A Taxonomy of Twists of Supersymmetric Yang–Mills Theory

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Abstract

We give a complete classification of twists of supersymmetric Yang-Mills theories in dimensions $2 \leq n \leq 10$. We formulate supersymmetric Yang-Mills theory classically using the BV formalism, and then we construct an action of the supersymmetry algebra using the language of L_∞ algebras. For each orbit in the space of square-zero supercharges in the supersymmetry algebra, under the action of the spin group and the group of R-symmetries, we give a description of the corresponding twisted theory. These twists can be described in terms of mixed holomorphic-topological versions of Chern-Simons and BF theory.

Contents

Introduction

In this paper we calculate supersymmetric twists of super Yang–Mills theories in dimension 2 through 10. Our main tools are the classical Batalin–Vilkovisky formalism, which eliminates the need for auxiliary fields to close the on-shell supersymmetry action, and a consistent use of dimensional reduction which allows us to deduce lower-dimensional statements from higher-dimensional statements.

Classical Field Theories

Let us begin with an informal discussion of classical field theories. A classical field theory is usually defined in terms of the data of the space of fields $\mathcal{F}$ equipped with an action functional. To incorporate gauge symmetries, one may either work with $\mathcal{F}$ as a stack or, as in the BRST formalism, with $\mathcal{F}$ as a Q-manifold, i.e. a graded manifold equipped with a square-zero vector field of cohomological degree 1 (the BRST differential). In the Batalin–Vilkovisky approach one considers instead the space of BV fields $\mathcal{E}$, which is equipped with a (-1)-shifted symplectic structure; this may be modeled by a QP-manifold [**Schwarz**]. Moreover, we assume that the Q-structure is Hamiltonian, i.e. that it is given by a Poisson bracket $\{S, -\}$ with respect to the BV action functional. Here $\mathcal{E}$ is supposed to be the derived critical locus of the action functional on $\mathcal{F}$.

In this paper we follow the approach developed in the works of Costello and Gwilliam [**CostelloBook**; **Book1**]. As the space of BV fields $\mathcal{E}$ is an infinite-dimensional manifold, it is difficult to work with it directly (for instance, to make sense of a (-1)-shifted symplectic structure). Instead, we zoom in on the neighborhood of a point where Q vanishes (i.e. we consider a given classical solution). We may then consider $\mathcal{E}$ as the space of sections of a graded vector bundle $E \to M$ over the spacetime manifold M. This allows us to work with finite-dimensional objects throughout. Namely, a (-1)-shifted symplectic structure on $\mathcal{E}$ boils down to a (-1)-shifted symplectic pairing $E \cong E^![-1]$, where $E^! = E^* \otimes \mathrm{Dens}_M$. We refer to Definition 1.2 for the precise definition of a classical field theory in the BV formalism that we use. Moreover, with this definition we may talk about weak equivalences of classical field theories (a notion inaccessible with QP-manifolds) which are simply maps of classical field theories inducing a quasi-isomorphism on $\mathcal{E}$. We call these *perturbative equivalences* (see Definition 1.7) to emphasize that we are working in a formal neighborhood of a given classical solution. For simplicity, throughout the paper we ignore issues of unitarity: in other words, we always consider complexified bundles of fields.

Classical Supersymmetric Field Theories

Now consider a classical field theory where the spacetime manifold is $M = \mathbb{R}^n$, and where the theory is moreover translation-invariant. Given the data of a spinorial representation Σ equipped with a symmetric pairing $\Gamma\colon \mathrm{Sym}^2(\Sigma) \to V = \mathbb{C}^n$, we may construct a super Lie algebra of supertranslations $\mathfrak{A} = \Pi\Sigma \oplus V$ with the only nonvanishing Lie bracket given by Γ. A supersymmetric classical field theory is then a translation-invariant classical field theory on $\mathbb{R}^n$ where the translation action on the fields is extended to an action of the super Lie algebra $\mathfrak{A}$. In addition, we may consider an R-symmetry group G_R which acts on Σ preserving Γ and the $\mathfrak{so}(n)$-action and also compatibly on the classical field theory.

In most literature on supersymmetry one simply tries to build an action of $\mathfrak{A}$ on the space of ordinary fields $\mathcal{F}$. However, one often runs into a problem that the supersymmetry action is only *on-shell*: the map from $\mathfrak{A}$ to vector fields $\mathrm{Vect}(\mathcal{F})$ preserves Lie brackets only on the critical locus of the action functional. The usual solution is to enlarge the space of fields by adding auxiliary fields with no kinetic terms on which there is an honest (*off-shell*) action of $\mathfrak{A}$. However, this choice may be not canonical. For instance, in 10d $\mathcal{N} = (1,0)$ super Yang–Mills one needs to break the Lorentz group $\mathrm{SO}(10)$ to $\mathrm{Spin}(7) \times \mathrm{SO}(2)$ to have an off-shell action of a subalgebra of $\mathfrak{A}$ where the odd part is 9-dimensional (instead of 16-dimensional) [**BaulieuBerkovitsBossardMartin**].

We instead take another approach pioneered by Baulieu, Bellon, Ouvry and Wallet [**BaulieuBV**]. Namely, one may canonically extend the supersymmetry action from ordinary fields $\mathcal{F}$ to the space of BV fields $\mathcal{E}$. The property of the action being on-shell now means that the map $\mathfrak{A} \to \mathrm{Vect}(\mathcal{E})$ preserves Lie brackets, but only up to homotopy. One may then try to incorporate these homotopies: to extend the Lie action to an L_∞ action. This contrasts with the auxiliary field approach of the previous paragraph, where one instead builds a resolution of the space of BV fields on which the supersymmetry Lie algebra acts strictly.

In this paper we consider supersymmetric Yang–Mills theories in dimensions 2 through 10. In dimensions 3 through 10 these may be obtained by dimensional reduction of the following theories: 10d $\mathcal{N} = (1,0)$ super Yang–Mills, 6d $\mathcal{N} = (1,0)$ super Yang–Mills, 4d $\mathcal{N} = 1$ super Yang–Mills and 3d $\mathcal{N} = 1$ super Yang–Mills. These theories depend on a choice of a Lie algebra $\mathfrak{g}$ equipped with a symmetric nondegenerate bilinear pairing. In addition, in dimensions 6, 4 and 3 we may add matter multiplets: in dimension 6 these depend on a choice of a symplectic $\mathfrak{g}$-representation (a hypermultiplet), in dimension 4 these depend on a choice of a $\mathfrak{g}$-representation (a chiral multiplet) and in dimension 3 these depend on a choice of an orthogonal $\mathfrak{g}$-representation. We do not consider superpotential and mass terms in this paper. Moreover, as we are working perturbatively, we ignore all topological terms (θ-terms).

The on-shell supersymmetry of pure super Yang–Mills theories in these dimensions can be proven by using a well-known relationship between composition algebras (e.g. division algebras) and supersymmetry (see Section 2.2) which goes back to the works [**Evans**; **KugoTownsend**]. For instance, we may construct the 10d $\mathcal{N} = (1,0)$ supersymmetry from the algebra of octonions $\mathbb{O}$, 6d $\mathcal{N} = (1,0)$ supersymmetry from the algebra of quaternions $\mathbb{H}$, 4d $\mathcal{N} = 1$ supersymmetry from the complex numbers $\mathbb{C}$ and 3d $\mathcal{N} = 1$ supersymmetry from the real numbers $\mathbb{R}$. Our treatment follows the work of Baez and Huerta [**BaezHuerta**] and we show how to extend the on-shell $\mathfrak{A}$-action to an L_∞-action using these ideas. As a new result, we also construct an L_∞-action on matter multiplets where the language of composition algebras turns out to be indispensable (see Section 3.2). Namely, for any real associative composition algebra $A_\mathbb{R}$ we simply need a complex $\mathfrak{g}$-representation equipped with an $A_\mathbb{R}$-module structure and a symmetric bilinear pairing. We have the following three cases:

- (**6d $\mathcal{N} = (1,0)$ supersymmetry**) For $A_\mathbb{R} = \mathbb{H}$ this forces $P = U \otimes W_+$, where U is a symplectic $\mathfrak{g}$-representation and W_+ is a 2d complex symplectic vector space (so that $\mathbb{H} \otimes_\mathbb{R} \mathbb{C} \cong \mathrm{End}(W_+)$).
- (**4d $\mathcal{N} = 1$ supersymmetry**) For $A_\mathbb{R} = \mathbb{C}$ this forces $P = R \oplus R^*$, where R is a $\mathfrak{g}$-representation.

- (**3d $\mathcal{N}=1$ supersymmetry**) For $A_{\mathbb{R}} = \mathbb{R}$ we simply have an orthogonal $\mathfrak{g}$-representation P.

In addition to the dimensional reduction of these four super Yang–Mills theories, there are also certain special super Yang–Mills theories with chiral supersymmetry: namely, 2d $\mathcal{N}=(1,0)$, $\mathcal{N}=(2,0)$ and $\mathcal{N}=(4,0)$ with matter as well as pure $\mathcal{N}=(\mathcal{N}_+,0)$ theories for any $\mathcal{N}_+$. We treat these separately (see Section 2.3), but again the language of composition algebras turns out to be convenient.

Supersymmetric Twists

The notion of supersymmetric twisting for a supersymmetric field theory was introduced by Witten [**WittenTQFT**]. The definitions we use in this paper will follow our previous work [**ElliottSafronov**], so let us briefly recall those.

Suppose Q is a square-zero supercharge, i.e. an odd element $Q \in \mathfrak{A}$ such that $[Q,Q]=0$. Then it gives rise to a square-zero odd symplectic vector field on the space of BV fields $\mathcal{E}$. In particular, we may modify the differential on $\mathcal{E}$ by the replacement $\mathrm{d} \mapsto \mathrm{d} + Q$. Working up to perturbative equivalence, this turns out to drastically simplify the theory as we will shortly see.

The original classical field theory carried a $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-grading, where $\mathbb{Z}$ is the cohomological (in the physics literature: ghost number) grading and $\mathbb{Z}/2\mathbb{Z}$ is the fermionic grading. We see that d has bidegree $(1,0)$ while Q has bidegree $(0,1)$. So, in general the twisted theory is only $\mathbb{Z}/2\mathbb{Z}$-graded (with respect to the total grading). To improve that, we may additionally consider a homomorphism $\alpha\colon \mathrm{U}(1) \to G_R$ into the R-symmetry group under which Q has weight 1 and such that the α-grading modulo 2 coincides with the fermionic grading. Then the α-grading gives rise to a $\mathbb{Z}$-grading on the twisted theory.

Finally, let us observe that the original classical field theory carried an action of $\mathrm{Spin}(n)$ by rotations of $\mathbb{R}^n$. But since Q is not preserved under $\mathrm{Spin}(n)$, this action does not survive in the twisted theory. To improve that, we may consider a group G with a *twisting homomorphism* $G \to \mathrm{Spin}(n) \times G_R$ under which Q is a scalar. Given such a twisting homomorphism, the twisted theory carries a G-action.

To summarize, supersymmetric twisting consists of the following three steps:

1. Choose a square-zero supercharge $Q \in \Sigma$ and modify the differential of the theory as $\mathrm{d} \mapsto \mathrm{d} + Q$.
2. Choose a group G together with a twisting homomorphism $G \to \mathrm{Spin}(n) \times G_R$ under which Q is scalar. To remove redundancy, we will assume $G \to \mathrm{Spin}(n)$ is an embedding.
3. Choose a homomorphism $\alpha\colon \mathrm{U}(1) \to G_R$ under which Q has weight 1 and such that the α-grading modulo 2 is the fermionic grading. This step may not be possible in general.

A classification of possible square-zero supercharges Q was previously done in [**ElliottSafronov**] and in this paper we use that classification to calculate the twist of super Yang–Mills theories on $\mathbb{R}^n$ in all dimensions.

Supersymmetric Twists and Supergravity

In this paper we only consider the case of global supersymmetry for super Yang–Mills theories on $\mathbb{R}^n$. In certain cases one may consider coupling of super Yang–Mills to supergravity in which case there is an interpretation of the twisting procedure as performing perturbation theory in a nontrivial supergravity background. Let us briefly explain this perspective.

A classical solution of supergravity consists, in particular, of the following data: a spacetime manifold M, a $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}} \to M$ equipped with a connection (spin connection), a G_R-bundle $P_R \to M$ equipped with a connection and a ghost for supertranslations $\eta \in \Gamma(M, (P_{\mathrm{Spin}} \times P_R) \times^{\mathrm{Spin}(n)\times G_R} \Sigma)$. The ghost η is bosonic: it lives in bidegree $(-1, 1)$ for the $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-grading, so it makes sense to give it a non-zero value. If we couple super Yang–Mills to supergravity, then the super Yang–Mills fields become sections of the associated bundles to $P_{\mathrm{Spin}} \times P_R$.

We have the following supergravity analogs of the data (Q, ϕ, α) for supersymmetric twisting:

- The supergravity analog of the choice of a square-zero supercharge Q is the value of the ghost η.
- The supergravity analog of the twisting homomorphism is a choice of G-bundle $P_G \to M$ with connection so that $P_{\mathrm{Spin}} \times P_R$ is induced via the homomorphism $G \to \mathrm{Spin}(n) \times G_R$.
- The supergravity analog of $\alpha\colon \mathrm{U}(1) \to G_R$, if a choice of trivial $\mathrm{U}(1)$-subbundle in G_R on which the connection restricts to zero.

Applications to Quantization

The quantization of gauge theories is notoriously subtle and requires a rich theory of renormalization. One attractive application of the descriptions of the twists of supersymmetric gauge theories that we provide is to study quantization in a setting where the machinery required for renormalization is much more rigid.

To rigorously study the quantization of supersymmetric Yang-Mills theory we can work with the mathematical theory of renormalization developed by Costello in [**CostelloBook**]. This theory of renormalization can been used to study field theories with and without supersymmetry. (Brian: many citations) (Chris: I'm not sure what you want to cite here.) In the context of (non-supersymmetric) Yang-Mills theory, for instance, it is shown that this theory recovers asymptotic freedom by an explicit analysis of the local counterterms present in the four-dimensional gauge theory [**EWY**].

In principle, the existence of local counterterms can be used to analyze the full untwisted supersymmetric gauge theories in a mathematically rigorous way. In practice, however, our approach to renormalization does not provide any significant advantage over traditional approaches used in QFT. However, a significant simplification happens at the level of the *twisted* supersymmetric Yang-Mills theories that we study in this work. To start with, for some examples (but not all), the twisted theory turns out to be a *topological field theory*. This occurs whenever the bracket $[Q, -]$ with the twisting supercharge surjects onto the space of translations. The theory of renormalization for topological theories can be handled using configuration spaces [**Kontsevich**; **AxelrodSinger**].

In the general setting of this paper, while not every twist results in a topological field theory, it does result in a theory in which some directions of spacetime behave topologically, and the remaining directions behave holomorphically. For a mixed holomorphic-topological translation invariant field theory of this type on $\mathbb{R}^n \times \mathbb{C}^d$, this means that at least half of the linearly independent translation invariant vector fields act on the field theory in a BRST exact way.

Inspired by the work of Costello and Li in [**BCOV1**] and Li in [**LiFeynman**; **LiVertex**], the foundations of renormalization for mixed holomorphic-topological field theories on Euclidean space has been developed in [**BWhol**]. The key result is that the renormalization for mixed holomorphic-topological theories is extremely well-behaved from an analytic perspective. It is shown in the cited work that, to first order in $\hbar$, the renormalization of mixed holomorphic/topological theory is *finite*. Furthermore, in [**LiVertex**], it is shown that in real dimension two this holds to all orders in $\hbar$.

These results yield a practical approach to the problem of mathematically characterizing the one-loop quantization of every twist of supersymmetric Yang-Mills theory. Furthermore, in all examples of theories obtained via twisting occuring in dimensions 8 and lower, not much is lost when asking for the one-loop quantization. The twisted gauge theories here are all either equivalent to BF-type theories (see Section 1.6.1) or deformations of such theories by a holomorphic differential operator. Such theories admit prequantizations (that is, they define families of effective field theories compatible under renormalization group flow), which are exact at one loop, meaning all higher order corrections vanish identically.

From this starting point, the first natural problem would be to verify whether these one-loop exact prequantizations define actual quantizations of the classical twisted field theory. That is, for each such theory, to compute the one-loop anomalies to the solution of the quantum master equation. This problem comes in two parts: first, to compute the one-loop anomaly to quantization of the theory on flat space $\mathbb{R}^n \times \mathbb{C}^d$, and second – in the case where we can use a twisting homomorphism to define the twisted theory on certain structured $(n+2d)$-manifolds, to calculate the corresponding one-loop anomaly on curved space (in other words, incorporating the computation of a gravitational anomaly). We plan to return to this question in future work.

The Relationship to Factorization Algebras

In our previous paper [**ElliottSafronov**], we discussed supersymmetric twisting with an emphasis not on the classical fields of a supersymmetric field theory, but instead on their classical or quantum *observables*. The factorization algebra formalism of Costello and Gwilliam [**Book1**; **Book2**] provides a model for the local structure of the observables in a general quantum field theory. In brief, for every open subset $U \subseteq M$ of the spacetime manifold, one associates a (possibly $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-graded) vector space $\mathrm{Obs}(U)$ of local observables on U. For any pair $U_1, U_2 \subseteq V$ of disjoint open subsets of an open set V, one associates a morphism

$$m^V_{U_1,U_2} \colon \mathrm{Obs}(U_1) \otimes \mathrm{Obs}(U_2) \to \mathrm{Obs}(V),$$

thought of as an *operator product* for local observables. These products should vary smoothly as one varies the open subsets U_1, U_2 and V. Starting with a classical field theory on M, defined using the BV formalism, one can build a factorization algebra $\mathrm{Obs}^{\mathrm{cl}}$ modelling the classical observables of the field theory. If the classical field theory carries the action of a group G, so does the associated factorization algebra. Furthermore, Costello and Gwilliam develop techniques for the quantization of such algebras of classical observables, using the theory of renormalization as discussed in the previous section.

In [**ElliottSafronov**] we studied the supersymmetric twisting procedure as applied to factorization algebras on $\mathbb{R}^n$ with an action of a supersymmetry algebra. If Q is a topological supercharge, then the Q-twist Obs^Q of a supersymmetric factorization algebra automatically satisfies a strong translation invariance condition: all translations must act homotopically trivially. In good circumstances, we can say even more. An $\mathbb{E}_n$*-algebra* is an algebra over the operad of little n-disks; in the language above, this can be obtained from a factorization algebra for which homotopy equivalent configurations $U_1, U_2 \subseteq V$ induce homotopy equivalent products.

Theorem ([**ElliottSafronov**])**.** If Q is a topological supercharge, and the operator $\mathrm{Obs}^Q(B_r(0)) \to \mathrm{Obs}^Q(B_R(0))$ associated to the inclusion of concentric balls is an equivalence, then the factorization algebra Obs^Q has the canonical structure of an $\mathbb{E}_n$-algebra.

The hypothesis of the theorem is automatically satisfied, for example, for superconformal theories, and should be concretely checkable in examples.

In the present work we classify twists of classical field theories, to which one can associate twisted factorization algebras of classical – and, if the appropriate anomalies vanish, quantum – observables in the sense of our previous work. In some (topological) examples, these define $\mathbb{E}_n$-algebras. In other examples, where the twist is not fully topological, the twisted local observables will define higher analogues of vertex algebras (as in, for instance, [**GwilliamWilliamsKM**]).

Summary of Twisted Super Yang–Mills Theories

In this section we will summarize the main results of the paper presented in Part II, where we calculate twists of super Yang–Mills theories in dimensions 2 through 10.

Let us begin by explaining what we mean by "calculation". Recall that for a Lie algebra $\mathfrak{g}$ there is a d-dimensional topological BF theory defined on a d-dimensional spacetime manifold M with the space of BV fields $\Omega^\bullet(M;\mathfrak{g})[1] \oplus$

$\Omega^\bullet(M;\mathfrak{g}^*)[d-2]$, where $\Omega^\bullet$ denotes the space of differential forms equipped with the de Rham differential d. If M is replaced by a complex manifold X, we may also consider its version with the space of fields $\Omega^{\bullet,\bullet}(X;\mathfrak{g})[1]\oplus\Omega^{\bullet,\bullet}(X;\mathfrak{g}^*)[2\dim(X)-2]$, where $\Omega^{\bullet,\bullet}$ is the space of differential forms equipped with the Dolbeault differential. Finally, we have yet another version, a *holomorphic BF theory*, with the space of BV fields $\Omega^{0,\bullet}(X;\mathfrak{g})[1]\oplus\Omega^{\dim(X),\bullet}(X;\mathfrak{g}^*)[\dim(X)-2]$, again equipped with the Dolbeault differential. We will denote the space of fields in these three examples as $T^*[-1]\operatorname{Map}(M_{\mathrm{dR}},B\mathfrak{g})$, $T^*[-1]\operatorname{Map}(X_{\mathrm{Dol}},B\mathfrak{g})$ and $T^*[-1]\operatorname{Map}(X,B\mathfrak{g})$ respectively (the notation is explained in Section 1.5). We may also combine these three examples into what we call a *generalized BF theory* with the spaces of fields $T^*[-1]\operatorname{Map}(X\times Y_{\mathrm{Dol}}\times M_{\mathrm{dR}},B\mathfrak{g})$ (see Definition 1.34 for more details).

Let us also recall that if $\mathfrak{g}$ is equipped with a symmetric bilinear nondegenerate pairing, we also have a 3-dimensional topological Chern–Simons theory. If we forgo $\mathbb{Z}$-gradings and work with $\mathbb{Z}/2\mathbb{Z}$-gradings, we may also consider a topological Chern–Simons theory in any odd dimension (see [**AlekseevMnev**] for a 1-dimensional version and [**BakGustavsson2**] for a 5-dimensional version). Just like for the BF theory, we also have two other versions which may be combined into a generalized Chern–Simons theory. Another direction we can generalize in is to replace the Lie algebra $\mathfrak{g}$ by a dg Lie algebra, in which case the BF theory itself becomes a particular example of the Chern–Simons theory.

Our goal will then be to show that a particular twist of super Yang–Mills is equivalent to a given generalized Chern–Simons theory. We summarize our results in two forms. In Tables 1, 2 and 3 we summarize all the possible twists of dimensional reductions of 10d $\mathcal{N}=(1,0)$ super Yang–Mills, 6d $\mathcal{N}=(1,0)$ super Yang–Mills and 4d $\mathcal{N}=1$ super Yang–Mills respectively. In Table 4 we summarize the twists of 2d supersymmetric Yang-Mills theory. Before these tables, we will give a short description of each twisted theory in a more physical language, with references to where in the literature it was previously considered.

Dimension 10

- $\mathcal{N}=(1,0)$ *holomorphic twist.* The unique twisted super Yang-Mills theory in 10 dimensions is equivalent, as a $\mathbb{Z}/2\mathbb{Z}$-graded theory, to 5d holomorphic Chern-Simons theory. It is SU(5) invariant, and can be defined on a Calabi-Yau 5-fold. This twist was first studied by Baulieu [**Baulieu**]. As is well-known [**GSanomaly**], the theory has a one-loop anomaly and does not admit a quantization.

Dimension 9

- $\mathcal{N}=1$ *minimal twist.* The unique twisted super Yang-Mills theory in 9 dimensions is equivalent, as a $\mathbb{Z}/2\mathbb{Z}$-graded theory, to a generalized version of Chern-Simons theory with one topological direction and four holomorphic directions. It is SU(4)-invariant, and can be defined on the product of a Calabi-Yau 4-fold and a flat 1-manifold (i.e. $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$).

Dimension 8

- $\mathcal{N}=1$ *holomorphic twist.* Super Yang-Mills theory in 8 dimensions admits three classes of twist. The minimal twist, by a holomorphic (or, equivalently, pure) supercharge, is equivalent to a holomorphic version of BF theory. This theory is U(4)-invariant, and can be defined on arbitrary complex 4-folds.
- $\mathcal{N}=1$ *intermediate twist.* This minimal twist admits a further twist by a rank 1 impure spinor, which is topological in two directions and holomorphic in the remaining three complex directions. This intermediate twisted theory is equivalent, as a $\mathbb{Z}/2\mathbb{Z}$-graded theory, to a generalized version of Chern-Simons theory. It is $\mathrm{SU}(3)\times\mathrm{Spin}(2;\mathbb{R})$-invariant, and can be defined on the product of a Calabi-Yau 3-fold and a Riemann surface.
- $\mathcal{N}=1$ *topological twist.* There is, finally, a novel topological twist in 8-dimensions, which does not arise via dimensional reduction from the theories above. This theory is perturbatively trivial, in the sense that the classical BV complex is contractible. It can, however, be realised as the generic point in a 1-parameter degeneration of the holomorphic twist. This theory is $\mathrm{Spin}(7;\mathbb{R})$-invariant, although the full one-parameter family

is only SU(4)-invariant. This theory was studied in [**AcharyaOLoughlinSpence**; **BaulieuKannoSinger**]; its equations of motion include the Spin(7)-instanton equation.

Dimension 7

- $\mathcal{N} = 1$ *minimal twist.* The twists of super Yang-Mills theory in 7 dimensions arise by dimensionally reducing the twists in 8 dimensions. The minimal twist, by a pure spinor, is equivalent to a generalized version of BF theory with one topological direction and three complex holomorphic directions. This theory is U(3)-invariant, and can be defined on the product of a complex 3-fold with a flat 1-manifold.
- $\mathcal{N} = 1$ *intermediate twist.* This minimal twist admits a further twist, by a rank 1 impure spinor, which is topological in three directions and holomorphic in the remaining two complex directions. This intermediate twisted theory is equivalent, as a $\mathbb{Z}/2\mathbb{Z}$-graded theory, to a generalized version of Chern-Simons theory. It is $\mathrm{SU}(2) \times \mathrm{Spin}(3;\mathbb{R})$-invariant, and can be defined on the product of a Calabi-Yau surface and a 3-manifold with spin structure.
- $\mathcal{N} = 1$ *topological twist.* The topological twist in 8-dimensions reduces to a topological twist in 7 dimensions. This theory is again perturbatively trivial, but can be realized as the generic point in a 1-parameter degeneration of the minimal twist. It is G_2-invariant, although the full one-parameter family is only SU(3)-invariant. This theory was also studied in [**AcharyaOLoughlinSpence**; **BaulieuKannoSinger**]; its equations of motion include the G_2-monopole equations.

Dimension 6

- $\mathcal{N} = (1,0)$ *and* $\mathcal{N} = (1,1)$ *holomorphic twist.* There is a holomorphic twist in 6 dimensions that can be defined for the $\mathcal{N} = (1,0)$ supersymmetric theory coupled to a hypermultiplet valued in a symplectic representation U. This twist is equivalent to holomorphic Chern-Simons theory coupled to a holomorphic symplectic boson valued in the representation U. It is generally only $\mathbb{Z}/2\mathbb{Z}$-graded, but can be promoted to a $\mathbb{Z}$-graded theory when $U = T^*R$ is of cotangent type. It is invariant for the action of SU(3) in general, and can be defined on Calabi-Yau 3-folds, and the unitary metalinear group MU(3) in the cotangent case, and can be defined on complex 3-folds with a choice of square root of the canonical bundle. When $U = T^*\mathfrak{g}$, the supersymmetry can be promoted to $\mathcal{N} = (1,1)$ supersymmetry, and the twist can be identified with a holomorphic BF theory coupled to adjoint holomorphic matter. This twist has also been calculated in forthcoming work of Dylan Butson [**Butson**].
- $\mathcal{N} = (1,1)$ *special rank* $(1,1)$ *twist.* In the $\mathcal{N} = (1,1)$ case there are two intermediate twists. The one by a supercharge of rank $(1,1)$ is equivalent to a generalized form of BF theory, with two topological directions and two complex holomorphic directions. This theory is $\mathrm{U}(2) \times \mathrm{Spin}(2;\mathbb{R})$-invariant, and can be defined on the product of a complex surface with a Riemann surface.
- $\mathcal{N} = (1,1)$ *rank* $(2,2)$ *twist.* The other intermediate twist, by a rank $(2,2)$ supercharge, is equivalent to a generalized form of Chern-Simons theory with four real directions and one complex holomorphic direction. This theory is only $\mathbb{Z}/2\mathbb{Z}$-graded. It is invariant for the action of $\mathrm{Spin}(4;\mathbb{R})$, and can be defined on the product of a flat Riemann surface and a spin 4-manifold.
- $\mathcal{N} = (1,1)$ *topological twist.* Finally, there is a topological twist in 6 dimensions obtained by dimensionally reducing the 7-dimensional topological twist. This theory is again perturbatively trivial, but can be realized as the generic point in a 1-parameter degeneration of the holomorphic twist. It is SU(3)-invariant, although the full one-parameter family is only $\mathrm{SU}(2) \times \mathrm{Spin}(2;\mathbb{R})$-invariant. This theory was also studied in [**AcharyaOLoughlinSpence**; **BaulieuKannoSinger**]; its equations of motion include the Donaldson-Thomas equations.

