TWISTED M-THEORY AND ITS PERTURBATIVE QUANTIZATION.

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Contents

1. Definition of twisted supergravity

1.1. **The classical BV theory.** In this section we define the central theory of study, within the Batalin–Vilkovisky formalism. The theory will be defined on any eleven-dimensional manifold of the form $X \times L$ where X is a Calabi–Yau five-fold with volume form Ω and L is a smooth oriented one-manifold.

Let V be a holomorphic vector bundle. If j is an integer, we let $\Omega^{0,j}(X, V)$ denote the space of anti-holomorphic Dolbeault forms of type j on with values in V. The $\overline{\partial}$ operator for V is of the form $\overline{\partial} : \Omega^{0,j}(X, V) \to \Omega^{0,j+1}(X)$. This operator is used to define the Dolbeault complex of V

$$\Omega^{0,\bullet}(X, V) = \left(\Omega^{0,j}(X, V)[-j] \ , \ \overline{\partial}\right)$$

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which is a free resolution for the sheaf of holomorphic sections of V. Applying this to the kth exterior power of the holomorphic tangent bundle $V = \wedge^k \mathrm{T}_X$, we obtain a resolution of the sheaf of k-linear polyvector fields which we denote by $\mathrm{PV}^{k,\bullet}(X) = \Omega^{0,\bullet}(X, \wedge^k \mathrm{T}_X)$.

Now, suppose X is a Calabi–Yau manifold with holomorphic volume form Ω. The holomorphic divergence operator extends to the Dolbeault complex to take the form

$$\partial_\Omega : \mathrm{PV}^{1,\bullet}(X) \to \mathrm{PV}^{0,\bullet}(X)$$

and is defined by the formula

$$\partial_\Omega(\mu) \wedge \Omega = L_\mu(\Omega)$$

where, on the right hand side, we mean the Lie derivative of Ω with respect to μ.

Definition 1.1. Let X be a Calabi–Yau five-fold and L a smooth one-dimensional manifold. *Eleven-dimensional twisted supergravity* on $X \times L$ is the $\mathbb{Z}/2$-graded BV theory on X whose fields are pairs

$$\mu \in \left(\Pi\mathrm{PV}^{1,\bullet}(X) \oplus \mathrm{PV}^{0,\bullet}(X)\right) \widehat{\otimes}\, \Omega^\bullet(L)$$

$$\gamma \in \left(\Pi\Omega^{0,\bullet}(X) \oplus \Omega^{1,\bullet}(X)\right) \widehat{\otimes}\, \Omega^\bullet(L)$$

The BV pairing is $\int (\gamma \vee \mu) \wedge \Omega$ and the BV action is

(1)

$$S = \int_{X\times L} \left[\gamma \vee \left(\overline{\partial}\mu + \partial_\Omega \mu + \mathrm{d}_{\mathrm{dR}}\mu\right)\right] \wedge \Omega + \frac{1}{2}\int_{X\times L} (\gamma \vee [\mu,\mu]) \wedge \Omega + \frac{1}{6}\int_{X\times L} (1+\mu)\gamma\partial\gamma\partial\gamma.$$

Proposition 1.2. *The action functional S satisfies the classical master equation*

$$\{S, S\} = 0.$$

1.1.1. *Divergence-free vector fields.* Suppose that X is a Calabi–Yau manifold of any dimension. One of the fundamental geometric objects in this paper will be the sheaf of divergence-free holomorphic vector fields on X. We will utilize a convenient semi-free resolution of this sheaf in the category of C^∞-modules.

Using the divergence operator, we define the double complex of sheaves

$$\mathrm{PV}^{1,\bullet}(X) \xrightarrow{\partial_\Omega} \mathrm{PV}^{0,\bullet}(X)[-1] \tag{2}$$

where the horizontal differential is the divergence operator and the vertical differential is the $\overline{\partial}$ operator (which we have left implicit).

It is immediate to check that the totalization of this double complex provides a free resolution of the sheaf of holomorphic divergence-free vector fields on X. Furthermore, there is a natural Lie bracket on this complex given by the Schouten–Nijenhuis bracket $\{-,-\}_{\mathrm{NS}}$ which is compatible with the usual Lie bracket of holomorphic vector fields. In other words, the complex is a sheaf of dg Lie algebras and the inclusion of holomorphic divergence-free vector fields is a quasi-isomorphism of sheaves of Lie algebras.

The space of differential forms on any smooth manifold M is a commutative dg algebra $\Omega^\bullet(M) = \left(\oplus_k \Omega^k(M)[-k], \mathrm{d}_{\mathrm{dR}}\right)$. By taking the tensor product with the dg Lie algebra (2) we obtain the dg Lie algebra

$$\mathcal{S} \overset{\text{def}}{=} \left(\mathrm{PV}^{1,\bullet}(X) \xrightarrow{\partial_\Omega} \Omega^{0,\bullet}(X)[-1]\right) \widehat{\otimes}\, \Omega^\bullet(M) \tag{3}$$

where X is a Calabi–Yau manifold and M is a smooth manifold. According to the direct sum decomposition, we will write sections of $\mathcal{S}$ as $\mu^1 + \mu^0$. The differential in this dg Lie algebra is of the form $\overline{\partial} \otimes 1 + \partial_\Omega \otimes 1 + 1 \otimes \mathrm{d}_{\mathrm{dR}}$, which we will abbreviate by $\overline{\partial} + \partial_\Omega + \mathrm{d}_{\mathrm{dR}}$. The bracket is defined using the bracket $\{\cdot,\cdot\}_{\mathrm{NS}}$ of polyvector fields together with the wedge product of forms on $\Omega^\bullet(M)$ by the formula

$$[\mu \otimes \alpha, \nu \otimes \beta] = \{\mu, \nu\}_{\mathrm{NS}} \otimes (\alpha \wedge \beta).$$

We remark that the structure maps of this Lie algebra, namely the differential and bracket, are given by differential and bidifferential operators, respectively. In this sense, $\mathcal{S}$ fits the definition of a *local dg Lie algebra* on the product manifold $X \times M$, see Definition **CG2**

Lemma 1.3. *A (perturbative) classical field theory in the BV formalism consists of two pieces of data:*

- *A local dg Lie algebra $(\mathcal{L}, \mathrm{d}, [\cdot,\cdot])$, where $\mathcal{L}$ is the sheaf of sections of a $\mathbb{Z}$-graded vector bundle L.*
- *An $\mathcal{L}$-invariant fiberwise non-degenerate map of vector bundles of degree (-3):*

$$\langle\cdot,\cdot\rangle \colon L \otimes L \to \mathrm{Dens}[-3]$$

where Dens *is the bundle of densities.*

Using the pairing, one reads of the BV action of the theory as

$$S(\phi) = \frac{1}{2}\int \langle \phi, \mathrm{d}\phi\rangle + \frac{1}{6}\int \langle \phi, [\phi,\phi]\rangle$$

where $\phi \in \mathcal{L}[1]$. The $\mathcal{L}$-invariance of the pairing together with the Jacobi identity for the dg Lie algebra $\mathcal{L}$ implies that S automatically satisfies the classical master equation $\{S, S\} = 0$. To prove Proposition 1.2 we will exhibit the theory in Definition 1.1 in terms of a local dg Lie algebra.

This local dg Lie algebra is built from the local dg Lie algebra $\mathcal{S}$ together with a certain module. Denote by $d \in \mathbb{Z}$ the sum $\dim_{\mathbb{C}}(X) + \dim_{\mathbb{R}}(M)$. Define the following sheaf of cochain complexes

$$\mathcal{M} \overset{\text{def}}{=} \left(\Omega^{0,\bullet}(X)[1] \xrightarrow{\partial} \Omega^{1,\bullet}(X)\right) \widehat{\otimes}\, \Omega^\bullet(M)[d].$$

The reason for the cohomological shifts will be apparent momentarily. According the direct sum decomposition, we will write sections of $\mathcal{M}$ as $\gamma^0 + \gamma^1$.

The sheaf $\mathcal{M}$ has the structure of a module over $\mathcal{S}$, defined by Lie derivative:

$$[\mu^1, \gamma^0 + \gamma^1] \stackrel{\text{def}}{=} L_{\mu^1}(\gamma^0) + L_{\mu^1}(\gamma^1).$$

Strictly speaking, the Lie derivative uses just the component of μ^1 and γ^i along X. Along the manifold M we simply take the wedge product of forms.

We now form the semi-direct product dg Lie algebra

$$\mathcal{S} \ltimes \mathcal{M}^![-3]. \tag{4}$$

Recall, that we are decomposing the sections as $(\mu, \gamma) = (\mu^1 + \mu^0, \gamma^0 + \gamma^1)$. One immediately verifies that this semi-direct product has the structure of a local dg Lie algebra. Additionally, there is a non-degenerate pairing defined using the holomorphic volume form Ω on X by the formula

$$(\mu^1 + \mu^0, \gamma^0 + \gamma^1) = \int_{X\times M} (\mu^1 \vee \gamma^1) \wedge \Omega + \int_{X\times M} (\mu^0\gamma^0)\Omega$$

which one can readily check is graded antisymmetric of degree (-3) due to our cohomological shifts in the definition of $\mathcal{M}$.

The associated BV action functional is of the form

$$S_0(\mu, \gamma) = \int_{X\times M} \left[\gamma \vee \left(\overline{\partial}\mu + \partial_\Omega \mu + \mathrm{d}_{\mathrm{dR}}\mu\right)\right] \wedge \Omega + \frac{1}{2}\int_{X\times M} (\gamma \vee [\mu, \mu]) \wedge \Omega. \tag{5}$$

The fact that this BV action functional satisfies the classical master equation

$$\{S_0, S_0\} = 0.$$

follows automatically from the axioms of the dg Lie algebra structure on (4).

