A Catalogue of Twists of Supersymmetric Gauge Theories

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Abstract

We give a complete classification of supersymmetric twists of super Yang-Mills theories with matter in all dimensions. Super Yang-Mills theories can be modelled classically using the BV formalism; we construct the supersymmetry algebra action using the language of L_∞ algebras, then for each class of square-zero supercharge we give a description of the corresponding twisted theory in terms of partially holomorphic versions of Chern-Simons and BF theory.

Contents

We obtain $\mathcal{N} = 2$ super Yang-Mills theory on $\mathbb{R}^4$ by dimensionally reducing $\mathcal{N} = (1,0)$ super Yang-Mills from $\mathbb{R}^6$, or $\mathcal{N} = 1$ super Yang-Mills from $\mathbb{R}^5$, with a hypermultiplet valued in a symplectic representation U. In these terms we can describe the BRST fields.

Fields: We can describe the BRST fields of $\mathcal{N} = 2$ super Yang-Mills with matter valued in the symplectic representation U by restricting the 5d $\mathcal{N} = 2$ fields from Section **??** to representations of the group $\mathrm{O}(4)$. In addition to the ghost c, the fields we obtain are

- $\mathfrak{g}$-valued Bosons: a gauge field $A \in \Omega^1(\mathbb{R}^4; \mathfrak{g})$, and a pair of scalar fields $(\phi_1, \phi_2) \in \Omega^1(\mathbb{R}^4; \mathfrak{g} \otimes \wedge^2(W))^2$.
- U-valued bosons: a W-valued scalar field $\phi \otimes v \in \Omega^0(\mathbb{R}^4; W \otimes U)$.
- $\mathfrak{g}$-valued fermions: a W-valued Dirac spinor $\lambda = (\lambda_+ \otimes u_+, \lambda_- \otimes u_-) \in \Omega^0(\mathbb{R}^4; (S_+ \otimes W \oplus S_- \otimes W^*) \otimes \mathfrak{g})$.
- U-valued fermions: a Dirac spinor $\psi = (\psi_+, \psi_-) \in \Omega^0(\mathbb{R}^4; (S_+ \oplus S_-) \otimes U)$.

Supersymmetry action: It will be enough for our purposes to describe only the action of a chiral supercharge.

Proposition 0.1. After reduction to $\mathrm{O}(4)$, the 6d $\mathcal{N} = (1,0)$ interaction terms $I^{(1)}$ and $I^{(2)}$ become

$$I^{(1)}_{\text{gauge}}(Q) = \int \mathrm{dvol}\Bigg(- ((\Gamma(Q_+, \lambda_-)(w_+, u_-), A^*) + \langle Q_+, \lambda_+ \rangle w_+ \wedge u_+, \phi_2^*) + \\ + \frac{1}{2}((\rho(F_A)Q_+, \lambda_+^*)(u_+^*, w_+) + (\rho(\mathrm{d}_A \phi_2(w_+ \wedge u_-^*))Q_+, \lambda_-^*)\Bigg)$$

$$I^{(1)}_{\text{matter}}(Q) = \int \mathrm{dvol}((((Q_+, \psi_+), \phi^*)(v^*, w_+) + \frac{1}{2}(\rho(\mathrm{d}_A\phi)Q_+, \psi_+^*)(v, w_+))$$

$$I^{(2)}_{\text{gauge}}(Q, Q) = \int \mathrm{dvol}\,(\frac{1}{4}(Q_+, Q_+)(\lambda_-, \lambda_-)(u_-, w_+)^2 - \frac{1}{2}(Q_+, \lambda_-^*)^2(w_+ \wedge u_-^*)^2 - (Q_+, Q_+)\phi_1(w_+ \wedge w_+)c^*)$$

$$I^{(2)}_{\text{matter}}(Q, Q) = \int \frac{1}{4}\mathrm{dvol}(Q_+, Q_+)(\psi_-^*, \psi_-^*)(w_-^* \wedge w_-^*)$$

if $Q = Q_+ \otimes w_+$ is a non-zero element of $S_+ \otimes W$. (Chris: missing thing here: trivialising those wedge pairs, or equivalently restricting $\Gamma(-,-)$ from 6d to 4d.)

