

Notes on 10d Super Yang-Mills

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1 Minimal Super Yang-Mills

Our starting point will be the theory complexifying the usual 10d super Yang-Mills theory. Fix a complex reductive gauge group G with Lie algebra \mathfrak{g} . The ordinary fields of super Yang-Mills theory on \mathbb{R}^{10} consist of a boson: a connection A on the trivial G -bundle, and a fermion: a \mathfrak{g} -valued section λ of the Weyl spinor bundle associated to the spinor representation S_+^{-1} . These fields are acted upon by the group of gauge transformations – G -valued functions on \mathbb{R}^{10} .

We can model the stack of fields modulo gauge transformations infinitesimally near the point 0 by the corresponding BRST complex. This is the local super Lie algebra

$$L_{\text{BRST}} = \Omega^0(\mathbb{R}^{10}; \mathfrak{g}) \rightarrow \Omega^1(\mathbb{R}^{10}; \mathfrak{g}) \oplus \Omega^0(\mathbb{R}^{10}; \Pi S_+ \otimes \mathfrak{g})$$

with the de Rham differential, placed in cohomological degrees 0 and 1, with bracket induced from the Lie bracket on \mathfrak{g} .

The action functional in 10d super Yang-Mills is given by

$$S(A, \lambda) = \int_{\mathbb{R}^{10}} \langle F_A \wedge *F_A + (\lambda, \not{D}_A \lambda) \rangle,$$

where $\langle - \rangle$ denotes an invariant pairing on \mathfrak{g} , and $(,)$ denotes a scalar-valued pairing $S_+ \otimes S_- \rightarrow \mathbb{C}$ (there will be a unique such pairing, up to rescaling, characterized by the condition that $(\rho(v)\lambda_1, \rho(v)\lambda_2) = (\lambda_1, \lambda_2)$ for each $v \in \mathbb{C}^{10}$, where ρ denotes Clifford multiplication).

We can re-encode this data in terms of the classical BV complex (Phil and I wrote this down in [EY18, Section 3.1]). This is the L_∞ -algebra whose underlying cochain complex takes the form

$$\Omega^0(\mathbb{R}^{10}; \mathfrak{g}) \xrightarrow{d} \Omega^1(\mathbb{R}^{10}; \mathfrak{g}) \xrightarrow{d*d} \Omega^9(\mathbb{R}^{10}; \mathfrak{g}) \xrightarrow{d} \Omega^{10}(\mathbb{R}^{10}; \mathfrak{g})$$

$$\Omega^0(\mathbb{R}^{10}; \Pi S_- \otimes \mathfrak{g}) \xrightarrow{*d} \Omega^{10}(\mathbb{R}^{10}; \Pi S_- \otimes \mathfrak{g}),$$

with degree -3 invariant pairing induced by the invariant pairing on \mathfrak{g} and the pairing $(,)$ between S_+ and S_- , and

¹If we didn't complexify we would instead consider $G_{\mathbb{R}}$ a compact connected Lie group, and a section of the Majorana-Weyl spinor bundle, which necessitates working in Lorentzian signature. I think for our purposes it's interesting enough to just consider the complexified theory and avoid signature issues. The complexified theory twists to holomorphic Chern-Simons theory with complex gauge group.

with degree 2 and 3 brackets given by the action of $\Omega^0(\mathbb{R}^{10}; \mathfrak{g})$ on everything along with

$$\begin{aligned}\ell_2^{\text{Bos}}: \Omega^1(\mathbb{R}^{10}; \mathfrak{g}) \otimes \Omega^1(\mathbb{R}^{10}; \mathfrak{g}) &\rightarrow \Omega^9(\mathbb{R}^{10}; \mathfrak{g}) \\ (A \otimes B) &\mapsto [A \wedge *dB] + [*dA \wedge B] + d * [A \wedge B] \\ \ell_2^{\text{Fer}}: \Omega^1(\mathbb{R}^{10}; \mathfrak{g}) \otimes \Omega^0(\mathbb{R}^{10}; S_+ \otimes \mathfrak{g}) &\rightarrow \Omega^{10}(\mathbb{R}^{10}; S_- \otimes \mathfrak{g}) \\ (A \otimes \lambda) &\mapsto *A\lambda\end{aligned}$$

in degree 2, and the map

$$\begin{aligned}\ell_3: \Omega^1(\mathbb{R}^{10}; \mathfrak{g}) \otimes \Omega^1(\mathbb{R}^{10}; \mathfrak{g}) \otimes \Omega^1(\mathbb{R}^{10}; \mathfrak{g}) &\rightarrow \Omega^9(\mathbb{R}^{10}; \mathfrak{g}) \\ (A \otimes B \otimes C) &\mapsto [A \wedge *[B \wedge C]] + [B \wedge *[C \wedge A]] + [C \wedge *[A \wedge B]]\end{aligned}$$

in degree 3.