The action S_0 is not the same as the action in Definition 1.1–they differ by the last term in Equation (1). Before moving on to this term, we mention a relationship of the BV theory described by S_0 to "BF" theory.

1.1.2. *Relationship to "BF" theory.* Associated to any local dg Lie algebra is a classical field theory in the BV formalism that one refers to as BF-theory and it is defined as follows. Denote the local dg Lie algebra by $\mathcal{A}$, it is the sheaf of sections of some $\mathbb{Z}$-graded vector bundle A. The fields of the associated BV theory are pairs

$$(A, B) \in \mathcal{A}[1] \oplus \mathcal{A}^![-2].$$

Here $\mathcal{A}^!$ denotes the sheaf of sections of the bundle $L^* \otimes \mathrm{Dens}$, where Dens is the bundle of densities. The action functional reads $S_{\mathrm{BF}} = \int BF_A = \int B(\mathrm{d}_{\mathcal{L}} A + \frac{1}{2}[A, A])$. In terms of the local Lie algebra description, the action is completely determined by the dg Lie algebra structure on $\mathcal{A} \ltimes \mathcal{A}^![-3]$. Here $\mathrm{d}_{\mathcal{A}}$ denotes the differential on $\mathcal{A}$ and $[\cdot, \cdot]$ is the Lie bracket. Notice that the equations of motion for the A-field is simply the Maurer–Cartan equation $\mathrm{d}_{\mathcal{A}} A + \frac{1}{2}[A, A] = 0$. For the B-field the equations of motion read $\mathrm{d}_{\mathcal{A}} B + [A, B] = 0$.

The theory described in the previous section is closely related to BF theory for the local Lie algebra $\mathcal{A} = \mathcal{S}$. Indeed, using the holomorphic volume form, one has an isomorphism of sheaves of cochain complexes

$$\Omega\colon \mathcal{M} \xrightarrow{\cong} \mathcal{S}^!. \tag{6}$$

So, at the level of fields, Ω determines an isomorphism of cochain complexes

$$\Omega\colon \mathcal{S} \oplus \mathcal{M}[-3] \xrightarrow{\cong} \mathcal{S} \oplus \mathcal{S}^![-3]. \tag{7}$$

On the left-hand side is the underlying space of fields of the theory defined in the last section (6), and on the right-hand side is the underlying space of fields of BF theory for $\mathcal{A} = \mathcal{S}$. However, this identification is *not* a map of local dg Lie algebras.

Nevertheless, the BV theory described by $\mathcal{S} \ltimes \mathcal{M}[-3]$ and BF theory are equivalent. The point is that there is an L_∞ morphism between the two underlying local dg Lie algebras, whose linear component agrees with the map (7).

Proposition 1.4. *There is an L_∞ map of local Lie algebras*

$$\Phi\colon \mathcal{S} \ltimes \mathcal{M}[-3] \to \mathcal{S} \ltimes \mathcal{S}^![-3]$$

whose linear term $\Phi^{(1)}$ is an isomorphism of cochain complexes that preserves the local pairings. In particular, the BV theory described by the action functional S_0 in (5) *and BF theory for the local Lie algebra $\mathcal{S}$ are equivalent.*

Proof. The L_∞ map is the identity on $\mathcal{S}$ and so arrises from a map of $\mathcal{S}$-modules

$$\Phi\colon \mathcal{M}[-3] \to \mathcal{S}^![-3]$$

that we denote by the same symbol.

One has

$$\mathcal{S}^![-3] = \left(\Omega^{5,\bullet}(X) \xrightarrow{\partial^!_\Omega} \Pi\Omega^{5,\bullet}(X, \mathrm{T}^*_X)\right) \widehat{\otimes}\, \Omega^\bullet(M)$$

where $\partial^!_\Omega$ sends $f\Omega$ to $(\partial f) \otimes \Omega$ (and is the identity on $\Omega^\bullet(M)$). The $\mathcal{S}$-module structure is the one by which $\mathrm{PV}^{1,\bullet}(X)$ acts by Lie derivative everywhere. Denote by $b^0\Omega + b^1\Omega$ the fields in the decomposition of $\mathcal{S}^!$ above.

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fix shifts

The L_∞ $\mathcal{L}$-module map $\Phi = \Phi^{(1)} + \Phi^{(2)}$ is defined by

$$\Phi^{(1)}(\gamma^0 + \gamma^1) = \gamma^0\Omega + \gamma^1\Omega \in \Omega^{5,\bullet}(X) \oplus \Pi\Omega^{5,\bullet}(X, \mathrm{T}^*_X)$$
$$\Phi^{(2)}(\mu^0 \otimes (\gamma^0 + \gamma^1)) = \mu^0\gamma^0\Omega + \mu^0\gamma^1 \in \Omega^{5,\bullet}(X) \oplus \Pi\Omega^{5,\bullet}(X, \mathrm{T}^*).$$

Let us sketch an argument that this is an L_∞ $\mathcal{S}$-module map. Suppose $\mu^1 \in \mathrm{PV}^{1,\bullet}(X)\widehat{\otimes}\Omega^\bullet(M)$. Then μ^1 acts on $\mathcal{M}$ by Lie derivative

$$\mu^1 \cdot (\gamma^0 + \gamma^1) = L_{\mu^1}(\gamma^0) + L_{\mu^1}(\gamma^1).$$

So,

$$\Phi^{(1)}(\mu^1 \cdot (\gamma^0 + \gamma^1)) - \mu^1 \cdot \Phi^{(1)}(\gamma^0 + \gamma^1) = (\gamma^0 + \gamma^1)L_{\mu^1}\Omega.$$

The right-hand side is precisely

$$\Phi^{(2)}((\partial_\Omega \mu^1) \otimes (\gamma^0 + \gamma^1))$$

which implies that $\Phi = \Phi^{(1)} + \Phi^{(2)}$ is an L_∞ module map.

$\square$

1.1.3. *Chern–Simons deformation.* We return back to the theory described by the action functional S_0 in (5). We will show that this action admits a deformation which puts it precisely in the form of the action in Definition 1.1. We explain how this deformation relates to the familiar Chern–Simons term in eleven-dimensional supergravity in Section 1.3.

As we defined it, S_0 is an action that makes sense on $X \times M$ where X is a Calabi–Yau manifold and M is a smooth manifold, both of arbitrary dimension. We are mostly interested in the case that X is a Calabi–Yau five-fold and $M = L$ is a real one-dimensional manifold. Also, for the deformed action to make sense, we must break the $\mathbb{Z}$-grading on the fields described by S_0 to a $\mathbb{Z}/2$-grading. Thus, as a $\mathbb{Z}/2$-graded space of fields we have

$$\begin{array}{rcl} \mu = \mu^1 + \mu^0 & \in & \left(\Pi \mathrm{PV}^{1,\bullet}(X) \oplus \mathrm{PV}^{0,\bullet}(X)\right) \widehat{\otimes} \Omega^\bullet(L) \\ \gamma = \gamma^0 + \gamma^1 & \in & \left(\Pi \Omega^{0,\bullet}(X) \oplus \Omega^{1,\bullet}(X)\right) \widehat{\otimes} \Omega^\bullet(L). \end{array}$$

The $\Pi(\cdot)$ denotes the $\mathbb{Z}/2$ parity shift.

The action S_0 describes the local dg Lie algebra structure on $\mathcal{S} \ltimes \mathcal{M}[-3]$ with differential $\overline{\partial} + \partial + \mathrm{d}_{dR}$ and brackets

$$[\mu_1, \mu_2] = \{\mu_1, \mu_2\}_{\mathrm{NS}}$$
$$[\mu^1, \gamma] = L_{\mu^1}\gamma.$$

Consider the following deformation of S_0 by the functional

$$J = \frac{1}{6} \int_{X \times L} (1 + \mu)\gamma \partial \gamma \partial \gamma.$$

This has the affect of deforming the dg Lie structure to the L_∞ structure described by the same differential and the following 2-ary $[\cdot, \cdot]$ and 3-ary operations $[\cdot, \cdot, \cdot]_3$ described by

$$\begin{array}{rcll} [\mu_1, \mu_2] & = & \{\mu_1, \mu_2\}_{\mathrm{NS}} & \\ [\mu, \gamma] & = & L_{\mu^1}\gamma & \\ [\gamma_1^1, \gamma_2^1] & = & \Omega \vee \left(\gamma_1^1 \partial \gamma_2^1 \pm \gamma_2^1 \partial \gamma_1^1\right) & \in \mathrm{PV}^{1,\bullet}(X) \widehat{\otimes} \Omega^\bullet(L). \\ [\gamma_1^1, \gamma_2^1, \mu^0]_3 & = & \mu^0 \Omega \vee \left(\gamma_1^1 \partial \gamma_2^1 \pm \gamma_2^1 \partial \gamma_1^1\right) & \in \mathrm{PV}^{1,\bullet}(X) \widehat{\otimes} \Omega^\bullet(L) \\ [\gamma_1^1, \gamma_2^1, \gamma_3^1]_3 & = & \Omega \vee (\gamma_1^1 \partial \gamma_2^1 \partial \gamma_3^1) & \in \Omega^{0,\bullet}(X) \widehat{\otimes} \Omega^\bullet(L). \end{array}$$

1.1.4. *A generalization of the theory.* We briefly discuss a generalization of this deformation of BF theory for the dg Lie algebra $\mathcal{S}(X) \otimes \Omega^\bullet(L)$ when X is any *odd dimensional* Calabi–Yau manifold of dimension at least three and L is a smooth one-dimensional manifold.

