorphic function that we describe in terms of the Dolbeault complex $\Omega^{0,\bullet}(\Sigma)$. This complex is, of course, a resolution of the sheaf of holomorphic functions. The fact that it is a resolution by smooth vector bundles will play an essential role momentarily.

The chiral boson is not a theory in the usual sense of the word, perturbatively or otherwise, as it is not described by an action functional: the equations of motion, namely that φ be holomorphic, do not arise as the variational problem of a classical action functional. Nevertheless, there is a way to formulate it in a slightly modified version of the BV formalism.

[1] We work in Euclidean signature throughout this paper.
[2] There is also a version of this self-duality condition for complex geometries, and we will try to avoid the use of the term "chiral" to avoid confusion.

Let's first consider a closely related theory, the (non-chiral) free boson, which does have a description in the BV formalism. The free boson is a two-dimensional conformal field theory whose fields consist of smooth maps

$$\varphi : \Sigma \to \mathrm{U}(1)$$

with action functional $\int_\Sigma \partial\varphi \wedge \overline{\partial}\varphi$. The equations of motion impose that φ is harmonic. Perturbatively, in the BV formalism, one can model this free theory by the following two-term cochain complex

$$\begin{array}{ccccc} & & \underline{0} & & \underline{1} \\ \mathcal{E}_{\mathrm{BV}} & = & \Omega^0(\Sigma) & \xrightarrow{\partial\overline{\partial}} & \Omega^2(\Sigma). \end{array}$$

This is the space of fields in the BV formalism. We can equip $\mathcal{E}$ with a degree (-1) antisymmetric non-degenerate pairing, which in this case is just given by multiplication and integration. That is

$$\omega_{\mathrm{BV}}(\varphi, \varphi^+) = \int \varphi\varphi^+$$

where $\varphi \in \Omega^0(\Sigma)$ and $\varphi^+ \in \Omega^2(\Sigma)$. This is the (-1)-symplectic form in the BV formalism.

Now, there is a natural map of cochain complexes

$$i : \Omega^{0,\bullet}(\Sigma) \to \mathcal{E}_{\mathrm{BV}}$$

which in degree zero is the identity map on smooth functions, and in degree one is defined by the holomorphic de Rham operator $\partial : \Omega^{0,1}(\Sigma) \to \Omega^2(\Sigma)$.

We can pull back the degree (-1) symplectic form ω on $\mathcal{E}$ to a two-form $i^*\omega$ on $\Omega^{0,\bullet}(\Sigma)$, which is closed because i is a cochain map. Explicitly, this two-form on the space $\Omega^{0,\bullet}(\Sigma)$ is $(i^*\omega)(\alpha, \alpha') = \int \alpha\partial\alpha'$. Since i is not a quasi-isomorphism, $i^*\omega$ is degenerate, and hence does not endow $\Omega^{0,\bullet}(\Sigma)$ with a BV structure.

We emphasize that the interpretation we are aiming for in this example is to view the chiral boson as being obtained from the free boson $(\mathcal{E}_{\mathrm{BV}}, \omega_{\mathrm{BV}})$ through the *constraint* that a harmonic function φ be holomorphic: $\overline{\partial}\varphi = 0$.

In analogy with ordinary symplectic geometry, we will refer to the data of a pair $(\mathcal{E}, \omega)$ where $\mathcal{E}$ is a graded space of fields, and ω is a closed two-form on $\mathcal{E}$, as a *presymplectic* BV theory. We make this precise in Definition 2.1, at least for the case of free theories. In the example of the chiral boson this pair is $(\Omega^{0,\bullet}(\Sigma), i^*\omega_{\mathrm{BV}})$.

The theory of the self-dual two-form in six-dimensions (more generally a self-dual $2k$-form in $4k+2$ dimensions) arises in an analogous fashion. There is an honest BV theory of a two-form on a Riemannian six-manifold which endows the theory of the self-dual two-form with the structure of a presymplectic BV theory. Among other examples, we give a precise formulation of the self-dual two-form in §2.

Returning to supersymmetry, we go on to formulate the $\mathcal{N} = (1,0)$ and $\mathcal{N} = (2,0)$ abelian tensor multiplets in the BV formalism. The main calculation we perform is of the *twists* of these theories. For background on twisting supersymmetric theories we refer to **WittenTwist**, and the later mathematical treatments in [13], [16].

In our examples, there are two classes of twists, characterized by how many directions are rendered "topological" and how many directions are rendered "holomorphic". For $\mathcal{N} = (1,0)$ supersymmetry there is a unique twist, and it is holomorphic in the sense that it depends holomorphically in all directions and exists on any complex 3-fold. For $\mathcal{N} = (2,0)$ supersymmetry there is a holomorphic twist that exists on complex 3-folds, and a further twist that is holomorphic in one complex direction and topological in four real directions. This non-minimal twist exists on manifolds of the form $\Sigma \times M$ where it behaves holomorphically on a Riemann surface Σ and topologically on a smooth four-manifold M. Like the untwisted theories, the twisted theories are formulated using the presymplectic BV formalism.

The key mathematical device we use to capture these supersymmetric theories, and their twists, is the theory of factorization algebras. Costello and Gwilliam have developed a systematic approach to the study of observables, of which local operators are a special case, in perturbative field theory. The general philosophy is that the observables of a perturbative (quantum) field theory have the structure of a *factorization algebra* on spacetime [4], [5]. Roughly, this factorization algebra of observables assigns to an open set U of spacetime a cochain complex $\mathrm{Obs}(U)$ of "observables supported on U". When two open sets U, V are disjoint, contained in some bigger open set W, the factorization algebra structure defines a rule of how to "multiply" observables $\mathrm{Obs}(U) \otimes \mathrm{Obs}(V) \to \mathrm{Obs}(W)$. For local operators, one should think of this as organizing the operator product expansion in a sufficiently coherent way.

In the ordinary BV formalism, the factorization algebra of observables has a very important structure, namely a Poisson bracket of cohomological degree $+1$ induced from the shifted symplectic form ω_{BV}. This is reminiscent of the Poisson structure on functions on an ordinary symplectic manifold.

In the case of a presymplectic manifold, the full algebra of functions does *not* carry such a bracket, yet there is a subalgebra of functions, called the *Hamiltonian* functions, that does. This issue persists in the presymplectic BV formalism, and some care must be taken to define a notion of observables that carries such a shifted Poisson structure. We tentatively solve this problem, and for special classes of free presymplectic BV theories we provide an appropriate notion of "Hamiltonian observables". The corresponding factorization algebra carries a shifted Poisson structure, which is a direct generalization of the work of Costello–Gwilliam that works to include presymplectic BV theories. [3]

Previous work. There has been an enormous amount of previous work in the physics literature on topics related to M5 branes and $\mathcal{N} = (2,0)$ superconformal theories in six dimensions, and any attempt to provide exhaustive references is doomed to fail. In light of this, our bibliography makes no pretense to be complete or even representative.

[3] The development of the theory of observables for more general presymplectic BV theories is joint work with Eugene Rabinovich.

The best we can offer is an extremely brief and cursory overview of some selected past literature, which may serve to orient the reader: The earliest approaches involved the study of relevant "black brane" solutions in eleven-dimensional supergravity theory [6].

For the physicist reader, we emphasize that we deal here with a formulation that is lacking, even at a purely classical level, in at least three respects. Firstly, we make no effort to formulate the theory non-perturbatively, even for gauge group $U(1)$; in a sense, our discussion deals only with the gauge group $\mathbb{R}$. (Some more speculative remarks about this, though, are given in §**??**.) Secondly, we start with a formulation which does not involve any coupling to eleven-dimensional supergravity, and makes no attempt to connect to the M5 brane, in the sense that we ignore the formulation of the theory in terms of a theory of maps. Associated issues (such as WZW terms and kappa symmetry) therefore make no appearance, although the connection to Kodaira–Spencer theory is indicated to show how we see our results as fitting into a larger story about twisted supergravity theories in the sense of Costello and Li [7], as mentioned above.

An outline of the paper. We begin in §2 by setting up a presymplectic version of the BV formalism for free theories. After stating some general results and reviewing a list of examples, the section culminates with a definition of the factorization algebra of observables of a presymplectic BV theory. In §3 we recall the necessary tools of six-dimensional supersymmetry and provide a definition of the $\mathcal{N} = (1,0)$ and $\mathcal{N} = (2,0)$ versions of the tensor multiplet in the presymplectic BV formalism. We provide a classification of the twists and give an explicit description of the action of the supersymmetry algebra on the presymplectic BV theory. We perform the calculation of the minimal twist of the tensor multiplet in §4, and of the non-minimal twist in §5. We touch back with string theory in §6, where we relate our twisted theories to the conjectural twist of Type IIB supergravity due to Costello–Li. Finally, in §7 we explore some consequences of our description of the twisted theories upon dimensional reduction, and perform some sanity checks with theories that are conjecturally obtained as the reduction of the theory on the M5 brane.

Conventions and notations.

- If $E \to M$ is a graded vector bundle on a smooth manifold M, then we define the new vector bundle $E^! = E^* \otimes \mathrm{Dens}_M$, where E^* is the linear dual and Dens_M is the bundle of densities on M. We denote by $\mathcal{E}$ the space of smooth sections of E, and $\mathcal{E}^!$ the space of sections of $E^!$. The notation $\mathcal{E}_c$ refers to the space of compactly supported sections of E. The notation $(\overline{\mathcal{E}}_c)$ $\overline{\mathcal{E}}$ refers to the space of (compactly supported) distributional sections of E.
- The sheaf of (smooth) p-forms will be denoted Ω^p and $\Omega^\bullet = \bigoplus \Omega^p[-p]$ is the $\mathbb{Z}$-graded sheaf of de Rham forms, with Ω^p in degree p. More generally, our grading conventions are cohomological, and are chosen such that the cohomological degree of a chain complex of differential forms is determined by the (total) form degree, but always taken to start with the lowest term of the complex in degree

zero. Thus Ω^p is a degree-zero object, $\Omega^{\leq p}$ is a chain complex with support in degrees zero to p, and $\Omega^{\geq p}(\mathbb{R}^d)$ begins with p-forms in degree zero and runs up to d-forms in degree $d-p$.

- On a complex manifold, we have the sheaves $\Omega^{i,\mathrm{hol}}$ of holomorphic forms of type $(i,0)$. The operator $\partial : \Omega^{i,\mathrm{hol}} \to \Omega^{i+1,\mathrm{hol}}$ is the holomorphic de Rham operator. The standard Dolbeault resolution of holomorphic i-forms is $(\Omega^{i,\bullet}, \overline{\partial})$ where $\Omega^{i,\bullet} = \oplus_k \Omega^{i,k}[-k]$ is the complex of Dolbeault forms of type $(i,\bullet)$ with (i,k) in cohomological degree $+k$.

2. A presymplectic Batalin–Vilkovisky formalism

In the standard Batalin–Vilkovisky (BV) formalism [8], one is interested in studying the (derived) critical locus of an action functional. On general grounds, derived critical loci are equipped with canonical (-1)-shifted symplectic structures [9]. In perturbation theory, where we work around a fixed classical solution, we can assume that the space of BV fields $\mathcal{E}$ are given as the space of sections of some graded vector bundle $E \to M$, where M is the spacetime. In this context, the (-1)-symplectic structure boils down to an equivalence of graded vector bundles $\omega : E \cong E^![-1]$.

We remind the reader that in the standard examples of "cotangent" perturbative BV theories, E is of the form

$$E = T^*[-1]F \overset{\text{def}}{=} F \oplus F^![-1], \tag{2}$$

where F is some graded vector bundle, which carries a natural (-1)-symplectic structure. The differential Q_{BV} is constructed such that

$$H^0(\mathcal{E}, Q_{\mathrm{BV}}) \cong \mathrm{Crit}(S), \tag{3}$$

i.e. so that the sheaf of chain complexes $(\mathcal{E}, Q_{\mathrm{BV}})$ is a model of the derived critical locus.

In general, we can think of the (-1)-symplectic structure ω as a two-form (with constant coefficients) on the infinite-dimensional linear space $\mathcal{E}$. Moreover, this two-form is of a very special nature: it arises locally on spacetime. For a more detailed introduction to the BV formalism its description of perturbative classical field theory, see [5], [10].

We will be interested in a generalization of the BV formalism, motivated by the classical theory of presymplectic geometry and its appearance in Dirac's theory of constrained systems in quantum mechanics. In ordinary geometry, a presymplectic manifold is a smooth manifold M equipped with a closed two-form $\omega \in \Omega^2(M)$, $\mathrm{d}\omega = 0$. Equivalently, ω can be viewed as a skew map of bundles $TM \to T^*M$. This is our starting point for the presymplectic version of the BV formalism in the derived and infinite dimensional setting of field theory.

2.1. **Presymplectic BV formalism.** We begin by introducing the presymplectic version of the BV formalism in terms of a two-form on the space of classical fields. This generalization shares many features with the usual BV setup: the two-form of degree (-1) on arises "locally" on spacetime, in the sense that it is defined by a differential operator acting on the fields. In this paper we are only concerned with free theories, so we immediately restrict our attention to this case.

It is important for us that our complexes are bigraded by the abelian group $\mathbb{Z} \times \mathbb{Z}/2$. We will refer to the integer grading as the *cohomological* or *ghost degree*, and the supplemental $\mathbb{Z}/2$ grading as *parity* or *fermion number*.

Before stating the definition of a free presymplectic BV theory, we set up the following notion about the skewness of a differential operator. Let E be a vector bundle on M and suppose $D : \mathcal{E} \to \mathcal{E}^![n]$ is a differential operator of degree n. The continuous linear dual of $\mathcal{E}$ is $\mathcal{E}^\vee = \overline{\mathcal{E}}^!_c$ (see §1). So, D defines the following composition

$$\overline{D} : \mathcal{E}_c \hookrightarrow \mathcal{E} \xrightarrow{D} \mathcal{E}^![n] \hookrightarrow \overline{\mathcal{E}}^![n].$$

The continuous linear dual of $\overline{D}$ is a linear map of the same form $\overline{D}^\vee : \mathcal{E}_c \to \overline{\mathcal{E}}^![n]$. We say the original operator D is *graded skew symmetric* if $\overline{D} = (-1)^{n+1}\overline{D}^\vee$.

Definition 2.1. A (perturbative) ***free presymplectic BV theory*** on a manifold M is a tuple $(E, Q_{\mathrm{BV}}, \omega)$ where:

- E is a finite-rank, $\mathbb{Z} \times \mathbb{Z}/2$-graded vector bundle on M, equipped with a differential operator

$$Q_{\mathrm{BV}} \in \mathrm{Diff}(\mathcal{E}, \mathcal{E})[1]$$

  of bidegree $(1, 0)$;
- a differential operator

$$\omega \in \mathrm{Diff}\left(\mathcal{E}, \mathcal{E}^!\right)[-1]$$

  of bidegree $(-1, 0)$;

which satisfy:

(1) the operator Q_{BV} satisfies $(Q_{\mathrm{BV}})^2 = 0$, and the resulting complex $(\mathcal{E}, Q_{\mathrm{BV}})$ is elliptic;
(2) the operator ω is graded skew symmetric with regard to the totalized $\mathbb{Z}/2$ grading;
(3) the operators ω and Q_{BV} are compatible: $[Q_{\mathrm{BV}}, \omega] = 0$.

We refer to the fields $\phi \in \mathcal{E}$ of cohomological degree zero as the "physical fields". For free theories, the linearized equations of motion can be read off as $Q_{\mathrm{BV}}\phi = 0$. As is usual in the BRST/BV formalism, gauge symmetries are imposed by the fields of cohomological degree -1.

The differential operator ω determines a bilinear pairing of the form

$$\int_M \omega : \mathcal{E}_c \times \mathcal{E}_c \to \mathrm{Dens}_M[-1] \xrightarrow{\int_M} \mathbb{C}[-1]$$

which endows the compactly supported sections $\mathcal{E}_c$ with the structure of a (-1)-shifted presymplectic vector space.

Of course, it should be clear that a (perturbative) free BV theory [5, Definition 7.2.1.1] is a free presymplectic BV theory such that ω is induced from a bilinear map of vector bundles which is fiberwise non-degenerate. The notion of a free presymplectic BV theory is thus a weakening of the more familiar definition. Indeed, when ω is an order zero differential operator such that $\omega : \mathcal{E} \xrightarrow{\cong} \mathcal{E}^![-1]$ is an isomorphism, the tuple $(E, Q_{\mathrm{BV}}, \omega)$ defines a free BV theory in the usual sense.

Remark 2.2. There are two natural ways to generalize Definition 2.1 that we do not pursue here:

– *Non-constant coefficient presymplectic forms*: More generally, one can ask that ω be given as a polydifferential operator of the form

$$\omega \in \prod_{n \geq 0} \mathrm{PolyDiff}(\mathcal{E}^{\otimes n} \otimes \mathcal{E}, \mathcal{E}^!)[-1].$$

The right-hand side is what one should think of as the space of "local" two-forms on $\mathcal{E}$.

– *"Interacting" presymplectic BV formalism*: Here, we require that $\mathcal{L} = \mathcal{E}[-1]$ be equipped with the structure of a local L_∞ algebra. Thus, the space of fields $\mathcal{E}$ should be thought of as the formal moduli space given by the classifying space $B\mathcal{L}$. In the situation above, the free theory corresponds to an abelian local L_∞ algebra, in which only the unary operation (differential) is nontrivial.

There is a natural compatibility between these two more general structures that is required. Using the description of the fields as the formal moduli space, one can view ω as a two-form $\omega \in \Omega^2(B\mathcal{L}) = \mathrm{C}^\bullet(\mathcal{L}, \wedge^2\mathcal{L}[1]^*)$. There is an internal differential on the space of two-forms given by the Chevalley–Eilenberg differential $\mathrm{d}_{\mathcal{L}}$ corresponding to the L_∞ structure. There is also an external, de Rham type, differential of the form $\mathrm{d}_{\mathrm{dR}} : \Omega^2(B\mathcal{L}) \to \Omega^3(B\mathcal{L})$. The compatibility of ω with the L_∞ structure is $\mathrm{d}_{\mathcal{L}}\omega = 0$ and $\mathrm{d}_{\mathrm{dR}}\omega = 0$. Since neither of these generalizations will play a role for us in this paper, we will omit formal definitions and hope that the idea is clear.

2.2. **Examples of presymplectic BV theories.** We proceed to give some examples of presymplectic BV theories, beginning with simple examples of degenerate pairings and proceeding to more ones more relevant to six-dimensional theories. The secondary goal of this section is to set up notation and terminology that will be used in the rest of the paper.

Example 2.3. Suppose (V, w) is a finite dimensional presymplectic vector space. That is, V is a finite dimensional vector space and $w : V \to V^*$ is a (degree zero) linear map which satisfies $w^* = -w$. Then, for any 1-manifold L, the elliptic complex

$$(\mathcal{E}, Q_{\mathrm{BV}}) = (\Omega^\bullet_L \otimes V, \ \mathrm{d}_{\mathrm{dR}})$$

is a presymplectic BV theory on L with

$$\omega = \mathrm{id}_{\Omega^\bullet} \otimes w : \Omega^\bullet_L \otimes V \to \Omega^\bullet_L \otimes V^* = \mathcal{E}^![-1].$$

Similarly, if Σ is a Riemann surface equipped with a spin structure $K_\Sigma^{\frac{1}{2}}$, then the elliptic complex

$$(\mathcal{E}, Q_{\mathrm{BV}}) = \left(\Omega_L^{0,\bullet} \otimes K_\Sigma^{\frac{1}{2}} \otimes V,\ \overline{\partial}\right)$$

is a presymplectic BV theory on Σ with

$$\omega = \mathrm{id}_{\Omega^{0,\bullet}\otimes K^{\frac{1}{2}}} \otimes w : \Omega_L^{0,\bullet} \otimes K_\Sigma^{\frac{1}{2}} \otimes V \to \Omega_L^{0,\bullet} \otimes K_\Sigma^{\frac{1}{2}} \otimes V^*.$$

Each theory in this example arose from an ordinary presymplectic vector space, which was also the source of the degeneracy of ω. The first example that is really intrinsic to field theory, and also relevant for the further discussion in this paper, is the following.

Example 2.4. Let Σ be a Riemann surface and suppose (W, h) is a finite dimensional vector space equipped with a symmetric bilinear form thought of as a linear map $h : W \to W^*$. Then

$$(\mathcal{E}, Q_{\mathrm{BV}}) = \left(\Omega_\Sigma^{0,\bullet} \otimes W,\ \overline{\partial}\right)$$

is a presymplectic BV theory with

$$\omega = \partial \otimes h : \Omega_\Sigma^{0,\bullet} \otimes W \to \Omega_\Sigma^{1,\bullet} \otimes W^* = \mathcal{E}^![-1].$$

We refer to this free presymplectic BV theory as the ***chiral boson*** with values in W, and will denote it by $\chi(0, W)$ (see next example). In the case that $W = \mathbb{C}$, we will simply denote this by $\chi(0)$.

Remark 2.5. While we did not require (W, h) to be nondegenerate in the above example, the theory is a genuinely presymplectic BV theory even if h is nondegenerate. This corresponds to the standard notion of the chiral boson in the physics literature, and we will have no cause to consider degenerate pairings h in what follows.

Example 2.6. Suppose X is a $(2k+1)$-dimensional complex manifold. Let $\Omega^{\bullet,\mathrm{hol}} = \left(\Omega^{\bullet,\mathrm{hol}}, \partial\right)$ be the holomorphic de Rham complex and let $\Omega^{\geq k+1,\mathrm{hol}}$ be the complex of forms of degree $\geq k+1$. By the holomorphic Poincaré lemma, $\Omega^{\geq k+1,\mathrm{hol}}$ is a resolution of the sheaf of holomorphic closed $(k+1)$-forms. Further, $\Omega^{\geq k+1,\mathrm{hol}}[-k-1]$ is a subcomplex of $\Omega^{\bullet,\mathrm{hol}}$ and there is a short exact sequence of sheaves of cochain complexes

$$\Omega^{\geq k+1,\mathrm{hol}}[-k-1] \to \Omega^{\bullet,\mathrm{hol}} \to \Omega^{\leq k,\mathrm{hol}}$$

which has a locally free resolution of the form

$$\Omega^{\geq k+1,\bullet}[-k-1] \to \Omega^{\bullet,\bullet} \to \Omega^{\leq k,\bullet}. \tag{4}$$

In this sequence, all forms are smooth and the total differential is $\partial + \overline{\partial}$ in each term. We use this quotient complex $\Omega^{\leq k,\bullet}$ to define another class of presymplectic BV theories.

