6d $\mathcal{N} = (1,0)$

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(Brian: This holds for $\Sigma = S_+ \otimes W_+$.)

Proposition 0.1. Suppose $Q_1, Q_2 \in \Sigma = S_+ \otimes W_+$ and $Q_3 \in S_-$. Then

$$(Q_1, Q_3)Q_2 + (Q_2, Q_3)Q_1 = \rho(\Gamma(Q_1, Q_2))Q_3$$

0.1 The (half) hyper multiplet

In this subsection, S_+/S_- denote the positive/negative irreducible spin representations of $\mathfrak{so}(6;\mathbb{C})$. The spinorial representation we will be interested in is $\Sigma = S_+ \otimes W_+$, where $\dim_{\mathbb{C}}(W_+) = 2$ is a two-dimensional multiplicity space. To define the half hyper multiplet we fix a complex, symplectic vector space U whose symplectic form we call ω_U. We combine the symplectic forms ω_U, ω_{W_+} to define a pairing

$$(-,-)_{W_+ \otimes U} := \omega_{W_+} \otimes \omega_U : (W_+ \otimes U)^{\otimes 2} \to \mathbb{C}.$$

The spinorial pairing $(-,-) : S_+ \otimes S_- \to \mathbb{C}$ together with the obvious symplectic form ω_{W_+} on W_+, and the symplectic form ω_U on U induces a pairing (that we denote by the same symbol)

$$(-,-)_U : (S_\pm \otimes U) \otimes (S_\mp \otimes U) \to \mathbb{C}.$$

Remark 0.2. The terminology "half" hyper multiplet is to be consistent with the terminology in the physics literature. The (full) hyper multiplet refers to a multiplet which depends on the choice of a (not necessarily symplectic) vector space R. The relationship between the two is obtained by looking at the cotangent space $U = T^*R$. That is, the half hyper multiplet with values in the symplectic vector space $U = T^*R$ (with its obvious symplectic structure) is the (full) hyper multiplet for R.

The field content for the 6-dimensional half hyper multiplet with values in U is:

- a scalar valued in $W_+ \otimes U$; $\phi \in C^\infty(\mathbb{R}^4; W_+ \otimes U)$;
- a negative Weyl spinor valued in U; $\psi \in C^\infty(\mathbb{R}^4; S_- \otimes U)$.

The space of BRST fields is thus

$$F = C^\infty(\mathbb{R}^4; W_+ \otimes U \oplus S_- \otimes U).$$

The BRST action is

$$S(\phi, \psi) = \int_{\mathbb{R}^4} -\frac{1}{2}(\mathrm{d}\phi \wedge *\mathrm{d}\phi)_{W_+ \otimes U} + \frac{1}{2}(\psi, \not{\partial}\psi)_U.$$

As usual, we denote the antifields by ϕ^* and ψ^*.

The action of supersymmetry is given by a linear and quadratic functional:

$$\begin{aligned} I^{(1)}(Q) &= \int (\phi^*, (Q,\psi))_{W_+\otimes U} + (\psi^*, \rho(\mathrm{d}\phi)Q)_U \\ I^{(2)}(Q_1,Q_2) &= \int \frac{1}{4}(\Gamma(Q_1,Q_2)_{W_+}, \Gamma(\psi^*,\psi^*)_U) \end{aligned}$$

where $Q, Q_1, Q_2 \in \Sigma = S_+ \otimes W_+$. We point out some conventions in the definition of $I^{(2)}$. Define $\Gamma(Q_1,Q_2)_{W_+} \in V$ as the image of $Q_1 \otimes Q_2$ under the composition

$$\Sigma \otimes \Sigma = (S_+ \otimes W_+) \otimes (S_+ \otimes W_+) \cong (S_+ \otimes S_+) \otimes (W_+ \otimes W_+) \xrightarrow{\Gamma\otimes\omega_{W_+}} V.$$

The following result states that these functionals encode an off-shell action of six-dimensional $\mathcal{N} = (1,0)$ supersymmetry on the half hypermultiplet valued in U. The proof is very similar to the case of the four-dimensional $\mathcal{N} = 1$ chiral multiplet.

