Notes on 10d Super Yang-Mills

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1 Minimal Super Yang-Mills

Our starting point will be the theory complexifying the usual 10d super Yang-Mills theory. Fix a complex reductive gauge group G with Lie algebra $\mathfrak{g}$. The ordinary fields of super Yang-Mills theory on $\mathbb{R}^{10}$ consist of a boson: a connection A on the trivial G-bundle, and a fermion: a $\mathfrak{g}$-valued section λ of the Weyl spinor bundle associated to the spinor representation S_+ [1]. These fields are acted upon by the group of gauge transformations – G-valued functions on $\mathbb{R}^{10}$.

We can model the stack of fields modulo gauge transformations infinitesimally near the point 0 by the corresponding BRST complex. This is the local super Lie algebra

$$L_{\mathrm{BRST}} = \Omega^0(\mathbb{R}^{10};\mathfrak{g}) \to \Omega^1(\mathbb{R}^{10};\mathfrak{g}) \oplus \Omega^0(\mathbb{R}^{10};\Pi S_+ \otimes \mathfrak{g})$$

with the de Rham differential, placed in cohomological degrees 0 and 1, with bracket induced from the Lie bracket on $\mathfrak{g}$.

The action functional in 10d super Yang-Mills is given by

$$S(A,\lambda) = \int_{\mathbb{R}^{10}} \langle F_A \wedge *F_A + (\lambda, \not{D}_A \lambda)\rangle,$$

where $\langle - \rangle$ denotes an invariant pairing on $\mathfrak{g}$, and $(,)$ denotes a scalar-valued pairing $S_+ \otimes S_- \to \mathbb{C}$ (there will be a unique such pairing, up to rescaling, characterized by the condition that $(\rho(v)\lambda_1, \rho(v)\lambda_2) = (\lambda_1, \lambda_2)$ for each $v \in \mathbb{C}^{10}$, where ρ denotes Clifford multiplication).

We can re-encode this data in terms of the classical BV complex (Phil and I wrote this down in [EY18, Section 3.1]). This is the L_∞-algebra whose underlying cochain complex takes the form

$$\Omega^0(\mathbb{R}^{10};\mathfrak{g}) \xrightarrow{\mathrm{d}} \Omega^1(\mathbb{R}^{10};\mathfrak{g}) \xrightarrow{\mathrm{d}*\mathrm{d}} \Omega^9(\mathbb{R}^{10};\mathfrak{g}) \xrightarrow{\mathrm{d}} \Omega^{10}(\mathbb{R}^{10};\mathfrak{g})$$

$$\Omega^0(\mathbb{R}^{10};\Pi S_- \otimes \mathfrak{g}) \xrightarrow{*\not{d}} \Omega^{10}(\mathbb{R}^{10};\Pi S_- \otimes \mathfrak{g}),$$

with degree -3 invariant pairing induced by the invariant pairing on $\mathfrak{g}$ and the pairing $(,)$ between S_+ and S_-, and

[1] If we didn't complexify we would instead consider $G_{\mathbb{R}}$ a compact connected Lie group, and a section of the Majorana-Weyl spinor bundle, which necessitates working in Lorentzian signature. I think for our purposes it's interesting enough to just consider the complexified theory and avoid signature issues. The complexified theory twists to holomorphic Chern-Simons theory with complex gauge group.

with degree 2 and 3 brackets given by the action of $\Omega^0(\mathbb{R}^{10};\mathfrak{g})$ on everything along with

$$\begin{aligned}\ell_2^{\mathrm{Bos}}\colon \Omega^1(\mathbb{R}^{10};\mathfrak{g})\otimes\Omega^1(\mathbb{R}^{10};\mathfrak{g}) &\to \Omega^9(\mathbb{R}^{10};\mathfrak{g})\\ (A\otimes B) &\mapsto [A\wedge *\mathrm{d}B]+[*\mathrm{d}A\wedge B]+\mathrm{d}*[A\wedge B]\\ \ell_2^{\mathrm{Fer}}\colon \Omega^1(\mathbb{R}^{10};\mathfrak{g})\otimes\Omega^0(\mathbb{R}^{10};S_+\otimes\mathfrak{g}) &\to \Omega^{10}(\mathbb{R}^{10};S_-\otimes\mathfrak{g})\\ (A\otimes\lambda) &\mapsto *\not{A}\lambda\end{aligned}$$