Suppose X is a Calabi–Yau manifold of complex dimension $2m+1$ where $m > 0$. Then, the fields of BF theory for the dg Lie algebra $\mathcal{S}(X) \otimes \Omega^\bullet(L)$ are of the form

$$\begin{aligned} \mu = \mu^1 + \mu^0 \quad &\in \quad \left(\mathrm{PV}^{1,\bullet}(X)[1] \oplus \mathrm{PV}^{0,\bullet}(X)\right) \widehat{\otimes} \, \Omega^\bullet(L) \\ \gamma = \gamma^0 + \gamma^1 \quad &\in \quad \left(\Omega^{0,\bullet}(X) \oplus \Omega^{1,\bullet}(X)[-1]\right) \widehat{\otimes} \, \Omega^\bullet(L)[dependsonm]. \end{aligned}$$

The deformation we consider is of the form

$$J_m(\gamma) = \frac{1}{m+1} \int_{X \times L} \gamma \underbrace{\partial\gamma \cdots \partial\gamma}_{m}.$$

That is, this deformation is $(m+1)$-linear and contains m holomorphic derivatives. As in the $m = 2$ (X is a five-fold) case that we discussed above, we see that J_m is closed for the differential $\{\widetilde{S}_{BF}, \cdot\}$ defining the BF theory.

Lemma 1.5. *The functional J_m is closed for the differential $\{\widetilde{S}_{BF}, \cdot\}$.*

A simple count reveals that the ghost degree of the functional J_m is $2m^2 - 2$. In particular, when $m \neq 1$, the above lemma implies that the action functional $\widetilde{S}_{BF} + J_m$ only defines a $\mathbb{Z}/2$-graded classical BV theory.

1.1.5. *The case of a CY three-fold.* When $m = 1$, hence X is a CY three-fold, the action $\widetilde{S}_{BF} + J_1$ actually defines a $\mathbb{Z}$-graded BV theory. Another feature of the case $m = 1$ is that the deformed theory is actually a *topological* field theory. It is instructive to spell out this case in more detail.

Using the isomorphism

$$(-) \vee \Omega : \mathrm{PV}^{i,\bullet}(X) \cong \Omega^{3-i,\bullet}(X)$$

we can identify the μ-fields of the theory in terms of differential forms

$$\alpha^2 + \alpha^3 \in \left(\Omega^{2,\bullet}(X)[1] \oplus \Omega^{3,\bullet}(X)\right) \widehat{\otimes} \, \Omega^\bullet(L)$$

by the formula $\alpha^{3-i} = \mu^i \vee \Omega$, for $i = 0, 1$. We will also write $\alpha^j = \gamma^j$ for $j = 0, 1$, so the remaining fields read

$$\alpha^0 + \alpha^1 \in \left(\Omega^{0,\bullet}(X)[3] \oplus \Omega^{1,\bullet}(X)[2]\right) \widehat{\otimes} \, \Omega^\bullet(L).$$

Let us write the theory as $\widetilde{S}_{BF} + cJ_1$ where c is a constant. We denote by $\mathcal{T}_c[X \times L]$ the resulting BV theory which depends on this coupling constant. Also, let $\mathcal{M}_c[X]$ denote the moduli space of solutions on the Calabi–Yau three-fold. Using this decomposition of the fields we can write the underlying *free* theory by the following action

$$\int_{X \times L} \left(\alpha^0(\overline{\partial} + \mathrm{d_{dR}})\alpha^3 + \alpha^1(\overline{\partial} + \mathrm{d_{dR}})\alpha^2\right) + \int_{X \times L} \alpha^0 \partial \alpha^2 + \frac{c}{2} \int_{X \times L} \alpha^1 \partial \alpha^1$$

Here, as above, ∂ denotes the holomorphic de Rham operator on X.

Now, for $c \neq 0$, we observe that this free action is equivalent to abelian seven-dimensional Chern–Simons theory on $X \times L$. Indeed, by the above formula the linearized BV complex of the theory is given by the following cochain complex

$$\left(\Omega^{\bullet,\bullet}(X) \mathbin{\widehat{\otimes}} \Omega^{\bullet}(L)[3]\ ,\ \overline{\partial} + \widehat{\partial} + \mathrm{d}_{\mathrm{dR}} + c\partial_{\Omega^1 \to \Omega^2}\right)$$

Here, $\widehat{\partial}$ denotes the components of the holomorphic de Rham differential $\Omega^{0,\bullet}(X) \to \Omega^{1,\bullet}(X)$ and $\Omega^{2,\bullet}(X) \to \Omega^{3,\bullet}(X)$. Also, $\partial_{\Omega^1 \to \Omega^2}$ denotes the component $\Omega^{1,\bullet}(X) \to \Omega^{2,\bullet}(X)$. When $c \neq 0$ we see that this cochain complex is isomorphic to the full (shifted) de Rham complex $\Omega^{\bullet}(X \times L)[3]$.

In the BV formalism, seven-dimensional Chern–Simons theory is described by the fields $\alpha^i \in \Omega^{i,\bullet}(X)\widehat{\otimes}\Omega^{\bullet}(L)[3-i]$, $i = 0, \ldots 3$ which we can now decompose as a de Rham type field $C^{\bullet} \in \Omega^{\bullet}(X \times L)[3]$ so that C^j has total de Rham degree j. The BV antipairing is the wedge and integrate pairing $\int C \wedge C'$ and the action reads

$$\frac{1}{2}\int_{X \times L} C\mathrm{d}C,$$

where d is the full de Rham differential on $X \times L$. Sometimes, this is referred to as "three-form Chern–Simons theory" since its fundamental field is a three-form C^3 and the equations of motion read $\mathrm{d}C^3 = 0$.

We have seen that when $c \neq 0$, the free theory underlying $\mathcal{T}_c[X \times L]$ is equivalent to seven-dimensional Chern–Simons theory. In particular, the moduli space of solutions $\mathcal{M}_c[X]$, for $c \neq 0$, has as its free limit the moduli space of circle 3-bundles on X with connection.

We can consider a different limit of $\mathcal{M}_c[X]$, namely the one where $c \to 0$. This is precisely the cotangent bundle to the moduli space of Calabi–Yau structures $\mathcal{M}_{CY}[X]$ on the three-fold X:

$$\mathcal{M}_0[X] = T^*\left(\mathcal{M}_{\mathrm{CY}}[X]\right).$$

In summary, we have seen that $\mathcal{M}_c[X]$ provides a roof of deformations between the moduli space of Calabi–Yau structures on X and the moduli of circle 3-bundles:

$$\begin{tikzcd} & \mathcal{M}_c[X] \arrow[dl, "c\to 0"'] \arrow[dr, "\text{free limit}", "c\neq 0"'] & \\ T^*\left(\mathcal{M}_{\mathrm{CY}}[X]\right) & & \left\{\begin{array}{c}\text{circle 3-bundles}\\ \text{on } X\end{array}\right\}. \end{tikzcd}$$

1.2. **Equations of motion.** We return to the case of the theory $\mathcal{T}$ on $X \times L$ where X is a Calabi–Yau five-fold and L is a one-manifold. In this section we analyze the equations of motion for the eleven-dimensional theory and provide partial moduli theoretic interpretation of the solution space.

The action functional of the theory in Definition **??** on $X \times L$ is of the form

$$\int_{X\times L} \gamma \vdash \left(\overline{\partial} + \mathrm{d}_{\mathrm{dR}} + \partial_\Omega\right) \mu \wedge \Omega + \frac{1}{2}\int_{X\times L} \gamma \vdash \{\mu,\mu\}_{\mathrm{NS}} \wedge \Omega + \frac{c}{3!}\int_{X\times L} \gamma\partial\gamma\partial\gamma.$$

Notice that just as in the last section we have included a coupling constant c into the term which deforms the classical BF type theory.

From this, we read off the equations of motion as

$$\begin{array}{rcl} \left(\overline{\partial} + \partial_\Omega + \mathrm{d}_{\mathrm{dR}}\right)\mu + \frac{1}{2}\{\mu,\mu\}_{\mathrm{NS}} + c\partial\gamma\partial\gamma \vdash \Omega & = & 0 \\ \left(\overline{\partial} + \partial_\Omega + \mathrm{d}_{\mathrm{dR}}\right)\gamma + L_\mu(\gamma) & = & 0. \end{array} \tag{8}$$

Here, $L_\mu(\gamma)$ denotes the Lie derivative of γ with respect to the vector field μ.

On $L = \mathbb{R}_t$ we proceed to describe the classical phase space of the model which is the space of solutions to the above equations at $t = 0$. Upon taking into account the gauge transformations, we will denote the resulting moduli space by $\mathcal{M}_c[X]$. When X is compact, the BV pairing in the bulk eleven-dimensional theory induces a symplectic structure on this moduli space.

Notice that just as in the case of the phase space of the 7-dimensional theory on a Calabi–Yau three-fold, at $c = 0$ this moduli space is precisely the cotangent bundle to the moduli space of Calabi–Yau five-folds $\mathcal{M}_0[X] = T^*(\mathcal{M}_{\mathrm{CY}}(X))$.

In the general case, we begin to decompose the two equations of motion above according to their form degree. When $c = 0$, the field $\mu^{1,1}$ is a Beltrami differential which provides the parameter describing the deformation of complex structure on the Calabi–Yau X. For $c \neq 0$, it makes an appearance in the following equation

$$\overline{\partial}\mu^{1,1} + \frac{1}{2}\{\mu^{1,1},\mu^{1,1}\}_{\mathrm{NS}} + c\partial\gamma^{1,0}\partial\gamma^{1,2} \vdash \Omega = 0. \tag{9}$$

(Notice that the term $\partial\gamma^{1,1}\partial\gamma^{1,1} = 0$ by form type reasons.)