Theorem 0.1. The functional $\mathfrak{S} = S + I^{(1)} + I^{(2)}$ satisfies the classical master equation

$$\mathrm{d}_{\mathrm{Lie}}\mathfrak{S} + \frac{1}{2}\{\mathfrak{S},\mathfrak{S}\} = 0. \tag{1}$$

We decompose the classical master equation (1) into the following equations:

$$\begin{array}{rcl} \{S, I^{(1)}\} &=& 0 \\ \{S, I^{(2)}\} + \mathrm{d}_{CE} I^{(1)} + \frac{1}{2}\{I^{(1)}, I^{(1)}\} &=& 0 \\ \mathrm{d}_{CE} I^{(2)} + \{I^{(1)}, I^{(2)}\} &=& 0 \\ \{I^{(2)}, I^{(2)}\} &=& 0 \end{array} \tag{2}$$

The last equation is automatically satisfied since $I^{(2)}$ is independent of ϕ and ψ.

The first equation in (2) states that the classical action for the chiral multiplet is supersymmetric.

Lemma 0.3. One has $\{S, I^{(1)}\}(Q) = 0$ for all $Q \in \Sigma$.

Proof. Observe

$$-\{\mathrm{d}\phi \wedge *\mathrm{d}\phi, \phi^*(Q,\psi)\} = -2\mathrm{d}(Q,\psi) \wedge *\mathrm{d}\phi.$$

and

$$\{(\psi, \partial\!\!\!/\psi), (\psi^*, \rho(\mathrm{d}\phi)Q)\} = 2(\rho(\mathrm{d}\phi)Q, \partial\!\!\!/\psi)\mathrm{d}^4x.$$

The sum of these two terms is zero by Lemma **??**. □

Next, we move on to the second equation in (2).

Lemma 0.4. One has

$$\{S, I^{(2)}\} + \mathrm{d}_{CE} I^{(1)} + \frac{1}{2}\{I^{(1)}, I^{(1)}\} = 0. \tag{3}$$

Proof. Evaluating the equation (3) on $v_1, v_2\mathfrak{iso}(V)$ reduces to the claim that (??) defines a strict Lie action. Evaluating on $v \in \mathfrak{iso}(V)$ and $Q \in \Sigma$, the claim reduces to the fact that $I^{(1)}$ is Poincaré invariant. So, the only nontrivial term to check is the evaluation on $Q_1, Q_2 \in \Sigma$.

The individual terms are:

$$\{I^{(1)}, I^{(1)}\}(Q_1, Q_2) = -\phi^*(Q_1, \rho(\mathrm{d}\phi)Q_2) - \phi^*(Q_2, \rho(\mathrm{d}\phi)Q_1) \\ - (\psi^*, \rho(\mathrm{d}(Q_1, \psi))Q_2) - (\psi^*, \rho(\mathrm{d}(Q_2, \psi))Q_1)$$

$$(\mathrm{d}_{CE} I^{(1)})(Q_1, Q_2) = -\phi^* L_{\Gamma(Q_1,Q_2)}(\phi) - (\psi^*, \Gamma(Q_1, Q_2).\psi)$$

and

$$\{S, I^{(2)}(Q_1, Q_2)\} = \frac{1}{2}\Gamma(Q_1, Q_2)\Gamma(\psi^*, \partial\!\!\!/\psi)$$

Note that we can rewrite this as $\frac{1}{2}\rho(\Gamma(Q_1, Q_2))\partial\!\!\!/\psi$.

We first collect all terms in Equation (3) proportional to ϕ^*:

$$-\frac{1}{2}(Q_1, \rho(\mathrm{d}\phi)Q_2) - \frac{1}{2}(Q_2, \rho(\mathrm{d}\phi)Q_1) - L_{\Gamma(Q_1,Q_2)}\phi.$$

By the Clifford identity (Brian: $v \wedge \Gamma(Q_1, Q_2) = (Q_1, \rho(v)Q_2)$) we observe that the first two terms cancel with the third term.