in degree 2, and the map

$$\begin{aligned}\ell_3\colon \Omega^1(\mathbb{R}^{10};\mathfrak{g})\otimes\Omega^1(\mathbb{R}^{10};\mathfrak{g})\otimes\Omega^1(\mathbb{R}^{10};\mathfrak{g}) &\to \Omega^9(\mathbb{R}^{10};\mathfrak{g})\\ (A\otimes B\otimes C) &\mapsto [A\wedge *[B\wedge C]]+[B\wedge *[C\wedge A]]+[C\wedge *[A\wedge B]]\end{aligned}$$

in degree 3.

1.1 On-Shell Supersymmetry Action

We can define an action of the 10d $\mathcal{N}=1$ supersymmetry algebra on the complex of BRST fields in this minimal super Yang-Mills theory. The Poincaré action is clear, and the action of the supersymmetry Q is generated – in the usual Physics notation – by the transformation [BSS77]

$$\begin{aligned}\delta_Q A &= \Gamma(Q,\lambda)\\ \delta_Q\lambda &= \not{F}_A Q,\end{aligned}$$

where the notation $\not{F}_A$ stands for the iterated Clifford multiplication $\not{F}_A = F_{ij}\gamma^i\gamma^j$. To check that this defines a supersymmetry action we need to check it's compatible with the brackets in the supersymmetry algebra. So, we calculate

$$\begin{aligned}[][\delta_{Q_1},\delta_{Q_2}]A &= (\Gamma(Q_2,\not{F}_AQ_1)-\Gamma(Q_1,\not{F}_AQ_2))\\ &= F_{ij}(Q_2\gamma^k\gamma^i\gamma^jQ_1 - Q_1\gamma^k\gamma^i\gamma^jQ_2)\\ &= F_{ij}(Q_2\gamma^k\gamma^i\gamma^jQ_1 - Q_2\gamma^j\gamma^i\gamma^kQ_1)\\ &= F_{ij}(Q_2\gamma^k\gamma^j\gamma^iQ_1 - Q_2\gamma^j\gamma^k\gamma^iQ_1)\\ &= F_{ij}\delta^{kj}(Q_2\gamma^iQ_1)\\ &= \delta_{[Q_1,Q_2]}A,\end{aligned}$$

where on the third line we used the fact that the pairing $\Gamma(-,-)$ is symmetric – i.e. that $\lambda_1\gamma^i\lambda_2=\lambda_2\gamma^i\lambda_1$ – three times, and on the fourth and fifth lines we used the Clifford relations. Similarly we can calculate

$$\begin{aligned}[][\delta_{Q_1},\delta_{Q_2}]\lambda &= (\not{F}_{\Gamma(Q_2,\lambda)}Q_1 - \not{F}_{\Gamma(Q_1,\lambda)}Q_2)\\ &= (Q_2\gamma_j\partial_i\lambda + [Q_2\gamma_i\lambda, Q_2\gamma_j\lambda])(\gamma^i\gamma^jQ_1) - (1\leftrightarrow 2)\\ &=\end{aligned}$$

Remark 1.1. I'm part way through trying to do these calculations myself, but for reference I found these calculations discussed in the master's thesis [Gui16] we've discussed before. For instance for the commutator of two supersymmetries acting on a spinor see equation 2.1.13.

2 Baulieu's 10 Model

Baulieu [Bau11] considers an extension of 10d Super-Yang Mills including an auxiliary $\mathfrak{g}$-valued scalar field h. The inclusion of this field breaks the manifest SO(10) symmetry to the subgroup SU(5), corresponding to a choice of complex structure on $\mathbb{R}^{10}$. Let's describe this classical theory in an explicitly SU(5)-invariant way. With respect to this choice of complex structure $\mathbb{C}^5$ becomes Calabi-Yau: we'll denote the holomorphic top-form by Ω and the map $\Omega^{p,q}(\mathbb{C}^5) \to \Omega^{p-1,q-1}(\mathbb{C}^5)$ induced from the Kähler structure by J.

