Notes on 10d Super Yang-Mills

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1 Minimal Super Yang-Mills

Our starting point will be the theory complexifying the usual 10d super Yang-Mills theory. Fix a complex reductive gauge group G with Lie algebra $\mathfrak{g}$. The ordinary fields of super Yang-Mills theory on $\mathbb{R}^{10}$ consist of a boson: a connection A on the trivial G-bundle, and a fermion: a $\mathfrak{g}$-valued section λ of the Weyl spinor bundle associated to the spinor representation S_+ [1]. These fields are acted upon by the group of gauge transformations – G-valued functions on $\mathbb{R}^{10}$.

We can model the stack of fields modulo gauge transformations infinitesimally near the point 0 by the corresponding BRST complex. This is the local super Lie algebra

$$L_{\mathrm{BRST}} = \Omega^0(\mathbb{R}^{10};\mathfrak{g}) \to \Omega^1(\mathbb{R}^{10};\mathfrak{g}) \oplus \Omega^0(\mathbb{R}^{10};\Pi S_+ \otimes \mathfrak{g})$$

with the de Rham differential, placed in cohomological degrees 0 and 1, with bracket induced from the Lie bracket on $\mathfrak{g}$.

The action functional in 10d super Yang-Mills is given by

$$S(A,\lambda) = \int_{\mathbb{R}^{10}} \langle F_A \wedge *F_A + (\lambda, \not{D}_A \lambda)\rangle,$$

where $\langle - \rangle$ denotes an invariant pairing on $\mathfrak{g}$, and $(,)$ denotes a scalar-valued pairing $S_+ \otimes S_- \to \mathbb{C}$ (there will be a unique such pairing, up to rescaling, characterized by the condition that $(\rho(v)\lambda_1, \rho(v)\lambda_2) = (\lambda_1, \lambda_2)$ for each $v \in \mathbb{C}^{10}$, where ρ denotes Clifford multiplication).

We can re-encode this data in terms of the classical BV complex (Phil and I wrote this down in [EY18, Section 3.1]). This is the L_∞-algebra whose underlying cochain complex takes the form

$$\Omega^0(\mathbb{R}^{10};\mathfrak{g}) \xrightarrow{\quad \mathrm{d} \quad} \Omega^1(\mathbb{R}^{10};\mathfrak{g}) \xrightarrow{\quad \mathrm{d}{*}\mathrm{d} \quad} \Omega^9(\mathbb{R}^{10};\mathfrak{g}) \xrightarrow{\quad \mathrm{d} \quad} \Omega^{10}(\mathbb{R}^{10};\mathfrak{g})$$

$$\Omega^0(\mathbb{R}^{10};\Pi S_- \otimes \mathfrak{g}) \xrightarrow{*\not{d}} \Omega^{10}(\mathbb{R}^{10};\Pi S_- \otimes \mathfrak{g}),$$

with degree -3 invariant pairing induced by the invariant pairing on $\mathfrak{g}$ and the pairing $(,)$ between S_+ and S_-, and

[1] If we didn't complexify we would instead consider $G_{\mathbb{R}}$ a compact connected Lie group, and a section of the Majorana-Weyl spinor bundle, which necessitates working in Lorentzian signature. I think for our purposes it's interesting enough to just consider the complexified theory and avoid signature issues. The complexified theory twists to holomorphic Chern-Simons theory with complex gauge group.

with degree 2 and 3 brackets given by the action of $\Omega^0(\mathbb{R}^{10};\mathfrak{g})$ on everything along with

$$\begin{aligned}\ell_2^{\mathrm{Bos}}\colon \Omega^1(\mathbb{R}^{10};\mathfrak{g})\otimes\Omega^1(\mathbb{R}^{10};\mathfrak{g}) &\to \Omega^9(\mathbb{R}^{10};\mathfrak{g})\\ (A\otimes B) &\mapsto [A\wedge *\mathrm{d}B]+[*\mathrm{d}A\wedge B]+\mathrm{d}*[A\wedge B]\\ \ell_2^{\mathrm{Fer}}\colon \Omega^1(\mathbb{R}^{10};\mathfrak{g})\otimes\Omega^0(\mathbb{R}^{10};S_+\otimes\mathfrak{g}) &\to \Omega^{10}(\mathbb{R}^{10};S_-\otimes\mathfrak{g})\\ (A\otimes\lambda) &\mapsto *\not{A}\lambda\end{aligned}$$