Let (W,h) be as in the previous example. (Following Remark 2.5, it may as well be nondegenerate.) The elliptic complex

$$(\mathcal{E}, Q_{\rm BV}) = \left(\Omega_X^{\leq k,\bullet} \otimes W[2k], \mathrm{d} = \partial + \overline{\partial}\right).$$

is a presymplectic BV theory with

$$\omega = \partial \otimes h : \Omega_X^{\leq k,\bullet} \otimes W[2k] \to \Omega_X^{\geq k+1,\bullet} \otimes W^*[k].$$

We denote this presymplectic BV theory by $\chi(2k, W)$, which we will refer to as the ***chiral*** $2k$***-form*** with values in W. In the case $W = \mathbb{C}$ we will simply denote this by $\chi(2k)$.

Example 2.7. Let M be a Riemannian $(4k+2)$-manifold, and (W,h) as above. The Hodge star operator $\star$ defines a decomposition

$$\Omega^{2k+1}(M) = \Omega_+^{2k+1}(M) \oplus \Omega_-^{2k+1}(M) \tag{5}$$

on the middle de Rham forms, where $\star$ acts by $\pm\sqrt{-1}$ on $\Omega_\pm^{2k+1}(M)$.

Consider the following exact sequence of sheaves of cochain complexes:

$$0 \to \Omega_-^{\geq 2k+1}[-2k-1] \to \Omega^\bullet \to \Omega_+^{\leq 2k+1} \to 0 \tag{6}$$

where

$$\Omega_+^{\leq 2k+1} = \left(\Omega^0 \xrightarrow{\mathrm{d}} \Omega^1[-1] \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^{2k}[-2k] \xrightarrow{\mathrm{d}_+} \Omega_+^{2k+1}[-2k-1]\right), \qquad \mathrm{d}_+ = \frac{1}{2}(\star + \sqrt{-1})\mathrm{d}, \tag{7}$$

and

$$\Omega_-^{\geq 2k+1} = \left(\Omega_-^{2k+1} \xrightarrow{\mathrm{d}} \Omega^{2k+2}[-1] \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^{4k+2}[-2k-1]\right) \tag{8}$$

Let

$$(\mathcal{E}, Q_{\rm BV}) = (\Omega_+^{\leq 2k+1} \otimes W[2k],\ \mathrm{d}\ ) \tag{9}$$

and

$$\omega = \mathrm{d} \otimes h : \Omega_+^{\leq 2k+1} \otimes W[2k] \to \Omega_-^{\geq 2k+1} \otimes W^*.$$

This data defines a presymplectic BV theory $\chi_+(2k, W)$ on any Riemannian $(4k+2)$-manifold, which we will refer to as the ***self-dual*** $2k$***-form*** with values in W. Again, in the case $W = \mathbb{C}$ we will simply denote this by $\chi_+(2k)$.

Remark 2.8. In general, the theories $\chi(2k)$ and $\chi_+(2k)$ are defined on different classes of manifolds; they can, however, be simultaneously defined when X is a complex manifold equipped with a Kähler metric. Even in this case, they are distinct theories, though their dimension reductions both agree with the usual chiral boson, see §**??**). In §4 we will show explicitly that the $\mathcal{N} = (1,0)$ tensor multiplet (which consists of $\chi_+(2)$ together with fermions and one scalar) becomes precisely $\chi(2)$ under a holomorphic twist.

There is, however, one case where the two theories coincide. A choice of metric on a Riemann surface determines a conformal class, which then corresponds precisely to a complex structure. As such, both of the theories $\chi(0)$ and $\chi_+(0)$ are always well-defined, and in fact agree; both are the theory of the chiral boson defined in Example 2.4.

We now recall a couple of examples of nondegenerate theories, for later convenience and to fix notation, that fit the definition of a standard free BV theory [5, Definition 7.2.1.1].

Example 2.9. Let M be a Riemannian manifold of dimension d. Let (W, h) be a complex vector space equipped with a non-degenerate symmetric bilinear pairing $h : W \cong W^*$. The theory $\Phi(0, W)$ of the ***free boson with values in*** W is the data

$$(\mathcal{E}, Q_{\mathrm{BV}}) = \left(\Omega^0(M) \otimes W \xrightarrow{\mathrm{d}\star\mathrm{d}\otimes\mathrm{id}_W} \Omega^d(M) \otimes W[-1]\right), \tag{10}$$

and $\omega = \mathrm{id}_{\Omega^0} \otimes h + \mathrm{id}_{\Omega^d} \otimes h$. Notice this is a BV theory, the (-1) presymplectic structure is non-degenerate.

Example 2.10. Let (W, h) be as in the previous example, $p \geq 0$ an integer, and suppose M is a Riemannian manifold of dimension $d \geq p$. The theory $\Phi(p, W)$ of ***free*** p***-form fields valued in*** W is defined [11] by the data

$$(\mathcal{E}, Q_{\mathrm{BV}}) = \left(\Omega^{\leq p} \otimes W[p] \xrightarrow{\mathrm{d}\star\mathrm{d}\otimes\mathrm{id}_W} \Omega^{\geq d-p} \otimes W[p-1]\right), \tag{11}$$

with (-1)-symplectic structure $\omega = \mathrm{id}_{\Omega^{\leq p}} \otimes h + \mathrm{id}_{\Omega^{\geq d-p}} \otimes h$. Notice again this is an honest BV theory, the presymplectic structure is non-degenerate. If $\alpha \in \mathcal{E}$ denotes a field, the classical action functional reads $\frac{1}{2}\int h(\alpha, \mathrm{d} \star \mathrm{d}\alpha)$.

This example clearly generalizes the free scalar field theory, and also does not depend in any way on our special choice of dimension. We will simply write $\Phi(p)$ for the case $W = \mathbb{C}$ when the spacetime M is understood.

Example 2.11. Let M be as in the last example, and suppose in addition it carries a spin structure compatible with the Riemannian metric. Let (R, w) be a complex vector space equipped with an antisymmetric non-degenerate bilinear pairing. The theory $\Psi_-(R)$ of ***chiral fermions valued in*** R is the data

$$(\mathcal{E}, Q_{\mathrm{BV}}) = \Gamma(\Pi S_- \otimes R) \xrightarrow{\partial\!\!\!/\otimes\mathrm{id}_R} \Gamma(\Pi S_+ \otimes R)[-1], \tag{12}$$

with (-1)-symplectic structure $\omega = \mathrm{id}_{S_+} \otimes w + \mathrm{id}_{S_-} \otimes w$.

We depart from the world of Riemannian manifolds to exhibit theories natural to the world of complex geometry that will play an essential role later on in the paper.

Example 2.12. Suppose X is a complex manifold of complex dimension 3 which is equipped with a square-root of its canonical bundle $K_X^{\frac{1}{2}}$. Let (S, w) be a $\mathbb{Z}/2$-graded vector space equipped with an even non-degenerate bilinear pairing. ***Abelian holomorphic Chern–Simons theory*** valued in S is the free BV theory whose complex of fields is

$$(\mathcal{E}, Q_{\mathrm{BV}}) = \Omega^{0,\bullet}(X, K_X^{\frac{1}{2}} \otimes S)[1]$$

with (-1)-symplectic structure $\omega = \mathrm{id}_{\Omega^{0,\bullet}} \otimes w$. This theory is naturally $\mathbb{Z} \times \mathbb{Z}/2$-graded and has action functional $\frac{1}{2}\int w(\alpha \wedge \overline{\partial}\alpha)$. Notice that the fields in cohomological degree zero consist of $\alpha \in \Omega^{0,1}(X, K_X^{\frac{1}{2}} \otimes S)$, and the equation of motion is $\overline{\partial}\alpha = 0$. This theory thus describes deformations of complex structure of the $\mathbb{Z}/2$-graded bundle $K_X^{\frac{1}{2}} \otimes S$. We denote this theory by $\Phi_{\mathrm{hCS}}(S)$. We will be most interested in the case $S = \Pi R$ where R is an ordinary (even) symplectic vector space, see Theorem 4.1.

2.3. **presymplectic BV theories and constraints.** Perturbative presymplectic BV theories stand in the same relationship to perturbative BV theories as presymplectic manifolds do to symplectic manifolds. Presymplectic structures obviously pull back along embeddings, whereas symplectic structures do not. There is thus always a preferred presymplectic structure on submanifolds of any (pre)symplectic manifold. In fact, this is the starting point for Dirac's theory of constrained mechanical systems **???**. BW: Add more discussion

Each of the examples of presymplectic BV theories we have given so far can be similarly understood as constrained systems relative to some symplectic BV theory.

Example 2.13 (The chiral boson and the free scalar). The chiral boson $\chi(0, W)$ on a Riemann surface Σ, from Example 2.4, can be understood as a constrained system relative to the free scalar $\Phi(0, W)$, see Example 2.9. At the level of the equations of motion this is obvious: the constrained system picks out the harmonic functions that are holomorphic.

In the BV formalism, this constraint is realized by the following diagram of sheaves on Σ:

(13)
$$\begin{tikzcd} \Omega^{0,0} \arrow[r, "\partial\bar{\partial}"] & \Omega^{1,1} \\ \Omega^{0,0} \arrow[u, "\mathrm{id}"] \arrow[r, "\bar{\partial}"] & \Omega^{0,1} \arrow[u, "\partial"'] \end{tikzcd}$$

It is evident that the diagram commutes, and that the vertical arrows define a cochain map upon tensoring with W:

(14)
$$\chi(0, W) \to \Phi(0, W).$$

Furthermore, a moment's thought reveals that the (-1)-shifted presymplectic pairing on $\chi(0, W)$ arises by pulling back the (-1)-shifted symplectic pairing on $\Phi(0, W)$.

Example 2.14 (The self-dual $2k$-form and the free $2k$-form). It is easy to form generalizations of the previous example. Consider the following diagram of sheaves on a Riemannian $(4k+2)$-manifold:

(15)
$$\begin{tikzcd}
\Omega^0 \arrow[r] & \cdots \arrow[r] & \Omega^{2k} \arrow[r, "\mathrm{d}*\mathrm{d}"] & \Omega^{2k+2} \arrow[r] & \cdots \arrow[r] & \Omega^{4k+2} \\
\Omega^0 \arrow[r] \arrow[u, "\mathrm{id}"] & \cdots \arrow[r] & \Omega^{2k} \arrow[r, "\mathrm{d}_+"] \arrow[u, "\mathrm{id}"] & \Omega^{2k+1}_+ \arrow[u, "\mathrm{d}"] & &
\end{tikzcd}$$

Just as above, the vertical arrows of this commuting diagram define a cochain map

(16)
$$\chi_+(2k, W) \to \Phi(2k, W),$$

under which the natural (-1)-shifted presymplectic structure of Example 2.7 arises by pulling back the (-1)-shifted symplectic form on $\Phi(2k, W)$.

If X is a complex manifold of complex dimension $2k+1$, the presymplectic BV theory of the chiral $2k$-form $\chi(2k)$ is defined, see Example 2.6. As a higher dimensional generalization of Example 2.13, $\chi(2k)$ can also be understood as a constrained system relative to theory of the free $2k$-form $\Phi(2k, W)$, see Example 2.10. It is an instructive exercise to construct the similar diagram that witnesses the presymplectic structure on the chiral $2k$-form $\chi(2k, W)$ by pullback from the ordinary (nondegenerate) BV structure on $\Phi(2k, W)$.

2.4. **The observables of a presymplectic BV theory.** The classical BV formalism, as formulated in [5], constructs a factorization algebra from a classical BV theory, which plays the role of functions on a symplectic manifold in the ordinary finite dimensional situation.

In symplectic geometry, functions carry a Poisson bracket. In the classical BV formalism there is a shifted version of Poisson algebras that play a similar role. By definition, a $\mathbb{P}_0$-algebra is a commutative dg algebra together with a graded skew-symmetric bracket of cohomological degree $+1$ which acts as a graded derivation with respect to the commutative product, see BW: ref??. Classically, the BV formalism outputs a $\mathbb{P}_0$-factorization algebra of classical observables [5, §5.2].

In this section, we will see that there is a $\mathbb{P}_0$-factorization algebra associated to a *presymplectic* BV theory, which agrees with the construction of [5] in the case that the presymplectic BV theory is nondegenerate. Unlike the usual situation, this algebra is not simply the functions on the space of fields, but consists of certain class of functions. We begin by recalling the situation in presymplectic mechanics.

To any presymplectic manifold (M, ω) one can associate a Poisson algebra. This construction generalizes the usual Poisson algebra of functions in the symplectic case, and goes as follows. Let $\mathrm{Vect}(M)$ be the Lie algebra of vector fields on M, and define the space of *Hamiltonian pairs*

(17)
$$\mathrm{Ham}(M, \omega) \subset \mathrm{Vect}(M) \oplus \mathcal{O}(M)$$

to be the linear subspace of pairs (X, f) satisfying $i_X \omega = \mathrm{d}f$. Correspondingly, we can define the space of *Hamiltonian functions* or *Hamiltonian vector fields* to be the image of $\mathrm{Ham}(M, \omega)$ under the obvious (forgetful) maps to $\mathcal{O}(M)$ or $\mathrm{Vect}(M)$ respectively. We will denote these spaces by $\mathcal{O}^\omega(M)$ and $\mathrm{Vect}^\omega(M)$. Notice that $\mathcal{O}^\omega(M)$ is the quotient of $\mathrm{Ham}(M,\omega)$ by the Lie ideal $\ker(\omega) \subset \mathrm{Ham}(M,\omega)$.

There is a bracket on $\mathrm{Ham}(M,\omega)$, defined by

$$[(X,f),(Y,g)] = ([X,Y], i_X i_Y(\omega)).$$

On the right-hand side the bracket $[-,-]$ is the usual Lie bracket of vector fields. Furthermore, there is a commutative product on $\mathrm{Ham}(M,\omega)$ defined by

$$(X,f)\cdot(Y,g) = (gX + fY, fg).$$

Together, they endow $\mathrm{Ham}(M,\omega)$ with the structure of a Poisson algebra. This Poisson bracket on Hamiltonian pairs induces a Poisson algebra structure on the algebra of Hamiltonian functions $\mathcal{O}^\omega(M)$.

In some situations, one can realize the Poisson algebra of Hamiltonian functions $\mathcal{O}^\omega(M)$ as functions on a particular symplectic manifold. Associated to the presymplectic form ω is the subbundle

$$\ker(\omega) \subseteq TM \tag{18}$$

of the tangent bundle. The closure condition on ω ensures that $\ker(\omega)$ is always involutive. If one further assumes that the leaf space $M/\ker(\omega)$ is a smooth manifold, then ω automatically descends to a symplectic structure along the quotient map $q : M \to M/\ker(\omega)$. Pulling back along this map determines an isomorphism of Poisson algebras

$$q^* : \mathcal{O}(M/\ker(\omega)) \xrightarrow{\cong} \mathcal{O}^\omega(M).$$

In particular, one can view the Poisson algebra of Hamiltonian functions as the $\ker(\omega)$-invariants of the algebra of functions $\mathcal{O}^\omega(M) = \mathcal{O}(M)^{\ker(\omega)}$. Notice that this formula makes sense without any conditions on the niceness of the quotient $M/\ker(\omega)$.

In our setting, the presymplectic data is given by a presymplectic BV theory. A natural problem is to define and characterize a version of Hamiltonian functions in this setting.

2.4.1. *The factorization algebra of observables.* As we've already mentioned, given a (nondegenerate) BV theory the work of [5] produces a factorization algebra of classical observables. If $(\mathcal{E},\omega,Q_{\mathrm{BV}})$ is the space of fields of a free BV theory on a manifold M then this factorization algebra $\mathrm{Obs}_{\mathcal{E}}$ assigns to the open set $U \subset M$ the cochain complex $\mathrm{Obs}_{\mathcal{E}}(U) = (\mathcal{O}^{sm}(\mathcal{E}(U)), Q_{\mathrm{BV}})$. Here $\mathcal{O}^{sm}(\mathcal{E}(U))$ refers to the "smooth" functionals on $\mathcal{E}(U)$, which by definition are[4]

$$\mathcal{O}^{sm}(\mathcal{E}(U)) = \mathrm{Sym}\left(\mathcal{E}^!_c(U)\right).$$

[4]Notice $\mathcal{E}^!_c(U) \hookrightarrow \mathcal{E}(U)^\vee$, so $\mathcal{O}^{sm}$ is a subspace of the space of all functionals on $\mathcal{E}(U)$.

Furthermore, since ω is an isomorphism, it induces a bilinear pairing

$$\omega^{-1} : \mathcal{E}^!_c \times \mathcal{E}^!_c \to \mathbb{C}[1].$$

By the graded Leibniz rule, this then determines a bracket

$$\{-,-\} : \mathcal{O}^{sm}(\mathcal{E}(U)) \times \mathcal{O}^{sm}(\mathcal{E}(U)) \to \mathcal{O}^{sm}(\mathcal{E}(U))[1]$$

endowing $\mathrm{Obs}_{\mathcal{E}}$ with the structure of a $\mathbb{P}_0$-factorization algebra, see [5, Lemma 5.3.0.1].

In this section, we turn our attention to defining the observables of a presymplectic BV theory, modeled on the notion of the algebra of Hamiltonian functions in the finite dimensional presymplectic setting. Suppose that $(\mathcal{E}, \omega, Q_{\mathrm{BV}})$ is a free presymplectic BV theory. The shifted presymplectic structure is defined by a differential operator

$$\omega : \mathcal{E} \to \mathcal{E}^![-1].$$

In order to implement the structures we recounted in the ordinary presymplectic setting, the first object we must come to terms with is the solution sheaf of this differential operator $\ker(\omega) \subset \mathcal{E}$.

In general $\ker(\omega)$ is not given as the smooth sections of a finite rank vector bundle, so it is outside of our usual context of perturbative field theory. However, suppose we could find a semi-free resolution $(\mathcal{K}^\bullet_\omega, D)$ by finite rank bundles

$$\ker(\omega) \xrightarrow{\simeq} (\mathcal{K}^\bullet_\omega, D)$$

which fits in a commuting diagram

$$\begin{tikzcd} \ker(\omega) \arrow[rr, "\simeq"] \arrow[dr] & & \mathcal{K}^\bullet_\omega \arrow[dl, "\pi"] \\ & \mathcal{E} & \end{tikzcd}$$

with the bottom left arrow the natural inclusion, and π is a linear differential operator. In the more general case where ω is not linear, we would require that $\mathcal{K}^\bullet_\omega$ have the structure of a dg Lie algebra resolving $\ker(\omega) \subset \mathrm{Vect}(\mathcal{E})$.

Given this data, the natural ansatz for the classical observables is the (derived) invariants of $\mathcal{O}(\mathcal{E})$ by $\mathcal{K}^\bullet_\omega$. A model for this is the Lie algebra cohomology:

$$\mathrm{C}^\bullet(\mathcal{K}^\bullet_\omega, \mathcal{O}(\mathcal{E})) = \mathrm{C}^\bullet(\mathcal{K}^\bullet_\omega \oplus \mathcal{E}[-1]).$$

In this free case that we are in, this cochain complex is isomorphic to functions on the dg vector space $\mathcal{K}^\bullet_\omega[1] \oplus \mathcal{E}$ where the differential is $D + Q_{\mathrm{BV}} + \pi$.

As in the case of the ordinary BV formalism, in the free case we can use the smoothed version of functions on fields.

Definition 2.15. Let $(\mathcal{E}, \omega, Q_{\mathrm{BV}})$ be a free presymplectic BV theory on M, and suppose $(\mathcal{K}^\bullet, D)$ is a semi-free resolution of $\ker(\omega) \subset \mathcal{E}$ as above. The cochain complex of ***classical observables supported on the open set*** $U \subset M$ is

$$\begin{aligned}\mathrm{Obs}^{\omega}_{\mathcal{E}}(U) &= \mathcal{O}^{sm}\left(\mathcal{K}^\bullet_\omega(U) \oplus \mathcal{E}(U)[-1], D + Q_{\mathrm{BV}} + \pi\right) \\ &= \left(\mathrm{Sym}\left((\mathcal{K}^\bullet_\omega)^!_c(U) \oplus \mathcal{E}^!_c(U)[1]\right), D + Q_{\mathrm{BV}} + \pi\right).\end{aligned}$$

By [4, Theorem 6.0.1] the assignment $U \mapsto \mathrm{Obs}^{\omega}_{\mathcal{E}}(U)$ defines a factorization algebra on M, which we will denote by $\mathrm{Obs}^{\omega}_{\mathcal{E}}$.

Example 2.16. Consider the chiral boson presymplectic BV theory $\chi(0)$, see Example 2.4, on a Riemann surface Σ. The kernel of $\omega = \partial$ is the sheaf of constant functions

$$\ker(\omega) = \underline{\mathbb{C}}_\Sigma \subset \Omega^{0,\bullet}(\Sigma).$$

By Poincaré's Lemma, the de Rham complex $\left(\Omega^\bullet_\Sigma, \mathrm{d}_{\mathrm{dR}} = \partial + \overline{\partial}\right)$ is a semi-free resolution of $\underline{\mathbb{C}}_\Sigma$. Thus, the classical observables are given as the Lie algebra cohomology of the abelian dg Lie algebra

$$\left(\Omega^\bullet_\Sigma \oplus \Omega^{0,\bullet}_\Sigma[-1], \mathrm{d}_{\mathrm{dR}} + \overline{\partial} + \pi\right)$$

where $\pi : \Omega^\bullet_\Sigma \to \Omega^{0,\bullet}_\Sigma$ is the projection. This dg Lie algebra is quasi-isomorphic to the abelian dg Lie algebra $\Omega^{1,\bullet}_\Sigma[-1]$, so the factorization algebra of classical observables is

$$\mathrm{Obs}^{\omega}_{\chi(0)} \simeq \mathcal{O}^{sm}(\Omega^{1,\bullet}_\Sigma) = \mathrm{Sym}\left(\Omega^{0,\bullet}_{\Sigma,c}[1]\right).$$

There are two special cases to point out.

