Twisted S-Duality

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Abstract

S-duality is a nontrivial self-duality of type IIB string theory that exchanges strong and weak coupling. We give a mathematically rigorous description of how S-duality acts on a low-energy supersymmetry-protected sector of IIB string theory, using a conjectural description of such protected sectors in terms of topological string theory. We then give some applications which are of relevance to Geometric Langlands Theory and the representation theory of the Yangian.

Contents

1 Introduction

S-duality was originally suggested as a strong-weak duality of 4-dimensional gauge theory [GNO77, MO77]: it says that a gauge theory with a given gauge group is equivalent to another gauge theory with its Langlands dual group as the gauge group and with the coupling constant inverted. Soon it was realized that gauge theory with $\mathcal{N} = 4$ supersymmetry is a better context to realize the suggestion [Osb79]. Although the original duality was suggested as a $\mathbb{Z}/2$-symmetry, it is then natural to extend it to $\mathrm{SL}(2, \mathbb{Z})$-symmetry as both the coupling constant and the theta angle are acted on nontrivially.

In the 1990s, it was discovered that S-duality can be extended to the context of string theory [SS93]. In particular, in [HT95] a conjecture was made that type IIB string theory has $\mathrm{SL}(2, \mathbb{Z})$-symmetry. After the conception of M-theory [Wit95b], this $\mathrm{SL}(2, \mathbb{Z})$-symmetry was found to admit a manifestation as the symmetries of a torus factor of a M-theory background [Sch95, Asp96].

From a mathematical perspective, one rather remarkable application of S-duality of 4-dimensional $\mathcal{N} = 4$ gauge theory is the work of Kapustin and Witten [KW07] where they argue that a version of the geometric Langlands correspondence is a special case of S-duality. Given that a special application of S-duality leads to such rich mathematics, it is natural to wonder what mathematical marvels the general phenomenon can detect. On the other hand, the stringy origin of S-duality has made such questions somewhat inaccessible to mathematicians. This leads to a natural question of how to mathematically understand its original context, find new examples of S-dual pairs, and accordingly make new mathematical conjectures.

In this paper, we answer the above call-to-action by making a mathematical definition of S-duality as a map on the space of closed string fields of type IIB at low energies. However, we only define it on certain supersymmetry-protected sectors that admit purely mathematical descriptions – one may call these protected sectors *twisted supergravity*. We remark that twisted supergravity is entirely contained in the massless sector of the physical theory, and that S-duality of massless IIB supergravity was physically understood as early as the 90s. In addition to mathematical rigor, the main merit of the formalism we develop here is that it allows for easier calculation of how S-duality acts on further deformations of twists of supersymmetric gauge theories. This allows for the recovery of old conjectures and formulation of new ones from a unified perspective: we demonstrate this point in Section 4.

The idea of twisting in the context of string theory and supergravity was introduced in a paper of Costello and Li [CL16]. This is somewhat similar to the idea of twisting supersymmetric field theory [Wit88] in that one extracts a sector that is easier to analyze. It is by now well-known that the idea of twists of supersymmetric field theories has been useful for mathematical applications, with the most famous example being mirror symmetry. On this note, our paper may be regarded as a first step of using the idea of twisted supergravity toward finding mathematical applications.

1.1 Summary

Now let us explain the contents of our paper in a bit more detail. We should recall string theorists' manifestation of $\mathrm{SL}(2,\mathbb{Z})$-symmetry of type IIB string theory using M-theory. According to this, the existence of S-duality follows from:

- the existence of type IIA and IIB superstring theories on a 10-dimensional manifold together with the following form of an equivalence, called T-duality
$$\mathbf{T}\colon \mathrm{IIA}[S^1_r \times M^9] \cong \mathrm{IIB}[S^1_{1/r} \times M^9]$$
- the existence of M-theory on a 11-dimensional manifold together with its reduction equivalence
$$\mathrm{red}_M\colon \mathrm{M}[S^1_{\mathrm{M}} \times M^{10}] \cong \mathrm{IIA}[M^{10}]$$
- the existence of an $\mathrm{SL}(2,\mathbb{Z})$-action on a torus $E_\tau = S^1_{r_1} \times S^1_{r_2}$ with $\tau = \frac{r_2}{r_1}$, and in turn, on M-theory on the background of the form $S^1_{r_1} \times S^1_{r_2} \times M^9$, that is, $\mathrm{M}[S^1_{r_1} \times S^1_{r_2} \times M^9]$

This leads to the following diagram:

$$\begin{tikzcd} \mathrm{M}[S^1_{\mathrm{M}} \times S^1_r \times M^9] \arrow[loop above, "\mathrm{SL}(2{,}\mathbb{Z})"] \arrow[r, "\mathrm{red}_M", "\simeq"'] & \mathrm{IIA}[S^1_r \times M^9] \arrow[r, "\mathbf{T}", "\simeq"'] & \mathrm{IIB}[S^1_{1/r} \times M^9] \arrow[loop above, "\mathrm{SL}(2{,}\mathbb{Z})"] \end{tikzcd}$$

Figure 1: S-duality of type IIB string theory from M-theory

Here the $\mathrm{SL}(2,\mathbb{Z})$-action on type IIB string theory is given by $\mathrm{SL}(2,\mathbb{Z})$-action on M-theory transferred through the T-duality and reduction map. In particular, the element $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ leads to the S-duality isomorphism $\mathbb{S}\colon \mathrm{IIB}[S^1_{1/r} \times M^9] \cong \mathrm{IIB}[S^1_{1/r} \times M^9]$. Of course, we don't know how to make sense of any of these stringy objects in a rigorous way.

However, Costello and Li [CL16] gave certain conjectural descriptions of twists of IIA and IIB superstring theory along with their low-energy limits, in terms of topological string theory and their associated closed string field theories. We review this in Subsection 2.3. Furthermore, Costello suggested a certain twist of M-theory. Thus, all three vertices in the above figure have certain protected sectors that admit mathematical descriptions.

Our task is then to find the desired isomorphisms and define a version of the S-duality isomorphism by composing them, and investigate consequences. The construction of the map is done in Section 3 and consequences are studied in Section 4. The resulting map on the space of closed string states or fields of twisted supergravity is what we call *twisted S-duality*.

In order to relate this to S-duality of gauge theory, we need to understand three additional ingredients:

- String theory admits D-branes as extended objects and the dynamics of open strings ending on them may be described by a gauge theory called the D-brane gauge theory, or world-volume theory. For instance, if we consider D3 branes in type IIB superstring theory, then the corresponding D-brane gauge theory is 4-dimensional $\mathcal{N} = 4$ gauge theory. After a twist of IIB superstring theory, the D-brane gauge theory in physical string theory becomes a twist of the D-brane gauge theory (see Subsection 2.1.3).

- S-duality exchanges extended objects in a certain way. We define the notion of a flux sourced by a D-brane, which is an element of the space of closed string states determined by a D-brane (see Subsection 2.4.2). Using this, we show that our proposed twisted S-duality map preserves D3 branes as expected (see Proposition 4.1 for a precise statement). We deduce from this the statement of the Dolbeault geometric Langlands conjecture.

- There exists a closed-open map from the closed string states to deformations of D-brane gauge theory (see Subsection 2.4.1). In particular, twists of 4-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory are realized as particular kinds of deformations, and their preimages under the closed-open map lie in the space of closed string fields. By looking at how S-duality acts on those closed string fields, we can check that which twists of D-brane gauge theory are S-dual to one another. In particular, the two twists which we call HT(A)-twist and HT(B)-twist are S-dual to each other. This amounts to recovering the de Rham geometric Langlands conjecture (see Subsection 4.2).

Moreover, our twisted S-duality map can be applied to arbitrary deformations of a gauge theory, producing new infinite family of dual pairs of deformations of a gauge theory. Some new conjectures are identified in Subsections 4.3 and 4.4. That we can identify nontrivial conjectures in this manner may be surprising to some readers. It is possible because we work in a sector where the dependence on a coupling constant is topological in nature; we consider a torus in the topological direction of the M-theory background and hence the dilaton field is topological. In particular, our analysis doesn't offer any deep insights regarding strong-weak duality.

Finally, we emphasize that the idea of studying how S-duality of type IIB string theory and S-duality on topological string theory are related was realized long before our work. Important works in this direction include the paper by Nekrasov, Ooguri, and Vafa [NOV04] and the paper on topological M-theory by Dijkgraaf, Gukov, Neitzke, and Vafa [DGNV05]. Our take of closed string field theory and supergravity based on BCOV theory [BCOV94, CL12] (see Subsection 2.3.1) and its applications to finding new dual pairs of 4-dimensional gauge theory seem different from their approach and we hope to understand precise relations to their work in the future.

1.2 Conventions

When we describe a field theory, we work with the BV formalism in a perturbative setting. We work with $\mathbb{Z}/2$-grading but may write $\mathbb{Z}$-grading in a way that it gives an expected cohomological degree when it can. Algebras are complexified unless otherwise mentioned; in particular, we see $\mathrm{GL}(N)$ when physicists expect to see $\mathrm{U}(N)$.

Acknowledgements

We thank Kevin Costello, Davide Gaiotto, and Si Li for their guidance on us learning aspects of string theory. Learning and even writing a paper about a part of string theory as mathematicians wasn't easy, so their help was invaluable. Especially this project wouldn't even have taken off without lectures on a related topic by Costello and following extensive discussions with him. This work was first announced at Aspen Center for Physics, which is supported by National Science Foundation grant PHY-1607611. We thank Dylan Butson, Jacques Distler, Brian Williams, and Edward Witten for critical questions and useful discussions.

2 Topological Strings and Twisted Supergravity

A goal of this paper is a mathematical description of how S-duality acts on certain supersymmetry-protected sectors of type IIB string theory and supergravity theory. In this section, we wish to establish a mathematical context for discussing such protected sectors. Conjecturally, these protected sectors are described by topological string theory.

In Subsection 2.1, we start with a brief mathematical treatment of topological string theory. Although this plays a mostly organizational role for our paper, this setting allows us to rigorously describe many maneuvers familiar to string theorists in terms of data attached to a Calabi-Yau category. In particular, closed string field theory and open string field theory arise as natural moduli problems determined by categorical data.

Having introduced such a framework, we wish to argue that it indeed describes protected sectors of the physical superstring theory. Ideally, there would be a twisting procedure that takes in the physical superstring theory as an input and produces the topological string theory we work with as an output. Unfortunately, a rigorous treatment of physical string theory and a mathematical codification of such a procedure is currently out of reach.

On the other hand, Costello and Li [CL16] define a class of so-called *twisted supergravity backgrounds* that conjecturally have the feature that the fields of supergravity in perturbation theory around such a background map to the closed string field theory of a topological string theory. Following the work of Costello and Li, we explain the twisting procedure for supergravity in Subsection 2.2 and moreover provide a conjectural description of twisted supergravity theory in terms of closed topological strings in Subsection 2.3. Then we move on to reviewing the relation between open and closed sectors in Subsection 2.4; in particular, we discuss that further deformations of gauge theories living on D-branes can be identified with closed string fields in the topological string.

Much of this section is a review of the exposition of [CL16], with a couple of constructions adapted for our purposes; the reader may find the original paper a useful supplement.

2.1 Topological String Theory

A large portion of topological string theory can be understood in terms of axiomatic 2-dimensional extended topological quantum field theory (TQFT). In fact, for our purposes we will think of topological string theory as such. While we do not use it in an essential way, the language can serve as a useful organizational device. The category that the 2d TQFT attaches to a point also determines two higher dimensional field theories. The first of these is closed string field theory, which contains twisted supergravity, and will be discussed in more detail in Subsection 2.3.2. The second of these is open string field theory and will be discussed in 2.1.3.

2.1.1 Review of 2d TQFT

Let us provide a brief review of 2-dimensional TQFT as relevant for our purpose. For more details, refer to the original articles [Cos07, Lur09]. A 2-dimensional TQFT is a symmetric monoidal functor $Z\colon \mathrm{Bord}_2 \to \mathrm{DGCat}$. Here, Bord_2 is a 2-category whose objects are 0-manifolds, 1-morphisms between objects are 1-manifolds with boundary the given 0-manifolds, and 2-morphisms are 2-manifolds with corresponding boundaries and corners, and DGCat denotes the 2-category of small DG categories. In fact, everything should be considered in an ∞-categorical context, but we suppress any mention of it, because our discussion is mostly motivational.

By the cobordism hypothesis, a fully extended framed 2d TQFT is determined by a fully dualizable object of DGCat, and an object of DGCat is known to be fully dualizable if and only if it is smooth and proper.[1] Additionally, we often wish to consider oriented theories, which are likewise determined by Calabi–Yau categories. In other words, a smooth proper Calabi–Yau category determines an oriented 2-dimensional extended TQFT [Cos07, Lur09]. Moreover, by only considering 2-manifolds where each connected component has at least one outgoing boundary component, one can similarly consider a version of oriented 2d TQFT determined by a not necessarily proper Calabi–Yau category [Lur09, Section 4.2]. In what follows, we will freely remove any compactness assumption on spaces at the price of working with the latter version.

Given a DG category $\mathcal{C}$, one can consider its Hochschild chains $\mathrm{Hoch}_\bullet(\mathcal{C})$ and Hochschild homology $\mathrm{HH}_\bullet(\mathcal{C})$. It is a well-known fact that Hochschild chains admit an action of a circle S^1. Now, note that in terms of the 2d framed TQFT $Z_\mathcal{C}$ determined by a smooth proper DG category $\mathcal{C}$, one has $\mathrm{Hoch}_\bullet(\mathcal{C}) = Z_\mathcal{C}(S^1_{\mathrm{cyl}})$, namely, what $Z_\mathcal{C}$ assigns to a circle S^1_{cyl} with the cylinder framing. This is the 2-framing induced from the canonical framing of S^1. From this perspective, the fact that one may rotate such a circle without changing the framing is responsible for the S^1-action on Hochschild chains. On the other hand, one may also consider Hochschild cochains $\mathrm{Hoch}^\bullet(\mathcal{C})$ and Hochschild cohomology $\mathrm{HH}^\bullet(\mathcal{C})$ of a DG category $\mathcal{C}$. Reasoning similar to the above explains why Hochschild cochains admit the structure of an $\mathbb{E}_2$-algebra. Indeed, by considering $Z_\mathcal{C}(S^1_{\mathrm{ann}})$ for a circle S^1_{ann} with the annulus framing, since a pair of pants diagram can be drawn in a way that respects the framing, $Z_\mathcal{C}(S^1_{\mathrm{ann}})$ is seen to have the desired algebra structure.

For a smooth proper Calabi–Yau DG category $\mathcal{C}$, or equivalently, an oriented 2d TQFT $Z_\mathcal{C}$ determined by it, these two spaces are identified up to a shift, in which case the two structures are combined to yield a BV algebra structure. Physically, we may think of such an identification between Hochschild chains and Hochschild cochains as a state-operator correspondence.

In our setting, we are interested in a particular type of topological string theory. Let M be a (compact) symplectic manifold of real dimension $2m$ and let X be a (compact) Calabi–Yau manifold of complex dimension n such that $2m + 2n = 10$. Then by a *mixed A-B topological string theory with target* $M \times X$, which we succinctly name as a topological string theory on $M_A \times X_B$, we mean the 2-dimensional oriented TQFT determined by the Calabi–Yau 5-category

$$\mathrm{Fuk}(M) \otimes \mathrm{Coh}(X),$$

where $\mathrm{Fuk}(M)$ refers to a Fukaya category of M and $\mathrm{Coh}(X)$ is the DG category of coherent sheaves. This Calabi–Yau 5-category should be thought of as the category of D-branes for the topological string theory on $M_A \times X_B$.

Remark 2.1. Note that we do not specify which version of Fukaya category we are considering here. In the end, the only case that is actually relevant for us is when $M = T^*S^1$ as appearing in Subsection 3.1, which admits an explicit description. Our current discussion is mostly to provide the context and motivation of what follows in a systematic way. In view of that, let us note that when the symplectic manifold M is of the form $M = T^*N$, the corresponding wrapped Fukaya category is, roughly speaking, equivalent to the category of modules over the algebra of chains $C_\bullet(\Omega_* N)$ where $\Omega_* N$ is the based loop space [Abo11].

2.1.2 Topological Closed String Field Theory

The next aim is to understand closed string field theory and its structures [WZ92, Zwi93] in the context of topological string theory. The following can be regarded as a motivation, as one can

[1] For the definition of a DG category being smooth and proper, see, for instance, [KS09]. It is also known that for a quasi-compact quasi-separated scheme X, the DG category $\mathrm{Perf}(X)$ of perfect complexes on X is smooth and proper if and only if X is smooth and proper in the classical sense, respectively.

take our discussion of BCOV theory in Subsection 2.3.1 as the starting point of a mathematical discussion for closed string field theory.

Let Z be a topological string theory in the above sense. Naively, one may think of the space of closed string states as $Z(S^1)$. However, in the physical theory, the worldsheet theory is a 2-dimensional conformal field theory is coupled to 2-dimensional gravity; in particular, the space of closed string states should be invariant under the group $\mathrm{Diff}(S^1)$ of diffeomorphisms of S^1 acting by reparametrizing the boundary components of the worldsheet. In our topological setting, this amounts to taking the S^1-invariant space $Z(S^1)^{S^1}$. In a categorical setting, for a DG category $\mathcal{C}$, we define its cyclic cochain complex to be $\mathrm{Cyc}^\bullet(\mathcal{C}) := \mathrm{Hoch}^\bullet(\mathcal{C})^{S^1}$. The claim is that there is a natural way to equip it with the structure of a degenerate field theory [BY16].

Here the space V^{S^1} of homotopy S^1-invariants for a cochain complex (V, d_V) with an action of S^1, or equivalently a cochain map $C_\bullet(S^1) \to \mathrm{End}(V)$, may be modeled by

$$V^{S^1} = (V[\![t]\!], d_V + tB),$$

where t is of cohomological degree 2 and $B\colon V \to V$ is an operator of cohomological degree -1, which is the image of the fundamental class $[S^1]$ under the map $C_\bullet(S^1) \to \mathrm{End}(V)$. In our case where $V = \mathrm{Hoch}^\bullet(\mathcal{C})$, the operator B is precisely Connes's B operator and we obtain that the shifted cyclic cochains $\mathrm{Cyc}^\bullet(\mathcal{C})[1]$ have the structure of an L_∞-algebra. Moreover, the Calabi–Yau structure of $\mathcal{C}$ yields a cyclic L_∞-algebra structure on the shifted cyclic cochains $\mathrm{Cyc}^\bullet(\mathcal{C})[1]$. The cyclic L_∞ structure may be thought of as coming from a realization of $\mathrm{Cyc}^\bullet(\mathcal{C})[1]$ as the -1 shifted tangent complex at $\mathcal{C}$ in the moduli of smooth proper Calabi-Yau d categories [BR20].

Moreover, a folklore theorem asserts that the formal neighborhood of $\mathcal{C}$ in the moduli of smooth proper Calabi-Yau d-categories is odd shifted Poisson. Indeed, consider the periodic cyclic cochains $\mathrm{pCyc}^\bullet(\mathcal{C})$, which can be identified with the Tate fixed points of $\mathrm{Hoch}^\bullet(\mathcal{C})$, where the space V^{Tate,S^1} of Tate fixed points for the S^1 action on (V, d_V) is modeled by

$$V^{\mathrm{Tate},S^1} = (V(\!(t)\!), d_V + tB).$$

Using the trace pairing and the residue pairing, $\mathrm{PCyc}^\bullet(\mathcal{C})$ has a canonical symplectic structure of degree $6-2d$ where d is the Calabi–Yau dimension of $\mathcal{C}$. Now from the canonical embedding $\mathrm{Cyc}^\bullet(\mathcal{C}) \to \mathrm{PCyc}^\bullet(\mathcal{C})$, the failure of the differential B preserving a splitting of $\mathrm{PCyc}^\bullet(\mathcal{C})$ into $\mathrm{Cyc}^\bullet(\mathcal{C})$ and its complement induces a Poisson structure on $\mathrm{Cyc}^\bullet(\mathcal{C})$ of degree $2d-5$. Note that it is only for $d = 3$ that naturally is equipped with a $\mathbb{P}_0$-structure in a $\mathbb{Z}$-graded sense.

All these combined, if one can find a local model $\mathcal{E}$ for $\mathrm{Cyc}^\bullet(\mathcal{C})[2]$, then it would have the structure of a degenerate field theory by construction. Topological closed string field theory ought to then be a BV quantization of $\mathcal{E}$.

Example 2.2. Consider a (compact) Calabi–Yau 5-fold X and the topological string theory determined by $\mathcal{C} = \mathrm{Coh}(X)$. We want to identify the corresponding closed string field theory. For more details, we refer the reader to Subsection 2.3.1.

We know $Z_{\mathcal{C}}(S^1)$ is identified with Hochschild cochains. By the HKR theorem (and the Kontsevich formality [Kon03] for an L_∞ equivalence after shifting), the Hochschild cochain complex is identified with the space of polyvector fields $\mathrm{PV}(X) = \bigoplus_{i,j} \mathrm{PV}^{i,j}(X)$ with the differential $\bar\partial$ (and the Lie bracket $[-,-]_{\mathrm{SN}}$, the Schouten–Nijenhuis bracket), where $\mathrm{PV}^{i,j}(X) = \Omega^{0,j}(X, \wedge^i T_X)$. Moreover, one finds $B = \partial\colon \mathrm{PV}^{i,j}(X) \to \mathrm{PV}^{i-1,j}(X)$. This leads to a description of closed string field theory of the topological string theory associated to $\mathrm{Coh}(X)$ as

$$\mathcal{E} = (\mathrm{PV}(X)[\![t]\!][2]; \bar\partial + t\partial, [-,-]_{\mathrm{SN}}).$$

Finally, from the symplectic pairing on $\mathrm{PV}(X)(\!(t)\!)$ given by

$$\omega(f(t)\mu, g(t)\nu) = (\mathrm{Res}_{t=0}\, f(t)g(-t)) \cdot \mathrm{Tr}(\mu\nu)$$

it induces the kernel $(\partial \otimes 1)\delta_{\mathrm{diag}}$ for the shifted Poisson structure on $\mathcal{E}$.

In fact, more is true in this example. As mentioned in Remark 2.11 and explained in the original paper [CL12], by a highly nontrivial L_∞ quasi-isomorphism, one can find another description of this degenerate field theory that admits a description in terms of a local action functional.

Remark 2.3. Hochschild (co)homology of a Fukaya category $\mathrm{Fuk}(M)$ on a general symplectic manifold M is known to be closely related to quantum cohomology $\mathrm{QH}(M)$ [Gan16, San17]. In particular, it is very hard to imagine a local cochain model which encodes all the non-perturbative information. Our interest will be restricted to the case where we consider $M_A \times X_B$ with $M = T^*N$: then remark 2.1 suggests that the closed string field theory should be given by $C_\bullet(LN) \otimes \mathcal{E}(X)$ where LN is the free loop space and $C_\bullet(LN)$ is identified as the space of Hochschild (co)chains of $C_\bullet(\Omega_* N)$-mod.

Remark 2.4. The invariant space V^{S^1} may be understood as a cochain complex $(\ker B, d_V)$ without introducing the t parameter. In the example above, this would lead to

$$\mathcal{E} = (\ker \partial \subset \mathrm{PV}(X)[2]; \bar{\partial}, [-,-]_{\mathrm{SN}}).$$

This description has a disadvantage that fields are required to satisfy a non-trivial differential equation; in particular, it would be much harder to perform quantization in this setting. On the other hand, at the classical level, it has the same amount of information. As our discussion of closed string field theory and supergravity is entirely at the classical level, we decide to work with this latter model. More discussion on this point can be found in Subsection 2.3.1.

2.1.3 Topological Open String Field Theory

We now turn to describing the open string field theory of a topological string theory. Given a topological string theory described by a smooth Calabi-Yau d category $\mathcal{C}$, we may think of an object $\mathcal{F} \in \mathcal{C}$ as describing a locus on spacetime on which open strings can end. The dynamics of open strings stretched from the support of $\mathcal{F}$ to itself are described by a field theory called the *D-brane gauge theory* or *world-volume theory*.

Given a smooth Calabi-Yau d category, one may consider its moduli of objects $\mathcal{M}_{\mathcal{C}}^{obj}$. This is a prestack associated to $\mathcal{C}$ whose R-points are given by $\mathrm{Hom}_{\mathrm{DGCat}}(\mathcal{C}, \mathrm{Perf} R)$; we think of this as defining a sort of universal open string field theory. The Calabi-Yau structure on $\mathcal{C}$ equips $\mathcal{M}_{\mathcal{C}}^{obj}$ with a $(2-d)$-shifted Poisson structure. Now given an object $\mathcal{F} \in \mathcal{C}$, we may consider the (-1)-shifted tangent complex $\mathbb{T}_{\mathcal{F}}[-1]\mathcal{M}_{\mathcal{C}}^{obj}$ - this is an L_∞ algebra with an odd pairing and defines the worldvolume theory of $\mathcal{F}$ as a perturbative BV theory. Furthermore, general results tell us that as L_∞-algebras $\mathbb{T}_{\mathcal{F}}[-1]\mathcal{M}_{\mathcal{C}}^{obj} = \mathrm{Ext}_{\mathcal{C}}(\mathcal{F}, \mathcal{F})$ where the L_∞-structure on the right hand side comes from skew-symmetrizing the natural A_∞-structure.

Let us explicate this in the example of the B-model on an odd-dimensional complex Calabi–Yau variety X, of dimension d. The corresponding Calabi–Yau category is $\mathrm{Coh}(X)$. The general story in the previous paragraph tells us that for a D-brane $\mathcal{F} \in \mathrm{Coh}(X)$, the algebra of open string states ending on it is given by the DG algebra $\mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{F})$. Moreover, a D-brane gauge theory is supposed to be a field theory living on the support of a D-brane, encoding the information of the algebra, and hence should in particular be local. In view of this, we consider the sheaf of DG algebras $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{Coh}(X)}(\mathcal{F}, \mathcal{F})$. A special case of interest for us is when we wrap N coincident D-branes along a complex submanifold Y, meaning that $\mathcal{F}$ is the trivial vector bundle of rank N over a subvariety Y of X. In this case, we take cohomology of the sheaf to obtain the exterior algebra $\wedge^\bullet N_{X/Y}$ where $N_{X/Y}$ is the normal bundle of Y in X. Then the corresponding D-brane gauge theory has space of fields given by

$$\mathcal{E} = \Omega^{0,\bullet}(Y, \wedge^\bullet N_{X/Y}) \otimes \mathfrak{gl}(N)[1]$$

From the fact that $\mathrm{Coh}(X)$ is a Calabi–Yau category, this space $\mathcal{E}$ has an induced structure of a classical BV theory with a symplectic pairing of degree $2-d$. Again we obtain a field theory in a $\mathbb{Z}$-graded matter exactly when $d=3$. Note that if X were even-dimensional, then the space $\mathcal{E}$ behaves like a field theory except that the symplectic pairing would be of even degree.

Example 2.5. Consider topological string theory on $\mathbb{C}^5_B$.

- Consider $\mathbb{C}^2 \subset \mathbb{C}^5$. Computing as above yields

$$\mathcal{E}^{\mathrm{Hol}}_{\mathrm{D3}} = \Omega^{0,\bullet}(\mathbb{C}^2)[\varepsilon_1, \varepsilon_2, \varepsilon_3] \otimes \mathfrak{gl}(N)[1],$$

  where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are odd variables that can be understood as parametrizing the directions normal to $\mathbb{C}^2$ in $\mathbb{C}^5$ and hence describing transverse fluctuations of the brane. This is the holomorphic twist of 4-dimensional $\mathcal{N}=4$ supersymmetric gauge theory with gauge group $\mathrm{GL}(N)$.

- Consider $\mathbb{C}^5 \subset \mathbb{C}^5$. Computing as above yields

$$\mathcal{E}^{\mathrm{Hol}}_{\mathrm{D9}} = \Omega^{0,\bullet}(\mathbb{C}^5) \otimes \mathfrak{gl}(N)[1].$$

  A result of Baulieu [Bau11] identifies this as a holomorphic twist of 10-dimensional $\mathcal{N}=1$ supersymmetric gauge theory with gauge group $\mathrm{GL}(N)$.

Next, let us consider a symplectic manifold M of dimension $2k$ with odd k as a target for A-type topological string theory. Then a D-brane should be given by a Lagrangian L of M. In this case, we should similarly compute its (derived) endomorphism algebra in the Fukaya category. In the current paper, we are mostly interested in the case of $M = \mathbb{R}^{2k}$ for which N coinciding D-branes on $L = \mathbb{R}^k \subset \mathbb{R}^{2k}$ yield a theory described by

$$\mathcal{E} = \Omega^\bullet(\mathbb{R}^k) \otimes \mathfrak{gl}(N)[1].$$

Again if k were even, then we would have gotten an even symplectic pairing on $\mathcal{E}$.

Example 2.6. Let L be a 3-manifold and take $M = T^*L$. Then we obtain Chern–Simons theory

$$\mathcal{E} = \Omega^\bullet(L) \otimes \mathfrak{gl}(N)[1]$$

as argued by [Wit95a].

In general, given a tensor product $\mathcal{C}_1 \otimes \mathcal{C}_2$ of Calabi–Yau categories, and an object $\mathcal{F}_1 \otimes \mathcal{F}_2 \in \mathcal{C}_1 \otimes \mathcal{C}_2$, we may compute $\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{C}_1 \otimes \mathcal{C}_2}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \cong \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{C}_1}(\mathcal{F}_1, \mathcal{F}_1) \otimes \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{C}_2}(\mathcal{F}_2, \mathcal{F}_2)$. Again, if it is an odd-dimensional Calabi–Yau category, as is for a topological string theory on $M_A \times X_B$, it would give a field theory. That is, given a D-brane in an arbitrary mixed A-B topological string theory, the corresponding D-brane gauge theory is described by a combination of the above two classes of examples with an odd symplectic pairing.

Example 2.7. Having N D-branes on $\mathbb{R}^2 \times \mathbb{C} \subset \mathbb{R}^4_A \times \mathbb{C}^3_B$ leads to

$$\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}} = \Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{gl}(N)[1],$$

which is the holomorphic-topological twist of 4d $\mathcal{N}=4$ gauge theory with gauge group $\mathrm{GL}(N)$.

2.2 Constructing Twisted Supergravity

One of the main utilities of string theory is that the low-energy dynamics of closed strings describe the dynamics of gravitons. Analogously, topological closed string field theory contains a version of gravity that governs a particular subclass of metric deformations present in physical gravity. For those topological string theories that conjecturally describe twists of superstrings, this version of gravity is known as twisted supergravity.

In this subsection, we review the construction of twisted supergravity following [CL16], with the goal of motivating the conjectural definitions and discussions in the next subsection. We also briefly describe the relationship between the twisting of supersymmetric field theories and supergravity, touching on the role that twisting homomorphisms for the former play in the latter context. This subsection is largely independent of the rest of the paper, and readers who are interested primarily in mathematical applications of our work may wish to skip ahead to the next subsection.

2.2.1 Type IIB Supergravity

The construction of twisted supergravity uses a description of type II supergravity in the BV-BRST formalism. Such a description of the full theory is both unwieldy and excessive; here we will give a partial description of the theory that includes the relevant ingredients for describing the twisting procedure. For concreteness, we will discuss the construction in the setting of type IIB supergravity on $\mathbb{R}^{10}$ – the construction for type IIA supergravity is completely analogous and the generalization to an arbitrary 10-manifold is straightforward.

We will work with supergravity in the first-order formalism. Roughly speaking, one may think of the theory as a gauge theory for a 10-dimensional supersymmetry algebra. Let us begin by recalling the definition of the relevant supersymmetry algebra.