Next, we collect all terms in Equation (3) proportional to ψ^*:

$$\begin{aligned} - \quad & \frac{1}{2}\rho(\mathrm{d}(Q_1, \psi))Q_2 - \frac{1}{2}\rho(\mathrm{d}(Q_2, \psi))Q_1 - \Gamma(Q_1, Q_2).\psi \\ + \quad & \frac{1}{2}\rho(\Gamma(Q_1, Q_2))\partial\!\!\!/\psi \end{aligned} \tag{4}$$

By Proposition **??** item (3), the first and second terms can be written as:

$$-\frac{1}{2}\rho(\mathrm{d}(Q_1, \psi))Q_2 - \frac{1}{2}\rho(\mathrm{d}(Q_2, \psi))Q_1 = -\frac{1}{2}\partial\!\!\!/\left((Q_1, \psi)Q_2 + (Q_2, \psi)Q_1\right).$$

Applying Proposition 0.1, to $Q_3 = \psi$, this becomes $-\frac{1}{2}\partial\!\!\!/\rho(\Gamma(Q_1, Q_2)\psi)$. Finally, by the Clifford identity, the sum of this term with the fourth term in (4) is precisely $\Gamma(Q_1, Q_2).\psi$. So the expression (4) is identically zero. □

Lemma 0.5.

$$\{I^{(1)}, I^{(2)}\}(Q_1, Q_2, Q_3) = 0$$

for every $Q_1, Q_2, Q_3 \in S_+ \oplus S_-$.

Proof. We have

$$\{I^{(1)}(Q_1), I^{(2)}(Q_2, Q_3)\} = \frac{1}{2}(\Gamma(Q_2, Q_3), \Gamma(\psi^*, \phi^* Q_1)) \\ = (\psi^*, \phi^* \rho(\Gamma(Q_2, Q_3))Q_1).$$

The expression $\{I^{(1)}, I^{(2)}\}(Q_1, Q_2, Q_3$ is obtained by cyclically symmetrizing the above expression. By Proposition **??** the cyclic symmetrization is identically zero. □

1 SCRAP

A general theory of matter for $\mathcal{N}=(1,0)$ supersymmetry arranges into a hypermultiplet which has the following BRST description. Let S_-^{6d} be the negative $\mathfrak{so}(6)$ irreducible spin representation and suppose R is a (complex) symplectic $\mathfrak{g}$-representation, with symplectic pairing labeled by $\langle -,-\rangle_R$. The BRST fields of the $(1,0)$ hypermultiplet consist of a $W\otimes R$-valued scalar

$$\phi \in \Omega^0(\mathbb{R}^6)\otimes W\otimes R$$

and an R valued negative Weyl spinor

$$\psi_- \in \Omega^0(\mathbb{R}^6)\otimes S_-^{6d}\otimes R.$$

Denote the induced symplectic pairing on $W\otimes R$ by $\langle -,-\rangle_{W\otimes R}$. The action of the free hypermultiplet is given by

$$S_{\text{matter}}(\phi,\psi_-) = \frac{1}{2}\int_{\mathbb{R}^6} \mathrm{d}^6x\ \langle \partial_{x_i}\phi, \partial_{x_i}\phi\rangle_{W\otimes R} + \frac{1}{2}\int_{\mathbb{R}^6}\mathrm{d}^6x\ \langle\psi_-,\not{\partial}\psi_-\rangle_R.$$

