Holomorphic M-theory and the SU(4)-invariant twist of type IIA

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BCOV with potentials refers to a modification of minimal BCOV theory where we impose certain constraints on the fields so as to make the Poisson BV structure of the theory invertible. These constraints amount to requiring that certain fields lie in the image of the divergence operator ∂, or better yet replacing ∂-closed fields in a summand $\mathrm{PV}^{d,\bullet}$ with all of $\mathrm{PV}^{d,\bullet}$ and using a fixed choice of splitting of $\partial : \mathrm{PV}^{d,\bullet} \to \mathrm{PV}^{d-1,\bullet}$ to rewrite $\mathrm{PV}^{d,\bullet} \cong \mathrm{im}\partial \oplus \ker \partial$.

Under the conjectures of Costello-Li that describe twisted type II supergravity in terms of BCOV theory, these primitives correspond to certain components of Ramond-Ramond fields, which are chosen as potentials for Ramond-Ramond field strengths.

1 Warm-up: Kodaira–Spencer theory on a Calabi–Yau surface

Let X be a Calabi–Yau surface. Minimal Kodaira–Spencer theory is a $\mathbb{Z}/2$-graded Poisson BV theory described by two fundamental sets of fields:

- An odd field given by a divergence-free holomorphic vector field μ^1.
- An even field given by a holomorphic function μ^0.

These fields combine to define a $\mathbb{Z}/2$-graded sheaf $\mathcal{E}^{\mathrm{hol}}$ on X. There is a Lie algebra structure the parity reversed sheaf $\Pi\mathcal{E}^{\mathrm{hol}}$ using the Lie bracket of holomorphic vector fields together with the natural action of holomorphic vector fields on holomorphic functions.

The sheaf $\mathcal{E}^{\mathrm{hol}}$ admits the following locally free description:

$$\begin{array}{cc} \underline{\text{odd}} & \underline{\text{even}} \\ & \\ & \mathrm{PV}^{0,\bullet} \\ & \\ \mathrm{PV}^{1,\bullet} & \xrightarrow{u\partial} u\mathrm{PV}^{0,\bullet}. \end{array}$$

We refer to this locally free description by $\mathcal{E}$. The Lie bracket on $\Pi\mathcal{E}^{\mathrm{hol}}$ described above extends to a Lie bracket on $\Pi\mathcal{E}$. Together with the differential this gives $\Pi\mathcal{E}$ the structure of a local dg Lie algebra.

The bundle $\mathcal{E}$ is equipped with an odd Poisson tensor defined by

$$\Pi = (\partial \otimes 1)\delta_{\mathrm{Diag}}.$$

We introduce another theory on the Calabi–Yau surface X that we call minimal Kodaira–Spencer theory *with potentials*. The underlying vector bundle is

$$\begin{array}{cc} \underline{\text{odd}} & \underline{\text{even}} \\ & \\ & \mathrm{PV}^{0,\bullet} \\ & \\ \mathrm{PV}^{2,\bullet}. & \end{array}$$

We will denote the resulting $\mathbb{Z}/2$-graded sheaf of cochain complexes by $\mathcal{E}_{\mathrm{pot}}$.

We interpret this as the theory of "potentials" of minimal Kodaira–Spencer theory in the following way. There is a map of bundles $\Phi : \mathcal{E}_{\mathrm{Pot}} \to \mathcal{E}$ which is the identity on $\mathrm{PV}^{0,\bullet}$ and given by $\partial : \mathrm{PV}^{2,\bullet} \to \mathrm{PV}^{1,\bullet}$ on the remaining component. It is immediate to see that Φ defines a map of sheaves of cochain complexes.

In fact, the parity shifted bundle $\Pi\mathcal{E}_{\mathrm{pot}}$ also has the structure of a local Lie algebra, and the map Φ intertwines these local Lie algebra structures.

To describe the local Lie algebra structure on minimal Kodaira–Spencer theory with potentials we use the Calabi–Yau form Ω to identify $\mathcal{E}_{\mathrm{Pot}}$ with the sheaf of cochain complexes

$$
\begin{array}{cc}
\underline{\text{odd}} & \underline{\text{even}} \\
 & \\
 & \Omega^{2,\bullet} \\
 & \\
\Omega^{0,\bullet}. &
\end{array}
$$

Now, note that any Calabi–Yau surface comes equipped with a holomorphic symplectic structure and there is a Poisson bracket defined on the sheaf of holomorphic functions. Since the bracket is defined in terms of holomorphic differential operators, it extends to a bracket on the Dobleault complex $\Omega^{0,\bullet}(X)$.

This further extends to a local Lie algebra structure on the semi-direct product

$$\Omega^{0,\bullet}(X) \ltimes \Pi\Omega^{2,\bullet}(X)$$

which describes the local Lie structure on $\Pi\mathcal{E}_{\mathrm{Pot}}$. It is immediate to verify that the map $\Phi : \mathcal{E}_{\mathrm{pot}} \to \mathcal{E}$ intertwines the two L_∞-structures.