Dimension 5

- $\mathcal{N}=1$ *and* $\mathcal{N}=2$ *minimal twist.* There is a minimal twist in 5 dimensions which can be defined with $\mathcal{N}=1$ supersymmetry in the presence of a U-valued hypermultiplet, obtained by dimensional reduction from 6d $\mathcal{N}=(1,0)$. This twist is equivalent to a generalized form of Chern-Simons theory with one topological direction and two complex holomorphic directions, coupled to a partially holomorphic symplectic boson. It is invariant for the metalinear group MU(2) for general U, and for U(2) when $U=T^*R$ is of cotangent type, and can be defined on the product of a flat 1-manifold and an arbitrary complex surface (with a choice of square root of the canonical bundle in the general case). This theory was studied by Källén and Zabzine [**KallenZabzine**]. If $U=T^*\mathfrak{g}$ the supersymmetry action can be promoted to the $\mathcal{N}=2$ supersymmetry algebra, and the twisted theory can be identified with a generalized BF theory valued in the adjoint quotient $\mathfrak{g}/\mathfrak{g}$.
- $\mathcal{N}=2$ *intermediate twist.* In the $\mathcal{N}=2$ case there is an intermediate twist with four invariant directions. This twist can be identified with a generalized BF theory with three topological directions and one complex holomorphic direction. It is invariant for the group $\mathrm{Spin}(2;\mathbb{R})\times\mathrm{Spin}(3;\mathbb{R})$ and can be defined on the product of a 2-manifold and a 3-manifold, each equipped with a spin structure. This twist was considered in [**ElliottPestun**] on manifolds of the form $\Sigma\times C\times S^1$, where Σ and C are Riemann surfaces, where the moduli space can be viewed as a multiplicative version of the Hitchin system.
- $\mathcal{N}=2$ *topological A twist.* There are two topological twists in 5d $\mathcal{N}=2$. One, which we refer to as the A-twist, arises by dimensionally reducing the topological twist of 6d $\mathcal{N}=(1,1)$ super Yang-Mills theory. It is perturbatively trivial, but can be realized non-trivially as the generic point in a 1-parameter degeneration of the intermediate twist. The full family is invariant for the group $\mathrm{Spin}(3;\mathbb{R})$. This theory was studied by Qiu and Zabzine, [**QiuZabzine**], where the equations of motion were shown to include the Haydys-Witten equations. Twisting homomorphisms in this situation were previously studied by Anderson [**Anderson**].
- $\mathcal{N}=2$ *topological B twist.* Finally, the other topological twist, associated to a rank 4 supercharge, can be identified with 5d Chern-Simons theory. This theory is only $\mathbb{Z}/2\mathbb{Z}$-graded, and it is invariant for the group $\mathrm{Spin}(3;\mathbb{R})$. This twist was identified in work of Geyer-Mülsch and of Bak-Gustavsson[**GeyerMuelsch**; **BakGustavsson1**; **BakGustavsson2**].

Dimension 4

- $\mathcal{N}=1$, $\mathcal{N}=2$ *and* $\mathcal{N}=4$ *holomorphic twist.* There is a holomorphic twist in dimension 4, which makes sense with $\mathcal{N}=1$ supersymmetry, coupled to a chiral multiplet valued in a representation R. This twist is equivalent to holomorphic BF theory coupled to a $\beta\gamma$-system valued in the representation R. It is invariant for the action of U(2), and can be defined on arbitrary complex surfaces. In the case where $R=U\oplus\mathfrak{g}$ for a symplectic representation U, the $\mathcal{N}=1$ supersymmetry can be lifted to $\mathcal{N}=2$ supersymmetry, and the twisted theory is modelled by a holomorphic BF theory now coupled to a holomorphic symplectic boson. In general this theory is only $\mathrm{M}U(2)$-invariant, and depends on a choice of square root of the canonical bundle. Finally, if $U=T^*\mathfrak{g}$, the supersymmetry lifts further to $\mathcal{N}=4$, and the twisted theory is modelled by a Dolbeault version of holomorphic BF theory – the moduli space of solutions to the equations of motion are represented by the shifted cotangent space to the moduli stack of G-Higgs bundles. This twist was studied by Johansen [**Johansen**].
- $\mathcal{N}=2$ *and* $\mathcal{N}=4$ *intermediate twist.* If $\mathcal{N}=2$ then there is an intermediate twist, with three invariant directions. This twist is equivalent to a generalized version of BF theory with two real directions and one complex holomorphic direction, coupled to a generalized version of a U-valued symplectic boson. This theory is $\mathrm{Spin}(2;\mathbb{R})\times\mathrm{Spin}(2;\mathbb{R})$-invariant, and can be defined on the product of two Riemann surfaces. This twist is often referred to as the *Kapustin twist* [**KapustinHolo**].
- $\mathcal{N}=2$ *and* $\mathcal{N}=4$ *topological A twist.* There is also a topological twist for $\mathcal{N}=2$, the *Donaldson twist*, which is perturbatively trivial, but can be realized non-trivially as the generic point in a 1-parameter degeneration of the holomorphic twist. This theory was first considered in [**WittenTQFT**], and the coupling to a hypermultiplet of cotangent type was studied in [**AlvarezLabastida**; **HyunParkPark**]. Its equations of motion include the Donaldson-Witten equations. If the supersymmetry is promoted to $\mathcal{N}=4$, then this theory is identified either with the theory studied by Vafa and Witten, [**VafaWitten**], or with the A-twist studied by Kapustin and

Witten [**KapustinWitten**], depending on the twisting homomorphism one uses to make the twisted theory $\mathrm{Spin}(4;\mathbb{R})$-invariant.

- $\mathcal{N}=4$ *topological B twist.* If $\mathcal{N}=4$, there are additional topological twists. The most important one is the Kapustin-Witten B-twist [**KapustinWitten**]. This theory is obtained by dimensional reduction from the B-twist of 5d $\mathcal{N}=2$, or by twisting by a special supercharge of rank $(2,2)$. It is equivalent to a topological BF theory. This theory is $\mathrm{Spin}(4;\mathbb{R})$-invariant using the Kapustin-Witten twisting homomorphism, and can be defined on arbitrary 4-manifolds.

- *More general* $\mathcal{N}=4$ *topological twists* In the $\mathcal{N}=4$ case, the B-twist fits into a larger family of topological twists, most of which are perturbatively trivial. A $\mathbb{CP}^{\nVdash}$-family of such twists, generically of rank $(2,2)$, was studied by Kapustin and Witten [**KapustinWitten**], which is invariant for $\mathrm{Spin}(4;\mathbb{R})$ using the Kapustin-Witten twisting homomorphism. There are additional topological twists, including those of rank $(1,2)$, which are only invariant for the group $\mathrm{Spin}(2;\mathbb{R})\times\mathrm{Spin}(2;\mathbb{R})$. An analysis of these twists using similar techniques to this paper, but with the aim of obtaining the full derived stack of solutions to the equations of motion, rather than only the perturbative classical field theory, was carried out in [**ElliottYoo1**].

Dimension 3

- $\mathcal{N}=2$*,* $\mathcal{N}=4$ *and* $\mathcal{N}=8$ *minimal twist.* There is a minimal twist in dimension 3 which can be defined with $\mathcal{N}=2$ supersymmetry, including a chiral multiplet valued in a representation R. This twist is perturbatively equivalent to a generalized form of BF theory with one real direction and one complex holomorphic direction, coupled to a generalized $\beta\gamma$-system valued in R. This theory is $\mathrm{U}(1)$-equivariant, and can be defined on the product of a Riemann surface and a flat 1-manifold. If the supersymmetry is promoted to $\mathcal{N}=4$, with $R=U\oplus\mathfrak{g}$ for a symplectic representation U, then the twisted theory is modelled by a generalized BF theory now coupled to a generalized symplectic boson. Finally, if $U=T^*\mathfrak{g}$ the supersymmetry lifts to $\mathcal{N}=8$, and the twisted theory can be identified with generalized BF theory valued in $T[1](\mathfrak{g}/\mathfrak{g})$.

- $\mathcal{N}=4$ *and* $\mathcal{N}=8$ *topological A twist.* If $\mathcal{N}$ is at least 4, this minimal twist can be deformed in two inequivalent ways. The first is a perturbatively trivial theory which we refer to as the A-twist, obtained by dimensional reduction from the 4d A-twist. This theory fits into a non-trivial family deforming the minimal twist, which is $\mathrm{U}(1)$-equivariant, and can be defined on the product of a Riemann surface and a flat 1-manifold. This twist was studied by Blau and Thompson [**BlauThompson1**], where it is referred to as a Casson invariant model, as well as Kapustin and Vyas [**KapustinVyas**] as a dimensional reduction of Donaldson-Witten theory.

- $\mathcal{N}=4$ *and* $\mathcal{N}=8$ *topological B twist.* The remaining topological twist is perturbatively equivalent to the 3d topological BF theory coupled to a symplectic boson valued in symplectic representation U. It is $\mathrm{Spin}(3;\mathbb{R})$-invariant, and can be defined on arbitrary 3-manifolds. This form of twist was studied by Rozansky and Witten, [**RozanskyWitten**] as well as by Blau and Thompson [**BlauThompson2**].

Dimension 2

- *Non-Chiral (*$\mathcal{N}=(2,2),(4,4),(8,8)$*) holomorphic twist.* There is a holomorphic twist in dimension 2 which can be defined with $\mathcal{N}=(2,2)$ supersymmetry, including a chiral multiplet valued in a representation R. This twist is perturbatively equivalent to a holomorphic BF theory coupled to a $\beta\gamma$-system valued in R. More specifically: to holomorphic BG theory valued in the tangent space $T[1]R/\mathfrak{g}$. This theory is $\mathrm{U}(1)$-equivariant, and can be defined on any Riemann surface. If the supersymmetry is promoted to $\mathcal{N}=(4,4)$, with $R=U\oplus\mathfrak{g}$ for a symplectic representation U, then the twisted theory can now be modelled by holomorphic BF theory coupled to a holomorphic symplectic boson. Finally, if $U=T^*\mathfrak{g}$ the supersymmetry lifts to $\mathcal{N}=8$, and the twisted theory can be identified with holomorphic BF theory valued in $T[1]T^*[2](\mathfrak{g}/\mathfrak{g}))$. The study of twists of 2d $\mathcal{N}=(2,2)$ supersymmetric field theories goes back to work of Eguchi and Yang [**EguchiYang**] and Witten [**Wittenmirror**].

- *Non-Chiral (*$\mathcal{N}=(2,2),(4,4),(8,8)$*) topological A twist.* In each of these cases, the minimal twist can again be deformed to a topological theory in two inequivalent ways. The first is a perturbatively trivial theory: the A-model valued in the cotangent space to the quotient stack $R/\mathfrak{g}$.

- *Non-Chiral ($\mathcal{N} = (2,2),(4,4),(8,8)$) topological B twist.* The remaining topological twist, the B-twist, is perturbatively equivalent to a 2d topological BF theory, coupled a topological matter valued in the representation R.

- *Chiral* $\mathcal{N} = (0, N_-)$ *holomorphic twist.* Chiral theories in 2-dimensions only admit a holomorphic twist. These twists can all be realised as holomorphic BF theory on $\mathbb{C}$, coupled to a holomorphic $\beta\gamma$-system (in the case where the chiral theory includes matter), or to a holomorphic free fermion (in the case where the chiral theory is pure gauge with $\mathcal{N} = (0, N_-)$ supersymmetry). Twisted $(0,2)$ σ-models were first studied by Witten in [**Wittenmirror**], and can be used to obtain the chiral algebra of chiral differential operators [**WittenCDO**].

d	$\mathcal{N}$	Twist	Description	Invariant Directions
10	$(1,0)$	Rank $(1,0)$	Holomorphic Chern-Simons Theory $\mathrm{Map}(\mathbb{C}^5, B\mathfrak{g})$	5 (holomorphic)
9	1	Rank 1	Generalized Chern-Simons Theory $\mathrm{Map}(\mathbb{C}^4 \times \mathbb{R}_{\mathrm{dR}}, B\mathfrak{g})$	5 (minimal)
8	1	Rank $(1,0)$ pure	Holomorphic BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^4, B\mathfrak{g})$	4 (holomorphic)
		Rank $(1,1)$	Generalized Chern-Simons Theory $\mathrm{Map}(\mathbb{C}^3 \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})$	5
		Rank $(1,0)$ impure	Perturbatively trivial (Spin(7) Instanton) $\mathrm{Map}(\mathbb{C}^4, B\mathfrak{g})_{\mathrm{dR}}$	8 (topological)
7	1	Rank 1 pure	Generalized BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^3 \times \mathbb{R}_{\mathrm{dR}}, B\mathfrak{g})$	4 (minimal)
		Rank 2	Generalized Chern-Simons Theory $\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}^3_{\mathrm{dR}}, B\mathfrak{g})$	5
		Rank 1 impure	Perturbatively trivial (G_2 Monopole) $\mathrm{Map}(\mathbb{C}^3 \times \mathbb{R}_{\mathrm{dR}}, B\mathfrak{g})_{\mathrm{dR}}$	7 (topological)
6	$(1,1)$	Rank $(1,0)$	Holomorphic BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^3, \mathfrak{g}/\mathfrak{g})$	3 (holomorphic)
		Rank $(1,1)$ special	Generalized BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})$	4
		Rank $(2,2)$	Generalized Chern-Simons Theory $\mathrm{Map}(\mathbb{C} \times \mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g})$	5
		Rank $(1,1)$ generic	Perturbatively trivial $\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})_{\mathrm{dR}}$	6 (topological)
5	2	Rank 1	Generalized BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$	3 (minimal)
		Rank 2 special	Generalized BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C} \times \mathbb{R}^3_{\mathrm{dR}}, B\mathfrak{g})$	4
		Rank 4	5d Chern-Simons Theory $\mathrm{Bun}_G(\mathbb{R}^5_{\mathrm{dR}})$	5 (topological)
		Rank 2 generic	Perturbatively trivial $\mathrm{Map}(\mathbb{C} \times \mathbb{R}^3_{\mathrm{dR}}, B\mathfrak{g})_{\mathrm{dR}}$	5 (topological)
4	4	Rank $(1,0)$	Holomorphic BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^2_{\mathrm{Dol}}, B\mathfrak{g})$	2 (holomorphic)
		Rank $(1,1)$	Generalized BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})$	3
		Rank $(2,2)$ special	BF Theory $T^*[-1]\mathrm{Map}(\mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g})$	4 (topological)
		Rank $(2,1)$	Perturbatively trivial $\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})_{\mathrm{dR}}$	4 (topological)
		Rank $(2,0)$	Perturbatively trivial $\mathrm{Map}(\mathbb{C}^2_{\mathrm{Dol}}, B\mathfrak{g})_{\mathrm{dR}}$	4 (topological)
		Rank $(2,2)$ generic	Perturbatively trivial $\mathrm{Map}(\mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g})_{\mathrm{dR}}$	4 (topological)
3	8	Rank 1	Generalized BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$	2 (minimal)
		Rank 2 (B)	BF Theory $T^*[-1]\mathrm{Map}(\mathbb{R}^3_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$	3 (topological)
		Rank 2 (A)	Perturbatively trivial $\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})_{\mathrm{dR}}$	3 (topological)

Table 1: Twists of Maximally Supersymmetric Pure Yang-Mills Theories with gauge group G (16 supercharges).

d	$\mathcal{N}$	Twist	Description	Invariant Directions
6	$(1,0)$	Rank $(1,0)$	Holomorphic BF Theory coupled to a holomorphic symplectic boson $\mathrm{Map}(\mathbb{C}^3, U/\!\!/G)$	3 (holomorphic)
5	1	Rank 1	Generalized BF Theory coupled to a generalized symplectic boson $\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}_{\mathrm{dR}}, U/\!\!/G)$	3 (minimal)
4	2	Rank $(1,0)$	Holomorphic BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^2, U/\!\!/G)$	2 (holomorphic)
		Rank $(1,1)$	Generalized BF Theory coupled to a generalized symplectic boson $\mathrm{Map}(\mathbb{C} \times \mathbb{R}^2_{\mathrm{dR}}, U/\!\!/G)$	3
		Rank $(2,0)$	Perturbatively trivial $\mathrm{Map}(\mathbb{C}^2, U/\!\!/G)_{\mathrm{dR}}$	4 (topological)
3	4	Rank 1	Generalized BF Theory coupled to a generalized symplectic boson $T^*[-1]\mathrm{Map}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, U/\!\!/G)$	2 (minimal)
		Rank 2 (B)	BF Theory coupled to a symplectic boson $\mathrm{Map}(\mathbb{R}^3_{\mathrm{dR}}, U/\!\!/G)$	3 (topological)
		Rank 2 (A)	Perturbatively trivial $\mathrm{Map}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, U/\!\!/G)_{\mathrm{dR}}$	3 (topological)

Table 2: Twists of Supersymmetric Yang-Mills Theories with gauge group G with a hypermultiplet valued in a symplectic representation U (8 supercharges).

d	$\mathcal{N}$	Twist	Description	Invariant Directions
4	1	Rank $(1,0)$	Holomorphic BF Theory coupled to R-matter $T^*[-1]\mathrm{Map}(\mathbb{C}^2, R/G)$	2 (holomorphic)
3	2	Rank 1	Generalized BF Theory coupled to R-matter $T^*[-1]\mathrm{Map}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, R/G)$	2 (minimal)

Table 3: Twists of Supersymmetric Yang-Mills Theories with gauge group G with a chiral multiplet valued in representation R (4 supercharges).

$\mathcal{N}$	Twist	Description	Invariant Directions
$(4,4)$	Rank $(1,0)$	Holomorphic BF theory coupled to a holomorphic symplectic boson $T^*[-1]\mathrm{Map}(\mathbb{C}, (U/\!\!/\mathfrak{g})_{\mathrm{Dol}})$	1 (holomorphic)
	Rank $(1,1)$ (B)	Topological BF theory coupled to a holomorphic symplectic boson $T^*[-1]\mathrm{Map}(\mathbb{R}^2_{\mathrm{dR}}, U/\!\!/\mathfrak{g})$	2 (topological)
	Rank $(1,1)$ (A)	Perturbatively trivial (A-model) $T^*[-1]\mathrm{Map}(\mathbb{C}, (U/\!\!/\mathfrak{g})_{\mathrm{dR}})$	2 (topological)
$(2,2)$	Rank $(1,0)$	Holomorphic BF theory coupled to R matter $T^*[-1]\mathrm{Map}(\mathbb{C}, (R/\mathfrak{g})_{\mathrm{Dol}})$	1 (holomorphic)
	Rank $(1,1)$ (B)	Topological BF theory coupled to R matter $T^*[-1]\mathrm{Map}(\mathbb{R}^2_{\mathrm{dR}}, (R/\mathfrak{g}))$	2 (topological)
	Rank $(1,1)$ (A)	Perturbatively trivial (A-model) $T^*[-1]\mathrm{Map}(\mathbb{C}, (R/\mathfrak{g})_{\mathrm{dR}})$	2 (topological)
$(0,\mathcal{N}_-)$	Rank $(1,0)$	Holomorphic BF theory coupled to $\mathcal{N}_- - 2$ free fermions $T^*[-1]\mathrm{Map}(\mathbb{C}, (\mathfrak{g} \otimes \mathbb{C}^{\mathcal{N}_- - 2})/\mathfrak{g})$	1 (holomorphic)
$(0,4)$	Rank $(1,0)$	Holomorphic BF theory coupled to a holomorphic symplectic boson $T^*[-1]\mathrm{Map}(\mathbb{C}, U/\!\!/\mathfrak{g})$	1 (holomorphic)
$(0,2)$	Rank $(1,0)$	Holomorphic BF theory coupled to R matter $T^*[-1]\mathrm{Map}(\mathbb{C}, R/\mathfrak{g})$	1 (holomorphic)

Table 4: Twists of Supersymmetric Yang-Mills Theories in two dimensions with gauge group G. When $\mathcal{N} = (0,2)$ and $(2,2)$ the theory includes a chiral multiplet valued in a representation R. When $\mathcal{N} = (0,4)$ and $(4,4)$ the theory includes a hypermultiplet valued in a symplectic representation U. We can promote the supersymmetry to $\mathcal{N} = (8,8)$ when $U = T^*\mathfrak{g}$, but no new twists occur.

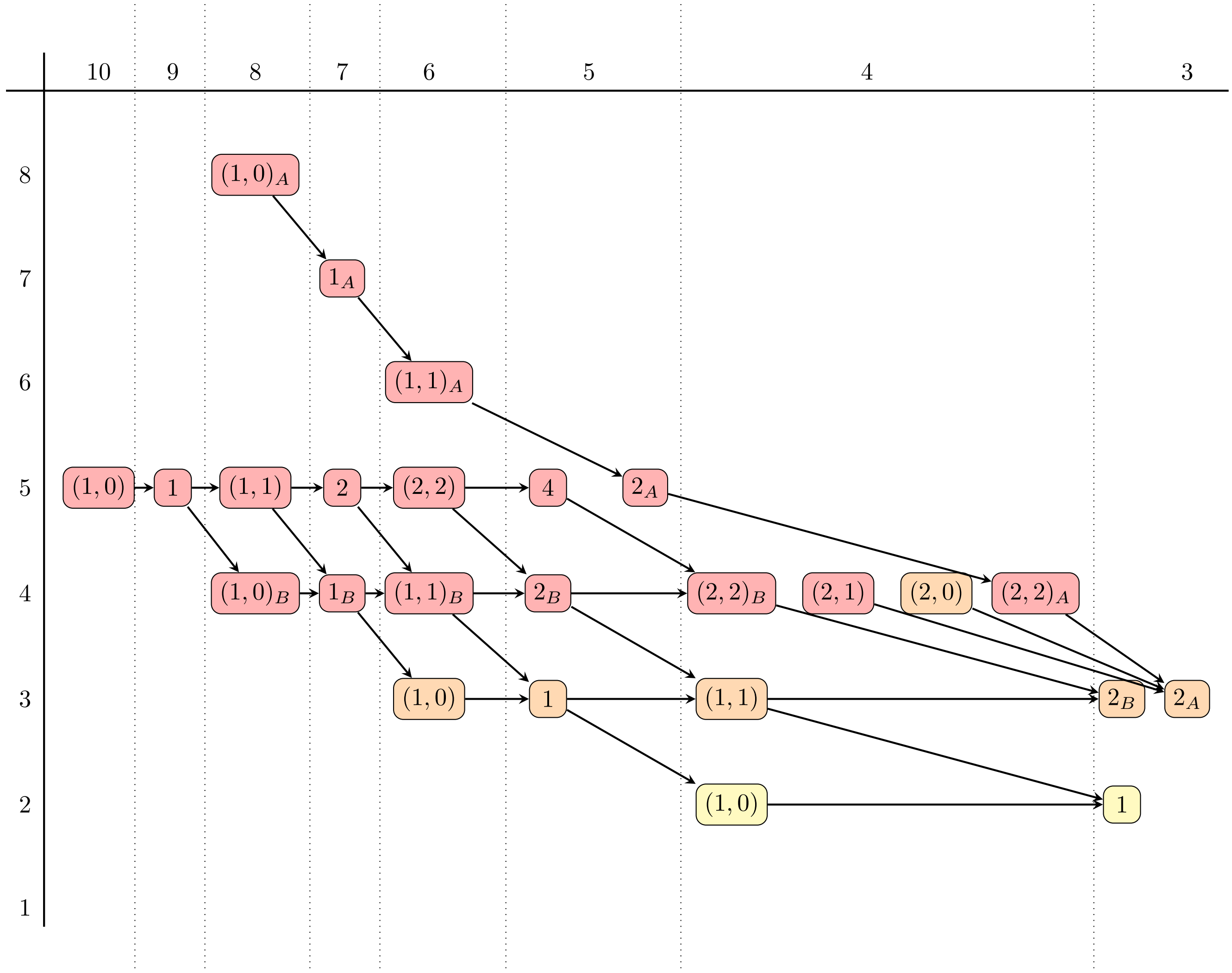

Figure 1: This figure shows the orbits of square-zero supercharges in each dimension, and how they relate to one another under dimensional reduction. The labels indicate each orbit: the number refers to the rank, and the subscript indicates the situations where the supercharges of a given rank split into multiplet orbits. Each column is labelled by a dimension, and each row by the number of invariant directions of the supercharge. Colours indicate the maximal supersymmetry algebra where the given supercharges live, so red indicates supercharges defined in algebras with 16 supercharges, orange those with 8 supercharges, and yellow those with 4 supercharges. There is an arrow whenever one twist dimensionally reduces to another twist one dimension lower.

Outline of the Paper

The remainder of the paper is divided into two parts. In Part I we set up the formalism that we will use when we study supersymmetric gauge theories and their twists. The first main ingredient is the Batalin-Vilkovisky formalism [**BatalinVilkovisky**] for classical field theory (Section 1), as developed by Costello and Gwilliam in [**CostelloBook**; **Book1**; **Book2**]. The other main ingredient is the systematic study of supersymmetry algebras and supersymmetric action functionals using normed division algebras (Section 2), following Baez and Huerta [**BaezHuerta**]. We use this formalism to prove in Section 3 that super Yang-Mills theories with matter in dimensions 10, 6, 4 and 3 are in fact supersymmetric, meaning that there is a well-defined L_∞ action of the supersymmetry algebra on the classical BV theory in question. We introduce the idea of dimensional reduction (Section 1.7) for classical field theories to show that supersymmetry action are well-defined in lower dimensions.

In Part II of the paper, we produce the classification of supersymmetric Yang-Mills theories in dimensions 2 to 10 systematically. We start with dimension 10 and work down by dimensional reduction. Each subsection is divided by the number of supersymmetries, and the orbits of square-zero supercharges by which we can twist. Twisted theories are characterized up to perturbative equivalence, including the residual Lorentz symmetry acting on each twisted theory.

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Part I

Supersymmetric Gauge Theory

1 The BV-BRST Formalism

In this section we will set up the homological formalism in which we study classical field theory: the BV-BRST formalism. Much of the material in this section is not original. We refer the reader to [**CostelloBook**; **Book2**] for more details on this perspective. We will conclude the section by describing a number of fundamental examples of classical field theories that are highly structured: mixed holomorphic-topological theories. We will also discuss the concept of *dimensional reduction* of a classical field theory on M along a fibration $M \to N$. We will use the idea of dimensional reduction to construct many of the supersymmetric field theories which we will consider in the next section.

1.1 Conventions

Throughout the paper we will frequently study obects, for instance vector bundles, equipped with a $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-grading. *Degree* will refer to the first (cohomological) grading and *odd* or *even* to the second (fermionic) grading. For an element x we denote by $|x| \in \mathbb{Z}/2\mathbb{Z}$ the total degree.

Given a vector bundle $E \to M$ we denote by $\mathcal{E}$ the topological vector space of smooth sections of E and by $\mathcal{E}_c$ the topological vector space of smooth compactly supported sections. We denote by $\mathcal{O}(\mathcal{E})$ (respectively $\mathcal{O}(\mathcal{E}_c)$) the completed algebra of symmetric functions on $\mathcal{E}$ (respectively $\mathcal{E}_c$). We denote by $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ the space of local functionals on $\mathcal{E}$ (see [**Book2**]). An element of $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ will be denoted symbolically by an expression of the form

$$\int_M f(\phi, \phi', \dots),$$

where f is a density on M depending on infinite jets of sections of E. Note, however, that the integral here is a formal symbol. The space of local functionals can be viewed as a subspace

$$\mathcal{O}_{\mathrm{loc}}(\mathcal{E}) \subset \mathcal{O}(\mathcal{E}_c)$$

where the integral symbol makes sense in earnest when applied to sections which are compactly supported. We denote by $\mathcal{O}^+_{\mathrm{loc}}(\mathcal{E}) \subset \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ the subspace of local functionals which are at least cubic.

Given two vector bundles E, F on M we can also make sense of the space of local functionals from E to F. By definition, this is

$$\mathrm{Fun}_{\mathrm{loc}}(\mathcal{E}, \mathcal{F}) = \prod_{n\geq 0} \mathrm{PolyDiff}(\mathcal{E}^{\times n}, \mathcal{F})_{S_n}$$

where $\mathrm{PolyDiff}(\mathcal{E}^{\times n}, \mathcal{F})$ denotes the space of polydifferential operators, and we take coinvariants for the obvious symmetric group action. When $\mathcal{F} = \mathcal{E}$, we refer to $\mathrm{Fun}_{\mathrm{loc}}(\mathcal{E}, \mathcal{E})$ as the space of local vector fields on E. There is a natural Lie bracket on $\mathrm{Fun}_{\mathrm{loc}}(\mathcal{E}, \mathcal{E})$ and a canonical action of this Lie algebra on local functionals.

