

CLASSIFICATION OF MASSLESS SUPERMULTIPLETS

1.

Let V be a complex vector space with a chosen spin structure. Let Σ be a spinorial representation of $\text{Spin}(V)$, $\Gamma: \text{Sym}^2(\Sigma) \rightarrow V$ the associated vector-valued pairing and T_Σ the supertranslation group with Lie algebra $\Pi\Sigma \oplus V$. The superPoincaré group is $\text{Spin}(V) \ltimes T_\Sigma$.

Recall that massless representations of the Poincaré group are parametrized in the following way. Split $V = V_0 \oplus \mathbf{C}e_+ \oplus \mathbf{C}e_-$, where e_+ and e_- are null vectors with $(e_+, e_-) = 1$. Then massless representations correspond to finite-dimensional representations of $\text{Spin}(V_0)$.

Remark 1.1. We ignore the continuous-spin representations. Alternatively, we only consider finite-dimensional representations of the little group.

We label irreducible representations of $\text{Spin}(2)$ by **helicity** which is a half-integer number.

Definition 1.2. Suppose $\dim(V_0) \geq 2$. A representation ρ of $\text{Spin}(V_0)$ is **admissible** if for some (equivalently, any) nondegenerate subspace $\mathbf{C}^2 \subset V_0$ the helicities appearing in ρ are at most 1 by absolute value.

In a similar way, massless representations of the superPoincaré group are parametrized as follows (see [Freed, Lecture 6] for details). Consider a quadratic form on Σ defined by $q(Q, Q) = (e_+, \Gamma(Q, Q))$ for $Q \in \Sigma$. Let Σ_0 be the radical of q and $\bar{\Sigma} = \Sigma/\Sigma_0$. Let $\text{Cl}(\bar{\Sigma}, q)$ be the associated Clifford algebra. Then massless representations correspond to finite-dimensional representations of $\text{Cl}(\bar{\Sigma}, q)$ with a compatible action of $\text{Spin}(V_0)$.

1.1. $d = 3$. We have $\text{Spin}(1) \cong \mathbf{Z}/2$. We denote its irreducible representations by \mathbf{C} and S which correspond to the massless scalar and massless spinor.

The spinorial representation is

$$\Sigma = (S \oplus S) \otimes W,$$

where W carries a nondegenerate symmetric bilinear form. We have

$$\bar{\Sigma} = S \otimes W.$$

There is a unique supermultiplet which contains $\dim(W)$ scalars and $\dim(W)$ spinors.

1.2. $d = 4$. The admissible irreducible representations of $\text{Spin}(2)$ are the trivial one-dimensional representation \mathbf{C} , the semi-spin representations S_\pm of helicity $\pm\frac{1}{2}$ and the half-vector representations V_\pm of helicity ± 1 .

The spinorial representation is

$$\Sigma = (S_+ \oplus S_-) \otimes W \oplus (S_+ \oplus S_-) \otimes W^*$$

for a complex vector space W .

The radical is $\Sigma_0 = S_+ \otimes W \oplus S_- \otimes W^*$, so

$$\bar{\Sigma} = S_- \otimes W \oplus S_+ \otimes W^*$$

with q the obvious quadratic form. In particular, it contains a Lagrangian subspace $S_+ \otimes W^*$, so irreducible massless representations of the superPoincaré group are parametrized by irreducible $\text{Spin}(2)$ -representations \mathbf{C}_h labeled by a half-integer h , so that the corresponding representation of the little group is

$$\mathbf{C}_h \otimes \wedge^\bullet(S_+ \otimes W^*).$$

This representation is admissible if $-1 \leq h \leq 1$ and $h + \mathcal{N}/2 \leq 1$. In particular, we must have $\mathcal{N} \leq 4$.

In this dimension we have to make sure our representations are CPT invariant. We will only deal with PT invariance. This corresponds to extending $\text{Spin}(2)$ to $\mathbf{Z}/2 \ltimes \text{Spin}(2)$, where $\mathbf{Z}/2$ acts on $\text{Spin}(2)$ by inversion.