We first fix some notation. Note that the Hodge star operator acting on $\Omega^5(\mathbb{R}^{10}) = \Omega^5(\mathbb{R}^{10}; \mathbb{C})$ squares to -1; we will use a subscript of $\pm$ to denote the $\pm i$-eigenspace of this action. Additionally, we decorate a space of forms with the subscript cc to denote the space of such forms with constant coefficients. Recall that the Lie algebra $\mathfrak{so}(10, \mathbb{C})$ has two irreducible spin representations $S_\pm$, each of complex dimension 16. Furthermore, we have an isomorphism of $\mathfrak{so}(10, \mathbb{C})$-representations $\mathrm{Sym}^2 S_+ \cong \mathbb{C}^{10} \oplus \Omega^5_{+,\mathrm{cc}}(\mathbb{R}^{10})$; let $\Gamma^+ \colon \mathrm{Sym}^2 S_+ \to \mathbb{C}^{10}$ denote the projection.

Definition 2.8. The *10-dimensional* $\mathcal{N} = (2, 0)$ *super-translation algebra* is the Lie superalgebra with underlying $\mathbb{Z}/2$-graded vector space

$$\mathcal{T}^{(2,0)} := \mathbb{C}^{10} \oplus \Pi(S_+ \otimes \mathbb{C}^2)$$

and bracket given as follows. Choose an inner product $\langle -, - \rangle$ on $\mathbb{C}^2$ and let $\{e_1, e_2\}$ denote an orthonormal basis. The bracket on odd elements is given by

$$[\psi_1 \otimes \alpha, \psi_2 \otimes \beta] = \Gamma^+(\psi_1, \psi_2)\langle \alpha, \beta \rangle.$$

The *10-dimensional* $\mathcal{N} = (2, 0)$ *supersymmetry algebra* is the Lie superalgebra given by the semidirect product

$$\mathfrak{siso}_{\mathrm{IIB}} := \mathfrak{so}(10, \mathbb{C}) \ltimes \mathcal{T}^{(2,0)}.$$

As remarked above, we will work in the first-order formalism, where supergravity is a theory with fundamental field a $\mathfrak{siso}_{\mathrm{IIB}}$-valued connection [CDP91]. The idea of describing gravity in this way may be unfamiliar, so let us first recall how this works in ordinary Einstein gravity. In the first-order formalism for (Euclidean) Einstein gravity on $\mathbb{R}^4$, the fundamental field is $A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{siso}(4)$ where $\mathfrak{siso}(4) = \mathfrak{so}(4) \ltimes \mathbb{R}^4$ denotes the Poincaré Lie algebra. Decomposing this into components, we find

- The component $e \in \Omega^1(\mathbb{R}^4) \otimes \mathbb{R}^4$ is the *vielbein* and encodes the metric as $g = (e \otimes e)$ where we used the standard inner product $(-,-)$ on $\mathbb{R}^4$.

- The component $\Omega \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{so}(4)$ is the *spin connection*.

The action of the theory takes the form $S(e, \Omega) = \int_{\mathbb{R}^4} e \wedge e \wedge F_\Omega$ where F_Ω denotes the curvature of Ω. Note that it is of first order – hence the name of the formulation.

Returning to the supergravity setting, the fundamental field is a $\mathfrak{siso}_{\mathrm{IIB}}$-valued connection, which we may locally express as a $\mathfrak{siso}_{\mathrm{IIB}}$-valued 1-form and decompose into components. This yields the following fields:

- The component $E \in \Omega^1(\mathbb{R}^{10}) \otimes \mathbb{C}^{10}$ is the vielbein. This encodes the metric as above.

- The component $\Omega \in \Omega^1(\mathbb{R}^{10}) \otimes \mathfrak{so}(10, \mathbb{C})$ is the *spin connection*.

- The component $\Psi \in \Omega^1(\mathbb{R}^{10}) \otimes \Pi(S^+ \otimes \mathbb{C}^2)$ is the *gravitino*.

The theory also includes other fields such as the B-field that we have not included here [CDP91].

Note that we have an action of the Lie algebra $C^\infty(\mathbb{R}^{10}, \mathfrak{siso}_{\mathrm{IIB}})$ on the above space of fields. We wish to treat the theory in the BV-BRST formalism, and to do so, we take a homotopy quotient of the space of fields by the action of $C^\infty(\mathbb{R}^{10}, \mathfrak{siso}_{\mathrm{IIB}})$ and then take the (-1)-shifted cotangent bundle. The resulting extended space of fields is $\mathbb{Z} \times \mathbb{Z}/2$-graded and contains the following

	-1	0	1	2
even	$\Omega^0(\mathbb{R}^{10}) \otimes \mathbb{C}^{10}$	$\Omega^1(\mathbb{R}^{10}) \otimes \mathbb{C}^{10}$	$\Omega^9(\mathbb{R}^{10}) \otimes \mathbb{C}^{10}$	$\Omega^{10}(\mathbb{R}^{10}) \otimes \mathbb{C}^{10}$
even	$\Omega^0(\mathbb{R}^{10}) \otimes \mathfrak{so}(10, \mathbb{C})$	$\Omega^1(\mathbb{R}^{10}) \otimes \mathfrak{so}(10, \mathbb{C})$	$\Omega^9(\mathbb{R}^{10}) \otimes \mathfrak{so}(10, \mathbb{C})$	$\Omega^{10}(\mathbb{R}^{10}) \otimes \mathfrak{so}(10, \mathbb{C})$
odd	$\Omega^0(\mathbb{R}^{10}) \otimes \Pi S$	$\Omega^1(\mathbb{R}^{10}) \otimes \Pi S$	$\Omega^9(\mathbb{R}^{10}) \otimes \Pi S$	$\Omega^{10}(\mathbb{R}^{10}) \otimes \Pi S$

where we have put $S = S_+ \otimes \mathbb{C}^2$, the $\mathbb{Z}$-grading is listed horizontally, and the $\mathbb{Z}/2$-grading vertically. We emphasize that this is a partial description of the theory that just includes fields needed in our construction. For a description of the full theory in the BV formalism, we refer the reader to [ESW18, Section 12.2].

Here

- The *bosonic ghost* is the field $q \in C^\infty(\mathbb{R}^{10}, \Pi S)$. This field has bidegree $(-1, 1)$ with respect to the $\mathbb{Z} \times \mathbb{Z}/2$-grading and will play a central role in the construction of twisted supergravity in the next subsection.

- The *ghost for diffeomorphisms* is the field $V \in \Omega^0(\mathbb{R}^{10}) \otimes \mathbb{C}^{10}$ and $V^* \in \Omega^{10}(\mathbb{R}^{10}) \otimes \mathbb{C}^{10}$ denotes its antifield.

- The field $\Psi^* \in \Omega^9(\mathbb{R}^{10}) \otimes \Pi S$ is the antifield to the gravitino.

On a curved spacetime (M^{10}, g) where g satisfies the supergravity equations of motion (i.e. Ricci flat), each entry above should be replaced with forms valued in an appropriate bundle on M.

Remark 2.9. Note that there is a nontrivial bracket between the diffeomorphism ghosts. If we replace $\mathbb{R}^{10}$ with a more general manifold, the diffeomorphism ghosts are going to be vector fields. The existence of this nontrivial bracket gives a sense in which first-order gravity is not a gauge theory, at least at face value. On the other hand, we are still going to colloquially refer to the action of $C^\infty(M; \mathfrak{siso}_{\mathrm{IIB}})$ as gauge transformations.

In addition to the bracket mentioned in the remark, the action includes the following terms:

- $\int_{\mathbb{R}^{10}} V^*[q,q]$, where $[-,-]$ denotes the bracket of $\mathfrak{siso}_{\mathrm{IIB}}$ extended $C^\infty(\mathbb{R}^{10})$-linearly.
- $\int_{\mathbb{R}^{10}} \Psi^* \nabla_g q$, where ∇_g is a metric connection on the trivial spinor bundle.

Note that varying the action functional with respect to V^* yields the equation of motion $[q,q]=0$. Therefore, it makes sense to take q-cohomology. Further, varying with respect to Ψ^* yields the equation of motion $\nabla_g q = 0$ so the bosonic ghost must be covariantly constant. Below we will use the subscript "cov" to refer to being covariantly constant on a possibly curved spacetime (M,g).

2.2.2 Twisting Supergravity

Let us now describe the construction of twisted supergravity. Afterwards, we describe some analogies with phenomena in supersymmetric and non-supersymmetric gauge theory to help orient the readers.

Definition 2.10. *Twisted supergravity* on $M = M^n$ by a supercharge $Q \in \Pi S$ is supergravity in perturbation theory around a solution to the equations of motion where the bosonic ghost q is required to be the covariantly constant scalar $d_Q \in C^\infty_{\mathrm{cov}}(M;\Pi S)[1]$. We say the twist is an *H-invariant twist* if $Q \in \Pi S$ is invariant under $H \subset \mathrm{Spin}(n)$.

That is, twisting supergravity simply amounts to choosing a particular vacuum around which to do perturbation theory. In that regard, it may be helpful to think of the following analogy with choosing a vacuum on the Coulomb branch of a supersymmetric gauge theory:

supersymmetric gauge theory	supergravity
G gauge group $\phi \in C^\infty(M,\mathfrak{g})$ scalar in vector multiplet	$\mathfrak{g} = \mathfrak{siso}_{\mathrm{IIB}}$ supersymmetry algebra $q \in C^\infty(M;\Pi S)[1]$ bosonic ghost
Coulomb branch $\phi_0 \in \mathfrak{g}$ or $\phi_0 \in C^\infty_{\mathrm{flat}}(M,\mathfrak{g})$ ask $\phi = \phi_0$	**twisted supergravity** $Q \in \Pi S \subset \mathfrak{siso}_{\mathrm{IIB}}$ or $d_Q \in C^\infty_{\mathrm{cov}}(M;\Pi S)[1]$ ask $q = d_Q$
broken gauge group $\mathrm{Stab}_G(\phi_0)$	broken SUSY algebra $\mathrm{Stab}_{\mathfrak{siso}_{\mathrm{IIB}}}(Q) := H^\bullet(\mathfrak{siso}_{\mathrm{IIB}}, Q)$

Here the subscript "flat" means that we take flat sections of a connection, on the background G-bundle, induced by the metric on M.

It is not essential that the gauge theory be supersymmetric to make the above analogy. To clarify this, let us provide a different analogy:

gauge theory with super gauge group	supergravity
$G = \mathrm{GL}(N\vert N)$ gauge group $\phi \in C^\infty(M,\Pi\mathfrak{gl}(N))[1]$ bosonic ghost	$\mathfrak{g} = \mathfrak{siso}_{\mathrm{IIB}}$ supersymmetry algebra $q \in C^\infty(M;\Pi S)[1]$ bosonic ghost
twisted gauge theory $\phi_0 \in \Pi\mathfrak{gl}(N) \subset \mathfrak{gl}(N\vert N)$ or $\phi_0 \in C^\infty_{\mathrm{flat}}(M,\Pi\mathfrak{gl}(N))$ ask $\phi = \phi_0$	**twisted supergravity** $Q \in \Pi S \subset \mathfrak{siso}_{\mathrm{IIB}}$ or $d_Q \in C^\infty_{\mathrm{cov}}(M;\Pi S)[1]$ ask $q = d_Q$
broken gauge group $\mathrm{Stab}_G(\phi_0)$	broken SUSY algebra $\mathrm{Stab}_{\mathfrak{siso}_{\mathrm{IIB}}}(Q) := H^\bullet(\mathfrak{siso}_{\mathrm{IIB}}, Q)$

Note that one cannot assign a non-zero vacuum expectation value to a fermionic element. However, if we have a fermionic component of an algebra, then its ghost is fermionic in the cohomological grading as well. This gives a bosonic ghost which can admit a non-zero vacuum expectation value so we can ask a field to be at that vacuum. The upshot is that a twist of supergravity by d_Q has residual supersymmetry action of $H^\bullet(\mathfrak{siso}_{\mathrm{IIB}}, Q)$. Moreover, by construction of supergravity theory, $H^\bullet(\mathfrak{siso}_{\mathrm{IIB}}, Q)$ should arise as fields of twisted supergravity. This idea is discussed in more detail in Subsection 2.3.3.

2.2.3 Gravitational Backgrounds from Twisting Homomorphisms

In this subsection we wish to relate twisted supergravity with the familiar procedure for twisting supersymmetric field theories. The main claim is that from a twisting homomorphism of a supersymmetric field theory, one can construct a twisted supergravity background such that coupling the supersymmetric field theory to the given background has the effect of twisting the supersymmetric field theory. That a world-volume theory of D-branes on a curved (non-twisted) supergravity background should be twisted was already known [BSV96].

Let us begin by outlining the general procedure. Given a square zero supercharge Q for a supersymmetric field theory in dimension n, let us consider the stabilizer subgroup $G(Q) := \mathrm{Stab}_{\mathrm{Spin}(n)\times G_R}(Q)$ of $\mathrm{Spin}(n) \times G_R$, that is, the largest subgroup for which Q is invariant. In practice, there exists a subgroup $H \subset \mathrm{Spin}(n)$ and a homomorphism $\rho\colon H \to G_R$ such that its graph is the group $G(Q)$. In this case, we call ρ a twisting homomorphism. The theory obtained by twisting with Q can then be defined on manifolds whose structure group is contained in H.

Now, let us restrict to the setting of world-volume theories of D-branes on type IIB string theory, namely, those with maximal supersymmetry. In this case, it is known that as we consider D_{2k-1}-brane on $M^n = M^{2k}$, the R-symmetry group is $G_R = \mathrm{Spin}(10-2k)$. Suppose the structure group of M is contained in H and let V denote the vector representation of the R-symmetry group G_R. We construct a supergravity background with the data of (M, ρ, V) as follows. Fixing a spin structure on M and using the assumption that the structure group of M is H, let P denote the H-reduction of the spin frame bundle of M. The claim is that the desired supergravity background is the 10-manifold $X = \mathrm{Tot}(P \times_H V)$, the total space of the V-bundle associated to P via ρ, together with a Calabi–Yau metric and a covariantly constant spinor. The existence of covariantly constant spinors is guaranteed by the Calabi–Yau structure if it exists, but as far as we are aware, it must be shown on a case-by-case basis that such a 10-manifold is Calabi–Yau.

We now carry out this procedure for the case of the Kapustin (or holomorphic-topological) twist of 4-dimensional $\mathcal{N} = 4$ gauge theory to identify the supergravity background. The twist will be invariant under a subgroup $\mathrm{Spin}(2) \times \mathrm{Spin}(2) \hookrightarrow \mathrm{Spin}(4)$; we identify this subgroup as $H = \mathrm{U}(1) \times \mathrm{U}(1)$ and think of a spin representation of $\mathrm{U}(1)$ to be of weight $\frac{1}{2}$. Then the Kapustin twisting homomorphism is given by

$$\rho\colon\ \mathrm{U}(1)\times\mathrm{U}(1) \to \mathrm{SU}(4), \qquad (\lambda,\mu) \mapsto \begin{pmatrix} \lambda^{1/2}\mu^{1/2} & & & \\ & \lambda^{-1/2}\mu^{1/2} & & \\ & & \lambda^{1/2}\mu^{-1/2} & \\ & & & \lambda^{-1/2}\mu^{-1/2} \end{pmatrix}$$

In the notation of our general procedure above, we have $H = \mathrm{U}(1) \times \mathrm{U}(1)$ so our twisted theory can be formulated on $M = \Sigma_1 \times \Sigma_2$ where the Σ_i are Riemann surfaces. Here, V is the vector representation of $\mathrm{Spin}(6)$. Note that its complexification $V_{\mathbb{C}}$, under the isomorphism $\mathrm{Spin}(6,\mathbb{C}) \cong \mathrm{SL}(4,\mathbb{C})$, is isomorphic to $\wedge^2 \mathbb{C}^4$ where $\mathbb{C}^4$ is the fundamental representation of $\mathrm{SL}(4,\mathbb{C})$. To determine X let us first decompose $V_{\mathbb{C}}$ as a representation of $\mathrm{im}\,\rho$. By the functoriality of restriction, we have that

$$\begin{aligned}\operatorname{Res}^{\operatorname{im}\rho}_{\mathrm{SL}(4,\mathbb{C})} \wedge^2\mathbb{C}^4 &\cong \wedge^2 \operatorname{Res}^{\operatorname{im}\rho}_{\mathrm{SL}(4,\mathbb{C})} \mathbb{C}^4 \\ &\cong \wedge^2(\mathbb{C}_{(\frac{1}{2},\frac{1}{2})} \oplus \mathbb{C}_{(-\frac{1}{2},\frac{1}{2})} \oplus \mathbb{C}_{(\frac{1}{2},-\frac{1}{2})} \oplus \mathbb{C}_{(-\frac{1}{2},-\frac{1}{2})}) \\ &\cong \mathbb{C}_{(0,1)} \oplus \mathbb{C}_{(1,0)} \oplus \mathbb{C}_{(0,0)} \oplus \mathbb{C}_{(0,0)} \oplus \mathbb{C}_{(-1,0)} \oplus \mathbb{C}_{(0,-1)}\end{aligned}$$

The representation $\wedge^2\mathbb{C}^4$ admits a real structure given by the Hodge star operator $\star$ which is a complex conjugate linear map satisfying $\star^2 = \mathrm{Id}$ on $\wedge^2\mathbb{C}^4$. If we write a complex basis of $\mathbb{C}^4$ as e_1, e_2, e_3, e_4, then the six real basis elements are given by

$$\begin{cases} e_1 \wedge e_2 + e_3 \wedge e_4 \\ \sqrt{-1}(e_1 \wedge e_2 - e_3 \wedge e_4) \end{cases} \qquad \begin{cases} e_1 \wedge e_3 + e_4 \wedge e_2 \\ \sqrt{-1}(e_1 \wedge e_3 - e_4 \wedge e_2) \end{cases} \qquad \begin{cases} e_1 \wedge e_4 + e_2 \wedge e_3 \\ \sqrt{-1}(e_1 \wedge e_4 - e_2 \wedge e_3) \end{cases}$$

Now from the $\mathrm{U}(1)$-weights of e_i, one can see that the left two pairs of them make vector representations while the third pair is the trivial representation. Choosing a spin structure on each of Σ_i and letting P denote $K_{\Sigma_1}^{1/2} \oplus K_{\Sigma_2}^{1/2}$, we have that $X = T^*(\Sigma_1 \times \Sigma_2) \times \mathbb{C}$. Let $X^\wedge$ denote $T^*_{\text{form}}(\Sigma_1 \times \Sigma_2) \times \mathbb{C}$ where the subscript is used to denote the formal neighborhood of the zero section. Since $\Sigma_1 \times \Sigma_2$ is Kähler, a result of Kaledin [Kal99] gives that $T^*_{\text{form}}(\Sigma_1 \times \Sigma_2)$ is hyperkähler, hence Calabi–Yau. Then $X^\wedge$ evidently is Calabi–Yau as well. Thus, we have constructed the desired gravitational background. As a consistency check, one observes that sections of the normal bundle of $\Sigma_1 \times \Sigma_2$ in X have spins that agree with those of the six adjoint scalars of 4-dimensional $\mathcal{N} = 4$ gauge theory after performing the Kapustin twist, that is, four 1-forms and two scalars.

2.3 Describing Twisted Supergravity

In the previous subsection, we discussed how one constructs twisted supergravity in some generality with the focus on twists of type IIB. However, given the complexity of 10-dimensional supergravity theories, working out a description of the twisted theory from first principles should be regarded as a difficult research question in and of itself. Work in this direction will appear in (Surya: forthcoming paper of Natalie and Kevin that doesn't have a title to my knoweldge?) Instead, we take a conjectural description of Costello and Li [CL16] as our starting point. Though these conjectures have not been proven at a mathematical level of rigour, several rigorous consistency checks have been performed.

A crucial ingredient in the conjectural description we will employ is *BCOV theory*; we introduce this theory along with some variants that are suited to our purposes in Subsection 2.3.1 below and define twisted supergravity theories in Subsection 2.3.2. We then describe explicitly in Subsection 2.3.3 how ghosts in twisted supergravity appear as residual symmetries in the conjectural descriptions. Finally, in Subsection 2.3.4, we discuss how one may think of the residual symmetries as giving further twists of supergravity theory.

2.3.1 BCOV Theory

The BCOV theory was originally introduced in the seminal work of Bershadsky–Cecotti–Ooguri–Vafa [BCOV94] under the name of Kodaira–Spencer theory of gravity for a Calabi–Yau 3-fold, as a string field theory for the topological B-model that describes deformations of complex structures. It was later generalized by Costello and Li [CL12] for an arbitrary Calabi–Yau manifold.

Let us briefly review the set-up. Let X be a d-dimensional Calabi–Yau manifold with a holomorphic volume form Ω_X. Consider the space $\mathrm{PV}^{i,j}(X) = \Omega^{0,j}(X, \wedge^i T_X)$, where T_X is the holomorphic tangent bundle of X. We introduce the space of polyvector fields on X given by $\mathrm{PV}(X) = \bigoplus_{i,j} \mathrm{PV}^{i,j}(X)$ where the summand $\mathrm{PV}^{i,j}(X)$ lives in cohomological degree $i+j$; accordingly, we write $|\mu| = i+j$ for $\mu \in \mathrm{PV}^{i,j}(X)$. Then $\mathrm{PV}(X)$ is a commutative differential graded algebra with the differential $\bar\partial$ and the product being the wedge product. Contracting with Ω_X yields an identification

$$(-) \vee \Omega_X \colon \mathrm{PV}^{i,j}(X) \cong \Omega^{d-i,j}(X).$$

This allows us to transfer the operator ∂ on $\Omega^{\bullet,\bullet}(X)$ to yield operator ∂ on $\mathrm{PV}^{\bullet,\bullet}(X)$. The result $\partial \colon \mathrm{PV}^{i,j}(X) \to \mathrm{PV}^{i-1,j}(X)$ is the divergence operator with respect to a holomorphic volume form Ω_X; it is a second-order differential operator of cohomological degree -1 so that

$$[\mu,\nu]_{\mathrm{SN}} := (-1)^{|\mu|-1}\Big(\partial(\mu\nu) - (\partial\mu)\nu - (-1)^{|\mu|}\mu(\partial\nu)\Big)$$

defines a Poisson bracket of degree -1. Here the sign factor is necessary to precisely match with the Schouten–Nijenhuis bracket, explaining the notation. It satisfies a shifted version of graded Lie algebra axioms in that the following hold:

$$[\alpha,\beta]_{\mathrm{SN}} = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta,\alpha]_{\mathrm{SN}}$$
$$[\alpha,[\beta,\gamma]_{\mathrm{SN}}]_{\mathrm{SN}} = [[\alpha,\beta]_{\mathrm{SN}},\gamma]_{\mathrm{SN}} + (-1)^{(|\alpha|-1)(|\beta|-1)}[\beta,[\alpha,\gamma]_{\mathrm{SN}}]_{\mathrm{SN}}.$$

In fact, $(\mathrm{PV}(X)[1], \bar\partial, [-,-]_{\mathrm{SN}})$ is a differential graded Lie algebra. We also consider the trace map

$$\mathrm{Tr}\colon \mathrm{PV}_c(X) \to \mathbb{C} \qquad \text{given by} \qquad \mu \mapsto \int_X (\mu \vee \Omega_X) \wedge \Omega_X$$

where $\mathrm{PV}_c(X)$ stands for compactly supported sections. Note that the trace map is non-trivial only on $\mathrm{PV}_c^{d,d}(X)$.

Consider $\mathrm{PV}(X)[2]$ so that the summand $\mathrm{PV}^{1,1}(X)$ is in degree 0. This part of the fundamental fields of the theory govern deformations of the complex structure on X, which is the reason why it was originally called Kodaira–Spencer gravity. As motivated in Subsection 2.1.2 and in particular Remark 2.4, we use $\mathcal{E}_{\mathrm{BCOV}}(X)$ to to denote the (shifted) cyclic L_∞-algebra

$$\mathcal{E}_{\mathrm{BCOV}}(X) = ((\ker\partial)(X) \subset \mathrm{PV}(X)[2]; \bar\partial, [-,-]_{\mathrm{SN}}; \mathrm{Tr})$$

together with the shifted Poisson structure given by the kernel $(\partial \otimes 1)\delta_{\mathrm{diag}}$. We sometimes write $\ker\partial = (\ker\partial)(X) \subset \mathrm{PV}(X)[2]$.

Note that the Poisson structure pairs components of $\mathrm{PV}^{0,\bullet}(X)$ pair with those of $\mathrm{PV}^{d-1,\bullet}(X)$, and more generally, $\mathrm{PV}^{i,\bullet}(X)$ with $\mathrm{PV}^{d-i-1,\bullet}(X)$. This suggests that $\oplus_{i\le d-1}\mathrm{PV}^{i,\bullet}(X)$ are the propagating fields and $\mathrm{PV}^{d,\bullet}(X)$ only plays the role of background fields. This motivates the introduction of *minimal BCOV theory* $\mathcal{E}_{\mathrm{mBCOV}}(X) \subset \mathcal{E}(X)$ by discarding non-propagating fields and restricting to the subspace $\ker\partial|_{\oplus_{i\le d-1}\mathrm{PV}^{i,\bullet}(X)} \subset \ker\partial$.

Remark 2.11. In [CL12], Costello and Li consider a resolution of the complex $(\ker\partial, \bar\partial)$ given by $(\mathrm{PV}(X)[\![t]\!][2], \bar\partial + t\partial)$. Here t is a formal variable of cohomological degree 2, interpreted as giving rise to gravitational descendants. They also introduce an interaction involving infinitely many terms that satisfies the classical master equation. As usual in the BV framework, this information can be encoded by an L_∞-algebra and they show that this is equivalent to the DG Lie algebra $(\mathrm{PV}(X)[\![t]\!][2], \bar\partial + t\partial, [-,-]_{\mathrm{SN}})$, which is what closed string field theory of the B-model should be, as motivated in Subsection 2.1.2. On the other hand, as explained in the introduction, we want to relate this to M-theory or 11-dimensional supergravity where we don't understand the meaning of gravitational descendants. This is one reason why we instead work with the model $(\ker\partial, \bar\partial, [-,-]_{\mathrm{SN}})$ for $\mathcal{E}_{\mathrm{BCOV}}$, and similarly for $\mathcal{E}_{\mathrm{mBCOV}}$, throughout the paper.

Finally, we introduce a modification of minimal BCOV theory. Consider the short exact sequence of cochain complexes

$$0 \longrightarrow \big(\ker \partial \subset \mathrm{PV}^{d,\bullet}(X)\big) \longrightarrow \mathrm{PV}^{d,\bullet}(X) \longrightarrow \big(\operatorname{im} \partial \subset \mathrm{PV}^{d-1,\bullet}(X)\big) \longrightarrow 0.$$

The following definition makes use of the choice of a splitting $C\colon \big(\operatorname{im}\partial \subset \mathrm{PV}^{d-1,\bullet}(X)\big) \to \mathrm{PV}^{d,\bullet}(X)$ or the induced non-canonical isomorphism

$$\phi_C\colon \mathrm{PV}^{d,\bullet}(X) \cong \Big(\operatorname{im}\partial \subset \mathrm{PV}^{d-1,\bullet}(X)\Big) \oplus \Big(\ker\partial \subset \mathrm{PV}^{d,\bullet}(X)\Big).$$

Definition 2.12. For the choice of a splitting $C\colon \big(\operatorname{im}\partial \subset \mathrm{PV}^{d-1,\bullet}(X)\big) \to \mathrm{PV}^{d,\bullet}(X)$, *minimal BCOV theory on $X = X^d$ with potential C* for $d = 2, 3$ is the following data:

- underlying space of fields given by the cochain complex

$$\mathcal{E}^{C}_{\mathrm{mBCOV}}(X) :=$$
$$\left(\bigoplus_{i \le d-2} \mathrm{PV}^{i,\bullet}(X) \cap \mathcal{E}_{\mathrm{mBCOV}}(X)\right) \oplus \Big((\operatorname{im}\partial \subset \mathrm{PV}^{d-1,\bullet}(X)) \oplus (\ker\partial \subset \mathrm{PV}^{d,\bullet}(X))\Big)[3-d]$$

- the shifted Poisson structure given by the kernel $(\partial \otimes 1)\delta_{\mathrm{diag}}$;
- the shifted L_∞-structure given by $\ell_1 = \bar\partial$, $\ell_2 = [-,-]^C_{\mathrm{SN}}$ and $\ell_{n\ge 3} = 0$, where $[-,-]^C_{\mathrm{SN}}$ is a certain modification of the Schouten–Nijenhuis bracket defined as follows:

  (1) For $\mu \in \mathrm{PV}^{0,\bullet}(X)$ and $\nu \in \operatorname{im}\partial \subset \mathrm{PV}^{d-1,\bullet}(X)$, we define

$$\begin{aligned} [\mu,\nu]^C_{\mathrm{SN}} &:= (-1)^{|\mu|-1}\partial[\mu, \phi_C^{-1}(\nu)]_{\mathrm{SN}} && \in \operatorname{im}\partial \subset \mathrm{PV}^{d-2,\bullet}(X) \\ [\nu,\mu]^C_{\mathrm{SN}} &:= \partial[\phi_C^{-1}(\nu), \mu]_{\mathrm{SN}} && \in \operatorname{im}\partial \subset \mathrm{PV}^{d-2,\bullet}(X) \end{aligned}$$

  (2) For $\mu \in \ker\partial \subset \mathrm{PV}^{1,\bullet}(X)$, $\nu \in (\operatorname{im}\partial \subset \mathrm{PV}^{d-1,\bullet}(X)) \oplus (\ker\partial \subset \mathrm{PV}^{d,\bullet}(X))$, we define

$$\begin{aligned} [\mu,\nu]^C_{\mathrm{SN}} &:= (-1)^{|\mu|-1}\phi_C[\mu, \phi_C^{-1}(\nu)]_{\mathrm{SN}} && \in (\operatorname{im}\partial \subset \mathrm{PV}^{d-1,\bullet}(X)) \oplus (\ker\partial \subset \mathrm{PV}^{d,\bullet}(X)) \\ [\nu,\mu]^C_{\mathrm{SN}} &:= \phi_C[\phi_C^{-1}(\nu), \mu]_{\mathrm{SN}} && \in (\operatorname{im}\partial \subset \mathrm{PV}^{d-1,\bullet}(X)) \oplus (\ker\partial \subset \mathrm{PV}^{d,\bullet}(X)) \end{aligned}$$

  (3) For $\mu, \nu \in \bigoplus_{i\le d-2} \mathrm{PV}^{i,\bullet}(X) \cap \mathcal{E}_{\mathrm{mBCOV}}(X)$, we define $[\mu,\nu]^C_{\mathrm{SN}} := [\mu,\nu]_{\mathrm{SN}}$;

  (4) All other brackets are zero.

  That this indeed defines a (shifted) DG Lie algebra is explained in Lemma 2.14.

In what follows, we may not explicitly denote the dependence on splitting for the ease of notation, but it is a crucial datum that should be fixed once and for all.

Remark 2.13. Before proving that it defines a (shifted) DG Lie algebra, we want to explain where the definition comes from. The definition can be viewed as replacing the summand $\ker\partial \subset \mathrm{PV}^{d-1,\bullet}(X)$ in the original minimal BCOV theory with the summand $\mathrm{PV}^{d,\bullet}(X)$ in the same degree, which we then write as $\big((\operatorname{im}\partial \subset \mathrm{PV}^{d-1,\bullet}(X)) \oplus (\ker\partial \subset \mathrm{PV}^{d,\bullet}(X))\big)$ via the choice of splitting C.