Finally, the theory $\mathcal{E}_{\mathrm{pot}}$ is a non-degenerate BV theory with BV pairing defined by the wedge-and-integrate pairing

$$\alpha, \beta \mapsto \int \alpha \wedge \beta.$$

Proposition 1.1. The map Φ determines a map of $\mathbb{P}_0$-factorization algebras on X:

$$\Phi^* : \mathrm{Obs}_{\mathcal{E}} \to \mathrm{Obs}_{\mathcal{E}_{\mathrm{Pot}}}.$$

2 BCOV theory with potentials on a CY4

Let X be a Calabi-Yau 4 fold. Minimal Kodaira-Spencer theory on X is a $\mathbb{Z}/2$-graded theory with the following fundamental fields:

- The even fields are a holomorphic function μ^0 and a ∂-closed holomorphic bivector μ^2.
- The odd fields are a divergence-free holomorphic vector field μ^1 and a ∂-closed holomorphic section μ^3 of $\wedge^3 T_X$.

The space of fields admits a locally free description obtained by including the “descendants". The descendants of the field μ^j will be denoted $u^k\mu^j$ where $k = 1, \ldots, j$. Here, $u^k\mu^j$ is a section of $\mathrm{PV}^{j-k,\bullet}$. The sheaf of cochain complexes $\mathcal{E}$ underlying minimal Kodaira–Spencer theory on X is

$$\begin{array}{ccccccc}
\underline{\text{odd}} & & \underline{\text{even}} & & \underline{\text{odd}} & & \underline{\text{even}} \\
 & & & & & & \mu^0 \in \mathrm{PV}^{0,\bullet} \\
 & & & & \sum u^k\mu^1 \in \mathrm{PV}^{1,\bullet} & \xrightarrow{u\partial} & u\mathrm{PV}^{0,\bullet} \\
 & & \sum u^k\mu^2 \in \mathrm{PV}^{2,\bullet} & \xrightarrow{u\partial} & u\mathrm{PV}^{1,\bullet} & \xrightarrow{u\partial} & u^2\mathrm{PV}^{0,\bullet} \\
\sum u^k\mu^3 \in \mathrm{PV}^{3,\bullet} & \xrightarrow{u\partial} & u\mathrm{PV}^{2,\bullet} & \xrightarrow{u\partial} & u^2\mathrm{PV}^{1,\bullet} & \xrightarrow{u\partial} & u^3\mathrm{PV}^{0,\bullet}
\end{array}$$

The differential on this sheaf of cochain complexes is given by $\overline{\partial} + u\partial$.

There is a local Lie algebra structure on $\Pi\mathcal{E}$ using the Schouten-Nijenhuis bracket $[-,-]_{\mathrm{Sch}}$ on polyvector fields. On the fields (including the descendants) it is defined by the formula

$$[u^k\mu^i, u^\ell\mu^j] = u^{k+\ell}[\mu^i, \mu^j]_{\mathrm{Sch}}.$$

Furthermore, the sheaf $\mathcal{E}$ is equipped with an odd Poisson tensor given by $\Pi = (\partial \otimes 1)\delta_{\mathrm{Diag}}$. Together, this data equips $\mathcal{E}$ with the structure of a $\mathbb{Z}/2$-graded Poisson BV theory.

As in the surface case, there is a closely related BV theory describing the "potentials" of minimal Kodaira–Spencer theory. The underlying sheaf of cochain complexes is

$$
\begin{array}{cccc}
\underline{\text{odd}} & \underline{\text{even}} & \underline{\text{odd}} & \underline{\text{even}} \\
 & & & \mu^0 \in \mathrm{PV}^{0,\bullet} \\
 & & \sum u^k \mu^1 \in \mathrm{PV}^{1,\bullet} \xrightarrow{\partial} & \mu^c \in \mathrm{PV}^{0,\bullet} \\
u^{-1}\gamma^3 + \gamma^3 \in u^{-1}\mathrm{PV}^{4,\bullet} \xrightarrow{u\partial} & \mathrm{PV}^{3,\bullet} & & \\
\gamma^4 \in \mathrm{PV}^{4,\bullet} & & &
\end{array}
$$

We will again refer to this sheaf as $\mathcal{E}_{\text{pot}}$.