(1) Suppose the shifted presymplectic form ω is an order zero differential operator. Then, $\ker(\omega)$ is a subbundle of $\mathcal{E}$, so there is no need to seek a resolution. Furthermore, in this case $\mathcal{E}/\ker(\omega)$ is also given as the sheaf of sections of a graded vector bundle $E/\ker(\omega)$, and ω descends to bundle isomorphism $\omega : E/\ker(\omega) \xrightarrow{\cong} (E/\ker(\omega))^!\,[-1]$.

In other words, $(\mathcal{E}/\ker(\omega), \omega, Q_{\mathrm{BV}})$ defines a (nondegenerate) free BV theory. The factorization algebra of the classical observables of the pre BV theory $\mathrm{Obs}^{\omega}_{\mathcal{E}}$ agrees with the factorization algebra of the BV theory $\mathcal{E}/\ker(\omega)$

$$\mathrm{Obs}_{\mathcal{E}/\ker(\omega)} = \left(\mathcal{O}^{sm}(\mathcal{E}/\ker(\omega), Q_{\mathrm{BV}}\right).$$

In this case, the observables inherit a $\mathbb{P}_0$-structure by [5, Lemma 5.3.0.1].

(2) This next case may seem obtuse, but fits in with many of the examples we consider. Suppose that the two-term complex defined by the presymplectic form ω:

$$\begin{array}{ccc} \underline{0} & & \underline{1} \\ & & \\ \mathcal{E} & \xrightarrow{\omega} & \mathcal{E}^![-1] \end{array}$$

is itself a semi-free resolution of $\ker(\omega)$. In this case, it is immediate to verify that the factorization algebra of observables is

$$\mathrm{Obs}^{\omega}_{\mathcal{E}} = \left(\mathcal{O}^{sm}(\mathcal{E}^![-1]), Q_{\mathrm{BV}}\right).$$

We mention that in this case $\mathrm{Obs}^{\omega}_{\mathcal{E}}$ is also endowed with a $\mathbb{P}_0$-structure defined directly by ω.

We can summarize the discussion in the two points above as follows.

Proposition 2.17. *If the presymplectic BV theory* $(\mathcal{E}, \omega, Q_{\mathrm{BV}})$ *satisfies* (1) *or* (2) *above then the classical observables* $\mathrm{Obs}^{\omega}_{\mathcal{E}}$ *form a* $\mathbb{P}_0$*-factorization algebra.*

Remark 2.18. We expect a definition of a $\mathbb{P}_0$-factorization algebra of observables associated to any (non-linear) presymplectic BV theory, though we do not pursue that here.

The chiral boson $\chi(0)$ satisfies condition (2), and so the factorization algebra of observables $\mathrm{Obs}^{\omega}_{\chi(0)} = \mathcal{O}(\Omega^{1,\bullet}_{\Sigma})$ carries a $\mathbb{P}_0$-structure. This $\mathbb{P}_0$-structure corresponds to the usual Poisson vertex algebra structure on the classical limit of the Heisenberg vertex algebra. BW: vertex refs?

3. The abelian tensor multiplet

We provide a definition of the (perturbative) abelian $\mathcal{N} = (2,0)$ tensor multiplet in the presymplectic BV formalism, together with the $\mathcal{N} = (1,0)$ tensor and hypermultiplets. As discussed in the previous section, in the BV formalism one must specify a (-1)-shifted symplectic (infinite dimensional) manifold, the fields, together with the data of a homological vector field which is compatible with the shifted symplectic form. The tensor multiplets in six dimensions are peculiar, because they only carry a presymplectic BV (shifted presymplectic) structure, as opposed to a symplectic one.

Roughly speaking, the fundamental fields of the tensor multiplet consist of a two-form field whose field strength is constrained to be self-dual, a scalar field valued in some R-symmetry representation, and fermions transforming in the positive spin representation of $\mathrm{Spin}(6)$. The degeneracy of the shifted symplectic structure arises from the presence of the self-duality constraint on the two-form in the multiplet, just as in the examples in §2.2.

We begin by defining the field content of each multiplet precisely and giving the presymplectic BV structure. The next step is to formulate the action of supersymmetry on the (1,0) and (2,0) tensor multiplets at the level of the BV formalism. Here, one makes use of the well-known linear transformations on physical fields that are given in the physics literature. See, for example, **vanProeyen**, **West** for the $\mathcal{N} = (2,0)$ multiplet; we will review these transformations below.

However, these transformations do not define an action of $\mathfrak{p}_{(2,0)}$ on the space of fields. In the physics terminology, they close only on-shell (and after accounting for gauge equivalence). In the BV formalism, this is rectified by extending the action to an L_∞ action on the BV fields. (See, just for example, [12] for an application of this technique.) For the hypermultiplet, this was performed explicitly in [13]; the hypermultiplet, however, is a symplectic BV theory in the standard sense.

For the tensor multiplet, supersymmetry also only exists on-shell; no strict Lie module structure can be given. The required L_∞-correction terms, however, present additional subtleties: the terms one would need to introduce to exhibit off-shell supersymmetry in the BV formalism are generally non-local, hence ill-defined in a naive sense.

For the purposes of computing the twist, this presents no issue, because the non-local correction terms play no role in defining the action of any square-zero supercharge. For holomorphic supercharges Q, the naive action is strict, and no higher L_∞ terms are needed to make use of the symmetry off-shell. To compute the non-minimal twist, we witness it as a further deformation of the minimal twist by second nilpotent supercharge Q'. It turns out that correction terms are required to have an off-shell action by Q' on the Q-twisted theory, but the relevant terms are in fact all local.

We will first recall the definitions of the relevant supersymmetry algebras; afterwards, we will construct the multiplets as free perturbative presymplectic BV theories, and go on to give the L_∞ module structure on the $\mathcal{N} = (2,0)$ tensor multiplet.

3.1. **Supersymmetry algebras in six dimensions.** Let $S_\pm \cong \mathbb{C}^4$ denote the complex 4-dimensional spin representations of $\mathrm{Spin}(6)$ and let $V \cong \mathbb{C}^6$ be the vector representation. There exist natural $\mathrm{Spin}(6)$-invariant isomorphisms

$$\wedge^2(S_\pm) \xrightarrow{\cong} V$$

and a non-degenerate $\mathrm{Spin}(6)$-invariant pairing

$$(-,-) : S_+ \otimes S_- \to \mathbb{C}.$$

The latter identifies $S_+ \cong (S_-)^*$ as $\mathrm{Spin}(6)$-representations. Under the exceptional isomorphism $\mathrm{Spin}(6) \cong SU(4)$, $S_\pm$ are identified with the fundamental and antifundamental representation respectively.

The odd part of the complexified six-dimensional $\mathcal{N} = (n,0)$ supersymmetry algebra is of the form

$$\Sigma_n = S_+ \otimes R_n,$$

where R_n is a $(2n)$-dimensional complex symplectic vector space whose symplectic form we denote by ω_R. There is thus a natural action of $\mathrm{Sp}(n)$ on R_n by the defining representation. Note that we can identify the dual $\Sigma_n^* = S_- \otimes R_n$ as representations of $\mathrm{Spin}(6) \times \mathrm{Sp}(n)$.

The full $\mathcal{N} = (n, 0)$ supertranslation algebra in six dimensions is the super Lie algebra

$$\mathfrak{t}_{(n,0)} = V \oplus \Pi\Sigma_n$$

with bracket

$$[-, -] = \wedge \otimes \omega_R : \wedge^2(\Pi\Sigma_n) \to V. \tag{19}$$

This algebra admits an action of $\mathrm{Spin}(6) \times \mathrm{Sp}(n)$, where the first factor is the group of (Euclidean) Lorentz symmetries and the second is called the R-symmetry group $G_R = \mathrm{Sp}(n)$. Extending the Lie algebra of $\mathrm{Spin}(6) \times \mathrm{Sp}(n)$ by this module produces the full $\mathcal{N} = (n, 0)$ super-Poincaré algebra, denoted $\mathfrak{p}_{(n,0)}$.

Remark 3.1. We can view $\mathfrak{p}_{(n,0)}$ as a graded Lie algebra by assigning degree zero to $\mathfrak{so}(6) \oplus \mathfrak{sp}(n)$, degree one to Σ_n, and degree two to V. In physics, this consistent $\mathbb{Z}$-grading plays the role of the conformal weight. Both this grading and the R-symmetry action become inner in the ***superconformal algebra***, which is the simple super Lie algebra

$$\mathfrak{c}_{(n,0)} = \mathfrak{osp}(8|n). \tag{20}$$

The abelian $\mathcal{N} = (2, 0)$ multiplet in fact carries a module structure for $\mathfrak{osp}(8|2)$; computing the holomorphic twist of this action should lead to an appropriate algebra acting by supervector fields on the holomorphic theory we compute below, which should then extend to an action of all holomorphic vector fields on an appropriate superspace, following the pattern of [14]. However, we leave this computation to future work.

For theories of physical interest, one considers $n = 1$ or 2. In the latter case, an accidental isomorphism identifies $\mathrm{Sp}(2)$ with $\mathrm{Spin}(5)$, which further identifies R_2 with the unique complex spin representation of $\mathrm{Spin}(5)$.

3.1.1. *Elements of square zero.* With an eye towards twisting, we recall the classification of square-zero elements in $\mathfrak{p}_{(n,0)}$ for $n = 1$ and 2, following [3], [15]. As above, we are interested in odd supercharges

$$Q \in \Pi\Sigma_n = \Pi S_+ \otimes R_n, \tag{21}$$

which satisfy the condition $[Q, Q] = 0$. Such supercharges define twists of a supersymmetric theory.

We will find it useful to refer to supercharges by their *rank* with respect to the tensor product decomposition (21) (meaning the rank of the corresponding linear map $R_n \to (S_+)^*$). It is immediate from the form of the supertranslation algebra that elements of rank one square to zero for any n.

When $n = 1$, it is also easy to see that any square-zero element must be of rank one, so that the space of such elements is isomorphic to the determinantal variety of rank-one matrices in $M^{4\times 2}(\mathbb{C})$. This can in turn be thought of as the image of the Segre embedding

$$\mathbb{P}^3 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7. \tag{22}$$

For $n = 2$, there are two distinct classes of such supercharges: those of rank one, which we will also refer to as minimal or holomorphic, and a certain class of rank-two elements, also called non-minimal or partially topological. A closer characterization of the two types of square-zero supercharges is the following:

Minimal (or holomorphic): A supercharge of this type is automatically square-zero. Moreover, such a supercharge has three invariant directions, and so the resulting twist is a holomorphic theory defined on complex three-folds. Similarly to the $n = 1$ case, the space of such elements is isomorphic to the determinantal variety of rank-one matrices in $M^{4\times 4}(\mathbb{C})$, which is the image of the Segre embedding

$$\mathbb{P}^3 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^{15}. \tag{23}$$

We remark that in the case $n = 2$, the supercharge Q of rank one defines a $\mathcal{N} = (1,0)$ subalgebra $\mathfrak{p}_{(1,0)} \cong \mathfrak{p}^Q_{(1,0)} \subset \mathfrak{p}_{(2,0)}$.

Non-minimal (or partially topological): Suppose $Q \in \Pi\Sigma_2$ is a rank-two supercharge (there is no such supercharge when $n = 1$). It can be written in the form

$$Q = \xi_1 \otimes r_1 + \xi_2 \otimes r_2. \tag{24}$$

Since $\wedge^2 S_+ \cong V$, such an element must satisfy a single quadratic condition

$$w(r_1, r_2) = 0 \tag{25}$$

in order to be of square zero. Such a supercharge has five invariant directions, and the resulting twist can be defined on the product of a smooth four-manifold with a Riemann surface. The space of all such supercharges is a subvariety of the determinantal variety of rank-two matrices in $M^{4\times 4}(\mathbb{C})$, cut out by this single additional quadratic equation. (Just as for the determinantal variety itself, its singular locus is precisely the space of rank-one (holomorphic) supercharges.)

We will compute the holomorphic twist below in §4 and the rank-two twist in §5. There, we will also recall some further details about nilpotent elements in $\mathfrak{t}_{(2,0)}$, showing how the non-minimal twist can be obtained as a deformation of a fixed minimal twist.

3.2. **Supersymmetry multiplets.** The two theories we are most interested in are the abelian $(1,0)$ and $(2,0)$ tensor multiplets. We define these here at the level of (perturbative, free) presymplectic BV theories, and then go on to discuss the $\mathcal{N} = (1,0)$ hypermultiplet, which will also play a role in what follows.

First, we define the $(1,0)$ theory. Recall that R_1 denotes the defining representation of $\mathrm{Sp}(1)$.

Definition 3.2. The six-dimensional ***abelian*** $\mathcal{N} = (1,0)$ ***tensor multiplet*** is the presymplectic BV theory $\mathcal{T}_{(1,0)}$ defined by the direct sum of presymplectic BV theories:

$$\mathcal{T}_{(1,0)} = \chi_+(2) \oplus \Psi_-(R_1) \oplus \Phi(0, \mathbb{C}), \tag{26}$$

defined on a Riemannian spin manifold M. This theory has a symmetry by the group $G_R = \mathrm{Sp}(1)$ which acts on R_1 by the defining representation and trivially on the summands $\chi_+(2)$, $\Phi(0,\mathbb{C})$.

This theory admits an action by the supertranslation algebra $\mathfrak{p}_{(1,0)}$, which will be constructed explicitly below in §3.3.

Note that the fields of cohomological degree zero together with their linear equations of motion are:

- a two-form $\beta \in \Omega^2(M)$, satisfying the linear constraint $\mathrm{d}_+(\beta) = 0 \in \Omega^3_+(M)$;
- a spinor $\psi \in \Omega^0(M, S_- \otimes R_1)$, satisfying the linear equation of motion $(\not{\partial} \otimes \mathrm{id}_{R_1})\psi = 0 \in \Omega^0(M, S_+ \otimes R_1)$;
- a scalar $\varphi \in \Omega^0(M)$, satisfying the linear equation of motion $\mathrm{d} \star \mathrm{d}\varphi = 0 \in \Omega^6(M)$.

Next, we define the $(2,0)$ theory. Recall, R_2 denotes the defining representation of $\mathrm{Sp}(2)$. Let W be the vector representation of $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$.

Definition 3.3. The six-dimensional ***abelian*** $\mathcal{N} = (2,0)$ ***multiplet*** is the presymplectic BV theory $\mathcal{T}_{(2,0)}$ defined by the direct sum of presymplectic BV theories:

$$\mathcal{T}_{(2,0)} = \chi_+(2) \oplus \Psi_-(R_2) \oplus \Phi(0, W). \tag{27}$$

defined on a Riemannian spin manifold. This theory has a symmetry by the group $G_R = \mathrm{Sp}(2)$ which acts on R_2 by the defining representation and W by the vector representation upon the identification $\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$. Note, $G_R = \mathrm{Sp}(2)$ acts trivially on the summand $\chi_+(2)$.

This theory admits an action by the supertranslation algebra $\mathfrak{p}_{(2,0)}$, which will be constructed explicitly below in §3.3.

Note that the fields of cohomological degree zero consist of

- a two-form $\beta \in \Omega^2(M)$, satisfying the linear constraint $\mathrm{d}_+(\beta) = 0 \in \Omega^3_+(M)$;
- a spinor $\psi \in \Omega^0(M, S_- \otimes R_2)$, satisfying the linear equation of motion $(\not{\partial} \otimes \mathrm{id}_{R_2})\psi = 0 \in \Omega^0(M, S_+ \otimes R_2)$;
- a scalar $\varphi \in \Omega^0(M, W)$, satisfying the linear equation of motion $(\mathrm{d} \star \mathrm{d} \otimes \mathrm{id}_W)\varphi = 0 \in \Omega^6(M, W)$.

Lastly, we discuss the six-dimensional $\mathcal{N} = (1,0)$ hypermultiplet.

Definition 3.4. Let R be a finite-dimensional symplectic vector space over $\mathbb{C}$, as above. The $\mathcal{N} = (1,0)$ ***hypermultiplet valued in*** R is the following free (nondegenerate) BV theory in six dimensions:

$$\mathcal{T}^{\text{hyp}}_{(1,0)}(R) = \Phi(0, R_1 \otimes R) \oplus \Psi_-(R) \tag{28}$$

The theory admits an action of the flavor symmetry group $\mathrm{Sp}(R)$. (Note that $R_1 \otimes R$ obtains a symmetric pairing from the tensor product of the symplectic pairings on R and R_1.)

Exhibiting each of these theories as an L_∞-module for the relevant supersymmetry algebra is the subject of the next subsection.

3.3. **The module structure.** The main goal of this section is to define an action of the $(2,0)$ supersymmetry algebra $\mathfrak{p}_{(2,0)}$ on the tensor multiplet $\mathcal{T}_{(2,0)}$. The action of the $(1,0)$ supersymmetry algebra on the constituent multiplets $\mathcal{T}_{(1,0)}$ and $\mathcal{T}^{\text{hyp}}_{(1,0)}(R_1')$ will then be obtained trivially by restriction.

This action is only defined up to homotopy, which means we will give a description of $\mathcal{T}_{(2,0)}$ as an L_∞-*module* over $\mathfrak{p}_{(2,0)}$. Such a structure is encoded by the data of a sequence of maps $\{\rho^{(j)}\}_{j\geq 1}$ of the form

$$\sum_{j\geq 1} \rho^{(j)} : \mathrm{Sym}^\bullet\left(\mathfrak{p}_{(2,0)}[1]\right) \otimes \mathcal{T}_{(2,0)} \to \mathcal{T}_{(2,0)}, \tag{29}$$

satisfying a list of compatibilities. Here

$$\rho^{(j)} : \mathfrak{p}_{(2,0)}^{\otimes j} \otimes \mathcal{T}_{(2,0)} \to \mathcal{T}_{(2,0)}, \tag{30}$$

and the homological degree of ρ^j is $1 - j$.

In the case at hand, $\rho^{(1)}$ will be given by the known supersymmetry transformations from the physics literature, extended in the standard way to the antifields where possible. As remarked above, these transformations themselves do not define a representation of $\mathfrak{p}_{(2,0)}$ on $\mathcal{T}_{(2,0)}$. However, they can be corrected by higher ρ's so as to define an L_∞ module structure. In fact, we will see that $\rho^{(j)} = 0$ for $j \geq 3$, so we will only need to work out the quadratic term $\rho^{(2)}$.

3.3.1. *The physical transformations.* The linear term $\rho^{(1)}$ consists of the standard transformations ρ^{phys} on the physical fields, together with certain transformations on the antifields obtained by standard techniques. We begin by reviewing the form of ρ^{phys}, which breaks up into the sum of four components:

$$\begin{aligned}
&\rho_V : V \otimes \mathcal{T}_{(2,0)} \to \mathcal{T}_{(2,0)} \\
&\rho_\Psi : \Sigma_2 \otimes \Psi_-(R_2) \to \chi_+(2) \oplus \Phi(0, W) \\
&\rho_\Phi : \Sigma_2 \otimes \Phi(0, W) \to \Psi_-(R_2) \\
&\rho_\chi : \Sigma_2 \otimes \chi_+(2) \to \Psi_-(R_2)
\end{aligned} \tag{31}$$

We will define each of these component maps in turn. First off, an element $X \in V \subset \mathrm{Vect}(\mathbb{R}^6)$ acts via the Lie derivative $L_X\alpha$, where α is any BV field. That is, $\rho_V(X \otimes \alpha) = L_X\alpha$.

The transformation of the physical fermion field (the component $(\Pi S_- \otimes R_2) \subset \Psi_-(R_2)$ in degree zero) is given by ρ_Ψ, which is defined as follows. Consider the isomorphism

$$(\Pi S_+ \otimes R_2) \otimes (\Pi S_- \otimes R_2) \cong \left(\mathbb{C} \oplus \wedge^2 V\right) \otimes \left(\mathbb{C} \oplus W \oplus \mathrm{Sym}^2(R_2)\right) \tag{32}$$

of $\mathrm{Spin}(6) \times \mathrm{Sp}(2)$ representations. It is clear by inspection that there are equivariant projection maps onto the irreducible representations $\wedge^2 V \otimes \mathbb{C}$ and $\mathbb{C} \otimes W$. These projections allow us to define ρ_Ψ as the composition of the following sequence of maps:

$$\begin{tikzcd}[ampersand replacement=\&]
 \& \& \Omega^0(\mathbb{R}^6, W) \arrow[d, "\subset"] \\
\Sigma_2 \otimes \Psi_-(R_2) \arrow[r, two heads] \arrow[urr, dotted, bend left, "\rho_{\Psi,0}"] \arrow[drr, dotted, bend right, "\rho_{\Psi,2}"'] \& (\Pi S_+ \otimes R_2) \otimes (\Pi S_- \otimes R_2) \arrow[r, dashed] \arrow[ur, two heads] \arrow[dr, two heads] \& \chi_+(2) \oplus \Phi(0, W). \\
 \& \& \Omega^2(\mathbb{R}^6, \mathbb{C}) \arrow[u, "\subset"']
\end{tikzcd} \tag{33}$$

Of course, this map is canonically decomposed as the sum of two maps (along the direct sum in the target), which we will later refer to as $\rho_{\Psi,0}$ and $\rho_{\Psi,2}$ respectively.