The field $\gamma^{1,0}$ also appears in the second class of equations of motion in (8) as

$$\overline{\partial}\gamma^{1,0} + \partial\gamma^{0,1} + L_{\mu^{1,1}}\gamma^{1,0} + L_{\mu^{1,0}}\gamma^{0,1} = 0.$$

We choose a gauge so that $\gamma^{0,1} = 0$, so that this equation reads

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Why is this allowed.

$$\overline{\partial}\gamma^{1,0} + L_{\mu^{1,1}}\gamma^{1,0} = 0. \tag{10}$$

Denote by $\beta^{2,0} = \partial\gamma^{1,0}$ the $(2,0)$-form obtained from $\gamma^{1,0}$ and consider the sequence of maps of holomorphic vector bundles

$$\wedge^{2,0}T^{*1,0} \xrightarrow{\beta^{2,0}\wedge} \wedge^4 T_X^{*1,0} \xrightarrow[\cong]{\Omega} T_X^{1,0}.$$

The image of this map is a holomorphic distribution $F \subset T_X^{1,0}$. It follows from the fact that $\partial\beta = 0$ that F defines a holomorphic foliation on X.

Using the foliation F, the holomorphic tangent bundle splits as $T^{1,0} = F \oplus Q$ where $Q = T_X^{1,0}/F$ is the quotient bundle. Using this splitting, we can split the components of

$\mu^{1,1}$ according to

$$T_X^{1,0} \otimes T_X^{*0,1} = (F \otimes \overline{F}^*) \oplus (F \otimes \overline{Q}^*) \oplus (Q \otimes \overline{F}^*) \oplus (Q \otimes \overline{Q}^*).$$

There is a component $\mu^{1,1}_{Q\overline{Q}}$ of $\mu^{1,1}$ which is a Beltrami differential for the quotient bundle Q. At the level of the quotient bundle Equation (9) becomes

$$\overline{\partial}\mu^{1,1}_{Q\overline{Q}} + \frac{1}{2}\{\mu^{1,1}_{Q\overline{Q}}, \mu^{1,1}_{Q\overline{Q}}\}_{\mathrm{NS}} = 0.$$

In other words $\mu^{1,1}_{Q\overline{Q}}$ is a Beltrami differential for this quotient bundle and hence determines a deformation of complex structure of $Q = T_X^{1,0}/F$.

So, after fixing the value of $\beta\partial\gamma^{1,0}$. The remaining components of $\mu^{1,1}$ determine deformations of complex structure of the bundle $F \subset T_X^{1,0}$. For instance, the $\mu^{1,1}_{F\overline{F}}$ determines a deformation of the complex structure on F. Furthermore, by Equation (10) we see that $\gamma^{1,0}$, hence $\beta = \partial\gamma^{1,0}$, is holomorphic for this deformed complex structure.

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state conjecture
about twist.

1.3. **Evidence for twisted supergravity.**

In parts of the remainder of the paper we will provide evidence, and consistency checks, that the eleven-dimensional theory that we have just defined is a candidate for the minimal twist of supergravity. In this section, we discuss a more direct relationship by exhibiting the fields of the theory we have outlined as components of the supergravity multiplet that are expected to survive in the twist.

We recall that our eleven-dimensional theory is only $\mathbb{Z}/2$-graded. This is consistent with the minimal twist of supergravity. Indeed, since the R-symmetry group is trivial there is no way to regrade the twisted theory in such a way that the BRST differential and twisting supercharge are of homogenous $\mathbb{Z}$-degree.

The conjecture implies that the even fields of our eleven dimensional theory should correspond to certain components of the fundamental fields supergravity that survive the minimal twist. The even components of the field μ and γ are of the form

$$\mu^{0,i;j}\ ,\ \mu^{1,i;j}\ ,\ \gamma^{0,i;j}\ ,\ \gamma^{1,i;j}$$

where $i+j$ is an even integer and $i = 0, \ldots 5$ and $j = 0, 1$. Here, as in the previous section, we have used the following index notation:

$$\begin{aligned} \mu^{k,i;j} &\in \mathrm{PV}^{k,i}(\mathbb{C}^5) \,\widehat{\otimes}\, \Omega^j(\mathbb{R}) \\ \gamma^{k,i;j} &\in \Omega^{k,i}(\mathbb{C}^5) \,\widehat{\otimes}\, \Omega^j(\mathbb{R}). \end{aligned}$$

Let us now turn to the field content of eleven-dimensional supergravity. In Euclidean signature, in the flat background, the fundamental fields of eleven-dimensional supergravity are

e	$\in$	$\Omega^1(\mathbb{R}^{11}) \otimes \mathbb{C}^{11}$	vielbien
ω	$\in$	$\Omega^1(\mathbb{R}^{11}) \otimes \mathfrak{so}(11)$	spin connection
C	$\in$	$\Omega^3(\mathbb{R}^{11})$	supergravity 3-form
ψ	$\in$	$\Pi\Omega^1(\mathbb{R}^{11}) \otimes S$	gravitino.

We note that the fields above have been complexified from the usual presentation of supergravity in terms of real fields. Here, S denotes the 128 dimensional complex spin representation of $\mathfrak{so}(11)$.

1.3.1. *Residual supersymmetry.* Recall, the (complexified) eleven-dimensional supertranslation algebra is a complex super Lie algebra of the form

$$\mathfrak{t}_{11d} = V \oplus \Pi S$$

where V is the fundamental representation of $\mathfrak{so}(11, \mathbb{C})$ and S is the irreducible (complex) spin representation. The only non-trivial Lie bracket is

$$[Q, Q'] = \Gamma_{\Omega^1}(Q \otimes Q')$$

where

$$\Gamma_{\Omega^1} : \mathrm{Sym}^2(S) \to V$$

is the unique $\mathfrak{so}(11, \mathbb{C})$-equivariant The super Poincaré algebra is

Brian finish

$$\mathfrak{siso}_{11d} = \mathfrak{so}(11, \mathbb{C}) \ltimes \mathfrak{t}_{11d}.$$

Introduce the cochain complex $\Omega^\bullet(\mathbb{R}^{11})$ of (complex valued) differential forms on $\mathbb{R}^{11}$. Let

$$\Gamma_{\Omega^2} : \mathrm{Sym}^2(S) \to \wedge^2 V \subset \Omega^2(\mathbb{R}^{11})$$

be the .

Brian finish

Definition 1.6. The super dg Lie algebra $\mathfrak{m2brane}$ is the central extension of $\mathfrak{siso}_{11d}$ by the cocycle

$$c_{M2} \in \mathrm{C}^2_{\mathrm{Lie}}\left(\mathfrak{siso}_{11d} \,;\, \Omega^\bullet(\mathbb{R}^{11})[2]\right)$$

defined by the formula $c_{M2}(Q, Q') = \Gamma_{\Omega^2}(Q \otimes Q') \in \Omega^2(\mathbb{R}^{11})$.

Notice that $\mathfrak{m2brane}$ is a $\mathbb{Z} \times \mathbb{Z}/2$-graded Lie algebra and the differential is of bidegree $(1, 0)$. Fix a rank one supercharge $Q \in S \subset \mathfrak{t}_{11d}$ satisfying $Q^2 = 0$. We will characterize the cohomology of the algebra $\mathfrak{m2brane}$ with respect to this supercharge. Before doing this, let us set up some notation.

Such a supercharge defines a maximal isotropic subspace $L \subset V$. As $\mathfrak{sl}(L; \mathbb{C}) = \mathfrak{sl}(5; \mathbb{C})$-modules the supersymmetry algebra decomposes as

$$L \oplus L^* \oplus \mathbb{C}_t$$

and $\mathfrak{so}(11, \mathbb{C})$ decomposes as

$$\wedge^2 L \oplus \wedge^2 L^* \oplus L \oplus L^*.$$

Furthermore, the spinorial representation can be identified with

$$S = \wedge^\bullet(L^*) = \mathbb{C} \oplus L^* \oplus \wedge^2 L^* \oplus \wedge^3 L^* \oplus \wedge^4 L^* \oplus \wedge^5 L^*.$$

The element Q lives in the first summand.

Lemma 1.7. *Fix a holomorphic supercharge $Q \in S$ and let $\mathfrak{m2brane}^Q$ be the Q-cohomology of $\mathfrak{m2brane}$. There is an isomorphism of $\mathbb{Z}/2$-graded Lie algebras*

$$\mathfrak{m2brane}^Q \simeq \mathfrak{sl}(5;\mathbb{C}) \ltimes \Pi(\wedge^2 L^* \oplus \mathbb{C} \cdot c)$$

Proof. According to the $\mathbb{Z} \times \mathbb{Z}/2$ bigrading, Q is degree $(0, \mathrm{odd})$. Since the differential on $\mathfrak{m2brane}$ is degree $(1, \mathrm{even})$, the total cohomology with respect to $\mathrm{d} + [Q, -]$ will only be $\mathbb{Z}/2$ graded.

We spell out the action of $[Q, -]$ on S according to the above decompositions. First, note that

$$[Q, -] \colon S \to L^* \oplus \mathbb{C}_t$$

is surjective and has 26-dimensional kernel. □

Consider now the eleven-dimensional theory $\mathcal{T} = \mathcal{T}_{\mathbb{C}^5 \times \mathbb{R}}$ defined on $\mathbb{C}^5 \times \mathbb{R}$. The BV action induces the structure of a dg Lie algebra on $\mathcal{T}[1]$.