In other words, we may understand the above DG Lie algebra structure as transported from a natural (shifted) DG Lie algebra structure on

$$\left(\bigoplus_{i\leq d-2} \mathrm{PV}^{i,\bullet}(X)\cap \mathcal{E}_{\mathrm{mBCOV}}(X)\right)\oplus \mathrm{PV}^{d,\bullet}(X)[3-d].$$

Let us describe the natural DG Lie algebra structure involving $\mathrm{PV}^{d,\bullet}(X)$. Note that the usual Schouten–Nijenhuis bracket would pair $\mathrm{PV}^{k,\bullet}(X)$ and $\mathrm{PV}^{d,\bullet}(X)$ to give an element of $\mathrm{PV}^{d+k-1,\bullet}(X)$, from which we know only the cases of $k=0,1$ can be nonzero. In general, if $\mu\in\ker\partial\subset\mathrm{PV}^{k,\bullet}(X)$, then we have the identities

$$[\mu,\partial\rho]_{\mathrm{SN}}=(-1)^{|\mu|-1}\partial[\mu,\rho]_{\mathrm{SN}}\qquad\text{and}\qquad \partial[\rho,\mu]_{\mathrm{SN}}=[\partial\rho,\mu]_{\mathrm{SN}}$$

which motivate the definition involving $\operatorname{im}\partial\subset\mathrm{PV}^{d-1,\bullet}(X)$. For (1), given that $\mathrm{PV}^{d,\bullet}(X)$ is placed in the degree as $\mathrm{PV}^{d-1,\bullet}(X)$ would have been in the usual BCOV theory, a nontrivial Lie bracket can only yield an element of $\mathrm{PV}^{d-2,\bullet}(X)$, which explains why the bracket with $\ker\partial\subset\mathrm{PV}^{d,\bullet}(X)$ vanishes. For (2), the bracket with $\ker\partial\subset\mathrm{PV}^{d,\bullet}(X)$ would be the same as the standard Schouten–Nijenhuis bracket except that the additional sign is still introduced because we changed its degree; see the following lemma.

Lemma 2.14. $(\mathcal{E}^C_{\mathrm{mBCOV}}(X);\bar\partial,[-,-]^C_{\mathrm{SN}})$ defines a shifted DG Lie algebra.

Proof. Here is a sketch of a proof for $d=3$, as the case of $d=2$ is similar but simpler. The only care to be taken is the case involving $\mu\in\phi_C(\mathrm{PV}^{d,\bullet}(X))$, as all the others are clear from the usual ones on $\mathrm{PV}(X)$ with the standard Schouten–Nijenhuis bracket.

First, we show that $[-,-]^C_{\mathrm{SN}}$ satisfies the shifted alternating property. Let $\nu\in\ker\partial\subset\mathrm{PV}^{3,\bullet}(X)$. If $\mu\in\mathrm{PV}^{0,\bullet}(X)$, then $[\mu,\nu]^C_{\mathrm{SN}}=0=[\nu,\mu]^C_{\mathrm{SN}}$. If $\mu\in\ker\partial\subset\mathrm{PV}^{1,\bullet}(X)$, then

$$[\mu,\nu]^C_{\mathrm{SN}}=(-1)^{|\mu|-1}[\mu,\nu]_{\mathrm{SN}}=(-1)^{|\mu|-1}(-1)^{(|\mu|-1)(|\nu|-1)+1}[\nu,\mu]_{\mathrm{SN}}=-(-1)^{(|\mu|-1)|\nu|}[\nu,\mu]^C_{\mathrm{SN}}$$

where $|-|$ denotes the usual polyvector degrees: in particular $|\nu|\equiv(\text{degree of }\nu\text{ in }\mathcal{E}^C_{\mathrm{mBCOV}})-1$ (mod 2) and hence the above is the desired alternating property. If $\nu=\partial\rho\in\operatorname{im}\partial\subset\mathrm{PV}^{2,\bullet}(X)$, then one finds

$$[\mu,\partial\rho]^C_{\mathrm{SN}}=(-1)^{|\mu|-1}\partial[\mu,\rho]_{\mathrm{SN}}=(-1)^{|\mu|-1}(-1)^{(|\mu|-1)(|\rho|-1)+1}\partial[\rho,\mu]_{\mathrm{SN}}=-(-1)^{(|\mu|-1)|\rho|}[\partial\rho,\mu]^C_{\mathrm{SN}}.$$

Second, we demonstrate that the Jacobi identity of the modified Schouten–Nijenhuis bracket follows from the Jacobi identity of the ordinary Schouten–Nijenhuis bracket. Let $\alpha\in\mathrm{PV}^{0,\bullet}(X)$, $\beta\in\ker\partial\subset\mathrm{PV}^{1,\bullet}(X)$. If $\gamma\in\ker\partial\subset\mathrm{PV}^{3,\bullet}(X)$, then

$$[\alpha,[\beta,\gamma]^C_{\mathrm{SN}}]^C_{\mathrm{SN}}-[[\alpha,\beta]^C_{\mathrm{SN}},\gamma]^C_{\mathrm{SN}}-(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta,[\alpha,\gamma]^C_{\mathrm{SN}}]^C_{\mathrm{SN}}=0$$

because each term involves the modified bracket between $\mathrm{PV}^{0,\bullet}(X)$ and $\ker\partial\subset\mathrm{PV}^{3,\bullet}(X)$ which is zero. On the other hand, if $\gamma=\partial\delta\in\operatorname{im}\partial\subset\mathrm{PV}^{2,\bullet}(X)$ for $\delta\in\mathrm{PV}^{3,\bullet}(X)$, then one finds

$$\begin{aligned}
&\big(\text{Jacobi identity of }[-,-]^C_{\mathrm{SN}}\text{ for }\alpha,\beta,\gamma\big)\\
&=[\alpha,[\beta,\gamma]^C_{\mathrm{SN}}]^C_{\mathrm{SN}}-[[\alpha,\beta]^C_{\mathrm{SN}},\gamma]^C_{\mathrm{SN}}-(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta,[\alpha,\gamma]^C_{\mathrm{SN}}]^C_{\mathrm{SN}}\\
&=(-1)^{|\beta|-1}[\alpha,\partial[\beta,\delta]_{\mathrm{SN}}]^C_{\mathrm{SN}}-[[\alpha,\beta]_{\mathrm{SN}},\gamma]^C_{\mathrm{SN}}-(-1)^{(|\alpha|-1)|\beta|}[\beta,\partial[\alpha,\delta]_{\mathrm{SN}}]^C_{\mathrm{SN}}\\
&=(-1)^{|\alpha|+|\beta|}\partial[\alpha,[\beta,\delta]_{\mathrm{SN}}]_{\mathrm{SN}}-(-1)^{|\alpha|+|\beta|}\partial[[\alpha,\beta]_{\mathrm{SN}},\delta]_{\mathrm{SN}}+(-1)^{|\alpha||\beta|}\partial[\beta,[\alpha,\gamma]_{\mathrm{SN}}]_{\mathrm{SN}}\\
&=(-1)^{|\alpha|+|\beta|}\partial\Big([\alpha,[\beta,\delta]_{\mathrm{SN}}]_{\mathrm{SN}}-[[\alpha,\beta]_{\mathrm{SN}},\delta]_{\mathrm{SN}}-(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta,[\alpha,\delta]_{\mathrm{SN}}]_{\mathrm{SN}}\Big)\\
&=(-1)^{|\alpha|+|\beta|}\partial\,(\text{Jacobi identity of }[-,-]_{\mathrm{SN}}\text{ for }\alpha,\beta,\delta)\;=\;0.
\end{aligned}$$

□

Remark 2.15. One can be more explicit for each dimension.

(1) In dimension 1, the same definition may be applied with the caveat that (1) should be ignored. Then the structure is completely induced from the one of $\mathrm{PV}^{1,\bullet}(X)$. However, the physical relevance of this theory is not clear in view of Remark 2.16. Hence we formally define $\mathcal{E}^C_{\mathrm{mBCOV}}(X) = \mathcal{E}_{\mathrm{mBCOV}}(X)$ when X is a 1-dimensional Calabi–Yau manifold.

(2) In dimension 2, one may heuristically think of this theory as (non-degenerate) BV theory. Recall that if we have usual BV theory described by a shifted DG Lie algebra $\mathcal{E}$ together with a (-1)-shifted symplectic pairing ω, then the corresponding action functional would be

$$S(\phi) = \int \frac{1}{2}\omega(\phi, \ell_1\phi) + \frac{1}{6}\omega(\phi, \ell_2(\phi,\phi)), \qquad \phi \in \mathcal{E}.$$

If we formally insert things in our case, with $\omega(-,-)$ understood as $\mathrm{Tr}((-)\wedge \partial^{-1}(-))$, then we would obtain

$$S(\phi) = \mathrm{Tr}\left(\frac{1}{2}\phi\wedge\bar\partial\,\partial^{-1}\phi + \frac{1}{6}\phi\wedge\partial^{-1}[\phi,\phi]^C_{\mathrm{SN}}\right), \qquad \phi\in\mathcal{E}^C_{\mathrm{mBCOV}}.$$

Now in view of Remark 2.13, we may identify the space of fields as $\mathrm{PV}^{0,\bullet}(X)[2]\oplus\mathrm{PV}^{2,\bullet}(X)[1]$, where an element of $\operatorname{im}\partial \subset \mathrm{PV}^{1,\bullet}(X)$ is now identified with an element of $\mathrm{PV}^{2,\bullet}(X)$ under ϕ_C. Then for a degree reason the action functional would become of the form

$$S(\phi) = \mathrm{Tr}\left(\alpha\wedge\bar\partial\,\beta + \frac{1}{2}\alpha\wedge[\beta,\partial\beta]_{\mathrm{SN}}\right), \qquad \alpha\in\mathrm{PV}^{0,\bullet}(X),\ \beta\in\mathrm{PV}^{2,\bullet}(X).$$

In other words, this becomes a $\mathbb{Z}/2$-graded BV theory described by a shifted DG Lie algebra $\mathcal{E} = \mathrm{PV}^{0,\bullet}(X)[2]\oplus\mathrm{PV}^{2,\bullet}(X)[1]$ with $\ell_1 = \bar\partial$, $\ell_2 = [-,\partial(-)]_{\mathrm{SN}}$, and the odd symplectic pairing given by the trace pairing. Moreover, one may note $[\beta_1,\partial\beta_2]_{\mathrm{SN}}\vee\Omega_X = \{\beta_1\vee\Omega_X, \beta_2\vee\Omega_X\}$ for $\beta_i\in\mathrm{PV}^{2,\bullet}(X)$ where $\{-,-\}$ is the holomorphic Poisson bracket obtained from the Calabi–Yau structure Ω_X on X. This observation will be used in Remark 3.5 when we compare 11d supergravity and type IIA supergravity under a circle reduction.

(3) In dimension 3, this theory is closely related to a version of BCOV theory as described in [CG18] in the context of twisted holography. For more discussion on this point and how our S-duality map intertwines with their description, we refer to Remark 3.20.

Remark 2.16. If $d > 3$, the above definition as given fails to be a (shifted) DG Lie algebra. Hence in what follows, by minimal BCOV theory on $X = X^d$ with potential C for $d > 3$, we only mean the underlying cochain complex (with the shifted Poisson structure) without any further operations. This is enough for the purpose of the current paper. On the other hand, the discussion above should be useful motivation in formulating the correct DG Lie algebra structure for $d > 3$; hence, we phrased the construction more generally despite only applying it to $d = 2, 3$.

To understand why we shouldn't expect the above definition to be meaningful for $d > 3$, and to get a feel for the necessary ingredients of a more general definition, it helps to identify the physical meaning of this modification. As explained in detail in Subsection 2.4.3 (see also Remark 2.26), we interpret this choice of splitting C as a choice of a potential for certain Ramond–Ramond (RR) field strengths when $d = 5$. This suggests that one should think of the minimal BCOV theory as twisted supergravity theory without any choice of RR forms and our definition as with a fixed choice of a potential. On the other hand, if $d > 3$, then for the democratic formulation of superstring/supergravity theory, one needs to make more choices of potentials. In particular, when $d = 4, 5$, one should make a choice of a potential in two summands, say splittings C_1 in degree d_1 and C_2 in degree d_2 such that $d_1 + d_2 \neq d - 1$. The last condition exactly corresponds to the fact that one cannot simultaneously make a choice of potentials for field strengths that are electro-magnetic dual to each other.

Note that when $X = \mathbb{C}^d$ is a flat space, we have $\ker \partial = \operatorname{im} \partial$ and hence

$$\mathcal{E}^C_{\mathrm{mBCOV}}(\mathbb{C}^d) \cong \mathcal{E}_{\mathrm{mBCOV}}(\mathbb{C}^d) \oplus (\ker \partial \subset \mathrm{PV}^{d,\bullet}(\mathbb{C}^d))[3-d].$$

Again, $\ker \partial \subset \mathrm{PV}^{d,\bullet}(\mathbb{C}^d)$ is placed with the same cohomological degree as $\ker \partial \subset \mathrm{PV}^{d-1,\bullet}(\mathbb{C}^d)$.

2.3.2 Definition of Closed String Field Theory and Twisted Supergravity

We now turn to giving descriptions of twisted closed string field theory and twisted supergravity.

Definition 2.17.

- The closed string field theory for the SU(5)-invariant twist of type IIB superstring theory on $\mathbb{C}^5$ is $\mathrm{IIB}_{\mathrm{cl}}[\mathbb{C}^5_B] := \mathcal{E}_{\mathrm{BCOV}}(\mathbb{C}^5)$.
- The closed string field theory for the SU(4)-invariant twist of type IIA superstring theory on $\mathbb{R}^2 \times \mathbb{C}^4$ is $\mathrm{IIA}_{\mathrm{cl}}[\mathbb{R}^2_A \times \mathbb{C}^4_B] := (\Omega^\bullet(\mathbb{R}^2), d) \otimes \mathcal{E}_{\mathrm{BCOV}}(\mathbb{C}^4)$.

Remark 2.18. These definitions are provided as a conjectural description of a twist of string theory in [CL16]. At the moment, it seems impossible to make a precise mathematical definition of string theory and hence the conjecture isn't mathematically posed. Therefore, we decide to take their conjectures as definitions and hence the starting point of our mathematical discussion. We emphasize that these conjectures have passed several consistency checks at a physical level of rigour.

Remark 2.19. One may consider the closed string field theory for a more general background. As explained in Subsection 2.1.2, nominally, one would hope to recover these as $\mathrm{Cyc}^\bullet(\mathcal{C})[2]$ for a topological string theory described by the Calabi–Yau category $\mathcal{C}$, that is, the cyclic L_∞-algebra structure on $\mathrm{Cyc}^\bullet(\mathcal{C})[1]$ from the Calabi–Yau structure together with a shifted Poisson structure. On the other hand, on a non-flat space, there is no local model for the A-model that captures the non-perturbative effects. Because we don't need such a general situation in the following discussion, we are content with the above definition.

Moreover, supergravity is supposed to be a theory of low-energy limit of closed string field theory where we don't see non-propagating fields in the B-model and non-perturbative information in the A-model. This suggests the following definitions which were also stated as conjectures in [CL16] (modulo our modification of using $\mathcal{E}^C_{\mathrm{mBCOV}}$ instead of $\mathcal{E}_{\mathrm{mBCOV}}$). Note that we discussed in Section 2.2 the constructions underlying these conjectural definitions.

Definition 2.20.

- Twists of type IIB supergravity on $M_A \times X_B$ for a symplectic $(8-4n)$-manifold M and a Calabi–Yau $(2n+1)$-fold X are of the form $\mathrm{IIB}_{\mathrm{SUGRA}}[M_A \times X_B] := (\Omega^\bullet(M), d) \otimes \mathcal{E}^C_{\mathrm{mBCOV}}(X)$.
- Twists of type IIA supergravity on $M_A \times X_B$ for a symplectic $(10-4n)$-manifold M and a Calabi–Yau $2n$-fold X are of the form $\mathrm{IIA}_{\mathrm{SUGRA}}[M_A \times X_B] := (\Omega^\bullet(M), d) \otimes \mathcal{E}^C_{\mathrm{mBCOV}}(X)$.

We will sometimes refer to the twist of IIB on X^5_B as *minimal twists*.

In other words, the twist of type II supergravity theory gives $(\Omega^\bullet(M), d) \otimes \mathcal{E}^C_{\mathrm{mBCOV}}(X)$ as long as M is a symplectic manifold and X is a Calabi–Yau manifold, but it does depend on whether it is of type IIA or IIB to see which backgrounds are allowed.

Remark 2.21. From the above conjectural descriptions, the main difference between twists of closed string field theory and supergravity theory is exactly given by background fields of the BCOV theory. Then it is a natural question to ask how to interpret those backgrounds fields. Partly motivated by this question, in a recent paper of the second author with W. He, S. Li, and X. Tang [HLTY19], it is suggested that those background fields should be understood as symmetry algebra in the BV framework. In some special case, the corresponding current observables turn out to yield infinitely many mutually commuting Hamiltonians of a dispersionless integrable hierarchy.

2.3.3 Residual Symmetries

In this subsection, we want to argue that our definition is reasonable by showing that $H^\bullet(\mathfrak{siso}_{\mathrm{IIB}}, Q)$ gives rise to fields of the twisted theory as motivated in Subsection 2.2.2.

In fact, in [CL16] a much stronger claim is argued on physical grounds: the fields of the SU(5)-invariant twist of type IIB supergravity map to the fields of BCOV theory. The key ingredient in the argument is that any theory that can be coupled to holomorphic Chern–Simons theory must have a map from its space of fields to the space of fields of BCOV theory; granting this claim, we must argue that twisted supergravity couples to holomorphic Chern–Simons theory. By a result of Baulieu [Bau11], holomorphic Chern–Simons theory is the minimal twist of the theory living on a D9 brane in type IIB string theory. One expects by string theory arguments that twisted supergravity should couple to this twist of the theory on a D9 brane just as supergravity couples to the world-volume theory of any supersymmetric D-brane in the physical string theory. Under this map, the fields of SU(5)-invariant twisted supergravity on $\mathbb{C}^5$ as defined in the previous subsection conjecturally map to the space of fields for a version of BCOV theory that we have called $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5_B]$. Likewise, one expects that the fields of further twists of IIB supergravity map to the theories $\mathrm{IIB}_{\mathrm{SUGRA}}[X_A \times Y_B]$ as we defined in Definition 2.20.

The above-mentioned map is difficult to describe mathematically, let alone explicitly. However, we will be able to describe it explicitly when restricted to a particular subspace of the fields of twisted supergravity. Consider a twisted supergravity background defined by specifying that the bosonic ghost takes constant value $q = d_Q$ for an odd $Q \in \mathfrak{siso}_{\mathrm{IIB}}$ such that $Q^2 = 0$. The d_Q-cohomology $H^\bullet(C^\infty(\mathbb{R}^{10}, \mathfrak{siso}_{\mathrm{IIB}}); d_Q)$ then describes precisely those gauge transformations which preserve the given background – these will appear as ghosts in the twisted theory. In particular restricting the putative map from the previous paragraph in the case of the SU(5)-invariant twist should yield a cochain map

$$\Xi\colon H^\bullet(\mathfrak{siso}_{\mathrm{IIB}}, Q) \to \mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5_B]$$

from the constant gauge transformations preserving the given background to the space of the closed string fields.

Let us explicitly describe the map Ξ. Fixing $\mathfrak{sl}(5, \mathbb{C}) \subset \mathfrak{so}(10, \mathbb{C})$, we wish to identify $Q_5 \in S_+ \otimes \mathbb{C}^2$ that are $\mathfrak{sl}(5, \mathbb{C})$-invariant and square to zero. Letting V_5 denote the fundamental representation of $\mathfrak{sl}(5, \mathbb{C})$, note that we have the following isomorphisms of $\mathfrak{sl}(5, \mathbb{C})$-representations:

- $\mathbb{C}^{10} \cong V_5 \oplus V_5^*$

- $S_+ \cong \mathbb{C} \oplus \wedge^2 V_5 \oplus \wedge^4 V_5 \cong \mathbb{C} \oplus \wedge^2 V_5 \oplus V_5^*$

where the last equality uses the $\mathfrak{sl}(5,\mathbb{C})$-invariant perfect pairing $V_5 \otimes \wedge^4 V_5 \to \wedge^5 V_5 \cong \mathbb{C}$.

Let $\Psi \in S_+$ be an element of the trivial summand. We choose an inner product $\langle -,-\rangle$ on $\mathbb{C}^2$ and orthonormal basis e_1, e_2 for $\mathbb{C}^2$. Then consider $Q_5 = \Psi \otimes e_1 \in S_+ \otimes \mathbb{C}^2$; this is clearly $\mathfrak{sl}(5,\mathbb{C})$-invariant. Let us check that it squares to zero. Note that the odd bracket on $\mathfrak{siso}_{\mathrm{IIB}}$ is a map of $\mathfrak{so}(10,\mathbb{C})$-representations, so by the functoriality of restriction, it must be a map of $\mathfrak{sl}(5,\mathbb{C})$-representations. Therefore, we see that $[Q_5,-]$ maps the $V_5^* \otimes \mathbb{C}e_1$ summand of $S_+ \otimes \mathbb{C}^2 \cong S_+ \otimes \mathbb{C}e_1 \oplus S_+ \otimes \mathbb{C}e_2$ isomorphically onto the $\wedge^4 V_5$ and kills all other summands of $S_+ \otimes \mathbb{C}^2$. This shows that Q_5 is in fact square-zero.

Now we wish to compute the cohomology of the complex

$$\mathfrak{so}(10,\mathbb{C}) \xrightarrow{[Q_5,-]} S_+ \oplus S_+ \xrightarrow{[Q_5,-]} V_5 \oplus V_5^* .$$

This is straightforward:

- As remarked above, $\mathrm{im}[Q_5,-] \subset \mathbb{C}^{10}$ is precisely V_5^*.
- Also from above, we see that $\ker[Q_5,-] \subset S_+ \oplus S_+$ is $\mathbb{C}^{\oplus 2} \oplus (\wedge^2 V_5)^{\oplus 2} \oplus V_5^*$. Furthermore, using the decomposition of $S_+ \otimes S_+$ as an $\mathfrak{so}(10,\mathbb{C})$ representation, we see that $\mathrm{im}[Q_5,-] \subset S_+ \otimes S_+$ is $\mathbb{C} \oplus \wedge^2 V_5$.
- By definition, $\ker[Q_5,-] \subset \mathfrak{so}(10,\mathbb{C})$ is just $\mathrm{Stab}(Q_5)$. This is a parabolic subalgebra with Levi factor isomorphic to $\mathfrak{sl}(5,\mathbb{C})$ as Q_5 was chosen to be $\mathfrak{sl}(5,\mathbb{C})$-invariant. The maximal nilpotent ideal in $\mathrm{Stab}(Q_5)$ is seen to be isomorphic to $\wedge^3 V_5$.

In sum we have proven the following:

Lemma 2.22. *$H^\bullet(\mathfrak{siso}_{\mathrm{IIB}}, Q_5)$ is the super Lie algebra with underlying $\mathbb{Z}/2\mathbb{Z}$-graded vector space*

$$\mathfrak{sl}(5,\mathbb{C}) \oplus \wedge^3 V_5 \oplus V_5 \oplus \Pi(\mathbb{C} \oplus \wedge^2 V_5 \oplus V_5^*)$$

where the Lie algebra structure is identified as a semidirect product of $\mathfrak{sl}(5,\mathbb{C}) \oplus \wedge^3 V_5 \subset \mathfrak{so}(10,\mathbb{C})$ with $V_5 \oplus \Pi(\mathbb{C} \oplus \wedge^2 V_5 \oplus V_5^*)$.

Now, we may give an explicit description of the cochain map $\Xi\colon H^\bullet(\mathfrak{siso}_{\mathrm{IIB}}, Q_5) \to \mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5]$. Let us suggestively choose a basis for V given by vectors of the form $\{\partial_i\}_{i=1,\cdots,5}$ and let $\{dz_i\}_{i=1,\cdots,5}$ denote the corresponding dual basis.

One can find the cochain map $\Xi\colon H^\bullet(\mathfrak{siso}_{\mathrm{IIB}}, Q_5) \to \mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5_B]$ as follows:

	$\mathfrak{siso}_{\mathrm{IIB}}$	$H^\bullet(\mathfrak{siso}_{\mathrm{IIB}}, Q_5) \longrightarrow \mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5_B]$
spin	$\mathfrak{so}(10,\mathbb{C})$	$\mathfrak{sl}(5,\mathbb{C}) \oplus \wedge^3 V_5 \longrightarrow \mathrm{PV}^{1,0}(\mathbb{C}^5) \oplus \mathrm{PV}^{3,0}(\mathbb{C}^5)$ $(A_{ij}, \partial_i\partial_j\partial_k) \longmapsto \left(\sum_{i,j} A_{ij} z_i \partial_{z_j}, \partial_{z_i}\partial_{z_j}\partial_{z_k}\right)$
translation	$V_5 \oplus V_5^*$	$V_5 \longrightarrow \mathrm{PV}^{1,0}(\mathbb{C}^5)$ $\partial_i \longmapsto \partial_{z_i}$
SUSY	$\Pi(S_+ \oplus S_+)$	$\Pi(\mathbb{C} \oplus \wedge^2 V_5 \oplus V_5^*) \longrightarrow \mathrm{PV}^{2,0}(\mathbb{C}^5) \oplus \mathrm{PV}^{0,0}(\mathbb{C}^5)$ $(1, \partial_i\partial_j, dz_i) \longmapsto (0, \partial_{z_i}\partial_{z_j}, z_i)$

By definition, square-zero elements of $\Xi(\mathbb{C} \oplus \wedge^2 V_5 \oplus V_5^*) \subset \mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5]$ describe further twists of the SU(5)-invariant twist of twisted supergravity.[2]

[2] An implicit assumption here is that the map res respects the DG Lie algebra structure. As discussed below, it doesn't in general, but it should still respect the bracket that takes two spinors to a vector. We can check that this is true by computing what that bracket looks like as a map of $\mathfrak{sl}(5,\mathbb{C})$ representations.

Remark 2.23. Note that the map Ξ we have described fails to be a map of DG Lie algebras. Indeed, given $\frac{\partial}{\partial z_i}, z_j \in \mathrm{PV}(X)$, we have $[\frac{\partial}{\partial z_i}, z_j]_{\mathrm{SN}} = \delta_{ij}$. However, we have no such bracket in the Q-cohomology of $\mathfrak{siso}_{\mathrm{IIB}}$ between those translations and supersymmetries that survive. To remedy this, [CL16] replace the super-translation algebra $\mathcal{T}^{(2,0)}$ with a certain form-valued central extension. We will introduce a version of this construction, adapted to our setting, in Subsection 3.5 below.

2.3.4 Further Twists by Residual Symmetries

In this subsection, which is independent of other parts of the paper, we explain the idea of further twists of twisted supergravity.

For us, Definition 2.20 gives all the definitions of twisted supergravity we need. On the other hand, one can ask if one can get other twists starting from the minimal twist of IIB supergravity, as well as the twist of IIA supergravity on $M_A^2 \times X_B^4$; that is, just as in the case of supersymmetric field theory, one may further twist a twisted supergravity theory by residual symmetries.

In the previous subsection, it is argued that there exist further twists of $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}_B^5]$ realized by constant coefficient holomorphic Poisson bivectors $\Pi \in \mathrm{PV}^{2,0}(X)$. Let us discuss if adding such a bivector indeed changes the theory as expected.

The following lemma is useful in describing such further twists of either theory.

Lemma 2.24. Let Ω denote the standard holomorphic symplectic on $\mathbb{C}^2$ and let $\Pi = \Omega^{-1}$ be the corresponding holomorphic Poisson bivector. Then the map $\Gamma(\mathbb{C}^2, T_{\mathbb{C}^2}) \cong \Gamma(\mathbb{C}^2, T^*_{\mathbb{C}^2})$ given by $\mu \mapsto \Omega(\mu, -)$ induces a quasi-isomorphism of cochain complexes

$$\big((\ker \partial)(\mathbb{C}^2), \bar\partial + [\Pi, -]_{\mathrm{SN}}\big) \cong \big(\Omega^\bullet(\mathbb{C}^2), d\big)$$

Proof. First, we note that the above map gives

$$(\mathrm{PV}^{\bullet,\bullet}(\mathbb{C}^2), \bar\partial) \cong (\Omega^{\bullet,\bullet}(\mathbb{C}^2), \bar\partial).$$

Next, we claim that

$$(\mathrm{PV}^{\bullet,\bullet}(\mathbb{C}^2), \bar\partial + [\Pi, -]_{\mathrm{SN}}) \cong (\Omega^{\bullet,\bullet}(\mathbb{C}^2), \bar\partial + \partial) \cong (\Omega^\bullet(\mathbb{C}^2), d)$$

or equivalently, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{PV}^{0,\bullet}(\mathbb{C}^2) \arrow[r, "\mathrm{Id}"] \arrow[d, "{[\Pi,-]_{\mathrm{SN}}}"'] & \Omega^{0,\bullet}(\mathbb{C}^2) \arrow[d, "\partial"] \\
\mathrm{PV}^{1,\bullet}(\mathbb{C}^2) \arrow[r, "\Omega\vee"] \arrow[d, "{[\Pi,-]_{\mathrm{SN}}}"'] & \Omega^{1,\bullet}(\mathbb{C}^2) \arrow[d, "\partial"] \\
\mathrm{PV}^{2,\bullet}(\mathbb{C}^2) \arrow[r, "\Omega^{\otimes 2}\vee"] & \Omega^{2,\bullet}(\mathbb{C}^2)
\end{tikzcd}$$

For concrete computation, we let $\Omega = du \wedge dv$ and hence $\Pi = \partial_u \wedge \partial_v$. For $f \in \mathrm{PV}^{0,\bullet}(\mathbb{C}^2)$, we have

$$\begin{aligned}\Omega([\partial_u \wedge \partial_v, f]_{\mathrm{SN}}, -) &= \Omega(\iota_{df}(\partial_u \wedge \partial_v), -) = (\partial_u f)\Omega(\partial_v, -) - (\partial_v f)\Omega(\partial_u, -) \\ &= (\partial_u f) du + (\partial_v f) dv = \partial f\end{aligned}$$

For $f\partial_u \in \mathrm{PV}^{1,\bullet}(\mathbb{C}^2)$, we have

$$\Omega^{\otimes 2}([\partial_u \wedge \partial_v, f\partial_u]_{\mathrm{SN}}, -) = \Omega^{\otimes 2}(\partial_u f \partial_u \partial_v, -) = \partial_u f du dv = \partial(f dv) = \partial(\Omega(f\partial_u)).$$

and similarly for $f\partial_v$. On $\mathrm{PV}^{2,\bullet}(\mathbb{C}^2)$, it is obvious. The claim is proved.

Finally, we want to argue that the operator ∂ on $\mathrm{PV}^{\bullet,\bullet}(\mathbb{C}^2)$ which we denote by ∂^{PV} to avoid confusion with ∂ on $\Omega^{\bullet,\bullet}(\mathbb{C}^2)$ is null-homotopic and hence $(\ker\partial)(\mathbb{C}^2)$ is still equivalent to $\mathrm{PV}^{\bullet,\bullet}(\mathbb{C}^2)$. We show this on $\Omega^{\bullet,\bullet}(\mathbb{C}^2)$ using the above isomorphism. Namely, consider an operator $\partial^{\mathrm{PV}}_{\Omega} \colon \Omega^{k,\bullet}(\mathbb{C}^2) \to \Omega^{k-1,\bullet}(\mathbb{C}^2)$ corresponding to ∂^{PV}, that is, given by $\alpha \mapsto \Omega^{\otimes(k-1)}(\partial^{\mathrm{PV}}\Pi^{\otimes k}(\alpha, -), -)$. Then one can similarly check that the operator $\partial^{\mathrm{PV}}_{\Omega}$ is made null-homotopic by contracting with Π, that is, $\partial^{\mathrm{PV}}_{\Omega} = [\partial, \iota_\Pi]$ where ι_Π is nontrivial only on $\Omega^{2,\bullet}(\mathbb{C}^2)$. □

From the lemma, it is clear that $(\mathrm{PV}(\mathbb{C}^2), \bar\partial)$, $(\ker\partial, \bar\partial)$, and $(\operatorname{im}\partial, \bar\partial)$ all become equivalent to $(\Omega^\bullet(\mathbb{C}^2), d) \cong (\Omega^\bullet(\mathbb{R}^4), d)$ upon adding $[\Pi, -]_{\mathrm{SN}}$ to their differential.

Now let us argue in a crucial example, why our definition of twisted supergravity for different backgrounds is compatible with Lemma 2.24.

Example 2.25. We would like to think of $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{R}^4_A \times \mathbb{C}^3_B]$ as the further twist of $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5_B]$ where we add $[\Pi, -]_{\mathrm{SN}}$ for $\Pi = \partial_u \wedge \partial_v$.