The transformation of the physical scalar field (the component $C^\infty(\mathbb{R}^6; W)$ in degree zero) is defined as follows. We observe that there is a map of $\mathrm{Spin}(6) \times \mathrm{Sp}(2)$ representations of the form

$$(\Pi S_+ \otimes R_2) \otimes (V \otimes W) \to S_- \otimes R_2, \tag{34}$$

which can be thought of (using the accidental isomorphism $B_2 \cong C_2$) as the tensor product of the six- and five-dimensional Clifford multiplication maps. $\rho_3^{(1)}$ can then be defined as the composition of the maps in the diagram

$$\begin{tikzcd}[ampersand replacement=\&]
\Sigma_2 \otimes \Phi(0, W) \arrow[r, two heads] \& (\Pi\Sigma_+ \otimes R_2) \otimes \Omega^0(\mathbb{R}^6, W) \arrow[r, "\mathrm{d}"] \& (\Pi\Sigma_+ \otimes R_2) \otimes \Omega^1(\mathbb{R}^6, W) \arrow[d, two heads] \\
 \& \& \Gamma(\Pi S_- \otimes R_2) \arrow[r, "\subset"] \& \Psi_-(R_2).
\end{tikzcd} \tag{35}$$

On the component $\Omega^2(\mathbb{R}^4) \subset \chi_+(2)$, the map ρ_χ is defined as follows. Recall that there is a projection map of $\mathrm{Spin}(6)$ representations

$$\pi : S_+ \otimes \wedge^3_-(V) \to S_-$$

obtained via the isomorphism $\wedge^3_-(V) \otimes S_+ \cong S_- \oplus [012]$.[5] This isomorphism is most easily seen using the accidental isomorphism with $\mathrm{SU}(4)$, where it can be derived using the standard rules for Young tableaux

[5]The notation refers to the Dynkin labels of type D_3.

and takes the form

$$\ydiagram{1,1} \otimes \ydiagram{1} \cong \ydiagram{1,1,1} \oplus \ydiagram{2,1}. \tag{36}$$

The map ρ_χ is then defined on physical fields by the following sequence of maps:

$$\begin{tikzcd}
\Sigma_2 \otimes \chi_+(2) \arrow[r, two heads] & (\Pi S_+ \otimes R_2) \otimes \Omega^2(\mathbb{R}^6) \arrow[r, "\mathrm{d}_-"] & (\Pi S_+ \otimes R_2) \otimes \Omega^3_-(\mathbb{R}^6) \arrow[d, two heads] & \\
 & & C^\infty(\mathbb{R}^6; \Pi S_- \otimes R_2) \arrow[r, "\subset"] & \Psi_-(R_2).
\end{tikzcd} \tag{37}$$

3.3.2. *Anti-maps.* We begin by briefly reviewing the construction of the "anti-maps" in the standard BV approach. The idea is that the action of a physical symmetry algebra $\mathfrak{g}$ is usually defined by a map

$$\rho : \mathfrak{g} \to \mathrm{Vect}(F) \tag{38}$$

that implements the physical symmetry transformations on the physical (BRST) fields, just as in the previous section. Of course there are strong conditions on ρ coming from, for example, the requirement of locality. In the BV formalism, there is additionally the requirement that the action of $\mathfrak{g}$ on the BV fields must preserve the shifted symplectic structure. There is an immediate way to extend the vector fields (38) to *symplectic* vector fields on the space $E = T^*[-1]F$ of BV fields: one can take the transformation laws of the antifields to be determined by the condition of preserving the shifted symplectic form. (In fact, such vector fields are always Hamiltonian in the standard case.) The induced transformations of the antifields are sometimes known as the *anti-maps* of the original transformations, and we will denote them with the superscript ρ^+.

For the translation action ρ_V, we have tacitly already included the anti-maps in our definition above. It remains to specify the anti-maps for the three maps defining the physical supersymmetry transformations. Two of these will be standard, but the anti-map for the presymplectic BV multiplet $\chi_+(2)$ requires a bit of care.

For the anti-map to ρ_Φ, no complexity appears: we can simply define it as the composition

$$\begin{tikzcd}
\Sigma_2 \otimes \Psi_-(R_2) \arrow[d, two heads] & \Omega^6(\mathbb{R}^6, W)[-1] \arrow[r, "\subset"] & \Phi(0, W). \\
(\Pi S_+ \otimes R_2) \otimes \Gamma(\Pi S_+[-1] \otimes R_2) \arrow[r, "\not\partial"] & (\Pi S_+ \otimes R_2) \otimes \Gamma(\Pi S_-[-1] \otimes R_2) \arrow[u, two heads] &
\end{tikzcd} \tag{39}$$

The anti-map to $\rho_{\Psi,0}$ is similarly straightforward, and can be expressed with the diagram

$$\Sigma_2 \otimes \Phi(0, W) \twoheadrightarrow (\Pi S_+ \otimes R_2) \otimes \Omega^6(\mathbb{R}^6, W)[-1] \xrightarrow{\cong} \Gamma(\Pi S_+[-1] \otimes R_2) \xrightarrow{\subset} \Psi_-(R_2). \tag{40}$$

The other two maps is determined by the nature of the pairing ω on $\chi_+(2)$. As such, the number of derivatives appearing is, at first glance, somewhat surprising. The anti-map to $\rho_{\Psi,2}$ takes the form

$$\begin{tikzcd} \Sigma_2 \otimes \chi_+(2) \arrow[d, two heads] & \Gamma(\Pi S_+[-1] \otimes R_2) \arrow[r, "\subset"] & \Psi_-(R_2). \\ (\Pi S_+ \otimes R_2) \otimes \Omega^3_+(\mathbb{R}^6)[-1] \arrow[r, "\mathrm{d}"] & (\Pi S_+ \otimes R_2) \otimes \Omega^4(\mathbb{R}^6)[-1] \arrow[u, two heads] & \end{tikzcd} \tag{41}$$

Finally, the anti-map to ρ_χ takes the form

$$\Sigma_2 \otimes \Psi_-(R_2) \twoheadrightarrow (\Pi S_+ \otimes R_2) \otimes \Gamma(\Pi S_+[-1] \otimes R_2) \twoheadrightarrow \Omega^3_+(\mathbb{R}^6)[-1] \xrightarrow{\subset} \chi_+(2). \tag{42}$$

The linear part of the L_∞ module structure can now be defined as the sum of the physical transformations and their anti-maps:

$$\rho^{(1)} = \rho^{\mathrm{phys}} + \rho^+_\Phi + \rho^+_{\Psi,0} + \rho^+_{\Psi,2} + \rho^+_\chi. \tag{43}$$

Proposition 3.5. *$\rho^{(1)}$ defines a collection of symplectic vector fields on $\mathcal{T}_{(2,0)}$.*

Proof. It is sufficient to demonstrate that the vector fields $\rho^{(1)}$ are Hamiltonian. We do this by giving a local Hamiltonian functional whose associated Hamiltonian vector field agrees with $\rho^{(1)}$. This functional is given by

$$\psi^+\epsilon \tag{44}$$

IAS: to be finished! Here $\epsilon \in (\Sigma_2)^\vee[1]$ is a generator of the Chevalley-Eilenberg complex for $\mathfrak{t}_{(2,0)}$. Of course, one can easily verify directly that the action of $\rho^{(1)}$ commutes with the map ω, which is sufficient to prove the claim. □

Proposition 3.6. *Viewed as a morphism from $\mathfrak{t}_{(2,0)}$ to $\mathrm{End}(\mathcal{T}_{(2,0)})$, $\rho^{(1)}$ is homotopy equivalent to a Lie map. That is, the map*

$$\begin{aligned} &\mu : \mathfrak{t}_{(2,0)} \otimes \mathfrak{t}_{(2,0)} \otimes \mathcal{T}_{(2,0)} \to \mathcal{T}_{(2,0)}, \\ &\qquad \alpha \otimes \beta \otimes f \mapsto \rho^{(1)}([\alpha,\beta], f) - \rho^{(1)}(\alpha, \rho^{(1)}(\beta, f)) \pm \rho^{(1)}(\beta, \rho^{(1)}(\alpha, f)) \end{aligned} \tag{45}$$

is nullhomotopic.

Proof. It is sufficient to consider the case when $\alpha, \beta \in \Sigma_2$, for which the sign in the above equation is negative. Furthermore, the first term simply produces the Lie derivative of any field in the direction $[\alpha, \beta]$. Since μ is an even degree-zero map, we can consider each degree and parity separately, beginning with the ghosts: here, it is easy to see that

$$\begin{aligned} \mu_{-2}(\alpha, \beta, \cdot) &= \mathcal{L}_{[\alpha,\beta]} : \Omega^0[2] \to \Omega^0[2], \\ \mu_{-1}(\alpha, \beta, \cdot) &= \mathcal{L}_{[\alpha,\beta]} : \Omega^1[1] \to \Omega^1[1], \end{aligned} \tag{46}$$

since the supersymmetry variations make no contribution. We next work out the action of μ_0 on the two-form field, which is given by

$$\mu_{0,\chi}(\alpha, \beta, \cdot) = \mathcal{L}_{(\alpha,\beta)} - \rho_\Psi(\alpha) \circ \rho_\chi(\beta) - \rho_\Psi(\beta) \circ \rho_\chi(\alpha) \tag{47}$$

as a map from $\Omega^2 \to \Omega^2 \oplus (\Omega^0 \otimes W)$. The map must be symmetric in the two factors of Σ_2; since Ω^2 is neutral under $\mathrm{Sp}(2)$ R-symmetry, the only possible contractions of $(R_2)^{\otimes 2}$ land in the trivial representation or in W, and both are antisymmetric. So the pairing on $(\Pi S_+)^{\otimes 2}$ must also be antisymmetric, showing that

$$\mu_{0,\chi}(\alpha, \beta, \cdot) = \mathcal{L}_{[\alpha,\beta]} - i_{[\alpha,\beta]}\mathrm{d}_- = \mathrm{d}i_{[\alpha,\beta]} + i_{[\alpha,\beta]}\mathrm{d}_+. \tag{48}$$

In degree one, there is also a unique equivariant map that can contribute: it is not difficult to show that

$$\mu_{1,\chi}(\alpha, \beta, \cdot) = \mathcal{L}_{[\alpha,\beta]} - i_{[\alpha,\beta]}d. \tag{49}$$

Since $[\alpha, \beta]$ is a constant vector field, the Lie derivative preserves the self-duality condition; from this, it follows via Cartan's formula that the anti-self-dual part of $i_{[\alpha,\beta]}d$ is equal to $d_- i_{[\alpha,\beta]}$, so that

$$\mu_{1,\chi}(\alpha, \beta, \cdot) = d_+ i_{[\alpha,\beta]}. \tag{50}$$

Similar arguments apply for the map μ_0 acting on the scalar field. One can check that

$$\mu_{0,\Phi}(\alpha, \beta, \cdot) = \mathcal{L}_{[\alpha,\beta]} - i_{[\alpha,\beta]}\mathrm{d} = 0, \tag{51}$$

after applying Cartan's magic formula. One can show by identical argument that $\mu_{1,\Phi}$ also vanishes.

We must now concern ourselves with off-diagonal terms acting on the even-parity portion of the theory; these degree-zero maps can go from the scalar to the tensor or vice versa, in degree zero or one. There are now two possible contractions of the supersymmetry generators that can contribute. Let us write, as above, $[\alpha, \beta] \in V \otimes \mathbb{C}$ for the bracket of two elements of Σ_2, and $\alpha * \beta \in V \otimes W$ for the other possible symmetric contraction obtained as a tensor product of two antisymmetric equivariant maps on S_+ and R_2 respectively.

In degree one, no such off-diagonal terms appear; there is no suitable equivariant map from $\Omega^6 \otimes W$ to $\Omega^3_+ \otimes \mathbb{C}$ with no differential operator. In the other direction, such a map exists, but is proportional to the square of the de Rham operator, and hence vanishes.

In degree zero, the map from $\Omega^0 \otimes W$ to $\Omega^2 \otimes \mathbb{C}$ must come from a contraction of the supersymmetry generators with the de Rham differential acting on the scalar. There is precisely one such map, which takes the form

$$\mu_{0,\Phi,\chi}(\alpha, \beta, \cdot) = (\alpha * \beta) \wedge d. \tag{52}$$

In the other direction, we would need to find a contraction of the supersymmetry generators with d_- acting on the two-form. But the only possible such equivariant map is antisymmetric, and thus forbidden. $\square$

$$
\begin{tikzcd}
\Omega^0 \otimes \mathbb{C} \arrow[r] \arrow[loop above, "i_{[\alpha,\beta]}d"] & \Omega^1 \otimes \mathbb{C} \arrow[r] \arrow[loop above, "i_{[\alpha,\beta]}d + d i_{[\alpha,\beta]}"] & \Omega^2 \otimes \mathbb{C} \arrow[r] \arrow[loop above, "i_{[\alpha,\beta]}d_+ + d i_{[\alpha,\beta]}"] \arrow[d, shift left, "\rho_\chi"] & \Omega^3_+ \otimes \mathbb{C} \arrow[loop above, "d_+ i_{[\alpha,\beta]}"] \arrow[d, shift right, "\rho^+_{\Psi,2}"'] \\
 & & S_- \otimes R_2 \arrow[r] \arrow[u, dashed, shift left, "\rho_{\Psi,2}"] \arrow[d, dashed, shift right, "\rho_{\Psi,0}"'] & S_+ \otimes R_2 \arrow[u, dashed, shift right, "\rho^+_\chi"'] \arrow[d, shift left, "\rho^+_\Phi"] \\
 & & \Omega^0 \otimes W \arrow[r, Rightarrow] \arrow[u, shift right, "\rho_\Phi"'] \arrow[uu, bend left, "(\alpha * \beta)\wedge d"] & \Omega^6 \otimes W \arrow[u, dashed, shift left, "\rho^+_{\Psi,0}"]
\end{tikzcd}
$$

FIGURE 1. The failure of $\rho^{(1)}$ to be a Lie map on the bosonic fields.

IAS: do fermion

3.3.3. *The L_∞ correction terms.* There are two obvious correction terms for the maps above. The first is of a standard form, and is simply $i_{[\alpha,\beta]}$, viewed as an endomorphism of degree -1 on $\chi(2)$. The other is the map $(\alpha * \beta) \wedge \cdot$, viewed as a map from $\Omega^0 \otimes W$ to $\Omega^1[1] \otimes \mathbb{C}$.

IAS: do fermion; where is other term with differential operator??

3.4. **Reduction of supersymmetry.** Of course, any $\mathcal{N} = (2,0)$ theory can be regarded as a theory with $\mathcal{N} = (1,0)$ supersymmetry by restriction to a fixed embedding

$$\mathfrak{p}_{(1,0)} \hookrightarrow \mathfrak{p}_{(2,0)}. \tag{53}$$

of a subalgebra. Such an embedding is equivalent to fixing a symplectic subspace $\mathbb{C}^2 \subset \mathbb{C}^4$. The R-symmetry group of $\mathfrak{p}_{(1,0)}$ is $G_R = \mathrm{Sp}(1)$, which can be identified with $SU(2)$; this leads to the further identification of R_1 with the fundamental representation of $SU(2)$ (or, equivalently, the spin representation of $\mathrm{Spin}(3)$). However, the choice of a symplectic subspace admits a larger stabilizer, namely $\mathrm{Sp}(1) \times \mathrm{Sp}(1)'$; one of these factors (the unprimed) plays the role of the $\mathcal{N} = (1,0)$ R-symmetry, and the primed factor becomes the flavor symmetry of the hypermultiplet, as will be explained below. We will similarly refer to representations of these groups as R_1 or R_1' respectively.

Such a decomposition gives rise to the following relationship between the $(1,0)$ and $(2,0)$ tensor multiplets:

Proposition 3.7. *With respect to a chosen $(1,0)$ subalgebra as above, the abelian $\mathcal{N} = (2,0)$ tensor multiplet decomposes as*

$$\mathcal{T}_{(2,0)} \to \mathcal{T}_{(1,0)} \oplus \mathcal{T}^{hyp}_{(1,0)}(R_1'). \tag{54}$$

Proof. At the level of the fields with R-symmetry action, this is a straightforward consequence of the representation-theoretic branching rules

$$R_2 \to R_1 \oplus R_1', \qquad W \to (R_1 \otimes R_1') \oplus \mathbb{C} \tag{55}$$

from $\mathrm{Sp}(2)$ to $\mathrm{Sp}(1) \times \mathrm{Sp}(1)'$. The compatibility of the action of the supersymmetry algebra follows from the action as constructed in the previous section, together with the action on the hypermultiplet. IAS: some more □

3.5. **Twisting supercharges.**

3.5.1. *The minimal scalar supercharge.*

Lemma 3.8. *Rank-one supercharges close off-shell.*

Proof. Examine $\delta^2 B$. Can examine R-symmetry independently. Both R-symmetry contractions are in the antisymmetric square of the R_2 representation, so vanish for rank-one elements. □

3.5.2. *The non-minimal scalar supercharge.*

4. The minimal twists

In this section we will compute the holomorphic twist of the abelian $\mathcal{N} = (1,0)$ and $(2,0)$ tensor multiplets, using the formulation and supersymmetry action developed in the preceding sections. We will begin by placing the theory on a Kähler manifold and decomposing the fields with respect to the Kähler structure; at the level of representation theory, this corresponds to recalling the branching rules from $\mathrm{SO}(6)$ to $\mathrm{U}(3)$ (more precisely, at the level of the double covers $\mathrm{MU}(3) \hookrightarrow \mathrm{Spin}(6)$), followed by a regrading. We will then deform the differential by a compatible holomorphic supercharge and discard acyclic portions of the resulting theory to obtain a description of the holomorphic twist as a presymplectic BV theory Our main result of this section is the following.

Theorem 4.1. *Let Q be a rank one supercharge in either of the supersymmetry algebras $\mathfrak{p}_{(1,0)}$ or $\mathfrak{p}_{(2,0)}$. In each case, one has the following description of the twist of the abelian tensor multiplet with $(1,0)$ and $(2,0)$ supersymmetry on a three-dimensional complex manifold X:*

- **(1,0)** *The holomorphic twist $\mathcal{T}^Q_{(1,0)}$ is equivalent to the $\mathbb{Z}$-graded presymplectic BV theory defined by the chiral 2-form:*

$$\mathcal{T}^Q_{(1,0)} \;\simeq\; \chi(2)$$

- **(2,0)** *The holomorphic twist $\mathcal{T}^Q_{(2,0)}$ is equivalent to the $\mathbb{Z} \times \mathbb{Z}/2$-graded presymplectic BV theory defined by the chiral 2-form plus abelian holomorphic CS with values in the odd symplectic vector space $\Pi R_1'$:*

$$\mathcal{T}^Q_{(2,0)} \;\simeq\; \chi(2) \oplus \Phi_{\mathrm{hCS}}(\Pi R_1'). \tag{56}$$

Moreover, this equivalence is $\mathrm{Sp}(1)'$-equivariant.

4.1. **Supersymmetric twisting.** In this section we briefly recall the procedure of twisting a supersymmetric field theory. For a more complete discussion see [13], [16], though we modify the construction very slightly (see [13, Remark 2.19]). As we've already mentioned, the key piece of data is that of a square-zero supercharge Q. Roughly, the twisted theory is given by deforming the classical BV operator Q_{BV} by Q.

In the cited references, the twisting procedure is performed starting with the data of a supersymmetric theory in the BV formalism. This means that one starts with the data of a classical theory in the BV formalism together with an (L_∞) action by the super Lie algebra of supertranslations.

In our context, there are two caveats to the construction of the twist. First, we do not have the data of a classical BV theory, but just of a presymplectic BV theory. Second, we have seen that the action by the six-dimensional supersymmetry algebras only hold upon imposing the classical equations of motion on the $(1,0)$ and $(2,0)$ tensor multiplets.

The first issue is not a serious problem. So long as the supersymmetry preserves the (-1)-shifted presymplectic structure, the twisted theory will still have the structure of a presymplectic BV theory.

The second issue is circumvented by observing that the twisted theory only depends on the piece of the action of supersymmetry generated by the twisting supercharge Q. For the minimal twists, which correspond to rank one supercharges, we have seen that this one-dimensional sub Lie algebra does act at the cochain level, without imposing the equations of motion, see Lemma 3.8.

The classical theory is a $\mathbb{Z} \times \mathbb{Z}/2$-graded theory, where the first grading is the cohomological degree and the second grading is the parity. By definition, a square zero supercharge Q is of bidegree $(0,1)$ yet the classical BV differential is of bidegree $(1,0)$. In order to make sense of the deformation $Q_{\mathrm{BV}} \rightsquigarrow Q_{\mathrm{BV}} + Q$ one could remember just the totalized $\mathbb{Z}/2$ grading, where both operators are odd.

Instead, one can use additional data to *regrade* the theory so that Q, Q_{BV} have the same homogenous degree. In addition to the action by supertranslations, a classical supersymmetric theory carries an action by the group $\mathrm{Spin}(n) \times G_R$, where G_R is the group of R-symmetries. For us, $n = 6$ and $G_R = \mathrm{Sp}(1)$ for $\mathcal{N} = (1,0)$ and $G_R = \mathrm{Sp}(2)$ for $\mathcal{N} = (2,0)$ supersymmetry.

- Given a square-zero supercharge Q, a ***regrading homomorphism*** is a homomorphism $\alpha : \mathrm{U}(1) \to G_R$ such that the weight of Q under α is $+1$.

Suppose $\mathcal{E} = (\mathcal{E}, Q_{\mathrm{BV}})$ is the cochain complex of fields of the classical theory, and for $\varphi \in \mathcal{E}$, denote by $|\varphi| = (p, q \mod 2) \in \mathbb{Z} \times \mathbb{Z}/2$ the bigrading. Given a regrading homomorphism α, we define a new $\mathbb{Z} \times \mathbb{Z}/2$-graded cochain complex of fields $\widetilde{\mathcal{E}}^\alpha = (\widetilde{\mathcal{E}}^\alpha, Q_{\mathrm{BV}})$ which agrees with $(\mathcal{E}, Q_{\mathrm{BV}})$ as a totalized $\mathbb{Z}/2$-graded cochain complex with new bigrading

$$|\varphi|_\alpha = |\varphi| + (\alpha(\varphi), \alpha(\varphi) \mod 2) \in \mathbb{Z} \times \mathbb{Z}/2$$

where $\alpha(\varphi)$ denotes the weight of the field $\varphi \in \mathcal{E}$ under α. Note that Q_{BV} and Q are both of bidegree $(1,0)$ as operators acting on the regraded fields $\widetilde{\mathcal{E}}^\alpha$. Our convention is that $\widetilde{\mathcal{E}}^\alpha$ denotes the cochain complex of fields that are regraded, but equipped with the original BV differential Q_{BV}. The shifted (pre) symplectic structure remains unchanged.