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??

Proposition 1.8. *The assignment defines a map of $\mathbb{Z}/2$-graded dg Lie algebras*

$$\mathfrak{m2brane} \to \mathcal{T}[1].$$

In particular, the Q-twisted algebra $\mathfrak{m2brane}^Q$ is a symmetry of eleven-dimensional theory on $\mathbb{C}^5 \times \mathbb{R}$.

2. Quantization of holomorphic-topological theories

2.1. **Effective renormalization.**

2.2. **The 11-dimensional theory.**

2.3. **The moduli space of quantizations.**

3. Relationship to the Type IIA string

In the remainder of the paper, we wish to establish various consistency checks corroborating the claim that the 11d theory of interest describes a twist of 11d supergravity. In this section, we demonstrate that this claim is consistent with a conjectural description of a twist of Type IIA supergravity due to **CLsugra**

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move earlier?

Before recalling this conjecture, we first fix some notation which is useful to describe the state space for the Type IIA string. For i, j integers define the space of (i,j)-polyvector fields to be

$$\mathrm{PV}^{i,j}(X) \stackrel{\text{def}}{=} \Omega^{0,j}(X, \wedge^i TX)$$

where TX is the holormorphic tangent bundle. Using the $\overline{\partial}$ operator for the holomorphic bundle $\wedge^i TX$ we obtain a cochain complex $\mathrm{PV}^{i,\bullet}(X) = \left(\oplus_j \mathrm{PV}^{i,j}(X)[-j], \overline{\partial}\right)$ which provides a free resolution of the sheaf of holomorphic polyvector fields $\mathrm{PV}^i_{\mathrm{hol}}(X)$ of type i.

There is a bracket on the space of holomorphic polyvector fields called the Nijenhuis–Schouten bracket. This bracket is defined using holomorphic pvdifferential operators, so extends to a bracket on the Dolbeault complex to define a bracket of the form

$$\{\cdot,\cdot\}_{\mathrm{NS}} \colon \mathrm{PV}^{i,j}(X) \times \mathrm{PV}^{k,\ell}(X) \to \mathrm{PV}^{i+k-1,j+\ell}(X).$$

This bracket endows the total complex

$$\mathrm{PV}^{\bullet,\bullet}(X)[1] = \left(\oplus_{i,j}\mathrm{PV}^{i,j}(X)[-i-j+1], \overline{\partial}\right) \tag{11}$$

with the structure of a dg Lie algebra. Here, we note that in this dg Lie algebra the space $\mathrm{PV}^{i,j}(X)$ lies in cohomological degree $i+j-1$.

When X is Calabi–Yau of complex dimension n, the holomorphic volume form Ω defines an isomorphism

$$\vee\Omega : \mathrm{PV}^{i,j}(X) \cong \Omega^{n-i,j}(X).$$

In turn, the holomorphic de Rham operator $\partial : \Omega^{p,j}(X) \to \Omega^{p+1,j}$ defines a holomorphic differential operator

$$\partial_\Omega : \mathrm{PV}^{i,j}(X) \to \mathrm{PV}^{i-1,j}(X).$$

This is the holomorphic analog of the divergence operator with respect to Ω.

There is compelling evidence **CLsugra** for the existence of a twist of the Type IIA string on the ten-manifold $M \times X$ where M is a real surface and X is a Calabi–Yau four-fold. Roughly, the twist behaves like the A-model topological string along M and the B-model topological string along X. The conjectural state space is

$$\Omega^\bullet(M) \,\widehat{\otimes}\, \mathrm{PV}^{\bullet,\bullet}(X).$$

The linear BRST differential on this state space is given by $\mathrm{d}_{\mathrm{dR}} + \overline{\partial}$. The fields of the closed string are given by the S^1-equivariant states, which is modeled **CLbcov** by

$$\Omega^\bullet(M) \,\widehat{\otimes}\, \mathrm{PV}^{\bullet,\bullet}(X)[\![u]\!]$$

where u is the equivariant parameter and the new linear BRST operator is $\mathrm{d}_{\mathrm{dR}} + \overline{\partial} + u\partial_\Omega$.

With this informal description of the twist of the Type IIA closed string in hand, one can deduce the following low energy limit of the description which gives a conjectural description of Type IIA supergravity.

Conjecture 3.1 (Costello-Li **CLsugra**). *Let (M, h_M) be a Kahler surface and (X, h_X) be a Calabi-Yau 4-fold. Consider perturbative type IIA supergravity on $M \times X$ around a background where:*

- *the graviton is set to $h_M + h_X$,*
- *the bosonic ghost for local supersymmetries is set to a covariantly constant square-zero spinor Q that is invariant for the* $\mathrm{SU}(4)$ *subgroup of isometries preserving $X \subset M \times X$,*
- *all other background fields are set to zero.*

In this background, the resulting theory is equivalent to the Kodaira–Spencer type theory whose fields are

$$\Omega^\bullet(M) \,\widehat{\otimes}\, \mathrm{PV}^{\bullet,\bullet}(X)[\![u]\!]$$

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his theory is
only $\mathbb{Z}/2$-graded,
right?

where u is a parameter of cohomological degree 2*.*

Type II supergravity around such backgrounds where the bosonic ghost takes a nonzero VEV is what is referred to as *twisted supergravity* in **CLsugra** In fact, the authors further conjecture that type IIA superstring theory on $M \times X$ around a particular Ramond-Ramond background is equivalent to a topological string theory given by the A-model on M and the B-model on X. This implies the above conjecture at the level of closed string field theory. Indeed the above description of twisted supergravity is a combination of Kodaira–Spencer theory on X with the zero-winding sector of Kahler gravity on M.

There is further evidence for this claim through studying the residual supersymmetry. Namely, they construct an L_∞ map from the Q-cohomology of the 10d $\mathcal{N} = (1,1)$ algebra to the fields of the above theory **CLsugra** realizing the former as a collection of ghosts in the latter. This allows us to make sense of further twists of the above twist.

The goal of this section is to identify the above conjectural twist of IIA supergravity with an S^1 reduction of our 11d theory. In order to do so, we will need to introduce a slight modification of the Kodaira-Spencer theory. We turn to this in the next subsection.

3.1. **Kodaira–Spencer theory with potentials.** One property of the description of the twist of type II supergravity in terms of Kodaira–Spencer theory is that the Ramond-Ramond fields of type II do not correspond to fundamental fields in Kodaira–Spencer theory. Rather it is certain components of the Ramond-Ramond field strengths that appear. However, in the identification of type IIA supergravity as the S^1 reduction of 11d supergravity, certain components of the C-field become components of the Ramond-Ramond 2-form. This suggests that in order to match the dimensional reduction of our 11d theory with a twist of type IIA, we must modify the description of the twist of type IIA to include certain components of Ramond-Ramond fields as potentials for those components of Ramond-Ramond field strengths that one finds in the twisted theory.

3.1.1. *Warm-up: four-dimensional Kodaira–Spencer theory.* In the BV formalism, minimal Kodaira–Spencer theory on X is a (degenerate) Poisson BV theory with space of fields given by

$$\begin{array}{ccc} \underline{\text{odd}} & & \underline{\text{even}} \\ & & \\ & & \mathrm{PV}^{0,\bullet} \\ & & \\ \mathrm{PV}^{1,\bullet} & \xrightarrow{u\partial} & u\mathrm{PV}^{0,\bullet}. \end{array}$$

We will denote this sheaf of $\mathbb{Z}/2$-graded cochain complexes by $\mathcal{E}_{\mathrm{KS}}$.

There is a local (dg) Lie algebra structure on the parity shifted object $\Pi\mathcal{E}_{\mathrm{KS}}$. The Lie bracket is defined using the Schouten-Nijenhuis bracket $[-,-]_{\mathrm{NS}}$ on polyvector fields and is given by the formula

$$[u^k\alpha, u^\ell\beta] = u^{k+\ell}[\alpha,\beta]_{\mathrm{NS}}.$$

where $k,\ell = 0,1$. Together with the differential this equips the parity shifted sheaf of cochain complexes $\Pi\mathcal{E}_{\mathrm{KS}}$ with the structure of a local dg Lie algebra.

The fields of minimal Kodaira–Spencer theory $\mathcal{E}_{\mathrm{KS}}$ is equipped with an odd Poisson tensor defined by

$$\Pi_{\mathrm{KS}} = (\partial \otimes 1)\delta_{\mathrm{Diag}}.$$

We introduce another theory on the Calabi–Yau surface X that we call minimal Kodaira–Spencer theory *with potentials*. The underlying vector bundle is

$$\begin{array}{cc} \underline{\text{odd}} & \underline{\text{even}} \\ & \mathrm{PV}^{0,\bullet} \\ \mathrm{PV}^{2,\bullet}. & \end{array}$$

We will denote the resulting $\mathbb{Z}/2$-graded sheaf of cochain complexes by $\mathcal{E}_{\mathrm{Pot}}$.

We interpret this as the theory of “potentials" of minimal Kodaira–Spencer theory in the following way. There is a map of sheaves of cochani complexes

$$\Phi : \mathcal{E}_{\mathrm{Pot}} \to \mathcal{E}_{\mathrm{KS}}$$

which is the identity on $\mathrm{PV}^{0,\bullet}$ and given by $\partial : \mathrm{PV}^{2,\bullet} \to \mathrm{PV}^{1,\bullet}$ on the remaining component. It is immediate to see that Φ defines a map of sheaves of cochain complexes. The theory $\mathcal{E}_{\mathrm{pot}}$ is equipped with a non-degenerate BV pairing defined by the wedge-and-integrate pairing

$$\omega_{\mathrm{pot}}(\alpha,\beta) = \int \alpha \wedge \beta.$$

It is immediate to verify that Φ intertwines the resulting bivector $\omega_{\mathrm{pot}}^{-1}$ and the Kodaira–Spencer Poisson bivector Π_{KS}.