First, consider how $\mathrm{PV}^{\leq 4,\bullet}(\mathbb{C}^5)$ decomposes into $\mathrm{PV}(\mathbb{C}^2_{u,v}) \otimes \mathrm{PV}(\mathbb{C}^3)$:

$\mathbb{C}^5$	on $\mathbb{C}^2_{u,v} \times \mathbb{C}^3$			
$\mathrm{PV}^{0,\bullet}$	$\mathrm{PV}^{0,\bullet} \otimes \mathrm{PV}^{0,\bullet}$			
$\mathrm{PV}^{1,\bullet}$	$\mathrm{PV}^{1,\bullet} \otimes \mathrm{PV}^{0,\bullet}$	$\mathrm{PV}^{0,\bullet} \otimes \mathrm{PV}^{1,\bullet}$		
$\mathrm{PV}^{2,\bullet}$	$\mathrm{PV}^{2,\bullet} \otimes \mathrm{PV}^{0,\bullet}$	$\mathrm{PV}^{1,\bullet} \otimes \mathrm{PV}^{1,\bullet}$	$\mathrm{PV}^{0,\bullet} \otimes \mathrm{PV}^{2,\bullet}$	
$\mathrm{PV}^{3,\bullet}$		$\mathrm{PV}^{2,\bullet} \otimes \mathrm{PV}^{1,\bullet}$	$\mathrm{PV}^{1,\bullet} \otimes \mathrm{PV}^{2,\bullet}$	$\mathrm{PV}^{0,\bullet} \otimes \mathrm{PV}^{3,\bullet}$
$\mathrm{PV}^{4,\bullet}$			$\mathrm{PV}^{2,\bullet} \otimes \mathrm{PV}^{2,\bullet}$	$\mathrm{PV}^{1,\bullet} \otimes \mathrm{PV}^{3,\bullet}$

As we add $[\Pi, -]_{\mathrm{SN}}$, the above lemma identifies the first three columns for $\mathrm{PV}(\mathbb{C}^2) \otimes \mathrm{PV}(\mathbb{C}^3)$ with $(\Omega^\bullet(\mathbb{R}^4), d) \otimes (\oplus_{i\leq 2} \mathrm{PV}^{i,\bullet}(\mathbb{C}^3))$ and the last column with $(\oplus_{j\leq 1} \mathrm{PV}^{j,\bullet}(\mathbb{C}^2) \otimes \mathrm{PV}^{3,\bullet}(\mathbb{C}^3), [\Pi, -]_{\mathrm{SN}})$.

Then, we consider $\mathcal{E}_{\mathrm{mBCOV}}(\mathbb{C}^5)$ by restricting to $(\ker\partial)(\mathbb{C}^5) = (\operatorname{im}\partial)(\mathbb{C}^5) \subset \mathrm{PV}(\mathbb{C}^5)$. It is clear that the first three columns become $(\Omega^\bullet(\mathbb{R}^4), d) \otimes \mathcal{E}_{\mathrm{mBCOV}}(\mathbb{C}^3)$. The claim is that combined with the additional term $\ker\partial \subset \mathrm{PV}^{5,\bullet}(\mathbb{C}^5)$ coming from the difference between $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5_B] = \mathcal{E}^C_{\mathrm{mBCOV}}(\mathbb{C}^5)$ and $\mathcal{E}_{\mathrm{mBCOV}}(\mathbb{C}^5)$, the last column is cohomologically trivial. To see, let us restrict our attention to $\mathrm{PV}(\mathbb{C}^2)$, because we will uniformly have $\ker\partial \subset \mathrm{PV}^{3,\bullet}$ for $\mathrm{PV}(\mathbb{C}^3)$. The last column provides $\mathrm{PV}^{0,\bullet} \oplus (\operatorname{im}\partial \subset \mathrm{PV}^{1,\bullet})$ and the additional term gives $(\ker\partial \subset \mathrm{PV}^{2,\bullet})$. Now, we note that $\big([\Pi, -]_{\mathrm{SN}} \colon \mathrm{PV}^{0,\bullet} \to (\operatorname{im}\partial \subset \mathrm{PV}^{1,\bullet}) \oplus (\ker\partial \subset \mathrm{PV}^{2,\bullet})\big)$ is identified with $\big((-) \wedge \partial_u \wedge \partial_v \colon \mathrm{PV}^{0,\bullet} \to \mathrm{PV}^{2,\bullet}\big)$ under the splitting in view of Remark 2.13, because $[\partial_u \wedge \partial_v, f]_{\mathrm{SN}} = \partial(f\partial_u \wedge \partial_v)$. In other words, $(\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5_B], [\Pi, -])$ is cohomologically equivalent to $(\Omega^\bullet(\mathbb{R}^4), d) \otimes \mathcal{E}_{\mathrm{mBCOV}}(\mathbb{C}^3)$.

In sum, the further twist of $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5_B]$ gotten by adding $[\Pi, -]$ for $\Pi = \partial_u \wedge \partial_v$ is not exactly $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{R}^4_A \times \mathbb{C}^3_B]$, but a version where $\mathcal{E}_{\mathrm{mBCOV}}(\mathbb{C}^3)$ appears in place of $\mathcal{E}^C_{\mathrm{mBCOV}}(\mathbb{C}^3)$.

Remark 2.26. Although our definition of twisted supergravity is slightly different from the original conjectural description of Costello and Li, up to cohomology we are just adding constant terms. One contribution of this paper is to argue that such a term is necessary to introduce in order for S-duality to act as an automorphism of our theory. This will be justified concretely in Remark 3.20 where we compare the S-duality action suggested by Costello and Gaiotto [CG18] and our S-duality map.

As mentioned in Remark 2.13, replacing $\mathcal{E}_{\mathrm{mBCOV}}$ by $\mathcal{E}^C_{\mathrm{mBCOV}}$ amounts to introducing certain Ramond–Ramond (RR) fields as primitives of RR field strengths (see the table in Subsection 2.4.3

where we define RR fields and field strengths in $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5_B]$ via analogy with physical type IIB supergravity). In view of this, the above example shows that even if one makes a choice of potentials for RR field strengths, such a choice becomes futile after a twist. It is left to a future work to understand that if we made a choice of potentials at two different degrees following Remark 2.16, then one could find a closer relationship between the two supergravity theories. That Lemma 2.24 does not preserve DG Lie algebra structures seems compatible with the expectation.

2.4 Coupling between Open and Closed Sectors

We now discuss the relations between the closed string sector (or supergravity) and open string sector (or D-brane gauge theory). We discuss how an element of the space of closed string states yields a deformation of D-brane gauge theory in Subsection 2.4.1 and how conversely having D-brane yields a closed string field as a flux sourced by the brane in Subsection 2.4.2. To help orient the reader, we have included in Subsection 2.4.3 a comparison of these mechanisms with analogous ones in Maxwell theory and physical supergravity theory.

2.4.1 Closed-Open Map

This subsection is based on [CL16, Subsection 7.2]. For more details, the readers are advised to refer to the original paper.

Consider the closed string field theory for SU(5)-invariant twist of type IIB string theory on X, described by $\mathcal{E}_{\mathrm{BCOV}}(X)$. As we have learned, when we consider D-branes of twisted type IIB string theory, we have a D-brane gauge theory living on them. Physically, whenever we have a BRST closed element of a closed string field theory, it yields a deformation of the D-brane gauge theory since those theories are coupled. This construction is implemented via the closed-open map. In our setting of twisted string theory or topological string theory, this can be understood in a conceptual way because we can consider the category of boundary conditions for the B-model, namely, $\mathrm{Coh}(X)$. The closed-open map then codifies the idea that a deformation of a category should induce deformations of the endomorphisms of every object. The following theorem can be understood as a closed-open map with the universal target $\mathrm{Coh}(X)$.

Theorem 2.27. [Kon03, WC12] Let X be a Calabi–Yau manifold. There is an equivalence of L_∞-algebras $(\mathrm{PV}(X)[\![t]\!][1], \bar{\partial}+t\partial, [-,-]_{\mathrm{SN}}) \to \mathrm{Cyc}^\bullet(\mathrm{Coh}(X))[1]$.

This L_∞-morphism is complicated to describe but fortunately we don't need to keep track of the higher maps for our purpose and will explicitly describe the map after taking cohomology.

The theorem succinctly encodes the coupling information between the closed string field theory and D-brane gauge theory. To see this, first note that for a D-brane $\mathcal{F} \in \mathrm{Coh}(X)$, one always has a map $\mathrm{Hoch}^\bullet(X) \to \mathrm{Hoch}^\bullet(\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{Coh}(X)}(\mathcal{F},\mathcal{F}))$. Identifying $\mathrm{Hoch}^\bullet(X)$ with $\mathrm{PV}(X)$ via the HKR theorem, the Calabi–Yau structure equips $\mathrm{PV}(X)$ with ∂ and $\mathrm{Hoch}^\bullet(\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{Coh}(X)}(\mathcal{F},\mathcal{F}))$ with the Connes B-operator in a compatible way, yielding a map of cochain complexes $\mathrm{PV}(X)[\![t]\!] \to \mathrm{Cyc}^\bullet(\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{Coh}(X)}(\mathcal{F},\mathcal{F}))$. Note that a cyclic cohomology class gives a first-order deformation as an A_∞-algebra with a trace pairing, which precisely gives a deformation of the gauge theory one would construct out of $\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{Coh}(X)}(\mathcal{F},\mathcal{F})$ for an odd Calabi–Yau manifold X. Then, the main content of the theorem is that with care about higher maps, this can be done in such a way that respects formal deformation theory.

As mentioned, our main interest is when X is a flat space and after we take cohomology. For example, consider $X = \mathbb{C}^5$ and D3 branes on $\mathbb{C}^2 \subset \mathbb{C}^5$ so that we obtain $\mathrm{Ext}^\bullet_{\mathrm{Coh}(\mathbb{C}^5)}(\mathcal{O}_{\mathbb{C}^2}, \mathcal{O}_{\mathbb{C}^2}) \cong \mathcal{O}(\mathbb{C}^2)[\varepsilon_1,\varepsilon_2,\varepsilon_3] = \mathcal{O}(\mathbb{C}^{2|3})$. Then after taking cohomology the map of interest is from the canonical map $\mathrm{PV}_{\mathrm{hol}}(\mathbb{C}^5) \to \mathrm{PV}_{\mathrm{hol}}(\mathbb{C}^{2|3})$. Here $\mathrm{PV}_{\mathrm{hol}} = \oplus_k \mathrm{PV}^k_{\mathrm{hol}}$ with $\mathrm{PV}^k_{\mathrm{hol}} = \mathrm{PV}^{k,0}_{\mathrm{hol}} \subset \mathrm{PV}^{k,0}$ is the space of holomorphic polyvector fields, that is, consisting of those which are in the kernel of $\bar{\partial}$. The map is the identity map on $\mathbb{C}^2$, whereas for the normal coordinates w_1, w_2, w_3 of $\mathbb{C}^2 \subset \mathbb{C}^5$, the map is given by a Fourier transform $w_i \mapsto \partial_{\varepsilon_i}$ and $\partial_{w_i} \mapsto \varepsilon_i$. Recall that having a formal parameter t together with additional differential $t\partial$ amounts to considering $\ker \partial$ in our formulation. Note in our model of twisted supergravity theory, we use a modified BCOV theory with additional $\ker \partial \subset \mathrm{PV}^{d,\bullet}(X)$, which should get sent to zero for a degree reason.

In other words, a first-order deformation of a D-brane gauge theory, which should be described by an element of a cyclic cohomology class, can be represented by a closed string state as desired. Note that the number of D-branes does not matter because $\mathfrak{gl}(N)$ is Morita-trivial and Hochschild cohomology is Morita-invariant. This is compatible with the expectation that a deformation given by a closed string state should work for arbitrary N in a uniform way.

Remark 2.28. Nothing in this discussion crucially depends on the fact that X is of dimension 5. Indeed, for our main application, we will consider a theory on $\mathbb{R}^4_A \times \mathbb{C}^3_B$ which corresponds to the case of $X = \mathbb{C}^3$.

2.4.2 Boundary States and Fields Sourced by D-branes

Just as in the physical string theory, branes in the topological string theory also source certain closed string fields that interact with the closed string sector. Mathematically, this means that fixing a D-brane should yield an element of the space of closed string states. Here, we will derive a procedure for computing such an element by examining some constraints on couplings between open and closed string field theories that are forced upon us from TQFT axiomatics.

Consider a mixed A-B topological string theory on $M \times X$ with category of D-branes $\mathcal{C}$ and fix a D-brane $\mathcal{F} \in \mathcal{C}$. To first order, a coupling between the D-brane gauge theory of $\mathcal{F}$ and the closed string field theory is given by a pairing

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{F}) \otimes \mathrm{Cyc}^\bullet(X) \to \mathbb{C}.$$

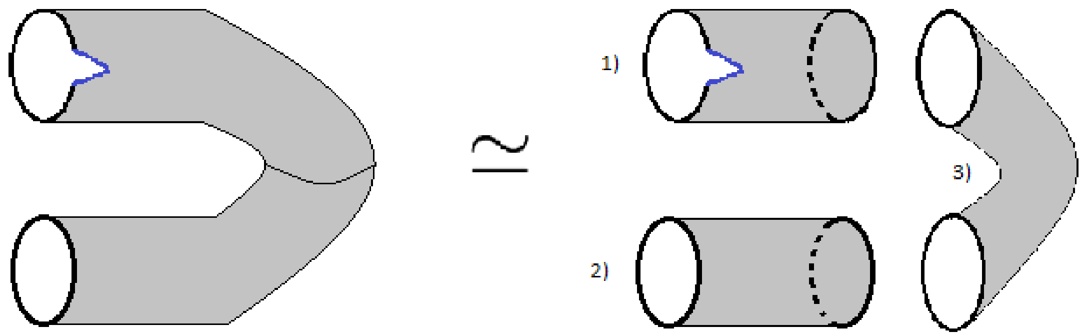
We may equivalently view this as an S^1-invariant map

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{F}) \otimes \mathrm{Hoch}^\bullet(X) \to \mathbb{C}$$

and using the identification between Hochschild chains and Hochschild cochains afforded by the Calabi–Yau structure of $\mathcal{C}$, we have a map

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{F}) \otimes \mathrm{Hoch}_\bullet(X) \to \mathbb{C}.$$

Now this latter map is exactly what the TQFT $Z_{\mathcal{C}}$ assigns to a world-sheet depicting a scattering process where the endpoints of an open string, which are labeled by the brane $\mathcal{F}$, fuse to yield a closed string, which then annihilates with another closed string. This is depicted in the left-hand side of the figure below, and the endpoints of the open string are depicted in blue.



Now, from the functoriality of $Z_{\mathcal{C}}$ with respect to compositions of cobordisms, we may compute $Z_{\mathcal{C}}$ of the left-hand side above, by computing $Z_{\mathcal{C}}$ of each of the pieces of the right-hand side above and composing them appropriately. Accordingly, we have that:

- applying $Z_{\mathcal{C}}$ to the cobordism labeled 1) above yields a map $\partial_{\mathrm{st}} : \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{F}) \to \mathrm{Hoch}_\bullet(X)$. We suggestively denote the image of $\mathrm{Id}_{\mathcal{F}}$ by $\mathrm{ch}(\mathcal{F})$; this is a mathematical codification of the *boundary state* associated to a boundary condition [MS06].
- applying $Z_{\mathcal{C}}$ to the cobordism labeled 2) above yields the identity map $\mathrm{Id}\colon \mathrm{Hoch}_\bullet(X) \to \mathrm{Hoch}_\bullet(X)$.
- applying $Z_{\mathcal{C}}$ to the cobordism labeled 3) above yields a pairing $\mathrm{Tr}\colon \mathrm{Hoch}_\bullet(X) \otimes \mathrm{Hoch}_\bullet(X) \to \mathbb{C}$.

Now composing the above, we see that the cobordism on the left-hand side above yields a map

$$\begin{aligned}\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{F}) \otimes \mathrm{Hoch}_\bullet(X) \to \mathrm{Hoch}_\bullet(X) \otimes \mathrm{Hoch}_\bullet(X) \to \mathbb{C}\\ \mathrm{Id}_{\mathcal{F}} \otimes \alpha \mapsto \mathrm{Tr}(\mathrm{ch}(\mathcal{F}) \otimes \alpha)\end{aligned}$$

Finally, appealing to the Calabi–Yau structure of $\mathcal{C}$ once more and letting Ω denote the isomorphism $\mathrm{Hoch}^\bullet(X) \cong \mathrm{Hoch}_\bullet(X)$, we see that the desired coupling must be a map

$$\begin{aligned}\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{F}) \otimes \mathrm{Hoch}^\bullet(X) \to \mathbb{C}\\ \mathrm{Id}_{\mathcal{F}} \otimes \mu \mapsto \mathrm{Tr}(\mathrm{ch}(\mathcal{F}) \otimes \Omega(\mu)).\end{aligned}$$

For concreteness, let us explicate the above in the case of the topological B-model with target a Calabi–Yau 5-fold X, that is, $\mathcal{C} = \mathrm{Coh}(X)$. In this case the HKR theorem gives us isomorphisms $\mathrm{Hoch}_\bullet(X) = \Omega^{-\bullet}(X)$, $\mathrm{Hoch}^\bullet(X) \cong \mathrm{PV}^{\bullet,\bullet}(X)$, the map ∂_{st} sends $\mathrm{Id}_{\mathcal{F}}$ to the ordinary Chern character of $\mathcal{F}$, the map Ω is given by contracting with the Calabi–Yau form Ω_X, and the pairing Tr is given by wedging and integrating. In sum, the above composition is the map

$$\begin{aligned}\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{Coh}(X)}(\mathcal{F}, \mathcal{F}) \otimes \mathrm{PV}^{\bullet,\bullet}(X) \to \mathbb{C}\\ \mathrm{Id}_{\mathcal{F}} \otimes \mu \mapsto \int_X \mathrm{ch}(\mathcal{F}) \wedge (\mu \vee \Omega_X)\end{aligned}$$

In particular, when $X = \mathbb{C}^5$ and $\mathcal{F} = \mathcal{O}_Y^N$ where $Y \subset \mathbb{C}^5$ is a subvariety of complex dimension k, then one has $\mathrm{ch}(\mathcal{F}) = N\delta_Y$, where δ_Y denotes the $(5-k, 5-k)$-form with distributional coefficients corresponding to the usual δ-function supported on Y. Thus, the right-hand side of the above functional becomes $N\int_Y \mu \vee \Omega_X$.

Now let us incorporate the action of $C_\bullet(S^1)$ on $\mathrm{Hom}(\mathrm{PV}^{\bullet,\bullet}(X), \mathbb{C})$ coming from rotating the closed string, in order to get a ∂-invariant functional. Explicitly, the action is given by precomposing with ∂^{-1}. The image of a map under this action will necessarily be S^1-invariant; thus, the desired functional introduced by the presence of the brane $\mathcal{F} = \mathcal{O}_Y^N$ is given by

$$I(\mu) = N\int_Y \partial^{-1}\mu \vee \Omega_X.$$

We note that this is only non-zero on the $(4-k,k)$-component of μ.

Let us see how considering a D-brane $\mathcal{F} = \mathcal{O}_Y^N$ and its coupling in the above sense modifies the equations of motion. For this purpose, we work with the formulation of BCOV theory without the interaction term; thus, the output of our derivation will be a linear approximation to the actual field sourced by a D-brane. The terms in the action functional involving the $\mu^{4-k,k}$ term are

$$\begin{aligned} &\int_X (\mu^{k,4-k}\,\bar\partial\,\partial^{-1}\mu^{4-k,k} \vee \Omega_X) \wedge \Omega_X + N\int_Y \partial^{-1}\mu^{4-k,k} \vee \Omega_X \\ = -&\int_X (\bar\partial\,\mu^{k,4-k}\partial^{-1}\mu^{4-k,k} \vee \Omega_X) \wedge \Omega_X + N\int_Y \partial^{-1}\mu^{4-k,k} \vee \Omega_X \end{aligned}$$

where in the second line we have integrated by parts. Varying with respect to $\partial^{-1}\mu^{4-k,k}$ yields the equations of motion

$$\bar\partial\,\mu^{k,4-k} \vee \Omega_X = N\delta_Y$$

Here we should think of $\mu^{k,4-k}$ satisfying the equations of motion as the field sourced by $\mathcal{O}_Y^N$ on X.

This motivates the following definition:

Definition 2.29. Let X be a Calabi–Yau variety and $\mathcal{F} \in \mathrm{Coh}(X)$ be a coherent sheaf. *A flux sourced by $\mathcal{F}$ on X* is a choice of a representative $F_\mathcal{F}$ for a class in $\mathrm{Hoch}^\bullet(X) \cong \mathrm{PV}(X)$ trivializing the Chern character $\mathrm{ch}(\mathcal{F}) \vee \Omega_X^{-1}$ and satisfying $\partial F_\mathcal{F} = 0$.

The only example we care about in this paper is the following:

Example 2.30. Let $X = \mathbb{C}^d$, $Y = \mathbb{C}^k \subset \mathbb{C}^d$, and $\mathcal{F} = \mathcal{O}_Y^N$. In this case, one can additionally ask the gauge fixing condition $\bar\partial^*\,F_\mathcal{F} = 0$. It turns out that

$$\bar\partial\,F_\mathcal{F} \vee \Omega_X = N\delta_Y, \qquad \partial F_\mathcal{F} = 0, \qquad \bar\partial^*\,F_\mathcal{F} = 0$$

uniquely characterize a representative, which is the so-called *Bochner–Martinelli kernel* [GH78].

This can be generalized to the case when the normal bundle $N_{X/Y}$ is trivial.

2.4.3 Comparison with Physical Supergravity Theory

Aspects of the above construction may be evocative of a familiar feature from other theories with (higher) gauge fields such as Maxwell theory or physical type II supergravity theories. In such theories, one may consider charged objects as we have done here; such objects source a flux, or field strength. In this subsection, which is independent of other parts of the paper, we explain the analogy with physical supergravity theory and in particular introduce RR forms and RR field strengths in the context of twisted supergravity.

The analogy is summarized in the following table:

	Maxwell theory on M^4	type IIB supergravity on X^{10}	$\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5_B]$
extended objects on worldvolume	particle of charge q $W \subset M^4$	N D_{2k-1}-branes $Y \subset X^{10}$	N D_{2k-1}-branes $\mathbb{C}^k \subset \mathbb{C}^5$
charged under	gauge field $A \in \Omega^1(M^4)$	RR $2k$-form $C^{(2k)} \in \Omega^{2k}(X^{10})$	RR $2k$-form $\mathcal{C}^{(2k)} \in \Omega^{k,k}(\mathbb{C}^5)$ where $\mathcal{C}^{(2k)} := (\partial^{-1}\mu^{4-k,k}) \vee \Omega$ for $\mu^{4-k,k} \in \mathrm{PV}^{4-k,k}(\mathbb{C}^5)$
flux or field strength	$F = dA \in \Omega^2(M^4)$	$G^{(2k+1)} = dC^{(2k)} \in \Omega^{2k+1}$	$d\mathcal{C}^{(2k)} \in \Omega^{2k+1}$ $= (\mathrm{Id} + \bar\partial\,\partial^{-1})\mu^{4-k,k} \vee \Omega$
duality	$F \leftrightarrow *F$	$G^{(10-2k-1)} = *G^{(2k+1)}$	"$\mathcal{G}^{(10-2k-1)} = *\mathcal{G}^{(2k+1)}$"
choice of potentials		if $G^{(2k+1)} = dC^{(2k)}$ $G^{(10-2k-1)} := *G^{(2k+1)}$	if $\mathcal{G}^{(2k+1)} = \bar\partial\partial^{-1}\mu^{4-k,k} \vee \Omega$ $\mathcal{G}^{(10-2k-1)} = \mu^{k,4-k} \vee \Omega$
modified action	$\int_{M^4} F \wedge *F$ $+q\int_W d^{-1}F + \cdots$	$\int_{X^{10}} G^{(2k+1)} \wedge G^{(10-2k-1)}$ $+N\int_Y d^{-1}G^{(2k+1)} + \cdots$	$\int_{\mathbb{C}^5} \mathcal{G}^{(2k+1)} \wedge \mathcal{G}^{(10-2k-1)}$ $+N\int_{\mathbb{C}^k} d^{-1}\mathcal{G}^{(2k+1)} + \cdots$
equation for sourced flux or field strength	$d * F = q\delta_W$	$dG^{(10-2k-1)} = N\delta_Y$	$\bar\partial\,\mu^{k,4-k} \vee \Omega_{\mathbb{C}^5} = N\delta_{\mathbb{C}^k}$

Note that unlike Maxwell theory where we treat A as a fundamental field and its field strength F as a derived notion, in the supergravity setting we regard the field strength itself as an element of the space of fields and call it the flux sourced by the brane. This may seem unusual, but is in accordance with the democractic formulation of string theory where the Ramond–Ramond (RR) field strengths $G^{(2k+1)}$ satisfying the constraint $G^{(2k+1)} = *G^{(10-2k-1)}$ are taken to be fundamental fields and one is free to choose potentials for exactly half of them [Tow95]. That is, when we use $G^{(2k+1)} = dC^{(2k)}$, the RR field strength $G^{(10-2k-1)}$ doesn't admit a description in terms of $C^{(10-2k-2)}$, but it is defined as $G^{(10-2k-1)} := *G^{(2k+1)}$. Physicists sometimes do not a priori make any choice and say that RR $2k$-form $C^{(2k)}$ is electro-magnetic dual to RR $(10-2k-2)$-form $C^{(10-2k-2)}$. In particular, the RR 5-form field strength $G^{(5)}$ is self-dual - this constraint must be dealt with carefully in discussing charge quantization. Once a choice of RR $2k$-form $C^{(2k)}$ is made one can add the term $\int_{Y^{2k}} C^{(2k)}$ to the action functional. It is then colloquial to say that the D_{2k-1}-brane is electrically charged under RR $2k$-form and and magnetically charged under the RR $(10-2k-2)$-form though for $k \neq 2$ only the electric field is part of the space of fields. In particular, we see that D3 branes are both electrically and magnetically charged under the RR 4-form.

In our twisted supergravity setting, in order for the democratic formulation to be respected, one should make a definition of $\mathcal{G}^{(2k+1)}$ and $\mathcal{G}^{(10-2k-1)}$ simultaneously when we choose a potential:

Definition 2.31. For $k \neq 2$, we define a pair of *Ramond–Ramond field strengths* $\mathcal{G}^{(2k+1)}$ and $\mathcal{G}^{(10-2k-1)}$

(1) *with the choice of a Ramond–Ramond* $2k$*-form* $\mathcal{C}^{(2k)} = \partial^{-1}\mu^{4-k,k} \vee \Omega_{\mathbb{C}^5}$ *as a potential*

$$\mathcal{G}^{(2k+1)} = \bar\partial\,\partial^{-1}\mu^{4-k,k} \vee \Omega_{\mathbb{C}^5} \quad \text{and} \quad \mathcal{G}^{(10-2k-1)} = \mu^{k,4-k} \vee \Omega_{\mathbb{C}^5}$$

(2) *with the choice of a Ramond–Ramond* $(10-2k-2)$*-form as a potential* to be

$$\mathcal{G}^{(2k+1)} = \mu^{4-k,k} \vee \Omega_{\mathbb{C}^5} \quad \text{and} \quad \mathcal{G}^{(10-2k-1)} = \bar\partial\,\partial^{-1}\mu^{k,4-k} \vee \Omega_{\mathbb{C}^5}$$

Note that with this definition and interpretation, the free part of the action of BCOV theory, $\int_X (\mu\,\bar\partial\,\partial^{-1}\mu \vee \Omega) \wedge \Omega$, is precisely of the form as expected from type IIB supergravity on X^{10}.

Remark 2.32. The flux or field strength can be integrated on a sphere linking the support of the charged object to yield the charge. Mathematically, the charge is often a characteristic class for the bundle carrying the gauge field, and charge quantization amounts to an integrality condition on the characteristic class [Fre02].

3 Twisted S-duality for $\mathrm{IIB}_{\mathrm{SUGRA}}[M^4_A \times X^3_B]$

3.1 T-duality

In this subsection, we explain a version of the usual T-duality between type IIA and type IIB superstring theories that holds in the protected sectors we have defined.

The idea of T-duality is simple; a string cannot detect a difference between a circle of radius r and a circle of radius $1/r$. Therefore, when one considers a string theory defined on a spacetime manifold with a factor of a circle S^1_r of radius r, it may often be identified with a seemingly different theory defined on a spacetime manifold with the circle replaced by a circle $S^1_{1/r}$ of radius $1/r$.

The main claim is that a shadow of the physically well-known T-duality between IIA and IIB string theories holds at the level of twisted closed string field theories. For instance, let us consider topological string theory on $\mathbb{R}^2_A \times X^4_B$. To have a factor of S^1, we consider $T^*S^1 \cong \mathbb{R} \times S^1$ at the place of $\mathbb{R}^2$; this yields topological strings on $(\mathbb{R} \times S^1)_A \times X^4_B$. Now we argue that this is equivalent to topological strings on $(\mathbb{C}^\times \times X^4)_B$. This would follow from an equivalence between A-model on $\mathbb{R} \times S^1$ and B-model on $\mathbb{C}^\times$ (see [AAEKO13]):

$$\mathrm{Fuk}_{\mathcal{W}}(T^*S^1) \simeq \mathrm{Coh}(\mathbb{C}^\times)$$

This is the underlying input for T-duality between topological string theories as we need.

An equivalence of categories induces an identification between Hochschild homologies that intertwines the natural S^1 actions. We summarize this in the following table:

	A-model on T^*S^1	B-model on $\mathbb{C}^\times$
$\mathrm{HH}_\bullet$	$\mathrm{SH}^{-\bullet}(T^*S^1) \cong H_\bullet(LS^1)$ $\cong \mathbb{C}[z, z^{-1}][\varepsilon]$	$\mathrm{HH}_\bullet(\mathbb{C}^\times) \cong \Omega^{-\bullet}(\mathbb{C}^\times)$ $\cong \mathbb{C}[z, z^{-1}][\frac{dz}{z}]$
Δ	$[S^1]\colon H_0(LS^1) \to H_1(LS^1)$ $f \mapsto (z\partial_z f)\varepsilon$	$\partial\colon \Omega^0(\mathbb{C}^\times) \to \Omega^{-1}(\mathbb{C}^\times)$ $f \mapsto (z\partial_z f) \cdot \frac{dz}{z}$

where Hochschild homology of wrapped Fukaya category of the cotangent bundle T^*S^1 is identified as symplectic cohomology $\mathrm{SH}^{-\bullet}(T^*S^1)$, or equivalently, homology $H_\bullet(LS^1)$ (see [Abo13]); here $LS^1 \cong S^1 \times \mathbb{Z}$ is the free loop space where $\mathbb{Z}$ encodes the winding number, S^1 encodes the initial position, and we write $\varepsilon = [S^1] \in H_1(S^1)$. Now identifying Hochschild homology and Hochschild cohomology using the Calabi–Yau structure, we obtain an isomorphism between $\mathbb{C}[z, z^{-1}][\varepsilon]$, where we abuse the notation to still write ε for the odd variable, and $\mathrm{PV}_{\mathrm{hol}}(\mathbb{C}^\times) \cong \mathbb{C}[z, z^{-1}][z\partial_z]$ such that the S^1-actions are still preserved. Note that under the identification $\ker\partial \subset \mathrm{PV}^1_{\mathrm{hol}}(\mathbb{C}^\times)$ corresponds to the 1-dimensional space $\mathbb{C}\langle z\partial_z\rangle$ and $\mathrm{im}\,\partial \subset \mathrm{PV}^0_{\mathrm{hol}}(\mathbb{C}^\times)$ corresponds to the space of non-constant Laurent polynomials which we denoteby $\mathbb{C}[z, z^{-1}] \setminus \mathbb{C}$.