The list of admissible supermultiplets:

- $\mathcal{N} = 1$. Chiral multiplet $\mathbf{C}^{\oplus 2} \oplus S_+ \oplus S_-$ and the $\mathcal{N} = 1$ vector multiplet $S_+ \oplus S_- \oplus V_+ \oplus V_-$.
- $\mathcal{N} = 2$. Half-hypermultiplet (aka $\mathcal{N} = 1$ chiral multiplet) and an $\mathcal{N} = 2$ vector multiplet $\mathbf{C}^{\oplus 2} \oplus (S_+ \oplus S_-)^{\oplus 2} \oplus V_+ \oplus V_-$.
- $\mathcal{N} = 4, \mathcal{N} = 3$. $\mathcal{N} = 4$ vector multiplet $\mathbf{C}^{\oplus 6} \oplus (S_+ \oplus S_-)^{\oplus 4} \oplus V_+ \oplus V_-$.

1.3. $d = 5$. The admissible irreducible representations of $\text{Spin}(3)$ are the trivial one-dimensional representation \mathbf{C} , the spin representation S and the vector representation V_3 .

The spinorial representation is

$$\Sigma = (S \oplus S) \otimes W,$$

where W is a complex symplectic vector space. The radical is

$$\Sigma_0 = S \otimes W$$

(say, the first summand), so

$$\bar{\Sigma} = S \otimes W.$$

The quadratic form is given by the product of the symplectic structure on W and the symplectic structure on S . Choose a Lagrangian subspace $L \subset W$. Then the massless irreducible representations of the superPoincaré group are parametrized by irreducible representations M of $\text{Spin}(3)$ so that the corresponding representation of the little group is

$$M \otimes \wedge^\bullet(S \otimes L).$$

The list of admissible supermultiplets:

- $\mathcal{N} = 1$. The chiral multiplet (aka half-hypermultiplet) $\mathbf{C}^{\oplus 2} \oplus S$ and the $\mathcal{N} = 1$ vector multiplet $\mathbf{C} \oplus S^{\oplus 2} \oplus V_3$.
- $\mathcal{N} = 2$. $\mathcal{N} = 2$ vector multiplet $\mathbf{C}^{\oplus 5} \oplus S^{\oplus 4} \oplus V_3$.

1.4. $d = 6$. The admissible irreducible representations of $\text{Spin}(4)$ are the trivial one-dimensional representation \mathbf{C} , the semi-spin representations S_\pm , the vector representation V_4 and the (anti) self-dual form representations $\text{Sym}^2(S_\pm)$. Note that we have a decomposition

$$\wedge^2 V_4 \cong \text{Sym}^2(S_+) \oplus \text{Sym}^2(S_-).$$

The spinorial representation is

$$\Sigma = (S_+ \oplus S_-) \otimes W_+ \oplus (S_+ \oplus S_-) \otimes W_-,$$

where W_\pm are symplectic vector spaces.

We have

$$\bar{\Sigma} = S_- \otimes W_+ \oplus S_+ \otimes W_-$$

with the quadratic form being the product of the symplectic form on S_- , S_+ and W_{\pm} . Pick Lagrangians $L_{\pm} \subset W_{\pm}$. Then the massless irreducible representations of the superPoincaré group are parametrized by an irreducible $\text{Spin}(4)$ -representation M , so that the corresponding representation of the little group is

$$M \otimes \wedge^{\bullet}(S_- \otimes L_+ \oplus S_+ \otimes L_-).$$

The list of admissible supermultiplets:

- $\mathcal{N} = (1, 0)$. The chiral multiplet (aka half-hypermultiplet) $\mathbf{C}^2 \oplus S_-$, the tensor multiplet $\mathbf{C} \oplus S_-^{\oplus 2} \oplus \text{Sym}^2(S_-)$ and the $\mathcal{N} = (1, 0)$ vector multiplet $S_+^{\oplus 2} \oplus V_4$.
- $\mathcal{N} = (1, 1)$. The $\mathcal{N} = (1, 1)$ vector multiplet $\mathbf{C}^{\oplus 4} \oplus (S_+ \oplus S_-)^{\oplus 2} \oplus V_4$.
- $\mathcal{N} = (2, 0)$. The $\mathcal{N} = (2, 0)$ tensor multiplet $\mathbf{C}^{\oplus 5} \oplus S_-^{\oplus 4} \oplus \text{Sym}^2(S_-)$.