There is one last step before performing the deformation of the classical differential by the supercharge Q in the regraded theory. In general, the symmetry group $\mathrm{Spin}(n) \times G_R$ will no longer act on the deformed theory since Q is generally not invariant under this group action.

- Let Q be a square-zero supercharge, and suppose $\iota : G \to \mathrm{Spin}(n)$ is a group homomorphism. A ***twisting homomorphism*** (relative to ι) is a homomorphism $\phi : G \to G_R$ such that Q is preserved under the product $\iota \times \phi : G \to \mathrm{Spin}(n) \times G_R$.

Given such a ϕ, we can restrict the regraded theory to a representation for the group G, which we will denote by $\phi^* \widetilde{\mathcal{E}}^\alpha$. We will refer to as the G-regraded theory. For simplicity, when α and ϕ are understood, we will denote this theory by $\widetilde{\mathcal{E}}$.

Given a square-zero supercharge Q, a regrading homomorphism α, and twisting homomorphism ϕ we can finally define a twist of a supersymmetric theory $\mathcal{E}$. It is the $\mathbb{Z} \times \mathbb{Z}/2$-graded theory whose underlying cochain complex of fields is

$$\mathcal{E}^Q = \left(\phi^* \widetilde{\mathcal{E}}^\alpha, Q_{\mathrm{BV}} + Q\right).$$

4.2. **Holomorphic decomposition.** Throughout the rest of this section we fix the data of a rank-one supercharge $Q \in \Sigma_1$ (which is automatically square-zero in $\mathfrak{p}_{(1,0)}$), and characterize the resulting twist of the $(1,0)$ tensor multiplet $\mathcal{T}_{(1,0)}$. As discussed in §3.1.1, such a Q defines a theory with three invariant directions, so we will refer to the twist as holomorphic. In addition to Q, to perform the twist we must prescribe a compatible pair of a twisting homomorphism ϕ and regrading homomorphism α.

Geometrically, the supercharge Q defines a complex structure $L = \mathbb{C}^3 \subset V = \mathbb{C}^6$ equipped with the choice of a holomorphic half-density on L.

Under the subgroup $\mathrm{MU}(3) \subset \mathrm{Spin}(6)$, the spin representations decompose as

$$S_+ = \det(L)^{\frac{1}{2}} \oplus L \otimes \det(L)^{-\frac{1}{2}} \quad , \quad S_- = \det(L)^{-\frac{1}{2}} \oplus L^* \otimes \det(L)^{\frac{1}{2}}. \tag{57}$$

‘ In particular, the odd part $\Sigma_1 = S_+ \otimes R_1$ of the super Lie algebra $\mathfrak{p}_{(1,0)}$ decomposes under the MU(3) as

$$\det(L)^{\frac{1}{2}} \otimes R_1 \oplus L \otimes \det(L)^{-\frac{1}{2}} \otimes R_1.$$

The holomorphic supercharge Q lies in the first factor.

Fix an embedding $\mathrm{U}(1) \subset G_R = \mathrm{Sp}(1)$ under which Q has weight $+1$. The twisting homomorphism is defined by the composition

$$\phi : \mathrm{MU}(3) \xrightarrow{\det^{\frac{1}{2}}} \mathrm{U}(1) \hookrightarrow \mathrm{Sp}(1).$$

Under this twisting homomorphism, the defining representation R_1 of $\mathrm{Sp}(1)$ splits as

$$R_1 = \det(L)^{-\frac{1}{2}} \oplus \det(L)^{\frac{1}{2}}. \tag{58}$$

Additionally, we fix the regrading homomorphism to be the inclusion

$$\alpha : \mathrm{U}(1) \hookrightarrow \mathrm{Sp}(1).$$

As outlined in §4.1, the data of ϕ and α allow us to consider the $G = \mathrm{MU}(3)$-regraded theory.

We observe that the odd part Σ_1 of $\mathfrak{p}_{(1,0)}$ decomposes under this twisting data as

$$\begin{array}{c|ccc} & -1 & 0 & 1 \\ \hline -2 & & & L \otimes \det(L)^{-1} \\ 0 & & & \mathbb{C} \cdot Q \\ 1 & L & & \\ 3 & \det(L) & & \end{array} \tag{59}$$

Here, the horizontal grading is by the ghost $\mathbb{Z}$-degree determined by α and the vertical grading is by spin $\mathrm{U}(1) \subset \mathrm{MU}(3)$. Note that Q lives in a scalar summand of ghost degree $+1$.

With respect to the twisting data, the $(1,0)$ theory has the following description.

Proposition 4.2. *The* $\mathrm{MU}(3)$*-regraded* $(1,0)$ *tensor multiplet* $\phi^* \widetilde{\mathcal{T}}^{\alpha}_{(1,0)}$ *decomposes as*

$$\phi^* \widetilde{\mathcal{T}}^{\alpha}_{(1,0)} = \chi_+(2) \oplus \widetilde{\Psi}_- \oplus \Phi(0, \mathbb{C}),$$

see Figure 2.

Under the regrading of the component $\Psi_-(R_1) \rightsquigarrow \widetilde{\Psi}_-$ we will use the following notation for the decomposition of the fields:

$$\Pi(\Omega^0 \otimes S_- \otimes R_1) \ni \psi_- \to \psi_-^{0,0} + \psi_-^{2,0} + \psi_-^{0,3} + \psi_-^{2,3} \in \Omega^{0,0}[1] \oplus \Omega^{2,0}[1] \oplus \Omega^{0,3}[-1] \oplus \Omega^{2,3}[-1].$$

And for the antifields:

$$\Pi(\Omega^0 \otimes S_+ \otimes R_1)[-1] \ni \psi_-^+ \to \psi_-^{+1,0} + \psi_-^{+3,0} + \psi_-^{+1,3} + \psi_-^{+3,3} \in \Omega^{1,0} \oplus \Omega^{3,0} \oplus \Omega^{1,3}[-2] \oplus \Omega^{3,3}[-2].$$

Proof. The components $\chi_+(2)$ and $\Phi(0, \mathbb{C})$ of $\mathcal{T}_{(1,0)}$ are acted on trivially by the R-symmetry group $G_R = \mathrm{Sp}(1)$, so we only need to focus on how $\Psi_-(R_1)$ is regraded. According to Equations (57) and (63), the physical fields decompose under the twisting homomorphism ϕ by:

$$\Pi\left(\Omega^0 \otimes S_- \otimes R_1\right) = \Pi(\Omega^{0,0} \oplus \Omega^{2,0}) \oplus \Pi(\Omega^{0,3} \oplus \Omega^{2,3}). \tag{60}$$

Similarly, the antifields decompose as

$$\Pi\left(\Omega^0 \otimes S_+ \otimes R_1\right)[-1] = \Pi(\Omega^{3,3} \oplus \Omega^{1,3})[-1] \oplus \Pi(\Omega^{3,0} \oplus \Omega^{1,0})[-1] \tag{61}$$

$$
\begin{tikzcd}[row sep=small, column sep=small]
\phi^*\widetilde{\mathfrak{T}}^{\alpha}_{(1,0)} & -2 & -1 & 0 & 1 & 2
\end{tikzcd}
$$

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & & & \Omega^{0,2} \arrow[dr, "\partial_\omega"] & & \\
 & & \Omega^{0,1} \arrow[ur] \arrow[dr] & & \Omega^{1,2}/\omega\Omega^{0,1} & \\
\chi_+(2)\ : & \Omega^{0} \arrow[ur] \arrow[dr] & & \Omega^{1,1} \arrow[ur, "\overline{\partial}_\omega"] \arrow[dr, "\partial^\omega"] & & \\
 & & \Omega^{1,0} \arrow[ur] \arrow[dr] & & \omega\Omega^{1,0} & \\
 & & & \Omega^{2,0} \arrow[ur, "\overline{\partial}^\omega"] \arrow[dr] & & \\
 & & & & \Omega^{3,0} &
\end{tikzcd}
$$

$$
\begin{tikzcd}[row sep=small, column sep=small]
\widetilde{\Psi}\ : & & \Omega^{0,0} \arrow[r] & \Omega^{1,0} & \Omega^{0,3} \arrow[r] & \Omega^{1,3} \\
 & & \Omega^{2,0} \arrow[r] \arrow[ur, "\partial^*"] & \Omega^{3,0} & \Omega^{2,3} \arrow[r] \arrow[ur, "\partial^*"] & \Omega^{3,3}
\end{tikzcd}
$$

$$
\begin{tikzcd}[row sep=small, column sep=small]
\Phi(0,\mathbb{C})\ : & & & \Omega^{0,0} \arrow[r, "\triangle"] & \Omega^{3,3} &
\end{tikzcd}
$$

FIGURE 2. The regraded $\mathcal{N} = (1,0)$ tensor multiplet. The unlabeled arrows denote the obvious ∂ or $\overline{\partial}$ operators.

The next step is to regrade the fields according to the homomorphism $\alpha : \mathrm{U}(1) \hookrightarrow \mathrm{Sp}(1) = G_R$. At the level of the decomposed fields in Equation (60), this $\mathrm{U}(1)$ acts by weight -1 on the first summand $\Omega^{0,0} \oplus \Omega^{2,0}$, and by weight $+1$ on the second summand $\Omega^{0,3} \oplus \Omega^{2,3}$. Thus, we see that the regraded fields of Equation (60) become

$$(\Omega^{0,0} \oplus \Omega^{2,0})[1] \oplus (\Omega^{0,3} \oplus \Omega^{2,3})[-1].$$

Similarly, the regraded anti-fields of Equation (61) become

$$(\Omega^{3,3} \oplus \Omega^{1,3})[-2] \oplus (\Omega^{3,0} \oplus \Omega^{1,0}).$$

It remains to identify the linear BV operator Q_{BV} in the regraded theory. This follows from the well-known decomposition of the Dirac operator, on a Kähler manifold:

$$
\begin{tikzcd}
S_- \otimes R_1 \arrow[rr, "\not{\partial}"] \arrow[d, equal] & & S_+ \otimes R_1 \arrow[d, equal] \\
\Omega^{0,0} \otimes K^{-\frac{1}{2}} \otimes R_1 \arrow[rr, "\partial"] & & \Omega^{1,0} \otimes K^{-\frac{1}{2}} \otimes R_1 \\
\Omega^{2,0} \otimes K^{-\frac{1}{2}} \otimes R_1 \arrow[rr, "\partial"] \arrow[urr, "\partial^*"] & & \Omega^{3,0} \otimes K^{-\frac{1}{2}} \otimes R_1 .
\end{tikzcd} \tag{62}
$$

$\square$

	$\mathcal{T}_{(1,0)}$	$\phi^*\widetilde{\mathcal{T}}^{\alpha}_{(1,0)}$
$\chi_+(2)$	$\beta \in \Omega^2$	$\beta \in \Omega^2 = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$
$\Psi_-(R_1)$	$\psi_- \in \Pi(S_- \otimes R_1)$	$\psi_- \in (\Omega^0 \oplus \Omega^{2,0})[1] \oplus (\Omega^{0,3} \oplus \Omega^{2,3})[-1]$
$\Phi(0,\mathbb{C})$	$\phi \in \Omega^0 \otimes W$	$\phi \in \Omega^0 \otimes W$

Table 1. The physical fields in the regraded $(1,0)$ theory.

We remark that while the homomorphism ϕ a priori endowed the regraded theory with a symmetry by the group $\mathrm{MU}(3)$, the description we gave above descends to theory with symmetry $\mathrm{U}(3)$.

In Table 1 we have summarized what happens to the physical fields (cohomological degree zero in the original theory) of the $(1,0)$ tensor multiplet in the regraded theory.

While the components $\chi_+(2), \Phi(0,\mathbb{C})$ were unaffected by both the twisting homomorphism ϕ and twisting data α, it will be useful to decompose these cochain complexes as $\mathrm{U}(3)$-representations. In its entirety, the regraded $(1,0)$ theory, together with its linear BV differential is summarized in Figure 2. Note that the unlabeled arrows in this figure represent the obvious ∂ and $\overline{\partial}$ operators. The operators ∂_ω, $\overline{\partial}_\omega$ denote the composition of the $\partial, \overline{\partial}$ operators, respectively, with the projection $\Omega^{1,2} \to \Omega^{1,2}/\omega\Omega^{0,1}$. The operators ∂^ω, $\overline{\partial}^\omega$ denote the orthogonal projection of the $\partial, \overline{\partial}$ operators, respectively, to the subspace $\omega\Omega^{1,0} \subset \Omega^{2,1}$. Finally, the ∂^* denotes the adjoint of ∂ with respect to the Kähler form ω.

4.3. **Proof of (1,0) part of Theorem 4.1.** With the holomorphic decomposition of the $(1,0)$ tensor multiplet in hand, we proceed to compute the twist by the rank one supercharge Q. Throughout the proof we refer to Figure 3, which uses the decomposition of the fields we found in the previous section. The black text denotes the fields in the component $\chi_+(2)$ of the tensor multiplet. The red text denotes the fields in the $\widetilde{\Psi}$ component, as in Proposition 4.2. Finally, the green text denotes the fields in the $\Phi(0,\mathbb{C})$ component. Each of the solid lines denotes the linear BV differential in the original, untwisted theory, see Figure 2.

We have labeled the differential generated by the supercharge Q by the dotted and dashed arrows, which we now proceed to justify. The dotted arrows $\cdots\cdots\rightarrow$ denote isomorphisms, and the dashed arrows $\dashrightarrow$ are given by the labeled differential operator. Throughout, we extensively refer to the notation in §3.3 where we constructed the action of supersymmetry on the tensor multiplet.

We begin with the component of the supersymmetry action which transforms a fermion into a scalar. In the notation of §3.3.1 this is the linear map $\rho_{\Psi,0}$. In the holomorphic decomposition of the fields we read off

$$\rho_{\Psi,0}(Q \otimes \psi_-) = \psi_-^{0,0} \in \Omega^{0,0} \subset \Phi(0,\mathbb{C}).$$

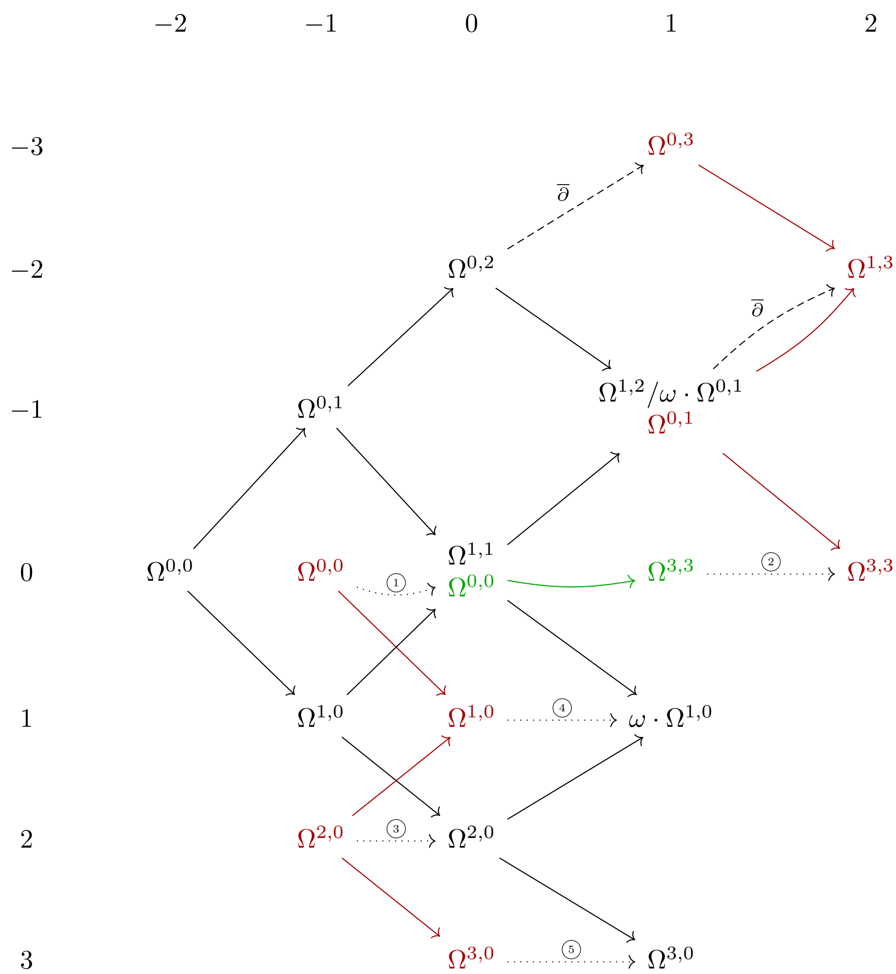

FIGURE 3. The holomorphic twist of the $\mathcal{N} = (1,0)$ tensor multiplet. The horizontal grading is the cohomological grading. The vertical grading is the weight with respect to $U(1) \subset U(3)$.

In standard physics notation, one would write this as $\delta_Q \phi = \psi_-^{0,0}$. This term accounts for the dotted arrow $\dashrightarrow$ in Figure 3 labeled 1. Similarly, on the anti-fields we have

$$\rho_{\Psi,0}(Q \otimes \phi^+) = \phi^+ \in \Omega^{3,3} \subset \widetilde{\Psi}_-[2]$$

which one could write as $\delta_Q \psi_-^+ = \phi^+$. This term accounts for the dotted arrow labeled 2.

We turn to the part of the supersymmetry which transforms $\widetilde{\Psi}_-$ into $\chi_+(2)$. . In the notation of §3.3.1 this is the linear map $\rho_{\Psi,2}$. In the holomorphic decomposition of the fields we read off

$$\rho_{\Psi,2}(Q \otimes \psi_-) = \psi_-^{2,0} \in \Omega^{2,0} \subset \chi_+(2).$$

This term accounts for the dotted arrow in Figure 3 labeled 3. On the anti-fields, the map $\rho_{\Psi,2}(Q\otimes -)$ is given by the composition

$$\rho_{\Psi,2}(Q\otimes -):\Omega^3_+ \twoheadrightarrow \Omega^{1,2}/\omega\Omega^{0,1} \xrightarrow{\overline{\partial}} \Omega^{1,3} \subset \widetilde{\Psi}_-[2]$$

This accounts for the dashed arrow $\Omega^{1,2}/\omega\Omega^{0,1} \dashrightarrow^{\overline{\partial}} \Omega^{1,3}$.

Next consider the supersymmetry which transforms $\chi_+(2)$ into $\widetilde{\Psi}_-$. In the notation of §3.3.1 this is the linear map ρ_χ. Acting on the physical fields, the nontrivial component is the composition

$$\rho_\chi(Q\otimes -):\Omega^2 \twoheadrightarrow \Omega^{0,2} \xrightarrow{\overline{\partial}} \Omega^{0,3} \subset \widetilde{\Psi}_-[1].$$

This accounts for the dashed arrow $\Omega^{0,2} \dashrightarrow^{\overline{\partial}} \Omega^{0,3}$. The anti map for this component of supersymmetry acts on $\Omega^{1,0}\oplus\Omega^{3,0}\subset\widetilde{\Psi}_-$ and reads

$$\rho_\chi(Q\otimes -):\Omega^{1,0}\oplus\Omega^{3,0} \xrightarrow{\cong} \omega\Omega^{1,0}\oplus\Omega^{3,0}\subset\chi_+(2)[1]$$

This accounts for the dotted arrows labeled 4 and 5.

To complete the proof, first notice that in position $(1,-1)$ we have applied the isomorphism $\omega^{-2}:\Omega^{2,3}\cong\Omega^{0,1}$ (for the text in red) given by the double contraction with the Kähler form. Following this, there is an obvious projection map from the total complex in Figure 3 to $\chi(2)$. It is immediate to check that this map preserves the degree (-1) presymplectic structures. Moreover, since all the dotted arrows are isomorphisms, the kernel of this map is acyclic, thus completing the proof.

4.4. **Holomorphic decomposition for the (2,0) theory.** In this section we finish the second part of Theorem 4.1 concerning the holomorphic twist of the $(2,0)$ tensor multiplet. Again, we fix the data of a rank one supercharge Q, this time viewed as an odd element of the super Lie algebra $\mathfrak{p}_{(2,0)}$.

Recall that the R-symmetry group of $(2,0)$ supersymmetry is $G_R=\mathrm{Sp}(2)$. As in the $(1,0)$ case, the supercharge Q defines a complex structure $L=\mathbb{C}^3\subset V=\mathbb{C}^6$ equipped with the choice of a holomorphic half-density on L. The twist carries a symmetry by the subgroup group $\mathrm{MU}(3)\subset\mathrm{Spin}(6)$ whose action is defined by the twisting homomorphism

$$\phi:\mathrm{MU}(3)\xrightarrow{\det^{\frac12}}\mathrm{U}(1)\xrightarrow{i\times 1}\mathrm{Sp}(1)\times\mathrm{Sp}(1)'\subset\mathrm{Sp}(2)=G_R.$$

Here, $i:\mathrm{U}(1)\hookrightarrow\mathrm{Sp}(1)$ denotes the embedding for which Q has weight $+1$. Also we use primes as in $\mathrm{Sp}(1)\times\mathrm{Sp}(1)'\subset\mathrm{Sp}(2)$ to differentiate between the two abstractly isomorphic groups.

Under the twisting homomorphism ϕ the defining representation R_2 of $\mathrm{Sp}(2)$ decomposes as

$$R_2=\det(L)^{-\frac12}\oplus\det(L)^{\frac12}\oplus R_1' \tag{63}$$

where $\mathrm{MU}(3)$ acts trivially on R_1'. The vector representation W of $\mathrm{Sp}(2) = \mathrm{Spin}(5)$ decomposes under ϕ as

$$W = \mathbb{C} \oplus \left(\det(L)^{-\frac{1}{2}} \oplus \det(L)^{\frac{1}{2}}\right) \otimes R_1'.$$

The regrading datum is specified by the homomorphism

$$\alpha : \mathrm{U}(1) \hookrightarrow \mathrm{Sp}(1) \xrightarrow{i \times 1} \mathrm{Sp}(1) \times \mathrm{Sp}(1)' \subset \mathrm{Sp}(2) = G_R.$$

Note that this factors through the twisting homomorphism we used in the $(1,0)$ case along the embedding $\mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(2)$.