We wish to apply this to supergravity, not closed string field theory, and we must take care in doing so. Recall that in our model of supergravity, which in particular doesn't capture any non-perturbative contributions, the fields supported on $(T^*S^1)_A$ yield a tensor factor of $\Omega^\bullet(T^*S^1)$. In particular, this is quasi-isomorphic to $\mathbb{C}[\varepsilon]$, but does not detect the non-constant part of $\mathbb{C}[z, z^{-1}]$. This means that on the B-side, we don't see how T-duality acts on a non-constant element of $\mathrm{PV}_{\mathrm{hol}}(\mathbb{C}^\times)$ at the level of supergravity. This point will be relevant in a later discussion (See Remark 3.14).

Remark 3.1. The previous paragraph can be explained in the following way as well. T-duality is supposed to exchange momentum and winding modes. As the $\bar{\partial}$-cohomology of $\mathrm{PV}(\mathbb{C}^\times)$ is identified as $\mathbb{C}[z, z^{-1}][z\partial_z]$, the minimal BCOV part $\mathrm{PV}^{0,\bullet}(\mathbb{C}^\times)$, which is low-energy limit, gives $\mathbb{C}[z, z^{-1}]$. On the other hand, for the A-side, given $H_\bullet(LS^1) \cong H_\bullet(S^1) \otimes H_\bullet(\mathbb{Z})$, the low-energy states can be thought of as where the winding number is zero, namely, $H_\bullet(S^1) \cong \mathbb{C}[\varepsilon]$. This is summarized as follows:

	A-model on T^*S^1	B-model on $\mathbb{C}^\times$
$\mathrm{HH}^\bullet$	$\mathbb{C}[z, z^{-1}] \otimes \mathbb{C}[\varepsilon]$	$\mathrm{PV}_{\mathrm{hol}}(\mathbb{C}^\times) \cong \mathbb{C}[z, z^{-1}] \otimes \mathbb{C}[z\partial_z]$
low-energy	$\mathbb{C}[\varepsilon]$	$\mathrm{PV}^0_{\mathrm{hol}}(\mathbb{C}^\times) \cong \mathbb{C}[z, z^{-1}]$
winding	$\mathbb{C}[z, z^{-1}]$	$\mathbb{C}[z\partial_z]$

From this point of view, once we start to discuss supergravity theory in the A-direction, as we only capture the low-energy part of the A-model, we cannot find the T-dual image of the low-energy part of $\mathrm{PV}^{0,\bullet}(\mathbb{C}^\times)$, namely, non-constant functions.

Below, we will consider T-duality between $\mathrm{IIA}_{\mathrm{SUGRA}}[(M^4 \times T^*S^1)_A \times X^2_B]$ and $\mathrm{IIB}_{\mathrm{SUGRA}}[M^4_A \times (\mathbb{C}^\times \times X^2)_B]$. Note that the above discussion involving BCOV and the minimal BCOV theory cannot be directly applied in this situation because our model for supergravity theories is based on the extended minimal BCOV theory $\mathcal{E}^C_{\mathrm{mBCOV}}$. Hence, we would need to find a correspondence between parts of $H_\bullet(LS^1) \otimes \mathcal{E}^C_{\mathrm{mBCOV}}(X)$ and $\mathcal{E}^C_{\mathrm{mBCOV}}(\mathbb{C}^\times \times X)$. Because $\mathcal{E}^C_{\mathrm{mBCOV}}(\mathbb{C}^\times \times X)$ is the one that plays the essential role in our story, we formulate the correspondence by matching up the terms of $\mathcal{E}^C_{\mathrm{mBCOV}}(\mathbb{C}^\times \times X)$ with the terms of $\mathcal{E}^C_{\mathrm{mBCOV}}(X)$ in the following manner:

Proposition 3.2. Suppose X is a Calabi–Yau 2-fold such that the natural map $(\mathrm{im}\, \partial \subset \mathrm{PV}^{1,\bullet}(X), \bar{\partial}) \to (\ker \partial \subset \mathrm{PV}^{1,\bullet}(X), \bar{\partial})$ is a quasi-isomorphism. The space of fields of extended minimal BCOV theory on $\mathbb{C}^\times \times X^2$

$$\mathcal{E}^C_{\mathrm{mBCOV}}(\mathbb{C}^\times \times X) = \mathrm{PV}^{0,\bullet} \oplus (\ker \partial \subset \mathrm{PV}^{1,\bullet}) \oplus (\mathrm{im}\, \partial \subset \mathrm{PV}^{2,\bullet}) \oplus (\ker \partial \subset \mathrm{PV}^{3,\bullet})$$

is in correspondence, upon taking $\bar{\partial}$-cohomology on the first factor, with the subset of

$$H_\bullet(LS^1) \otimes \mathcal{E}^C_{\mathrm{mBCOV}}(X) = \mathbb{C}[z, z^{-1}][\varepsilon] \otimes \big(\mathrm{PV}^{0,\bullet}(X) \oplus (\mathrm{im}\, \partial \subset \mathrm{PV}^{1,\bullet}(X)) \oplus (\ker \partial \subset \mathrm{PV}^{2,\bullet}(X))\big)$$

consisting of a direct sum of the following three terms:

(1) $\big(\mathbb{C}[z, z^{-1}] \oplus \mathbb{C}\langle\varepsilon\rangle\big) \otimes \mathrm{PV}^{0,\bullet}(X)$;

(2) $\big(\mathbb{C}[z, z^{-1}] \oplus \mathbb{C}\langle\varepsilon\rangle\big) \otimes \big(\mathrm{im}\, \partial \subset \mathrm{PV}^{1,\bullet}(X)\big)$;

(3) $\big(\mathbb{C}[z, z^{-1}]\langle\varepsilon\rangle\big) \otimes \big(\ker \partial \subset \mathrm{PV}^{2,\bullet}(X)\big)$.

Note that if $X = \mathbb{C}^2$ or if X is compact so that the Hodge-to-de Rham spectral sequence degenerates, then the condition is satisfied.

Proof. In this proof, we use tensor product of nuclear spaces as, for example, explained in [Cos11a, Appendix 2]. The crucial property we use is that for manifolds M, N, $C^\infty(M)\otimes C^\infty(N) = C^\infty(M\times N)$ and a similar statement holds for sections of vector bundles on M and N.

The first term

$$\mathrm{PV}^{0,\bullet}(\mathbb{C}^\times \times X) = \mathrm{PV}^{0,\bullet}(\mathbb{C}^\times)\otimes \mathrm{PV}^{0,\bullet}(X)$$

corresponds to $\mathbb{C}[z,z^{-1}]\otimes \mathrm{PV}^{0,\bullet}(X)$, as $\mathrm{PV}^0_{\mathrm{hol}}(\mathbb{C}^\times) = \mathbb{C}[z,z^{-1}]$.

The second term

$$\big(\ker\partial \subset \mathrm{PV}^{1,\bullet}\big) = \big(\mathrm{PV}^{0,\bullet}(\mathbb{C}^\times)\otimes(\ker\partial\subset \mathrm{PV}^{1,\bullet}(X))\big)\oplus\big((\ker\partial\subset \mathrm{PV}^{1,\bullet}(\mathbb{C}^\times))\otimes \mathrm{PV}^{0,\bullet}(X)\big)$$

corresponds to $\mathbb{C}[z,z^{-1}]\otimes\big(\ker\partial\subset \mathrm{PV}^{1,\bullet}(X)\big)$ and $\mathbb{C}\langle\varepsilon\rangle\otimes \mathrm{PV}^{0,\bullet}(X)$. This is the place we use the assumption to identify $\mathbb{C}[z,z^{-1}]\otimes\big(\ker\partial\subset \mathrm{PV}^{1,\bullet}(X)\big)$ with $\mathbb{C}[z,z^{-1}]\otimes\big(\mathrm{im}\,\partial\subset \mathrm{PV}^{1,\bullet}(X)\big)$.

The third term

$$\big(\mathrm{im}\,\partial\subset \mathrm{PV}^{2,\bullet}\big) = \big((\mathrm{im}\,\partial\subset \mathrm{PV}^{0,\bullet}(\mathbb{C}^\times))\otimes(\ker\partial\subset \mathrm{PV}^{2,\bullet}(X))\big)\oplus\big((\ker\partial\subset \mathrm{PV}^{1,\bullet}(\mathbb{C}^\times))\otimes(\mathrm{im}\,\partial\subset \mathrm{PV}^{1,\bullet}(X))\big)$$

corresponds to $\big(\mathbb{C}[z,z^{-1}]\setminus\mathbb{C}\big)\langle\varepsilon\rangle\otimes\big(\ker\partial\subset \mathrm{PV}^{2,\bullet}(X)\big)$ and $\mathbb{C}\langle\varepsilon\rangle\otimes\big(\mathrm{im}\,\partial\subset \mathrm{PV}^{1,\bullet}(X)\big)$.

The last term

$$\big(\ker\partial\subset \mathrm{PV}^{3,\bullet}(\mathbb{C}^\times\times X)\big) = (\ker\partial\subset \mathrm{PV}^{1,\bullet}(\mathbb{C}^\times))\otimes(\ker\partial\subset \mathrm{PV}^{2,\bullet}(X))$$

corresponds to $\mathbb{C}\langle\varepsilon\rangle\otimes\big(\ker\partial\subset \mathrm{PV}^{2,\bullet}(X)\big)$. $\square$

Combined with Remark 3.1, this suggests that in a supergravity setting, we can perform T-duality only in the form of

$$\begin{aligned}
\mathbb{C}[\varepsilon]\otimes \mathrm{PV}^{0,\bullet}(X) &\longleftrightarrow \mathbb{C}[z\partial_z]\otimes \mathrm{PV}^{0,\bullet}(X)\\
\mathbb{C}[\varepsilon]\otimes\big(\mathrm{im}\,\partial\subset \mathrm{PV}^{1,\bullet}(X)\big) &\longleftrightarrow \mathbb{C}[z\partial_z]\otimes\big(\mathrm{im}\,\partial\subset \mathrm{PV}^{1,\bullet}(X)\big)\\
\mathbb{C}\langle\varepsilon\rangle\otimes\big(\ker\partial\subset \mathrm{PV}^{2,\bullet}(X)\big) &\longleftrightarrow \mathbb{C}\langle z\partial_z\rangle\otimes\big(\ker\partial\subset \mathrm{PV}^{2,\bullet}(X)\big)
\end{aligned}$$

using the isomorphism $\mathbb{C}[\varepsilon]\cong\mathbb{C}[z\partial_z]$. We denote these isomorphisms by $\mathbf{T}$ for future reference and this is the version of T-duality we will use in our construction of twisted S-duality below.

3.2 A $G_2\times \mathrm{SU}(2)$-invariant Twist of 11d Supergravity

The main claim of this subsection is that there is a shadow of the usual relation between 11-dimensional supergravity and type IIA supergravity even after a twist.

We learned the following definition of a twist of 11-dimensional supergravity theory from Costello in the context of his paper on the subject [Cos16].

Definition 3.3. The $G_2\times\mathrm{SU}(2)$*-invariant twist of 11-dimensional supergravity* on $M\times X$, where M is a G_2-manifold and X is a Calabi–Yau 2-fold, is a BV theory described as follows: the space of fields with its shifted L_∞-algebra structure is

$$11\mathrm{d}[M_A\times X_B] := (\Omega^\bullet(M)\otimes\Omega^{0,\bullet}(X)[1];\ \ell_1 = d_M\otimes 1 + 1\otimes\bar\partial_X,\ \ell_2 = \{-,-\},\ \ell_n = 0 \text{ for } n\geq 3)$$

where $\{-,-\}$ denotes the Poisson bracket with respect to the holomorphic volume form Ω_X on X canonically extended to the entire space, together with an odd invariant pairing given by wedging and integrating against the holomorphic volume form on X.

We want to emphasize that this definition is very much conjectural, but we will argue that all these conjectural descriptions of Costello–Li [CL16] and Costello [Cos16] are compatible with expectations from string theory once we introduce some modification as explained in Remark 2.26. For instance, Corollary 3.6 shows that this conjectural description of 11-dimensional supergravity on S^1_M is equivalent to the conjectural description of type IIA supergravity.

The following lemma plays a crucial role in establishing our claim. This is motivated by a similar result in [Cos16, Section 13].

Lemma 3.4. There is an equivalence of DG Lie algebras

$$\big(\Omega^\bullet(S^1) \otimes \Omega^{0,\bullet}(X);\ \ell_1 = d_{S^1} + \bar\partial_X,\ \ell_2 = \{-,-\}\big) \simeq \mathcal{E}^C_{\mathrm{mBCOV}}(X)[-1].$$

Proof. We first identify $\Omega^\bullet(S^1) \cong \mathbb{C}[\varepsilon]$ where ε is of odd degree. Note $\Omega^{0,\bullet}(X) = \mathrm{PV}^{0,\bullet}(X)$ and $\Omega^{0,\bullet}(X) \cong \mathrm{PV}^{2,\bullet}(X)$ by $\alpha \mapsto \alpha \wedge \Pi_X$ where we used the holomorphic symplectic form Ω_X and the induced bivector Π_X on X. Now we define a quasi-isomorphism of cochain complexes

$$\Phi\colon \Omega^{0,\bullet}(X)[\varepsilon] \to \mathrm{PV}^{0,\bullet}(X)[1] \oplus (\operatorname{im}\partial \subset \mathrm{PV}^{1,\bullet}(X)) \oplus (\ker\partial \subset \mathrm{PV}^{2,\bullet}(X))$$

by

$$\begin{aligned} \Omega^{0,\bullet}(X)\varepsilon &\to \mathrm{PV}^{0,\bullet}(X)[1] \\ \alpha\varepsilon &\mapsto \alpha \end{aligned} \qquad\qquad \begin{aligned} \Omega^{0,\bullet}(X) &\to (\operatorname{im}\partial \subset \mathrm{PV}^{1,\bullet}(X)) \oplus (\ker\partial \subset \mathrm{PV}^{2,\bullet}(X)) \\ \alpha &\mapsto \phi_C(\alpha \wedge \Pi_X) \end{aligned}$$

where $\phi_C\colon \mathrm{PV}^{2,\bullet}(X) \to (\operatorname{im}\partial \subset \mathrm{PV}^{1,\bullet}(X)) \oplus (\ker\partial \subset \mathrm{PV}^{2,\bullet}(X))$ is the identification determined by the choice of splitting C in the definition of $\mathcal{E}^C_{\mathrm{mBCOV}}(X)$.

We claim that this map also respects the Lie bracket. It is useful to note

$$\{\alpha,\beta\} = (-1)^{|\alpha|-1}[\partial\alpha \wedge \Pi_X, \beta \wedge \Pi_X]_{\mathrm{SN}} \vee \Omega_X = [\alpha \wedge \Pi_X, \partial\beta \wedge \Pi_X]_{\mathrm{SN}} \vee \Omega_X, \qquad \alpha,\beta \in \Omega^{0,\bullet}(X)$$

which follows from $(\partial\alpha \wedge \Pi_X) \wedge (\beta \wedge \Pi_X) = 0$. We need to check the compatibility of the map with brackets in the following three cases:

- For $\alpha\varepsilon, \beta\varepsilon \in \Omega^{0,\bullet}(X)\varepsilon$, their bracket vanishes as $\varepsilon^2 = 0$. The modified Schouten–Nijenhuis bracket on $\mathrm{PV}^{0,\bullet}(X)$ agrees with the ordinary Schouten–Nijenhuis bracket which vanishes.
- For $\alpha\varepsilon \in \Omega^{0,\bullet}(X)\varepsilon$, $\beta \in \Omega^{0,\bullet}(X)$, we have

  $$\begin{aligned} \Phi(\{\alpha\varepsilon,\beta\}) = \Phi(\{\alpha,\beta\}\varepsilon) = \{\alpha,\beta\} &= [\alpha \wedge \Pi_X, \partial\beta \wedge \Pi_X]_{\mathrm{SN}} \vee \Omega_X \\ &= [\alpha, \partial\beta \wedge \Pi_X]_{\mathrm{SN}} = (-1)^{|\alpha|-1}\partial[\alpha, \beta \wedge \Pi_X]_{\mathrm{SN}} \\ &= [\alpha, \phi_C(\beta \wedge \Pi_X)]^C_{\mathrm{SN}} = [\Phi(\alpha\varepsilon), \Phi(\beta)]^C_{\mathrm{SN}} \end{aligned}$$

  where on the second line, we observed that $[-, \partial\beta \wedge \Pi_X]_{\mathrm{SN}}$ intertwines the identification $(-) \wedge \Pi_X \colon \Omega^{0,\bullet}(X) \cong \mathrm{PV}^{2,\bullet}(X) \colon (-) \vee \Omega_X$.
- Finally, consider $\alpha,\beta \in \Omega^{0,\bullet}(X)$. Let us prove $\Phi(\{\alpha,\beta\}) = [\Phi(\alpha),\Phi(\beta)]^C_{\mathrm{SN}}$ by separately discussing the following two cases.

  First, if $\partial\beta \wedge \Pi_X = 0$, then $\{\alpha,\beta\} = 0$ and hence $\Phi(\{\alpha,\beta\}) = 0$. The claim is

  $$[\Phi(\alpha),\Phi(\beta)]^C_{\mathrm{SN}} = [\phi_C(\alpha \wedge \Pi_X), \phi_C(\beta \wedge \Pi_X)]^C_{\mathrm{SN}} = [\phi_C(\alpha \wedge \Pi_X), \beta \wedge \Pi_X]_{\mathrm{SN}} = 0.$$

  If $\partial\alpha \wedge \Pi_X \neq 0$, then $[\phi_C(\alpha \wedge \Pi_X), \beta \wedge \Pi_X]_{\mathrm{SN}} = [\partial\alpha \wedge \Pi_X, \beta \wedge \Pi_X]_{\mathrm{SN}} = (-1)^{|\alpha|-1}\{\alpha,\beta\} \wedge \Pi_X$. If $\partial\alpha \wedge \Pi_X = 0$, then it follows for a degree reason.

Second, if $\partial\beta \wedge \Pi_X \neq 0$, then we have

$$\begin{aligned}\Phi(\{\alpha,\beta\}) &= \phi_C\left(\{\alpha,\beta\}\wedge\Pi_X\right) = (-1)^{|\alpha|-1}\phi_C([\partial\alpha\wedge\Pi_X, \beta\wedge\Pi_X]_{\mathrm{SN}} \vee \Omega_X \wedge \Pi_X) \\ &= (-1)^{|\alpha|-1}\phi_C\left([\phi_C(\alpha\wedge\Pi_X), \beta\wedge\Pi_X]_{\mathrm{SN}}\right) = [\phi_C(\alpha\wedge\Pi_X), \phi_C(\beta\wedge\Pi_X)]^C_{\mathrm{SN}} \\ &= [\Phi(\alpha),\Phi(\beta)]^C_{\mathrm{SN}}\end{aligned}$$

□

Remark 3.5. Recall Remark 2.15 where $\mathcal{E}^C_{\mathrm{mBCOV}}(X)$ is identified as $\mathbb{Z}/2$-graded BV theory with $\mathcal{E}[-1] = \mathrm{PV}^{0,\bullet}(X)[1] \oplus \mathrm{PV}^{2,\bullet}(X)$ with action functional $S(\alpha,\beta) = \int \alpha\wedge\bar\partial\beta + \frac{1}{2}\alpha\wedge\{\beta,\beta\}$ for $\alpha \in \mathrm{PV}^{0,\bullet}(X)$ and $\beta \in \mathrm{PV}^{2,\bullet}(X)$. It is exactly identified with 2-dimensional BV theory defined by $\Omega^{0,\bullet}(\mathbb{C}^2)[\varepsilon]$ with $|\varepsilon| = -1$. Indeed, this is essentially the statement argued in [Cos16, Section 13].

The following corollary demonstrates that the usual relationship between 11-dimensional supergravity and type IIA supergravity holds at the level of twisted theories. Let us fix a G_2 structure on $M^6 \times S^1$. For instance, M^6 may be a Calabi–Yau 3-fold.

Corollary 3.6. There is an equivalence of DG Lie algebras

$$\mathrm{red}_M \colon \ 11\mathrm{d}[(M^6\times S^1)_A \times X^2_B] \to \mathrm{IIA}_{\mathrm{SUGRA}}[M^6_A \times X^2_B].$$

Proof. This follows from the above lemma and definition because the map is the identity in the direction of M^6. □

Remark 3.7. (Surya: An example is worked out in [Pol98] section 4) Since the reduction of 11d supergravity on any circle should yield IIA supergravity, it is natural to ask if reducing M-theory on a circle $S^1 \subset X$ yields a twist of IIA supergravity theory and if so which one. For simplicity, let us consider a flat space $11\mathrm{d}[\mathbb{R}^7_A \times (\mathbb{C}\times\mathbb{C}^\times)_B]$ with $S^1 \subset \mathbb{C}^\times$. To answer this question, we choose $\mathrm{SU}(3) \subset G_2$ with which we write $\mathbb{R}^7 \cong \mathbb{R}\times\mathbb{C}^3_{\mathrm{CY}}$ so that $\mathbb{C}^3_{\mathrm{CY}}$ is thought of having a corresponding Calabi–Yau structure. Then we expect that such a reduction yields $\mathrm{IIA}[(\mathbb{R}\times\mathbb{R}_{\geq 0})_A \times (\mathbb{C}^3_{\mathrm{CY}}\times\mathbb{C})_B]$ where $\mathbb{C}$ is equipped with a linear superpotential.

Indeed, reducing along a holomorphic circle is sensitive to the radius of the circle. It is well-known to physicists that M-theory on S^1 is equivalent to type IIA string theory with a number of D0 branes. Denoting the 11d metric as $g_{\mu\nu}$, where $\mu,\nu = 0,\cdots,10$ with the index 10 corresponding to S^1 direction, the location of the D0 branes in IIA is given by a particular representative for the class Poincaré dual (Phil: our space is non-compact?) to $d * d(g_{a,10}dx^a), a = 0,\cdots,9$; this is determined by noting the relation between the 11d metric and the Ramond–Ramond 1-form of type IIA supergravity, under which the D0 brane is charged. We note that the particular representative must satisfy a BPS-type bound on its volume. In particular, if the 11-dimensional metric were flat, then there would be no D0 brane introduced by S^1 reduction.

In addition $11\mathrm{d}[\mathbb{R}^7_A \times (\mathbb{C}\times\mathbb{C}^\times)_B]$ still depends nontrivially on g_{1010} and includes field configurations where the off-diagonal g_{a10} components may be nonzero. (Phil: What does this even mean in our twisted setting?) Indeed the equations of motion allow the metric to evolve in such a way that $\mathbb{C}\times\mathbb{C}^\times$ fibres over $\mathbb{R}^7$ nontrivially. Thus, we arrive at the conclusion that the twist of IIA after reducing must support nontrivial configurations of D0 branes. The only twist of type IIA supergravity that supports D0 branes is the SU(4)-invariant twist, namely, the one living on a background of the form $\mathbb{R}^2_A \times \mathbb{C}^4_B$ as claimed.

Furthermore, to see that the linear superpotential must be present, we make the following argument using M5 branes. In physical 11d supergravity on a background of the form $M^7 \times \mathbb{R}^4$

where M^7 is a G_2 manifold, the supersymmetric cycles which M5 branes may wrap are given by $N^4 \times \mathbb{R}^2$ where $N^4 \subset M^7$ is a coassociative 4-fold with respect to the G_2 structure on M^7. In our setting, under $\mathbb{R}^7 \cong \mathbb{R} \times \mathbb{C}^3_{\mathrm{CY}}$, a holomorphic surface $\mathbb{C}^2 \subset \mathbb{C}^3_{\mathrm{CY}}$ yields a coassociative 4-fold $\{0\} \times \mathbb{C}^2 \subset \mathbb{R} \times \mathbb{C}^3_{\mathrm{CY}}$ (see, e.g., [Joy07, Section 12.2.1]) and we consider $\{0\} \times \mathbb{C}^\times \subset \mathbb{C} \times \mathbb{C}^\times$ to model $\mathbb{R}^2 \subset \mathbb{R}^4$. Now, we expect that the world-volume theory of a stack of M5 branes wrapping $\mathbb{R}^4 \times \{0\} \times \mathbb{C}^\times \subset \mathbb{R}^7 \times \mathbb{C} \times \mathbb{C}^\times$ or

$$\begin{array}{cl} \text{background} & \mathbb{R} \times \mathbb{C}^3_{\mathrm{CY}} \times \mathbb{C} \times \mathbb{C}^\times \\ \text{M5 branes} & \{0\} \times \mathbb{C}^2 \times \{0\} \times \mathbb{C}^\times \end{array}$$

is the holomorphic-topological twist of the 6d $\mathcal{N} = (2,0)$ superconformal field theory. It is further expected that reducing this twist of the 6d theory on $S^1 \subset \mathbb{C}^\times$ yields an A-type twist of 5d $\mathcal{N} = 2$ gauge theory whose solutions to the equations of motion may be described by $\underline{\mathrm{Map}}((\mathbb{R}_{\geq 0})_{\mathrm{dR}} \times \mathbb{C}^2_{\mathrm{Dol}}, BG)_{\mathrm{dR}}$.[3] An A-type twist of the world-volume theory of a stack of D-branes is gotten by introducing a linear superpotential in the $\mathbb{C}$-direction transverse to the world-volume, so this tells us that reducing the ambient theory on a holomorphic circle must introduce a linear superpotential.

3.3 Closed String Fields under Twisted S-duality

As described in the introduction, one way physicists think of S-duality of type IIB string theory comes from the fact that IIB theory on a circle is equivalent to M-theory on a torus. Via this equivalence, the S-duality is just the action of $S \in \mathrm{SL}(2,\mathbb{Z})$ on M-theory on a torus. This situation is neatly summarized in Figure 1. We have given mathematical descriptions of protected sectors of the closed string field theories of each of the vertices in Figure 1, and have described versions of the T-duality map $\mathbf{T}$ and the reduction map red_M in the context of these protected sectors. Thus, we may describe a mathematically rigorous version of Figure 1 and use this to give a definition of S-duality on $\mathrm{IIB}_{\mathrm{SUGRA}}[M^4_A \times X^3_B]$. More precisely, we have the following diagram

$$\begin{tikzcd} 11\mathrm{d}[(M^4 \times \mathbb{R} \times S^1_M \times S^1)_A \times X^2_B] \arrow[loop above, "\mathrm{SL}(2,\mathbb{Z})"] \arrow[r, "\mathrm{red}_M", "\simeq"'] & \mathrm{IIA}_{\mathrm{SUGRA}}[(M^4 \times T^*S^1)_A \times X^2_B] \arrow[r, "\mathbf{T}"] & \mathrm{IIB}_{\mathrm{SUGRA}}[M^4_A \times (\mathbb{C}^\times \times X^2)_B] \arrow[loop above, "\mathrm{SL}(2,\mathbb{Z})"] \end{tikzcd}$$

where

- The map red_M is the reduction along S^1_M discussed in Subsection 3.2.
- The map $\mathbf{T}$ denotes T-duality as defined in Subsection 3.1. In particular, X^2 is a Calabi–Yau 2-fold as discussed for Proposition 3.2 throughout this subsection unless otherwise mentioned.
- The $\mathrm{SL}(2,\mathbb{Z})$ action on $S^1_M \times S^1$ in 11-dimensional supergravity is given by the natural action of $\mathrm{SL}(2,\mathbb{Z})$ on $\varepsilon_M, \varepsilon$ where we make the identification $\Omega^\bullet(S^1_M \times S^1) \simeq \mathbb{C}[\varepsilon_M, \varepsilon]$. In particular, $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ sends $\varepsilon_M \mapsto -\varepsilon$ and $\varepsilon \mapsto \varepsilon_M$.

[3] As a consistency check of this claim, note that by the AGT correspondence [AGT10], subjecting the holomorphic-topological twist of the 6d $\mathcal{N}$=(2,0) theory to a B-type Ω-deformation on $\mathbb{C}^2$ yields a 2d theory whose observables are the affine $\mathcal{W}$-algebra. Further reducing on a holomorphic circle should yield the mode algebra of the affine $\mathcal{W}$-algebra. By works of Grojnowski [Gro95] and Nakajima [Nak97] (for the abelian case) and Schiffmann–Vasserot [SV13] and Maulik–Okounkov [MO12] (more generally), this algebra acts on the (equivariant) cohomology of the moduli of instantons of $\mathbb{C}^2$, which is exactly the Hilbert space of the above twist of 5d $\mathcal{N} = 2$ theory. We expect that a more direct check of this claim can be performed by comparing residual supersymmetries.

In particular, we would like to define the S-duality map by

$$\mathbb{S} = \mathbf{T} \circ \mathrm{red}_M \circ S \circ \mathrm{red}_M^{-1} \circ \mathbf{T}^{-1}$$

Remark 3.8. Recall that this map cannot be defined everywhere on $\mathrm{IIB}_{\mathrm{SUGRA}}[M_A^4 \times (\mathbb{C}^\times \times X)_B]$. First of all, as we noted in Proposition 3.2, T-duality is not defined on a non-constant part of $\mathrm{PV}^{0,\bullet}(\mathbb{C}^\times)$ in $\mathcal{E}^C_{\mathrm{mBCOV}}(\mathbb{C}^\times \times X)$. Moreover, it is not defined on $1 \otimes \mathrm{PV}^{2,\bullet}(X) \subset \mathrm{PV}(\mathbb{C}^\times \times X)$ of $\mathcal{E}^C_{\mathrm{mBCOV}}(\mathbb{C}^\times \times X)$ either because its T-dual wouldn't be a part of type IIA supergravity. This restriction seems necessary as soon as we decided not to work at a non-perturbative level. In fact, we expect that these two non-accessible parts are precisely S-dual to each other in a non-perturbative setting in view of Definition 3.13 and Remark 3.14.

Remark 3.9. Note that we may in fact define an action of all of $\mathrm{SL}(2,\mathbb{C})$, which is the complexification of the expected action of $\mathrm{SL}(2,\mathbb{R})$ on supergravity. We expect an integral structure to emerge upon consideration of nonperturbative effects; while our discussion won't touch on this explicitly, we will only derive formulas for the action of $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{C})$.

We can use the above map to derive explicit formulas for how S-duality acts on various parts of the space of fields of supergravity theory. Note that while the reduction map red_M is an equivalence of DG Lie algebras, the map $\mathbf{T}$ is essentially given by an isomorphism $\mathbb{C}[z\partial_z] \cong \mathbb{C}[\varepsilon]$. Therefore, we will proceed in cases, depending on the polyvector degree and the support of the closed string field. In what follows, we will use a fixed splitting C used in defining $\mathrm{IIA}_{\mathrm{SUGRA}}[(M^4 \times T^*S^1)_A \times X^2_B]$ and the isomorphism $\phi_C \colon \mathrm{PV}^{2,\bullet}(X) \cong \left(\mathrm{im}\,\partial \subset \mathrm{PV}^{1,\bullet}(X)\right) \oplus \left(\ker \partial \subset \mathrm{PV}^{2,\bullet}(X)\right)$ unless otherwise mentioned. Also, every map in sight is a map of cochain complexes; in particular, in the B-model direction, any element is $\bar\partial$-closed. All the proofs are straightforward from what we have discussed in previous subsections.

