QUANTIZATION OF THE UNCONSTRAINED THEORY

Consider the $\mathbb{Z}/2$-graded theory on $\mathbb{C}^5 \times \mathbb{R}$ with two sets of fundamental fields:

$$\mu \in \Pi\Omega^\bullet(\mathbb{R}) \,\widehat{\otimes}\, \mathrm{P.V.}^{1,\bullet}(\mathbb{C}^5)$$
$$\gamma \in \Omega^\bullet(\mathbb{R}) \,\widehat{\otimes}\, \Omega^{1,\bullet}(\mathbb{C}^5).$$

We will denote the full space of fields of the theory by $\mathcal{E}$.

This is a non-degenerate BV theory with pairing defined by

$$\int \gamma \wedge (\mu \vdash \Omega).$$

The action is

$$S = \int \gamma\left(\mathrm{d}\mu + \frac{1}{2}[\mu,\mu]\right) \vdash \Omega + \frac{1}{3}\int \gamma\partial\gamma\partial\gamma.$$

where $[\cdot,\cdot]$ denotes the Lie bracket of vector fields.

The first term describes a theory of BF type based on the local Lie algebra

$$\mathcal{L} = \Omega^\bullet(\mathbb{R}) \,\widehat{\otimes}\, \mathrm{P.V.}^{1,\bullet}(\mathbb{C}^5).$$

With the term $\frac{1}{3}\int \gamma\partial\gamma\partial\gamma$ we will view this theory as a deformation of a BF type theory.

Let $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ denote the cochain complex of local functionals. It is equipped with the classical BRST operator

$$\{S, -\} = \mathrm{d} + \mathrm{d}_{\mathrm{CE}} + \delta$$

where d_{CE} denotes (roughly) the Chevalley–Eilenberg differential for holomorphic vector fields and δ is the operator $\left\{\frac{1}{3}\int \gamma\partial\gamma\partial\gamma\ ,\ \cdot\ \right\}$.

1. One-loop Feynman diagrams

To construct the one-loop quantization we must first construct a one-loop effective family. This is a family of functionals $\{I[L]\}_{L>0}$ which satisfies the renormalization group equations.

The 11-dimensional theory on $\mathbb{C}^5 \times \mathbb{R}$ that we consider is a mixed holomorphic-topological theory. Heuristically, this means that the theory depends holomorphically on $\mathbb{C}^5$ and topologically on $\mathbb{R}$.

Theorem 1.1 (**GWR**)**.** *For any mixed holomorphic-topological theory on $\mathbb{C}^n \times \mathbb{R}^m$, $n, m \geq 0$, there exists a translation-invariant effective family $\{I[L]\}$ which to first order in $\hbar$ is finite.*

We briefly recount the main elements involved in the proof of this theorem. The result relies on the existence of a propagator corresponding to the "holomorphic-topological" gauge that enjoys particularly desirable analytic properties. At the level of the free theory, the linear BRST operator for the 11-dimensional theory in question is of the form

$$\overline{\partial} + \partial_\Omega + \mathrm{d}_{\mathrm{dR}}.$$

The first term denotes the $\overline{\partial}$-operator corresponding to a graded holomorphic vector bundle built from polyvector fields on $X = \mathbb{C}^5$. The second term term is the holomorphic differential operator action by the divergence operator on the same bundle of polyvector fields. The third term is the de Rham operator along $\mathbb{R}$.

The "holomorphic-topological gauge" refers to the choice of the following gauge fixing operator

$$Q^{\mathrm{GF}} = \overline{\partial}^* + \mathrm{d}^*_{\mathrm{dR}}.$$

The heat kernel is

$$\begin{aligned} K_t(z,t;w,s) = & \frac{1}{(2\pi i)^{11/2}} e^{|z-w|^2/4t + (t-s)^2/4t} \times \\ & \sum_{i=1}^{5} (\partial_{z^i} - \partial_{w^i}) \otimes (\mathrm{d}z^i - \mathrm{d}w^j) \otimes \prod_{j=1}^{5} (\mathrm{d}\overline{z}^j - \mathrm{d}\overline{w}^j) \otimes (\mathrm{d}t - \mathrm{d}s) \end{aligned}$$