The ordinary fields A and λ of super Yang-Mills will decompose according to the decomposition of the vector and Weyl spinor representations of SO(10) into irreducible SU(5)-representations. So, explicitly A splits up into fields

$$A_{1,0} + A_{0,1} \in \Omega^{1,0}(\mathbb{C}^5; \mathfrak{g}) \oplus \Omega^{0,1}(\mathbb{C}^5; \mathfrak{g}),$$

and λ splits up into fields

$$\chi + \psi + B \in \Pi(\Omega^0(\mathbb{C}^5; \mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C}^5; \mathfrak{g}) \oplus \Omega^{0,2}(\mathbb{C}^5; \mathfrak{g})).$$

In terms of these fields, and including the auxiliary field h, the action functional becomes

$$S(A_{1,0}, A_{0,1}, \chi, \psi, B, h) = \int \langle (B \wedge \overline{\partial}_{A_{0,1}} B) + J^2(F_{2,0} \wedge F_{0,2})\Omega + \|h\|^2 \Omega + hJ(F_{1,1})\Omega + J(\chi \wedge (\overline{\partial}_{A_{0,1}} \psi))\Omega + J^2(B \wedge (\partial_{A_{1,0}} \psi))\Omega \rangle.$$

One derives this action functional by decomposing the 10d super Yang-Mills action functional into SU(5) irreducible component fields, then introducing a Lagrange multiplier h to eliminate the $F_{1,1}^2$ term.

2.1 Off-Shell Action of a Scalar Supersymmetry

BW: I realized I'm still confused with the role of Baulieu's h-field. I really suspect it's different than the usual auxiliary field story. It seems like Baulieu doesn't even write down the apparent off-shell SUSY module structure.
BW: I think without ever introducing an auxiliary field we can obtain the following module structure.

Let $\mathcal{O}_{loc}^{BRST}$ and $\mathcal{O}_{loc}^{BV}$ be the local functionals for the BRST and BV fields, respectively. The BRST operator endows $\mathcal{O}_{loc}^{BRST}$ with the structure of a cochain complex, and the BV operator together with the BV bracket endow $\mathcal{O}_{loc}^{BV}[-1]$ with the structure of a dg Lie algebra. Note that there is a map of dg Lie algebras

$$\mathcal{O}_{loc}^{BV}[-1] \to \mathrm{End}(\mathcal{O}_{loc}^{BRST})$$

sending a functional I to the endomorphism $\{I, -\}$.

The cursory definition of the linear map

$$\mathfrak{g}_{\mathcal{N}=1} \to \mathrm{End}(\mathcal{O}_{loc}^{BRST})$$

is not a map of Lie algebras. It fails to preserve the Lie bracket by a term proportional to the equations of motion in the field λ. I claim that Baulieu's introduction of the h-field does not resolve this issue. (We can see this by counting fermion number. The space where the failure for this to be a Lie map is S_+, yet we are only adding a scalar h in the auxiliary, so there's no way.

Instead, what one should try is the following. (This doesn't ever use a holomorphic language, so maybe I'm making a silly mistake.) There is an obvious lift of the linear map $\mathfrak{g}_{\mathcal{N}=1} \to \mathrm{End}(\mathcal{O}_{loc}^{BRST})$ to the BV complex

$$\mathfrak{g}_{\mathcal{N}=1} \to \mathcal{O}_{loc}^{BV}[-1].$$

This is still, of course, not a map of dg Lie algebras. However, I claim that there is an L_∞ correction to this map.

Fix a basis $\{Q_\alpha\}$ of S_+ and write a general element of the form $Q_\alpha = \epsilon^\alpha Q_\alpha$. The putative, linear action, of the element Q on the BV complex is through functionals of the form

$$\int \epsilon \lambda^* \not{F}_A$$

where λ^* denotes the anti-field to λ. I claim that we can correct the action by adding a quadratic term to the action that sends a pair $Q_1 \otimes Q_2$ to the functional

$$\int \epsilon_1 \lambda^* \epsilon_2 \lambda^*.$$

In terms of an L_∞ action, I'm saying that there is the usual linear map

$$I^{(1)} : \mathfrak{g}_{\mathcal{N}=1} \to \mathcal{O}_{loc}^{BV}$$

plus an L_∞ correction of the form:

$$I^{(2)} : \mathfrak{g}_{\mathcal{N}=1}^{\otimes 2} \to \mathcal{O}_{loc}^{BV}[-2].$$

You may ask why we have to stop there, but I don't have a great conceptual answer besides the computational fact that it seems like it works out. I claim that $I^{(1)} + I^{(2)}$ determines an off-shell action of $\mathfrak{g}_{\mathcal{N}=1}$ on the BV complex.