In fact, the parity shifted bundle $\Pi\mathcal{E}_{\mathrm{pot}}$ also has the structure of a local Lie algebra, and the map Φ intertwines these local Lie algebra structures.

To describe the local Lie algebra structure on minimal Kodaira–Spencer theory with potentials we use the Calabi–Yau form Ω to identify $\mathcal{E}_{\mathrm{Pot}}$ with the sheaf of cochain complexes

$$\begin{array}{cc} \underline{\text{odd}} & \underline{\text{even}} \\ & \Omega^{2,\bullet} \\ \Omega^{0,\bullet}. & \end{array}$$

Now, note that any Calabi–Yau surface comes equipped with a holomorphic symplectic structure and there is a Poisson bracket defined on the sheaf of holomorphic functions. Since the bracket is defined in terms of holomorphic differential operators, it extends to a bracket on the Dobleault complex $\Omega^{0,\bullet}(X)$.

This further extends to a local Lie algebra structure on the semi-direct product

$$\Omega^{0,\bullet}(X) \ltimes \Pi\Omega^{2,\bullet}(X)$$

which describes the local Lie structure on $\Pi\mathcal{E}_{\text{Pot}}$. It is immediate to verify that the map $\Phi : \mathcal{E}_{\text{pot}} \to \mathcal{E}$ intertwines the two L_∞-structures.

This equips $\mathcal{E}_{\text{pot}}$ with the structure of an interacting (non-degenerate) BV theory. Its relationship to (minimal) Kodaira–Spencer theory can be summarized as follows.

Proposition 3.2. *The map $\Phi : \mathcal{E}_{\text{pot}} \to \mathcal{E}_{\text{KS}}$ is a map of Poisson BV theories. In particular, it determines a map of $\mathbb{P}_0$-factorization algebras on X:*

$$\Phi^* : \text{Obs}_{\mathcal{E}_{\text{KS}}} \to \text{Obs}_{\mathcal{E}_{\text{Pot}}}.$$

3.1.2. *Eight-dimensional Kodaira–Spencer theory.* Let X be a Calabi-Yau 4 fold. Minimal Kodaira-Spencer theory on X is a $\mathbb{Z}/2$-graded theory with the following fundamental fields:

- The even fields are a holomorphic function μ^0 and a ∂-closed holomorphic bivector μ^2.
- The odd fields are a divergence-free holomorphic vector field μ^1 and a ∂-closed holomorphic section μ^3 of $\wedge^3 T_X$.

The space of fields admits a locally free description obtained by including the "descendants". The descendants of the field μ^j will be denoted $u^k\mu^j$ where $k = 1, \ldots, j$. Here, $u^k\mu^j$ is a section of $\text{PV}^{j-k,\bullet}$. The sheaf of cochain complexes $\mathcal{E}$ underlying minimal Kodaira–Spencer theory on X is

$$\begin{tikzcd}
\underline{\text{odd}} & \underline{\text{even}} & \underline{\text{odd}} & \underline{\text{even}} \\
 & & & \mu^0 \in \text{PV}^{0,\bullet} \\
 & & \sum u^k\mu^1 \in \text{PV}^{1,\bullet} \arrow[r, "u\partial"] & u\text{PV}^{0,\bullet} \\
 & \sum u^k\mu^2 \in \text{PV}^{2,\bullet} \arrow[r, "u\partial"] & u\text{PV}^{1,\bullet} \arrow[r, "u\partial"] & u^2\text{PV}^{0,\bullet} \\
\sum u^k\mu^3 \in \text{PV}^{3,\bullet} \arrow[r, "u\partial"] & u\text{PV}^{2,\bullet} \arrow[r, "u\partial"] & u^2\text{PV}^{1,\bullet} \arrow[r, "u\partial"] & u^3\text{PV}^{0,\bullet}
\end{tikzcd}$$

The differential on this sheaf of cochain complexes is given by $\overline{\partial} + u\partial$.

There is a local Lie algebra structure on $\Pi\mathcal{E}$ using the Schouten-Nijenhuis bracket $\{-,-\}_{\mathrm{NS}}$ on polyvector fields. On the fields (including the descendants) it is defined by the formula

$$[u^k\mu^i, u^\ell\mu^j] = u^{k+\ell}\{\mu^i, \mu^j\}_{\mathrm{NS}}.$$

The space of fields of minimal Kodaira–Spencer theory $\mathcal{E}_{\mathrm{KS}}$ is equipped with an odd Poisson tensor defined by

$$\Pi_{\mathrm{KS}} = (\partial \otimes 1)\delta_{\mathrm{Diag}}.$$

Together with the local Lie algebra structure, this data equips $\mathcal{E}_{\mathrm{KS}}$ with the structure of a $\mathbb{Z}/2$-graded Poisson BV theory.

As in the surface case, there is a closely related BV theory describing the "potentials" of minimal Kodaira–Spencer theory. The underlying sheaf of cochain complexes is

$$\begin{array}{cccc}
\underline{\text{odd}} & \underline{\text{even}} & \underline{\text{odd}} & \underline{\text{even}} \\
 & & & \eta^0 \in \mathrm{PV}^{0,\bullet} \\
 & & \mu^1 + u\mu^0 \in \mathrm{PV}^{1,\bullet} & \xrightarrow{u\partial} u\mathrm{PV}^{0,\bullet} \\
u^{-1}\gamma^4 + \gamma^3 \in u^{-1}\mathrm{PV}^{4,\bullet} & \xrightarrow{u\partial} \mathrm{PV}^{3,\bullet} & & \\
\beta^4 \in \mathrm{PV}^{4,\bullet} & & &
\end{array}$$

We will again refer to this sheaf as $\mathcal{E}_{\mathrm{pot}}$. Note that we can identify

$$\mathcal{E}_{\mathrm{pot}} \cong \Pi T^*(\mathrm{PV}^{0,\bullet} \oplus (\mathrm{PV}^{1,\bullet} \to u\mathrm{PV}^{0,\bullet})).$$

Definition 3.3. Let X be a Calabi-Yau 4-fold, and let $\mathcal{L}$ denote the local dg Lie algebra $\mathrm{PV}^{0,\bullet} \oplus (\mathrm{PV}^{1,\bullet} \to u\mathrm{PV}^{0,\bullet})$ with Lie bracket given by the Schouten bracket. Kodaira-Spencer with potentials on X is the $\mathbb{Z}/2$-graded BV theory given by deforming the BF theory of $\mathcal{L}$ by the local functional

$$J(\nu^0, \gamma^3) = \frac{1}{3}\int_X \left((\nu^0\partial\gamma^3\partial\gamma^3) \vdash \Omega\right) \wedge \Omega$$

To facillitate the calculations in this section, it will be useful to explicate the Lie structure. We have the following brackets:

- The dg Lie structure on $\mathcal{L}$ is given by the following brackets:

$$\begin{aligned}
\mathrm{PV}^{1,\bullet} \otimes \mathrm{PV}^{0,\bullet} \to \mathrm{PV}^{0,\bullet}, &\quad [\mu^1, \nu^0] = \{\mu^1, \nu^0\}_{NS} \\
\mathrm{PV}^{1,\bullet} \otimes \mathrm{PV}^{1,\bullet} \to \mathrm{PV}^{1,\bullet}, &\quad [\mu^1, \mu^1] = \{\mu^1, \mu^1\}_{NS} \\
\mathrm{PV}^{1,\bullet} \otimes u\mathrm{PV}^{0,\bullet} \to u\mathrm{PV}^{0,\bullet}, &\quad [\mu^1, u\mu^0] = u\{\mu^1, \mu^0\}_{NS}.
\end{aligned}$$

- Next we have the action of $\mathcal{L}$ on $\Pi\mathcal{L}^*$, given by the brackets:

$$\mathrm{PV}^{4,\bullet} \otimes \mathrm{PV}^{0,\bullet} \to \mathrm{PV}^{3,\bullet}, \quad [\beta^4, \nu^0] = \{\beta^4, \nu^0\}_{\mathrm{NS}}$$
$$\mathrm{PV}^{3,\bullet} \otimes \mathrm{PV}^{1,\bullet} \to \mathrm{PV}^{3,\bullet}, \quad [\gamma^3, \mu^1] = \{\gamma^3, \mu^1\}_{\mathrm{NS}} \tag{12}$$
$$u^{-1}\mathrm{PV}^{4,\bullet} \otimes u\mathrm{PV}^{0,\bullet} \to \mathrm{PV}^{3,\bullet}, \quad [u^{-1}\gamma^4, u\mu^0] = \{\gamma^4, \mu^0\}_{\mathrm{NS}}. \tag{13}$$

- The deformation $J(\nu^0, \gamma^3)$ is given by the brackets:

$$\mathrm{PV}^{3,\bullet} \otimes \mathrm{PV}^{3,\bullet} \to \mathrm{PV}^{4,\bullet}, \quad [\gamma^3, \tilde{\gamma}^3] = \frac{1}{2}(\{\partial\gamma^3, \tilde{\gamma}^3\}_{\mathrm{NS}} + (-1)^{|\gamma^3|}\{\gamma^3, \partial\tilde{\gamma}^3\}_{\mathrm{NS}})$$
$$\mathrm{PV}^{3,\bullet} \otimes \mathrm{PV}^{0,\bullet} \to \mathrm{PV}^{1,\bullet}, \quad [\gamma^3, \nu^0] = \{\partial\gamma^3, \nu^0\}_{\mathrm{NS}}.$$