1.2 Classical BV Theories

The classical BV (Batalin-Vilkovisky) formalism [**BatalinVilkovisky**] is a model for classical field theory from the Lagrangian perspective. In brief the classical BV formalism produces a local model for the critical locus of an action functional, but considered in the derived sense. That is, given a space $\mathcal{F}$ of fields and an action functional with derivative $\mathrm{d}S$, one considers not just the usual locus in $\mathcal{F}$ of fields with $\mathrm{d}S(\phi) = 0$, but the derived intersection $\mathrm{dCrit}(S) = \mathcal{F} \cap^h_{T^*\mathcal{F}} \Gamma_{\mathrm{d}S}$ of the zero section in $T^*\mathcal{F}$ with the graph of $\mathrm{d}S$. The formalism we describe below can be interpreted as an abstract formalism for modelling the tangent complex at a point to a derived critical locus $\mathrm{dCrit}(S)$.

Definition 1.1. A ***free BV theory*** on a manifold M is the data of:

- a finite rank $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-graded vector bundle $E \to M$ equipped with an even differential operator of cohomological degree $+1$
$$Q_{\mathrm{BV}} \colon \mathcal{E} \to \mathcal{E}[1]$$
such that (1): $Q_{\mathrm{BV}}^2 = 0$ and (2): the pair $(\mathcal{E}, Q_{\mathrm{BV}})$ is an elliptic complex;
- a map of bundles
$$\omega \colon E \otimes E \to \mathrm{Dens}_M[-1]$$
that is
  (1) fiberwise nondegenerate,
  (2) graded skew symmetric, and
  (3) satisfies $\int_M \omega(e_0, Q_{\mathrm{BV}} e_1) = (-1)^{|e_0|} \int_M \omega(Q_{\mathrm{BV}} e_0, e_1)$ where e_i are compactly supported sections of E .

We call $\mathcal{E}$ the ***space of BV fields***, and we call the complex (E, Q_{BV}) the ***classical BV complex***. The pairing ω equips the algebra of local functionals on E with a ***BV bracket*** (see [**CostelloBook**])

$$\{-,-\} \colon \mathcal{O}_{\mathrm{loc}}(\mathcal{E}) \times \mathcal{O}_{\mathrm{loc}}(\mathcal{E}) \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E}).$$

This bracket is of cohomological degree $+1$. This bracket is a graded version of the so-called Soloviev bracket [**Soloviev**] defined on the ∞-jets, as described in Section 4 of [**GetzlerBracket**].

We explain how to define the BV bracket in our context. First note that there is a linear map

$$\mathrm{d}_{\mathrm{dR}} \colon \mathcal{O}_{\mathrm{loc}}(\mathcal{E}) \to \mathrm{Fun}_{\mathrm{loc}}(\mathcal{E}, \mathcal{E}^!)$$

defined as follows. A local functional $F \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ can be written as an equivalence class of a sum of densities of the form

$$D_1(-) \cdots D_n(-)\Omega$$

where D_i is a differential operator $D_i \colon \mathcal{E} \to C^\infty_M$ and Ω is a density on M. Without loss of generality, suppose F is of this form. Then, we can view F as a functional in $\mathcal{O}(\mathcal{E}_c)$ by the assignment

$$\phi \mapsto \int_M D_1(\phi) \cdots D_n(\phi)\Omega$$

where ϕ denotes a compactly supported section. Define the symmetric multilinear map

$$\begin{array}{cccl} \mathrm{d}_{\mathrm{dR}}F & : & \mathcal{E}_c^{\times(n-1)} & \to \mathcal{E}^\vee \\ & & (\phi_1,\ldots,\phi_{n-1}) & \mapsto D_1(\phi_1)\cdots D_{n-1}(\phi_{n-1})D_n(-) + \{\text{symmetric terms}\}. \end{array}$$

Integrating by parts, we see that for any $(n-1)$-tuple $(\phi_1,\ldots,\phi_{n-1}) \in \mathcal{E}_c^{n-1}$ the linear functional $(\mathrm{d}_{\mathrm{dR}}F)(\phi_1,\ldots,\phi_{n-1})$ is an element of $\mathcal{E}^!$. This implies that $\mathrm{d}_{\mathrm{dR}}F \in \mathrm{Fun}_{\mathrm{loc}}(\mathcal{E},\mathcal{E}^!)$.

The non-degenerate pairing ω determines a bundle isomorphism $\omega\colon E \cong E^![-1]$ and hence an isomorphism of local functions

$$\omega\colon \mathrm{Fun}_{\mathrm{loc}}(\mathcal{E},\mathcal{E}^!) \cong \mathrm{Fun}_{\mathrm{loc}}(\mathcal{E},\mathcal{E}[1]).$$

We recognize the right hand side as the space of local vector fields placed in a shifted cohomological degree. In total, we see that a local functional F determines a local vector field by applying this isomorphism to $\mathrm{d}_{\mathrm{dR}}F$:

$$X_F := \omega \circ \mathrm{d}_{\mathrm{dR}}(F) \in \mathrm{Fun}_{\mathrm{loc}}(\mathcal{E},\mathcal{E}[1]).$$

This is the Hamiltonian vector field corresponding to F. Finally, the BV bracket between local functionals F, G is defined by

$$\{F,G\} = X_F(G).$$

The BV bracket enjoys the graded skew symmetry property

$$\{F,G\} = (-1)^{|F||G|}\{G,F\}$$

as well as the graded Jacobi identity. This bracket together wtih Q_{BV} endows $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1]$ with the structure of a dg Lie algebra. Since the space of local functionals is not an algebra, the bracket does not satisfy any type of Leibniz rule.

Now, we will include interactions in the BV picture.

Definition 1.2. A ***classical BV field theory*** (or simply, classical field theory) is a free BV theory (E, Q, ω) equipped with an even functional

$$I \in \mathcal{O}^+_{\mathrm{loc}}(\mathcal{E})$$

of cohomological degree zero satisfying the Maurer-Cartan equation

$$Q_{\mathrm{BV}}I + \frac{1}{2}\{I,I\} = 0.$$

Given a classical field theory $(E, Q_{\mathrm{BV}}, \omega, I)$ we denote by

$$S = \frac{1}{2}\int_M \omega(e, Q_{\mathrm{BV}}e) + I \in \mathcal{O}_{\mathrm{loc}}(E)$$

the BV action of the theory.

The local functional S satisfies the *classical master equation*

$$\{S,S\} = 0.$$

In fact, given a degree (-1) nondegenerate pairing ω on E, prescribing the data of a classical field theory (namely a pair (Q_{BV}, I) satisfying the Maurer-Cartan equation) is equivalent to prescribing a local functional $S \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ that is at least quadratic and satisfies the classical master equation. The operator Q_{BV} is the BV bracket with the quadratic part of the BV action S, and the cubic and higher terms in S coincide with I.

Because of this, we will sometimes refer to the triple (E, S, ω) instead of the quadruple$(E, Q_{\mathrm{BV}}, I, \omega)$ as the data of a classical field theory.

Remark 1.3. We will also consider $\mathbb{Z}/2\mathbb{Z}$***-graded classical field theories*** which are defined as before, but where E has only a single $\mathbb{Z}/2\mathbb{Z}$-grading and, correspondingly, Q_{BV} is simply an odd operator.

Remark 1.4. The data of a classical BV theory can be equivalently encoded in an elliptic L_∞ algebra $L = E[-1]$ equipped with a cyclic structure, i.e. a symplectic isomorphism $L \cong L^![-3]$ [**Book2**].

We will also sometimes consider $\mathbb{C}[t]$-families of classical field theories. These will be defined as follows.

Definition 1.5. A $\mathbb{C}[t]$-family of classical BV theories is a quadruple $(E, Q_{\mathrm{BV}}, \omega, I)$ as before, but where now the classical BV differential is t-dependent. In other words, it is given as a map

$$Q_{\mathrm{BV}} \colon \mathcal{E} \to \mathcal{E} \otimes \mathbb{C}[t][1],$$

where, for each $t_0 \in \mathbb{C}$, the quadruple $(E, Q^{t_0}_{\mathrm{BV}}, \omega, I)$

$$Q^{t_0}_{\mathrm{BV}} = \mathrm{ev}_{t_0} \circ Q_{\mathrm{BV}} \colon \mathcal{E} \to \mathcal{E} \otimes \mathbb{C}[t][1] \to \mathcal{E}[1]$$

defines a classical field theory.

Next, we formulate the notion of a morphism, and an equivalence, of classical BV theories.

Definition 1.6. A ***morphism*** $\Phi \colon (E, Q_{\mathrm{BV}}, \omega, I) \rightsquigarrow (E', Q'_{\mathrm{BV}}, \omega', I')$ of classical field theories over the same manifold M is a collection $\Phi = \sum_{n\geq 1}^{\infty} \Phi_n$ of poly-differential operators $\Phi_n \colon \mathrm{Sym}^n(E) \to E'$, that intertwine the differentials $Q_{\mathrm{BV}}, Q'_{\mathrm{BV}}$, the pairings ω, ω', and the interactions I, I'. A morphism is a ***perturbative equivalence*** if the map $\Phi_1 \colon (\mathcal{E}, Q_{\mathrm{BV}}) \to (\mathcal{E}', Q'_{\mathrm{BV}})$ is a quasi-isomorphism.

The interpretation of this definition is that Φ is a non-linear map between the bundles of BV fields, and Φ_n is its n^{th} Taylor coefficient.

Definition 1.7. A classical field theory is ***perturbatively trivial*** if it is perturbatively equivalent to the zero theory ($E = 0$).

We will now describe two primitive examples of equivalences of classical field theories which will be useful in simplifying our descriptions of twisted theories. First, we consider the process of integrating out an auxiliary field.

Proposition 1.8. Fix a volume form dvol_M on M. Suppose (E, ω, S) is a classical field theory, where $E \cong E_0 \oplus (\mathcal{O}_M \oplus \mathrm{Dens}_M[-1])$ with the symplectic pairing ω given by a sum of a symplectic pairing ω_0 on E_0 and the standard symplectic pairing on the second summand. Denote by ϕ a section of $\mathcal{O}_M$ and by ϕ^* a section of $\mathrm{Dens}_M[-1]$. Suppose the BV action is

$$S = S_0 + \frac{1}{2} \int \mathrm{dvol}_M (\phi^2 - 2\phi S_1),$$

where S_0 is a local functional independent of ϕ, ϕ^* and S_1 is a $\mathcal{O}_M$-valued polydifferential operator which is independent of ϕ. Then the theory (E, Q, ω, I) is perturbatively equivalent to the theory (E_0, ω_0, S') with the BV action $S' = S_0 - S_1^2/2$, where the equivalence sets $\phi = S_1$ and $\phi^* = 0$.

Remark 1.9. In terms of the classical BV complex, this proposition tells us that if a classical BV complex is of the form

$$\begin{tikzcd}
\cdots & \underline{0} & \underline{1} & \cdots \\
\cdots \arrow[r] & E_0^0 \arrow[r, "Q_0"] \arrow[dr] & E_0^1 \arrow[r] & \cdots \\
& \mathcal{O}_M \arrow[r, "\mathrm{dvol}"] \arrow[ur] & \mathrm{Dens}_M &
\end{tikzcd}$$

where the bottom map multiplies a function by the volume element, then we can replace it with a quasi-isomorphic cochain complex consisting of only the first line, provided we make a suitable modification of the classical action functional.

Proof. Concretely, suppose that the linear part of S_1 is given by an operator Q_1, and the interacting part of S_1 is given by a functional $I_1 = \sum_{n=1}^{\infty} I_1^{(n)}$. The desired equivalence $\Phi\colon (E,\omega,S) \to (E_0,\omega_0,S')$ is given by the natural projection $\Phi = \Phi_1\colon E \to E_0$. The quasi-inverse $\Psi\colon (E_0,\omega_0,S') \to (E,\omega,S)$ is defined as follows. First $\Psi_1(e) = (e, Q_1(e), 0) \in E$. For $n > 1$, define

$$\begin{aligned} \Psi_n\colon \operatorname{Sym}^n(E_0) &\to E \\ e_1 \otimes \cdots \otimes e_n &\mapsto (0, I_1^{(n)}(e_1,\ldots,e_n), 0). \end{aligned}$$

These Ψ_n manifestly intertwines the pairings ω and ω'. To see that they intertwine the action functionals, we observe that

$$\begin{aligned} S(\Psi(e)) &= S(e, S_1(e), 0) \\ &= S_0(e) + \frac{1}{2}\,\mathrm{dvol}_M \int (S_1(e)^2 - 2S_1(e)^2) \\ &= S_0(e) - \frac{1}{2} S_1(e)^2 \\ &= S'(e). \end{aligned}$$

□

We may also remove a trivial BRST doublet.

Proposition 1.10. Let (E_0,ω_0,S_0) be a classical BV theory and $F \to M$ a graded vector bundle. Consider the theory (E,ω,S) with underlying graded vector bundle

$$E = E_0 \oplus \big(F \oplus F^![-1]\big) \oplus \big(F^! \oplus F[-1]\big)$$

whose sections we denote by $e_0 + \phi + \phi^* + \psi + \psi^*$ according to the above decomposition. The shifted symplectic form ω is given by the sum of ω_0 plus the standard degree $+1$ pairings between $F, F^![-1]$ and $F^!, F[-1]$. Suppose further that the local functional

$$S = S_0 + \int \phi\psi^* - \int \phi I_\phi - \int \psi^* I_{\psi^*} - \int \phi^* I_{\phi^*} - \int \psi I_\psi$$

satisfies the classical master equation, where $I_\phi, I_{\psi^*}, I_{\phi^*}, I_\psi$ are polydifferential operators on fields valued in $F^!$, F, F, $F^!$ respectively, and which are independent of ϕ and ψ^*. Then, the classical BV theory (E,ω,S) is perturbatively equivalent to the BV theory (E_0,ω_0,S') where S' is given by setting $\phi = I_{\psi^*}, \phi^* = 0$ and $\psi^* = I_\phi, \psi = 0$ in the original action functional S.

Remark 1.11. For the classical BV theory (E,ω,S) as in the proposition, the linearized BV differential defines the following cochain complex of fields:

$$\begin{tikzcd}
\cdots & \underline{-1} & \underline{0} & \underline{1} & \underline{2} & \cdots \\
\cdots \arrow[r] & E_0^{-1} \arrow[r] \arrow[dr, dotted] \arrow[ddr, dotted] & E_0^0 \arrow[r] \arrow[dr, dotted] \arrow[ddr, dotted] & E_0^1 \arrow[r] & E_0^2 \arrow[r] & \cdots \\
& & F_\phi \arrow[ur, dotted] \arrow[dr] & F^!_{\phi*} \arrow[ur, dotted] & & \\
& & F^!_{\psi^*} \arrow[uur, dotted] \arrow[ur] & F_\psi \arrow[uur, dotted] & &
\end{tikzcd}$$

where the subscripts match the notation for the fields in the statement above [1]. The top line is the underlying cochain complex of the theory with fields E_0. The arrows $F_\phi \to F_\psi$ and $F^!_{\psi^*} \to F^!_{\phi^*}$ are given by the identity. The

[1] Note that we are writing F as if it is concentrated in a single cohomological degree, but the proposition applies for any graded vector bundle as in the statement of the proposition.

dotted arrows represent terms in the differential arising from $I_\phi, I_{\psi^*}, I_{\phi^*}, I_\psi$. The above proposition implies we can replace this cochain complex of fields with a quasi-isomorphic complex consisting of only the first line, provided we make a suitable modification of the classical action functional.

Proof. Concretely, we'll write $\sum_{n\geq 1} I_\phi^{(n)}$ and $\sum_{n\geq 1} I_{\psi^*}^{(n)}$ for the Taylor expansions of I_ϕ and I_{ψ^*} respectively. The desired equivalence $\Phi\colon (E,\omega,S) \to (E_0,\omega_0,S')$ is given by the natural projection $\Phi = \Phi_1\colon E \to E_0$. The quasi-inverse $\Psi\colon (E_0,\omega_0,S') \to (E,\omega,S)$ is defined as follows. The linear term is $\Psi_1(e) = (e, I_\phi^{(1)}(e), 0, 0, I_{\psi^*}^{(1)}(e))$, and for $n > 1$ we have

$$\Psi_n(e_1 \otimes \cdots \otimes e_n) = (0, I_\phi^{(n)}(e_1,\ldots,e_n), 0, I_{\psi^*}^{(n)}(e_1,\ldots,e_n)).$$

The maps Ψ_n manifestly intertwine the pairings on E_0 and E, since the image of Ψ_n lands in an isotropic summand of the $E_1 \oplus E_1^![-1] \oplus E_1^! \oplus E_1[-1]$ part of E. Also, by construction, the Ψ_n intertwine the action functionals, since

$$\begin{aligned} S(F(e)) &= S_0(e) + \frac{1}{2}\int_M \omega(I_{\psi^*}(e), I_\phi(e) - I_\phi(e)) + \omega(I_\phi(e), I_{\psi^*}(e) - I_{\psi^*}(e)) \\ &= S'(e). \end{aligned}$$

□

1.3 Symmetries in the Classical BV Formalism

In this section we define what it means for a (super) Lie algebra to act on a classical field theory (see also [**Book2**] for a related discussion). Let $(E, Q_{\mathrm{BV}}, \omega, I)$ be a classical field theory and let $\mathfrak{g}$ be a super Lie algebra. We will define $\mathfrak{g}$-equivariant local observables in the classical field theory by introducing $\mathfrak{g}$-valued background fields into our classical field theory, and extending the action functional to a functional that involves these background fields, but still satisfies the classical master equation. We begin by defining an appropriate version of the Chevalley-Eilenberg cochain complex.

Definition 1.12. The ***Chevalley-Eilenberg complex*** for the Lie algebra $\mathfrak{g}$, with coefficients in $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$, will be defined as follows. Consider the graded vector space

$$C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E})) = \bigoplus_n \mathrm{Hom}(\wedge^n \mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))[-n]$$

parametrizing multilinear maps $f\colon \mathfrak{g}^{\otimes n} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ which satisfy the antisymmetry property

$$f(x_1,\ldots,x_i,x_{i+1},\ldots,x_n) = (-1)^{|x_1||x_2|+1} f(x_1,\ldots,x_{i+1},x_i,\ldots,x_n)$$

where $x_j \in \mathfrak{g}$. The Chevalley-Eilenberg differential is given, following the sign conventions of [**SafronovCoisoInt**], by the formula

$$(\mathrm{d}_{\mathrm{CE}} f)(x_1,\ldots,x_n) = \sum_{i<j} (-1)^{|x_i|\sum_{p=1}^{i-1}|x_p| + |x_j|\sum_{p=1,p\neq i}^{j-1}|x_p| + i + j + |f|} f([x_i,x_j], x_1, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_n).$$

The complex is additionally equipped with a degree $+1$ BV bracket via the formula

$$\{f,g\}(x_1,\ldots,x_{k+l}) = \sum_{\sigma\in S_{k,l}} \mathrm{sgn}(\sigma)(-1)^{\epsilon+\epsilon_1}\{f(x_{\sigma(1)},\ldots,x_{\sigma(k)}), g(x_{\sigma(k+1)},\ldots,x_{\sigma(k+l)})\},$$

where $S_{k,l}$ is the set of (k,l)-shuffles, ϵ is the usual Koszul sign and

$$\epsilon_1 = |g|k + \sum_{i=1}^k |x_{\sigma(i)}|(l + |g|).$$

The operator Q_{BV} on $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ extends $C^\bullet(\mathfrak{g})$-linearly to an operator on $C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))$ by the rule

$$(Q_{\mathrm{BV}} f)(x_1, \ldots, x_n) = Q_{\mathrm{BV}} f(x_1, \ldots, x_n)$$

where $f\colon \mathfrak{g}^{\otimes n} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$. The differentials d_{CE} and Q_{BV} are compatible in the sense that $(\mathrm{d}_{\mathrm{CE}} + Q_{\mathrm{BV}})^2 = 0$ making $C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))$ into a cochain complex with total differential $\mathrm{d}_{\mathrm{CE}} + Q_{\mathrm{BV}}$. Via the BV bracket, the shift of this cochain complex $C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))[-1]$ is a dg Lie algebra. This shifted cotangent complex will model equivariant local observables in our classical field theory, but to finish defining the $\mathfrak{g}$ action we must define the equivariant version of the classical interaction. This is defined as follows.

Definition 1.13. Let $(E, Q_{\mathrm{BV}}, \omega, I)$ be a classical field theory. An ***action*** of a super Lie algebra $\mathfrak{g}$ on $(E, Q_{\mathrm{BV}}, \omega, I)$ is an element of cohomological degree zero

$$I_{\mathfrak{g}} = \sum_{k \geq 0} I_{\mathfrak{g}}^{(k)} \text{ in } C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E})),$$

where $I_{\mathfrak{g}}^{(k)}$ is a multilinear map $\mathfrak{g}^{\otimes k} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$, that satisfies the following three conditions:

(a) $I_{\mathfrak{g}}^{(0)} = I$.

(b) For each $k \geq 1$ and $x_1, \ldots, x_k \in \mathfrak{g}$ the local functional $I_{\mathfrak{g}}^{(k)}(x_1, \ldots, x_k)$ is at least quadratic in the fields.

(c) $I_{\mathfrak{g}}$ satisfies the Maurer–Cartan equation:

$$(\mathrm{d}_{\mathrm{CE}} + Q_{\mathrm{BV}}) I_{\mathfrak{g}} + \frac{1}{2}\{I_{\mathfrak{g}}, I_{\mathfrak{g}}\} = 0.$$

Remark 1.14. We have seen that a classical BV theory can also be presented in terms of a BV action $S \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ satisfying the classical master equation $\{S, S\} = 0$. One can also formulate actions of a Lie algebra on a classical theory in these terms. The data of an action of a Lie algebra $\mathfrak{g}$ on a classical field theory (E, ω, S) is equivalent to the choice of a local functional $\mathfrak{S}_{\mathfrak{g}} = \sum_k \mathfrak{S}_{\mathfrak{g}}^{(k)} \in C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))$, at least quadratic in the fields, such that $\mathfrak{S}_{\mathfrak{g}}^{(0)} = S$ and such that the classical master equation

$$\mathrm{d}_{\mathrm{CE}} \mathfrak{S}_{\mathfrak{g}} + \frac{1}{2}\{\mathfrak{S}_{\mathfrak{g}}, \mathfrak{S}_{\mathfrak{g}}\} = 0$$

is satisfied.

Remark 1.15. We have defined an action of a Lie algebra on a classical field theory in terms of a Noether current $I_{\mathfrak{g}}$. Such data gives rise to an L_∞ action of $\mathfrak{g}$ on the space of fields $\mathcal{E}$ in the following way. By the Maurer-Cartan equation, the operator $\mathrm{d}_{\mathrm{CE}} + Q_{\mathrm{BV}} + \{I_{\mathfrak{g}}, -\}$ defines a differential on the graded vector space $\mathcal{O}(\mathfrak{g}[1] \oplus \mathcal{E})$. By assumption that the Noether current is at least quadratic in the fields, we see that this differential defines a family of maps

$$\mathfrak{g}^{\otimes k} \otimes \mathcal{E}^{\otimes \ell} \to \mathcal{E}$$

combining to give $\mathcal{E}$ the structure of an L_∞-module for $\mathfrak{g}$.

We may also define actions of supergroups on classical field theories. The action of a supergroup G is more data than the action of a super Lie algebra $\mathfrak{g}$: it includes the infinitesimal action of the Lie algebra $\mathfrak{g}$, along with an action of G on the fields exponentiating this infinitesimal action. That is, we make the following definition.

Definition 1.16. Let $(E, Q_{\mathrm{BV}}, \omega, I)$ be a classical field theory, and let G be a supergroup acting on spacetime M. An ***action*** of G on $(E, Q_{\mathrm{BV}}, \omega, I)$ is given by the following data:

- An action of G on $\mathcal{E}$ compatible with the G-action on M.
- An (elliptic) action $I_{\mathfrak{g}}$ of its super Lie algebra $\mathfrak{g}$ with $I_{\mathfrak{g}}^{(k)} = 0$ for $k \geq 2$

These are required to satisfy the following conditions:

- The G-action on $\mathcal{E}$ preserves the symplectic pairing ω, the differential Q and the interaction term I.
- For every $x \in \mathfrak{g}$, the vector field $X_{I^{(1)}_{\mathfrak{g}}(x)}$ on $\mathcal{E}$ coincides with the infinitesimal action of $\mathfrak{g}$ on $\mathcal{E}$.

Remark 1.17. While we allow for L_∞ actions of Lie algebras, we will only consider strict actions of Lie groups in the present work.

1.4 From BRST to BV

We will now explain how to build classical BV theories from more traditional data: that of the *usual* fields of a classical field theory, together with the usual action functional and the action of gauge transformations. These data can be packaged into what's known as a BRST theory, where fermionic fields (referred to as ghosts) are introduced to generate the infinitesimal gauge transformations, in the following way.

Definition 1.18. A ***classical BRST theory*** on a manifold M consists of the following data:

- a $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-graded vector bundle F together with the structure of a local L_∞ algebra on the shift $F[-1]$.
- A local functional $S_{\mathrm{BRST}} \in \mathcal{O}_{\mathrm{loc}}(\mathcal{F})$ of polynomial degree ≥ 2.

Together, these data must satisfy the equation

$$Q_{\mathrm{BRST}} S_{\mathrm{BRST}} = 0,$$

where Q_{BRST} is the Chevalley–Eilenberg differential defined by the L_∞ structure on $F[-1]$.

We call $\mathcal{F}$ the ***space of BRST fields***.

Remark 1.19. In the most typical examples, the bundle F is concentrated in $\mathbb{Z}$-degrees -1 and 0. In this case, sections in degree 0 are thought of as physical fields, and ghosts – sections in degree -1 – are thought of as generators of the infinitesimal gauge symmetry. The action of gauge transformations on fields is then encoded by the Lie structure.

From a classical BRST theory $(\mathcal{F}, S_{\mathrm{BRST}})$, one can construct a classical BV theory as follows. Let $\{\ell_k\}_{k\geq 1}$ be the L_∞ structure maps underlying the local Lie algebra $F[-1]$.

First, we define the free BV theory. Split $S_{\mathrm{BRST}} = S^{\mathrm{free}}_{\mathrm{BRST}} + I_{\mathrm{BRST}}$, where $I_{\mathrm{BRST}} \in \mathcal{O}^+_{\mathrm{loc}}(\mathcal{F})$ and $S^{\mathrm{free}}_{\mathrm{BRST}}$ is a quadratic local functional which we may view as defining a map

$$S^{\mathrm{free}}_{\mathrm{BRST}} \colon F \to F^!.$$

The underlying bundle of the BV theory is

$$E = F \oplus F^![-1].$$

The BV pairing ω on E is defined in terms of the natural pairing between F and $F^!$. The differential of the free BV theory is

$$Q_{\mathrm{BV}} = \ell_1 + S^{free}_{\mathrm{BRST}}.$$

The interacting theory is constructed as follows. First, note that for $k \geq 2$ the L_∞ structure maps $\{\ell_k\}_{k\geq 2}$ on $\mathcal{F}$ pull back to multilinear maps on $\mathcal{E}$ via the obvious projection $p \colon \mathcal{E} \to \mathcal{F}$. These structure maps assemble into a local functional $I_F \in \mathcal{O}^+_{\mathrm{loc}}(\mathcal{E})$ defined by

$$I_F(e) = \sum_{k\geq 2} \frac{1}{(k+1)!} \int_M \omega_F(e, (p^*\ell_k)(e, \ldots, e))$$

which is linear along $\mathcal{F}^!$. Likewise, the BRST action I_{BRST} pulls back to $\mathcal{E}$, and we define the BV interaction as the sum

$$I_{\mathrm{BV}} = I_F + p^* I_{\mathrm{BRST}} \in \mathcal{O}^+_{\mathrm{loc}}(\mathcal{E}).$$

Lemma 1.20. Suppose (F, S_{BRST}) is a classical BRST theory such that $(\mathcal{E}, Q)$ defined above is an elliptic complex. Then $(E, Q_{\mathrm{BV}}, \omega, I)$ is a classical BV theory.

We refer to the classical BV theory $(E, Q_{\mathrm{BV}}, \omega, I)$ as *the BV theory associated to the BRST theory* $\mathcal{F}$. In the case where $S_{\mathrm{BRST}} = 0$ we refer to the associated BV theory as being of *cotangent type*, which we will denote by $T^*[-1]\mathcal{F}$.

Remark 1.21. In general, multiple BRST theories can give rise to the same BV theory. A BV theory $(E, Q_{\mathrm{BV}}, \omega, I)$ is of cotangent type as long as there is *some* F with $S_{\mathrm{BRST}} = 0$ producing the given theory using the construction above. Theories of cotangent type can still have interesting, non-trivial action functionals, encoded by the L_∞ structure on F.