In addition to $\mathrm{MU}(3)$, the twist enjoys a global symmetry by the group $\mathrm{Sp}(1)'$. Moreover, these actions commute for the trivial reason that $\mathrm{MU}(3)$ acts trivially on $\mathrm{Sp}(1)'$. Using Equation (57), we observe that, after applying the twisting homomorphism ϕ, the odd part Σ_2 of the super Lie algebra $\mathfrak{p}_{(2,0)}$ transforms under $\mathrm{MU}(3) \times \mathrm{Sp}(1)' \subset \mathrm{Spin}(6) \times \mathrm{Sp}(2)$ as:

(64)

	-1	0	1
3	$\det(L)$		
$5/2$			
2			
$3/2$		$\det(L)^{\frac{1}{2}} \otimes \Pi R_1'$	
1	L		
$1/2$			
0			$\mathbb{C} \cdot Q$
$-1/2$		$L \otimes \det(L)^{-\frac{1}{2}} \otimes \Pi R_1'$	
-1			
$-3/2$			
-2			$L \otimes \det(L)^{-1}$

In this table, the vertical grading organizes spin number, and the horizontal grading is by ghost $\mathbb{Z}$-degree. The terms involving R_1' are all odd with respect to the new $\mathbb{Z}/2$-grading.

The holomorphic supercharge Q lies in the red summand. Its only nonzero bracket occurs with the supercharges in L represented in green above, using the degree-zero pairing on the R-symmetry space. As remarked above, this bracket witnesses a nullhomotopy of the translations in L with respect to the holomorphic supercharge.

$$
\begin{tikzcd}[column sep=small]
\widetilde{\mathcal{T}}^{\mathrm{hyp}}(R_1') & -1 & 0 & 1 & 2 \\
\hline
 & & \Omega^{0,3}(K^{\frac{1}{2}} \otimes R_1') \arrow[r, "\overline{\partial}^*"] & \Omega^{0,2}(K^{\frac{1}{2}} \otimes R_1') & \\
 & & \Omega^{0,1}(K^{\frac{1}{2}} \otimes R_1') \arrow[r, "\overline{\partial}^*"] \arrow[ur, "\overline{\partial}"] & \Omega^{0}(K^{\frac{1}{2}} \otimes R_1') & \\
\hline
 & \Omega^{0}(K^{\frac{1}{2}} \otimes R_1') \arrow[r, "\triangle"] & \Omega^{0}(K^{\frac{1}{2}} \otimes R_1') & & \\
 & & & \Omega^{0,3}(K^{\frac{1}{2}} \otimes R_1') \arrow[r, "\triangle"] & \Omega^{0,3}(K^{\frac{1}{2}} \otimes R_1')
\end{tikzcd}
$$

FIGURE 4. The subcomplex $\widetilde{\mathcal{T}}^{\mathrm{hyp}}(R_1')$ of the regraded $(2,0)$ tensor multiplet, see Proposition 4.3. The top complex is the result of regrading the fermions in the $(1,0)$ hypermultiplet, and the bottom complex is the result of regrading the bosons in the $(1,0)$ hypermultiplet.

In Proposition 3.7 we described the $\mathrm{Sp}(1) \times \mathrm{Sp}(1)'$ decomposition of the $(2,0)$ tensor multiplet as a sum of the $(1,0)$ tensor multiplet plus the $(1,0)$ hypermultiplet valued in the symplectic representation R_1':

$$\mathcal{T}_{(2,0)} = \mathcal{T}_{(1,0)} \oplus \mathcal{T}^{\mathrm{hyp}}_{(1,0)}(R_1')$$

Analogously, accounting for the twisting data ϕ, α just introduced we have the following description of the regraded $(2,0)$ tensor multiplet.

Proposition 4.3. *As a $\mathbb{Z} \times \mathbb{Z}/2$-graded theory, the regraded $(2,0)$ tensor multiplet $\phi^*\widetilde{\mathcal{T}}^{\alpha}_{(2,0)}$ as*

$$\phi^*\widetilde{\mathcal{T}}^{\alpha}_{(2,0)} = \phi^*\widetilde{\mathcal{T}}^{\alpha}_{(1,0)} \oplus \Pi\widetilde{\mathcal{T}}^{\mathrm{hyp}}(R_1')$$

where $\phi^*\widetilde{\mathcal{T}}^{\alpha}_{(1,0)}$ is the regraded $(1,0)$ tensor multiplet as in Proposition 4.2 and $\widetilde{\mathcal{T}}^{\mathrm{hyp}}(R_1')$ is the free BV theory whose complex of fields is displayed in Figure 4.

In Figure 4, the operator $\overline{\partial}^*$ denotes the adjoint of $\overline{\partial}$ corresponding to the standard Kähler form on $\mathbb{C}^3$. Under the regrading $\mathcal{T}^{\mathrm{hyp}}(R_1') = \Phi(0, R_1') \oplus \Psi_-(R_1') \rightsquigarrow \Pi\widetilde{\mathcal{T}}^{\mathrm{hyp}}(R_1')$, we will denote the decomposition of the fields as:

$$\Phi(0, R_1') \ni \phi' = \phi^{\frac{3}{2},0} + \phi^{\frac{3}{2},3} \in \Omega^0(K^{\frac{1}{2}} \otimes R_1')[1] \oplus \Omega^{0,3}(K^{\frac{1}{2}} \otimes R_1')[-1] \tag{65}$$

for the scalars and

$$\Psi_-(R_1') \ni \psi_-' = \psi_-^{\frac{3}{2},3} + \psi_-^{\frac{3}{2},1} \in \Omega^{0,3}(K^{\frac{3}{2}} \otimes R_1') \oplus \Omega^{0,1}(K^{\frac{1}{2}} \otimes R_1') \tag{66}$$

for the fermions. A similar decomposition holds for the anti-fields.

Proof. The $\mathcal{N} = (2,0)$ multiplet splits as a sum of three complexes

$$\mathcal{T}_{(2,0)} = \chi_+(2) \oplus \Psi_-(R_2) \oplus \Phi(0, W).$$

As in the case of the $\mathcal{N} = (1,0)$ multiplet, the component $\chi_+(2)$ is not charged under the R-symmetry group $G_R = \mathrm{Sp}(2)$.

The physical fields of $\Psi_-(R_2)$ decompose under the twisting homomorphism ϕ as:

$$\Pi\left(\Omega^0 \otimes S_- \otimes R_2\right) = \Big(\Pi(\Omega^{0,0} \oplus \Omega^{2,0}) \oplus \Pi(\Omega^{0,3} \oplus \Omega^{2,3})\Big) \oplus \Pi\Big(\Omega^0 \otimes S_- \otimes R_1'\Big). \tag{67}$$

The first component in parentheses contributes to the regraded $\mathcal{N} = (1,0)$ tensor as in Proposition 4.2. The second component

$$\Omega^0 \otimes S_- \otimes R_1' = \Omega^0(K^{-\frac{1}{2}} \otimes R_1') \oplus \Omega^{0,1}(K^{\frac{1}{2}} \otimes R_1')$$

contributes to the regraded hypermultiplet $\Pi\widetilde{\mathcal{T}}^{\mathrm{hyp}}(R_1')$. There is a similar decomposition for the anti-fields in $\Psi_-(R_2)$.

Next, the physical fields of the scalar theory $\Phi(0, W)$ decompose as

$$\Omega^0 \otimes W = \Omega^0 \oplus \Omega^0 \otimes \left(K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}\right) \otimes R_1' \tag{68}$$

The first summand, the single copy of smooth functions Ω^0, contributes to the regraded $(1,0)$ tensor multiplet. The second summand contributes to $\Pi\widetilde{\mathcal{T}}^{\mathrm{hyp}}(R_1')$. There is a similar decomposition for the anti-fields in $\Phi(0, W)$.

By Proposition 4.2, upon regrading, we see that the components $\chi_+(2)$, the first summand of 67, and the first summand of (68), combine to give the regraded $(1,0)$ tensor multiplet.

Of the remaining terms, the only component which is acted upon nontrivially by $\mathrm{Sp}(2)$ is the second summand in (68) (and the corresponding antifields). Under α, we see that the factor proportional to $K^{\frac{1}{2}}$ has weight -1 and the factor $K^{\frac{1}{2}}$ has weight $+1$. It remains to check that the BV differential decomposes as stated, but this is nearly identical to the proof of Proposition 4.2. □

4.5. **Proof of (2,0) part of Theorem 4.1.** We now complete the proof of Theorem 4.1, which involves deforming the regraded theory described in Proposition 4.3 by the holomorphic supercharge Q. Throughout this section we refer to the description of the twisted theory in Figure 5.

According to Proposition 4.3, the Q-twisted theory splits as a sum of two complexes

$$\mathcal{T}^{Q}_{(1,0)} \oplus \Pi\widetilde{\mathcal{T}}^{\mathrm{hyp}}(R_1')^Q \tag{69}$$

where $\mathcal{T}^{Q}_{(1,0)}$ is the Q-twist of the $(1,0)$ tensor multiplet and $\widetilde{\mathcal{T}}^{\mathrm{hyp}}(R_1')^Q$ is the theory obtained by deforming the regraded hypermultiplet $\widetilde{\mathcal{T}}^{\mathrm{hyp}}(R_1')$ by Q.

In Figure 5, the black solid arrows represent the twist of the $\mathcal{N} = (1,0)$ tensor multiplet, as we computed in §4.3 which corresponds to the first summand $\mathcal{T}^{Q}_{(1,0)}$ in (69). The red text refers to the regraded hypermultiplet $\widetilde{\mathcal{T}}^{\mathrm{hyp}}(R_1')$. The red solid arrows represent the underlying classical BV differential of the regraded hypermultiplet. Note that we use the shorthand notation $\Omega^{\pm\frac{3}{2},\ell}(R_1')$ to mean the Dolbeault forms of type

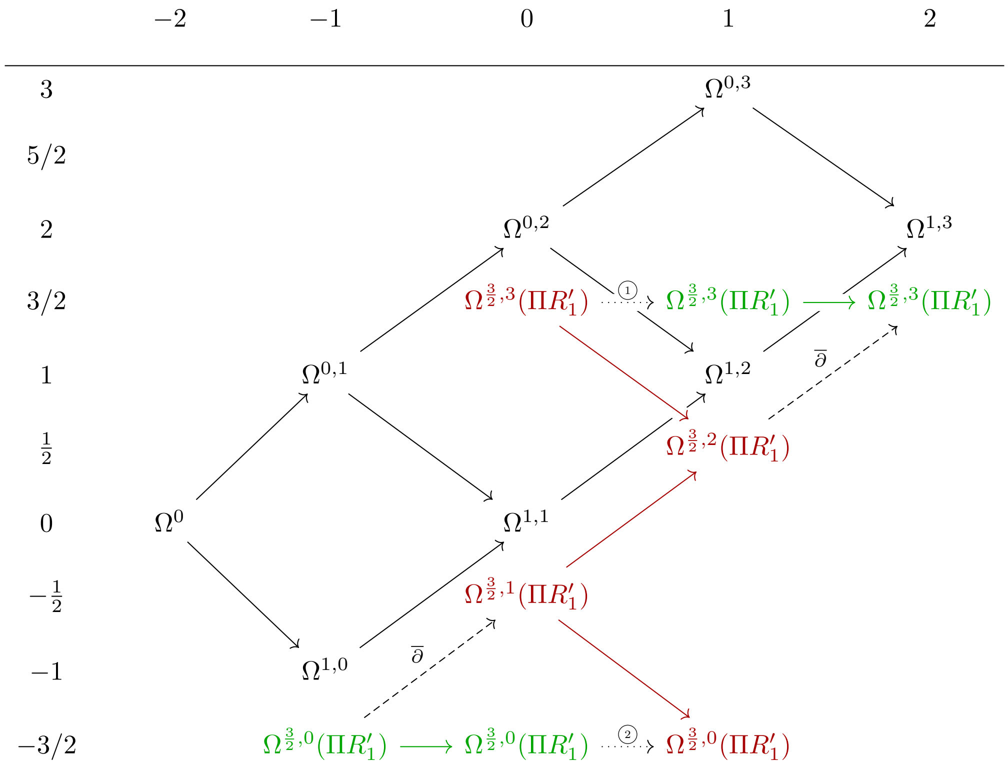

FIGURE 5. The holomorphically twisted $\mathcal{N}=(2,0)$ theory $\mathcal{T}^{Q}_{(2,0)}$. The horizontal grading is the cohomological $\mathbb{Z}$-grading. Note that the green and red text sits in *odd* $\mathbb{Z}/2$-degree. The vertical grading is the weight with respect to $\mathrm{U}(1)\subset\mathrm{MU}(3)$.

$(0,\ell)$ valued in the holomorphic vector bundle $K^{\pm\frac12}\otimes R_1'$. We have labeled the differentials generated by the holomorphic supercharge Q acting on the hypermultiplet by the dotted and dashed arrows. As in the $\mathcal{N}=(1,0)$ case, the dotted arrows $\dashrightarrow$ denote isomorphisms, and the dashed arrows $\dashrightarrow$ are given by the labeled differential operator, which we now proceed to characterize. Again, we refer to the notation in §3.3 where we constructed the action of supersymmetry on the tensor multiplet.

We begin with the component of the supersymmetry action which transforms a fermion into a scalar. In the notation of §3.3.1 this is the linear map $\rho_{\Psi,0}$. In the holomorphic decomposition, see Equation (66), of the fields we read off

$$\rho_{\Psi,0}(Q\otimes\psi_-')=\psi_-^{\frac32,3}\in\Omega^{\frac32,3}\subset\Phi(0,R_1')[1],$$

This term accounts for the dotted arrow $\dashrightarrow$ in Figure 5 labeled (1). Similarly, on the anti-fields we have

$$\rho_{\Psi,0}\left(Q\otimes\phi^{+'}\right)=\phi^{+\frac32,0}\in\Omega^{\frac32,0}\subset\Psi_-(R_1'),$$

see the notation of Equation (65). This term accounts for the dotted arrow labeled (2).

Next, we look at the component of supersymmetry which transforms a scalar into a fermion. In the notation of §3.3.1 this is the linear map ρ_Φ. In the holomorphic decomposition of fields we have

$$\rho_\Phi(Q \otimes \phi') = \overline{\partial}\phi^{\frac{3}{2},0} \in \Omega^{\frac{3}{2},1} \subset \Psi_-(R_1').$$

Similarly, on the anti-fields we have

$$\rho_\Phi(Q \otimes \psi_-^{'+}) = \overline{\partial}\psi^{+\frac{3}{2},2} \in \Omega^{\frac{3}{2},3} \subset \Phi(0, R_1').$$

These maps account for each of the dashed arrows in Figure 5.

Arguing similarly to the $(1,0)$ case, we observe that there is an obvious projection map from the total complex in Figure 5 to $\chi(2) \oplus \Phi_{\mathrm{hCS}}(\Pi R_1')$. (On $\chi(2)$ it is the same map as in the $(1,0)$ case). It is immediate to check that this map preserves the degree (-1) presymplectic structures. Moreover, since all the dotted arrows are isomorphisms, the kernel of this map is acyclic, which completes the proof.

4.5.1. *An alternative description.* There is an alternative to the twisting data (ϕ, α) in the case of the $(2,0)$ tensor multiplet. The key difference is that this variation admits a smaller global symmetry group. Note that the theory described in the previous section carries a global symmetry by the group $\mathrm{MU}(3) \times \mathrm{Sp}(1)'$, even *after* twisting. This alternative twist breaks this global $\mathrm{Sp}(1)'$ symmetry completely, but further descends the $\mathrm{MU}(3)$-action to an action by $\mathrm{U}(3)$.

The reason this twist enjoys a smaller symmetry group is because it depends on the choice of a polarization of the 2-dimensional symplectic vector space R_1'. Such a polarization determines an embedding $i' : \mathrm{U}(1) \hookrightarrow \mathrm{Sp}(1)'$ which we now fix.

Define the new twisting homomorphism by the composition

$$\widetilde{\phi} : \mathrm{MU}(3) \xrightarrow{\det^{\frac{1}{2}}} \mathrm{U}(1) \xrightarrow{\mathrm{diag}} \mathrm{U}(1) \times \mathrm{U}(1) \xrightarrow{i \times i'} \mathrm{Sp}(1) \times \mathrm{Sp}(1)' \subset \mathrm{Sp}(2).$$

As in the previous section, $i : \mathrm{U}(1) \to \mathrm{Sp}(1)$ denotes the homomorphism for which Q has weight $+1$.

Additionally, we have the regrading homomorphism

$$\widetilde{\alpha} : \mathrm{U}(1) \xrightarrow{\mathrm{diag}} \mathrm{U}(1) \times \mathrm{U}(1) \xrightarrow{i \times i'} \mathrm{Sp}(1) \times \mathrm{Sp}(1)' \subset \mathrm{Sp}(2).$$

With this choice of a regrading homomorphism, the twisted theory $\mathcal{T}^Q_{(2,0)}$ is concentrated in even $\mathbb{Z}/2$-degree and hence defines a $\mathbb{Z}$-graded theory. Aside from this, the only part of the calculation that changes is the subcomplex defined by the green and red text of Figure 5, which we will henceforth denote by $\mathcal{A} \subset \mathcal{T}^Q_{(2,0)}$.

For example, in the original description of the twist the scalar field lives in $\Pi\Omega^{\frac{3}{2},0}(R_1')[1]$. According to this new twisting data this becomes

$$\Pi\Omega^{\frac{3}{2},0}(R_1')[1] \oplus \Pi\Omega^{\frac{3}{2},3}(R_1')[-1] \rightsquigarrow \left(\Omega^{0,0} \oplus \Omega^{3,0}[2]\right) \oplus \left(\Omega^{3,0}[-2] \oplus \Omega^{3,3}[1]\right).$$

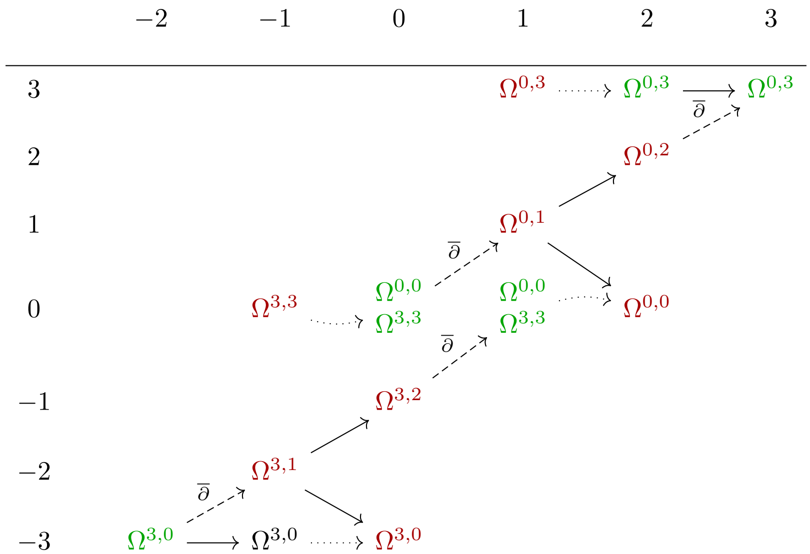

FIGURE 6. The description of the subcomplex $\mathcal{A} \subset \mathcal{T}^{Q}_{(2,0)}$ using the alternative twisting data.

Similarly, using the original twisting data, the fermion field lives in $\Pi\Omega^{32,1} \oplus \Pi\Omega^{\frac{3}{2},3}$. According to this new twisting data this becomes

$$\Pi\Omega^{\frac{3}{2},1} \oplus \Pi\Omega^{\frac{3}{2},3} \rightsquigarrow \left(\Omega^{0,1}[-1] \oplus \Omega^{3,1}[1]\right) \oplus \left(\Omega^{0,3}[-1] \oplus \Omega^{3,3}[1]\right).$$

In total, using this alternative twisting data, the green and red subcomplex of the diagram in Figure 5, which we denote $\mathcal{A}$, is displayed in Figure 4.5.1. As before the solid arrows denote the differentials in the original untwisted theory. The dotted arrows $\cdots\cdots\!\!>$ denote isomorphisms and the dashed arrows $- - - - \!\!>$ arrows are given by the labeled differential operators induced by the action by Q. The green text labels the components arising from the scalar part of the untwisted theory, the red text labels the components arising from the fermion.

We recognize that the complex of Figure 4.5.1 admits a cochain map to the $\beta\gamma$ system on $\mathbb{C}^3$. Since the dotted arrows are isomorphisms, this cochain map is a quasi-isomorphism. The following proposition follows from tracing through the presymplectic BV structures, which is completely similar to the previous calculations.

Proposition 4.4. *Using the twisting data* $(\widetilde{\phi}, \widetilde{\alpha})$, *the holomorphic twist of the* $(2,0)$ *tensor multiplet is equivalent to the* $\mathbb{Z}$*-graded presymplectic BV theory,*

$$\chi(2) \oplus \beta\gamma(\mathbb{C}).$$

Moreover, this equivalence is $\mathrm{U}(3)$*-equivariant.*

For reference later on, we provide a description of the holomorphic twist using $(\widetilde{\phi}, \widetilde{\alpha})$ in Figure **??**.

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & -2 & -1 & 0 & 1 & 2 & 3 \\
-3 & & & & \Omega^{0,3} \arrow[dr] & & \\
-2 & & & \Omega^{0,2} \arrow[ur] \arrow[dr] & \Omega^{3,3} & \Omega^{1,3} & \\
-1 & & \Omega^{0,1} \arrow[ur] \arrow[dr] & \Omega^{3,2} \arrow[ur] & \Omega^{1,2} \arrow[ur] & & \Omega^{0,3} \\
0 & \Omega^{0,0} \arrow[ur] \arrow[dr] & \Omega^{3,1} \arrow[ur] & \Omega^{1,1} \arrow[ur] & & \Omega^{0,2} \arrow[ur] & \\
1 & \Omega^{3,0} \arrow[ur] & \Omega^{1,0} \arrow[ur] & & \Omega^{0,1} \arrow[ur] & & \\
2 & & & \Omega^{0,0} \arrow[ur] & & &
\end{tikzcd}
$$

FIGURE 7. An alternative description of the holomorphic twist of the $\mathcal{N} = (2,0)$ multiplet.
BW: I'm not sure what the spins are doing here...