Like in the case of Kodaira–Spencer theory on a complex surface, there is a map of sheaves of cochain complexes $\Phi^{(1)} : \mathcal{E}_{\mathrm{pot}} \to \mathcal{E}_{\mathrm{KS}}$ given by testing

$$\begin{pmatrix} \nu^0 \\ \mu^1 + u\mu^0 \\ u^{-1}\gamma^4 + \gamma^3 \\ \beta^4 \end{pmatrix} \mapsto \begin{pmatrix} \nu^0 \\ \mu^1 + u\mu^0 \\ \partial\gamma^3 \\ \partial\beta^4 \end{pmatrix}$$

This fails to be a map of local dg Lie algebras, but the induced map on ∂-cohomology is indeed a map of local Lie algebras. Therefore, we still have the following:

Proposition 3.4. *There is a map of local L_∞-algebras $\mathcal{E}_{\mathrm{pot}} \to \mathcal{E}_{\mathrm{KS}}$ extending $\Phi^{(1)}$.*

Proof. Consider the collection of linear maps $\{\Phi^{(n)} : \mathcal{E}_{\mathrm{Pot}}^{\otimes n} \to \mathcal{E}_{\mathrm{KS}}[n-1]\}$ where $\Phi^{(1)}$ is given as above, $\Phi^{(2)}$ is given by

$$\begin{pmatrix} \nu^0 \\ \mu^1 + u\mu^0 \\ u^{-1}\gamma^4 + \gamma^3 \\ \beta^4 \end{pmatrix} \otimes \begin{pmatrix} \tilde{\nu}^0 \\ \tilde{\mu}^1 + u\tilde{\mu}^0 \\ u^{-1}\tilde{\gamma}^4 + \tilde{\gamma}^3 \\ \tilde{\beta}^4 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ \{\mu^0, \tilde{\gamma}^3\}_{\mathrm{NS}} - (-1)^{(|\tilde{\mu}^0|-1)(|\gamma^3|-1)}\{\tilde{\mu}^0, \gamma^3\}_{\mathrm{NS}} \\ 0 \end{pmatrix}$$

and $\Phi^{(n)} = 0$ for all $n \neq 1, 2$

We claim that the $\Phi^{(n)}$ define an L_∞ map. To this end, there are three kinds of conditions to check.

- A 2-linear condition which expresses that $\Phi^{(2)}$ renders the failure of $\Phi^{(1)}$ to preserve the brackets homotopically trivial. That is, for $a, b \in \mathcal{E}_{\mathrm{Pot}}$,

$$\Phi^{(1)}([a,b]) - [\Phi^{(1)}(a), \Phi^{(1)}(b)] = d_{\mathrm{KS}}\Phi^{(2)}(a,b) - \Phi^{(2)}(d_{\mathrm{Pot}}a, b) - (-1)^{|a|}\Phi^{(2)}(a, d_{\mathrm{Pot}}b).$$

  Note that $\Phi^{(1)}$ in fact preserves all brackets other than the brackets 12, 13. It is immediate from the expression for $\Phi^{(2)}$ that the right-hand side of **??** vanishes for a, b in the domain of brackets other than 12, 13. Therefore, we need only check that **??** in two cases.

Consider $\gamma^3 \otimes \tilde{\mu}^1 \in \mathrm{PV}^{3,\bullet} \otimes \mathrm{PV}^{1,\bullet}$. We have that

$$\begin{aligned}
\Phi^{(1)}([\gamma^3, \tilde{\mu}^1]) &- \{\Phi^1(\gamma^3), \Phi^1(\tilde{\mu}^1)\}_{\mathrm{NS}} \\
&= \partial\{\gamma^3, \tilde{\mu}^1\}_{\mathrm{NS}} - \{\partial\gamma^3, \tilde{\mu}^1\}_{\mathrm{NS}} \\
&= \{\partial\gamma^3, \tilde{\mu}^1\}_{\mathrm{NS}} + (-1)^{|\gamma^3|-1}\{\gamma^3, \partial\tilde{\mu}^1\}_{\mathrm{NS}} - \{\partial\gamma^3, \tilde{\mu}^1\}_{\mathrm{NS}} \\
&= (-1)^{|\gamma^3|-1}\{\gamma^3, \partial\tilde{\mu}^1\}_{\mathrm{NS}}
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
d_{\mathrm{KS}}\Phi^{(2)}(\gamma^3, \tilde{\mu}^1) - \Phi^{(2)}(d_{\mathrm{Pot}}\gamma^3, \tilde{\mu}^1) &- (-1)^{|a|}\Phi^{(2)}(\gamma^3, d_{\mathrm{Pot}}\tilde{\mu}^1) \\
&= -(-1)^{|\gamma^3|}\Phi^{(2)}(\gamma^3, u\partial\tilde{\mu}^1) \\
&= -(-1)^{|\gamma^3|-1}(-1)^{(|\partial\tilde{\mu}^1|-1)(|\gamma^3|-1)}\{\partial\tilde{\mu}^1, \gamma^3\}_{\mathrm{NS}} \\
&= (-1)^{|\gamma^3|-1}\{\gamma^3, \partial\tilde{\mu}^1\}_{\mathrm{NS}}
\end{aligned}$$

Next consider $u^{-1}\gamma^4 \otimes u\tilde{\mu}^0 \in u^{-1}\mathrm{PV}^{4,\bullet} \otimes u\mathrm{PV}^0$. We have that

$$\begin{aligned}
\Phi^{(1)}([u^{-1}\gamma^4, u\tilde{\mu}^0]) - \{\Phi^1(u^{-1}\gamma^4), \Phi^1(\tilde{u}\mu^0)\}_{\mathrm{NS}} &= \partial\{\gamma^4, \tilde{\mu}^0\}_{\mathrm{NS}} \\
&= \{\partial\gamma^4, \tilde{\mu}^0\}_{\mathrm{NS}}
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
d_{\mathrm{KS}}\Phi^{(2)}(u^{-1}\gamma^4, \tilde{\mu}^0) - \Phi^{(2)}(d_{\mathrm{Pot}}u^{-1}\gamma^4, \tilde{\mu}^0) &- (-1)^{|u^{-1}\gamma^4|}\Phi^{(2)}(u^{-1}\gamma^4, d_{\mathrm{Pot}}u\tilde{\mu}^0) \\
&= \Phi^{(2)}(\partial\gamma^4, u\tilde{\mu}^0) \\
&= -(-1)^{(|\tilde{\mu}^0|-1)(|\partial\gamma^4|-1)}\{\tilde{\mu}^0, \partial\gamma^4\}_{\mathrm{NS}} \\
&= \{\partial\gamma^4, \tilde{\mu}^0\}_{\mathrm{NS}}
\end{aligned}$$

Therefore the 2-linear condition holds.

- Next we have a 3-linear condition; since $\Phi^{(3)} = 0$, the condition becomes

$$0 = \sum_{\sigma \in S_3} \{\Phi^{(2)}(x_{\sigma(1)}, x_{\sigma(2)}), \Phi^{(1)}(x_{\sigma(k)})\}_{\mathrm{NS}} + \sum_{\sigma \in S_3} \Phi^{(2)}([x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}).$$

  This should vanish because of the Jacobi identity for the Schouten-Nijenhius bracket.
- Finally we have a 4-linear condition; since $\Phi^{(3)} = \Phi^{(4)} = 0$, the condition reads

$$0 = \sum_{\sigma \in S_4} \{\Phi^{(2)}(x_{\sigma(1)}, x_{\sigma(2)}), \Phi^{(2)}(x_{\sigma(3)}, x_{\sigma(4)})\}_{\mathrm{NS}}.$$

□

Surya
signs in last two conditions? comment on why they vanish

Corollary 3.5. *The $\{\Phi^{(n)}\}$ determine a map of Poisson BV theories $\mathcal{E}_{pot} \to \mathcal{E}_{KS}$. In particular it determines a map of $\mathbb{P}_0$-factorization algebras $\Phi^* : \mathrm{Obs}_{\mathcal{E}_{\mathrm{KS}}} \to \mathrm{Obs}_{\mathcal{E}_{\mathrm{Pot}}}$.*

3.2. **The Type IIA topological string.**

3.3. **Reduction of twisted supergravity.**

3.3.1. *Calabi–Yau compactifications.*

3.4. **Twisted supergravity on a three-fold.**

4. The non-minimal G2 twist

Brian
some general background on G2 twist, state main result

We consider the eleven-dimensional theory on the space

$$X \times Z \times \mathbb{R}$$

where Z is a hyper-Kähler surface and X is a Calabi–Yau three-fold. Denote by Ω_Z the holomorphic volume form on Z and Ω_X the holomorphic volume form on X. We will work in a background where the $(1,0)$-component of the field γ satisfies the following equations:

$$\begin{aligned} \partial\gamma^{1,0} &= \Omega_Z \\ \bar\partial\gamma^{1,0} + \mathrm{d}_{\mathrm{dR}}\gamma^{1,0} &= 0. \end{aligned} \tag{14}$$

The second equation implies that $\gamma^{1,0}$ is constant along $\mathbb{R}$ and holomorphic along $X \times Z$. The first equation says that $\gamma^{1,0}$ is a trivialation of the holomorphic volume form on Z.

To see that this is a consistent background of the eleven-dimensional theory we must check that $\gamma^{1,0}$ satisfies the appropriate equations of motion.