If the fields of the classical BRST theory are denoted by ϕ, we denote their antifields in the classical BV theory by ϕ^*, so that

$$\{\phi(x), \phi^*(y)\} = \{\phi^*(y), \phi(x)\} = \delta(x - y).$$

(Brian: Looks like you're using a bracket we haven't defined. We've only defined brackets on local functionals, but I think when you write $\phi(x)$ you mean a point-like observable, which is far from local.) (Chris: I don't think we need to say this so precisely. I think it's enough to say that a homogeneous splitting $F = \bigoplus_i \Phi_i$ induces a splitting $F^! = \bigoplus_i \Phi_i^!$, and if we denote a general element of the summand Φ_i by ϕ then we will denote a general element of the summand $\Phi_i^!$ by ϕ^*.)

1.5 Local Formal Moduli Problems

It will be convenient for us to represent spaces of fields in classical field theories in terms of mapping spaces. In this section we explain their meaning, while in the rest of the paper it will be just a convenient notation.

Recall that a formal moduli problem is a functor from connective dg Artinian algebras $(R, \mathfrak{m})$ to simplicial sets which satisfies a derived version of Schlessinger's condition. We refer to [**DAGX**; **PridhamFMP**; **Toen**] for more details.

For instance, if $\mathfrak{g}$ is a complex L_∞ algebra, we have a formal moduli problem $B\mathfrak{g}$ defined by

$$(B\mathfrak{g})(R, \mathfrak{m}) = \mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m}),$$

where $\mathrm{MC}(\mathfrak{g} \otimes \mathfrak{m})$ is the simplicial set of Maurer–Cartan elements. The main result of [**DAGX**; **PridhamFMP**] is that the functor B defines an equivalence of ∞-categories between L_∞ algebras and formal moduli problems. The inverse functor sends a formal moduli problem $\mathcal{M}$ to the L_∞-algebra $T_{\mathcal{M},*}[-1]$, the shifted tangent complex of $\mathcal{M}$ at the basepoint. This important result will only serve as a motivation.

Let V be a $\mathfrak{g}$-representation. Then we may construct an L_∞ algebra

$$L_{V,\mathfrak{g}} = \mathfrak{g} \oplus V[-1]$$

with the only nontrivial brackets coming from the L_∞ brackets on $\mathfrak{g}$ and the action map of $\mathfrak{g}$ on V. We introduce the notation

$$V/\mathfrak{g} := BL_{V,\mathfrak{g}}.$$

Example 1.22. If $\mathfrak{g}$ is an L_∞ algebra, it has an adjoint representation $\mathfrak{g}$. We define the ***n*-shifted tangent bundle** of $B\mathfrak{g}$ to be

$$T[n]B\mathfrak{g} = \mathfrak{g}[n+1]/\mathfrak{g}.$$

Example 1.23. Suppose $\mathfrak{g}$ is an L_∞ algebra which is bounded as a complex and has finite-dimensional graded pieces. Then $\mathfrak{g}^*$ is a coadjoint representation of $\mathfrak{g}$. We define the ***n*-shifted cotangent bundle** of $B\mathfrak{g}$ to be

$$T^*[n]B\mathfrak{g} = \mathfrak{g}^*[n-1]/\mathfrak{g}.$$

Definition 1.24. Let $\mathfrak{g}$ be an L_∞ algebra. A ***$\mathbb{G}_m$-action*** on a formal moduli problem $B\mathfrak{g}$ is a weight grading $\mathfrak{g} = \bigoplus_{m\in\mathbb{Z}} \mathfrak{g}(m)$ compatible with the L_∞ structure.

Example 1.25. Suppose $\mathfrak{g}$ is an L_∞ algebra and V is a $\mathfrak{g}$-representation. Then $V/\mathfrak{g}$ carries a $\mathbb{G}_m$-action: the underlying L_∞ algebra $\mathfrak{g}\oplus V[-1]$ carries a grading where $\mathfrak{g}$ has weight 0 and $V[-1]$ has weight 1. For instance, $T[n]B\mathfrak{g}$ and $T^*[n]B\mathfrak{g}$ carry $\mathbb{G}_m$-actions.

Now suppose L is a local L_∞ algebra on a manifold M. For every open subset $U\subset M$ we have a formal moduli problem

$$(BL)(U) = BL(U),$$

i.e. $\mathcal{L}$ defines a presheaf BL of formal moduli problems on M. The following definition was introduced in [**Book2**].

Definition 1.26. A ***local formal moduli problem on*** M is a presheaf of formal moduli problems on M represented by a local L_∞ algebra.

Remark 1.27. In [**Book2**] an extra assumption of ellipticity is required for the local L_∞ algebras considered. It is only relevant for quantization which we do not consider in this paper, so for simplicity we will not require ellipticity (though in fact all examples we consider will end up being elliptic).

Example 1.28. Let X, Y be complex manifolds, M a smooth manifold and $B\mathfrak{g}$ a formal moduli problem represented by a complex L_∞ algebra $\mathfrak{g}$. Then we may define the following local formal moduli problem on $X\times Y\times M$. Let $\Omega_X^{0,\bullet}$ be the graded vector bundle of $(0,n)$-forms on X, $\Omega_Y^{\bullet,\bullet}$ be the graded vector bundle of (p,q)-forms on Y and $\Omega_M^\bullet$ a graded vector bundle of differential forms on M. We may then consider a graded vector bundle

$$L = \Omega_X^{0,\bullet}\otimes\Omega_Y^{\bullet,\bullet}\otimes\Omega_M^\bullet\otimes\mathfrak{g}$$

on $X\times Y\times M$. It carries a differential given by the sum $\overline{\partial}_X + \overline{\partial}_Y + \mathrm{d}_M + \mathrm{d}_{\mathfrak{g}}$. It also carries a local L_∞ structure which uses the L_∞ structure on $\mathfrak{g}$ and the wedge product of differential forms. We then define

$$\mathrm{Map}(X\times Y_{\mathrm{Dol}}\times M_{\mathrm{dR}}, B\mathfrak{g}) := BL.$$

Remark 1.29. A smooth complex algebraic variety X gives rise to derived stacks X_{Dol} and X_{dR} defined by Simpson [**Simpson**; **PTVV**]. So, given smooth complex algebraic varieties X, Y, M and a derived stack F we may consider the mapping stack

$$\mathrm{Map}(X\times Y_{\mathrm{Dol}}\times M_{\mathrm{dR}}, F).$$

If $F = BG$ is the classifying stack of a complex algebraic group, the mapping stack $\mathrm{Map}(X\times Y_{\mathrm{Dol}}\times M_{\mathrm{dR}}, BG)$ has a natural geometric interpretation: its closed points correspond to algebraic principal G-bundles P on $X\times Y\times M$, with a Higgs field on each slice $P|_{\{x\}\times Y\times\{m\}}$ and a flat connection on each slice $P|_{\{x\}\times\{y\}\times M}$. Example 1.28 is an analogous construction in the world of formal moduli problems, and can be thought of as modelling the formal neighborhood of the trivial bundle in this moduli stack.

Example 1.30. Consider $X, Y, M, \mathfrak{g}$ as in Example 1.28 and suppose E is a line bundle on $X\times Y\times M$, equipped with a holomorphic structure along $X\times Y$ and a flat connection along M. Moreover, assume $B\mathfrak{g}$ carries a $\mathbb{G}_m$-action. We then have a local L_∞ algebra

$$L = \bigoplus_m \Omega_X^{0,\bullet}\otimes\Omega_Y^{\bullet,\bullet}\otimes\Omega_M^\bullet\otimes\mathfrak{g}(m)\otimes E^{\otimes m}$$

on $X\times Y\times M$. We define

$$\mathrm{Sect}(X\times Y_{\mathrm{Dol}}\times M_{\mathrm{dR}}, B\mathfrak{g}\times_{\mathbb{G}_m} L) := BL.$$

As in Examples 1.22 and 1.23, we may define shifted tangent and cotangent bundles of a local formal moduli problem which give more examples.

Lemma 1.31. Consider $X, Y, M, \mathfrak{g}$ as in Example 1.28. The local formal moduli problem $T^*[n]\,\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g})$ is isomorphic to the local formal moduli problem

$$\mathrm{Sect}\left(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, T^*[n + \dim(X) + 2\dim(Y) + \dim(M)]B\mathfrak{g} \times_{\mathbb{G}_m} (K_X \otimes \mathrm{Dens}_M)\right),$$

where K_X is the canonical bundle of X and Dens_M is the line bundle of densities on M.

Proof. The claim follows from the following isomorphisms of graded vector bundles:

$$\begin{aligned}
\left(\Omega_X^{0,\bullet}\right)^! &\cong \Omega_X^{0,\bullet} \otimes K_X[\dim(X)] \\
\left(\Omega_Y^{\bullet,\bullet}\right)^! &\cong \Omega_Y^{\bullet,\bullet}[2\dim(Y)] \\
\left(\Omega_M^{\bullet}\right)^! &\cong \Omega_M^{\bullet} \otimes \mathrm{Dens}_M[\dim(M)].
\end{aligned}$$

□

Given a local formal moduli problem, we may talk about shifted symplectic structures [**PTVV**] on it. In this paper will only be interested in a strict notion as follows.

Definition 1.32. Let $\mathcal{M}$ be a local formal moduli problem on M represented by a local L_∞ algebra L. A ***strict*** n***-shifted symplectic structure*** on $\mathcal{M}$ is a pairing $\omega\colon L \otimes L \to \mathrm{Dens}_M[n-2]$ satisfying the following conditions:

1. It is fiberwise nondegenerate.
2. It is graded skew symmetric.
3. The pairing $\mathcal{L}_c \otimes \mathcal{L}_c \to \mathbb{C}$ defined by
$$\alpha \otimes \beta \mapsto \int_M \omega(\alpha \otimes \beta)$$
is an invariant pairing on the L_∞ algebra $\mathcal{L}_c$.

Remark 1.33. A classical BV field theory as defined in Section 1.2 is equivalent to the data of a local formal moduli problem on the spacetime manifold M equipped with a strict (-1)-shifted symplectic structure.

1.6 Examples of Classical Field Theories

In this section we give some examples of classical field theories we will use in our classification of twisted supersymmetric field theories. All theories we consider in this section are $\mathbb{Z}$-graded, i.e. the space E of fields will be completely even with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading.

1.6.1 Generalized BF Theory

Our first example will generalize the fundamental example of *BF theory* to a not entirely topological context. Ordinarily, BF theory describes the classical BRST theory on a d-manifold M with fields given by a G-gauge field A and a $\mathfrak{g}$-valued $(d-2)$-form B, with action functional

$$S(A, B) = \int_M \langle B \wedge F_A \rangle.$$

This theory is, in fact, of cotangent type, where the base of the cotangent includes A and its antifield, and the fiber includes B and its antifield. This basic setup can be generalized to a setting where M need not be entirely topological, and where $\mathfrak{g}$ may be a more general L_∞ algebra, in the following way.

Definition 1.34. Let X and Y be complex manifolds and let M be a smooth oriented manifold. Fix a complex L_∞ algebra $\mathfrak{g}$. The ***generalized BF theory*** is the BV theory associated to the following BRST theory:

- The spacetime is the smooth manifold $X \times Y \times M$.
- The bundle of BRST fields is the $\mathbb{Z}$-graded bundle $F = \Omega^{0,\bullet}_X \otimes \Omega^{\bullet,\bullet}_Y \otimes \Omega^\bullet_M \otimes \mathfrak{g}[1]$. The shift $F[-1]$ is equipped with a natural local L_∞ algebra structure from $\mathfrak{g}$.
- The BRST action is $S_{\mathrm{BRST}} = 0$.

We denote the space of BV fields by $T^*[-1]\,\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g})$.

Let us unpack the definition. Let $d = \dim_{\mathbb{C}}(X) + 2\dim_{\mathbb{C}}(Y) + \dim(M)$. Then the bundle of BV fields is

$$E = \Omega^{0,\bullet}_X \otimes \Omega^{\bullet,\bullet}_Y \otimes \Omega^\bullet_M \otimes \mathfrak{g}[1] \oplus \Omega^{\dim(X),\bullet}_X \otimes \Omega^{\bullet,\bullet}_Y \otimes \Omega^\bullet_M \otimes \mathfrak{g}^*[d-2],$$

where we denote the two fields by A and B. The BV action is

$$S = \int_{X\times Y\times M} \langle B \wedge (\overline{\partial}_X + \overline{\partial}_Y + \mathrm{d}_{\mathrm{dR},M})A\rangle + \sum_{k\geq 1} \frac{1}{k!}\int_{X\times Y\times M} \langle B \wedge \ell_k(A,\ldots,A)\rangle,$$

where $\langle -,-\rangle$ is the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ and where ℓ_k denote the k^{th} component of the L_∞ structure on $\mathfrak{g}$.

Example 1.35. For $X = Y = \mathrm{pt}$ and $\mathfrak{g}$ an ordinary Lie algebra we recover the usual topological BF theory with the BV action

$$S = \int_M \left\langle B \wedge \left(\mathrm{d}_{\mathrm{dR}}A + \frac{1}{2}[A\wedge A]\right)\right\rangle.$$

We will see many BF theories as the output when we twist supersymmetric gauge theories. In fact, a special case of the definition above also arises when twisting theories of matter. We will refer to as this as a generalized $\beta\gamma$ system, where the definition will extend that of the usual 2d $\beta\gamma$ system.

Definition 1.36. Let X and Y be complex manifolds and let M be a smooth manifold. Fix a complex vector space V. The ***generalized $\beta\gamma$ system*** valued in V is the BV theory associated to the following BRST theory:

- The spacetime is the smooth manifold $X \times Y \times M$.
- The bundle of BRST fields is the $\mathbb{Z}$-graded bundle $F = \Omega^{0,\bullet}_X \otimes \Omega^{\bullet,\bullet}_Y \otimes \Omega^\bullet_M \otimes V$. $F[-1]$ is equipped with an L_∞ structure with $\ell_k = 0$ for $k \geq 2$.
- The BRST action is $S_{\mathrm{BRST}} = 0$.

Remark 1.37. This is indeed a special case of generalized BF theory: the generalized $\beta\gamma$ system appears as generalized BF theory with the L_∞ algebra given by $\mathfrak{g} = V[-1]$, with trivial L_∞ structure.

We may also couple a $\beta\gamma$ system to a more general generalized BF theory as follows.

Example 1.38. Let X, Y, M be as before. Suppose $\mathfrak{g}$ is a dg Lie algebra and V is a dg representation of $\mathfrak{g}$. Consider the dg Lie algebra

$$\mathcal{L} = \mathfrak{g} \oplus V[-1]$$

with the only nontrivial brackets $\mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ given by the Lie bracket on $\mathfrak{g}$ and $\mathfrak{g} \otimes V \to V$ given by the $\mathfrak{g}$-action on V. The space of fields in the corresponding generalized BF theory will be denoted using the formal moduli problem notation as

$$\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, T^*[d](V/\mathfrak{g})) := \mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, T^*[d]\mathcal{L}),$$

where $d = \dim_{\mathbb{C}}(X) + 2\dim_{\mathbb{C}}(Y) + \dim(M) - 1$.

The following is obvious from the definition.

Proposition 1.39. Let $\mathfrak{g}$ be a dg Lie algebra, and consider the generalized BF theory on $\mathbb{R}^{2n_1+2n_2+n_3}$ for Lie algebra $\mathfrak{g}$ with the space of BV fields

$$\mathcal{E} = \mathrm{Map}(\mathbb{C}^{n_1} \times (\mathbb{C}^{n_2})_{\mathrm{Dol}} \times (\mathbb{R}^{n_3})_{\mathrm{dR}}, T^*[d]B\mathfrak{g}).$$

Then it carries an action of $\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{SO}(n_3)$ given by the pullback action on differential forms on $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{R}^{n_3}$.

Remark 1.40. In fact, the $\mathrm{SO}(n_3)$-action given by the previous proposition extends to a homotopically trivial action in the sense of [**ElliottSafronov**].

1.6.2 Generalized Chern–Simons Theory

The next class of examples of classical BV theories will be generalizations of Chern-Simons theory. Unlike the example of the generalized BF theory, these theories are not generally of cotangent type.

Definition 1.41. Let X and Y be complex manifolds and let M be a smooth oriented manifold. Fix an L_∞ algebra $\mathfrak{g}$. We assume that X is equipped with a holomorphic volume form $\Omega_X \in \Omega^{\dim(X),0}(X)$ and that $\mathfrak{g}$ is equipped with a nondegenerate invariant symmetric pairing $\langle -, - \rangle \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}[\dim_{\mathbb{C}}(X) + 2\dim_{\mathbb{C}}(Y) + \dim(M) - 3]$. The ***generalized Chern–Simons theory*** is the following classical BV theory:

- The spacetime is the smooth manifold $X \times Y \times M$.
- The bundle of BV fields is the $\mathbb{Z}$-graded bundle $E = \Omega_X^{0,\bullet} \otimes \Omega_Y^{\bullet,\bullet} \otimes \Omega_M^\bullet \otimes \mathfrak{g}[1]$.
- $Q_{\mathrm{BV}} = \overline{\partial}_X + \overline{\partial}_Y + \mathrm{d}_{\mathrm{dR},M} + \ell_1$.
- The pairing $\omega \colon E \otimes E \to \mathrm{Dens}_M[-1]$ is given by the combination of the wedge product of differential forms, wedging with Ω_X and the pairing $\langle -, - \rangle$ on $\mathfrak{g}$.
- The interaction term is
$$I = \sum_{k \geq 2} \frac{1}{(k+1)!} \int_{X \times Y \times M} \Omega_X \wedge \langle A \wedge \ell_k(A, \ldots, A) \rangle.$$

We denote the space of BV fields by $\mathcal{E} = \mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g})$.

We may also consider a $\mathbb{Z}/2\mathbb{Z}$-graded version of the above theory where $\mathfrak{g}$ is merely $\mathbb{Z}/2\mathbb{Z}$-graded.

Example 1.42. For $X = Y = \mathrm{pt}$, M a 3-manifold and $\mathfrak{g}$ an ordinary Lie algebra we recover the usual 3-dimensional Chern-Simons theory with the BV action

$$S = \int_M \left(\frac{1}{2} \langle A \wedge \mathrm{d}_{\mathrm{dR}} A \rangle + \frac{1}{6} \langle A \wedge [A \wedge A] \rangle \right).$$

More generally, if $X = Y = \mathrm{pt}$ and M is any d-dimensional manifold where d is odd, we recover d-dimensional Chern-Simons theory. This has the same BV action, where now A is a (not necessarily homogeneous) differential form on M. If d is not 3 this theory is only $\mathbb{Z}/2\mathbb{Z}$-graded.

Example 1.43. For $Y = M = \mathrm{pt}$, X a Calabi-Yau 3-fold and $\mathfrak{g}$ an ordinary Lie algebra we recover the holomorphic Chern–Simons theory with the BV action

$$S = \int_X \Omega_X \wedge \left(\frac{1}{2} \langle A \wedge \overline{\partial} A \rangle + \frac{1}{6} \langle A \wedge [A \wedge A] \rangle \right).$$

As in the previous example, this still makes sense if X is a Calabi-Yau d-fold with d odd, as a $\mathbb{Z}/2\mathbb{Z}$-graded theory.

Example 1.44. If $\mathfrak{h}$ is an L_∞ algebra, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*[d-3]$ carries a natural L_∞ structure given by combining the original L_∞ structure on the first term and the coadjoint action of the first term on the second term. The L_∞ algebra $\mathfrak{g}$ carries a natural symmetric pairing of degree $d-3$ given by the obvious pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$. Generalized Chern–Simons theory for $\mathfrak{g}$ in this case recovers the generalized BF theory from Definition 1.34.

Example 1.45. Let X, Y, M be as before and denote $d = \dim_{\mathbb{C}}(X) + 2\dim_{\mathbb{C}}(Y) + \dim(M)$. Suppose $\mathfrak{g}$ is a dg Lie algebra and V is a $\mathfrak{g}$-representation equipped with a $(d-1)$-shifted symplectic structure $V \otimes V \to \mathbb{C}[d-1]$. Consider the dg Lie algebra

$$\mathcal{L} = \mathfrak{g} \oplus V[-1] \oplus \mathfrak{g}^*[d-3]$$

with the brackets $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ given by the Lie bracket on $\mathfrak{g}$, $\mathfrak{g} \otimes V \to V$ given by the $\mathfrak{g}$-action on V, $\mathfrak{g} \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ given by the coadjoint action and $\mu \colon V \otimes V \to \mathfrak{g}^*[d-1]$ defined by $(\mu(v,w), x)_{\mathfrak{g}} = ([x,v], w)_V$. The dg Lie algebra $\mathcal{L}$ carries nondegenerate invariant symmetric pairing of cohomological degree $d-3$ given by pairing $\mathfrak{g}$ and $\mathfrak{g}^*$ and pairing V with itself. The space of fields in the corresponding Chern–Simons theory will be denoted by

$$\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, V /\!\!/ \mathfrak{g}) := \mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathcal{L}).$$

We will also sometimes use the following variant of the generalized Chern–Simons theory (see also [**GinzburgRozenblyum**]).

Definition 1.46. Let m be a positive integer. Suppose X is a complex manifold equipped with an m-th root $K_X^{1/m}$ of the canonical bundle, $Y, M, \mathfrak{g}$ are as before and $\mathfrak{g}$ is equipped with a weight grading $\mathfrak{g} = \oplus_n \mathfrak{g}(n)$ for which the symmetric pairing $\langle -, - \rangle$ on $\mathfrak{g}$ has weight m. We may consider a version of the generalized Chern–Simons theory with the bundle of BV fields

$$E = \bigoplus_n \Omega_X^{0,\bullet} \otimes \Omega_Y^{\bullet,\bullet} \otimes \Omega_M^\bullet \otimes \mathfrak{g}(n) \otimes K_X^{n/m}[1]$$

and the symplectic pairing ω and the action functional as before where we do not use the holomorphic volume form on X.

We denote the space of fields by $\mathcal{E} = \mathrm{Sect}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g} \times_{\mathbb{G}_m} K_X^{1/m})$.

Example 1.47. Consider the setting of Example 1.45. The dg Lie algebra $\mathcal{L}$ carries a weight grading, where we consider $\mathfrak{g}$ in weight 0, V in weight 1 and $\mathfrak{g}^*$ in weight 2. With respect to this weight grading the bilinear pairing on $\mathfrak{g}$ has weight 2, so we may define the corresponding generalized Chern–Simons theory with the space of fields

$$\mathrm{Sect}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, (U \otimes K_X^{1/2}) /\!\!/ \mathfrak{g}).$$

As with generalized BF theory, generalized Chern–Simons theory carries a natural rotation action by linear automorphism groups of spacetime.

Proposition 1.48. Suppose $\mathfrak{g}$ is a dg Lie algebra equipped with a nondegenerate invariant symmetric pairing of degree $n_1 + 2n_2 + n_3 - 3$. Consider the generalized Chern–Simons theory on $\mathbb{R}^{2n_1+2n_2+n_3}$ with the space of BV fields

$$\mathcal{E} = \mathrm{Map}(\mathbb{C}^{n_1} \times (\mathbb{C}^{n_2})_{\mathrm{Dol}} \times (\mathbb{R}^{n_3})_{\mathrm{dR}}, B\mathfrak{g}).$$

Then it carries an action of $\mathrm{SU}(n_1) \times \mathrm{U}(n_2) \times \mathrm{SO}(n_3)$ given by the pullback action on differential forms on $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{R}^{n_3}$.

We may slightly enhance the previous proposition if we are in the setting of Example 1.45 and we choose a square root of the canonical bundle. We define the unitary metalinear group to be

$$\mathrm{MU}(n) = \mathrm{U}(n) \times_{\mathrm{U}(1)} \mathrm{U}(1),$$

where $\mathrm{U}(n) \to \mathrm{U}(1)$ is the determinant map and $\mathrm{U}(1) \to \mathrm{U}(1)$ is the map $z \mapsto z^2$. We denote by $\det^{1/2} \colon \mathrm{MU}(n) \to \mathrm{U}(1)$ the projection on the second factor; this may be thought of as a square root of the determinant representation of $\mathrm{U}(n)$. The natural $\mathrm{U}(n)$-action on $\mathbb{C}^n$ lifts to an $\mathrm{MU}(n)$-action on the bundle of half-densities $K_{\mathbb{C}^n}^{1/2} \to \mathbb{C}^n$.

Proposition 1.49. Suppose $\mathfrak{g}$ is a dg Lie algebra and V a $\mathfrak{g}$-representation equipped with a $(n_1+2n_2+n_3-1)$-shifted symplectic structure. Consider the generalized Chern–Simons theory on $\mathbb{R}^{2n_1+2n_2+n_3}$ with the space of BV fields

$$\mathcal{E} = \mathrm{Map}(\mathbb{C}^{n_1} \times (\mathbb{C}^{n_2})_{\mathrm{Dol}} \times (\mathbb{R}^{n_3})_{\mathrm{dR}}, (V \otimes K^{1/2}_{\mathbb{C}^{n_1}}) /\!\!/ \mathfrak{g}).$$

Then it carries an action of $\mathrm{MU}(n_1) \times \mathrm{U}(n_2) \times \mathrm{SO}(n_3)$ given by the pullback action on differential forms on $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{R}^{n_3}$.

Example 1.50. There is a special case of this, which one might label the "holomorphic symplectic boson" [**SWchar**]. This is the case when $n_1 = 1$, $n_2 = n_3 = 0$, so V is a $\mathfrak{g}$ representation with an ordinary symplectic structure. If $V = T^*R$, we are considering the classical cotangent field theory whose classical solutions to the equations of motion model the cotangent bundle to G-bundles on a Riemann surface with a section of the associated bundle to R.

1.6.3 Generalized Hodge Theory

Generalized BF theories can be naturally deformed to theories which are perturbatively trivial, but which arise as shadows of non-trivial non-perturbative theories. These will often appear as topological twists of supersymmetric field theories, the most famous example being the 2d A-model. By a deformation we will mean a $\mathbb{C}[t]$-family of classical BV theories which reduce to the given theory at $t = 0$.

Given an L_∞ algebra $\mathfrak{g}$ we denote by $\mathfrak{g}_{\mathrm{Hod}}$ the $\mathbb{C}[t]$-linear L_∞ algebra

$$\mathfrak{g}_{\mathrm{Hod}} = \mathbb{C}[t] \otimes (\mathfrak{g} \oplus \mathfrak{g}[1])$$

with the L_∞ brackets coming from the L_∞ brackets on $\mathfrak{g}$ in the first term, where we consider $\mathfrak{g}[1]$ as the adjoint representation of $\mathfrak{g}$. The differential is given by the original differential on $\mathfrak{g}$ plus the term $t\,\mathrm{id}$ from the second summand to the first summand. The terminology here comes from Simpson's Hodge stack [**Simpson**]: if X is a smooth scheme, or more generally a derived Artin stack, one can define a derived stack X_{Hod} over $\mathbb{A}^1$, where the fiber at $0 \in \mathbb{A}^1$ is the Dolbeault stack X_{Dol} of X, essentially the 1-shifted tangent bundle, and the fiber at a non-zero point is equivalent to the de Rham stack X_{dR} of X, which has contractible tangent complex.

If $\mathfrak{g}$ carries a nondegenerate invariant symmetric pairing of degree d, so does $\mathfrak{g}_{\mathrm{Hod}}$.

Definition 1.51. Let X and Y be complex manifolds and M a smooth oriented manifold. Fix an L_∞ algebra $\mathfrak{g}$. We assume X is equipped with a holomorphic volume form $\Omega_X \in \Omega^{\dim(X),0}(X)$ and $\mathfrak{g}$ is equipped with a nondegenerate invariant symmetric pairing $\langle -,- \rangle \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}[\dim_{\mathbb{C}}(X) + 2\dim_{\mathbb{C}}(Y) + \dim(M) - 3]$. The ***generalized Hodge theory*** is the $\mathbb{C}[t]$-family of classical BV theories, as in Definition 1.5, given by the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$.

Proposition 1.52. The $t = 0$ specialization of the generalized Hodge theory with the space of fields $\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$ is isomorphic to the generalized BF theory with the space of fields $T^*[-1]\,\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g})$.

The specialization of the generalized Hodge theory at $t \neq 0$ is perturbatively trivial.

Proof. At $t = 0$ we get

$$\mathfrak{g}_{\mathrm{Hod}}|_{t=0} \cong \mathfrak{g} \oplus \mathfrak{g}[1] \cong \mathfrak{g} \oplus \mathfrak{g}^*[d-1],$$

where we use the symmetric bilinear pairing on $\mathfrak{g}$ in the second isomorphism. The first claim then follows from Example 1.44.