Proposition 3.10 (S-duality on polyvector fields of degree 0)**.** Consider

$$1 \otimes F \in \left(\mathbb{C}[z\partial_z] \cap \mathrm{PV}^{0,\bullet}(\mathbb{C}^\times_z)\right) \otimes \mathrm{PV}^{0,\bullet}(X).$$

Then one has

$$\mathbb{S}(1 \otimes F) = -z\partial_z \wedge \phi_C(F \wedge \Pi_X) \in \left(\mathbb{C}[z\partial_z] \cap \mathrm{PV}^{1,\bullet}(\mathbb{C}^\times_z)\right) \otimes \phi_C\left(\mathrm{PV}^{2,\bullet}(X)\right).$$

Proof. We compute $\mathbb{S}(1 \otimes F) = (\mathbf{T} \circ \mathrm{red}_M \circ S \circ \mathrm{red}_M^{-1} \circ \mathbf{T}^{-1})(1 \otimes F)$ in steps:

$$\begin{aligned}
\mathbb{S}(1 \otimes F) &= (\mathbf{T} \circ \mathrm{red}_M \circ S \circ \mathrm{red}_M^{-1} \circ \mathbf{T}^{-1})(1 \otimes F) \\
&= (\mathbf{T} \circ \mathrm{red}_M \circ S \circ \mathrm{red}_M^{-1})(1 \otimes F) && \text{by } \mathbb{C}[\varepsilon] \otimes \mathrm{PV}^{0,\bullet}(X) \overset{\mathbf{T}}{\cong} \mathbb{C}[z\partial_z] \otimes \mathrm{PV}^{0,\bullet}(X) \\
&= (\mathbf{T} \circ \mathrm{red}_M \circ S)(\varepsilon_M \otimes F) && \text{by } \mathbb{C}\langle \varepsilon_M \rangle \otimes \Omega^{0,\bullet}(X) \overset{\mathrm{red}_M}{\cong} \mathrm{PV}^{0,\bullet}(X) \\
&= (\mathbf{T} \circ \mathrm{red}_M)(-\varepsilon \otimes F) && \text{by } \mathbb{C}[\varepsilon_M, \varepsilon] \otimes \Omega^{0,\bullet}(X) \overset{S}{\cong} \mathbb{C}[\varepsilon_M, \varepsilon] \otimes \Omega^{0,\bullet}(X) \\
&= \mathbf{T}(-\varepsilon \otimes \phi_C(F \wedge \Pi_X)) && \text{by } \Omega^{0,\bullet}(X) \overset{\mathrm{red}_M}{\cong} \phi_C\left(\mathrm{PV}^{2,\bullet}(X)\right) \\
&= -z\partial_z \wedge \phi_C(F \wedge \Pi_X) && \text{by } \mathbb{C}\langle \varepsilon \rangle \otimes \phi_C\left(\mathrm{PV}^{2,\bullet}(X)\right) \overset{\mathbf{T}}{\cong} \mathbb{C}\langle z\partial_z \rangle \otimes \phi_C\left(\mathrm{PV}^{2,\bullet}(X)\right)
\end{aligned}$$

□

By extending the above $\Omega^\bullet(M^4)$-linearly, this gives an identification of parts of $\mathrm{IIB}_{\mathrm{SUGRA}}[M_A^4 \times (\mathbb{C}^\times \times X^2)_B]$, namely,

$$\Omega^\bullet(M^4) \otimes \mathbb{C}\langle 1 \rangle \otimes \mathrm{PV}^{0,\bullet}(X) \cong \Omega^\bullet(M^4) \otimes \mathbb{C}\langle -z\partial_z \rangle \otimes \phi_C\left(\mathrm{PV}^{2,\bullet}(X)\right).$$

Proposition 3.11 (S-duality on polyvector fields of degree 1)**.** $\mathbb{S}$ acts as the identity on:

- $\mathbb{C}\langle 1\rangle \otimes \operatorname{im}\partial \subset \big(\mathbb{C}[z\partial_z] \cap \mathrm{PV}^{0,\bullet}(\mathbb{C}^\times_z)\big) \otimes \mathrm{PV}^{1,\bullet}(X)$.
- $\mathbb{C}\langle z\partial_z\rangle \otimes \mathrm{PV}^{0,\bullet}(X) \subset \big(\mathbb{C}[z\partial_z] \cap \mathrm{PV}^{1,\bullet}(\mathbb{C}^\times_z)\big) \otimes \mathrm{PV}^{0,\bullet}(X)$

Proof. We consider the above two cases separately.

- Let $\mu \in \operatorname{im}\partial \subset \mathrm{PV}^{1,\bullet}(X)$ be arbitrary. By definition, $\mu = \phi_C(g \wedge \Pi_X)$ for a unique element $g \in \mathrm{PV}^{0,\bullet}(X)$. Then

$$\begin{aligned}\mathbb{S}(1\otimes\mu) &= (\mathbf{T}\circ\mathrm{red}_M\circ S\circ\mathrm{red}_M^{-1}\circ\mathbf{T}^{-1})(1\otimes\mu) = (\mathbf{T}\circ\mathrm{red}_M\circ S\circ\mathrm{red}_M^{-1})(1\otimes\mu)\\ &= (\mathbf{T}\circ\mathrm{red}_M\circ S)(1\otimes g) = (\mathbf{T}\circ\mathrm{red}_M)(1\otimes g)\\ &= \mathbf{T}(1\otimes\phi_C(g\wedge\Pi_X)) = \mathbf{T}(1\otimes\mu) = 1\otimes\mu\end{aligned}$$

- Let $z\partial_z \otimes F \in \mathbb{C}\langle z\partial_z\rangle \otimes \mathrm{PV}^{0,\bullet}(X)$ be arbitrary. Then we have that

$$\begin{aligned}\mathbb{S}(z\partial_z\otimes F) &= (\mathbf{T}\circ\mathrm{red}_M\circ S\circ\mathrm{red}_M^{-1}\circ\mathbf{T}^{-1})(z\partial_z\otimes F) = (\mathbf{T}\circ\mathrm{red}_M\circ S\circ\mathrm{red}_M^{-1})(\varepsilon\otimes F)\\ &= (\mathbf{T}\circ\mathrm{red}_M\circ S)(\varepsilon_M\varepsilon\otimes F) = (\mathbf{T}\circ\mathrm{red}_M)(-\varepsilon\varepsilon_M\otimes F)\\ &= (\mathbf{T}\circ\mathrm{red}_M)(\varepsilon_M\varepsilon\otimes F) = \mathbf{T}(\varepsilon\otimes F) = z\partial_z\otimes F\end{aligned}$$

□

Proposition 3.12 (S-duality on polyvector fields of degrees 2,3)**.** Consider

$$z\partial_z \otimes \mu \in \big(\mathbb{C}[z\partial_z]\cap\mathrm{PV}^{1,\bullet}(\mathbb{C}^\times_z)\big)\otimes\phi_C(\mathrm{PV}^{2,\bullet}(X)).$$

Then we have

$$\mathbb{S}(z\partial_z\otimes\mu) = 1\otimes F \in \big(\mathbb{C}[z\partial_z]\cap\mathrm{PV}^{0,\bullet}(\mathbb{C}^\times_z)\big)\otimes\mathrm{PV}^{0,\bullet}(X)$$

where $\mu = \phi_C(F\wedge\Pi_X)$ uniquely determines $F \in \mathrm{PV}^{0,\bullet}(X)$.

Proof. We note

$$\begin{aligned}\mathbb{S}(z\partial_z\otimes\mu) &= (\mathbf{T}\circ\mathrm{red}_M\circ S\circ\mathrm{red}_M^{-1}\circ\mathbf{T}^{-1})(z\partial_z\otimes\mu) = (\mathbf{T}\circ\mathrm{red}_M\circ S\circ\mathrm{red}_M^{-1})(\varepsilon\otimes\mu)\\ &= (\mathbf{T}\circ\mathrm{red}_M\circ S)(\varepsilon\otimes F) = (\mathbf{T}\circ\mathrm{red}_M)(\varepsilon_M\otimes F) = \mathbf{T}(1\otimes F) = 1\otimes F\end{aligned}$$

□

For our application, we would like to apply this formula in a setting where $\mathbb{C}^\times$ is replaced by $\mathbb{C}$. Morally, such a formula is gotten by precomposing $\mathbb{S}$ with a restriction map and post-composing with a map that takes the radius of $\mathbb{C}^\times$ to ∞. This effectively amounts to replacing $z\partial_z$ by ∂_z. Furthermore, we can give a more uniform conjectural description for how S-duality acts on an arbitrary element of $\mathrm{IIB}_{\mathrm{SUGRA}}[M^4_A \times X^3_B]$. Again, it acts trivially on $\Omega^\bullet(M^4)$, so it is enough to describe the action on $\mathcal{E}^C_{\mathrm{mBCOV}}(X)$. Note that we used the splitting C for type IIB supergravity or the induced isomorphism $\phi_C\colon\ \mathrm{PV}^{3,\bullet}(X) \cong \big(\operatorname{im}\partial\subset\mathrm{PV}^{2,\bullet}(X)\big)\oplus\big(\ker\partial\subset\mathrm{PV}^{3,\bullet}(X)\big)$ here.

Definition 3.13. Let (X,Ω_X) be a Calabi–Yau 3-fold with a holomorphic volume form Ω_X. The *twisted S-duality* $\mathbb{S}$ on $\mathcal{E}^C_{\mathrm{mBCOV}}(X)$ is defined by the following:

- Given $F \in \mathrm{PV}^{0,\bullet}(X)$,

$$\mathbb{S}(F) = \phi_C(F\wedge\Omega_X^{-1}) \in \phi_C(\mathrm{PV}^{3,\bullet}(X));$$

- Given $\mu \in \ker \partial \subset \mathrm{PV}^{1,\bullet}(X)$,

$$\mathbb{S}(\mu) = \mu \in \ker \partial \subset \mathrm{PV}^{1,\bullet}(X);$$

- Given $\nu \in \phi_C(\mathrm{PV}^{3,\bullet}(X))$

$$\mathbb{S}(\nu) = -F \in \mathrm{PV}^{0,\bullet}(X)$$

  where F is such that $\nu = \phi_C(F \wedge \Omega_X^{-1})$.

In other words, in the end, we conjecture that the twisted S-duality on $\mathrm{IIB}[M_A^4 \times X_B^3]$ is induced from the Calabi–Yau volume form Ω_X on X in this simple way.

Remark 3.14. Unlike the formulas we derived from first principles in Propositions 3.10, 3.11, and 3.12 which relied only on the low-energy fields of our closed string field theories, Definition 3.13 is sensitive to stringy effects. For instance, consider $\alpha \in \mathrm{PV}^{0,\bullet}(\mathbb{C}^\times) \subset \mathrm{PV}(\mathbb{C}^\times \times \mathbb{C}^2)$. Then by Remark 3.1, T-duality along $\mathbb{C}^\times$ takes α to a closed string field in the A-model that only depends on winding modes. Such closed string fields are not in the supergravity approximation.

Therefore, in order to derive our conjectural description, we need a suitable enhancement of our twist of 11d supergravity that includes more than just the lowest energy fields. We expect such an enhancement to be subtle – it should likely involve a deformation of the de Rham complex on a G_2 manifold as an $\mathbb{E}_3$-algebra, where the deformed product involves counts of associative 3-folds [Joy18]. Comparison with the usual curve counting invariants in the context of topological string theory is summarized in the following table:

	(A-model) topological string	twisted M-theory
spacetime = $\mathbb{R}^4 \times M$	M : CY 3-fold	M : G_2 manifold
extended objects supersymmetric cycles	F1 string Σ J-holomorphic curves in M	M2 brane S associative 3-folds in M
theory on an extended object	A-model on Σ with target M	3d theory on S with target M
states/observables structure at cochain level	quantum cohomology $\mathrm{QH}(M)$ counting J-holomorphic curves $\mathbb{E}_2$-algebra	M2 cohomology $\mathrm{M2}(M)$[4] counting associative 3-folds $\mathbb{E}_3$-algebra

The proposed S-duality map in Definition 3.13 in fact respects the DG Lie algebra structure on $\mathrm{IIB}_{\mathrm{SUGRA}}[M_A^4 \times X_B^3]$.

Theorem 3.15. The action of $\mathbb{S}$ preserves the DG Lie algebra structure on $\mathrm{IIB}_{\mathrm{SUGRA}}[M_A^4 \times X_B^3]$.

Proof. Since $\mathbb{S}$ acts as the identity on components of fields supported on M^4 and on all form components, it is clear that $\mathbb{S}$ commutes with the differential. Therefore, we need only check that $\mathbb{S}$ preserves the modified Schouten bracket of $\mathcal{E}^C_{\mathrm{mBCOV}}(X^3)$. We have the following cases;

- Let $F \in \mathrm{PV}^{0,\bullet}(X)$ and $\mu \in \ker \partial \subset \mathrm{PV}^{1,\bullet}(X)$. Then

$$\mathbb{S}\left([F,\mu]^C_{\mathrm{SN}}\right) = \mathbb{S}\left([F,\mu]_{\mathrm{SN}}\right) = \phi_C([F,\mu]_{\mathrm{SN}} \wedge \Omega_X^{-1}).$$

[4] There is a sense in which quantum cohomology is a misnomer because as a deformation of de Rham cohomology $H_{\mathrm{dR}}(M)$ of M, it is from a stringy effect. Hence string cohomology (or perhaps F1 cohomology) may have been a better name for quantum cohomology. However, string cohomology now referes to a different construction so we are stuck with established terminology. In view of this remark, we opt to call the corresponding deformation of de Rham cohomology of a G_2 manifold M2 cohomology.

On the other hand, we have that

$$[\mathbb{S}(F), \mathbb{S}(\mu)]^C_{\mathrm{SN}} = [\phi_C(F \wedge \Omega_X^{-1}), \mu]^C_{\mathrm{SN}} = \phi_C[F \wedge \Omega_X^{-1}, \mu]_{\mathrm{SN}}.$$

These two agree because $[-, \mu]_{\mathrm{SN}}$ intertwines the identification $(-) \wedge \Omega_X^{-1} \colon \mathrm{PV}^{0,\bullet}(X) \cong \mathrm{PV}^{3,\bullet}(X)$.

- Let $F \in \mathrm{PV}^{0,\bullet}(X)$ and $\mu \in \phi_C(\mathrm{PV}^{3,\bullet}(X))$, say $\mu = \phi_C(G \wedge \Omega_X^{-1})$ for $G \in \mathrm{PV}^{0,\bullet}(X)$. It is easy to see that both $\mathbb{S}\left([F, \mu]^C_{\mathrm{SN}}\right)$ and $[\mathbb{S}(F), \mathbb{S}(\mu)]^C_{\mathrm{SN}}$ vanish if $\mu \in \ker \partial \subset \mathrm{PV}^{3,\bullet}(X)$. Hence we suppose $\mu = \phi_C(G \wedge \Omega_X^{-1}) = \partial(G \wedge \Omega_X^{-1})$. Then

  $$\mathbb{S}\left([F, \mu]^C_{\mathrm{SN}}\right) = \mathbb{S}\left([F, \partial(G \wedge \Omega_X^{-1})]^C_{\mathrm{SN}}\right) = (-1)^{|F|-1}\mathbb{S}\left(\partial[F, G \wedge \Omega_X^{-1}]_{\mathrm{SN}}\right) = (-1)^{|F|-1}\partial[F, G \wedge \Omega_X^{-1}]_{\mathrm{SN}}$$

  On the other hand, we have

  $$[\mathbb{S}(F), \mathbb{S}(\mu)]^C_{\mathrm{SN}} = [\phi_C(F \wedge \Omega_X^{-1}), -G]^C_{\mathrm{SN}}.$$

  One can check that if $\partial(F \wedge \Omega_X^{-1}) = 0$, then both terms vanish. Suppose $\partial(F \wedge \Omega_X^{-1}) \neq 0$. Then the first term is

  $$\begin{aligned}
  &(-1)^{|F|-1}\partial[F, G \wedge \Omega_X^{-1}]_{\mathrm{SN}} \\
  &= (-1)^{|F|-1}(-1)^{|F|-1}\partial\left(\partial(F \wedge G \wedge \Omega_X^{-1}) - \partial F \wedge G \wedge \Omega_X^{-1} - (-1)^{|F|}F \wedge \partial(G \wedge \Omega_X^{-1})\right) \\
  &= (-1)^{|F|-1}\partial\left(F \wedge \partial(G \wedge \Omega_X^{-1})\right) \;=\; (-1)^{|F|-1}\partial\left(F \wedge \iota_{\Omega_X^{-1}}\partial_{\mathrm{dR}}G\right) \\
  &= -\partial\left(\iota_{\Omega_X^{-1}}(F \wedge \partial_{\mathrm{dR}}G)\right) \;=\; -\iota_{\Omega_X^{-1}}\left(\partial_{\mathrm{dR}}(F \wedge \partial_{\mathrm{dR}}G)\right) \;=\; -\iota_{\Omega_X^{-1}}\left(\partial_{\mathrm{dR}}F \wedge \partial_{\mathrm{dR}}G\right)
  \end{aligned}$$

  where ∂_{dR} is the holomorphic de Rham differential applied to $\Omega^{0,\bullet}(X)$, $\iota_{\Omega_X^{-1}} \colon \Omega(X) \to \mathrm{PV}(X)$ is the contraction with Ω_X^{-1}, and $\wedge$ stands for the wedge product of forms or polyvector fields. The second term is

  $$\begin{aligned}
  &[\phi_C(F \wedge \Omega_X^{-1}), -G]^C_{\mathrm{SN}} \;=\; -\partial[F \wedge \Omega_X^{-1}, G]_{\mathrm{SN}} \\
  &= -(-1)^{|F\wedge\Omega_X^{-1}|-1}\partial\left(\partial(F \wedge \Omega_X^{-1} \wedge G) - \partial(F \wedge \Omega_X^{-1}) \wedge G - (-1)^{|F\wedge\Omega_X^{-1}|}F \wedge \Omega_X^{-1} \wedge \partial G\right) \\
  &= (-1)^{|F|}\partial\left(\partial(F \wedge \Omega_X^{-1}) \wedge G\right) \;=\; (-1)^{|F|}\partial\left((\iota_{\Omega_X^{-1}}\partial_{\mathrm{dR}}F) \wedge G\right) \\
  &= (-1)^{|F|}\partial\left(\iota_{\Omega_X^{-1}}(\partial_{\mathrm{dR}}F \wedge G)\right) \;=\; (-1)^{|F|}\iota_{\Omega_X^{-1}}\left(\partial_{\mathrm{dR}}(\partial_{\mathrm{dR}}F \wedge G)\right) \;=\; -\iota_{\Omega_X^{-1}}\left(\partial_{\mathrm{dR}}F \wedge \partial_{\mathrm{dR}}G\right).
  \end{aligned}$$

  where all the degrees are the usual polyvector degrees. Note that the two coincide as desired.

- Let $\mu \in \ker \partial \subset \mathrm{PV}^{1,\bullet}(X)$ and $\nu \in \phi_C(\mathrm{PV}^{3,\bullet}(X))$, say $\nu = \phi_C(G \wedge \Omega_X^{-1})$. Then note

  $$\mathbb{S}\left([\mu, \nu]^C_{\mathrm{SN}}\right) = \mathbb{S}(\phi_C[\mu, G \wedge \Omega_X^{-1}]_{\mathrm{SN}}) = \mathbb{S}\left(\phi_C\left([\mu, G]_{\mathrm{SN}} \wedge \Omega_X^{-1}\right)\right) = -[\mu, G]_{\mathrm{SN}} = [\mathbb{S}(\mu), \mathbb{S}(\nu)]^C_{\mathrm{SN}}.$$

$\square$

3.4 Twisted $\mathrm{SL}(2,\mathbb{Z})$ Action

Now note that we have an action of the entire group $\mathrm{SL}(2,\mathbb{Z})$ on $11\mathrm{d}[(M^4 \times \mathbb{R} \times S^1_M \times S^1)_A \times X^2_B]$. Hence we can apply a strategy similar to the above to derive formulas for how the entire duality group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathrm{IIB}_{\mathrm{SUGRA}}[M^4_A \times (\mathbb{C}^\times \times X)_B]$. That is, we define an operator

$$\mathbb{T} = \mathbf{T} \circ \mathrm{red}_M \circ T \circ \mathrm{red}_M^{-1} \circ \mathbf{T}^{-1}$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ acts on 11-dimensional supergravity by $\varepsilon_M \mapsto \varepsilon_M, \varepsilon \mapsto \varepsilon_M + \varepsilon$.

Proposition 3.16 (T element on polyvector fields of degrees 0,1)**.** $\mathbb{T}$ acts as the identity on

- $\mathbb{C}[z\partial_z] \otimes \mathrm{PV}^{0,\bullet}(X) \subset \mathrm{PV}(\mathbb{C}^\times_z) \otimes \mathrm{PV}^{0,\bullet}(X)$.
- $\mathbb{C}\langle 1\rangle \otimes \mathrm{im}\,\partial \subset \big(\mathbb{C}[z\partial_z] \cap \mathrm{PV}^{0,\bullet}(\mathbb{C}_z)\big) \otimes \mathrm{PV}^{1,\bullet}(X)$

Proof. We consider each of the above cases separately.

- Let $1 \otimes F \in \mathbb{C}[z\partial_z] \otimes \mathrm{PV}^{0,\bullet}(X)$ be arbitrary. Then we have that

$$\begin{aligned}\mathbb{T}(1 \otimes F) &= (\mathbf{T} \circ \mathrm{red}_M \circ T \circ \mathrm{red}_M^{-1} \circ \mathbf{T}^{-1})(1 \otimes F) = (\mathbf{T} \circ \mathrm{red}_M \circ T \circ \mathrm{red}_M^{-1})(1 \otimes F)\\ &= (\mathbf{T} \circ \mathrm{red}_M \circ T)(\varepsilon_M \otimes F) = (\mathbf{T} \circ \mathrm{red}_M)(\varepsilon_M \otimes F)\\ &= \mathbf{T}(1 \otimes F) = 1 \otimes F\end{aligned}$$

  Similarly, let $z\partial_z \otimes F \in \mathbb{C}[z\partial_z] \otimes \mathrm{PV}^{0,\bullet}(X)$ be arbitrary. Then we have that

$$\begin{aligned}\mathbb{T}(z\partial_z \otimes F) &= (\mathbf{T} \circ \mathrm{red}_M \circ T \circ \mathrm{red}_M^{-1} \circ \mathbf{T}^{-1})(z\partial_z \otimes F) = (\mathbf{T} \circ \mathrm{red}_M \circ T \circ \mathrm{red}_M^{-1})(\varepsilon \otimes F)\\ &= (\mathbf{T} \circ \mathrm{red}_M \circ T)(\varepsilon_M\varepsilon \otimes F) = (\mathbf{T} \circ \mathrm{red}_M)(\varepsilon_M(\varepsilon_M + \varepsilon) \otimes F)\\ &= (\mathbf{T} \circ \mathrm{red}_M)(\varepsilon_M\varepsilon \otimes F) = \mathbf{T}(\varepsilon \otimes F) = z\partial_z \otimes F\end{aligned}$$

  where we have used $\varepsilon_M^2 = 0$.

- Let $\mu \in \mathrm{im}\,\partial \subset \mathrm{PV}^{1,\bullet}(X)$ be arbitrary. Note $\mu = \phi_C(g \wedge \Pi_X)$ for a unique element $g \in \mathrm{PV}^{0,\bullet}(X)$. Then we have

$$\begin{aligned}\mathbb{T}(1 \otimes \mu) &= (\mathbf{T} \circ \mathrm{red}_M \circ T \circ \mathrm{red}_M^{-1} \circ \mathbf{T}^{-1})(1 \otimes \mu) = (\mathbf{T} \circ \mathrm{red}_M \circ T \circ \mathrm{red}_M^{-1})(1 \otimes \mu)\\ &= (\mathbf{T} \circ \mathrm{red}_M \circ T)(g) = (\mathbf{T} \circ \mathrm{red}_M)(g) = \mathbf{T}(1 \otimes \mu) = 1 \otimes \mu\end{aligned}$$

□

Proposition 3.17 (T element on polyvector fields of degrees 2,3)**.** If $\mu \in \phi_C\left(\mathrm{PV}^{2,\bullet}(X)\right)$, then

$$\mathbb{T}(z\partial_z \otimes \mu) = 1 \otimes F + z\partial_z \otimes \mu \in \mathbb{C}[z\partial_z] \otimes \mathrm{PV}(X)$$

where $\mu = \phi_C(F \wedge \Pi_X)$.

Proof. We wish to compute

$$\begin{aligned}\mathbb{T}(z\partial_z \otimes \mu) &= (\mathbf{T} \circ \mathrm{red}_M \circ T \circ \mathrm{red}_M^{-1} \circ \mathbf{T}^{-1})(z\partial_z \otimes \mu) = (\mathbf{T} \circ \mathrm{red}_M \circ T \circ \mathrm{red}_M^{-1})(\varepsilon \otimes \mu)\\ &= (\mathbf{T} \circ \mathrm{red}_M \circ T)(\varepsilon \otimes F) = (\mathbf{T} \circ \mathrm{red}_M)((\varepsilon_M + \varepsilon) \otimes F)\\ &= \mathbf{T}(1 \otimes F + \varepsilon \otimes \mu) = 1 \otimes F + z\partial_z \otimes \mu\end{aligned}$$

□

As for $\mathbb{S}$, we can define $\mathbb{T}$ on an arbitrary Calabi–Yau 3-fold X.

Definition 3.18. Let (X, Ω_X) be a Calabi–Yau 3-fold with a holomorphic volume form Ω_X. We define $\mathbb{T}$ on $\mathcal{E}^C_{\mathrm{mBCOV}}(X)$ by the following:

- Given $F \in \mathrm{PV}^{0,\bullet}(X)$,
$$\mathbb{T}(F) = F \in \mathrm{PV}^{0,\bullet}(X);$$
- Given $\mu \in \ker \partial \subset \mathrm{PV}^{1,\bullet}(X)$,
$$\mathbb{T}(\mu) = \mu \in \ker \partial \subset \mathrm{PV}^{1,\bullet}(X);$$
- Given $\nu \in \phi_C(\mathrm{PV}^{3,\bullet}(X))$
$$\mathbb{T}(\nu) = \nu + G \in \mathrm{PV}(X)$$
where G is such that $\nu = \phi_C(G \wedge \Omega_X^{-1})$.

Theorem 3.19. Let (X, Ω_X) be a Calabi–Yau 3-fold. Then $\mathbb{S}$ and $\mathbb{T}$ generate the action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathcal{E}^C_{\mathrm{mBCOV}}(X)$ as a DG Lie algebra and hence $\mathrm{IIB}_{\mathrm{SUGRA}}[M^4_A \times X^3_B]$.

Proof. First, we need to check $\mathbb{S}^2 = -\mathrm{Id}$ and $(\mathbb{S}\mathbb{T})^3 = -\mathrm{Id}$; the first is apparent and the second follows from noting $\mathbb{S}|_{\ker \partial \subset \mathrm{PV}^{1,\bullet}(X)} = \mathrm{Id} = \mathbb{T}|_{\ker \partial \subset \mathrm{PV}^{1,\bullet}(X)}$ and the following chain

$$F \xrightarrow{\mathbb{T}} F \xrightarrow{\mathbb{S}} \mu \xrightarrow{\mathbb{T}} \mu + F \xrightarrow{\mathbb{S}} -F + \mu \xrightarrow{\mathbb{T}} \mu \xrightarrow{\mathbb{S}} -F$$

where $F \in \mathrm{PV}^{0,\bullet}(X)$ and $\mu = \phi_C(F \wedge \Omega_X^{-1}) \in \phi_C(\mathrm{PV}^{3,\bullet}(X))$.

Thanks to Theorem 3.15, it remains to show that the action of $\mathbb{T}$ preserves the DG Lie algebra structure. We prove in a similar way as follows:

- Let $F \in \mathrm{PV}^{0,\bullet}(X)$ and $\mu \in \ker \partial \subset \mathrm{PV}^{1,\bullet}(X)$. Then
$$\mathbb{T}\left([F, \mu]^C_{\mathrm{SN}}\right) = \mathbb{T}\left([F, \mu]_{\mathrm{SN}}\right) = [F, \mu]_{\mathrm{SN}} = [\mathbb{T}(F), \mathbb{T}(\mu)]_{\mathrm{SN}} = [\mathbb{T}(F), \mathbb{T}(\mu)]^C_{\mathrm{SN}}.$$
- Let $F \in \mathrm{PV}^{0,\bullet}(X)$ and $\nu \in \phi_C(\mathrm{PV}^{3,\bullet}(X))$, say $\nu = \phi_C(G \wedge \Omega_X^{-1})$ for $G \in \mathrm{PV}^{0,\bullet}(X)$.
$$\mathbb{T}\left([F, \nu]^C_{\mathrm{SN}}\right) = [F, \nu]^C_{\mathrm{SN}} = [F, \nu + G]^C_{\mathrm{SN}} = [\mathbb{T}(F), \mathbb{T}(\nu)]^C_{\mathrm{SN}}$$
- Let $\mu \in \ker \partial \subset \mathrm{PV}^{1,\bullet}(X)$ and $\nu \in \phi_C(\mathrm{PV}^{3,\bullet}(X))$, say $\nu = \phi_C(G \wedge \Omega_X^{-1})$. Then note
$$\mathbb{T}([\mu, \nu]^C_{\mathrm{SN}}) = [\mu, \nu]^C_{\mathrm{SN}} + H$$
where $[\mu, \nu]^C_{\mathrm{SN}} = \phi_C(H \wedge \Omega_X^{-1})$. On the other hand, we have
$$[\mathbb{T}(\mu), \mathbb{T}(\nu)]^C_{\mathrm{SN}} = [\mu, \nu + G]^C_{\mathrm{SN}} = [\mu, \nu]^C_{\mathrm{SN}} + [\mu, G]_{\mathrm{SN}}$$
Now it remains to show $H = [\mu, G]_{\mathrm{SN}}$, that is, $[\mu, \phi_C(G \wedge \Omega_X^{-1})]^C_{\mathrm{SN}} = \phi_C([\mu, G]_{\mathrm{SN}} \wedge \Omega_X^{-1})$. Note
$$\begin{aligned}
[\mu, \phi_C(G \wedge \Omega_X^{-1})]^C_{\mathrm{SN}} &= (-1)^{|\mu|-1} \phi_C[\mu, G \wedge \Omega_X^{-1}]_{\mathrm{SN}} = (-1)^{|\mu|-1}(-1)^{(|\mu|-1)|G|} \phi_C[G \wedge \Omega_X^{-1}, \mu]_{\mathrm{SN}} \\
&= (-1)^{(|\mu|-1)(|G|-1)} \phi_C\left([G, \mu]_{\mathrm{SN}} \wedge \Omega_X^{-1}\right) \\
&= (-1)^{(|\mu|-1)(|G|-1)}(-1)^{(|\mu|-1)(|G|-1)} \phi_C\left([\mu, G]_{\mathrm{SN}} \wedge \Omega_X^{-1}\right) \\
&= \phi_C\left([\mu, G]_{\mathrm{SN}} \wedge \Omega_X^{-1}\right)
\end{aligned}$$
where we used the fact that $[-, \mu]_{\mathrm{SN}}$ intertwines the identification $(-) \wedge \Omega_X^{-1} \colon \mathrm{PV}^{0,\bullet}(X) \cong \mathrm{PV}^{3,\bullet}(X)$.

$\square$

Remark 3.20. In their work on holography [CG18], Costello and Gaiotto give a presentation of BCOV theory on a Calabi–Yau 3-fold that exhibits a manifest $\mathrm{SL}(2,\mathbb{C})$ symmetry and argue this comes from S-duality of type IIB string theory. Here we argue our constructions are consistent with their claims.