1.1 On-Shell Supersymmetry Action

We can define an action of the 10d $\mathcal{N} = 1$ supersymmetry algebra on the complex of BRST fields in this minimal super Yang-Mills theory. The Poincaré action is clear, and the action of the supersymmetry Q is generated – in the usual Physics notation – by the transformation [BSS77]

$$\begin{aligned}\delta_Q A &= \Gamma(Q, \lambda) \\ \delta_Q \lambda &= \not{F}_A Q,\end{aligned}$$

where the notation \not{F}_A stands for the iterated Clifford multiplication $\not{F}_A = F_{ij}\gamma^i\gamma^j$. To check that this defines a supersymmetry action we need to check it's compatible with the brackets in the supersymmetry algebra. So, we calculate

$$\begin{aligned}[\delta_{Q_1}, \delta_{Q_2}]A &= (\Gamma(Q_2, \not{F}_A Q_1) - \Gamma(Q_1, \not{F}_A Q_2)) \\ &= F_{ij}(Q_2\gamma^k\gamma^i\gamma^j Q_1 - Q_1\gamma^k\gamma^i\gamma^j Q_2) \\ &= F_{ij}(Q_2\gamma^k\gamma^i\gamma^j Q_1 - Q_2\gamma^j\gamma^i\gamma^k Q_1) \\ &= F_{ij}(Q_2\gamma^k\gamma^j\gamma^i Q_1 - Q_2\gamma^j\gamma^k\gamma^i Q_1) \\ &= F_{ij}\delta^{kj}(Q_2\gamma^i Q_1) \\ &= \delta_{[Q_1, Q_2]}A,\end{aligned}$$

where on the third line we used the fact that the pairing $\Gamma(-, -)$ is symmetric – i.e. that $\lambda_1\gamma^i\lambda_2 = \lambda_2\gamma^i\lambda_1$ – three times, and on the fourth and fifth lines we used the Clifford relations. Similarly we can calculate

$$\begin{aligned}[\delta_{Q_1}, \delta_{Q_2}]\lambda &= (\not{F}_{\Gamma(Q_2, \lambda)} Q_1 - \not{F}_{\Gamma(Q_1, \lambda)} Q_2) \\ &= (Q_2\gamma_j\partial_i\lambda + [Q_2\gamma_i\lambda, Q_2\gamma_j\lambda])(\gamma^i\gamma^j Q_1) - (1 \leftrightarrow 2) \\ &= \end{aligned}$$

Remark 1.1. I'm part way through trying to do these calculations myself, but for reference I found these calculations discussed in the master's thesis [Gui16] we've discussed before. For instance for the commutator of two supersymmetries acting on a spinor see equation 2.1.13.

2 Baulieu's 10 Model

Baulieu [Bau11] considers an extension of 10d Super-Yang Mills including an auxiliary \mathfrak{g} -valued scalar field h . The inclusion of this field breaks the manifest $\text{SO}(10)$ symmetry to the subgroup $\text{SU}(5)$, corresponding to a choice of

complex structure on \mathbb{R}^{10} . Let's describe this classical theory in an explicitly $SU(5)$ -invariant way. With respect to this choice of complex structure \mathbb{C}^5 becomes Calabi-Yau: we'll denote the holomorphic top-form by Ω and the map $\Omega^{p,q}(\mathbb{C}^5) \rightarrow \Omega^{p-1,q-1}(\mathbb{C}^5)$ induced from the Kähler structure by J .

The ordinary fields A and λ of super Yang-Mills will decompose according to the decomposition of the vector and Weyl spinor representations of $SO(10)$ into irreducible $SU(5)$ -representations. So, explicitly A splits up into fields

$$A_{1,0} + A_{0,1} \in \Omega^{1,0}(\mathbb{C}^5; \mathfrak{g}) \oplus \Omega^{0,1}(\mathbb{C}^5; \mathfrak{g}),$$

and λ splits up into fields

$$\chi + \psi + B \in \Pi(\Omega^0(\mathbb{C}^5; \mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C}^5; \mathfrak{g}) \oplus \Omega^{0,2}(\mathbb{C}^5; \mathfrak{g})).$$

In terms of these fields, and including the auxiliary field h , the action functional becomes

$$S(A_{1,0}, A_{0,1}, \chi, \psi, B, h) = \int \langle (B \wedge \bar{\partial}_{A_{0,1}} B) + J^2(F_{2,0} \wedge F_{0,2})\Omega + \|h\|^2\Omega + hJ(F_{1,1})\Omega + J(\chi \wedge (\bar{\partial}_{A_{0,1}} \psi))\Omega + J^2(B \wedge (\partial_{A_{1,0}} \psi))\Omega \rangle.$$

One derives this action functional by decomposing the 10d super Yang-Mills action functional into $SU(5)$ irreducible component fields, then introducing a Lagrange multiplier h to eliminate the $F_{1,1}^2$ term.

2.1 Off-Shell Action of a Scalar Supersymmetry

3 Homotopy Data

3.1 Homotopy Transfer of the Scalar Supersymmetry

References

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