At $t \neq 0$ the L_∞ algebra $\mathfrak{g}_{\mathrm{Hod}}$ becomes acyclic, which proves the second claim. □

1.7 Dimensional Reduction

In this section we formulate the procedure of dimensional reduction of a classical field theory. Fix a submersion $p\colon M \to N$ equipped with a fiberwise volume form, i.e. an isomorphism $p^*\mathrm{Dens}_N \cong \mathrm{Dens}_M$. The idea is that the *dimensional reduction* of a classical field theory on M along the submersion p is the theory obtained by restricting to those fields which are constant along the fibers of p. We will begin with an abstract definition of dimensional reduction, then prove that if $M = N \times \mathbb{R}^k$, and we consider field theories which are translation invariant along the fiber, then this procedure is well-defined.

Definition 1.53. We say that a classical field theory $(E_N, Q_N, \omega_N, I_N)$ on N is a ***dimensional reduction*** along p of the classical field theory $(E_M, Q_M, \omega_M, I_M)$ on a manifold M if one is given the data of a linear isomorphism $p^*E_N \cong E_M$ of the bundles of BV fields satisfying the following conditions:

- The diagram

$$\begin{tikzcd} p^*E_N \otimes p^*E_N \arrow[r, "\omega_N"] \arrow[d, "\sim"] & p^*\mathrm{Dens}_N[-1] \arrow[d, "\sim"] \\ E_M \otimes E_M \arrow[r, "\omega_M"] & \mathrm{Dens}_M[-1] \end{tikzcd}$$

  is commutative.

- The diagram

$$\begin{tikzcd} \mathcal{E}_N \arrow[r, "Q_N"] \arrow[d, "p^*"] & \mathcal{E}_N[1] \arrow[d, "p^*"] \\ \mathcal{E}_M \arrow[r, "Q_M"] & \mathcal{E}_M[1] \end{tikzcd}$$

  is commutative.

- Under the map $p^*\colon \mathcal{E}_N \to \mathcal{E}_M$ we have $p^*I_M = I_N$.

We have an obvious notion of isomorphisms of dimensional reductions: these are linear isomorphisms of classical field theories on N which are compatible with the isomorphisms $p^*E_N \cong E_M$. Thus, the collection of dimensional reductions of a given classical field theory on M forms a groupoid. In fact, when dimensional reduction makes sense, this groupoid is always contractible.

Proposition 1.54. Suppose $(E_M, Q_M, \omega_M, I_M)$ is a classical field theory on M and $p\colon M \to N$ is a homotopy equivalence. Then the groupoid of dimensional reductions of $(E_M, Q_M, \omega_M, I_M)$ is either contractible or empty.

Suppose $M = N \times \mathbb{R}$ and choose a translation-invariant density along the $\mathbb{R}$ direction. If the original classical field theory is translation-invariant along the $\mathbb{R}$ direction, dimensional reductions exist.

Proof. **Uniqueness**. We begin by showing that any two dimensional reductions are isomorphic and moreover that such an isomorphism is unique if it exists. Since $p\colon M \to N$ is a homotopy equivalence, the functor p^* establishes an isomorphism between the category of graded vector bundles on N and on M. In a similar way, p^* establishes an equivalence between the category of graded vector bundles E_N on N equipped with a nondegenerate pairing $E_N \otimes E_N \to \mathrm{Dens}_N[-1]$ and a similar category for M.

Since $\mathcal{E}_N \to \mathcal{E}_M$ is injective, the diagram

$$\begin{tikzcd} \mathcal{E}_N \arrow[r, "Q_N"] \arrow[d, "p^*"] & \mathcal{E}_N[1] \arrow[d, "p^*"] \\ \mathcal{E}_M \arrow[r, "Q_M"] & \mathcal{E}_M[1] \end{tikzcd}$$

uniquely determines Q_N from Q_M. Moreover, the condition $p^* I_M = I_N$ uniquely determines N.

Existence. Now suppose $(E_M, Q_M, \omega_M, I_M)$ is translation-invariant along the $\mathbb{R}$ direction. Translation invariance provides the descent datum to construct the bundle of fields E_N on N equipped with a nondegenerate pairing ω_N. Moreover, it shows that the differential Q_M preserves the subspace $\mathcal{E}_N \hookrightarrow \mathcal{E}_M$. The restriction of I_M under the same embedding is independent of the $\mathbb{R}$ factor by translation invariance, so $I_N = p^* I_M$ is again a local functional. □

Remark 1.55. Therefore, it makes sense to talk about "the" dimensional reduction of a classical field theory along the projection $p\colon N \times \mathbb{R} \to N$: there exists a dimensional reduction which is unique up to a canonical isomorphism.

We will now describe dimensional reductions of some of the previously discussed BV theories. Throughout this section we let X and Y be complex manifolds and M be a smooth manifold. We focus on BV theories described as formal mapping spaces whose sources are formal manifolds of the form

$$X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}. \tag{1}$$

We first consider the case in which M is of the form $M' \times \mathbb{R}$ and we reduce along the projection

$$p_{\mathrm{dR}}\colon X \times Y \times (M' \times \mathbb{R}) \to X \times Y \times M'.$$

In this case, we will only need to know the dimensional reduction of generalized Chern-Simons theory.

Proposition 1.56. Fix an L_∞ algebra $\mathfrak{g}$ equipped with a nondegenerate invariant pairing as in Definition 1.41 and consider the corresponding generalized Chern–Simons theory

$$\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times (M' \times \mathbb{R})_{\mathrm{dR}}, B\mathfrak{g}).$$

Its dimensional reduction along the projection p_{dR} is equivalent to the generalized BF theory

$$T^*[-1]\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M'_{\mathrm{dR}}, B\mathfrak{g}).$$

Proof. To simplify the notation in the proof, we assume $X, Y, M' = \mathrm{pt}$, though the argument in the general case is identical. Then $\mathfrak{g}$ carries a (-2)-shifted pairing $\langle -, - \rangle$. In particular, the generalized BF theory

$$T^*[-1]\,\mathrm{Map}(\mathrm{pt}, B\mathfrak{g}) = T^*[-1]B\mathfrak{g}$$

has the bundle of BV fields $\mathfrak{g}[1] \oplus \mathfrak{g}^*[-2]$. We may identify it with $\mathfrak{g}[1] \oplus \mathfrak{g}$, where the pairing ω_N pairs the two factors using $\langle -, - \rangle$.

We may identify $p^*(\mathfrak{g}[1] \oplus \mathfrak{g}) \cong \Omega^\bullet_{\mathbb{R}} \otimes \mathfrak{g}[1]$ as vector bundles on $\mathbb{R}$. Under this identification the integration pairing ω_M on differential forms reduces to the pairing ω_N. The de Rham differential vanishes on translation-invariant forms which shows a compatibility of dimensional reduction with the differentials Q_{BV}. Finally, in both cases the interaction term comes from the L_∞ structure on $\mathfrak{g}$. □

Next, we consider dimensional reduction along a holomorphic direction. First, we set up some notation.

Let $V_{\mathbb{R}}$ be a real vector space equipped with a nondegenerate symmetric bilinear pairing and an orientation. The symmetric bilinear pairing trivializes $\det(V_{\mathbb{R}})^{\otimes 2}$ and the orientation allows us to obtain a trivialization of $\det(V_{\mathbb{R}})$, i.e. a real volume form. We denote by $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. Note that V inherits a nondegenerate Hermitian form from the symmetric bilinear pairing on $V_{\mathbb{R}}$. Also, since

$$\det(V) \cong \det(V_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$$

the real volume form on $V_{\mathbb{R}}$ determines a complex volume form on V.

Complexification yields a group homomorphism

$$\mathrm{SO}(V_{\mathbb{R}}) \longrightarrow \mathrm{SU}(V) \tag{2}$$

such that the real projection $\mathrm{Re}\colon V \to V_{\mathbb{R}}$ is $\mathrm{SO}(V_{\mathbb{R}})$-equivariant.

As in Equation (1), we assume X is a complex manifold of the form $X' \times V$ and consider the dimensional reduction along the map

$$p_{\overline{\partial}} : (X' \times V) \times Y \times M \to X' \times Y \times (M \times V_{\mathbb{R}})$$

induced by $\mathrm{Re} : V \to V_{\mathbb{R}}$.

Proposition 1.57. Let $X, Y, M, \mathfrak{g}$ be as before and $V_{\mathbb{R}}, V$ as above. Fix an L_∞ algebra $\mathfrak{g}$ equipped with a nondegenerate invariant pairing as in Definition 1.41 and consider the generalized Chern–Simons theory

$$\mathrm{Map}((X' \times V) \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g}).$$

Its dimensional reduction along the projection $p_{\overline{\partial}}$ is equivalent to the generalized Chern–Simons theory

$$\mathrm{Map}(X' \times Y_{\mathrm{Dol}} \times (M \times V_{\mathbb{R}})_{\mathrm{dR}}, B\mathfrak{g}).$$

The equivalence is $\mathrm{SO}(V_{\mathbb{R}})$-equivariant.

Proof. We may assume $X', Y, M = \mathrm{pt}$ as in the previous proof.

We have an isomorphism of vector bundles on V:

$$\Omega^{0,\bullet}_V \cong \mathrm{Sym}^\bullet_{\mathbb{C}}(\underline{V}[-1])$$

and similarly an isomorphism of bundles on $V_{\mathbb{R}}$:

$$\Omega^\bullet_{V_{\mathbb{R}}} \cong \mathrm{Sym}^\bullet_{\mathbb{R}}(\underline{V}_{\mathbb{R}}[-1]) \otimes_{\mathbb{R}} \mathbb{C}.$$

Under the composition

$$\Omega^\bullet(V_{\mathbb{R}}; \mathbb{C}) \xrightarrow{p^*} \Omega^\bullet(V) \to \Omega^{0,\bullet}(V)$$

defined by pulling back forms along p and projecting onto $(0, \bullet)$-forms, the map $\Omega_V \wedge (-)\colon \Omega^{0,\bullet}_V \to \Omega^{\dim(V),\bullet}_V \to \Omega^{\dim(V),\dim(V)}_V = \mathrm{Dens}_V$ which projects onto the top component, reduces to the map $\Omega^\bullet(V_{\mathbb{R}}; \mathbb{C}) \to \Omega^{\dim(V)}_{V_{\mathbb{R}}} = \mathrm{Dens}_{V_{\mathbb{R}}}$ which also projects onto the top component. This shows that the BV pairings of the original and dimensionally reduced theory are compatible. □

As a corollary of this result we obtain the reduction of generalized Hodge theory.

Corollary 1.58. Let $X', Y, M, V_{\mathbb{R}}, V$ be as before. Fix an L_∞ algebra equipped with a non-degenerate pairing as in Definition 1.51 and consider the generalized Hodge theory

$$\mathrm{Map}((X' \times V) \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}}).$$

Its dimensional reduction along the projection $p_{\overline{\partial}}$ is equivalent to the generalized Hodge theory

$$\mathrm{Map}(X' \times Y_{\mathrm{Dol}} \times (M \times V_{\mathbb{R}})_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}}).$$

The equivalence is $\mathrm{SO}(V_{\mathbb{R}})$-equivariant.

By essentially the same argument as in the proof of Proposition 1.57, we obtain the reduction of generalized BF theory.

Proposition 1.59. Let $X', Y, M, V_{\mathbb{R}}, V$ be as before. Fix an L_∞ algebra $\mathfrak{g}$ and consider the generalized BF theory

$$T^*[-1]\,\mathrm{Map}((X' \times V) \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g}).$$

Its dimensional reduction along the projection $p_{\overline{\partial}}$ is equivalent to the generalized BF theory

$$T^*[-1]\,\mathrm{Map}(X' \times Y_{\mathrm{Dol}} \times (M \times V_{\mathbb{R}})_{\mathrm{dR}}, B\mathfrak{g}).$$

This equivalence is $\mathrm{SO}(V_{\mathbb{R}})$-equivariant.

Finally, we take Y to be a complex manifold of the form $Y' \times \mathbb{C}$, where V is complex k-dimensional and consider the projection

$$p_{\mathrm{Dol}} : X \times (Y' \times \mathbb{C}) \times M \to X \times Y' \times (M \times \mathbb{R})$$

induced by $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$.

Proposition 1.60. Let X, M, Y' be as above. Fix an L_∞ algebra $\mathfrak{g}$ and consider the generalized BF theory

$$T^*[-1]\mathrm{Map}(X \times (Y' \times \mathbb{C})_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g}).$$

Its dimensional reduction along p_{Dol} is equivalent to the generalized BF theory

$$T^*[-1]\mathrm{Map}(X \times Y'_{\mathrm{Dol}} \times (M \times \mathbb{R})_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$$

This equivalence is $\mathrm{SO}(V_{\mathbb{R}})$-equivariant.

Proof. Note that there is an isomorphism of L_∞ algebras

$$\Omega^{\bullet,\bullet}(Y' \times \mathbb{C}; \mathfrak{g}) = \Omega^{\bullet,\bullet}(Y') \otimes \Omega^{0,\bullet}(\mathbb{C}) \otimes \mathfrak{g}[\epsilon]$$

where ϵ is a parameter of degree $+1$. The result then follows from Proposition 1.59 applied to the L_∞ algebra $\mathfrak{g}[\epsilon]$. ☐

2 Supersymmetry

Having set up the formalism behind classical field theories in the BV and BRST formalisms, we will introduce the other main formal ingredient of this paper: the supersymmetry action. So, we will discuss the classification of supersymmetry algebras, the notion of a supersymmetric field theory, and the idea of a *twist* of a supersymmetric field theory, extending work of the first two authors in [**ElliottSafronov**]. We will introduce the classification of supersymmetry algebras using the division algebra perspective of Baez and Huerta [**BaezHuerta**], which will be useful for the classification of super Yang-Mills theories in Section 3 below.

2.1 Supersymmetry Algebras

In this section we will recall the framework for supersymmetry algebras and their classification following Deligne [**DeligneSpinors**] and our previous work [**ElliottSafronov**], we refer there for more details.

Let $V_{\mathbb{R}}$ be a finite-dimensional real vector space of dimension $n = \dim_{\mathbb{R}}(V_{\mathbb{R}})$ equipped with an orientation and a nondegenerate symmetric bilinear form. Denote by $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. Consider the Lie algebra $\mathfrak{so}(V)$. Let us recall the following facts:

- If n is odd, $\mathfrak{so}(V)$ has a distinguished fundamental representation called the ***spin*** representation S.

- If n is even, $\mathfrak{so}(V)$ has a pair of distinguished fundamental representations called the ***semi-spin*** representations S_+ and S_-.

Definition 2.1. A ***spinorial representation*** Σ is a finite sum of spin or semi-spin representations of $\mathfrak{so}(V)$.

So, in odd dimensions we have $\Sigma = S \otimes W$ and in even dimensions we have $\Sigma = S_+ \otimes W_+ \oplus S_- \otimes W_-$, where W denotes a multiplicity space.

We have an embedding $\mathrm{U}(n) \subset \mathrm{SO}(2n, \mathbb{R})$ which lifts to an embedding $\mathrm{MU}(n) \subset \mathrm{Spin}(2n, \mathbb{R})$. If we denote by L the standard n-dimensional representation of $\mathrm{U}(n)$, then the semi-spin representations of $\mathrm{Spin}(2n, \mathbb{R})$ restrict to $\mathrm{MU}(n)$ as

$$S_+ \cong \det(L)^{-1/2} \otimes \wedge^{\mathrm{even}} L, \qquad S_- \cong \det(L)^{-1/2} \otimes \wedge^{\mathrm{odd}} L.$$

Definition 2.2. Fix a spinorial representation Σ and a nondegenerate $\mathfrak{so}(V)$-equivariant pairing $\Gamma \colon \mathrm{Sym}^2(\Sigma) \to V$. The ***supertranslation Lie algebra*** is the $\mathfrak{so}(V)$-equivariant super Lie algebra $\mathfrak{A} = \Pi\Sigma \oplus V$ whose only nontrivial bracket is given by Γ.

For a given spinorial representation, the pairing Γ is typically unique up to a scale, so a supertranslation Lie algebra is specified by fixing a spinorial representation. In turn, a spinorial representation is determined by the dimension of the multiplicity space, so we will talk about $\mathcal{N}$ or $(\mathcal{N}_+, \mathcal{N}_-)$ supertranslation Lie algebras, where the numbers are specified as follows.

- If $n \equiv 0, 1, 3, 4 \pmod 8$, we let $\mathcal{N} = \dim(W)$.
- If $n \equiv 2 \pmod 8$, we let $\mathcal{N}_\pm = \dim(W_\pm)$.
- If $n \equiv 5, 7 \pmod 8$, we let $2\mathcal{N} = \dim(W)$.
- If $n \equiv 6 \pmod 8$, we let $2\mathcal{N}_\pm = \dim(W_\pm)$.

Fix the following data:

- A spinorial representation Σ of $\mathfrak{so}(V)$.
- An $\mathfrak{so}(V)$-equivariant nondegenerate pairing $\Gamma \colon \mathrm{Sym}^2(\Sigma) \to V$.
- A Lie group G_R, the ***group of R-symmetries***, which acts on Σ by $\mathfrak{so}(V)$-equivariant automorphisms preserving Γ.

Note that the supertranslation Lie algebra $\mathfrak{A}$ is a $\mathrm{Spin}(V_\mathbb{R}) \times G_R$-equivariant super Lie algebra. We will sometimes want to refer to the infinitesimal version of this action.

Definition 2.3. Let $\mathfrak{A}$ be a supertranslation algebra. The corresponding ***supersymmetry algebra*** is the Lie algebra $(\mathfrak{so}(V) \oplus \mathfrak{g}_R) \ltimes \mathfrak{A}$.

We will now define the fundamental notion of a supersymmetric field theory. Consider a spacetime manifold $M = V_\mathbb{R}$. Let $\mathrm{ISO}(V_\mathbb{R}) = \mathrm{Spin}(V_\mathbb{R}) \ltimes V_\mathbb{R}$ be the ***Poincaré group*** which acts by affine transformations on M.

Definition 2.4. A classical field theory (E, S, ω) is ***supersymmetric*** if $E \to M$ is an $\mathrm{ISO}(V_\mathbb{R}) \times G_R$-equivariant vector bundle and the infinitesimal strict action of the translation Lie algebra V on the classical theory is extended to a $\mathrm{Spin}(V_\mathbb{R}) \times G_R$-equivariant L_∞ action of the supertranslation Lie algebra $\mathfrak{A}$ on the classical theory.

2.2 Composition Algebras and Minimal Supersymmetry

We will now recall a relationship between certain “minimal” supersymmetry algebras and composition algebras. Our treatment will essentially follow that of Baez and Huerta [**BaezHuerta**].

Let A be a unital (possibly non-associative) complex algebra equipped with an antiinvolution $\sigma\colon A \to A$. We make the following assumptions:

1. The map $\mathrm{Re}(x) = x \mapsto (x + \sigma(x))/2$ defines a projector onto the subspace of A spanned by the unit.
2. By the previous assumption we have a quadratic form $x\sigma(x)\colon A \to \mathbb{C}$. We assume that it is nondegenerate.

In fact, the data of the antiinvolution σ may equivalently be encoded in the data of a non-degenerate multiplicative norm $x \mapsto x\sigma(x)$, i.e. A is a real composition algebra [**SpringerVeldkamp**].

For a 2×2-matrix M with entries in A we define its hermitian adjoint $M^\dagger$ by transposing the matrix and applying σ to the entries. We define V to be the space of 2×2 Hermitian matrices with values in A, so that $\dim(V) = \dim(A)+2$. V carries a nondegenerate quadratic form given by $M \mapsto -\det(M)$. Moreover, it carries an orthogonal involution $\widetilde{M} = M - \mathrm{Tr}(M)\cdot 1$.

Given a left A-module, we may turn it into a right A-module via the antiinvolution $\sigma\colon A \to A$. Since A is a Frobenius algebra, we have the following basic construction

Lemma 2.5. Suppose M is a left A-module and N a right A-module equipped with a nondegenerate pairing $(-,-)\colon M\otimes N \to \mathbb{C}$ satisfying $(am, n) = (m, na)$ for every $a\in A$, $m \in M$ and $n\in N$. Then there is a unique map $(-,-)^A\colon M\otimes N \to A$ of (A,A)-bimodules whose real part is $(-,-)$.

Consider two isomorphic spinorial representations $\Sigma = A\oplus A$ and $\Sigma^* = A\oplus A$; we will equip the sum of these representations with an action of the Clifford algebra. The action on the two summands Σ and Σ^* will be different, hence the different notation. We define action maps

$$\rho\colon V\otimes\Sigma \to \Sigma^*, \qquad \rho\colon V\otimes\Sigma^* \to \Sigma$$

given, respectively, by

$$\rho(M)Q = MQ, \qquad \rho(M)Q = \widetilde{M}Q.$$

The following is proved in [**BaezHuerta**].

Proposition 2.6. The action maps $V\otimes\Sigma\to\Sigma^*$ and $V\otimes\Sigma^*\to\Sigma$ satisfy the Clifford relation

$$\rho(M)\rho(M) = -\det(M)\cdot 1.$$

So, $\Sigma\oplus\Sigma^*$ forms a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford module.

We have a nondegenerate scalar spinorial pairing

$$\Sigma\otimes\Sigma^* \longrightarrow \mathbb{C}$$

given by

$$(Q_1, Q_2) = \mathrm{Re}(Q_1^\dagger Q_2).$$

It obviously satisfies $(Q_1\sigma(a), Q_2) = (Q_1, Q_2 a)$. The extension to an A-valued pairing provided by Lemma 2.5 is given by

$$(Q_1, Q_2)^A = Q_1^\dagger Q_2.$$

By duality we obtain maps $\Gamma\colon \mathrm{Sym}^2(\Sigma) \to V$ and $\Gamma\colon \mathrm{Sym}^2(\Sigma^*) \to V$ given, respectively, by

$$\Gamma(Q_1, Q_2) = Q_1\widetilde{Q_2^\dagger + Q_2}Q_1^\dagger, \qquad \Gamma(Q_1, Q_2) = Q_1 Q_2^\dagger + Q_2 Q_1^\dagger.$$

We will now state two important properties of Γ and the spinorial pairing. The following statement was proved in [**BaezHuerta**] (see also [**Schray**] for the case $\dim(V) = 10$).

Theorem 2.7. Suppose A is alternative, i.e. $a\otimes b\otimes c \mapsto (ab)c - a(bc)$ is completely antisymmetric. For $Q_1, Q_2, Q_3 \in \Sigma$ we have

$$\rho(\Gamma(Q_1, Q_2))Q_3 + \rho(\Gamma(Q_2, Q_3))Q_1 + \rho(\Gamma(Q_2, Q_3))Q_1 = 0.$$

If we moreover assume A is associative, there is a relationship between the scalar spinorial pairing and Γ.

Theorem 2.8. Suppose A is associative. For $Q_1, Q_2 \in \Sigma$ and $Q_3 \in \Sigma^*$ we have

$$Q_1(Q_2, Q_3)^A + Q_2(Q_1, Q_3)^A = \rho(\Gamma(Q_1, Q_2))Q_3.$$

Proof. The right-hand side is

$$(Q_1 Q_2^\dagger + Q_2 Q_1^\dagger)Q_3$$

which by associativity can be rewritten as

$$Q_1(Q_2^\dagger Q_3) + Q_2(Q_1^\dagger Q_3)$$

which is the left-hand side. □

We will be interested in the following examples:

1. (**3d** $\mathcal{N} = 1$ **supersymmetry**) $A = \mathbb{C}$. We have $\dim(V) = 3$ and Σ is the spin representation of $\mathrm{Spin}(3, \mathbb{C})$.
2. (**4d** $\mathcal{N} = 1$ **supersymmetry**) $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[x]/(x^2 + 1)$ with $\sigma(x) = -x$. Moreover, $\dim(V) = 4$ and $\Sigma = S_+ \oplus S_-$ is the sum of semi-spin representations.
3. (**6d** $\mathcal{N} = (1, 0)$ **supersymmetry**) $A = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{End}(W_+)$, where W_+ is a two-dimensional symplectic vector space with σ given by the dual operator. Moreover, $\dim(V) = 6$ and $\Sigma = S_+ \otimes W_+$.
4. (**10d** $\mathcal{N} = (1, 0)$ **supersymmetry**) $A = \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$. We have $\dim(V) = 10$ and $\Sigma = S_+$ is a semi-spin representation.

All four examples are alternative, while the first three examples are moreover associative.

2.3 Two-dimensional Chiral Supersymmetry

In the previous section we have related composition algebras to minimal supersymmetry algebras in dimensions 3, 4, 6 and 10. In this section we explain a different relationship between composition algebras and supersymmetry algebras, this time in the case of 2d $\mathcal{N} = (\mathcal{N}_+, 0)$ supersymmetry.

Recall that in the case $\dim(V) = 2$ we have two one-dimensional semi-spin representations S_+, S_-. Moreover, we have an isomorphism

$$V \cong S_+^{\otimes 2} \oplus S_-^{\otimes 2}$$

and a pairing $(-,-)\colon S_+ \otimes S_- \to \mathbb{C}$, both of which are $\mathfrak{so}(V)$-equivariant. We denote the embeddings $S_\pm^{\otimes 2} \subset V$ by $\Gamma_\pm$, so that

$$(\Gamma_+(s,s), \Gamma_-(f,f)) = 2(s,f)^2. \tag{3}$$

Let W be a complex vector space of dimension $\mathcal{N}_+$ equipped with a nondegenerate symmetric bilinear pairing. We consider the spinorial representation

$$\Sigma = S_+ \otimes W$$

and its dual

$$\Sigma^* = S_- \otimes W.$$

The Clifford action $V \otimes S_+ \to S_-$ is defined so that

$$\rho(\Gamma_-(f,f))s = 2(s,f)f$$

and similarly for $V \otimes S_- \to S_+$.

Proposition 2.9. For $v, w \in V$ and $s \in S_+ \oplus S_-$ we have

$$\rho(v)\rho(w)s + \rho(w)\rho(v)s = 2(v,w)s.$$

Proof. It is enough to prove the claim for $s \in S_+$, $w \in S_+^{\otimes 2}$ and $v \in S_-^{\otimes 2}$. Assume $w = \Gamma_+(s,s)$ and $v = \Gamma_-(f,f)$ for $f \in S_-$. Then we have

$$\begin{aligned}\rho(\Gamma_+(s,s))\rho(\Gamma_-(f,f))s &= 2(s,f)\rho(\Gamma_+(s,s))f \\ &= 4(s,f)^2 s.\end{aligned}$$

But by (3) we have

$$(\Gamma_+(s,s), \Gamma_-(f,f)) = 2(s,f)^2$$

which proves the claim. $\square$

The Clifford action $V \otimes S_\pm \to S_\mp$ extends in an obvious way to a Clifford action $V \otimes \Sigma \to \Sigma^*$ and $V \otimes \Sigma^* \to \Sigma$ given by the identity on the W component. Thus, $\Sigma \oplus \Sigma^*$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford module.

The spaces Σ, Σ^* are equipped with $\mathfrak{so}(V)$-equivariant nondegenerate pairings $\Gamma\colon \mathrm{Sym}^2(\Sigma) \to V$ defined by

$$\Gamma(s \otimes q_1, s \otimes q_2) = \Gamma_+(s,s)(q_1,q_2)$$

and $\Gamma\colon \mathrm{Sym}^2(\Sigma^*) \to V$ defined similarly.

Proposition 2.10. For any $v \in V$ and $Q_1, Q_2 \in \Sigma$ or $Q_1, Q_2 \in \Sigma^*$ we have

$$(v, \Gamma(Q_1, Q_2)) = (\rho(v)Q_1, Q_2).$$

Proof. It is enough to prove the claim with $Q_1, Q_2 \in \Sigma$. Assume $v = \Gamma_-(f,f)$ for some $f \in S_-$, $Q_1 = s \otimes q_1$ and $Q_2 = s \otimes q_2$. Then the left-hand side is

$$(\Gamma_-(f,f), \Gamma_+(s,s))(q_1,q_2) = 2(s,f)^2(q_1,q_2).$$

The right-hand side is

$$\begin{aligned}(\rho(\Gamma_-(f,f)s \otimes q_1, s \otimes q_2) &= (2(s,f)f \otimes q_1, s \otimes q_2) \\ &= 2(s,f)^2(q_1,q_2).\end{aligned}$$

$\square$

An important property of two-dimensional chiral supersymmetry is the following analog of Theorem 2.7.