We begin by recalling the construction in [CG18], adapted for comparison. The authors begin by rewriting the space of fields of BCOV theory on X by replacing $\ker\partial \subset \mathrm{PV}^{2,\bullet}(X)$ by $\operatorname{im}\partial$ or $\mathrm{PV}^{3,\bullet}(X)$. In this way, if we write $\ker\partial \subset \mathrm{PV}^{p,q}(X)$ as $\ker^{p,q}$, then the space of fields is

$$\begin{tikzcd}
\mathrm{PV}^{0,0} \arrow[r] & \mathrm{PV}^{0,1} \arrow[r] & \mathrm{PV}^{0,2} \arrow[r] & \mathrm{PV}^{0,3} & & & \\
& & \ker^{1,0} \arrow[r] & \ker^{1,1} \arrow[r] & \ker^{1,2} \arrow[r] & \ker^{1,3} & \\
& & & & \mathrm{PV}^{3,0} \arrow[r] & \mathrm{PV}^{3,1} \arrow[r] & \mathrm{PV}^{3,2} \arrow[r] & \mathrm{PV}^{3,3}
\end{tikzcd}$$

where the fields of the theory in degree zero are $\alpha^{1,1} \in \ker^{1,1}$, $\alpha^{0,2} \in \mathrm{PV}^{0,2}$, and $\gamma^{3,0} \in \mathrm{PV}^{3,0}$. Making this replacement, the ghost number zero part of the action functional of the theory reads

$$\begin{aligned}
&\tfrac{1}{2}\operatorname{Tr}(\bar\partial\,\alpha^{1,1}\wedge\partial^{-1}\alpha^{1,1})) && +\tfrac{1}{6}\operatorname{Tr}(\alpha^{1,1}\wedge\alpha^{1,1}\wedge\alpha^{1,1}) \\
&+\operatorname{Tr}(\alpha^{0,2}\wedge\bar\partial\,\gamma^{3,0}) && +\operatorname{Tr}(\alpha^{0,2}\wedge\alpha^{1,1}\wedge\partial\gamma^{3,0}).
\end{aligned}$$

The first line above is the action for so-called type I BCOV theory on X (for more detailed discussion, see [CL19]) and the second line describes another field theory coupled to type I BCOV via the second term in the second line. Let us analyze the coupling term. Note that for any $\alpha,\beta \in \ker\partial$, we have that $\partial(\alpha\wedge\beta) = [\alpha,\beta]_{\mathrm{SN}}$. Now, as in Remark 2.15, we can make the following heuristic observations:

$$\operatorname{Tr}(\alpha^{0,2}\wedge\bar\partial\,\gamma^{3,0}) = \operatorname{Tr}(\alpha^{0,2}\wedge\bar\partial\,\partial^{-1}\alpha^{2,0})$$

and

$$\begin{aligned}
\operatorname{Tr}(\alpha^{0,2}\wedge\alpha^{1,1}\wedge\partial\gamma^{3,0}) =& \frac{1}{2}\operatorname{Tr}(\alpha^{0,2}\wedge\partial^{-1}[\alpha^{1,1},\partial\gamma^{3,0}]_{\mathrm{SN}} + \alpha^{1,1}\wedge\partial^{-1}[\alpha^{0,2},\partial\gamma^{3,0}]_{\mathrm{SN}}) \\
=& -\frac{1}{2}\operatorname{Tr}\left(\alpha^{0,2}\wedge\partial^{-1}\partial[\alpha^{1,1},\gamma^{3,0}]_{\mathrm{SN}} + \alpha^{1,1}\wedge\partial^{-1}\partial[\alpha^{0,2},\gamma^{3,0}]_{\mathrm{SN}}\right) \\
=& -\frac{1}{2}\operatorname{Tr}\left(\alpha^{0,2}\wedge\partial^{-1}\partial[\alpha^{1,1},\partial^{-1}\alpha^{2,0}]_{\mathrm{SN}} + \alpha^{1,1}\wedge\partial^{-1}\partial[\alpha^{0,2},\partial^{-1}\alpha^{2,0}]_{\mathrm{SN}}\right) \\
=& \frac{1}{2}\operatorname{Tr}\left(\alpha^{0,2}\wedge\partial^{-1}[\alpha^{1,1},\alpha^{2,0}]^{C}_{\mathrm{SN}} + \alpha^{1,1}\wedge\partial^{-1}[\alpha^{0,2},\alpha^{2,0}]^{C}_{\mathrm{SN}}\right).
\end{aligned}$$

In other words, we see that the action functional agrees with what one would have written from our definition of $\mathcal{E}^{C}_{\mathrm{mBCOV}}(X)$.

Contracting against the Calabi–Yau form Ω_X, we may write $\gamma^{3,\bullet} = \gamma^{0,\bullet}\wedge\Omega_X^{-1}$ where $\gamma^{0,\bullet} \in \Omega^{0,\bullet}(X)[2]$. Now consider a $\mathbb{C}^\times$-action under which $\gamma^{3,0}$ has weight 1 and $\alpha^{0,2}$ has weight -1. By way of this, we shift the cohomological and fermionic degrees of the fields. Doing so and renaming things, the fields look like

$$\Pi\left(\Omega^{0,0} \longrightarrow \Omega^{0,1} \longrightarrow \Omega^{0,2} \longrightarrow \Omega^{0,3}\right)$$

$$\ker^{1,0} \longrightarrow \ker^{1,1} \longrightarrow \ker^{1,2} \longrightarrow \ker^{1,3}$$

$$\Pi\left(\Omega^{0,0} \longrightarrow \Omega^{0,1} \longrightarrow \Omega^{0,2} \longrightarrow \Omega^{0,3}\right)$$

where the fields of degree zero are $\alpha^{1,1}$ as before, together with two fermionic $(0,1)$-forms $\gamma^{0,1}, \alpha^{0,1} \in \Omega^{0,1}(X) \otimes \Pi\,\mathbb{C}$. Having rewritten BCOV theory in this way, Costello and Gaiotto note that the theory has a manifest global $\mathrm{SL}(2,\mathbb{C})$ symmetry permuting the $\alpha^{0,\bullet}$ and $\gamma^{0,\bullet}$ fields.

It is clear from the above description that from the point of view of the original space of fields of BCOV theory on X, the proposed $\mathrm{SL}(2,\mathbb{C})$ action amounts to the canonical identification of $\mathrm{PV}^{0,\bullet}(X)$ and $\mathrm{PV}^{3,\bullet}(X)$ using the Calabi–Yau volume form, possibly with a sign. Note that this is clearly our proposed twisted $\mathrm{SL}(2,\mathbb{C})$ action, so in a sense we provided a context for their conjecture in terms of protected sectors of M-theory and type II string theories and checked the case of $\mathbb{C}^\times \times X^2$, where X^2 is as in Proposition 3.2, based on the description of twisted supergravity theories.

We in fact expect a stronger statement to be true: the fields of $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{R}^4_A \times X^3_B]$ should localize to $\mathcal{E}^C_{\mathrm{mBCOV}}(X)$ upon subjecting the theory to an Ω-deformation along $\mathbb{R}^4$. This point will be addressed in future work.

3.5 S-duality Equivariant Residual Symmetries

In this subsection, we argue that our S-duality map is compatible with expected S-duality on certain space of residual supersymmetry algebra. (Surya: add an introduction to this section)

Let us begin by providing some physics background. Recall the $\mathcal{N} = (2,0)$ super-translation algebra and the $\mathcal{N} = (2,0)$ supersymmetry algebra, that is,

$$\mathcal{T}^{(2,0)} = \mathbb{C}^{10} \oplus \Pi(S_+ \otimes \mathbb{C}^2) \qquad \text{and} \qquad \mathfrak{siso}_{\mathrm{IIB}} = \mathfrak{so}(10,\mathbb{C}) \ltimes \mathcal{T}^{(2,0)}.$$

It is a classic fact due to [Tow95] that form-valued central extensions of $\mathcal{T}^{(2,0)}$ correspond to D-branes, the NS5-brane, and the F1-string. This essentially follows from Noether's theorem: that is, given any of the above extended objects, one has an action of $\mathcal{T}^{(2,0)}$ on the world-volume theory on strings or branes, and one would like the map from the corresponding currents of $\mathcal{T}^{(2,0)}$ to (a shift of) the observables of the world-volume theory to be a map of DG Lie algebras. This is in general not true unless we replace the current algebra with a central extension thereof – this is where the central extensions in consideration originate. Replacing $\mathcal{T}^{(2,0)}$ with this central extension in our first-order description of type IIB supergravity would yield a much more uniform description of the theory.

Remark 3.21. Indeed, the higher gauge fields such as the B-field and RR-forms are realized as coming from those components of the fundamental field that are valued in the central piece. (Phil: Can we say things about B-fields?) In fact, a more modern perspective [FSS15] reformulates each of these form-valued central extensions as a one-dimensional central extension as L_∞-algebras. From this perspective, the fundamental field of supergravity is a 1-form valued in a Lie n-algebra – this locally describes an n-connection on a bundle of n-groups which is the correct global nature of the B-field and RR-forms.

It is further known that the central extensions of $\mathfrak{siso}_{\mathrm{IIB}}$ are related by a $\mathbb{Z}/4$-action, reflecting the action of S-duality on branes. Our goal is to compare our S-duality map with a certain residual supersymmetry algebra. Recall that in Subsection 2.3.3, we realized that the cohomology of $\mathfrak{siso}_{\mathrm{IIB}}$ with respect to an $\mathfrak{sl}(5,\mathbb{C})$-invariant square-zero supercharge Q_5 has a canonical map to the $\mathfrak{sl}(5,\mathbb{C})$-invariant twist of type IIB supergravity theory, that is,

$$\Xi\colon H^\bullet(\mathfrak{siso}_{\mathrm{IIB}}, Q_5) \to \mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5_B].$$

Recall that it is not a map of DG Lie algebras. Now, we simultaneously introduce two modifications. One is to consider a fore-mentioned universal extension that admits the action of S-duality. The other is to take cohomology with respect to an $\mathfrak{sl}(3,\mathbb{C})$-invariant square-zero supercharge Q_3 to match with $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{R}^4_A \times \mathbb{C}^3_B]$. Then with an induced S-duality action on the cohomology of the extension of $\mathfrak{siso}_{\mathrm{IIB}}$ with respect to Q_3, we should have an S-duality equivariant map of DG Lie algebras from this cohomology to the fields of $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{R}^4_A \times \mathbb{C}^3_B]$, fitting in the following table:

(residual) supersymmetry algebra	supergravity theory
$\mathfrak{siso}_{\mathrm{IIB}}$	physical IIB supergravity
$H^\bullet(\mathfrak{siso}_{\mathrm{IIB}}, Q_5)$	$\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5_B]$
$H^\bullet(\mathfrak{superstring}_{\mathrm{IIB}}, Q_3)$	$\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{R}^4_A \times \mathbb{C}^3_B]$

Physically speaking, since $\Xi\left(H^\bullet(\mathfrak{siso}_{\mathrm{IIB}}, Q_5)\right) \subset \mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{C}^5]$ describes certain ghosts in the twisted theory, we expect the image of the cohomology of this extended algebra to include some components of higher ghosts of higher gauge fields. Then the S-duality equivariance of the embedding would then prove that our S-duality map agrees with how S-duality acts on higher ghosts in the physical string theory.

We now pursue a version of this construction: we will describe a two-dimensional central extension incorporating the central extensions corresponding to the F1 and D1 strings under the dictionary above, and an action of $\mathrm{SL}(2,\mathbb{C})$ on the graded vector space underlying the centrally-extended algebra. To describe this central extension, we first fix some notation. The $\mathcal{N}=(2,0)$ supertranslation algebra $\mathcal{T}^{(2,0)} = \mathbb{C}^{10} \oplus \Pi(S_+ \otimes \mathbb{C}^2)$ is equipped with a nontrivial Lie bracket from $\Gamma^+\colon \mathrm{Sym}^2 S_+ \to \mathbb{C}^{10}$. We fix an inner product $\langle -,- \rangle$ on the $\mathbb{C}^2$ factor of $\mathcal{T}^{(2,0)}$ and orthonormal basis e_1, e_2 for $\mathbb{C}^2$. The supersymmetry algebra $\mathfrak{siso}_{\mathrm{IIB}} = \mathfrak{so}(10,\mathbb{C}) \ltimes \mathcal{T}^{(2,0)}$ has additional Lie bracket from the natural action of $\mathfrak{so}(10,\mathbb{C})$ on $\mathcal{T}^{(2,0)}$.

Definition 3.22. We let $\mathfrak{superstring}_{\mathrm{IIB}}$ denote the super L_∞-algebra with underlying $\mathbb{Z}\times\mathbb{Z}/2$-graded vector space

$$\mathfrak{superstring}_{\mathrm{IIB}} = \mathfrak{siso}_{\mathrm{IIB}} \oplus (\Omega^\bullet(\mathbb{R}^{10})[1] \otimes \mathbb{C}^2_S) = \mathfrak{so}(10,\mathbb{C}) \oplus \mathbb{C}^{10} \oplus \Pi(S_+ \otimes \mathbb{C}^2) \oplus (\Omega^\bullet(\mathbb{R}^{10})[1] \otimes \mathbb{C}^2_S)$$

and with the bracket on $\Pi(S_+ \otimes \mathbb{C}^2)$ determined as follows. Choose a basis F, D for $\mathbb{C}^2_S$ and let $\mathrm{F1}, \mathrm{D1} \in \mathrm{Sym}^2(\mathbb{C}^2)$ such that in the e_1, e_2 basis for $\mathbb{C}^2$, one has $\mathrm{F1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathrm{D1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then the bracket is given by

$$[\psi_1 \otimes e_i, \psi_2 \otimes e_j] = [\psi_1 \otimes e_i, \psi_2 \otimes e_j]_T + [\psi_1 \otimes e_i, \psi_2 \otimes e_j]_F + [\psi_1 \otimes e_i, \psi_2 \otimes e_j]_D$$

where

$$\begin{aligned}
[\psi_1 \otimes e_i, \psi_2 \otimes e_j]_T &:= \Gamma^+(\psi_1 \otimes \psi_2)\langle e_i, e_j \rangle \\
[\psi_1 \otimes e_i, \psi_2 \otimes e_j]_F &:= \Gamma^+(\psi_1 \otimes \psi_2) \otimes \mathrm{F1}(e_i, e_j) F \\
[\psi_1 \otimes e_i, \psi_2 \otimes e_j]_D &:= \Gamma^+(\psi_1 \otimes \psi_2) \otimes \mathrm{D1}(e_i, e_j) D
\end{aligned}$$

where the image of Γ^+ may be thought of as a constant coefficient $\mathbb{C}$-valued 1-form $\Omega^1_{\mathrm{cc}}(\mathbb{R}^{10})$.

Remark 3.23. Let us begin by unpacking some identifications that are implicit in the above definition. Above, as Γ^+ is valued in the space of constant coefficient 1-forms $\Omega^1_{cc}(\mathbb{R}^{10})$, the brackets $[-,-]_F$, $[-,-]_T$ determine elements of $C^2(\mathcal{T}^{(2,0)}, \Omega^1_{cc}(\mathbb{R}^{10}))$. There is a map

$$C^2(\mathcal{T}^{(2,0)}, \Omega^1_{cc}(\mathbb{R}^{10})) \to C^3(\mathcal{T}^{(2,0)})$$
$$\mu \mapsto (\psi_1 \wedge \psi_2 \wedge v \mapsto \mu(\psi_1 \wedge \psi_2)(v))$$

under which each of $[-,-]_F$, $[-,-]_T$ becomes an element that pairs 2 spinors with a translation. In [SSS09], it is established that these elements of $C^3(\mathcal{T}^{(2,0)})$ are closed, and hence determine $\mathbb{C}[1]$-valued central extensions of $\mathcal{T}^{(2,0)}$ as an L_∞-algebra with the extra bracket $l_3\colon (\mathcal{T}^{(2,0)})^{\otimes 3} \to \mathbb{C}[1]$ corresponding to the element. In our definition of $\mathfrak{superstring}_{\mathrm{IIB}}$ above, we resolve $\mathbb{C}[1]$ so we see an $\Omega^\bullet(\mathbb{R}^{10})[1]$-valued extension. Moreover, the L_∞-structure on $\mathfrak{superstring}_{\mathrm{IIB}}$ is induced from the one determined by the cocycle in $C^3(\mathcal{T}^{(2,0)})$ by homotopy transfer. We will use a particular model for this transferred L_∞-structure, where translations act on $\Omega^1(\mathbb{R}^{10}) \otimes \mathbb{C}^2$ by Lie derivative, and on all other summand of the central piece trivially. This is natural in the gravity setting where translation already acts as a Lie derivative on sections of any bundle.

For each choice of $\mathfrak{sl}(3,\mathbb{C}) \subset \mathfrak{so}(10,\mathbb{C})$ we have an $\mathrm{SL}(2,\mathbb{C})$-action on $\mathfrak{superstring}_{\mathrm{IIB}}$ defined as follows. Denoting by V_3, the fundamental representation of $\mathfrak{sl}(3,\mathbb{C})$, note that

$$\mathrm{Res}^{\mathfrak{sl}(3,\mathbb{C})}_{\mathfrak{so}(10,\mathbb{C})} S_+ = \mathbb{C}^{\oplus 4} \oplus V_3^{\oplus 2} \oplus V_3^{*\oplus 2}.$$

Letting $\mathbb{C}^2_{V_3^*}$ denote the multiplicity space for the V_3^*-summand in the above decomposition, the desired $\mathrm{SL}(2,\mathbb{C})$ is the diagonal inside the $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ where a factor naturally acts on $\mathbb{C}^2_{V_3^*}$ and the other on $\mathbb{C}^2_S$. (Phil: Where does this come from? Especially the action on $\mathbb{C}^2_{V_3^*}$?)

Recall the choice of $\mathfrak{sl}(5,\mathbb{C}) \subset \mathfrak{so}(10,\mathbb{C})$ made in the proof of Lemma 2.22 with which, in particular, we find a decomposition $S_+ \cong \mathbb{C} \oplus \wedge^2 V_5 \oplus V_5^*$ as a representation of $\mathfrak{sl}(5,\mathbb{C})$ and choose

$$Q_5 = \Psi \otimes e_1 \in \mathbb{C} \otimes \mathbb{C}^2 \subset S_+ \otimes \mathbb{C}^2\,.$$

In what follows we fix the copy of $\mathfrak{sl}(3,\mathbb{C}) \subset \mathfrak{sl}(5,\mathbb{C})$ consisting of the lower 3×3 block. Let

$$Q_3 = Q_5 + \partial_1 \wedge \partial_2 \otimes e_2 \in S_+ \otimes \mathbb{C}^2;$$

this is a square zero supercharge invariant under the chosen $\mathfrak{sl}(3,\mathbb{C})$. We now compute $H^\bullet(\mathfrak{superstring}_{\mathrm{IIB}}, Q_3)$ and demonstrate that the above action of $\mathrm{SL}(2,\mathbb{C})$ descends.

Lemma 3.24. $H^\bullet(\mathfrak{superstring}_{\mathrm{IIB}}, Q_3)$ is the super Lie algebra whose underlying $\mathbb{Z}/2$-graded vector space is

$$\left(\mathfrak{sl}(3,\mathbb{C}) \oplus W \oplus \wedge^3 V\right) \ltimes \mathbb{C}\langle \partial_3, \partial_4, \partial_5\rangle \oplus \Pi\left(\mathbb{C}\langle \partial_3\partial_4, \partial_3\partial_5, \partial_4\partial_5\rangle \oplus \mathbb{C}\langle dz_3, dz_4, dz_5\rangle \oplus \mathbb{C}\langle c_F, c_D\rangle\right).$$

Here W is an 8-dimensional subspace of $\mathfrak{sl}(5,\mathbb{C})$ complementary to the copy of $\mathfrak{sl}(3,\mathbb{C}) \subset \mathfrak{sl}(5,\mathbb{C})$ consisting of the lower right 3×3 diagonal block. The brackets are as follows:

- the semidirect product structure comes from realizing $\mathfrak{sl}(3,\mathbb{C}) \oplus W$ as a subalgebra of $\mathfrak{sl}(5,\mathbb{C})$ and using the semidirect product structure from Lemma 2.22.
- the commutator of two elements in $\Pi\left(\mathbb{C}\langle \partial_3\partial_4, \partial_3\partial_5, \partial_4\partial_5\rangle \oplus \mathbb{C}\langle dz_3, dz_4, dz_5\rangle\right)$ is the commutator viewed as elements of $\Pi(S_+ \otimes \mathbb{C}^2)$.

- the elements c_F, c_D are central.
- Let $(-,-)$ be the standard Hermitian inner product on $\mathbb{C}^{10}$. Given $v \in \mathbb{C}\langle \partial_3, \partial_4, \partial_5\rangle$, $\psi \in \mathbb{C}\langle \partial_3\partial_4, \partial_3\partial_5, \partial_4\partial_5\rangle \oplus \mathbb{C}\langle dz_3, dz_4, dz_5\rangle \oplus \mathbb{C}\langle c_F, c_D\rangle$,
$$[v, \psi] = f(v, \Gamma^+(\Psi \otimes \psi + \partial_1\partial_2 \otimes \psi))$$
where we identify the image of Γ^+ with $\mathbb{C}^{10}$.

Proof. The following proof is a modification of the proof of Lemma 6.5.1 in [CL16]. We wish to compute the cohomology of the following $\mathbb{Z}/2$-graded complex:

$$\mathfrak{so}(10,\mathbb{C}) \xrightarrow{\begin{pmatrix} [Q_3,-]_T \\ 0 \\ 0 \end{pmatrix}} \begin{matrix} S_+ \otimes \mathbb{C}^2 \\ \oplus \\ F\Omega^0 \\ \oplus \\ D\Omega^0 \end{matrix} \xrightarrow{\begin{pmatrix} [Q_3,-]_T & & \\ [Q_3,-]_F & d & \\ [Q_3,-]_D & & d \end{pmatrix}} \begin{matrix} \mathbb{C}^{10} \\ \oplus \\ F\Omega^1 \\ \oplus \\ D\Omega^1 \end{matrix} \xrightarrow{\begin{pmatrix} 0 & d & \\ 0 & & d \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} d & \\ & d \end{pmatrix}} \begin{matrix} F\Omega^{10} \\ \oplus \\ D\Omega^{10} \end{matrix} .$$

This is a straightforward if lengthy calculation. We first compute cohomology with respect to the differential where the terms involve bracketing with Q_5 and the de Rham differential. This lets us use part of the proof of Lemma 2.22. (Surya: Need to make comment about collapse of spectral sequence for the total complex of this bicomplex) (Phil: or three steps?) Doing so, the result is:

$$\mathfrak{sl}(5,\mathbb{C}) \oplus \wedge^3 V_5 \xrightarrow{\begin{pmatrix} [\partial_1 \wedge \partial_2 \otimes e_1, -]_T \\ 0 \\ 0 \end{pmatrix}} (\mathbb{C} \oplus \wedge^2 V_5 \oplus V_5^*) \oplus (\mathbb{C}\, c_F \oplus \mathbb{C}\, c_D) \xrightarrow{\begin{pmatrix} [\partial_1 \wedge \partial_2, -]_T & \end{pmatrix}} V_5 .$$

Here c_F, c_D represent $-1 \in F\Omega^0, D\Omega^0$ respectively and are central.

The cohomology of this complex is as follows.

- Choose a basis of $\mathfrak{sl}(5,\mathbb{C})$ consisting of the elementary matrices, that is, E_{ij} with $i \neq j$ and $E_{ii} - E_{i+1,i+1}$. Then the elements E_{12}, E_{21} as well as any basis elements only involving $E_{3,*}, E_{4,*}, E_{5,*}$ all survive to cohomology. This is exactly the subspace $\mathfrak{sl}(3,\mathbb{C}) \oplus W$ as claimed. Further, the entire $\wedge^3 V_5$ summand survives to cohomology.
- The kernel at the second term consists of $\mathbb{C} \oplus \wedge^2 V \oplus \mathbb{C}\langle dz_3, dz_4, dz_5\rangle \oplus \mathbb{C}\, c_F \oplus \mathbb{C}\, c_D$. The image from $\mathfrak{sl}(5,\mathbb{C})$ is the 7-dimensional subspace of $\wedge^2 V$ consisting of elements of the form $\partial_1\partial_*, \partial_2\partial_*$. Thus, the cohomology at the second term is given by
$$\mathbb{C}\langle \partial_3\partial_4, \partial_3\partial_5, \partial_4\partial_5\rangle \oplus \mathbb{C}\langle dz_3, dz_4, dz_5\rangle \oplus \mathbb{C}\langle c_F, c_D\rangle$$
- The image of the last map is $\mathbb{C}\langle \partial_1, \partial_2\rangle$ so the cohomology is $\mathbb{C}\langle \partial_3, \partial_4, \partial_5\rangle$.

Note that since we have computed the Q_3-cohomology as the cohomology of a central extension of the Q_5-cohomology, we may appeal to Lemma 2.22 to see that the first three brackets are as claimed. Indeed, the odd bracket from the lemma restricts to a map of $\mathfrak{sl}(3,\mathbb{C})$ representations so must have image in $\mathbb{C}\langle \partial_3, \partial_4, \partial_5\rangle$. We need only compute $[v, \psi]$ for $v \in \mathbb{C}\langle \partial_3, \partial_4, \partial_5\rangle$, $\psi \in \mathbb{C}\langle \partial_3\partial_4, \partial_3\partial_5, \partial_4\partial_5\rangle \oplus \mathbb{C}\langle dz_3, dz_4, dz_5\rangle \oplus \mathbb{C}\langle c_F, c_D\rangle$. To do so, we need to choose $[Q_3, -]$-invariant representatives in $\mathfrak{superstring}_{\mathrm{IIB}}$ for the class of ψ.

To this end, define
$$L(\psi) = \psi \otimes e_2 - F \int_0^x \Gamma^+(\Psi \otimes \psi + \partial_1\partial_2 \otimes \psi) \in S_+ \otimes e_2 \oplus F\Omega^0$$

(Surya: There doesn't seem to be an obvious way to correct this to get something closed + other confusions) where again we view Γ^+ as landing in constant coefficient 1-forms on $\mathbb{R}^{10}$. This is in fact a closed element of $\mathfrak{superstring}_{\mathrm{IIB}}$:

$$\begin{pmatrix} [Q_3,-]_T & & \\ [Q_3,-]_F & d & \\ [Q_3,-]_D & & d \end{pmatrix} \begin{pmatrix} \psi\otimes e_2 \\ -\int_0^x \Gamma^+(\Psi\otimes\psi+\partial_1\partial_2\otimes\psi) \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ \Gamma^+((\Psi+\partial_1\partial_2)\otimes\psi) - d\int_0^x \Gamma^+(\Psi\otimes\psi+\partial_1\partial_2\otimes\psi) \\ 0 \end{pmatrix} = 0$$

where we have used that $[Q_3,-]_T$ and $[Q_3,-]_D$ vanish on elements of $S_+\otimes e_2$.

Now we compute in $\mathfrak{superstring}_{\mathrm{IIB}}$:

$$\begin{aligned} [v, L(\psi)] &= [v, \Psi\otimes\psi+\partial_1\partial_2\otimes\psi] - F[v, \int_0^x \Gamma^+(\Psi\otimes\psi+\partial_1\partial_2\otimes\psi)] \\ &= 0 - F(v, \Gamma^+(\Psi\otimes\psi+\partial_1\partial_2\otimes\psi)) \\ &= c_F(v, \Gamma^+(\Psi\otimes\psi+\partial_1\partial_2\otimes\psi) \text{ in cohomology} \end{aligned}$$

In the second equality above, we have identified the $\mathbb{C}$-valued linear function on $\mathbb{R}^{10}$ given by the integral of a constant coefficient 1-form in the image of Γ^+ with an element of $\mathbb{C}^{10}$.

This completes the proof. □

Note that $\mathbb{C}\langle\partial_3\partial_4,\partial_3\partial_5,\partial_4\partial_5\rangle \cong \wedge^2 V_3 \cong V_3^*$, so we may identify

$$\mathbb{C}\langle\partial_3\partial_4,\partial_3\partial_5,\partial_4\partial_5\rangle \oplus \mathbb{C}\langle dz_3, dz_4, dz_5\rangle \cong V_3^*\otimes\mathbb{C}^2 .$$

It is then clear that the $\mathrm{SL}(2,\mathbb{C})$ action on $\mathfrak{superstring}_{\mathrm{IIB}}$ descends to the diagonal $\mathrm{SL}(2,\mathbb{C})$ inside the $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ naturally acting on $\mathbb{C}^2\oplus\mathbb{C}\langle c_F, c_D\rangle$.

Lemma 3.25. The descended $\mathrm{SL}(2,\mathbb{C})$ action preserves the bracket on $H^\bullet(\mathfrak{superstring}_{\mathrm{IIB}}, Q_3)$ from lemma 3.24.

Proof. □

We now have the following map $H^\bullet(\mathfrak{superstring}_{\mathrm{IIB}}, Q_3) \to \mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{R}^4_A\times\mathbb{C}^3_B]$

	$\mathfrak{superstring}_{\mathrm{IIB}}$	$H^\bullet(\mathfrak{superstring}_{\mathrm{IIB}}, Q_3) \longrightarrow \mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{R}^4_A\times\mathbb{C}^3_B]$
spin	$\mathfrak{so}(10,\mathbb{C})$	$\mathfrak{sl}(5,\mathbb{C})\oplus\wedge^3 V\oplus\mathfrak{sl}(3,\mathbb{C})\oplus W\oplus\wedge^3 V \longrightarrow \Omega^\bullet(\mathbb{R}^4)\otimes\mathrm{PV}^{1,0}(\mathbb{C}^3)$ $(A_{ij}, \partial_i\partial_j\partial_k) \longmapsto 1\otimes\sum_{i,j} A_{ij} z_i \partial_{z_j}$
translation	$V\oplus V^*$	$\mathbb{C}\langle\partial_3,\partial_4,\partial_5\rangle \longrightarrow \Omega^\bullet(\mathbb{R}^4)\otimes\mathrm{PV}^{1,0}(\mathbb{C}^3)$ $\partial_i \longmapsto \partial_{z_i}$
SUSY	$\Pi(S_+\oplus S_+)$	$\Pi(\mathbb{C}\oplus\mathbb{C}\langle\partial_i\partial_j\rangle_{3\le i,j\le 5}\oplus\mathbb{C}\langle dz_k\rangle_{3\le k\le 5}) \to \Omega^\bullet(\mathbb{R}^4)\otimes\left(\mathrm{PV}^{2,0}\oplus\mathrm{PV}^{1,0}\right)$ $(a, \partial_i\partial_j, dz_k) \longmapsto (\partial_{z_i}\partial_{z_j}, z_k)$
higher ghosts	$\Omega^\bullet(\mathbb{R}^{10})[1]\otimes\mathbb{C}^2_S$	$\mathbb{C}\langle f, d\rangle \longrightarrow \Omega^\bullet(\mathbb{R}^4)\otimes(\mathrm{PV}^{0,0}(\mathbb{C}^3)\oplus\mathrm{PV}^{3,0}(\mathbb{C}^3))$ $(f,d) \longmapsto (1, \partial_{z_3}\partial_{z_4}\partial_{z_5})$

Lemma 3.26. The above map is a map of dg Lie algebras and intertwines the action of $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ on $H^\bullet(\mathfrak{superstring}_{\mathrm{IIB}}, Q_3)$ with the action of $\mathbb{S}$ on $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{R}^4_A\times\mathbb{C}^3_B]$.

Proof. To see that the map is Lie algebra homomorphism, by Remark **??** we need only check that the bracket involving the central terms is preserved. This is clear because we have the bracket $\{z_i, \partial_{z_j}\} = \delta_{ij}$ in $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{R}^4_A \times \mathbb{C}^3_B]$.

For the second statement, we begin by explicating the action of S under the isomorphism $\mathbb{C}\langle \partial_3\partial_4, \partial_3\partial_5, \partial_4\partial_5\rangle \oplus \mathbb{C}\langle dz_3, dz_4, dz_5\rangle \cong V_3^* \otimes \mathbb{C}^2$. Choosing a basis for the latter consisting of $v_i \otimes x_j$ with $i = 3, 4, 5$ and $j = 1, 2$, the isomorphism at hand sends

$$\partial_3\partial_4 \mapsto v_5 \otimes x_1,\ \partial_3\partial_5 \mapsto -v_4 \otimes x_1,\ \partial_4\partial_5 \mapsto v_3 \otimes x_1,\ dz_i \mapsto v_i \otimes x_2.$$

Thus we have that S acts as in the following table:

$S(\partial_3\partial_4) = -dz_5$	$S(dz_5) = \partial_3\partial_4$
$S(\partial_3\partial_5) = dz_4$	$S(dz_4) = -\partial_3\partial_5$
$S(\partial_4\partial_5) = -dz_3$	$S(dz_3) = \partial_4\partial_5$

Thus, we readily see that the third row in the table preceding the statement of the lemma intertwines S with $\mathbb{S}$.

Finally, on the central piece, $\mathbb{C}\langle c_F, c_D\rangle$ it is clear that S acts as $c_F \mapsto c_D, c_D \mapsto -c_F$. Thus, it's clear that the last row in the above table is S-equivariant.