Lemma 4.1. *Any field $\gamma^{1,0}$ satisfying the equations in (14) satisfies the Maurer–Cartan equation of the eleven-dimensional theory.*

Proof. We must show that $\gamma^{1,0}$ satisfies the following Maurer–Cartan equation

$$Q_{\mathrm{BRST}}\gamma^{1,0} + \frac{1}{2}[\gamma^{1,0}, \gamma^{1,0}] = 0$$

where Q_{BRST} is the linear BRST operator and $[-,-]$ is the Lie bracket defining the interacting piece of the eleven-dimensional theory.

Recall, the linear BRST differential is of the form

$$\bar\partial + \partial_\Omega + \partial_{\Omega^0\to\Omega^1} + \mathrm{d}_{\mathrm{dR}}$$

Here, ∂_Ω is the divergence operator which only acts on the μ-type fields and $\partial_{\Omega^0\to\Omega^1}$ is the first piece of the holomorphic de Rham operator acting on $\Omega^{0,\bullet}(X \times Z)$. Since $(\bar\partial + \mathrm{d}_{\mathrm{dR}})\gamma^{1,0} = 0$ by assumption, we see that $\gamma^{1,0}$ is closed for the linear BRST operator.

The only component of the Lie bracket involving two fields of type γ is of the form

$$[\gamma,\gamma] = (\partial\gamma \wedge \partial\gamma) \vee (\Omega_X \wedge \Omega_Z) \in \mathrm{PV}^{1,\bullet}(X \times Z) \mathbin{\widehat{\otimes}} \Omega^\bullet(L).$$

Since $\partial\gamma^{1,0}$ is a $(2,0)$ form along Z, we see that $\partial\gamma^{1,0} \wedge \partial\gamma^{1,0} = 0$. We conclude that $\gamma^{1,0}$ satisfies the Maurer–Cartan equation. $\square$

Next, we expand the action functional near the background where $(1,0)$ component of γ takes the value $\gamma^{1,0}$ where $\gamma^{1,0}$ satisfies equations (14). This will generate new kinetic and interacting terms which we can extract by inserting a formal parameter δ and expressing the action functional in terms of the deformed field $\widetilde{\gamma} = \gamma + \delta\gamma^{1,0}$.

There are two interaction terms in the original theory. The first is

$$\frac{1}{2}\int_{X\times Z\times L} (\gamma \vee \{\mu,\mu\}) \wedge (\Omega_Z \wedge \Omega_X) \tag{15}$$

and the second is

$$\frac{1}{6}\int_{X\times Z\times L} \gamma\partial\gamma\partial\gamma. \tag{16}$$

We can integrate Equation (15) by parts to put it in the form $\frac{1}{2}\int_{X\times Z\times L}[(\partial\gamma)\vee(\mu\wedge\mu)]$ where $\mu\wedge\mu$ is the wedge product of polvector fields.Expanding this expression around $\gamma \to \widetilde{\gamma} = \gamma + \delta\gamma^{1,0}$ we obtain

Brian
there might be some factors I'm being sloppy with here

$$\frac{1}{2}\int (\gamma \vee \{\mu,\mu\}) \wedge (\Omega_Z \wedge \Omega_X) + \frac{\delta}{2}\int [\Omega_Z \vee (\mu\wedge\mu)] \wedge (\Omega_Z \wedge \Omega_X).$$

Here, we have used the identity $\partial\gamma^{1,0} = \Omega_Z$.

Next, we expand Equation (16) around $\gamma \to \widetilde{\gamma} = \gamma + \delta\gamma^{1,0}$. This becomes

$$\frac{1}{6}\int \gamma\partial\gamma\partial\gamma + \frac{\delta}{2}\int (\gamma\partial\gamma)\wedge\Omega_Z.$$

Notice that there are no δ^2 terms since $\partial\gamma^{1,0}\partial\gamma^{1,0} = 0$.

So far, we have written everything in terms of action functionals. There is a completely equivalent statement in terms of the resulting dg Lie algebra structure describing the eleven-dimensional theory in this background which we summarize as follows.

Lemma 4.2. *The dg Lie algebra describing the eleven-dimensional theory placed in a background where the* $(1,0)$ *component of* γ *takes the value* $\gamma^{1,0}$ *satisfying (14) is isomorphic to the the dg Lie algebra whose differential is*

$$Q_{\mathrm{BRST}} + \frac{1}{2}\left\{\int [\Omega_Z \vee (\mu\wedge\mu)]\wedge(\Omega_Z\wedge\Omega_X) + \int (\gamma\partial\gamma)\wedge\Omega_Z,\ -\right\} \tag{17}$$

and whose Lie bracket is unchanged. Here, Q_{BRST} *is the original linear BRST differential of the eleven-dimensional theory.*

This lemma followed directly from our analysis of the way the action functional of the theory decomposed around this particular background. The next result we state is an equivalence with a theory which exists on any product of manifolds $Z \times M$ where Z is a hyper-Kähler surface as above but now M is any smooth seven-dimensional manifold.

Proposition 4.3. *The eleven-dimensional theory on* $X\times Z\times L$ *placed in the background where the* $(1,0)$ *component of* γ *takes the value* $\gamma^{1,0}$ *satisfying (14) is equivalent to the* $\mathbb{Z}/2$*-graded theory whose fields are* $\alpha \in \Omega^{0,\bullet}(Z)\,\widehat{\otimes}\,\Omega^{\bullet}(X\times L)[1]$ *and whose action functional* *reads*

$$\frac{1}{2}\int \alpha \mathrm{d}\alpha + \frac{1}{6}\int \alpha\{\alpha,\alpha\}$$

where $\{-,-\}$ *is the Poisson bracket on* $\Omega^{0,\bullet}(Z)$.

As stated in this proposition, it is clear that the theory does not depend on the complex structure on the Calabi–Yau three-fold X even though this is not *a priori* obvious from the description in Lemma 4.2. The theory only depends on the smooth structure on $M = X \times L$.

Brian
cite Surya and Phil, Chris and B, Kevin

Proof. To unpack this new differential (17) in Lemma 4.2 we need to reidentify the fields of the eleven-dimensional theory. Using the holomorphic volume forms Ω_X and Ω_Z we make the following identifications

$$\begin{aligned} \mathrm{PV}^{1,\bullet}(X\times Z) &= \mathrm{PV}^{1,\bullet}(X) \,\widehat{\otimes}\, \mathrm{PV}^{0,\bullet}(Z) \oplus \mathrm{PV}^{0,\bullet}(X) \,\widehat{\otimes}\, \mathrm{PV}^{1,\bullet}(Z) \\ &\cong \Omega^{2,\bullet}(X) \,\widehat{\otimes}\, \Omega^{0,\bullet}(Z) \oplus \Omega^{0,\bullet}(X) \,\widehat{\otimes}\, \Omega^{1,\bullet}(Z) \end{aligned} \tag{18}$$

From Lemma 4.2 we see that the new terms in the linear BRST differential arise from the quadratic terms in the action

$$\int [\Omega_Z \vee (\mu\wedge\mu)] \wedge (\Omega_Z \wedge \Omega_X) + \int (\gamma\partial\gamma)\wedge\Omega_Z. \tag{19}$$

Recall that the linear BRST complex of the original eleven-dimensional theory is obtained by taking the tensor product of the cochain complexes $\mathrm{PV}^{1,\bullet}(X\times Z) \xrightarrow{\partial_\Omega} \mathrm{PV}^{0,\bullet}(X\times Z)$ and $\Omega^{0,\bullet}(X\times Z) \xrightarrow{\partial} \Omega^{1,\bullet}(X\times Z)$ with the de Rham complex $\Omega^\bullet(L)$ on the one-manifold L. Using the decomposition (18) we see that the deformed linear BRST complex is the tensor product of the de Rham complex $\Omega^\bullet(L)$ with the cochain complex

$$\cong \quad \begin{tikzcd}
\Omega^{0,\bullet}(X)\,\widehat{\otimes}\,\Omega^{1,\bullet}(Z) \arrow[dr, "\partial_Z"] & \\
& \Omega^{3,\bullet}(X)\,\widehat{\otimes}\,\Omega^{2,\bullet}(Z) \\
\Omega^{2,\bullet}(X)\,\widehat{\otimes}\,\Omega^{0,\bullet}(Z) \arrow[ur, "\partial_X"'] & \\
& \Omega^{1,\bullet}(X)\,\widehat{\otimes}\,\Omega^{0,\bullet}(Z) \arrow[ul, dashed, bend right, "\partial_X"] \\
\Omega^{0,\bullet}(X)\,\widehat{\otimes}\,\Omega^{0,\bullet}(Z) \arrow[ur, "\partial_X"] \arrow[dr, "\partial_Z"'] & \\
& \Omega^{0,\bullet}(X)\,\widehat{\otimes}\,\Omega^{1,\bullet}(Z) \arrow[uuuuul, dashed, bend left]
\end{tikzcd}$$

Here, the dashed arrow along the outside of the diagram corresponds to the BV antibracket with the first term in (19). Note that this is the identity morphism on $\Omega^{0,\bullet}(X)\,\widehat{\otimes}\,\Omega^{1,\bullet}(Z)$.

The other dashed arrow corresponds to the BV antibracket with the second term in (19). It is given by the holomorphic de Rham operator $\partial_X : \Omega^{1,\bullet}(X) \to \Omega^{2,\bullet}(X)$.

Since the outer dashed arrow is an isomorphism, we see that this cochain map is a quasi-isomorphism. It is immediate to check that this quasi-isomorphism intertwines the Lie brackets.

Brian

cochain homotopy finish

□