Theorem 2.11. For $Q_1, Q_2, Q_3 \in \Sigma$ we have

$$\rho(\Gamma(Q_1, Q_2))Q_3 = 0.$$

Proof. Indeed, $\Gamma(Q_1, Q_2)$ lies in $S_+^{\otimes 2} \subset V$, but the nonzero Clifford action is given by

$$S_-^{\otimes 2} \otimes (S_+ \otimes W) \longrightarrow S_- \otimes W.$$

□

We will now fix a composition algebra A with an antiinvolution σ as in Section 2.2 and set $W = A$. The nondegenerate symmetric bilinear pairing $a_1, a_2 \mapsto \mathrm{Re}(a_1\sigma(a_2))$ on A endows W with a pairing. Both Σ and Σ^* are right A-modules and the Clifford actions $V \otimes \Sigma \to \Sigma^*$ and $V \otimes \Sigma^* \to \Sigma$ are maps of right A-modules.

Since Σ and Σ^* are right A-modules, by Lemma 2.5 we may extend the scalar spinorial pairing to an A-valued pairing $\Sigma \otimes \Sigma^* \to A$ by

$$(s_1 \otimes q_1, s_2 \otimes q_2)^A = (s_1, s_2)\sigma(q_1)q_2.$$

Theorem 2.12. For $Q_1, Q_2 \in \Sigma$ and $Q_3 \in \Sigma^*$ we have

$$Q_1(Q_2, Q_3)^A + Q_2(Q_1, Q_3)^A = \rho(\Gamma(Q_1, Q_2))Q_3.$$

Proof. Pick basis elements $s \in S_+$ and $f \in S_-$, so that

$$Q_1 = s \otimes q_1, \qquad Q_2 = s \otimes q_2, \qquad Q_3 = f \otimes q_3.$$

The right-hand side is

$$\begin{aligned}(q_1, q_2)\rho(\Gamma_+(s, s))f \otimes q_3 &= 2(s, f)s \otimes (q_1, q_2)q_3 \\ &= (s, f)s \otimes (q_1\sigma(q_2) + q_2\sigma(q_1))q_3.\end{aligned}$$

We have

$$Q_1(Q_2, Q_3)^A = s \otimes q_1(s, f)(\sigma(q_2)q_3),$$

so the left-hand side is

$$s(s, f) \otimes (q_1(\sigma(q_2)q_3) + q_2(\sigma(q_1)q_3)).$$

By associativity of A the two expressions are equal. □

2.4 Supersymmetric Twisting

The idea of twisting, originally developed by Witten [**WittenTQFT**], is to modify a classical BV theory by deforming the differential Q_{BV} by the action of a square-zero fermionic symmetry.

Definition 2.13. A ***square-zero supercharge*** is a nonzero element $Q \in \Sigma$ such that $\Gamma(Q, Q) = 0$. Its ***number of invariant directions*** is the dimension of the image of $\Gamma(Q, -)\colon \Sigma \to V$.

It is shown in [**ElliottSafronov**] that the number d of invariant directions is at least $n/2$. We will use the following adjectives for square-zero supercharges depending on d:

- A supercharge Q is ***topological*** if $d = n$.
- A supercharge Q is ***holomorphic*** if n is even and $d = n/2$.
- A supercharge Q is ***minimal*** if n is odd and $d = (n+1)/2$.

In the intermediate case we refer to Q as a ***holomorphic-topological*** (alternatively, partially topological) supercharge. The collection of all square-zero supercharges in dimensions 2 through 10 (where one restricts to supersymmetries with at most 16 supercharges) was studied in [**ElliottSafronov**] and [**EagerSaberiWalcher**]. In particular, the orbits of square-zero supercharges under the R-symmetry group, $\mathrm{Spin}(V)$ and the obvious scaling action of $\mathbb{C}^\times$ are shown in Figure 1.

Let (E, S, ω) be a supersymmetric classical field theory. Recall, this means we have a Maurer-Cartan element

$$I_{\mathfrak{A}} = \sum_{k \geq 0} I_{\mathfrak{A}}^{(k)} \in C^\bullet(\mathfrak{A}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))$$

where $I_{\mathfrak{A}}^{(k)} \colon \mathfrak{A}^{\otimes k} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ as in Definition 1.13 and the classical field theory has an action of the R-symmetry group G_R.

Definition 2.14. Suppose (E, S, ω) is a supersymmetric classical field theory and Q a square-zero supercharge. The ***Q-twisted classical field theory*** is the $\mathbb{Z}/2\mathbb{Z}$-graded classical field theory with the same bundle of BV fields and the symplectic pairing ω and the BV action

$$S^Q = S + \sum_{k \geq 0} I_{\mathfrak{A}}^{(k)}(Q, \ldots, Q).$$

Given additional data, we may enhance the classical field theory.

Definition 2.15. Let $Q \in \Sigma$ be a square-zero supercharge. A homomorphism $\alpha \colon U(1) \to G_R$ is ***compatible*** with Q if Q has weight 1 and the α-grading on mod 2 on E coincides with the fermionic grading.

Given such an α we may consider a new $\mathbb{Z}$-grading on E given by the sum of the cohomological grading and the grading given by α. The map $I_{\mathfrak{A}}^{(k)} \colon \mathfrak{A}^{\otimes k} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ is G_R-equivariant, so the element $I_{\mathfrak{A}}^{(k)}(Q, \ldots, Q)$ has α-grading k. But it also has cohomological degree $-k$. In other words, the twisted action S^Q has total degree zero, so (E, S^Q, ω) is a $\mathbb{Z}$-graded classical field theory.

Definition 2.16. Let $Q \in \Sigma$ be a square-zero supercharge and suppose $G \to \mathrm{Spin}(V_{\mathbb{R}})$ is a fixed group homomorphism. A ***twisting homomorphism*** is a homomorphism $\phi \colon G \to G_R$ such that Q is preserved under the composite $G \to \mathrm{Spin}(V_{\mathbb{R}}) \times G_R$.

The classical field theory (E, S, ω) carries a $\mathrm{Spin}(V_{\mathbb{R}}) \times G_R$-action. However, the Q-twisted theory (E, S^Q, ω) does not in general carry a $\mathrm{Spin}(V_{\mathbb{R}}) \times G_R$-action since the elements $I_{\mathfrak{A}}^{(k)}(Q, \ldots, Q)$ are not in general invariant under $\mathrm{Spin}(V_{\mathbb{R}}) \times G_R$. However, given a twisting homomorphism ϕ we see that $I_{\mathfrak{A}}^{(k)}(Q, \ldots, Q)$ is preserved under G, so (E, S^Q, ω) carries a G-action.

2.5 Dimensional Reduction of Supersymmetric Theories

Suppose $V_{\mathbb{R}} = \mathbb{R}^n$ as before and choose a subspace $W_{\mathbb{R}} \subset V_{\mathbb{R}}$, so that $V_{\mathbb{R}} = W_{\mathbb{R}} \oplus W_{\mathbb{R}}^\perp$. We denote $W = W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

Fix a spinorial representation Σ of $\mathfrak{so}(V)$, a nondegenerate pairing $\Gamma_V \colon \mathrm{Sym}^2(\Sigma) \to V$ and a group of R-symmetries G_V. This datum generates a supersymmetry algebra, which we'll denote by $\mathfrak{A}$. We have a natural embedding

$$\mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp) \subset \mathfrak{so}(V),$$

so Σ restricts to a spinorial $\mathfrak{so}(W)$ representation. We define the dimensionally reduced Γ-pairing as the composite

$$\Gamma_W\colon \operatorname{Sym}^2(\Sigma) \xrightarrow{\Gamma_V} V \to W,$$

where the last map is the orthogonal projection onto W. Finally, we have a new R-symmetry group

$$G_W = G_V \times \operatorname{Spin}(W_{\mathbb{R}}^{\perp}).$$

This datum generates a supersymmetry algebra $\mathfrak{A}'$ in dimension $\dim(W_{\mathbb{R}})$ as defined in Section 2.1.

Recall from Proposition 1.54 that the dimensional reduction of a classical field theory along the projection $p\colon V_{\mathbb{R}} \to W_{\mathbb{R}}$ exists and is unique. We have the following generalization of this statement to supersymmetric theories.

Proposition 2.17. Suppose (E, Q, ω, I) is an $\mathfrak{A}$-supersymmetric classical field theory on $V_{\mathbb{R}}$. Then its dimensional reduction along the projection $p\colon V_{\mathbb{R}} \to W_{\mathbb{R}}$ has a unique $\mathfrak{A}'$-supersymmetric structure, compatible with the supersymmetry on $V_{\mathbb{R}}$ in the sense that $p^* I^{(i)}_{V_{\mathbb{R}}} = I^{(i)}_{W_{\mathbb{R}}}$.

Proof. This follows from the proof of Proposition 1.54 by coupling the theory (E, Q, ω, I) to auxiliary fields generated by the representation Σ. $\square$

The following proposition is an immediate consequence of Proposition 2.17 and Definition 2.14.

Proposition 2.18. Fix a twisting datum (Q, α), where $\Gamma_V(Q, Q) = 0$. Then the dimensional reduction of the twist of the classical field theory E is isomorphic to the twist of the dimensional reduction of E.

3 Supersymmetric Yang–Mills Theories

In this section we construct supersymmetry action on super Yang–Mills theories. We have the following variants of the super Yang–Mills theory depending on $\dim(\Sigma)$:

- (16 supercharges). This theory exists in dimensions 2 through 10 and it depends on a Lie algebra $\mathfrak{g}$.
- (8 supercharges). This theory exists in dimensions 2 through 6 and it depends on a Lie algebra $\mathfrak{g}$ together with a symplectic $\mathfrak{g}$-representation U.
- (4 supercharges). This theory exists in dimensions 2 through 4 and it depends on a Lie algebra $\mathfrak{g}$ together with a $\mathfrak{g}$-representation R.
- (2 supercharges). This theory exists in dimensions 2 through 3 and it depends on a Lie algebra $\mathfrak{g}$ together with an orthogonal $\mathfrak{g}$-representation P.

There are a few additional possibilities that occur in dimension 2.

- ($\mathcal{N}_+$ supercharges, chiral supersymmetry). This theory exists in dimension 2 and depends on a Lie algebra $\mathfrak{g}$.
- (4 supercharges, chiral supersymmetry). This theory exists in dimension 2 and depends on a Lie algebra $\mathfrak{g}$ together with a symplectic $\mathfrak{g}$-representation U.
- (2 supercharges, chiral supersymmetry). This theory exists in dimension 2 and depends on a Lie algebra $\mathfrak{g}$ together with a $\mathfrak{g}$-representation R.
- (1 supercharge, chiral supersymmetry). This theory exists in dimension 2 and depends on a Lie algebra $\mathfrak{g}$ together with an orthogonal $\mathfrak{g}$-representation P.

In each case the lower-dimensional theories are obtained by dimensional reduction from the theory in the highest dimension: for instance, 7d $\mathcal{N}=1$ super Yang–Mills (16 supercharges) is obtained by dimensional reduction from 10d $\mathcal{N}=(1,0)$ super Yang–Mills. So, it will be enough to construct the supersymmetry action in these highest-dimensional theories.

3.1 Pure Super Yang–Mills Theory

We begin with a description of certain pure supersymmetric Yang–Mills theories. Let $V_{\mathbb{R}}=\mathbb{R}^n$ be a real vector space of dimension n equipped with a nondegenerate symmetric bilinear pairing and V its complexification. Fix a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford module $\Sigma\oplus\Sigma^*\to\mathbb{C}$ with the associated Γ-pairings

$$\Gamma\colon \mathrm{Sym}^2(\Sigma)\to V, \qquad \Gamma\colon \mathrm{Sym}^2(\Sigma^*)\to V$$

defined as in Appendix A. We make the following assumption on this setup.

Assumption 3.1. For $Q_1,Q_2,Q_3\in\Sigma$ we have

$$\rho(\Gamma(Q_1,Q_2))Q_3+\rho(\Gamma(Q_2,Q_3))Q_1+\rho(\Gamma(Q_3,Q_1))Q_2=0.$$

Explicitly, we consider the following cases:

- (**2d** $\mathcal{N}=(\mathcal{N}_+,0)$ **supersymmetry**) We have $\dim(V)=2$ and $\Sigma=S_+\otimes W$ for some complex vector space W equipped with a nondegenerate symmetric bilinear pairing. Assumption 3.1 is satisfied by Theorem 2.11.
- (**3d** $\mathcal{N}=1$ **supersymmetry**) We have $\dim(V)=3$ and $\Sigma=S$. Assumption 3.1 is satisfied by Theorem 2.7.
- (**4d** $\mathcal{N}=1$ **supersymmetry**) We have $\dim(V)=4$ and $\Sigma=S_+\oplus S_-$. Assumption 3.1 is satisfied by Theorem 2.7.
- (**6d** $\mathcal{N}=(1,0)$ **supersymmetry**) We have $\dim(V)=6$ and $\Sigma=S_+\otimes W_+$ for a two-dimensional complex symplectic vector space W_+. Assumption 3.1 is satisfied by Theorem 2.7.
- (**10d** $\mathcal{N}=(1,0)$ **supersymmetry**) We have $\dim(V)=10$ and $\Sigma=S_+$. Assumption 3.1 is satisfied by Theorem 2.7.

Let $\mathfrak{g}$ be a Lie algebra equipped with a nondegenerate symmetric bilinear pairing. The fields of the Yang–Mills theory are as follows:

- A connection $A\in\Omega^1(V_{\mathbb{R}};\mathfrak{g})$ on the trivial bundle.
- A spinor $\lambda\in\Gamma(V_{\mathbb{R}};\Pi\Sigma\otimes\mathfrak{g})$.
- A ghost field $c\in\Omega^0(V_{\mathbb{R}};\mathfrak{g}[1])$.

Denote by $F_A=\mathrm{d}A+\frac{1}{2}[A\wedge A]$ the curvature of A and let $\not{d}_A$ be the twisted Dirac operator obtained from Γ (see Appendix A).

Definition 3.2. The BRST theory for classical supersymmetric Yang-Mills theory has underlying $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-graded bundle:

$$F_{\mathrm{gauge}}=\Omega^1(V_{\mathbb{R}};\mathfrak{g})\oplus\Gamma(V_{\mathbb{R}};\Pi\Sigma\otimes\mathfrak{g})\oplus\Omega^0(V_{\mathbb{R}};\mathfrak{g}[1])$$

whose sections we denote by (A,λ,c). The dg Lie structure on $F_{\mathrm{gauge}}[-1]$ has differential given by the de Rham differential $\mathrm{d}::\Omega^0(V_{\mathbb{R}};\mathfrak{g})\to\Omega^1(V_{\mathbb{R}};\mathfrak{g})$ and bracket

$$[-,-] \quad : \quad \Omega^0(V_{\mathbb{R}};\mathfrak{g})\otimes\left(\Omega^1(V_{\mathbb{R}};\mathfrak{g})\oplus\Gamma(V_{\mathbb{R}};\Sigma\otimes\mathfrak{g})\oplus\Omega^0(V_{\mathbb{R}};\mathfrak{g})\right) \quad\to\quad \Omega^1(V_{\mathbb{R}};\mathfrak{g})\oplus\Gamma(V_{\mathbb{R}};\Sigma\otimes\mathfrak{g})\oplus\Omega^0(V_{\mathbb{R}};\mathfrak{g})$$

defined by $[c, A+\lambda+c'] = [c,A]+[c,\lambda]+[c,c']$. The BRST action is defined by

$$S_{\mathrm{BRST}}(A,\lambda) = \int_{V_{\mathbb{R}}} \mathrm{dvol}\left(-\frac{1}{4}(F_A,F_A)+\frac{1}{2}(\lambda, \not{d}_A\lambda)\right).$$

The BV theory of supersymmetric Yang-Mills is the BV theory associated to this BRST theory. By definition, the fields are identified with sections of the bundle $T^*[-1]F_{\mathrm{gauge}} = F_{\mathrm{gauge}} \oplus F^!_{\mathrm{gauge}}[-1]$. If we denote by (A^*, λ^*, c^*) the anti-fields, the full BV action takes the form:

$$S_{\mathrm{gauge}} = \int_{V_{\mathbb{R}}} \mathrm{dvol}\left(-\frac{1}{4}(F_A,F_A)+\frac{1}{2}(\lambda, \not{d}_A\lambda) + (\mathrm{d}_A c, A^*) + ([c,\lambda],\lambda^*) + \frac{1}{2}([c,c],c^*)\right). \tag{4}$$

To simplify the notation, the pairing on $\mathfrak{g}$ from now on will be implicit.

The Poincaré group acts, in the sense of Definition 1.16, on Yang–Mills theory on $\mathbb{R}^n$. Indeed, there is an obvious Poincaré action on fields where we use that Σ is a representation of $\mathrm{Spin}(V_{\mathbb{R}})$. The corresponding Hamiltonian is given by

$$I^{(1)}_{\mathrm{gauge}}(v) = \int_{V_{\mathbb{R}}} \mathrm{dvol}\left((L_v A, A^*) - (v.\lambda, \lambda^*) - (v.c)c^*\right), \tag{5}$$

for $v \in \mathfrak{iso}(V)$, where $v.\lambda$ contains both a derivative and the $\mathfrak{so}(V)$ action on Σ.

We will now construct an elliptic L_∞ action of the super Lie algebra $\mathfrak{A}$ on the theory. Following Definition 1.13, we have to prescribe a collection of functionals $I^{(1)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}, \ldots$, where $I^{(k)}_{\mathrm{gauge}} \colon \mathfrak{A}^{\otimes k} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$, together satisfying the classical master equation. The supersymmetry action we construct will extend the Poincaré action from (5), so we just have to specify the values of $I^{(k)}_{\mathrm{gauge}}$ on the supersymmetry generators in Σ. The action of supersymmetry is given by a linear and a quadratic functional

$$I^{(1)}_{\mathrm{gauge}}(Q) = \int_{V_{\mathbb{R}}} \mathrm{dvol}\left(-(\Gamma(Q,\lambda),A^*) + \frac{1}{2}(\rho(F_A)Q,\lambda^*)\right) \tag{6}$$

$$I^{(2)}_{\mathrm{gauge}}(Q_1,Q_2) = \int_{V_{\mathbb{R}}} \mathrm{dvol}\left(\frac{1}{4}(\Gamma(Q_1,Q_2),\Gamma(\lambda^*,\lambda^*)) - \frac{1}{2}(Q_1,\lambda^*)(Q_2,\lambda^*) - \iota_{\Gamma(Q_1,Q_2)}Ac^*\right). \tag{7}$$

The following theorem summarizes the fact that super Yang–Mills theory is indeed supersymmetric in the sense of Definition 2.4.

Theorem 3.3. The functional $\mathfrak{S}_{\mathrm{gauge}} = S_{\mathrm{gauge}} + I^{(1)}_{\mathrm{gauge}} + I^{(2)}_{\mathrm{gauge}} \in C^\bullet(\mathfrak{A}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_{\mathrm{gauge}}))$ satisfies the classical master equation

$$\mathrm{d}_{\mathrm{CE}}\mathfrak{S}_{\mathrm{gauge}} + \frac{1}{2}\{\mathfrak{S}_{\mathrm{gauge}}, \mathfrak{S}_{\mathrm{gauge}}\} = 0.$$

Thus, $\mathfrak{S}_{\mathrm{gauge}}$ defines an elliptic L_∞ action of the super Lie algebra $\mathfrak{A}$ on super Yang–Mills theory and so super Yang–Mills theory is supersymmetric.

The rest of the section will be devoted to the proof of the above theorem. The classical master equation decomposes into the following equations:

$$\begin{aligned}
\{S_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}\} &= 0 \\
\{S_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\} + \mathrm{d}_{\mathrm{CE}} I^{(1)}_{\mathrm{gauge}} + \frac{1}{2}\{I^{(1)}_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}\} &= 0 \\
\mathrm{d}_{\mathrm{CE}} I^{(2)}_{\mathrm{gauge}} + \{I^{(1)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\} &= 0 \\
\{I^{(2)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\} &= 0.
\end{aligned}$$

Note that the last equation is automatically satisfied since $I^{(2)}_{\mathrm{gauge}}$ is independent of λ and c. The rest of the claims will be proved in a sequence of Lemmas. To simplify the expressions, we drop the integrals.

Lemma 3.4. One has $\{S_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}(Q)\} = 0$.

Proof. Let us decompose $S_{\mathrm{gauge}} = \sum_{i=1}^5 S^i_{\mathrm{gauge}}$ into individual summands.

The first term gives

$$\begin{aligned}\{S^1_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}(Q)\} &= -\frac{1}{2}(\mathrm{d}_A\Gamma(Q,\lambda), F_A) \\ &= -\frac{1}{2}(-1)^{n-1}(*\mathrm{d}_A * F_A, \Gamma(Q,\lambda)).\end{aligned}$$

The second term gives

$$\begin{aligned}\{S^2_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}(Q)\} &= -\frac{1}{2}(\lambda, \rho(\Gamma(Q,\lambda))\lambda) + \frac{1}{2}(\rho(F_A)Q, \not{d}_A\lambda) \\ &= -\frac{1}{2}(\Gamma(Q,\lambda), \Gamma(\lambda,\lambda)) - \frac{1}{2}(\lambda, \not{d}_A(\rho(F_A)Q)) \\ &= -\frac{1}{2}(\Gamma(Q,\lambda), \Gamma(\lambda,\lambda)) - \frac{1}{2}(-1)^n(\lambda, \rho(*\mathrm{d}_A * F_A)Q),\end{aligned}$$

where we have used Proposition A.3 and the Bianchi identity in the last line.

By (28) and Assumption 3.1 we have $(\Gamma(Q,\lambda), \Gamma(\lambda,\lambda)) = 0$, so $\{S^1_{\mathrm{gauge}} + S^2_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}(Q)\} = 0$.

Finally, $\{S^3_{\mathrm{gauge}} + S^4_{\mathrm{gauge}} + S^5_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}(Q)\} = 0$ due to gauge-invariance of $I^{(1)}(Q)$. □

Remark 3.5. The previous Lemma expresses the fact that the pure super Yang–Mills action is supersymmetric; this was proven by Baez and Huerta in [**BaezHuerta**], and our proof essentially follows the proof in loc. cit.

Lemma 3.6. One has

$$\{S_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\} + \mathrm{d}_{\mathrm{CE}} I^{(1)}_{\mathrm{gauge}} + \frac{1}{2}\{I^{(1)}_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}\} = 0.$$

Proof. Evaluating the equation

$$\{S_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\} + \mathrm{d}_{\mathrm{CE}} I^{(1)}_{\mathrm{gauge}} + \frac{1}{2}\{I^{(1)}_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}\} = 0$$

on $v_1, v_2 \in \mathfrak{iso}(V)$, the claim reduces to the fact that (5) defines a strict Lie action. Evaluating it on $v \in \mathfrak{iso}(V)$ and $Q \in \Sigma$, the claim reduces to the fact that $I^{(1)}_{\mathrm{gauge}}$ is Poincaré-invariant. So, the only nontrivial check is for $Q_1, Q_2 \in \Sigma$.

The individual terms are

$$\begin{aligned}\frac{1}{2}\{I^{(1)}_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}\}(Q_1,Q_2) &= -\{I^{(1)}_{\mathrm{gauge}}(Q_1), I^{(1)}_{\mathrm{gauge}}(Q_2)\} \\ &= -\frac{1}{2}(\rho(\mathrm{d}_A\Gamma(Q_1,\lambda))Q_2, \lambda^*) + \frac{1}{2}(\Gamma(Q_2, \rho(F_A)Q_1), A^*) \\ &\quad - \frac{1}{2}(\rho(\mathrm{d}_A\Gamma(Q_2,\lambda))Q_1, \lambda^*) + \frac{1}{2}(\Gamma(Q_1, \rho(F_A)Q_2), A^*),\end{aligned}$$

$$\begin{aligned}(\mathrm{d}_{\mathrm{CE}} I^{(1)}_{\mathrm{gauge}})(Q_1,Q_2) &= I^{(1)}_{\mathrm{gauge}}(\Gamma(Q_1,Q_2)) \\ &= (L_{\Gamma(Q_1,Q_2)}(A), A^*) - (\Gamma(Q_1,Q_2).\lambda, \lambda^*) - (\Gamma(Q_1,Q_2).c)c^*\end{aligned}$$

and

$$\begin{aligned}\{S_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}(Q_1,Q_2)\} = &-\frac{1}{2}(Q_2,\lambda^*)(Q_1,\not{d}_A\lambda+[c,\lambda^*]) - \frac{1}{2}(Q_1,\lambda^*)(Q_2,\not{d}_A\lambda+[c,\lambda^*]) \\ &+\frac{1}{2}(\Gamma(Q_1,Q_2),\Gamma(\lambda^*,\not{d}_A\lambda+[c,\lambda^*])) + \iota_{\Gamma(Q_1,Q_2)}(\mathrm{d}_A c)c^* - (\mathrm{d}_A\iota_{\Gamma(Q_1,Q_2)}A, A^*) \\ &+([\lambda,\iota_{\Gamma(Q_1,Q_2)}A],\lambda^*) - [\iota_{\Gamma(Q_1,Q_2)}A, c]c^*\end{aligned}$$

The total coefficient in front of A^* is

$$\frac{1}{2}\Gamma(Q_1,\rho(F_A)Q_2) + \frac{1}{2}\Gamma(Q_2,\rho(F_A)Q_1) + L_{\Gamma(Q_1,Q_2)}A - \mathrm{d}_A\iota_{\Gamma(Q_1,Q_2)}A.$$

Using Proposition A.1 we get that the sum of the first two terms is $-\iota_{\Gamma(Q_1,Q_2)}F_A$ which cancels the last two terms.

The total coefficient in front of c^* is

$$-\Gamma(Q_1,Q_2).c + \iota_{\Gamma(Q_1,Q_2)}(\mathrm{d}_A c) - [\iota_{\Gamma(Q_1,Q_2)}A, c] = 0.$$

The total coefficient in front of λ^* is

$$\begin{aligned}&-\frac{1}{2}\rho(\mathrm{d}_A\Gamma(Q_1,\lambda))Q_2 - \frac{1}{2}\rho(\mathrm{d}_A\Gamma(Q_2,\lambda))Q_1 - \Gamma(Q_1,Q_2).\lambda \\ &+\frac{1}{2}\rho(\Gamma(Q_1,Q_2))\not{d}_A\lambda - \frac{1}{2}(Q_2,\not{d}_A\lambda)Q_1 - \frac{1}{2}(Q_1,\not{d}_A\lambda)Q_2 + [\lambda,(\Gamma(Q_1,Q_2),A)]\end{aligned}$$

Using Proposition A.2 the first, second, fifth and sixth terms combine to

$$-\frac{1}{2}\not{d}_A\rho(\Gamma(Q_1,\lambda))Q_2 - \frac{1}{2}\not{d}_A\rho(\Gamma(Q_2,\lambda))Q_1$$

which is equal to $\frac{1}{2}\not{d}_A\rho(\Gamma(Q_1,Q_2))\lambda$ by Assumption 3.1. Using the Clifford relation this term cancels the rest of the terms. ☐

Evaluating the equation

$$\mathrm{d}_{\mathrm{CE}}I^{(2)}_{\mathrm{gauge}} + \{I^{(1)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\} = 0$$

on $v_1,v_2,v_3\in\mathfrak{iso}(V)$ or on $v_1,v_2\in\mathfrak{iso}(V)$ and $Q\in\Sigma$ we automatically get zero. Evaluating it on $v\in\mathfrak{iso}(V)$ and $Q_1,Q_2\in\Sigma$ we get Poincaré-invariance of $I^{(2)}_{\mathrm{gauge}}$.

Lemma 3.7.

$$\{I^{(1)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\}(Q_1,Q_2,Q_3) = 0$$

for every $Q_1,Q_2,Q_3\in\Sigma$.

Proof. We have

$$\begin{aligned}\{I^{(1)}_{\mathrm{gauge}}(Q_1), I^{(2)}_{\mathrm{gauge}}(Q_2,Q_3)\} = &-\iota_{\Gamma(Q_2,Q_3)}\Gamma(Q_1,\lambda)c^* - \frac{1}{2}(\Gamma(Q_2,Q_3),\Gamma(\rho(A^*)Q_1,\lambda^*)) \\ &+\frac{1}{2}(Q_2,\rho(A^*)Q_1)(Q_3,\lambda^*) + \frac{1}{2}(Q_3,\rho(A^*)Q_1)(Q_2,\lambda^*).\end{aligned}$$

$\{I^{(1)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\}(Q_1,Q_2,Q_3)$ is obtained by cyclically symmetrizing the above expression. By Assumption 3.1 the cyclic symmetrization of the term with c^* is zero. The Clifford relation implies that

$$\begin{aligned}\frac{1}{2}(\Gamma(Q_2,Q_3),\Gamma(\rho(A^*)Q_1,\lambda^*)) &= -\frac{1}{2}(\Gamma(Q_2,Q_3),\Gamma(\rho(A^*)\lambda^*,Q_1)) + (\Gamma(Q_2,Q_3),A^*)(Q_1,\lambda^*) \\ &= -\frac{1}{2}(\rho(\Gamma(Q_2,Q_3))Q_1,\rho(A^*)\lambda^*) + (\Gamma(Q_2,Q_3),A^*)(Q_1,\lambda^*).\end{aligned}$$

Therefore, again using Assumption 3.1 we see that the cyclic symmetrization of the terms with A^* vanishes. ☐

3.2 Coupling to Matter Multiplets

In this section we describe the coupling of super Yang–Mills theory to matter valued in a $\mathfrak{g}$-representation P, i.e. the supersymmetric gauged linear σ-models. Our description of the supersymmetry of the matter multiplet is inspired by the presentation of the supersymmetric nonlinear σ-models by Deligne and Freed in [**DeligneFreed**].