The propagator is

$$P_{\epsilon<L}(z,t;w,s) = \int_{T=\epsilon}^{L} Q^{\mathrm{GF}} K_t(z,t;w,s) \mathrm{d}T.$$

2. The quantum master equation

The anomaly to satisfying the scale L quantum master equation is

$$\Theta[L] = QI[L] + \hbar \triangle_L I[L] + \frac{1}{2}\{I[L], I[L]\}_L.$$

Here $\triangle_L$ is the well-defined regularized scale L BV Laplacian, and $\{-,-\}_L$ is a regularized version of the classical BV bracket. The $\hbar \to 0$, $L \to 0$ limit of the above equation is precisely the classical master equation. An effective family $\{I[L]\}$ is a quantum field theory if it satisfies the scale L QME for every $L > 0$. For more complete details we refer to **Book2**.

In general, not every effective theory satisfies the QME. The *scale L anomaly* to satisfying the QME describes the failure of $I[L]$ to satisfy the scale L QME. Since $I[L]$ is filtered by powers of $\hbar$, so is the anomaly. For theories of cotangent type as considered in this paper, $I[L]$ truncates at order $\hbar$, hence we only need to consider the $\hbar$-linear anomaly which we denote by $\hbar\Theta[L]$. The $L \to 0$ limit of $\Theta[L]$ is defined and determines a cohomological degree $+1$ *local functional*

$$\Theta \overset{\text{def}}{=} \lim_{L\to 0} \Theta[L] \in \mathcal{O}_{\text{loc}}(\mathcal{E}).$$

Moreover, Θ is closed for the classical differential $\{S, -\}_{\text{BV}}$, hence determines a cohomology class

$$[\Theta] \in H^1(\mathcal{O}_{\text{loc}}(\mathcal{E}), \{S, -\}_{\text{BV}}),$$

see **CostelloBook**.

2.1. **The obstruction deformation complex.** There is a decreasing filtration on $\mathcal{O}_{\text{loc}}(\mathcal{E})$ whose pth layer $F^p\mathcal{O}_{\text{loc}}(\mathcal{E})$ consists of local functionals which are at least p-linear in the field γ. BW: I'm worried that this isn't a complete filtration.

Roughly, the pth layer in the associated graded of this filtration is the local cohomology of holomorphic vector fields on $\mathbb{C}^5$ with values in a particular local module corresponding to p-linear local functionals of the field γ.

- At the zeroth layer $p = 0$, the associated graded is the local Lie algebra cochains $\mathrm{C}^\bullet_{\text{loc}}(\mathcal{L})$. By an argument similar to BW: Brian and Chris, we see that at this stage there the class of the anomaly can be identified with a class in $H^{12}(\mathfrak{w}_5)$ where $\mathfrak{w}_5$ is the Lie algebra of vector fields on the formal 5-disk. This cohomology is trivial $H^{12}(\mathfrak{w}_5) = H^{13}(\mathrm{BU}(5)) = 0$. BW: check this first isomorphism, there might be a connecting differential
- BW: at the next layer it seems like we should be looking at $H^\bullet(\mathfrak{w}_5, \mathfrak{w}_5)$.

2.2. **Characterizing the anomaly cocycle.** A general result of Costello **CostelloWittengenus** states that the one-loop anomaly of any BV theory reduces to the weight of a sum of wheel graphs. Furthermore, using the holomorphic-topological gauge, the types of wheels that appear are those with a fixed number of vertices as we will soon explain.

To state the following result, we fix some notation. If Γ is a graph with a distinguished internal edge e, let $W_{\Gamma,e}(P_{\epsilon<L}, K_\epsilon, I)$ denote the following modified weight. Instead of placing $P_{\epsilon<L}$ at each internal edge, we place K_ϵ at the edge labeled e and $P_{\epsilon<L}$ on the remaining internal edges.

Lemma 2.1. *The one-loop anomaly cocycle $\Theta \in \mathcal{O}_{\text{loc}}(\mathcal{E})$ is cohomologous to the $L \to 0$ limit of the expression*

$$\lim_{\epsilon\to 0} \sum_{\Gamma \in \text{Wheel}_{7,e}} W_\Gamma(P_{\epsilon<L}, K_\epsilon, I).$$

Here the sum is over all wheels with seven vertices equipped with a distinguished edge e.