□

4 Applications

In this section, we present some applications of our constructions. We focus on the case of flat spaces for technical reasons. We argue that several interesting deformations of 4-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory are S-dual to each other. The deformations in question will be further deformations of the holomorphic-topological twist (also known as Kapustin twist after [Kap06]). For $G = \mathrm{GL}(N)$ this is precisely the theory living on a stack of N D3 branes wrapping $\mathbb{R}^2 \times \mathbb{C}$ in topological string theory on $\mathbb{R}^4_A \times \mathbb{C}^3_B$ (see Example 2.7). The deformations of interest are:

- The HT(A) and HT(B) twists of 4d $\mathcal{N} = 4$ gauge theory. These are further twists of the Kapustin twist that are relevant for the Geometric Langlands theory as analyzed in [EY18, EY19].
- Different types of superconformal deformations of 4d $\mathcal{N} = 4$ gauge theory as is realized in the supergravity setting via AdS/CFT correspondence. We also explain how these arise as homotopies trivializing the action of certain rotations on the background in the flavor of the Ω-deformation.
- A quadratic superpotential transverse to the world-volume of the D3 branes. This deforms holomorphic-topological twist of 4-dimensional $\mathcal{N} = 4$ theory with gauge group $\mathrm{GL}(N)$ into 4-dimensional Chern–Simons theory with gauge group $\mathrm{GL}(N|N)$.

The desired claims will follow in two steps:

1. We first check that the holomorphic-topological twist is preserved under S-duality. To do so, we check that the stack of D3 branes is mapped to itself, which is as expected from physical string theory. This amounts to checking that the field sourced by the D3 branes in the sense of Definition 2.29 is preserved by S-duality. This is also an expected result from the work of Kapustin [Kap06] but we argue it from a stringy perspective.

2. We then check that the claimed deformations are exchanged under S-duality. This amounts to checking that the preimages of these deformations under the closed-open map are mapped to each other under S-duality.

4.1 S-duality of a D3 Brane

In this subsection, we argue that a stack of D3 branes wrapping $\mathbb{R}^2 \times \mathbb{C}$ in type IIB supergravity theory on $\mathbb{R}^4_A \times \mathbb{C}^3_B$ is preserved under S-duality and conclude that this is related to a version of the geometric Langlands correspondence.

4.1.1 S-duality on a Field Sourced by D3 Branes

As indicated above, the argument will proceed by analyzing the equation for the field sourced by the D3 branes.

Proposition 4.1. The field sourced by D3 branes wrapping $\mathbb{R}^2 \times \mathbb{C} \subset \mathbb{R}^4_A \times \mathbb{C}^3_B$ is preserved under S-duality.

Proof. Let us first consider the minimal twist of type IIB supergravity theory on $\mathbb{C}^5$. For future reference, we say that $\mathbb{C}^5$ has coordinates u, v, z, w_1, w_2 and fix a holomorphic volume form $\Omega_{\mathbb{C}^5} = du \wedge dv \wedge dz \wedge dw_1 \wedge dw_2$. Suppose we have a stack of N D3 branes supported at $v = w_1 = w_2 = 0$. By Definition 2.29, the field F sourced by the N D3 branes is a $(2,2)$-polyvector $F \in \ker \partial$ satisfying the equation

$$\bar{\partial} F \vee \Omega_{\mathbb{C}^5} = N \delta_{v=w_1=w_2=0}.$$

A solution of this equation is given by the Bochner–Martinelli kernel

$$F_{\mathrm{sol}} = N \frac{\partial_u \wedge \partial_z}{(|v|^2 + |w_1|^2 + |w_2|^2)^3} (\bar{v} d\bar{w}_1 \wedge d\bar{w}_2 + \bar{w}_1 d\bar{w}_2 \wedge d\bar{v} + \bar{w}_2 d\bar{v} \wedge d\bar{w}_2).$$

We are omitting an overall factor to get the same normalization as in Definition 2.29, as our argument holds regardless of the normalization. The reader may find formulas with the correct normalization in [GH78].

Now consider the further twist of IIB supergravity theory $\mathrm{IIB}_{\mathrm{SUGRA}}[\mathbb{R}^4_A \times \mathbb{C}^3_B]$. We may think of it as gotten by making $\mathbb{C}^2_{u,v}$ noncommutative, i.e. by turning on the Poisson bivector $\partial_u \wedge \partial_v$ (see Example 2.25). Then the field sourced by the D3 branes now should satisfy the deformed equation

$$(\bar{\partial} + [\partial_u \wedge \partial_v, -]_{\mathrm{SN}}) F \vee \Omega_{\mathbb{C}^5} = N \delta_{v=w_1=w_2=0},$$

where $[-,-]_{\mathrm{SN}}$ denotes the Schouten–Nijenhuis bracket on $\mathrm{PV}^{\bullet,\bullet}(\mathbb{C}^2)$ up to a sign. We claim that F_{sol} is in the kernel of $[\partial_u \wedge \partial_v, -]_{\mathrm{SN}}$ so that it is still a solution to the deformed equation. To show this, note

$$[\partial_u \wedge \partial_v, -]_{\mathrm{SN}} = \partial_u \wedge \partial_v(-) \pm \partial_u(-) \wedge \partial_v$$

and then F_{sol} is evidently annihilated by both of these terms separately. This establishes the claim.

Under the isomorphism $\Omega^\bullet(\mathbb{R}^4) \cong \mathrm{PV}^{\bullet,\bullet}(\mathbb{C}^2_{u,v})$ induced by the standard symplectic form on $\mathbb{C}^2$, the above becomes

$$\begin{aligned} F_{\mathrm{sol}} &= N \frac{dv \wedge \partial_z}{(|v|^2 + |w_1|^2 + |w_2|^2)^3} (\bar{v} d\bar{w}_1 \wedge d\bar{w}_2 + \bar{w}_1 d\bar{w}_2 \wedge d\bar{v} + \bar{w}_2 d\bar{v} \wedge d\bar{w}_1) \\ &\in (\Omega^1(\mathbb{R}^4) \oplus \Omega^2(\mathbb{R}^4)) \otimes \mathbb{C}\langle \partial_z \rangle \otimes \mathrm{PV}^{0,\bullet}(\mathbb{C}^2_{w_1,w_2}) \end{aligned}$$

Now, by Definition 3.13, $\mathbb{S}$ acts as the identity on F_{sol}. $\square$

Remark 4.2. We proved the proposition for the flat space $\mathbb{R}^2 \times \mathbb{C} \subset \mathbb{R}^4_A \times \mathbb{C}^3_B$ because it is the most familiar and fundamental setting. The same argument works for $\mathbb{R}^2 \times \mathbb{C}^\times \subset \mathbb{R}^4_A \times \mathbb{C}^\times_B \times \mathbb{C}^2_B$ using Proposition 3.11, which is based only on first principle arguments.

4.1.2 Dolbeault Geometric Langlands Correspondence

We now explain the relation of the above with the so-called Dolbeault geometric Langlands correspondence. A large part of what follows is a summary of the second author's joint work with C. Elliott [EY18]. For more details and contexts, one is advised to refer to the original article. (Phil: Need to revisit things related to GL and KW after understanding things better because the twists are not the same as KW's.)

Let C be a smooth projective curve and G be a reductive group over $\mathbb{C}$ (and we write $\check{G}$ for its Langlands dual group). The best hope version of the geometric Langlands duality asserts an equivalence of DG categories

$$\mathrm{D}(\mathrm{Bun}_G(C)) \simeq \mathrm{QCoh}(\mathrm{Flat}_{\check{G}}(C)).$$

where $\mathrm{D}(\mathrm{Bun}_G(C))$ is the category of D-modules on the space $\mathrm{Bun}_G(C)$ of G-bundles on C and $\mathrm{QCoh}(\mathrm{Flat}_{\check{G}}(C))$ is of quasi-coherent sheaves on the space $\mathrm{Flat}_{\check{G}}(C)$ of flat $\check{G}$-connections on C. This is the *de Rham geometric Langlands* correspondence.

Kapustin and Witten [KW07] studied certain $\mathbb{CP}^1$-family of topological twists of 4-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory. To find the relation with the geometric Langlands correspondence, they realized that two twists should play particularly important roles. These are what are called the *A-twist* and *B-twist*, because they become A-model and B-model after certain compactification. Namely, if the 4-dimensional spacetime manifold X is of the form $X = \Sigma \times C$, then compactification along C leads to A-model on Σ with target moduli space of Higgs bundles on C and B-model on Σ with target the moduli space of flat connections on C. Then studying the categories of boundary conditions of S-dual theories leads to a version of the geometric Langlands correspondence. However, this is most naturally seen as depending only on the topology of C (which led to the exciting program of *Betti geometric Langlands* correspondence [BZN16]), as opposed to the algebraic structure of C which the original program is about.

In [EY18], a framework was introduced to capture the algebraic structure of the moduli spaces of solutions to the equations of motion. Moreover, it was suggested that when $X = \Sigma \times C$ one can study *holomorphic-topological twist* where the dependence is topological on Σ and holomorphic on C and the following theorem was proven. Here we write $\mathrm{EOM}(M) = \mathrm{EOM}^G(M)$ for the moduli space of solutions of a twisted gauge theory with group G on a spacetime manifold M and use subscripts to denote which twist we have used.

Theorem 4.3. [EY18]

$$\mathrm{EOM}_{\mathrm{HT}}(\Sigma \times C) = T^*_{\mathrm{form}}[-1]\, \underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}} \times C_{\mathrm{Dol}}, BG) = T^*_{\mathrm{form}}[-1]\, \underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}}, \mathrm{Higgs}_G(C))$$

where $\mathrm{Higgs}_G(Y) := \underline{\mathrm{Map}}(Y_{\mathrm{Dol}}, BG)$ is the moduli space of G-Higgs bundles on Y, where $Y_{\mathrm{Dol}} := T_{\mathrm{form}}[1]Y$ is the Dolbeault stack of Y, Y_{dR} is the de Rham stack of Y, and $T^*_{\mathrm{form}}[-1](-)$ stands for the formal completion of the (-1)-shifted cotangent bundle along the zero section.

In this language, a B-model with target X would be described by $T^*_{\mathrm{form}}[-1]\, \underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}}, X)$, so the result can be summarized as stating that compactifying the holomorphic-topological twist along C yields the B-model with target $\mathrm{Higgs}_G(C)$. Categorified geometric quantization of B-model is studied in the sequel [EY19] where its relation with the moduli space of vacua is also investigated in detail. For the purpose of this paper, one can take the category of boundary conditions of the B-model with target X to be $\mathrm{QCoh}(X)$ as is common in the context of homological mirror symmetry.

Now recall that the holomorphic-topological twist is exactly what we see by putting D3 branes on $\mathbb{R}^2 \times \mathbb{C} \subset \mathbb{R}^4_A \times \mathbb{C}^3_B$. The globalization data needed to consider a theory on a non-flat space $\Sigma \times C$ comes from a twisting homomorphism. Then the fact that the field sourced by D3 branes on $\mathbb{R}^2 \times \mathbb{C}$ is preserved under S-duality suggests that the holomorphic-topological twist must be self-dual under S-duality for $G = \mathrm{GL}(N)$. In other words, our S-duality result expects a nontrivial conjectural equivalence $\mathrm{QCoh}(\mathrm{Higgs}_G(C)) \simeq \mathrm{QCoh}(\mathrm{Higgs}_G(C))$ for $G = \mathrm{GL}(N)$. This is compatible with the conjectural equivalence $\mathrm{QCoh}(\mathrm{Higgs}_G(C)) \simeq \mathrm{QCoh}(\mathrm{Higgs}_{\check{G}}(C))$ for a general reductive group G, which is what is known as the *classical limit of geometric Langlands* or *Dolbeault geometric Langlands* correspondence of Donagi and Pantev [DP12].

4.2 S-duality and De Rham Geometric Langlands Correspondence

We claim that our S-duality map in fact predicts the de Rham geometric Langlands correspondence as well. We begin by recalling a result from [EY18].

Theorem 4.4. [EY18]

$$\begin{aligned}
\mathrm{EOM}_{\mathrm{HT(A)}}(\Sigma \times C) &= \underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}} \times C_{\mathrm{Dol}}, BG)_{\mathrm{dR}} &&= T^*_{\mathrm{form}}[-1]\,\underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}}, \mathrm{Bun}_G(C)_{\mathrm{dR}}) \\
\mathrm{EOM}_{\mathrm{HT(B)}}(\Sigma \times C) &= T^*_{\mathrm{form}}[-1]\,\underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}} \times C_{\mathrm{dR}}, BG) &&= T^*_{\mathrm{form}}[-1]\,\underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}}, \mathrm{Flat}_G(C))
\end{aligned}$$

Here the notation is different from the one of [EY18]. We use the notation HT(A) and HT(B) to emphasize that those are realized as further twists of the holomorphic-topological twist. By considering category of boundary conditions or taking the category of quasi-coherent sheaves of the target of the B-model, we obtain $D(\mathrm{Bun}_G(C)) := \mathrm{QCoh}(\mathrm{Bun}_G(C)_{\mathrm{dR}})$ and $\mathrm{QCoh}(\mathrm{Flat}_G(C))$ respectively for $G = \mathrm{GL}(N)$. Note that these are the main protagonists of the de Rham geometric Langlands correspondence and we did not need to invoke A-model or analytic dependence of the moduli space. This was the main point of [EY18].

In this paper we argue that these two different twists are indeed S-dual pairs, thereby recovering the de Rham geometric Langlands correspondence from our framework.

Again, the way we globalize to consider the spacetime of the form $\Sigma \times C$ amounts to fixing a twisting homomorphism. The claim is that, modulo globalization which is carefully discussed in the original paper, we can see that these two twists are indeed realized as deformations of the holomorphic-topological twist and that those two deformations are S-dual to each other.

To see this, recall that N D3-branes on $\mathbb{R}^2 \times \mathbb{C} \subset \mathbb{R}^4_A \times \mathbb{C}^3_B$, yields the holomorphic-topological twist of $\mathrm{GL}(N)$ gauge theory described by

$$\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}} = \Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{gl}(N)[1] \qquad \text{with differential} \qquad d_{\mathbb{R}^2} \otimes 1 + 1 \otimes \bar{\partial}_{\mathbb{C}}$$

In [EY18], it is argued that HT(A)-twist is given by ∂_{ε_1} and HT(B)-twist is given by $\varepsilon_2 \partial_z$ where z is a coordinate of $\mathbb{C}$. This is summarized in the following table:

theory	perturbative local	non-perturbative global
$\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}$	$\Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{g}$	$T^*_{\mathrm{form}}[-1]\,\underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}}, \mathrm{Higgs}_G(C))$
$\mathcal{E}^{\mathrm{HT(A)}}_{\mathrm{D3}}$	$\Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{g}$ with ∂_{ε_1}	$T^*_{\mathrm{form}}[-1]\,\underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}}, \mathrm{Bun}_G(C)_{\mathrm{dR}})$
$\mathcal{E}^{\mathrm{HT(B)}}_{\mathrm{D3}}$	$\Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{g}$ with $\varepsilon_2 \partial_z$	$T^*_{\mathrm{form}}[-1]\,\underline{\mathrm{Map}}(\Sigma_{\mathrm{dR}}, \mathrm{Flat}_G(C))$

Remark 4.5. Heuristically speaking, one can rewrite

$$T^*_{\text{form}}[-1]\,\underline{\text{Map}}(\Sigma_{\text{dR}}, \text{Higgs}_G(C)) \simeq \underline{\text{Map}}(\Sigma_{\text{dR}}, \underline{\text{Map}}(C_{\text{Dol}}, BG))_{\text{Dol}}$$

where ε_i is responsible for each Dolbeault stack. With our choice of convention, ε_1 is for the outer Dol and ε_2 is for the inner Dol. Then ∂_{ε_1} is exactly what deforms the outer Dol to dR. Moreover, the twisting homomorphism makes ε_2 as dz so $\varepsilon_2\partial_z$ becomes the ∂-operator, which deforms C_{Dol} to C_{dR}. This explains why those deformations realize the desired global descriptions.

Finally we need to show that these two deformations are dual to each other under our S-duality map. This is very easy.

Proposition 4.6. The HT(A) and HT(B) twists are mapped to each other under S-duality.

Proof. Under the closed-open map, the preimages of the deformations ∂_{ε_1} and $\varepsilon_2\partial_z$ are superpotentials w_1 and the Poisson tensor $\partial_{w_2} \wedge \partial_z$, where w_i denote holomorphic coordinates transverse to the world-volume of the D3 branes. Now, by Definition 3.13, we see that

$$\mathbb{S}(w_1) = \partial_{w_2} \wedge \partial_z.$$

The overall sign doesn't matter for the twists, so we are done. $\square$

Remark 4.7. As before, Proposition 3.10 implies that the statement is true as written for $\mathbb{R}^2\times\mathbb{C}^\times \subset \mathbb{R}^4_A \times \mathbb{C}^\times_B \times \mathbb{C}^2_B$ without invoking Definition 3.13.

4.3 S-duality on Superconformal Symmetries

In this subsection, we study the next simplest case which is a quadratic polynomial and find S-duality on superconformal symmetries of D3 brane gauge theory in twisted IIB supergravity.

In the context of AdS/CFT correspondence of twisted IIB supergravity theory $\text{IIB}_{\text{SUGRA}}(\mathbb{C}^5_B)$ in the presence of a D3 brane on $\mathbb{C}^2 \subset \mathbb{C}^5$, Costello and Li [CL16, Section 9] identified superconformal symmetries of the D-brane gauge theory on $\mathbb{C}^2$, namely, the holomorphic twist of 4d $\mathcal{N} = 4$ theory. Concretely, if we write coordinates $\mathbb{C}^2_{u,z} \subset \mathbb{C}^5_{u,v,z,w_1,w_2}$, then these consist of a type of six elements

$$uv, \quad uw_1, \quad uw_2, \quad zv, \quad zw_1, \quad zw_2 \quad \in \quad \text{PV}^0_{\text{hol}}(\mathbb{C}^5) \subset \text{PV}^0_{\text{hol}}(\mathbb{C}^5 \setminus \mathbb{C}^2)$$

and another type of three elements of $\text{PV}^2_{\text{hol}}(\mathbb{C}^5) \subset \text{PV}^2_{\text{hol}}(\mathbb{C}^5 \setminus \mathbb{C}^2)$ given by

$$\partial_v(u\partial_u + z\partial_z - w_1\partial_{w_1} - w_2\partial_{w_2}), \quad \partial_{w_1}(u\partial_u + z\partial_z - v\partial_v - w_2\partial_{w_2}), \quad \partial_{w_2}(u\partial_u + z\partial_z - v\partial_v - w_1\partial_{w_1}).$$

To compare it with our setting where we take a further twist to have $\text{IIB}_{\text{SUGRA}}(\mathbb{R}^4_A \times \mathbb{C}^3_B)$, we restrict our attention to $\mathbb{C}^3_{z,w_1,w_2} \subset \mathbb{C}^5$. Then the residual superconformal symmetries are $zw_1, zw_2 \in \text{PV}^0_{\text{hol}}(\mathbb{C}^3)$ and $\partial_{w_1}(z\partial_z - w_2\partial_{w_2}), \partial_{w_2}(z\partial_z - w_1\partial_{w_1}) \in \text{PV}^2_{\text{hol}}(\mathbb{C}^3)$.

The next proposition says that these residual symmetries are precisely S-dual to each other in the following way:

Proposition 4.8. Under the twisted S-duality map $\mathbb{S}$, we obtain the following correspondence (up to sign)

$$\begin{aligned} zw_1 \quad &\longleftrightarrow \quad \partial_{w_2}(z\partial_z - w_1\partial_{w_1}) \\ zw_2 \quad &\longleftrightarrow \quad \partial_{w_1}(z\partial_z - w_2\partial_{w_2}). \end{aligned}$$

Proof. It follows from the following simple computations:

$$\mathbb{S}(zw_1) = w_1\partial_{w_1}\partial_{w_2} - z\partial_z\partial_{w_2} = \partial_{w_2}(z\partial_z - w_1\partial_{w_1})$$

$$\mathbb{S}(zw_2) = w_2\partial_{w_1}\partial_{w_2} + z\partial_z\partial_{w_1} = -\partial_{w_1}(z\partial_z - w_2\partial_{w_2}).$$

□

Remark 4.9. In this remark, we provide another interpretation of these deformations.[5] The following claims are conjectural at the moment and discussions of the precise mathematical framework needed to articulate the nature of relevant objects is beyond the scope of the current paper.

It was observed in [SW19] that $z\partial_{\varepsilon_2}$ in our notation corresponds to the superconformal deformation of [BLLPRvR15] in the context of 4-dimensional $\mathcal{N} = 2$ chiral algebra, after performing a holomorphic twist. On the other hand, the same chiral algebra was recently understood [But20, Jeo19, OY19] in terms of a certain Ω-background in the holomorphic-topological twist of the 4d $\mathcal{N} = 2$ theory [Kap06]. This leads to a natural question of how to understand our deformations in terms of Ω-backgrounds.

Let us analyze this situation in the holomorphic-topological twist of 4-dimensional $\mathcal{N} = 4$ theory, understood as a special case of $\mathcal{N} = 2$ theory. One claim is that the HT(B)-twist $\varepsilon_2\partial_z$ from the previous example and the deformation $z\partial_{\varepsilon_2}$ combine to give an Ω-background in the B-twist [Yag14]. It is then natural to wonder about the nature of its S-dual image. We claim that the HT(A)-twist ∂_{ε_1} and the deformation $\varepsilon_1\varepsilon_2\partial_{\varepsilon_2} + \varepsilon_1 z\partial_z$ similarly combine to give an Ω-background on the A-twist in the original sense of Nekrasov [Nek03]. That is, we have an analogy

$$(\text{A-type } \Omega\text{-background}) : \varepsilon_1\varepsilon_2\partial_{\varepsilon_2} + \varepsilon_1 z\partial_z = (\text{B-type } \Omega\text{-background}) : z\partial_{\varepsilon_2}$$

or more precisely,

	A-type	B-type
twist of $\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}}$	$\Omega^\bullet(\mathbb{R}^2)[\varepsilon_2] \otimes \left(\Omega^{0,\bullet}(\mathbb{C})\varepsilon_1 \xrightarrow{\partial_{\varepsilon_1}} \Omega^{0,\bullet}(\mathbb{C})\right)$	$\Omega^\bullet(\mathbb{R}^2)[\varepsilon_1] \otimes \left(\Omega^{0,\bullet}(\mathbb{C}) \xrightarrow{\varepsilon_2\partial_z} \Omega^{0,\bullet}(\mathbb{C})\varepsilon_2\right)$
Ω-background	$\mathcal{L}_{z\partial_z} = z\partial_z + \varepsilon_2\partial_{\varepsilon_2}$	$\mathcal{L}_{z\partial_z} = z\partial_z + \varepsilon_2\partial_{\varepsilon_2}$
deformation as homotopy for Ω	$\varepsilon_1\varepsilon_2\partial_{\varepsilon_2} + \varepsilon_1 z\partial_z$ on $\mathbb{C}[z, \varepsilon_1, \varepsilon_2]$ $[\partial_{\varepsilon_1}, \varepsilon_1\varepsilon_2\partial_{\varepsilon_2} + \varepsilon_1 z\partial_z] = z\partial_z + \varepsilon_2\partial_{\varepsilon_2}$	$z\partial_{\varepsilon_2}$ on $\mathbb{C}[z, \varepsilon_2]$ $[\varepsilon_2\partial_z, z\partial_{\varepsilon_2}] = z\partial_z + \varepsilon_2\partial_{\varepsilon_2}$

(Surya: This paragraph should be revised; Omega background constructions on D-branes ought to be induced by Omega background constructions in supergravity) Let us comment on significance of this claim. Note that our description of deformations of a D-brane gauge theory is in terms of Hochschild cohomology or cyclic cohomology; this description is universal so abstractly speaking one should be able to encode any deformation in this manner. In particular, the Ω-background yields a deformation of a D-brane gauge theory, so it must admit a description in terms of Hochschild or cyclic cohomology, though such a realization may not be manifest. Moreover, in the B-type case, our way of encoding the Ω-background as a class in cyclic cohomology exactly corresponds to the superconformal deformation; the fact that this cyclic cohomology class is in fact an Ω-background in disguise recovers the known equivalence between two different descriptions of the 4d $\mathcal{N} = 2$ chiral algebras. Furthermore, our S-duality map implies that we should be able to describe the A-type Ω-background analogously, via a cyclic cohomology class whose S-dual implements the B-type Ω-background.

[5] We are grateful to Dylan Butson and Brian Williams for very enlightening discussions on the topic of this remark.

Remark 4.9 can be used to fill in details of a derivation that was sketched by Costello in his talk [Cos17] at 2017 String-Math conference of why the Yangian appears in the algebra of monopole operators in the A-twist of 3-dimensional $\mathcal{N} = 4$ theory as in [BDG17, BFN19]. Here the A-twist means the twist where the algebra of local operators parametrizes the Coulomb branch. A discussion on related topics is also given in the work of Costello and Yagi [CY18].

We recall Costello's derivation, adapted to our context. Consider IIB$[\mathbb{R}^4_A \times \mathbb{C}^3_B]$ with the following brane configuration:

	0	1	2	3	4	5	6	7	8	9
	u		v		z		w_1		w_2	
K D5		$\times$	$\times$		$\times$	$\times$	$\times$	$\times$		
N D3	$\times$	$\times$			$\times$	$\times$				

We also turn on a closed string field of the form zw_2.

Let us first consider the stack of D5 branes. The world-volume theory of the stack of D5 branes is a holomorphic-topological twist of 6-dimensional $\mathcal{N} = (1,1)$ supersymmetric gauge theory. Computing the self-Ext of this brane, we find that the space of fields is given by

$$\mathcal{E} = \Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C}^2_{z,w_1})[\varepsilon_2] \otimes \mathfrak{gl}(K)[1].$$

As the image of the closed string field under the closed-open map is the deformation $z\partial_{\varepsilon_2}$, turning this on leads to

$$\begin{aligned}&\Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C}_{w_1}) \otimes \left(\Omega^{0,\bullet}(\mathbb{C}_z)\varepsilon_2 \xrightarrow{z\partial_{\varepsilon_2}} \Omega^{0,\bullet}(\mathbb{C}_z)\right) \otimes \mathfrak{gl}(K)[1]\\ \cong\ &\Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C}_{w_1}) \otimes \mathfrak{gl}(K)[1]\end{aligned}$$

In sum, the D-brane gauge theory on the D5 branes becomes 4d Chern–Simons theory on $\mathbb{R}^2 \times \mathbb{C}_{w_1}$ with gauge group $\mathrm{GL}(K)$. In fact, what we have described is exactly the construction of [CY18] in our chosen protected sector of type IIB theory.

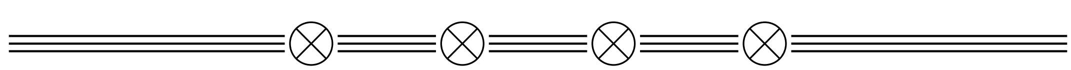
Now let us consider what happens to the D3 branes. Of course, the D-brane gauge theory of the D3 branes is precisely $\mathcal{E}^{\mathrm{HT}}_{\mathrm{D3}} = \Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C}_z)[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{gl}(N)[1]$. Now turning on the deformation $z\partial_{\varepsilon_2}$ yields the complex

$$\begin{aligned}&\Omega^\bullet(\mathbb{R}^2) \otimes \left(\Omega^{0,\bullet}(\mathbb{C}_z)\varepsilon_2 \xrightarrow{z\partial_{\varepsilon_2}} \Omega^{0,\bullet}(\mathbb{C}_z)\right) \otimes \mathbb{C}[\varepsilon_1] \otimes \mathfrak{gl}(N)[1]\\ \cong\ &\Omega^\bullet(\mathbb{R}^2)[\varepsilon_1] \otimes \mathfrak{gl}(N)[1]\end{aligned}$$

Thus, we find 2-dimensional BF theory with gauge group $\mathrm{GL}(N)$.

A similar calculation shows that the bi-fundamental strings stretched between the two stacks of branes yields free fermions as a 1-dimensional defect living on the line, corresponding to the direction 1 in the table. Thus, we find exactly the topological string set-up of [IMZ18], where it is shown that the operators of the coupled system that live on the line generate a quotient of the Yangian of $\mathfrak{gl}(K)$.

We wish to analyze the effect of S-duality on the above setup. Physically, it is known that D5 branes become NS5 branes after S-duality. Therefore, acting on the set-up by S-duality yields a Hanany–Witten brane cartoon whose low-energy dynamics is described by a 3-dimensional $\mathcal{N} = 4$ linear quiver gauge theory with quiver like



where $\bigotimes$ represents an NS5 brane and horizontal lines represent D3 branes; the picture is the case when $N = 3$ and $K = 4$.

Checking the claim of [Cos17] then amounts to checking that the local operators of this 3d $\mathcal{N} = 4$ theory, deformed by the closed string field $\varepsilon_1\varepsilon_2\partial_{\varepsilon_2} + \varepsilon_1 z\partial_z$, give the same quotient of the Yangian of $\mathfrak{gl}(K)$. We intend to return to this question, and a broader analysis of 3-dimensional $\mathcal{N} = 4$ quiver gauge theories in our context, elsewhere.

4.4 S-duality on 4d Chern–Simons Theory

As another quadratic polynomial, we consider

$$\mathbb{S}(w_1 w_2) = w_1\partial_z\partial_{w_1} - w_2\partial_z\partial_{w_2}$$

so that we know $\partial_{\varepsilon_1}\partial_{\varepsilon_2}$ is S-dual to $\pi = \varepsilon_1\partial_{\varepsilon_1}\partial_z - \varepsilon_2\partial_{\varepsilon_2}\partial_z$.

First, to understand the consequence of adding the deformation $\partial_{\varepsilon_1}\partial_{\varepsilon_2}$ to the holomorphic-topological twist $\Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C})[\varepsilon_1, \varepsilon_2] \otimes \mathfrak{gl}(N)[1]$, we note that $(\mathbb{C}[\varepsilon_1, \varepsilon_2], \partial_{\varepsilon_1}\partial_{\varepsilon_2})$ is the Clifford algebra $\mathrm{Cl}(\mathbb{C}^2) \cong \mathrm{End}(\mathbb{C}^{1|1})$, and hence the deformed theory is

$$\mathcal{E} = \Omega^\bullet(\mathbb{R}^2) \otimes \Omega^{0,\bullet}(\mathbb{C}) \otimes \mathfrak{gl}(N|N)[1]$$

also known as the 4d Chern–Simons theory with gauge group given by the supergroup $\mathrm{GL}(N|N)$.

On the other hand, π gives a peculiar noncommutative deformation of the twisted theory. The original category $\mathrm{Coh}(\mathrm{Higgs}_G(C))$ of boundary conditions of the 4-dimensional theory with $G = \mathrm{GL}(N)$ reduced along C is now deformed to $\mathrm{Coh}(\mathrm{Higgs}_G(C), \pi)$.

Remark 4.10. Costello suggested that this deformation can be explicitly constructed in terms of difference modules. In particular, he suggested that when $C = E$ is an elliptic curve, then the deformed category is the category of coherent sheaves on the moduli space $\mathrm{Higgs}_\lambda(E)$ of rank N vector bundles F on E together with a homomorphism $F \to T_\lambda^* F$ where $T_\lambda \colon E \to E$ is the translation by λ. Indeed, when $\lambda = 0$, this recovers the moduli space $\mathrm{Higgs}_G(E)$ of G-Higgs bundles on E, explaining the notation.

A general principle tells us that the category of line defects of a theory acts on the category of boundary conditions. In this case, we understand the category of line defects of 4-dimensional Chern–Simons theory on a Calabi–Yau curve C, namely, $\mathbb{C}$, $\mathbb{C}^\times$, or E, [Cos13, CWY17, CWY18] as monoidal category of representations of the Yangian, the quantum loop group, the elliptic quantum group for $\mathrm{GL}(N|N)$, respectively.

Hence from duality one can conjecture that the category of line defects of 4d Chern–Simons theory for $\mathrm{GL}(N|N)$ acts on the category of boundary conditions $\mathrm{Coh}(\mathrm{Higgs}_G(C), \pi)$. It would be interesting to make this conjecture more precise along the line of suggestion of Costello and investigate it further.

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