The action of supersymmetry on the 6-dimensional matter theory is encoded by a linear and quadratic functional:

$$I^{(1)}(Q) = \int \langle \phi^*, (Q,\psi_-)\rangle_{W\otimes R} + \int\langle\psi_-^*,\rho(\mathrm{d}\phi)Q\rangle_R$$
$$I^{(2)}(Q_1\otimes Q_2) = \int\langle\psi_-^*,\rho(\Gamma(Q_1,Q_2))\psi_-\rangle_R$$

where we recall that the pairing is of the form $\Gamma:\Sigma^{\otimes 2} = (S_+\otimes W)^{\otimes 2}\to V$. (Brian: I'm also using $(S_+)^* = S_-$.) These functionals provide an off-shell action of $\mathcal{N}=(1,0)$ supersymmetry on the free hypermultiplet, as the following proposition shows. Below, $\mathfrak{L}_{\text{matter}}$ is the abelian local Lie algebra describing the free hypermultiplet, and $\mathfrak{A}$ is the $(0,1)$ supertranslation algebra. As always, I_{Poin} is the action functional encoding ordinary Poincaré invariance.

Proposition 1.1 ([SWchar]). The functional

$$\mathfrak{S}_{\text{matter}} = S_{\text{matter}} + I_{\text{Poin}} + I^{(1)} + I^{(2)} \in \mathrm{C}^\bullet_{\text{Lie}}(\mathfrak{A})\otimes \mathrm{C}^\bullet_{\text{loc}}(\mathfrak{L}_{\text{matter}})[-1]$$

satisfies the Maurer-Cartan equation

$$\left(\mathrm{d}_{\text{Lie}}\mathfrak{S}_{\text{matter}} + \frac{1}{2}\{\mathfrak{S}_{\text{matter}},\mathfrak{S}_{\text{matter}}\}\right) = 0.$$

Proof. (Brian: Do this already! It's almost identical to the proof in 4d, just need to do a bit of translation.) □

Next, we move on to the 6-dimensional $\mathcal{N}=(1,0)$ super Yang-Mills. The theory has BRST fields given by a ghost c, a 6-dimensional gauge field A, and a Lie algebra valued spinor

$$\lambda_+\in\Omega^0(\mathbb{R}^6)\otimes S_+^{6d}\otimes W\otimes\mathfrak{g}$$

where W is a (complex) 2-dimensional symplectic vector space. The tensor product $S_+^{6d}\otimes W$ forms the 8-dimensional symplectic Weyl representation. The action of supersymmetry on the 6-dimensional matter theory is encoded by a linear and quadratic functional:

$$I^{(1)}_{\text{gauge}}(Q) = \int\langle A^*,\Gamma(Q,\lambda_+)\rangle + \langle\lambda^*,\not{F}_A Q\rangle$$
$$I^{(2)}_{\text{gauge}}(Q_1\otimes Q_2) = \int\left\langle \lambda^*\ ,\ \rho(\Gamma(Q_1,Q_2))\lambda^* + \frac{1}{2}\left((Q_2,\lambda^*)Q_1 + (Q_1,\lambda^*)Q_2\right)\right\rangle.$$

These functionals define an off-shell action of the $\mathcal{N}=(0,1)$ supersymmetry on the vector multiplet, which we proved above in Theorem , and state below as a proposition. Below, $\mathfrak{L}_{\text{gauge}}$ denotes the local Lie algebra describing super Yang-Mills, and $S_{\text{BV,gauge}}$ is the full BV action.

implies the following proposition for $6d$

Proposition 1.2 (special case of Theorem 1)**.** The functional

$$\mathfrak{S}_{\text{gauge}} = S_{\text{BV,gauge}} + I_{\text{Poin}} + I^{(1)}_{\text{gauge}} + I^{(2)}_{\text{gauge}} \in \mathrm{C}^\bullet_{\text{Lie}}(\mathfrak{A}) \otimes \mathrm{C}^\bullet_{\text{loc}}(\mathfrak{L}_{\text{gauge}})[-1]$$

satisfies the Maurer-Cartan equation

$$\left(\mathrm{d}_{\text{Lie}}\mathfrak{S}_{\text{gauge}} + \frac{1}{2}\{\mathfrak{S}_{\text{gauge}}, \mathfrak{S}_{\text{gauge}}\}\right) = 0.$$