At the level of generators, we are proposing a new transformation law of the form

$$\begin{aligned} \delta A &= \mathrm{d}c + \Gamma(\epsilon, \lambda) \\ \delta\lambda &= [c, \lambda] + \epsilon \not{F}_A \pm \#\epsilon(\epsilon\lambda^*) \\ \delta\lambda^* &= \not{\partial}\lambda + [c, \lambda^*] + \epsilon \not{A}^*. \end{aligned}$$

I've included terms coming from the BV operator (which have no ϵ dependence). The only new term is the last one in the transformation law for λ, $\epsilon(\epsilon\lambda^*)$.

(Chris: I'm a little confused by a few things: can I ask for clarification?

1. Firstly, while I understand the map $(I \mapsto \{I, -\})\colon \mathcal{O}_{loc}^{BV}[-1] \to \mathrm{End}(\mathcal{O}_{loc}^{BV})$, I don't understand how it lifts to $\mathrm{End}(\mathcal{O}_{loc}^{BRST})$. For instance (maybe a little heuristically, but I hope what I'm trying to say makes sense), let α be a scalar field and let α^* be its antifield. Let $I = (\alpha^*)^2$ in $\mathcal{O}_{loc}^{BV}$. Then $\{(\alpha^*)^2, \alpha\} = 2\alpha^*$ is a non-trivial element of $\mathcal{O}_{loc}^{BV}$ which is not in the image of $\mathcal{O}_{loc}^{BRST}$, which says that my naïve guess for how to define your map doesn't work, so I must be supposed to do something more clever?

2. I wanted to clarify something about your "counting fermion number" argument. I thought that this argument only applied to the action of a whole supersymmetry algebra, not a single square zero supercharge. That is, if we're trying to construct off-shell BRST supersymmetry then the space of BRST fields should yield a representation of the supersymmetry algebra. Representations $V_0 \oplus V_1$ of the supersymmetry algebra have to have the same number of bosonic and fermionic degrees of freedom – i.e. $\dim V_0 = \dim V_1$, because we can choose a supertranslation Q which squares to a translation. So if α denotes the supersymmetry algebra action, the composite of $\alpha(Q)|_{V_0}\colon V_0 \to V_1$ and $\alpha(Q)|_{V_1}\colon V_1 \to V_0$ is equal to the action of a translation, which is an isomorphism, and therefore $\alpha(Q)$ must also be an isomorphism, so $\dim V_0 = \dim V_1$. However this argument doesn't apply anymore if you just want an action of a single square-zero odd symmetry. Maybe you have something else in mind though?

3. Could you include some of the calculation that the map you define at the end is an L_∞ map? This seems like it might be exactly what we're looking for.

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3 Homotopy Data

3.1 Homotopy Transfer of the Scalar Supersymmetry

References

[Bau11] L. Baulieu. "SU(5)-invariant decomposition of ten-dimensional Yang–Mills supersymmetry". *Physics Letters B* 698.1 (2011), pp. 63–67. arXiv: `1009.3893 [hep-th]`.

[BSS77] L. Brink, J. Schwarz, and J. Scherk. “Supersymmetric Yang-Mills theories”. *Nuclear Phys. B* 121.1 (1977), pp. 77–92. URL: `https://doi.org/10.1016/0550-3213(77)90328-5`.

[EY18] C. Elliott and P. Yoo. “Geometric Langlands Twists of $N = 4$ Gauge Theory from Derived Algebraic Geometry”. *Advances in Theoretical and Mathematical Physics* 22.3 (2018). arXiv: `1507.03048 [math-ph]`.

[Gui16] M. Guillen. “$D = 10$ Super Yang-Mills, $D = 11$ Supergravity and the Pure Spinor Superfield Formalism”. MA thesis. Instituto de Física Teórica Universidade Estadual Paulista, 2016.