Consider as before $V_{\mathbb{R}}$ and a Clifford module $\Sigma \oplus \Sigma^*$ satisfying Assumption 3.1. In addition, fix a complex associative composition algebra A equipped with an antiinvolution σ as in Section 2.2. Suppose $\Sigma \oplus \Sigma^*$ carries a compatible right A-module structure. Let $(-,-)^A \colon \Sigma \otimes \Sigma^* \to A$ be the correspoding A-valued pairing given by Lemma 2.5. We make the following additional assumption.

Assumption 3.8. For $Q_1, Q_2 \in \Sigma$ and $Q_3 \in \Sigma^*$ we have

$$Q_1(Q_2, Q_3)^A + Q_2(Q_1, Q_3)^A = \rho(\Gamma(Q_1, Q_2))Q_3.$$

Explicitly, we consider the following cases:

- (**2d** $\mathcal{N} = (1, 0)$ **supersymmetry**) $A = \mathbb{C}$. Assumption 3.8 is satisfied by Theorem 2.12.
- (**2d** $\mathcal{N} = (2, 0)$ **supersymmetry**) $A = \mathbb{C}[x]/(x^2 + 1)$. Assumption 3.8 is satisfied by Theorem 2.12.
- (**2d** $\mathcal{N} = (4, 0)$ **supersymmetry**) $A = \mathrm{End}(Z)$. Assumption 3.8 is satisfied by Theorem 2.12.
- (**3d** $\mathcal{N} = 1$ **supersymmetry**) $A = \mathbb{C}$. Assumption 3.8 is satisfied by Theorem 2.8.
- (**4d** $\mathcal{N} = 1$ **supersymmetry**) $A = \mathbb{C}[x]/(x^2 + 1)$. Assumption 3.8 is satisfied by Theorem 2.8.
- (**6d** $\mathcal{N} = (1, 0)$ **supersymmetry**) $A = \mathrm{End}(Z)$. Assumption 3.8 is satisfied by Theorem 2.8.

Let P be a left A-module equipped with a $\mathbb{C}$-valued nondegenerate symmetric bilinear pairing such that

$$(av, w) = (v, \sigma(a)w).$$

Moreover, assume P carries a $\mathfrak{g}$-action commuting with the A-module structure and preserving the bilinear pairing. Explicitly, for $A = \mathbb{C}, \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ we get the following data:

- $A = \mathbb{C}$. We are looking for a $\mathfrak{g}$-representation P equipped with a nondegenerate symmetric bilinear pairing.
- $A = \mathbb{C}[x]/(x^2 + 1)$. A left A-module P splits as $P = P_+ \oplus P_-$, where x acts as $\pm i$ on $P_\pm$. Note that with respect to the right A-action x acts as $\mp i$ on $P_\pm$. So, the symmetric bilinear pairing identifies $P_+ \cong P_-^*$. In other words, the data boils down to a $\mathfrak{g}$-representation R, so that $P = R \oplus R^*$.
- $A = \mathrm{End}(Z)$. A left A-module is necessarily of the form $P \cong Z \otimes U$, where A just acts on Z. Compatibility of the orthogonal pairing on P with the A-action implies that it is given by a product of the symplectic pairing on Z and a symplectic pairing on U. So, the data boils down a symplectic $\mathfrak{g}$-representation U.

We are going to construct a theory on $V_{\mathbb{R}}$ describing a matter multiplet valued in P. The BRST fields are given as follows:

- a scalar $\phi \in \Gamma(V_{\mathbb{R}}; P)$;
- a spinor $\psi \in \Gamma(V_{\mathbb{R}}; \Pi\Sigma^* \otimes_A P)$.

As usual, we denote the antifields by $\phi^* \in \Gamma(V_\mathbb{R}; \Pi P)$ and $\psi^* \in \Gamma(V_\mathbb{R}; \Sigma \otimes_A P)$.

We extend the pairings on P and between Σ and Σ^* to a pairing between $\Sigma \otimes_A P$ and $\Sigma^* \otimes_A P$ in the following way. Given $\sum_i \tilde{s}_i \otimes v_i \in \Sigma^* \otimes_A P$ and $\sum_j s_j \otimes w_j \in \Sigma \otimes_A P$, their pairing is

$$\sum_{i,j} \mathrm{Re}((v_i, w_j)^A (s_j, \tilde{s}_i)^A), \tag{8}$$

where we extend both pairings to A-valued pairings using Lemma 2.5. We may also extend the Γ-pairing to a map

$$\Gamma\colon \mathrm{Sym}^2(\Sigma^* \otimes_A P) \to V$$

defined by the property

$$(v, \Gamma(\psi_1, \psi_2)) = (\psi_1, \rho(v)\psi_2), \qquad v \in V,\ \psi_i \in \Sigma^* \otimes_A P.$$

The BV action for the matter multiplet is

$$S_{\mathrm{matter}} = \int_{V_\mathbb{R}} \mathrm{dvol} \left(\frac{1}{2}(\mathrm{d}_A\phi, \mathrm{d}_A\phi) + (\psi, \not{\mathrm{d}}_A\psi) + 2(\lambda\phi, \psi) + (c\psi, \psi^*) - (c\phi, \phi^*) \right),$$

where we use the pairing (8) in the second term.

It is Poincaré-invariant with the corresponding Hamiltonian

$$I^{(1)}_{\mathrm{matter}}(v) = \int_{V_\mathbb{R}} \mathrm{dvol}\left((L_v A, \phi^*) - (v.\psi, \psi^*)\right), \tag{9}$$

for $v \in \mathfrak{iso}(V)$.

The action of supersymmetry is given by a linear and quadratic functional:

$$I^{(1)}_{\mathrm{matter}}(Q) = \int_{V_\mathbb{R}} \mathrm{dvol} \left(((Q, \psi), \phi^*) + \frac{1}{2}(\rho(\mathrm{d}_A\phi)Q, \psi^*) \right) \tag{10}$$

$$I^{(2)}_{\mathrm{matter}}(Q_1, Q_2) = \frac{1}{4}\int_{V_\mathbb{R}} \mathrm{dvol}(\Gamma(Q_1, Q_2), \Gamma(\psi^*, \psi^*)) \tag{11}$$

where $Q, Q_1, Q_2 \in \Sigma$.

We consider the full action of the super Yang–Mills theory

$$S_{\mathrm{BV}} = S_{\mathrm{gauge}} + S_{\mathrm{matter}}$$

together with a supersymmetry action functionals

$$I^{(1)} = I^{(1)}_{\mathrm{gauge}} + I^{(1)}_{\mathrm{matter}}, \qquad I^{(2)} = I^{(2)}_{\mathrm{gauge}} + I^{(2)}_{\mathrm{matter}}.$$

The following result states that these functionals encode an off-shell action of the supersymmetry algebra.

Theorem 3.9. The functional $\mathfrak{S} = S_{\mathrm{BV}} + I^{(1)}_{\mathrm{BV}} + I^{(2)}_{\mathrm{BV}}$ satisfies the classical master equation

$$\mathrm{d}_{\mathrm{CE}}\mathfrak{S} + \frac{1}{2}\{\mathfrak{S}, \mathfrak{S}\} = 0. \tag{12}$$

The rest of this section is devoted to the proof of Theorem 3.9. The classical master equation (12) decomposes into a sequence of equations

$$\begin{aligned} \{S_{\mathrm{BV}}, I^{(1)}\} &= 0 \\ \{S_{\mathrm{matter}}, I^{(2)}\} + \mathrm{d}_{\mathrm{CE}} I^{(1)}_{\mathrm{matter}} + \{I^{(1)}_{\mathrm{gauge}}, I^{(1)}_{\mathrm{matter}}\} + \tfrac{1}{2}\{I^{(1)}_{\mathrm{matter}}, I^{(1)}_{\mathrm{matter}}\} &= 0 \\ \mathrm{d}_{\mathrm{CE}} I^{(2)}_{\mathrm{matter}} + \{I^{(1)}_{\mathrm{matter}}, I^{(2)}_{\mathrm{matter}}\} &= 0 \\ \{I^{(2)}_{\mathrm{matter}}, I^{(2)}_{\mathrm{matter}}\} &= 0 \end{aligned} \tag{13}$$

The last equation is automatically satisfied since $I^{(2)}$ is independent of the fields ϕ,ψ,A,λ.

The first equation in (13) states that the classical action is supersymmetric.

Lemma 3.10. One has $\{S_{\mathrm{BV}}, I^{(1)}\}(Q)=0$ for all $Q\in\Sigma$.

Proof. Let us decompose $S_{\mathrm{matter}}=\sum_{i=1}^5 S^i_{\mathrm{matter}}$ into individual summands.

The first term gives

$$\begin{aligned}\{S^1_{\mathrm{matter}}, I^{(1)}(Q)\} &= -(\mathrm{d}_A\phi, \mathrm{d}_A(Q,\psi)) + (\Gamma(Q,\lambda)\phi, \mathrm{d}_A\phi)\\ &= \mathrm{d}_A^*\mathrm{d}_A\phi(Q,\psi) + (\Gamma(Q,\lambda)\phi, \mathrm{d}_A\phi).\end{aligned}$$

The second term gives

$$\begin{aligned}\{S^2_{\mathrm{matter}}, I^{(1)}(Q)\} &= -(\psi, \not{\mathrm{d}}_A\rho(\mathrm{d}_A\phi)Q) - (\psi, \rho(\Gamma(Q,\lambda))\psi)\\ &= -(\psi, \rho(F_A)Q)\phi - \mathrm{d}_A^*\mathrm{d}_A\phi(\psi,Q) - (\psi,\rho(\Gamma(Q,\lambda))\psi),\end{aligned}$$

where we have used Proposition A.3 in the second line.

The third term gives

$$\begin{aligned}\{S^3_{\mathrm{matter}}, I^{(1)}(Q)\} &= ((\rho(F_A)Q)\phi,\psi) + 2(\lambda(Q,\psi),\psi) - (\lambda\phi, \rho(\mathrm{d}_A\phi)Q)\\ &= ((\rho(F_A)Q)\phi,\psi) + (\rho(\Gamma(Q,\lambda))\psi,\psi) - (\Gamma(\lambda\phi,Q), \mathrm{d}_A\phi),\end{aligned}$$

where we have used Assumption 3.8 in the middle term and (28) in the last term. It is then obvious that

$$\{S^1_{\mathrm{matter}} + S^2_{\mathrm{matter}} + S^3_{\mathrm{matter}}, I^{(1)}(Q)\} = 0.$$

Finally, the terms $\{S^4_{\mathrm{matter}} + S^5_{\mathrm{matter}} + S^3_{\mathrm{gauge}}, I^{(1)}(Q)\}$ are zero due to gauge-invariance of $I^{(1)}(Q)$ while the rest of the terms are zero by Lemma 3.4. ☐

Next, we move on to the second equation in (13).

Lemma 3.11. One has

$$\{S_{\mathrm{matter}}, I^{(2)}\} + \mathrm{d}_{\mathrm{CE}} I^{(1)}_{\mathrm{matter}} + \{I^{(1)}_{\mathrm{gauge}}, I^{(1)}_{\mathrm{matter}}\} + \frac{1}{2}\{I^{(1)}_{\mathrm{matter}}, I^{(1)}_{\mathrm{matter}}\} = 0. \tag{14}$$

Proof. Evaluating expression (14) on $v_1, v_2\in\mathfrak{iso}(V)$ reduces to the claim that (9) defines a strict Lie action. Evaluating on $v\in\mathfrak{iso}(V)$ and $Q\in\Sigma$, the claim reduces to the fact that $I^{(1)}$ is Poincaré-invariant. So, the only nontrivial term to check is the evaluation on $Q_1,Q_2\in\Sigma$.

The individual terms are:

$$\begin{aligned}\frac{1}{2}\{I^{(1)}_{\mathrm{matter}}, I^{(1)}_{\mathrm{matter}}\}(Q_1,Q_2) =& -\{I^{(1)}_{\mathrm{matter}}(Q_1), I^{(1)}_{\mathrm{matter}}(Q_2)\}\\ =& -\frac{1}{2}(Q_1, \rho(\mathrm{d}_A\phi)Q_2)\phi^* - \frac{1}{2}(Q_2,\rho(\mathrm{d}_A\phi)Q_1)\phi^*\\ &+\frac{1}{2}(\rho(\mathrm{d}(Q_1,\psi))Q_2,\psi^*) + \frac{1}{2}(\rho(\mathrm{d}(Q_2,\psi))Q_1,\psi^*),\end{aligned}$$

$$\begin{aligned}\{I^{(1)}_{\mathrm{gauge}}, I^{(1)}_{\mathrm{matter}}\}(Q_1,Q_2) =& -\{I^{(1)}_{\mathrm{gauge}}(Q_1), I^{(1)}_{\mathrm{matter}}(Q_2)\} - \{I^{(1)}_{\mathrm{gauge}}(Q_2), I^{(1)}_{\mathrm{matter}}(Q_1)\}\\ =& -\frac{1}{2}(\rho(\Gamma(Q_1,\lambda))(\phi Q_2),\psi^*) - \frac{1}{2}(\rho(\Gamma(Q_2,\lambda))(\phi Q_1),\psi^*),\end{aligned}$$

$$(\mathrm{d}_{\mathrm{CE}} I^{(1)}_{\mathrm{matter}})(Q_1, Q_2) = L_{\Gamma(Q_1,Q_2)}(\phi)\phi^* - (\Gamma(Q_1, Q_2).\psi, \psi^*),$$

and

$$\{S_{\mathrm{matter}}, I^{(2)}(Q_1, Q_2)\} = \frac{1}{2}\Gamma(Q_1, Q_2)\Gamma(\psi^*, \not{d}_A\psi - 2\lambda\phi + c\psi^*) - ((\iota_{\Gamma(Q_1,Q_2)}A)\psi, \psi^*) + ((\iota_{\Gamma(Q_1,Q_2)}A)\phi, \phi^*)$$

We first collect all terms in equation (14) proportional to ϕ^*:

$$-\frac{1}{2}(Q_1, \rho(\mathrm{d}_A\phi)Q_2) - \frac{1}{2}(Q_2, \rho(\mathrm{d}_A\phi)Q_1) + L_{\Gamma(Q_1,Q_2)}\phi + (\iota_{\Gamma(Q_1,Q_2)}A)\phi.$$

By (28) we observe that the first two terms cancel with the last two terms.

Next, we collect all terms in equation (14) containing ψ^* and ψ:

$$\frac{1}{2}\not{d}_A Q_2(Q_1\psi)^A + \frac{1}{2}\not{d}_A Q_1(Q_2, \partial_i\psi)^A - \Gamma(Q_1, Q_2).\psi + \frac{1}{2}\rho(\Gamma(Q_1, Q_2))\not{d}_A\psi - (\iota_{\Gamma(Q_1,Q_2)}A)\psi. \tag{15}$$

Applying Assumption 3.8 to $Q_3 = \psi$, the first two terms become $\frac{1}{2}\not{d}_A\rho(\Gamma(Q_1, Q_2)\psi)$. Finally, by the Clifford identity the sum of this term with the fourth term in (15) is precisely $\iota_{\Gamma(Q_1,Q_2)}\mathrm{d}_A\psi$ which cancels the remaining terms. □

Lemma 3.12.

$$\{I^{(1)}_{\mathrm{matter}}, I^{(2)}_{\mathrm{matter}}\}(Q_1, Q_2, Q_3) = 0$$

for every $Q_1, Q_2, Q_3 \in \Sigma$.

Proof. We have

$$\begin{aligned}\{I^{(1)}_{\mathrm{matter}}(Q_1), I^{(2)}_{\mathrm{matter}}(Q_2, Q_3)\} &= \frac{1}{2}(\Gamma(Q_2, Q_3), \Gamma(\psi^*, \phi^* Q_1)) \\ &= (\psi^*, \phi^*\rho(\Gamma(Q_2, Q_3))Q_1).\end{aligned}$$

The expression $\{I^{(1)}_{\mathrm{matter}}, I^{(2)}_{\mathrm{matter}}\}(Q_1, Q_2, Q_3)$ is obtained by cyclically symmetrizing the above expression. By Assumption 3.1 the cyclic symmetrization is identically zero. □

Part II

Classification of Twists

In the following sections we fix a complex Lie algebra $\mathfrak{g}$ equipped with a symmetric bilinear invariant nondegenerate pairing, which should be thought of as the complexified Lie algebra of the gauge group.

4 Dimension 10

The 10-dimensional supersymmetry algebra has odd part $\Sigma \cong S_+ \otimes W_+ \oplus S_- \otimes W_-$, where S_+, S_- are the 16-dimensional semi-spin representations of $\mathrm{Spin}(10, \mathbb{C})$, and where W_+ and W_- are complex vector spaces equipped with nondegenerate symmetric bilinear pairings. There are Yang–Mills theories with $\mathcal{N} = (1, 0)$ or $\mathcal{N} = (0, 1)$ supersymmetries. We concentrate on the first case, the second case being identical. So, we fix $W_+ = \mathbb{C}$ and $W_- = 0$.

4.1 $\mathcal{N} = (1,0)$ Super Yang–Mills

We consider $\mathcal{N} = (1,0)$ super Yang–Mills theory on $M = \mathbb{R}^{10}$ with the Euclidean metric.

This theory admits a unique twist:

- A square-zero supercharge $Q \neq 0 \in \Sigma$ has 5 invariant directions and does not admit a compatible homomorphism α. So, it gives rise to a $\mathbb{Z}/2\mathbb{Z}$-graded holomorphic theory. Such a supercharge is stabilized by $G = \mathrm{SU}(5) \subset \mathrm{Spin}(10, \mathbb{C})$.

4.1.1 Holomorphic Twist

Let $Q \in \Sigma$ be a square-zero supercharge. The image of $\Gamma(Q, -)\colon \Sigma \to V$ is a complex Lagrangian subspace $L \subset V$. Denote by $\sigma\colon V \to V$ the complex conjugation induced by the real structure $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Since the bilinear form on $V_{\mathbb{R}}$ is positive-definite, $L \cap \sigma(L) = 0$. In other words, L defines a (linear) complex structure on $V_{\mathbb{R}}$. Moreover, we may canonically identify $\sigma(L) \cong L^*$.

Remark 4.1. It is important here that we are working in the complexified setting. While it makes sense to study a real form of 10d supersymmetric Yang-Mills theory in Lorentzian signature associated to a real Lie algebra $\mathfrak{g}_{\mathbb{R}}$, where the fermions are valued in the Majorana-Weyl spinor bundle, in this real supersymmetry algebra there are no square-zero supercharges, since a square-zero supercharge induces a 5-dimensional isotropic subspace of $\mathbb{R}^{10}$, which only exists in the split signature $5+5$ (in which case there is no real structure for the Weyl spinor representation). So necessarily, twists of maximal super Yang-Mills theory cannot be compatible with any unitary structure.

Let $\mathrm{ML}(L)$ be the metalinear group of L. Under the embedding $\mathrm{ML}(L) \subset \mathrm{Spin}(V)$, the semi-spin representation $\Sigma = S_+$ decomposes as

$$\Sigma = \det(L)^{1/2} \oplus \wedge^2 L^* \otimes \det(L)^{1/2} \oplus L \otimes \det(L)^{-1/2}.$$

$Q \in \Sigma$ lies in the first summand, so the choice of Q is equivalent to the choice of a (linear) Kähler structure L on $V_{\mathbb{R}}$ together with a complex half-density on L. The square of this half-density defines a Calabi–Yau structure on M.

We will now rewrite the fields and the action in terms of the Calabi–Yau structure. Let $\omega \in \Omega^{1,1}(M)$ be the Kähler form, $\Omega \in \Omega^{5,0}(M)$ the holomorphic volume form and $\Lambda\colon \Omega^{p+1,q+1}(M) \to \Omega^{p,q}(M)$ the dual Lefschetz operator. We denote the real volume form on M by

$$\mathrm{dvol} = \frac{\omega^5}{5!}.$$

The vector representation decomposes as

$$\Omega^1(M) \cong \Omega^{1,0}(M) \oplus \Omega^{0,1}(M),$$

the semi-spin representation S_+ decomposes as

$$\Omega^0(M; S_+) \cong \Omega^{1,0}(M) \oplus \Omega^{0,2}(M) \oplus \Omega^0(M)$$

and the semi-spin representation S_- decomposes as

$$\Omega^0(M; S_-) \cong \Omega^{0,1}(M) \oplus \Omega^{2,0}(M) \oplus \Omega^0(M).$$

Under this decomposition the scalar pairing $S_+ \otimes S_- \to \mathbb{C}$ corresponds to the wedge product of individual components post-composed with Λ. Under the above decompositions the Clifford multiplication of a vector $A = A_{1,0} + A_{0,1}$ and a spinor $\lambda = \rho + B + \chi \in S_+$ is given by

$$\rho(A)\lambda = (A_{0,1}\chi + \Lambda(A_{1,0} \wedge B), A_{1,0} \wedge \rho + *(A_{0,1} \wedge B \wedge \Omega), \Lambda(A_{0,1} \wedge \rho)) \in S_-.$$

Fields: The BRST fields are given by:

- Gauge fields $A_{1,0} \in \Omega^{1,0}(M;\mathfrak{g})$, $A_{0,1} \in \Omega^{0,1}(M;\mathfrak{g})$.
- Fermions $\rho \in \Omega^{1,0}(M;\Pi\mathfrak{g})$, $B \in \Omega^{0,2}(M;\Pi\mathfrak{g})$, $\chi \in \Omega^{0}(M;\Pi\mathfrak{g})$.
- A ghost field $c \in \Omega^0(M;\mathfrak{g})[1]$.

We denote their antifields by $A^*_{1,0}, A^*_{0,1}, \rho^*, B^*, \chi^*, c^*$.

The BV action of the theory is obtained from (4) by decomposing it in terms of the above fields. To write it out we will need an expression for the Hodge star operator on Kähler manifolds, see [**Huybrechts**].

Proposition 4.2. Let (M,ω) be a Kähler d-fold and decompose

$$\Omega^2(M) = \Omega^{2,0}(M) \oplus \Omega^{0,2}(M) \oplus (\mathbb{C}\omega \oplus \Omega^{1,1}_{\perp}(M)).$$

Then

1. The spaces $\Omega^{2,0}(M) \oplus \Omega^{0,2}(M)$, $\mathbb{C}\omega$ and $\Omega^{1,1}_{\perp}(M)$ are mutually orthogonal.
2. For a form $\alpha \in \Omega^{2,0}(M) \oplus \Omega^{0,2}(M)$ we have
$$*\alpha = \frac{1}{(d-2)!}\alpha \wedge \omega^{d-2}.$$
3. For $\alpha \in \Omega^{1,1}_{\perp}(M)$ we have
$$*\alpha = -\frac{1}{(d-2)!}\alpha \wedge \omega^{d-2}.$$
4. For $\alpha \in \mathbb{C}\omega$ we have
$$*\alpha = \frac{1}{(d-1)!}\alpha \wedge \omega^{d-2}.$$

Corollary 4.3. Let M be a Kähler d-fold and $F = F_{2,0} + F_{1,1} + F_{0,2}$ a two-form. Then

$$F \wedge *F + \frac{1}{(d-2)!} F \wedge F \wedge \omega^{d-2} = \left(4(F_{2,0}, F_{0,2}) + (\Lambda F_{1,1})^2\right) \frac{\omega^d}{d!}.$$

Since we are working near the trivial connection, the topological term $\int F \wedge F \wedge \omega^3$ is exact, so we will drop it. The BV action of the twisted theory S_{BV} is then the sum of the following terms:

$$S_{\mathrm{BRST}} = \int \mathrm{dvol} \left(-(F_{2,0}, F_{0,2}) - \frac{1}{4}(\Lambda F_{1,1})^2 + \chi \Lambda(\overline{\partial}_{A_{0,1}}\rho) + (B, \partial_{A_{1,0}}\rho) \right) + \frac{1}{2} B \wedge \overline{\partial}_{A_{0,1}} B \wedge \Omega \tag{16}$$

$$S_{\mathrm{anti}} = \int \mathrm{dvol} \left((\partial_{A_{1,0}} c, A^*_{1,0}) + (\overline{\partial}_{A_{0,1}} c, A^*_{0,1}) + ([\rho, c], \rho^*) + [\chi, c]\chi^* + ([B,c], B^*) + \frac{1}{2}[c,c]c^* \right) \tag{17}$$

$$I^{(1)} = \int \mathrm{dvol} \left(-(\rho, A^*_{1,0}) + (F_{0,2}, B^*) + \frac{1}{2}\Lambda F_{1,1}\chi^* \right) \tag{18}$$

$$I^{(2)} = -\frac{1}{4}\int \mathrm{dvol}(\chi^*)^2. \tag{19}$$

Theorem 4.4. The holomorphic twist of 10d $\mathcal{N} = (1,0)$ super Yang–Mills on $M = \mathbb{R}^{10}$ is perturbatively equivalent to the holomorphic Chern–Simons theory on $M \cong \mathbb{C}^5$ with the space of fields $\mathrm{Map}(M, B\mathfrak{g})$. Moreover, the equivalence is SU(5)-equivariant.

Proof. First, we may eliminate χ and χ^* using Proposition 1.8. So, the above theory is perturbatively equivalent to the theory without χ and χ^* with the BV action

$$\begin{aligned} S_{\text{no }\chi} = &\int \mathrm{dvol}\left(-(F_{2,0}, F_{0,2}) + (B, \partial_{A_{1,0}}\rho)\right) + \frac{1}{2} B \wedge \overline{\partial}_{A_{0,1}} B \wedge \Omega \\ &+ \int \mathrm{dvol}\left((\partial_{A_{1,0}} c, A^*_{1,0}) + (\overline{\partial}_{A_{0,1}} c, A^*_{0,1}) + ([\rho, c], \rho^*) + ([B, c], B^*) + \frac{1}{2}[c,c]c^*\right) \\ &+ \int \mathrm{dvol}(-(\rho, A^*_{1,0}) + (F_{0,2}, B^*)) \end{aligned}$$

Next, we have a term $\int \mathrm{dvol}\, \rho \wedge A^*_{1,0}$ in the action, i.e. $(\rho, A_{1,0})$ is a trivial BRST doublet, so by Proposition 1.10 we may remove it. The above theory then becomes perturbatively equivalent to the theory without fields $\rho, \rho^*, A_{1,0}, A^*_{1,0}$ and with the BV action

$$S_0 = \int \frac{1}{2} B \wedge \overline{\partial}_{A_{0,1}} B \wedge \Omega + \mathrm{dvol}\left((\overline{\partial}_{A_{0,1}} c, A^*_{0,1}) + ([B, c], B^*) + \frac{1}{2}[c,c]c^* + (F_{0,2}, B^*)\right)$$

Up to rescaling of the antifields, it coincides with the BV action for holomorphic Chern–Simons (see Section 1.6.2). $\square$

Remark 4.5. A similar claim was previously proved by Baulieu [**Baulieu**] by adding an auxiliary field to 10d $\mathcal{N} = (1, 0)$ super Yang–Mills.

5 Dimension 9

The 9-dimensional supersymmetry algebra has odd part $\Sigma \cong S \otimes W$, where S is the 16-dimensional spin representation of $\mathrm{Spin}(9, \mathbb{C})$ and W is a complex vector space equipped with a nondegenerate symmetric bilinear pairing. There is a Yang–Mills theory with $\mathcal{N} = 1$ supersymmetry, so we fix $W = \mathbb{C}$.

5.1 $\mathcal{N} = 1$ Super Yang–Mills

We consider $\mathcal{N} = 1$ super Yang–Mills theory on $M = \mathbb{R}^9$ equipped with the Euclidean metric.

This theory admits a unique twist:

- A square-zero supercharge $Q \neq 0 \in \Sigma$ has 5 invariant directions and does not admit a compatible homomorphism α. So, it gives rise to a $\mathbb{Z}/2\mathbb{Z}$-graded holomorphic theory. Such a supercharge is stabilized by $G = \mathrm{SU}(4) \subset \mathrm{Spin}(9, \mathbb{C})$.