1.5. $d = 7$. The only irreducible admissible representations of $\text{Spin}(5)$ are the trivial one-dimensional representation \mathbf{C} , the 4d spin representation S and the 5d vector representation V_5 .

The spinorial representation is

$$\Sigma = (S \oplus S) \otimes W,$$

where W is a symplectic vector space. We have

$$\bar{\Sigma} = S \otimes W$$

with the quadratic form the product of the symplectic structure on S and the symplectic structure on W . Pick a Lagrangian subspace $L \subset W$. Then the irreducible representation of the little group is

$$M \otimes \wedge^{\bullet}(S \otimes L),$$

where M is an irreducible representation of $\text{Spin}(5)$.

The list of admissible supermultiplets:

- $\mathcal{N} = 1$. The vector multiplet $\mathbf{C}^{\oplus 3} \oplus S^{\oplus 2} \oplus V_5$.

1.6. $d = 8$. The only irreducible admissible representations of $\text{Spin}(6)$ are the trivial one-dimensional representation \mathbf{C} , the 4d semi-spin representations S_{\pm} and the 6d vector representation V_6 . The spinorial representation is

$$\Sigma = (S_+ \oplus S_-) \otimes (W \oplus W^*)$$

for a complex vector space W . We have

$$\bar{\Sigma} = S_+ \otimes W \oplus S_- \otimes W^*$$

with the quadratic form given by a pairing $S_+ \otimes S_- \rightarrow \mathbf{C}$. Then the irreducible representation of the little group is

$$M \otimes \wedge^{\bullet}(S_+ \otimes W),$$

where M is an irreducible representation of $\text{Spin}(6)$.

The list of admissible supermultiplets:

- $\mathcal{N} = 1$. The vector multiplet $\mathbf{C}^{\oplus 2} \oplus S_+ \oplus S_- \oplus V_6$.

1.7. $d = 9$. The only admissible irreducible representations of $\text{Spin}(7)$ are the trivial one-dimensional representation \mathbf{C} , the 8d spin representation S and the 7d vector representation V_7 . The spinorial representation is

$$\Sigma = (S \oplus S) \otimes W,$$

where W carries a nondegenerate symmetric bilinear form. We have

$$\bar{\Sigma} = S \otimes W.$$

Note that in the case $\dim(W)$ odd there is no $\text{Spin}(7)$ -invariant Lagrangian subspace in $\bar{\Sigma}$.

The list of admissible supermultiplets:

- $\mathcal{N} = 1$. The vector multiplet $\mathbf{C} \oplus S \oplus V$.

1.8. $d = 10$. The only admissible irreducible representations of $\text{Spin}(8)$ are the trivial one-dimensional representation \mathbf{C} , the semi-spin representations S_{\pm} and the 8d vector representation V_8 . The spinorial representation is

$$\Sigma = (S_+ \oplus S_-) \otimes W_+ \oplus (S_+ \oplus S_-) \otimes W_-,$$

where W_{\pm} carry nondegenerate symmetric bilinear forms. We have

$$\bar{\Sigma} = S_- \otimes W_+ \oplus S_+ \otimes W_-.$$

The list of admissible supermultiplets:

- $\mathcal{N} = (1, 0)$. The vector multiplet $S_- \oplus V$.

REFERENCES

- [Freed] D. Freed, Classical field theory and supersymmetry, <https://web.ma.utexas.edu/users/dafr/pcmi.pdf>.