4.6. **The twisted factorization algebras.** In §2.4.1 we have defined a notion of Hamiltonian observables for certain classes of presympletic BV theories. For a holomorphic supercharge Q, each of the twisted presymplectic BV theories $\mathcal{T}^Q_{(1,0)}$, $\mathcal{T}^Q_{(2,0)}$ and $\widetilde{\mathcal{T}}^Q_{(2,0)}$ satisfy Condition (2) in §2.4.1. So, in each of these cases we obtain a $\mathbb{P}_0$-factorization algebra of Hamiltonian observables.

The twist of the $(1,0)$ theory $\mathcal{T}^Q_{(1,0)}$ is defined on any complex 3-fold X. We denote the corresponding factorization algebra of observables on X by $\mathrm{Obs}_{(1,0)}$, with the supercharge Q understood. We can describe this $\mathbb{P}_0$-factorization algebra explicitly as follows. Recall $\mathcal{T}^Q_{(1,0)} \simeq \chi(2)$ which, as a cochain complex, is $\Omega^{\leq 1,\bullet}[2]$ equipped with the differential $\overline{\partial} + \partial$. Keeping track of shifts, one has $\chi(2)^! = \Omega^{\geq 2,\bullet}[2]$, again equipped with the differential $\overline{\partial} + \partial$. Thus, the factorization algebra is described by

$$\mathrm{Obs}_{(1,0)} = \left(\mathcal{O}^{sm}(\Omega^{\geq 2,\bullet}[1]), \overline{\partial} + \partial\right)$$

where $\mathcal{O}^{sm}$ denotes the "smooth" functionals as defined in §2.4.1. Explicitly, to an open set $U \subset X$, the factorization algebra assigns the cochain complex

$$\mathrm{Obs}_{(1,0)}(U) = \Big(\mathrm{Sym}\left(\Omega_c^{\leq 1,\bullet}(U)[3]\right), \overline{\partial} + \partial\Big).$$

With this description in hand, the $\mathbb{P}_0$-structure is also easy to interpret. Given two linear observables $\mathcal{O}, \mathcal{O} \in \Omega_c^{\leq 1,\bullet}(U)[3]$, the $\mathbb{P}_0$-bracket is

$$\{\mathcal{O}, \mathcal{O}'\} = \int_U \mathcal{O}\partial\mathcal{O}'. \tag{70}$$

The bracket extends to non-linear observables by the graded Leibniz rule.

We will not explicitly need to mention the factorization algebra associated to the twist of the $(2,0)$ theory $\mathcal{T}^Q_{(2,0)}$. However, we will study the factorization algebra associated to its alternative twist $\widetilde{\mathcal{T}}^Q_{(2,0)}$, which we will denote by $\mathrm{Obs}_{(2,0)}$. Similarly to the $(1,0)$ case, we obtain the following explicit description of this factorization algebra. To an open set $U \subset X$, it assigns the cochain complex

$$\mathrm{Obs}_{(2,0)}(U) = \left(\mathrm{Sym}\left(\Omega_c^{\leq 1,\bullet}(U)[3] \oplus \Omega_c^{3,\bullet}(U)[3] \oplus \Omega_c^{0,\bullet}(U)[1]\right), \overline{\partial} + \partial \right).$$

The $\mathbb{P}_0$-bracket on linear observables is again straightforward. The first linear factor is the same as in the $(1,0)$ case. The second two linear factors are the linear observables of the $\beta\gamma$ system on $\mathbb{C}^3$. For linear observables in $\Omega^{\leq 1,\bullet}(U)[3]$ it is given by the same formula as in (70). The only other nonzero bracket between linear observables occurs between elements $\mathcal{O} \in \Omega_c^{3,\bullet}(U)[3]$ and $\mathcal{O}' \in \Omega_c^{0,\bullet}(U)[1]$ where it is given by

$$\{\mathcal{O}, \mathcal{O}'\} = \int_U \mathcal{O}\mathcal{O}'.$$

5. The non-minimal twist

We have classified in §3.1 the possible twisting supercharges of the $(2,0)$ supersymmetry algebra. We found that they were characterized by the rank of the supercharge, which for a non-trivial square-zero element could be either one or two. The minimal, rank one, case was studied in the last section. We now turn to the further, non-minimal, twist of the $(2,0)$ theory.

Before computing the twist, it is instructive to get a handle on the explicit data involved in choosing a non-minimal twisting supercharge. As a $\mathrm{Spin}(6)\times\mathrm{Sp}(2)$-module, the odd part of the supertranslation algebra $\mathfrak{p}_{(2,0)}$ is $\Sigma_2 \cong \Pi S_+ \otimes R_2$. It is thus easy to compute the stabilizer of a chosen rank-one supercharge, which is the product of the respective stabilizers of fixed vectors in S_+ and R_2 separately. This is the subgroup $\mathrm{MU}(3) \times \mathrm{Sp}(1)' \times \mathrm{U}(1) \subset \mathrm{Spin}(6) \times \mathrm{Sp}(2)$. As representations of the stabilizer, S_+ and R_2 decompose as

$$S_+ = \det(L)^{\frac{1}{2}} \oplus L \otimes \det(L)^{-\frac{1}{2}}, \qquad R_2 = \mathbb{C}^{-1} \oplus (R_1')^0 \oplus \mathbb{C}^{+1}. \tag{71}$$

Here, the superscripts $\mathbb{C}^{\pm 1}$ denote the charges under $\mathrm{U}(1)$.

We can thus consider the following diagram representing the decomposition of Σ_2 as a $\mathrm{MU}(3) \times \mathrm{U}(1) \subset \mathrm{Spin}(6) \times \mathrm{Sp}(2)$ representation:

$$\begin{array}{ccc} \det(L)^{\frac{1}{2}} \otimes \mathbb{C}^{-1} & \det(L)^{\frac{1}{2}} \otimes (R_1')^0 & \det(L)^{\frac{1}{2}} \otimes \mathbb{C}^{+1} \\ \\ L \otimes \det(L)^{-\frac{1}{2}} \otimes \mathbb{C}^{-1} & L \otimes \det(L)^{-\frac{1}{2}} \otimes (R_1')^0 & L \otimes \det(L)^{-\frac{1}{2}} \otimes \mathbb{C}^{+1} \end{array} \tag{72}$$

The holomorphic supercharge is indicated in red (note that we have not yet applied any twisting homomorphism). Its only nonzero bracket occurs with the supercharges in $L \otimes \det(L)^{-\frac{1}{2}} \otimes \mathbb{C}^{+1}$, represented in green above, using the degree-zero pairing on the R-symmetry space. As remarked above, this bracket witnesses a nullhomotopy of the translations in L with respect to the holomorphic supercharge. The other bracket map of interest to us pairs the supercharges represented in blue with themselves, via the map

$$(L \otimes \det(L)^{-\frac{1}{2}} \otimes (R_1')^0)^{\otimes 2} \to \wedge^2 L \otimes \det(L)^{-1} \otimes \wedge^2 R_1' \cong L^\vee. \tag{73}$$

Remark 5.1. This equivariant decomposition makes clear the structure of the tangent space to the nilpotence variety at a holomorphic supercharge. The dimension of the normal bundle is 3, represented by the component colored green above; all other supercharges anticommute with Q, and therefore define first-order deformations, which are tangent vectors to the nilpotence variety. The dimension of the tangent space at a holomorphic supercharge is thus 12, although the projective variety is in fact only 10-dimensional. The fibers of the tangent bundle are "too large" because the holomorphic locus is in fact the singular locus of the variety. In fact, as remarked above, the singular locus (or space of holomorphic supercharges) is a copy of $\mathbb{P}^3 \times \mathbb{P}^3$, consisting of four-by-four matrices of rank one; its tangent space is spanned by the black entries in the diagram (64). The red entry is Q itself, representing the tangent direction along the affine cone of the projective variety.

The deformations represented by the blue elements are of interest here; they generate the non-minimal twist (and therefore represent deforming away from the holomorphic locus of the nilpotence variety, into the locus of nonminimal supercharges). However, not all such infinitesimal deformations give rise to finite deformations of Q; geometrically, this corresponds to the fact that the nilpotence variety is singular, and not all vectors in the algebraic tangent space correspond to paths in the variety. Since the nilpotence conditions are quadratic, though, this can be checked at order two: for a deforming supercharge $Q' \in L \otimes \det(L)^{-\frac{1}{2}}$, we just need the condition that

$$[Q', Q'] = 0$$

inside $\mathfrak{p}_{(2,0)}$. Examining the bracket map discussed above shows immediately that the deforming supercharges with zero self-bracket are precisely the rank-one elements:

$$Q' = \alpha \otimes w : \quad \alpha \in L \otimes \det(L)^{-\frac{1}{2}}, \ w \in (R_1')^0. \tag{74}$$

The data of Q' has an especially nice interpretation through the lens of the holomorphic twist. We recall the alternative twisting homomorphism $\widetilde{\phi}$ from §4.5.1. Notice that this twisting homomorphism breaks the $\mathrm{Sp}(1)'$ symmetry by fixing a polarization of R_1'. Further, upon twisting by $\widetilde{\phi}$ the relevant component of the spinor representation decomposes under $\mathrm{MU}(3)$ as

$$L \otimes \det(L)^{-\frac{1}{2}} \rightsquigarrow L \otimes \det(L)^{-1} \oplus L.$$

Without loss of generality we can assume Q' lies in the first factor. Thus, from the perspective of the holomorphic twist, the datum of a further nonminimal twist therefore consists precisely of a polarization of the symplectic vector space R_1', together with a nonzero translation invariant section of $\wedge^2 T^{1,0}\mathbb{C}^3$, where $T^{1,0}\mathbb{C}^3$ is the holomorphic tangent bundle.

5.1. **Twisting data.** We proceed to describe the twisting data natural to the non-minimal twist. To compute the twist we will use a similar twisting homomorphism and an identical regrading homomorphism to that of the §4.5.1.

The twisting supercharge is of the form

$$Q_{\mathrm{nm}} := Q + Q'$$

where Q is the minimal supercharge lying in the red component of (72) and Q' is a rank one supercharge lying in the blue component of (72). The twisting homomorphism is defined by the composition

$$\phi_{\mathrm{nm}} : \mathrm{U}(2) \times \mathrm{U}(1) \to \mathrm{U}(3) \times \mathrm{U}(3) \xrightarrow{\det^{\frac{1}{2}} \times \det^{\frac{1}{2}}} \mathrm{U}(1) \times \mathrm{U}(1) \xrightarrow{(i,(i')^{-1})} \mathrm{Sp}(1) \times \mathrm{Sp}(1)' \hookrightarrow \mathrm{Sp}(2).$$

The first map is the block diagonal embedding of $(A, x) \in \mathrm{U}(2) \times \mathrm{U}(1)$ into $\mathrm{U}(3)$ via $\begin{pmatrix} A & 0 \\ 0 & x \end{pmatrix}$ in the first factor and via $\begin{pmatrix} A & 0 \\ 0 & x^{-1} \end{pmatrix}$ into the second factor. Also, $i : \mathrm{U}(1) \to \mathrm{Sp}(1)$ is the unique homomorphism for which Q has weight $+1$ and $i' : \mathrm{U}(1) \to \mathrm{Sp}(1)$ is defined by the polarization determined by Q'.

Additionally, we have the regrading homomorphism

$$\alpha_{\mathrm{nm}} : \mathrm{U}(1) \xrightarrow{\mathrm{diag}} \mathrm{U}(1) \times \mathrm{U}(1) \xrightarrow{i \times i'} \mathrm{Sp}(1) \times \mathrm{Sp}(1)' \subset \mathrm{Sp}(2).$$

It is a direct calculation to verify that ϕ_{nm}, α_{nm} constitute twisting data for the the nonminimal twisting supercharge $Q + Q'$. Notice that Q and Q' commute. Also, we have already compute the twist by the minimal supercharge Q, albeit with a slightly different twisting homomorphism. Using this twisting data we obtain the following result which is proved BW: blah blah.

Proposition 5.2. *Using the twisting data* $(\phi_{\mathrm{nm}}, \alpha_{\mathrm{nm}})$*, the* Q_{nm}*-twist of the* $(2,0)$ *tensor multiplet is described by the solid arrows in Figure 8.*

5.2. **Symmetries of the holomorphic twist.** Let $Q \in \Sigma_2$ be a holomorphic supercharge. By construction, at the level of the twisted theory we break the symmetry by the $(2,0)$ super Poincaré algebra $\mathfrak{p}_{(2,0)}$ to its Q-cohomology. Upon regrading and applying the twisting data of §4.5.1 to the Q-cohomology of $\mathfrak{p}_{(2,0)}$, this gives us an action of a $\mathbb{Z}$-graded Lie algebra $\mathfrak{p}^Q_{(2,0)}$ on the holomorphic twist of the $(2,0)$ theory.

We are interested in further twists of the $(2,0)$ theory as classified in §3.1 which we can think of as Maurer–Cartan elements in the twisted supersymmetry algebra $\mathfrak{p}^Q_{(2,0)}$. It turns out that these Maurer–Cartan elements actually lie in a subalgebra $\mathfrak{g}^Q \subset \mathfrak{p}^Q_{(2,0)}$ that we now define.

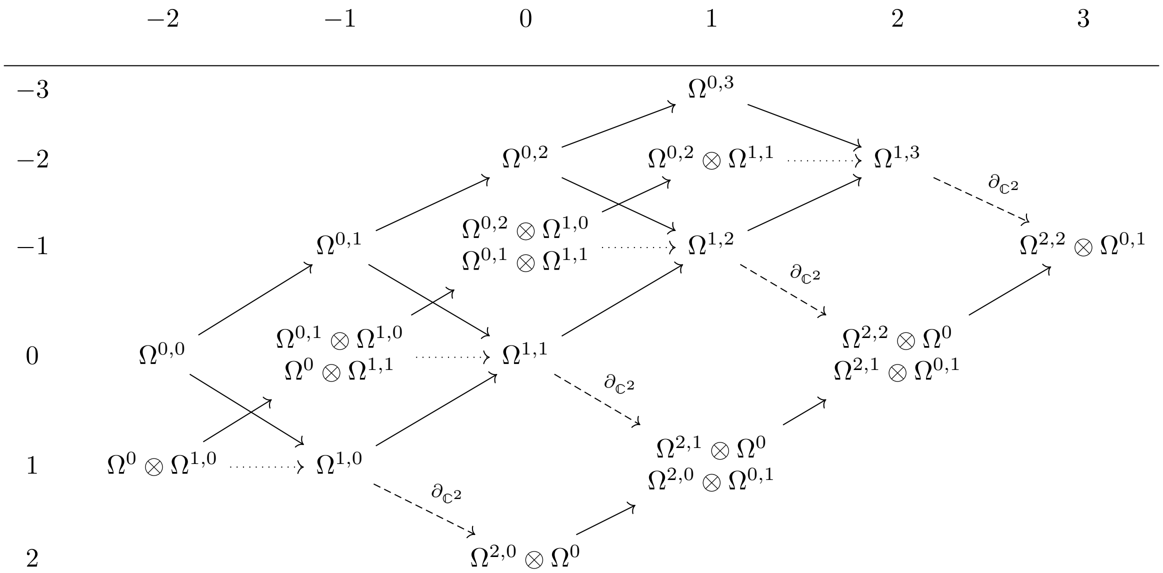

FIGURE 8. The solid arrows represent the holomorphic twist of the $\mathcal{N} = (2,0)$ multiplet using the non-minimal twisting data. The notation $\Omega^{i,j}$ refers to forms of type (i,j) on $\mathbb{C}^3 = \mathbb{C}^2 \times \mathbb{C}$. The notation $\Omega^{i,j} \otimes \Omega^{k,\ell}$ refers to forms on $\mathbb{C}^3 = \mathbb{C}^2 \times \mathbb{C}$ which are of type (i,j) on $\mathbb{C}^2$ and of type (k,ℓ) on $\mathbb{C}$.

Let L be a three-dimensional complex vector space. Let

$$\mathfrak{g}^Q = L^*[1] \oplus L \oplus \wedge^2 L[-1]$$

whose elements we will denote by (ω, X, π). This space has a $\mathbb{Z}$-graded Lie algebra structure whose bracket is defined by pairing L^* with $\wedge^2 L$ as in $[\omega, \pi] = \langle \omega, \pi \rangle \in L$. Here $\langle \cdot, \cdot \rangle$ denotes the $\mathrm{U}(L)$-invariant pairing $\wedge^2(L) \otimes L^* \to L$. BW: say why this is a subalgebra of poincare

We define an action of $\mathfrak{g}^Q$ on the holomorphic twist of the $(2,0)$ theory $\widetilde{\mathcal{T}}^Q_{(2,0)}$ (using the alternative twisting data of §4.5.1). Recall, as a graded object on $\mathbb{C}^3$ we have the decomposition

$$\widetilde{\mathcal{T}}^Q_{(2,0)} = \Omega^{0,\bullet}[2] \oplus \Omega^{1,\bullet}[1] \oplus \Omega^{3,\bullet}[2] \oplus \Omega^{0,\bullet}$$

and the corresponding fields we will write as $(\mathsf{c}, \mathsf{A}, \beta, \gamma)$. Recall, there is the internal $\overline{\partial}$ operator and also the linear BRST operator $c \mapsto \partial \mathsf{c} = \mathsf{A}$ where ∂ is the holomorphic de Rham operator on $\mathbb{C}^3$.

The action of $\mathfrak{g}^Q$ on $\widetilde{\mathcal{T}}^Q_{(2,0)}$ is an action only up to homotopy, which we will encode through an L_∞ action of the form $\rho_Q = \rho_Q^{(1)} + \rho_Q^{(2)}$ where $\rho_Q^{(1)}$ is linear and $\rho_Q^{(2)}$ is quadratic.

Explicitly, the linear term

$$\rho_Q^{(1)} : \mathfrak{g}^Q \otimes \widetilde{\mathcal{T}}^Q_{(2,0)} \to \widetilde{\mathcal{T}}^Q_{(2,0)}$$

is defined on $X \in L$ by $\rho_Q(X)\alpha = L_X\alpha$, where α is any field, and on the remaining part of the algebra by the formulas:

$$\begin{array}{rclcl}\rho_Q(\omega)\mathsf{A} & = & \omega \wedge \partial \mathsf{A} & \in & \Omega^{3,\bullet}_{\beta} \quad , \quad \rho_Q(\pi)\mathsf{A} \;=\; \langle \pi, \partial \mathsf{A}\rangle \;\in\; \Omega^{0,\bullet}_{\gamma} \\ \rho_Q(\omega)\gamma & = & \omega \wedge \gamma & \in & \Omega^{1,\bullet}_{\mathsf{A}} \quad , \quad \rho_Q(\pi)\beta \;=\; \langle \pi, \beta\rangle \;\in\; \Omega^{1,\bullet}_{\mathsf{A}}.\end{array}$$

Whenever it appears, the symbol $\langle \cdot, \cdot \rangle$ refers to the obvious $\mathrm{U}(L)$-invariant pairing.

The quadratic term is of the form

$$\rho^{(2)}_Q : (\mathfrak{g}^Q)^{\otimes 2} \otimes \widetilde{\mathfrak{T}}^Q_{(2,0)} \to \widetilde{\mathfrak{T}}^Q_{(2,0)}[-1]$$

and is defined by

$$\rho_Q(\omega \otimes \pi \otimes \mathsf{A}) \quad = \quad \iota_{\langle \omega, \pi\rangle}\mathsf{A} \quad \in \quad \Omega^{0,\bullet}_{\mathsf{c}}.$$

Proposition 5.3. *The maps $\rho^{(1)}_Q$ and $\rho^{(2)}_Q$ define an L_∞ action of $\mathfrak{g}^Q$ on $\widetilde{\mathfrak{T}}^Q_{(2,0)}$. Morerover, this action preserves the shifted presymplectic structure on $\widetilde{\mathfrak{T}}^Q_{(2,0)}$.*

Proof. For simplicity we denote $\rho^{(1)}_Q(\xi)\alpha = \xi \cdot \alpha$ for any $\xi \in \mathfrak{g}^Q$.

To verify that ρ_Q is an L_∞ action we need to verify the following identities:

(1) $\omega \cdot (\pi \cdot \gamma) + \pi \cdot (\omega \cdot \gamma) = \langle \omega, \pi \rangle \cdot \gamma$.
(2) $\omega \cdot (\pi \cdot \beta) + \pi \cdot (\omega \cdot \beta) = \langle \omega, \pi \rangle \cdot \beta$.
(3) $\omega \cdot (\pi \cdot \mathsf{A}) + \pi \cdot (\omega \cdot \mathsf{A}) - \langle \omega, \pi \rangle \cdot \mathsf{A} = \partial \rho^{(2)}_Q(\omega \otimes \pi)\mathsf{A}$.
(4) $\omega \cdot (\pi \cdot \mathsf{c}) + \pi \cdot (\omega \cdot \mathsf{c}) - \langle \omega, \pi \rangle \cdot \mathsf{c} = -\rho^{(2)}_Q(\omega \otimes \pi)\partial \mathsf{c}$.

Items (1) and (2) are straightforward. For instance:

$$\omega \cdot (\pi \cdot \gamma) + \pi \cdot (\omega \cdot \gamma) = \langle \pi, \omega \wedge \partial \gamma \rangle = \iota_{\langle \omega, \pi\rangle}\partial\gamma = \langle \omega, \pi \rangle \cdot \gamma$$

The L_∞ relation in item (3) follows from the following direct calculation.

Lemma 5.4. *For any ω, π, A one has the relation*

$$\omega \cdot (\pi \cdot \mathsf{A}) - \pi \cdot (\omega \cdot \mathsf{A}) = \iota_{\langle \omega, \pi \rangle}\partial \mathsf{A}.$$

Item (3) then follows from Cartan's magic formula $\partial\iota_{\langle \omega, \pi\rangle}\mathsf{A} + \iota_{\langle \omega, \pi\rangle}\partial\mathsf{A} = \langle \omega, \pi \rangle \cdot \mathsf{A}$.