in degree 2, and the map

$$\begin{aligned}\ell_3\colon \Omega^1(\mathbb{R}^{10};\mathfrak{g})\otimes\Omega^1(\mathbb{R}^{10};\mathfrak{g})\otimes\Omega^1(\mathbb{R}^{10};\mathfrak{g}) &\to \Omega^9(\mathbb{R}^{10};\mathfrak{g})\\ (A\otimes B\otimes C) &\mapsto [A\wedge *[B\wedge C]]+[B\wedge *[C\wedge A]]+[C\wedge *[A\wedge B]]\end{aligned}$$

in degree 3.

1.1 On-Shell Supersymmetry Action

We can define an action of the 10d $\mathcal{N}=1$ supersymmetry algebra on the complex of BRST fields in this minimal super Yang-Mills theory. The Poincaré action is clear, and the action of the supersymmetry Q is generated – in the usual Physics notation – by the transformation [BSS77]

$$\begin{aligned}\delta_Q A &= \Gamma(Q,\lambda)\\ \delta_Q\lambda &= \not{F}_A Q,\end{aligned}$$

where the notation $\not{F}_A$ stands for the iterated Clifford multiplication $\not{F}_A = F_{ij}\gamma^i\gamma^j$. To check that this defines a supersymmetry action we need to check it's compatible with the brackets in the supersymmetry algebra. So, we calculate

$$\begin{aligned}[\delta_{Q_1},\delta_{Q_2}]A &= (\Gamma(Q_2,\not{F}_A Q_1)-\Gamma(Q_1,\not{F}_A Q_2))\\ &= F_{ij}(Q_2\gamma^k\gamma^i\gamma^j Q_1 - Q_1\gamma^k\gamma^i\gamma^j Q_2)\\ &= F_{ij}(Q_2\gamma^k\gamma^i\gamma^j Q_1 - Q_2\gamma^j\gamma^i\gamma^k Q_1)\\ &= F_{ij}(Q_2\gamma^k\gamma^j\gamma^i Q_1 - Q_2\gamma^j\gamma^k\gamma^i Q_1)\\ &= F_{ij}\delta^{kj}(Q_2\gamma^i Q_1)\\ &= \delta_{[Q_1,Q_2]}A,\end{aligned}$$

where on the third line we used the fact that the pairing $\Gamma(-,-)$ is symmetric, i.e. $\lambda_1\gamma^i\lambda_2=\lambda_2\gamma^i\lambda_1$, three times, and on the fourth and fifth lines we used the Clifford relations. Similarly we can calculate

$$\begin{aligned}[\delta_{Q_1},\delta_{Q_2}]\lambda &= (\not{F}_{\Gamma(Q_2,\lambda)}Q_1 - \not{F}_{\Gamma(Q_1,\lambda)}Q_2)\\ &=\end{aligned}$$

2 Baulieu's 10 Model

2.1 Off-Shell Action of a Scalar Supersymmetry

3 Homotopy Data

3.1 Homotopy Transfer of the Scalar Supersymmetry

References

[BSS77] L. Brink, J. Schwarz, and J. Scherk. "Supersymmetric Yang-Mills theories". *Nuclear Phys. B* 121.1 (1977), pp. 77–92. URL: `https://doi.org/10.1016/0550-3213(77)90328-5`.

[EY18] C. Elliott and P. Yoo. "Geometric Langlands Twists of $N = 4$ Gauge Theory from Derived Algebraic Geometry". *Advances in Theoretical and Mathematical Physics* 22.3 (2018). arXiv: `1507.03048 [math-ph]`.