Twisting data: There are three classes of square-zero supercharge in the 3d $\mathcal{N} = 2$ supersymmetry algebra, distinguished by the ranks of the two summands $(Q_+, Q_-) \in S_+ \otimes W \oplus S_- \otimes W^*$.

- Rank $(1,0)$ and $(0,1)$ supercharges automatically square to zero. The corresponding twists are holomorphic, and coincide with the 4d $\mathcal{N} = 1$ twists discussed above.
- Rank $(2,0)$ and $(0,2)$ supercharges also automatically square to zero. The corresponding twists are topological (the *Donaldson twist*).
- Rank $(1,1)$ square-zero supercharges have three invariant directions.

We will identify the latter two twists as further deformations of the minimal twist, which we can describe similarly to what we saw for $\mathcal{N} = 1$.

Theorem 0.2. The minimal twist of 4d $\mathcal{N} = 2$ super Yang-Mills theory with gauge group G and symplectic matter representation U on a Calabi-Yau surface S is perturbatively equivalent to the generalized BF theory coupled to a higher holomorphic symplectic U-valued boson, with space of fields $\mathrm{Map}(S, T^*[1](U /\!/ \mathfrak{g}))$.

Proof. To show this, we start with the 5d $\mathcal{N} = 1$ minimal twist from Theorem **??** and dimensionally reduce in the de Rham direction L_{dR}. That is, we apply Proposition **??** to the theorem obtained in Theorem **??**. □

We'll first consider the deformation of the holomorphic twist to the topological twist corresponding to a rank $(2,0)$ supercharge.

Theorem 0.3. The deformation of the holomorphic twist to the Donaldson twist of 4d $\mathcal{N} = 2$ super Yang-Mills theory with gauge group G and symplectic matter representation U is perturbatively equivalent, as a 1-parameter family of theories, to the Hodge family $\mathrm{Map}(S, (U /\!/ \mathfrak{g})_{\mathrm{Hod}})$.

Proof. Suppose the family of generically rank $(2,0)$ supercharge splits as a sum of two rank 1 supercharges as $Q_{\mathrm{hol}} + tQ$, where $t \in \mathbb{C}$. It is enough to understand the deformation of the holomorphically twisted action functional, which we wrote down in the proof of Theorem **??**, by $I^{(1)}(tQ) + I^{(2)}(tQ, tQ)$ using Proposition **??**.

(Chris: that term should correspond to an isomorphism between the two copies of $\mathfrak{g} \oplus \mathfrak{g}[-1] \oplus U[-1]$ in the BV complex. Fields that survive will be $c, A_{0,1}, \phi_1, \phi_2, \psi_+$, then a single scalar from ϕ, a single $(0,1)$-form from λ_- and a single scalar from λ_+, along with anti-fields. The terms that will contribute to the deformation of the differential will be an $A^*\lambda_-$ and a $\phi_2^*\lambda_+$ term from $I^{(1)}_{\mathrm{gauge}}$, a $\phi^*\psi_+$ term from $I^{(1)}_{\mathrm{matter}}$, a $c^*\phi_1$ term from $I^{(2)}_{\mathrm{gauge}}$, and a $\psi_-^*\psi_-^*$ term from $I^{(2)}_{\mathrm{matter}}$.) □

We now address the rank $(1,1)$ twist. This is more straightforward: we can understand it by dimensional reduction from the 5d holomorphic twist, this time in a non-invariant direction.

Theorem 0.4. The rank $(1,1)$ twist of 4d $\mathcal{N} = 2$ super Yang-Mills theory with gauge group G and symplectic matter representation U on a product $C_1 \times C_2$ of two Riemann surfaces, is perturbatively equivalent to the generalized BF theory coupled to a U-valued higher holomorphic boson with space of fields $\mathrm{Map}(C_1 \times C_{2\mathrm{dR}}, U /\!/ \mathfrak{g})$. This theory is generally only $\mathbb{Z}/2\mathbb{Z}$-graded.

Proof. We start with the 5d $\mathcal{N} = 1$ holomorphic twist, as in Theorem **??**, and dimensionally reduce in one of the two non-invariant directions. That is, we apply Proposition **??**. □