Finally, notice the first two terms in item (4) are identically zero. This relation follows from the standard formula for the Lie derivative $\iota_{\langle \omega, \pi\rangle}\partial \mathsf{c} = \langle \omega, \pi \rangle \cdot \mathsf{c}$.

□

6. Comparison to Kodaira–Spencer gravity

In this section we document a relationship between the twist of the tensor multiplet with a holomorphic theory defined on Calabi–Yau manifolds that has roots in string theory and theories of supergravity. This theory, which we will refer to as Kodaira–Spencer theory, is gravitational in the sense that it describes variations of the Calabi–Yau structure, and was first introduced in **BCOV** as the closed string field theory describing the B-twisted topological string on three-folds. Work of Costello–Li **CLBCOV1**, **CLBCOV2**, **CLBCOVtype1** has began to systematically exhibit the relationship of Kodaira–Spencer theory on more general manifolds to twists of other classes of string theories and theories of supergravity. BW: point to sergei's topological m theory

We consider Kodaira–Spencer theory on a Calabi–Yau three-fold X, and we denote by Ω the nowhere vanishing holomorphic volume form. Denote by $\mathrm{PV}^{i,j}(X) = \Gamma(X, \wedge^i T_X \otimes \wedge^j \overline{T})$ the jth term in the Dolbeault resolution of polyvector fields of type i. The fields of Kodaira–Spencer theory are

$$\mathcal{T}_{\mathrm{KS}} \overset{\mathrm{def}}{=} \mathrm{PV}^{\bullet,\bullet}(X)[[t]][2].$$

Here t denotes a formal parameter of degree $+2$. The gradings are such that the degree of the component $t^k \mathrm{PV}^{i,j}$ is $i + j + 2k - 2$. The complex of fields carries the differential

$$Q_{\mathrm{KS}} = \overline{\partial} + t\partial_\Omega$$

where ∂_Ω fits into the diagram

$$\begin{tikzcd} \mathrm{PV}^{i,j}(X) \arrow[r, "\partial_\Omega"] \arrow[d, "\Omega"', "\cong"] & \mathrm{PV}^{i-1,j}(X) \arrow[d, "\cong"', "\Omega"] \\ \Omega^{3-i,j}(X) \arrow[r, "\partial"] & \Omega^{4-i,\bullet}(X). \end{tikzcd}$$

for $i \geq 1$. Note that ∂_Ω is an operator of degree -1 on $\mathcal{T}_{\mathrm{KS}}$, so that $\overline{\partial} + t\partial_\Omega$ is an operator of homogenous degree $+1$. The fields of Kodaira–Spencer theory are not the sections of a finite rank vector bundle, but we will pick out certain subspaces of fields which are the sections of a finite rank bundle.

Kodaira–Spencer theory fits into the BV formalism as a (degenerate) Poisson BV theory **CLBCOV1**. For a precise definition of a Poisson BV theory see **ButsonYoo**. The degree $+1$ Poisson bivector Π_{KS} on $\mathcal{T}_{\mathrm{KS}}$ which endows $\mathcal{T}_{\mathrm{KS}}$ with a Poisson BV structure is defined by

$$\Pi_{\mathrm{KS}} = (\partial \otimes 1)\delta_\Delta \in \overline{\mathcal{T}}_{\mathrm{KS}}(X) \widehat{\otimes} \overline{\mathcal{T}}_{\mathrm{KS}}(X).$$

Here, δ_Δ is the Dirac delta-function on the diagonal in $X \times X$.

Any Poisson BV theory defines a $\mathbb{P}_0$-factorization algebra of observables **ButsonYoo**. For the free limit of Kodaira–Spencer theory this $\mathbb{P}_0$-factorization algebra is completely explicit. To an open set $U \subset X$ one assigns the cochain complex:

$$\mathrm{Obs}_{\mathrm{KS}}(U) = \left(\mathrm{Sym}\left(\mathcal{T}^!_{\mathrm{KS},c}(U)\right) \ , \ Q_{\mathrm{KS}} \right).$$

The BV bracket is defined via contraction with Π_{KS}. We denote the resulting $\mathbb{P}_0$-factorization algebra for Kodaira–Spencer theory by $\mathrm{Obs}_{\mathrm{KS}}$.

There are variations of the theory obtained by looking at certain subcomplexes of $\mathcal{T}_{\mathrm{KS}}$ and by restricting the $\mathbb{P}_0$-bivector. They are called: *minimal* Kodaira–Spencer theory, denoted by $\widetilde{\mathcal{T}}_{\mathrm{KS}}$; *Type I* Kodaira–Spencer theory, denoted $\mathcal{T}_{\mathrm{Type\ I}}$; and *minimal Type I* Kodaira–Spencer theory, denoted $\widetilde{\mathcal{T}}_{\mathrm{KS}}$. They fit into the following diagram of embeddings of complexes of fields:

$$\begin{tikzcd}
& \widetilde{\mathcal{T}}_{\mathrm{KS}} \arrow[dr] & \\
\widetilde{\mathcal{T}}_{\mathrm{Type\ I}} \arrow[ur] \arrow[dr] & & \mathcal{T}_{\mathrm{KS}} \\
& \mathcal{T}_{\mathrm{Type\ I}} \arrow[ur] &
\end{tikzcd}$$

The corresponding $\mathbb{P}_0$ factorization algebras of classical observables will be denoted $\widetilde{\mathrm{Obs}}_{\mathrm{KS}}$, $\mathrm{Obs}_{\mathrm{Type\ I}}$, and $\widetilde{\mathrm{Obs}}_{\mathrm{Type\ I}}$ (whose definitions we recall below).

The goal of this section is relate Kodaira–Spencer theory to the twists of the $(1,0)$ and $(2,0)$ superconformal theories using factorization algebras. Recall that in §2.4 we showed that *presymplectic BV* theories, such as the chiral $2k$-form $\chi(2k)$, admit a $\mathbb{P}_0$-factorization algebra consisting of the "Hamitlonian" observables. We have provided a detailed description of the factorization algebras associated to the holomorphic twists of the $(1,0)$ and $(2,0)$ theories in §4.6. The main result is the following.

Theorem 6.1. *Let X be a Calabi–Yai three-fold and Q be a holomorphic supercharge. The following statements are true regarding the holomorphic twists $\mathcal{T}^{Q}_{(2,0)}$ and $\mathcal{T}^{Q}_{(1,0)}$ of the $\mathcal{N}=(2,0)$ and $\mathcal{N}=(1,0)$ tensor multiplets, respectively:*

(1) There is a sequence of morphisms of complexes of fields:

$$\begin{tikzcd}
& \beta\gamma(\mathbb{C}) \oplus \Omega^{\geq 2,\bullet}[1] \arrow[dr, "f"] & \\
\mathcal{T}^{Q}_{(2,0)} \arrow[ur, "g"] & & \widetilde{\mathcal{T}}_{\mathrm{KS}}.
\end{tikzcd} \tag{75}$$

which induces a morphism of $\mathbb{P}_0$-factorization algebras on X:

$$\widetilde{\mathrm{Obs}}_{\mathrm{KS}} \to \mathrm{Obs}_{(2,0)}$$

whose fiber is a locally constant factorization algebra.

(2) There is a sequence of morphisms of complexes of fields:

$$\begin{tikzcd}
& \Omega^{\geq 2,\bullet}[1] \arrow[dr, "f", "\cong"'] & \\
\mathcal{T}^{Q}_{(1,0)} \arrow[ur, "g"] & & \mathcal{T}_{\mathrm{Type\ I}}.
\end{tikzcd} \tag{76}$$

which induces a quasi-isomorphism of $\mathbb{P}_0$-factorization algebras on X:

$$\widetilde{\mathrm{Obs}}_{\text{Type I}} \xrightarrow{\simeq} \mathrm{Obs}_{(1,0)}$$

These result may be summarized as follows. For the $(1,0)$ theory, one finds that the factorization algebra of Hamiltonian observables of the presymplectic BV theory $\mathcal{T}^{Q}_{(1,0)}$ is equivalent to the free limit of the observables of Type I Kodaira–Spencer theory. For the $(2,0)$ theory, the observables of the presymplectic BV theory $\mathcal{T}^{Q}_{(2,0)}$ differ from the free limit of the observables of minimal Kodaira–Spencer theory by a locally constant factorization algebra. This locally constant part has been explained in **SuryaYoo** BW: finish.

6.1. **Minimal theory.** Many of the fields in the complex $\mathcal{T}_{\mathrm{KS}}$ are invisible to the shifted Poisson structure we have just introduced. There is a piece of $\mathcal{T}_{\mathrm{KS}}$ that "sees" the Poisson bracket, called the minimal theory. The fields of the minimal theory form the subcomplex of fields of full Kodaira–Spencer theory $\widetilde{\mathcal{T}}_{\mathrm{BCOV}} \subset \mathrm{PV}^{\bullet,\bullet}(X)[[t]][2]$ defined by

$$\widetilde{\mathcal{T}}_{\mathrm{KS}} \overset{\mathrm{def}}{=} \bigoplus_{i+k\leq 2} t^k \mathrm{PV}^{i,\bullet}[-i-2k+2].$$

The shifted Poisson tensor Π_{KS} restricts to one on this subcomplex, thus defining another Poisson BV theory whose fields are $\widetilde{\mathcal{T}}_{\mathrm{KS}}$.

6.1.1. *Proof of part (1) of Theorem 6.1.* This is a direct calculation. Observe that the minimal fields decompose into six graded summands:

$$\widetilde{\mathcal{T}}_{\mathrm{KS}} = \mathrm{PV}^{0,\bullet}[2] \oplus \mathrm{PV}^{1,\bullet}[1] \oplus t\mathrm{PV}^{0,\bullet} \oplus \mathrm{PV}^{2,\bullet} \oplus t\mathrm{PV}^{1,\bullet}[-1] \oplus t^2\mathrm{PV}^{0,\bullet}[-2].$$

and the differential takes the form:

$$\begin{array}{ccccc}
\underline{-2} & \underline{-1} & \underline{0} & \underline{1} & \underline{2} \\
\mathrm{PV}^{0,\bullet} & & & & \\
& \mathrm{PV}^{1,\bullet} & \xrightarrow{t\partial_\Omega} t\mathrm{PV}^{0,\bullet} & & \\
& & \mathrm{PV}^{2,\bullet} & \xrightarrow{t\partial_\Omega} t\mathrm{PV}^{1,\bullet} & \xrightarrow{t\partial_\Omega} t^2\mathrm{PV}^{0,\bullet}
\end{array} \tag{77}$$

Using the Calabi–Yau form Ω we can identify each line above with some complex of differential forms. For the first line, we have $\mathrm{PV}^{0,\bullet} \overset{\Omega}{\cong} \Omega^{3,\bullet}$. The second line is isomorphic to the cochain complex $\Omega^{\geq 2,\bullet}[1]$, where ∂_Ω is identified with the holomorphic de Rham operator. This is the standard resolution of closed two-forms up to a shift. Similarly, the third line is isomorphic to $\Omega^{\geq 1,\bullet}$. This is the standard resolution for closed one-forms.

In total, the cochain complex of minimal Kodaira–Spencer theory $\mathcal{T}_{\mathrm{KS}}$ is isomorphic to

$$\Omega^{3,\bullet}[2] \oplus \Omega^{\geq 2,\bullet}[1] \oplus \Omega^{\geq 1,\bullet}.$$

We define the morphism f in the first diagram (75) of Theorem 6.1. Recall, the cochain complex of fields of the $\beta\gamma$ system with values in $\mathbb{C}$ is

$$\Omega^{0,\bullet} \oplus \Omega^{3,\bullet}[2].$$

On the components $\Omega^{3,\bullet}[2]$ and $\Omega^{\geq 2,\bullet}[1]$, we take f to be the identity morphism. On the component $\Omega^{0,\bullet}$ we take f to be the holomorphic de Rham operator

$$f = \partial : \Omega^{0,\bullet} \to \Omega^{\geq 1,\bullet}.$$

Using the description of the holomorphic twist in §4.5.1, we have identified the minimal twist of the $\mathcal{N} = (2,0)$ theory with $\mathcal{T}^{Q}_{(2,0)} \cong \chi(2) \oplus \beta\gamma(\mathbb{C})$. The morphism g is defined to be the identity on the $\beta\gamma(\mathbb{C})$ component. On $\chi(2) = \Omega^{\leq 1,\bullet}[2]$ the morphism g is also given by the holomorphic de Rham operator

$$g = \partial : \Omega^{\leq 1,\bullet}[2] \to \Omega^{\geq 2,\bullet}[1].$$

To finish the proof, we introduce an intermediate factorization algebra that we think of as the observables associated to the Poisson BV theory $\beta\gamma(\mathbb{C}) \oplus \Omega^{\geq 2,\bullet}[1]$. Let $\mathcal{F}$ be the factorization algebra which assigns to $U \subset X$ the cochain complex

$$\mathcal{F}(U) = \left(\mathrm{Sym}\left(\beta\gamma^!_c(U) \oplus \Omega^{\leq 1,\bullet}_c(U)[3]\right), \overline{\partial}_{\beta\gamma} + \overline{\partial} + \partial\right).$$

The maps f, g induce maps of factorization algebras

$$\mathrm{Obs}_{\mathrm{KS}} \xrightarrow{f^*} \mathcal{F} \xrightarrow{g^*} \mathrm{Obs}_{(2,0)}$$

Following the description of $\mathrm{Obs}_{(2,0)}$ given in §4.6, we observe that the map g^* is a quasi–isomorphism. The result follows from the fact that the kernel of f is the sheaf of constant functions $\underline{\mathbb{C}}$.

6.2. **Type I theory.** Type I Kodaira–Spencer theory has underlying complex of fields

$$\mathcal{T}_{\text{Type I}} = \bigoplus_{i+k = \text{ odd}} t^k \mathrm{PV}^{i,\bullet}[-i-2k-2].$$

This theory is related to the Type I superstring, see **CLtypeI**.

The complex of fields of minimal Type I Kodaira–Spencer theory $\widetilde{\mathcal{T}}_{\text{Type I}}$ is the intersection of the fields of the minimal theory with the Type I theory. The only polyvector fields that appear are of arity zero and one, so that:

$$\widetilde{\mathcal{T}}_{\text{Type I}} = \mathrm{PV}^{1,\bullet}[1] \oplus t\mathrm{PV}^{0,\bullet}$$

As before, the differential is the internal $\overline{\partial}$ operator plus the operator $t\partial_\Omega$ which maps the first component to the second. Notice that $\widetilde{\mathcal{T}}_{\text{Type I}} \subset \widetilde{\mathcal{T}}_{\text{KS}}$ as the middle line in diagram (77).

The proof of part (2) of Theorem 6.1 is more direct than the last section. We have already explained how to identify $\widetilde{\mathcal{T}}_{\text{Type I}}$ with the resolution of closed two-forms $\Omega^{\geq 2,\bullet}[1]$. This is the isomorphism f in diagram (76).

The morphism g in diagram (76) is the holomorphic de Rham operator ∂. The same argument as in the last section shows that $g \circ f$ defines the desired quasi-isomorphism

$$(g \circ f)^* : \widetilde{\mathrm{Obs}}_{\text{Type I}} \xrightarrow{\simeq} \mathrm{Obs}_{(1,0)}$$

7. Dimensional reduction

In this section, let E be an elliptic curve and Y a complex surface. We consider the holomorphic twists of the $(1,0)$ and $(2,0)$ theories on the complex three-fold $Y \times E$. Recall, in §4.6 we have defined the factorization algebra of classical observables of the holomorphic $(1,0)$ theory $\mathrm{Obs}_{(1,0)}$ and of the holomorphic twist (using the alternative twisting homomorphism of §4.5.1) of the $(2,0)$ theory $\mathrm{Obs}_{(2,0)}$. We look at the dimensional reduction of these factorization algebras along the elliptic curve E, meaning we consider the pushforward along the projection map $Y \times E \to Y$.

Upon reduction along E, we find a relationship of the factorization algebras $\mathrm{Obs}_{(1,0)}$ and $\mathrm{Obs}_{(2,0)}$ to the factorization algebras associated to the holomorphic twists of pure 4d $\mathcal{N} = 2$ and $\mathcal{N} = 4$ Yang–Mills theory for the abelian one-dimensional Lie algebra.

Following [13], [16], we recall the description of the holomorphic twist of supersymmetric Yang–Mills in four-dimensions. Each of these holomorphic twists exists on any complex surface Y.

For the case of $4d$ $\mathcal{N} = 2$, the holomorphic twist is described by the underlying complex of fields

$$\Omega^{0,\bullet}(Y)[\varepsilon][1] \oplus \Omega^{2,\bullet}(Y)[\varepsilon][1] \tag{78}$$

where ε is a formal parameter of degree $+1$. This theory is free and is equipped with the linear BRST operator given by the $\overline{\partial}$-operator. The degree (-1) pairing on the space of fields is given by the integration pairing along Y together with the Berezin integral in the odd ε direction. That is, given fields $A + \varepsilon A'$ and $B + \varepsilon B'$ where $A, A' \in \Omega^{0,\bullet}(Y)$, $B, B' \in \Omega^{2,\bullet}(Y)$, the pairing is

$$(A + \varepsilon A', B + \varepsilon B') \mapsto \int_Y (AB' + A'B).$$

Since the pure supersymmetric Yang–Mills theory for an abelian Lie algebra is a free theory we consider the "smooth" version $\mathcal{O}^{sm}$ of the classical observables just as in §**??**. We denote the associated factorization algebra of classical observables by $\mathrm{Obs}_{4d\ \mathcal{N}=2}$. Via the degree (-1) pairing this factorization algebra is equipped with a $\mathbb{P}_0$-structure.

The description of the holomorphic twist of $4d$ $\mathcal{N}=4$ supersymmetric Yang–Mills theory for abelian Lie algebra is similar. The underlying complex of fields is

$$\Omega^{0,\bullet}(Y)[\varepsilon,\delta][1] \oplus \Omega^{2,\bullet}(Y)[\varepsilon,\delta][2] \tag{79}$$

The degree (-1) pairing is given by the integration pairing along Y together with the Berezin integral in the odd ε, δ directions.

Proposition 7.1. *Let $\pi : Y \times E \to Y$ be the projection.*

- *There is a morphism of $\mathbb{P}_0$-factorization algebras on Y*
$$\pi_* \operatorname{Obs}_{(1,0)} \to \operatorname{Obs}_{4d\ \mathcal{N}=2}$$
whose cofiber is a locally constant factorization algebra with trivial $\mathbb{P}_0$-structure.
- *There is a morphism of $\mathbb{P}_0$-factorization algebras on Y*
$$\pi_* \operatorname{Obs}_{(2,0)} \to \operatorname{Obs}_{4d\ \mathcal{N}=4}$$
whose cofiber is a locally constant factorization algebra with trivial $\mathbb{P}_0$-structure.

Proof. We consider the $(1,0)$ case first. Following the description in §4.6, the factorization algebra $\operatorname{Obs}_{(1,0)}$ is given by the "smooth" functionals on the sheaf of cochain complexes $\Omega^{\geq 2,\bullet}[1]$ on $Y \times E$. Since E is formal, there is a quasi-isomorphism $\mathbb{C}[\varepsilon] \xrightarrow{\simeq} \Omega^{0,\bullet}(E)$. Here, ε is a chosen generator for the sheaf of sections of the anti-holomorphic canonical bundle on E. Thus, there is a quasi-isomorphism of sheaves on Y:

$$\Omega^{2,\bullet}(Y)[\varepsilon] \oplus \mathrm{d}z\Omega^{\geq 1,\bullet}(Y)[\varepsilon] \xrightarrow{\simeq} \pi_*\Omega^{\geq 2,\bullet}.$$

Here, $\mathrm{d}z$ denotes the holomorphic volume form on the elliptic curve.

The sheaf of cochain complexes $\Omega^{\geq 1,\bullet}(Y)$ is a resolution for the sheaf of closed one-forms on the complex surface Y. The ∂-operator determines a map of cochain complexes $\partial : \Omega^{0,\bullet}(Y) \to \Omega^{\geq 1,\bullet}(Y)$ whose kernel is the sheaf of constant functions.

Putting this together, we find that there is a map of sheaves of cochain complexes on Y:

$$\Omega^{0,\bullet}(Y)[\varepsilon][1] \oplus \Omega^{2,\bullet}(Y)[\varepsilon][1] \xrightarrow{\partial} \pi_*\Omega^{\geq 2,\bullet}[1].$$

We recognize the left-hand side as the complex of fields underlying the holomorphic twist of $4d$ $\mathcal{N}=2$. Applying the functor of taking the "smooth" functionals $\mathcal{O}^{sm}(-)$ we obtain the first statement of the proposition. It is immediate to verify that this map intertwines the $\mathbb{P}_0$-structures.

The second statement is not much harder. Recall, the complex of fields of the holomorphic twist of the $(2,0)$ theory is obtained by adjoining the $\beta\gamma$ system on the three-fold $Y \times E$ to the holomorphic twist of the

$(1,0)$ theory. As a sheaf on $Y \times E$, the complex of fields of the $\beta\gamma$ system is

$$\Omega^{0,\bullet} \oplus \Omega^{3,\bullet}[2].$$

Pushing forward along π the complex becomes

$$\Omega^{0,\bullet}(Y)[\varepsilon] \oplus \mathrm{d}z\Omega^{2,\bullet}(Y)[\varepsilon][2].$$

Notice that this is a symplectic BV theory with the wedge and integrate pairing. The statement then follows from the observation that there is an isomorphism of symplectic BV theories

$$\left(\Omega^{0,\bullet}(Y)[\varepsilon][1] \oplus \Omega^{2,\bullet}(Y)[\varepsilon][1]\right) \oplus \left(\Omega^{0,\bullet}(Y)[\varepsilon] \oplus \mathrm{d}z\Omega^{2,\bullet}(Y)[\varepsilon][2]\right)$$

with the holomorphic twist of $4d$ $\mathcal{N} = 4$ as in (79). $\square$

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