There is a local Lie algebra structure described as follows. (Brian: It's mostly the Schouten bracket, but there is the additional "weird" bracket

$$[\gamma^3, \gamma^{3'}] = [\partial\gamma^3, \gamma^{3'}] \pm [\gamma^3, \partial\gamma^{3'}].$$

)

There is a map of sheaves of cochain complexes $\Phi : \mathcal{E}_{\text{pot}} \to \mathcal{E}$ given by the identity map on $\mathrm{PV}^{1,\bullet}$ and $\mathrm{PV}^{0,\bullet}$ and the ∂ operator on $\mathrm{PV}^{3,\bullet}$ and $\mathrm{PV}^{4,\bullet}$. Explicitly, in formulas

$$
\begin{array}{ccccc}
\Phi(t^k\mu^i) & = & u^k\mu^i & \in & u^k\mathrm{PV}^{i,\bullet} \quad i = 0,1 \\
\Phi(u^{-1}\gamma^3) & = & 0 & & \\
\Phi(\gamma^3) & = & \partial\gamma^3 & \in & \mathrm{PV}^{2,\bullet} \\
\Phi(\gamma^4) & = & \partial\gamma^4 & \in & \mathrm{PV}^{3,\bullet}.
\end{array}
$$

Together with the wedge and integrate pairing, $\mathcal{E}_{\text{pot}}$ has the structure of a nondegenerate BV theory.

In fact, we have the following, analogous to the case of a Calabi-Yau surface:

Proposition 2.1. The map Φ determines a map of $\mathbb{P}_0$-factorization algebras on X:

$$\Phi^* : \mathrm{Obs}_{\mathcal{E}} \to \mathrm{Obs}_{\mathcal{E}_{\text{Pot}}}.$$

3 Dimensional Reduction

Let's consider the 11-dimensional theory on the manifold

$$\mathbb{R} \times \mathbb{C}^\times \times \mathbb{C}^4.$$

The fields decompose as follows. For the "base" direction, fields we labeled by μ, we have

$$\mu = \begin{pmatrix} \mu_{01} \\ \mu_{10} \end{pmatrix} + u\mu^0_{11d} \in \left(\begin{matrix} \mathrm{PV}^{0,\bullet}(\mathbb{C}^\times) \otimes \mathrm{PV}^{1,\bullet}(\mathbb{C}^4) \\ \oplus \mathrm{PV}^{1,\bullet}(\mathbb{C}^\times) \otimes \mathrm{PV}^{0,\bullet}(\mathbb{C}^4) \end{matrix} \to u\mathrm{PV}^{0,\bullet}(\mathbb{C}^\times \times \mathbb{C}^4) \right) \hat{\otimes} \Omega^\bullet(\mathbb{R})[1].$$

For the "fiber" direction, fields we labeled by γ, we have

$$\gamma = u^{-1}\gamma^0_{11d} + \begin{pmatrix} \gamma_{01} \\ \gamma_{10} \end{pmatrix} \in \left(u^{-1}\Omega^{0,\bullet}(\mathbb{C}^\times \times \mathbb{C}^4) \to \begin{matrix} \Omega^{0,\bullet}(\mathbb{C}^\times) \otimes \Omega^{1,\bullet}(\mathbb{C}^4) \\ \oplus \Omega^{1,\bullet}(\mathbb{C}^\times) \otimes \Omega^{0,\bullet}(\mathbb{C}^4) \end{matrix} \right) \hat{\otimes} \Omega^\bullet(\mathbb{R}).$$

The "zero modes" of the reduction along the circle $S^1 \subset \mathbb{C}^\times$ will be denoted by the same symbols. (Brian: write down this sheaf on $\mathbb{C}^4 \times \mathbb{R}^2$.)

Define the map of sheaves of cochain complexes

$$\Psi : \pi_* \mathcal{E}_{11d} \to \mathcal{E}_{\mathrm{pot}} \hat{\otimes} \Omega^\bullet(\mathbb{R}^2)$$

by

$$\begin{array}{rcccl} \Psi(\mu_{01}) & = & \mu_{01} & \in & \mathrm{PV}^{1,\bullet}(\mathbb{C}^4) \hat{\otimes} \Omega^\bullet(\mathbb{R}^2) \\ \Psi(\mu_{10}) & = & \mu_{10} \vee \Omega^{-1}_{\mathbb{C}^4} & \in & \mathrm{PV}^{4,\bullet}(\mathbb{C}^4) \hat{\otimes} \Omega^\bullet(\mathbb{R}). \end{array}$$

Proposition 3.1. There is a homomorphism of L_∞-algebras from the $\bar{\partial}_{\mathbb{C}^\times}$-cohomology of M theory on $\mathbb{R} \times \mathbb{C}^\times \times \mathbb{C}$ to $\Omega^\bullet(\mathbb{R}^2) \otimes \mathcal{E}_{Pot}$ given by

- $[\gamma_{10}] \mapsto \mu^0 \in \mathrm{PV}^{0,\bullet} \subset \mathcal{E}_{Pot}$.
- $[\mu_{01}] \mapsto \mu^1 \in \mathrm{PV}^{1,\bullet}(\mathbb{C}^4) \subset \mathcal{E}_{Pot}$.
- $[\mu^0] \mapsto \mu^c \in \mathrm{PV}^{0,\bullet}(\mathbb{C}^4) \subset \mathcal{E}_{Pot}$.
- $[\gamma_{01}] \mapsto \mu^3 = \gamma_{01} \vee \Omega^{-1}_{\mathbb{C}^4} \in \mathrm{PV}^{3,\bullet} \subset \mathcal{E}_{Pot}$ where $\Omega_{\mathbb{C}^4}$ denotes the holomorphic volume form on $\mathbb{C}^4$.
- $[\mu_{10}] \mapsto \mu^4 = \mu_{10} \vee \Omega^{-1}_{\mathbb{C}^4} \in \mathrm{PV}^{4,\bullet} \subset \mathcal{E}_{Pot}$.

preserving the relevant pairings.

That is, the reduction of the holomorphic M theory on a holomorphic circle should be the SU(4) invariant twist of IIA.

Remark 3.2. Note that the field $\mu^c \in \mathcal{E}_{Pot}$ is nonpropating. It must pair against a field in PV^3. In order for the field μ^0 in M-theory to propagate, it must pair against a field $\gamma^0 \in \Omega^{0,\bullet}$ which is the ghost for the gauge transformation for γ^1. This suggests an enlargement of $\mathcal{E}_{Pot}$ where we add an additional field in PV^3 which will be the image of the ghost of γ^1 in M-theory under dimensional reduction.

(Surya: When we discussewd dimensional reduction on 9/22 over Zoom, I accidentally switched the roles of γ_{10} and μ_{10}. I realized this was wrong - the resulting map cannot be a Lie map. Indeed, there is certainly a nontrivial bracket between γ_{10} and γ_{10} in M-theory, from expanding the $\gamma\partial\gamma\partial\gamma$ term. However, there is no nontrivial bracket between $\mathrm{PV}^{3,\bullet}$ and $\mathrm{PV}^{4,\bullet}$ in $\mathcal{E}_{\mathrm{Pot}}$)

4 Compactification to five dimensions

Now, consider placing the 11-dimensional theory on the manifold

$$X \times Y \times \mathbb{R}$$

where X is compact Calabi–Yau three-fold and Y is a Calabi–Yau surface. We consider the compactification of the theory along the projection map

$$\pi : X \times Y \times \mathbb{R} \to Y \times \mathbb{R}.$$

Recall, the space of fields of the 11-dimensional theory is

$$T^*[-1]\left(\mathrm{PV}^{\leq 1,\bullet}(X \times Y)\hat{\otimes}\Omega^\bullet(\mathbb{R})[1]\right). \tag{1}$$

Since X is compact and Calabi–Yau, we have a sequence of quasi-isomorphisms

$$\mathrm{PV}^{j,\bullet}(X) \cong_{\Omega_X} \Omega^{3-j,\bullet}(X) \simeq H^{3-j,\bullet}(X).$$

The first isomorphism is simply contraction with the Calabi–Yau form $\Omega_X \in \Omega^{3,hol}(X)$ and the second quasi-isomorphism follows from formality of X.

In particular, at the level of sheaves one has a quasi-isomorphism

$$\pi_*\left(\mathrm{PV}^{\leq 1,\bullet}(X \times Y)\hat{\otimes}\Omega^\bullet(\mathbb{R})[1]\right) \simeq H^{\geq 2,\bullet}(X) \otimes \mathrm{PV}^{\leq 1,\bullet}(Y)\hat{\otimes}\Omega^\bullet(\mathbb{R})[1].$$

This describes the “base" direction, fields we labeled by μ, of the space of fields (1) upon compactification.

We extract the piece of the above sheaf which involves the top cohomology $H^{3,3}(X) \cong H^6(X) \cong \mathbb{C}$. This is the sheaf

$$\mathrm{PV}^{\leq 1,\bullet}(Y)\hat{\otimes}\Omega^\bullet(\mathbb{R})[1]. \tag{2}$$

(Recall, we are only working with sheaves of $\mathbb{Z}/2$-graded cochain complexes.)

Likewise, the "fiber" direction of (1) becomes, after compactification:

$$\pi_*\left(\Omega^{\leq 1,\bullet}(X \times Y)\hat{\otimes}\Omega^\bullet(\mathbb{R})\right) \simeq H^{\leq 1,\bullet}(X) \otimes \Omega^{\leq 1,\bullet}(X)\hat{\otimes}\Omega^\bullet(\mathbb{R}).$$

Under the BV pairing, the piece of this sheaf of cochain complexes with (2) is the part involving $H^0(X) \cong \mathbb{C}$. This is precisely

$$\Omega^{\leq 1}(X)\hat{\otimes}\Omega^\bullet(\mathbb{R}).$$