We may identify the odd part of the 9d $\mathcal{N} = 1$ supersymmetry algebra with the odd part of the 10d $\mathcal{N} = (1, 0)$ supersymmetry algebra. Under this identification a supercharge Q squares to zero in 9d iff it squares to zero in 10d.

5.1.1 Minimal Twist

Let $Q \in \Sigma$ be a square-zero supercharge. Denote the image of $\Gamma(Q, -)\colon \Sigma \to V$ by $L^\perp \subset V$. Its orthogonal complement L is maximal isotropic and $L^\perp/L$ is one-dimensional. Since the bilinear form on $V_{\mathbb{R}}$ is positive-definite, $L \cap \sigma(L) = 0$. Moreover, $N = L^\perp \cap \sigma(L^\perp) \subset V$ is a σ-stable one-dimensional subspace, we let $N_{\mathbb{R}}$ be the σ-invariants of N. Therefore, we get a decomposition

$$V = L \oplus \sigma(L) \oplus N,$$

where $L^\perp = L \oplus N$.

Under the embedding $\mathrm{ML}(L) \subset \mathrm{Spin}(V)$ the spin representation $\Sigma = S$ decomposes as

$$\Sigma = \wedge^\bullet L \otimes \det(L)^{-1/2}$$

and the supercharge Q lies in the one-dimensional subspace $\det(L)^{1/2} \subset \Sigma$. Therefore, the choice of Q is equivalent to the choice of a one-dimensional subspace $N_\mathbb{R} \subset V_\mathbb{R}$ and a complex structure on $V_\mathbb{R}/N_\mathbb{R}$ together with a complex half-density.

It will be convenient to perform a computation of the twist in a slightly more general setting which will be useful for lower-dimensional computations.

Suppose L is a complex vector space equipped with a Hermitian structure and a complex half-density. Suppose $N_\mathbb{R} = \mathbb{R}^{5-\dim(L)}$ equipped with a Euclidean metric and a spin structure. Denote by $N = N_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ its complexification which carries a complex half-density. Let $V_\mathbb{R} = L \times N$ (a 10-dimensional real vector space). By the results of Section 4.1.1, there is a canonical square-zero supercharge $Q \in \Sigma$ determined by the complex structure on $L \times N$ and a complex half-density.

The dimensional reduction of 10d super Yang–Mills on $L \times N$ along $\mathrm{Re}\colon N \to N_\mathbb{R}$ is by definition the $(5+\dim(L))$-dimensional super Yang–Mills on $L \times N_\mathbb{R}$. Since $N \cong N_\mathbb{R} \oplus iN_\mathbb{R}$, this theory carries an action of the R-symmetry group $G_R = \mathrm{Spin}(N_\mathbb{R})$. We consider a twisting homomorphism $\phi\colon \mathrm{SU}(L) \times \mathrm{Spin}(N_\mathbb{R}) \to G_R = \mathrm{Spin}(N_\mathbb{R})$ given by the projection onto the second factor under which Q is preserved.

Theorem 5.1. The twist of $(5+\dim(L))$-dimensional super Yang–Mills on $L \times N_\mathbb{R}$ by Q is perturbatively equivalent to the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(L \times (N_\mathbb{R})_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{SU}(L) \times \mathrm{Spin}(N_\mathbb{R})$-equivariant.

Proof. By Theorem 4.4 the twist of 10d $\mathcal{N} = 1$ super Yang–Mills on $L \times N$ by Q is perturbatively equivalent to the holomorphic Chern–Simons theory. Moreover, the equivalence is $\mathrm{SU}(L) \times \mathrm{SU}(N)$-equivariant. By Proposition 1.57 we get that the dimensional reduction of holomorphic Chern–Simons on $L \times N$ along $\mathrm{Re}\colon N \to N_\mathbb{R}$ is isomorphic to the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(L \times N_\mathbb{R}, B\mathfrak{g})$ and this isomorphism is $\mathrm{SU}(L) \times \mathrm{SO}(N_\mathbb{R})$-equivariant, where $\mathrm{SO}(N_\mathbb{R})$ acts on N via the homomorphism (2).

Therefore, we just need to establish that the $\mathrm{Spin}(N_\mathbb{R})$-action on the twisted $(5+\dim(L))$-dimensional super Yang–Mills obtained using the twisting homomorphism coincides with the $\mathrm{Spin}(N_\mathbb{R})$-action on the generalized Chern–Simons theory. The $\mathrm{Spin}(N_\mathbb{R})$-action on the fields of $(5+\dim(L))$-dimensional super Yang–Mills is obtained via the homomorphism

$$\mathrm{Spin}(N_\mathbb{R}) \xrightarrow{\text{diagonal}} \mathrm{Spin}(N_\mathbb{R}) \times \mathrm{Spin}(N_\mathbb{R}) \hookrightarrow \mathrm{Spin}(N_\mathbb{R} \oplus N_\mathbb{R}),$$

where the diagonal embedding comes from the identity map to the partial Lorentz group $\mathrm{Spin}(N_\mathbb{R})$ and the twisting homomorphism, i.e. the identity map, to the R-symmetry group $G_R = \mathrm{Spin}(N_\mathbb{R})$. The $\mathrm{SO}(N_\mathbb{R})$-action on the fields of the generalized Chern–Simons theory is given by the composite

$$\mathrm{SO}(N_\mathbb{R}) \xrightarrow{(2)} \mathrm{SU}(N) \longrightarrow \mathrm{SO}(N_\mathbb{R} \oplus N_\mathbb{R})$$

The claim then follows from the commutativity of the diagram

$$\begin{tikzcd} \mathrm{SO}(N_\mathbb{R}) \arrow[r, "\text{diagonal}"] \arrow[d, "(2)"] & \mathrm{SO}(N_\mathbb{R}) \times \mathrm{SO}(N_\mathbb{R}) \arrow[d] \\ \mathrm{SU}(N) \arrow[r] & \mathrm{SO}(N_\mathbb{R} \oplus N_\mathbb{R}). \end{tikzcd}$$

□

We will now concentrate on the 9-dimensional case.

Theorem 5.2. The minimal twist of 9d $\mathcal{N}=1$ super Yang–Mills on $M=\mathbb{C}^4\times\mathbb{R}$ is perturbatively equivalent to the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(\mathbb{C}^4\times\mathbb{R}_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is SU(4)-equivariant.

Proof. Any square-zero supercharge in the 9-dimensional supersymmetry algebra is square-zero in the 10-dimensional supersymmetry algebra. The claim follows from Theorem 5.1 applied to $L=\mathbb{C}^4$. □

6 Dimension 8

The 8-dimensional supersymmetry algebra has odd part $\Sigma\cong S_+\otimes W\oplus S_-\otimes W^*$, where S_+, S_- are the 8-dimensional semi-spin representations of $\mathrm{Spin}(8,\mathbb{C})$ and W is a complex vector space. The semi-spin representations carry nondegenerate symmetric bilinear pairings $S_\pm\otimes S_\pm\to\mathbb{C}$. There is a Yang–Mills theory with $\mathcal{N}=1$ supersymmetry, so we fix $W=\mathbb{C}$.

6.1 $\mathcal{N}=1$ Super Yang–Mills

We consider $\mathcal{N}=1$ super Yang–Mills theory on $M=\mathbb{R}^8$ with the Euclidean metric. It admits R-symmetry group $G_R=\mathrm{Spin}(2,\mathbb{C})$ which acts with weight $1/2$ on W and weight $-1/2$ on W^*.

This theory admits three twists:

- Supercharges $(Q,0)$ and $(0,Q)$ with $(Q,Q)_{S_\pm}=0$. These are holomorphic. Moreover, we have an embedding $\alpha\colon \mathrm{U}(1)\hookrightarrow\mathrm{Spin}(2,\mathbb{C})$ under which they have weight 1, so they give rise to a $\mathbb{Z}$-graded holomorphic theory. Such a supercharge is stabilized by $G=\mathrm{SU}(4)\subset\mathrm{Spin}(8,\mathbb{C})$. We have a twisting homomorphism $\phi\colon \mathrm{MU}(4)\xrightarrow{\det^{1/2}}\mathrm{U}(1)\xrightarrow{\alpha} G_R$, so the twisted theory carries an action of MU(4).
- Supercharges $(Q,0)$ and $(0,Q)$ with $(Q,Q)_{S_\pm}\neq 0$. These are topological. As before, we may choose a compatible homomorphism α, so they give rise to a $\mathbb{Z}$-graded topological theory. Such a supercharge is stabilized by $\mathrm{Spin}(7,\mathbb{R})\subset\mathrm{Spin}(8,\mathbb{C})$.
- Square-zero supercharges (Q_+,Q_-) where both $Q_\pm$ are nonzero. These have 5 invariant directions and do not admit a compatible homomorphism α, so they give rise to a $\mathbb{Z}/2\mathbb{Z}$-graded theory. We have $(Q_\pm,Q_\pm)_{S_\pm}=0$. The supercharge Q_+ is stabilized by $\mathrm{SU}(4)\subset\mathrm{Spin}(8,\mathbb{R})$. The supercharge Q_- is stabilized by $\mathrm{SU}(3)\subset\mathrm{SU}(4)\subset\mathrm{Spin}(8,\mathbb{R})$. We have a twisting homomorphism $\phi\colon\mathrm{SU}(3)\times\mathrm{Spin}(2,\mathbb{R})\to G_R=\mathrm{Spin}(2,\mathbb{C})$ given by projection onto the second factor, so the twisted theory in fact carries an action of $\mathrm{SU}(3)\times\mathrm{Spin}(2,\mathbb{R})$.

6.1.1 Holomorphic Twist

Suppose $Q\in S_+$ such that $(Q,Q)_{S_+}=0$. As in Section 4.1.1, the data of such Q is equivalent to the data of a Kähler structure L on $V_\mathbb{R}$ together with a complex half-density on L.

We consider the twisting homomorphism $\det^{1/2}\colon\mathrm{MU}(4)\to\mathrm{Spin}(2,\mathbb{C})$ under which Q becomes scalar. Moreover, we have an embedding $\alpha\colon\mathrm{U}(1)\subset\mathrm{Spin}(2,\mathbb{C})$, so the theory is $\mathbb{Z}$-graded and carries an MU(4)-action. In fact, this action will factor through U(4) by a direct observation.

Fields: The BRST fields are given by:

- Gauge fields $A_{1,0} \in \Omega^{1,0}(M;\mathfrak{g})$ and $A_{0,1} \in \Omega^{0,1}(M;\mathfrak{g})$.
- Scalar fields $a \in \Omega^{4,0}(M;\mathfrak{g})[2]$ and $\widetilde{a} \in \Omega^{0,4}(M;\mathfrak{g})[-2]$.
- Fermions $\chi \in \Omega^0(M;\mathfrak{g})[-1]$, $B \in \Omega^{0,2}(M;\mathfrak{g})[-1]$, $\widetilde{\chi} \in \Omega^{0,4}(M;\mathfrak{g})[-1]$, $\rho \in \Omega^{1,0}(M;\mathfrak{g})[1]$ and $C \in \Omega^{3,0}(M;\mathfrak{g})[1]$.
- A ghost field $c \in \Omega^0(M;\mathfrak{g})[1]$.

Theorem 6.1. The holomorphic twist of 8d $\mathcal{N}=1$ super Yang–Mills on $M=\mathbb{R}^8$ is perturbatively equivalent to the holomorphic BF theory on $M \cong \mathbb{C}^4$ with the space of fields $T^*[-1]\operatorname{Map}(M, B\mathfrak{g})$. Moreover, the equivalence is U(4)-equivariant.

Proof. 8d $\mathcal{N}=1$ super Yang–Mills theory is obtained by dimensionally reducing 10d $\mathcal{N}=1$ super Yang–Mills theory. Under dimensional reduction the 10d fields from Section 4.1.1 decompose as follows:

$$\begin{aligned} A_{1,0} &\rightsquigarrow A_{1,0}, \widetilde{a} \\ A_{0,1} &\rightsquigarrow A_{0,1}, a \\ \rho &\rightsquigarrow \rho, \widetilde{\chi} \\ B &\rightsquigarrow B, C. \end{aligned}$$

The claim about the underlying $\mathbb{Z}/2\mathbb{Z}$-graded theories follows by applying dimensional reduction (Proposition 1.56) to the computation of the minimal twist of 9d $\mathcal{N}=1$ super Yang–Mills (Theorem 5.2). We are left to check that the equivalence respects the gradings and the U(4)-action. Indeed, the equivalence given by Theorem 5.2 eliminates fields $A_{1,0}, \widetilde{a}, \rho, \chi, \widetilde{\chi}$ and hence the underlying local L_∞ algebra after the twist becomes

$$\Omega^0(\mathbb{C}^4;\mathfrak{g})_c \longrightarrow \Omega^{0,1}(\mathbb{C}^4;\mathfrak{g})_{A_{0,1}} \longrightarrow \Omega^{0,2}(\mathbb{C}^4;\mathfrak{g})_B \longrightarrow \Omega^{0,3}(\mathbb{C}^4;\mathfrak{g})_{C^*} \longrightarrow \Omega^{0,4}(\mathbb{C}^4;\mathfrak{g})_{a^*}$$
$$\oplus$$
$$\Omega^{4,0}(\mathbb{C}^4;\mathfrak{g})_a \longrightarrow \Omega^{4,1}(\mathbb{C}^4;\mathfrak{g})_C \longrightarrow \Omega^{4,2}(\mathbb{C}^4;\mathfrak{g})_{B^*} \longrightarrow \Omega^{4,3}(\mathbb{C}^4;\mathfrak{g})_{A_{0,1}^*} \longrightarrow \Omega^{4,4}(\mathbb{C}^4;\mathfrak{g})_{c^*}$$

concentrated in cohomological degrees $-1,\dots,4$. These fields have the same degrees as in the holomorphic BF theory. □

6.1.2 Topological Twist

Next we discuss the case of the topological twist. We are going to prove that it is perturbatively trivial. In fact, it will be useful to study a degeneration of the topological twist to a holomorphic twist and decribe the corresponding family of twisted theories.

Let $V_{\mathbb{R}} = \mathbb{R}^8$ and $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Fix a Kähler structure on $\mathbb{R}^8$ and denote by $L \subset V$ the i-eigenspace of the complex structure. Moreover, fix a complex volume form on L. Under $\mathrm{SU}(L) \subset \mathrm{Spin}(V)$ the semi-spin representation S_+ decomposes as

$$S_+ \cong \mathbb{C}Q_0 \oplus \wedge^2 L \oplus \mathbb{C}\overline{Q}_0.$$

The scalar spinorial pairing $S_+ \otimes S_+ \to \mathbb{C}$ is given by pairing the outer terms with each other and $\wedge^2 L$ with itself using the complex volume form on L. Consider a family of square-zero supercharges

$$Q_t = Q_0 + t\overline{Q}_0 \in S_+ \tag{20}$$

for $t \in \mathbb{C}$. We have

$$(Q_t, Q_t) = t,$$

so at $t = 0$ we have a holomorphic supercharge and and $t \neq 0$ we have a topological supercharge.

We will use the notation for fields of 8d $\mathcal{N} = 1$ super Yang–Mills from Section 6.1.1. Using the Calabi–Yau structure we will regard $C \in \Omega^{0,1}(\mathbb{C}^4; \mathfrak{g})[1]$ and $\widetilde{\chi} \in \Omega^0(\mathbb{C}^4; \mathfrak{g})[-1]$. First, we are going to write the functionals (6) and (7) in terms of these fields.

Proposition 6.2. The functionals $I^{(1)}$ and $I^{(2)}$ (see (6) and (7)) in terms of the fields of 8d $\mathcal{N} = 1$ super Yang–Mills are

$$\begin{aligned} I^{(1)}(Q_t) &= \int \mathrm{dvol}\left(-(\rho, A^*_{1,0}) - t(C, A^*_{0,1}) - (\widetilde{\chi} + t\chi)\widetilde{a}^*\right) \\ &\quad + \int \mathrm{dvol}\left((F_{0,2}, B^*) + t\Omega^{-1}F_{2,0} \wedge B^* + (\overline{\partial}_{A_{0,1}}a, C^*) + (t\partial_{A_{1,0}}a, \rho^*) + \frac{1}{2}\Lambda F_{1,1}(\chi^* - t\widetilde{\chi}^*) + \frac{1}{2}[a, \widetilde{a}](\chi^* + t\widetilde{\chi}^*)\right) \\ I^{(2)}(Q_t, Q_t) &= \int \mathrm{dvol}\left(t\chi^*\widetilde{\chi}^* + \frac{t}{2}\Omega^{-1}B^* \wedge B^* - \frac{1}{4}(\chi^* + t\widetilde{\chi}^*)^2 + tac^*\right). \end{aligned}$$

The action of the twisted 8d super Yang–Mills theory is given by

$$S_{Q_t} = S_{\mathrm{BRST}} + S_{\mathrm{anti}} + I^{(1)}(Q_t) + I^{(2)}(Q_t),$$

where S_{BRST} and S_{anti} are given by (16) and (17) respectively.

We have a homomorphism $\alpha\colon \mathrm{U}(1) \to G_R = \mathrm{Spin}(2, \mathbb{R})$ with respect to which Q_t has weight 1, so the Q_t-twisted theory will have a $\mathbb{Z}$-grading.

Theorem 6.3. The twist of 8d $\mathcal{N} = 1$ super Yang–Mills with respect to the family Q_t of square-zero supercharges is perturbatively equivalent to the holomorphic Hodge theory $\mathrm{Map}(\mathbb{C}^4, B\mathfrak{g}_{\mathrm{Hod}})$. Moreover, this equivalence is $\mathrm{SU}(4)$-equivariant.

Proof. The proof proceeds as in the proof of Theorem 4.4 with slight modifications.

Observe that the quadruple of fields $\{\chi^*, \chi, \widetilde{\chi}^*, \widetilde{\chi}\}$ has the same Poisson brackets as the quadruple $\{\chi^* - t\widetilde{\chi}^*, \chi, \widetilde{\chi}^*, \widetilde{\chi} + t\chi\}$. Therefore, we may eliminate the fields $\chi^* - t\widetilde{\chi}^*, \chi$ using Proposition 1.8. We then have trivial BRST doublets $\{\widetilde{\chi} + t\chi, \widetilde{a}\}$ and $\{\rho, A_{1,0}\}$ which may be eliminated using Proposition 1.10. We are left with the action

$$S_{BF} + \int \mathrm{dvol}\left(-t(C, A^*_{0,1}) + tac^* + \frac{t}{2}\Omega^{-1}B^* \wedge B^*\right),$$

where S_{BF} is the action functional of the holomorphic twist at $t = 0$. Since the extra terms are quadratic in the fields, the claim is reduced to a comparison of the underlying local L_∞ algebra of the twisted theory and that of the holomorphic Hodge theory. The former is given by (cf. the proof of Theorem 6.1)

$$\begin{tikzcd} & \Omega^0(\mathbb{C}^4; \mathfrak{g})_c \arrow[r] & \Omega^{0,1}(\mathbb{C}^4; \mathfrak{g})_{A_{0,1}} \arrow[r] & \Omega^{0,2}(\mathbb{C}^4; \mathfrak{g})_B \arrow[r] & \Omega^{0,3}(\mathbb{C}^4; \mathfrak{g})_{C^*} \arrow[r] & \Omega^{0,4}(\mathbb{C}^4; \mathfrak{g})_{a^*} \\ \Omega^0(\mathbb{C}^4; \mathfrak{g})_a \arrow[r] \arrow[ur, dashed, "t\,\mathrm{id}"] & \Omega^{0,1}(\mathbb{C}^4; \mathfrak{g})_C \arrow[r] \arrow[ur, dashed, "t\,\mathrm{id}"] & \Omega^{0,2}(\mathbb{C}^4; \mathfrak{g})_{B^*} \arrow[r] \arrow[ur, dashed, "t\,\mathrm{id}"] & \Omega^{0,3}(\mathbb{C}^4; \mathfrak{g})_{A^*_{0,1}} \arrow[r] \arrow[ur, dashed, "t\,\mathrm{id}"] & \Omega^{0,4}(\mathbb{C}^4; \mathfrak{g})_{c^*} \arrow[ur, dashed, "t\,\mathrm{id}"] & \end{tikzcd}$$

which is exactly the local L_∞ algebra of the holomorphic Hodge theory. ☐

Corollary 6.4. The topological twist of 8d $\mathcal{N} = 1$ super Yang–Mills is perturbatively trivial.

Proof. The topological twist of 8d $\mathcal{N} = 1$ super Yang–Mills is the twist by Q_t with $t \neq 0$. By Theorem 6.3 it is equivalent to the $t \neq 0$ specialization of the holomorphic Hodge theory which by Proposition 1.52 is perturbatively trivial. ☐

6.1.3 Partially Topological Twist

Finally we discuss the case of the partially topological supercharge $(Q_+, Q_-) \in \Sigma$. We consider the twisting homomorphism $\phi\colon \mathrm{SU}(3) \times \mathrm{Spin}(2, \mathbb{R}) \to G_R = \mathrm{Spin}(2, \mathbb{C})$ given by projection on the second factor.

Theorem 6.5. The partially topological twist of 8d $\mathcal{N} = 1$ super Yang–Mills is perturbatively equivalent to the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(\mathbb{C}^3 \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{SU}(3) \times \mathrm{Spin}(2, \mathbb{R})$-equivariant.

Proof. Since Q_+ and Q_- satisfy $(Q_\pm, Q_\pm)_{S_\pm} = 0$, they lift to a square-zero supercharge in the 10-dimensional supersymmetry algebra. The claim follows from Theorem 5.1 applied to $L = \mathbb{C}^3$. □

7 Dimension 7

The 7-dimensional supersymmetry algebra has odd part $\Sigma \cong S \otimes W$, where S is the 8-dimensional spin representation of $\mathrm{Spin}(7, \mathbb{C})$ and W is a complex symplectic vector space. The spin representation carries a nondegenerate symmetric bilinear pairing $S \otimes S \to \mathbb{C}$. There is a Yang–Mills theory with $\mathcal{N} = 1$ supersymmetry, so we fix $W = \mathbb{C}^2$.

7.1 $\mathcal{N} = 1$ Super Yang–Mills

We consider $\mathcal{N} = 1$ super Yang–Mills theory on $M = \mathbb{R}^7$ with the Euclidean metric. It admits R-symmetry group $G_R = \mathrm{Spin}(3, \mathbb{C})$ with W the two-dimensional spin representation.

This theory admits three twists:

- Rank 1 supercharges $Q = \alpha \otimes w \in S \otimes W$, where $(\alpha, \alpha)_S = 0$. These are minimal, i.e. the number of invariant directions is 4. We have a homomorphism $\alpha\colon \mathrm{U}(1) \to G_R = \mathrm{Spin}(3, \mathbb{C})$ under which they have weight 1. We also have a twisting homomorphism $\phi\colon \mathrm{MU}(3) \xrightarrow{\det^{1/2}} \mathrm{U}(1) \to \mathrm{Spin}(3, \mathbb{C})$, so the twisted theory is $\mathbb{Z}$-graded and carries an action of $\mathrm{MU}(3)$.
- Rank 1 supercharges $Q = \alpha \otimes w \in S \otimes W$, where $(\alpha, \alpha)_S \neq 0$. These are topological and stabilized by $G_2 \subset \mathrm{Spin}(7, \mathbb{C})$. We have a homomorphism $\alpha\colon \mathrm{U}(1) \to G_R$ under which they have weight 1.
- Square-zero supercharges Q of rank 2. These have 5 invariant directions and do not admit a compatible homomorphism α, so they give rise to a $\mathbb{Z}/2\mathbb{Z}$-graded theory. We have a twisting homomorphism $\phi\colon \mathrm{SU}(2) \times \mathrm{Spin}(3, \mathbb{R}) \to G_R = \mathrm{Spin}(3, \mathbb{C})$ given by projection on the second factor, so the theory carries an action of $\mathrm{SU}(2) \times \mathrm{Spin}(3, \mathbb{R})$.

7.1.1 Minimal Twist

Denote the image of $\Gamma(Q, -)\colon \Sigma \to V$ by $L^\perp$, so that its orthogonal complement $L \subset V$ is a 3-dimensional isotropic subspace. As in Section 5.1.1, the data of a partially topological supercharge is equivalent to the choice of a one-dimensional subspace $N_\mathbb{R} \subset V_\mathbb{R}$ and a complex structure on $V_\mathbb{R}/N_\mathbb{R}$ together with a half-density.

It will be convenient to perform a computation of the twist in a slightly more general setting which will be useful for lower-dimensional computations.

Suppose L is a complex vector space equipped with a Hermitian structure. Suppose $N_\mathbb{R} = \mathbb{R}^{4-\dim(L)}$ equipped with a Euclidean metric and a spin structure. Denote by $N = N_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ its complexification which carries a complex half-density. Let $V_\mathbb{R} = L \times N$ (an 8-dimensional real vector space). We set $W^8 = \det(L)^{-1/2}$ as a representation of $\mathrm{MU}(L)$. By the results of Section 6.1.1, there is a canonical square-zero supercharge $Q \in \Sigma$ determined by the complex structure on $L \times N$ and a complex half-density on N.

The dimensional reduction of 8d $\mathcal{N} = 1$ super Yang–Mills on $L \times N$ along $\mathrm{Re}\colon N \to N_\mathbb{R}$ is by definition the $(4 + \dim(L))$-dimensional super Yang–Mills on $L \times N_\mathbb{R}$ with 16 supercharges. Since $N \cong N_\mathbb{R} \oplus iN_\mathbb{R}$, this theory carries an action of the R-symmetry group $G_R = \mathrm{Spin}(2, \mathbb{C}) \times \mathrm{Spin}(N_\mathbb{R})$. We consider the grading $\alpha\colon \mathrm{U}(1) \to G_R$ given by the embedding into the first copy of $\mathrm{U}(1)$ and a twisting homomorphism

$$\phi\colon \mathrm{MU}(L) \times \mathrm{Spin}(N_\mathbb{R}) \xrightarrow{\det^{1/2} \times \mathrm{id}} G_R = \mathrm{Spin}(2, \mathbb{R}) \times \mathrm{Spin}(N_\mathbb{R}).$$

Theorem 7.1. The twist of $(4+\dim(L))$-dimensional super Yang–Mills on $L \times N_\mathbb{R}$ by Q is perturbatively equivalent to the generalized BF theory with the space of fields $T^*[-1]\,\mathrm{Map}(L \times (N_\mathbb{R})_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{MU}(L) \times \mathrm{Spin}(N_\mathbb{R})$-equivariant.

Proof. By Theorem 6.1 the twist of 8d $\mathcal{N} = 1$ super Yang–Mills on $L \times N$ by Q is perturbatively equivalent to the holomorphic BF theory. Moreover, the equivalence is $\mathrm{MU}(L) \times \mathrm{SU}(N)$-equivariant. By Proposition 1.59 we get that the dimensional reduction of holomorphic BF theory on $L \times N$ along $\mathrm{Re}\colon N \to N_\mathbb{R}$ is isomorphic to the generalized BF theory with the space of fields $T^*[-1]\,\mathrm{Map}(L \times N_\mathbb{R}, B\mathfrak{g})$ and this isomorphism is $\mathrm{MU}(L) \times \mathrm{SO}(N_\mathbb{R})$-equivariant, where $\mathrm{SO}(N_\mathbb{R})$ acts on N via the homomorphism (2). $\square$

The 7-dimensional result immediately follows.

Theorem 7.2. The minimal twist of 7d $\mathcal{N} = 1$ super Yang–Mills on $M = \mathbb{C}^3 \times \mathbb{R}$ is perturbatively equivalent to the generalized BF theory with the space of fields $T^*[-1]\,\mathrm{Map}(\mathbb{C}^3 \times \mathbb{R}_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{U}(3)$-equivariant.

7.1.2 Topological Twist

Next we study the topological twist. As in the case of the minimal twist, we will perform a computation applicable in lower dimensions as well.

Let L be a complex vector space equipped with a Hermitian structure and a complex half-density. Suppose $N_\mathbb{R} = \mathbb{R}^{4-\dim(L)}$ equipped with a Euclidean metric and a spin structure. Denote by $N = N_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ its complexification which carries a complex half-density. Let $V_\mathbb{R} = L \times N$, a real 8-dimensional vector space equipped with a complex structure and a complex half-density. Using results of Section 6.1.2 we obtain a family Q_t of 8d square-zero supercharges.

The dimensional reduction of 8d $\mathcal{N} = 1$ super Yang–Mills on $L \times N$ along $\mathrm{Re}\colon N \to N_\mathbb{R}$ gives $(4 + \dim(L))$-dimensional super Yang–Mills on $L \times N_\mathbb{R}$ with 16 supercharges. The R-symmetry group is $G_R = \mathrm{Spin}(N_\mathbb{R} \oplus \mathbb{R}^2)$ and we consider the grading given by the homomorphism $\alpha\colon \mathrm{U}(1) \hookrightarrow \mathrm{Spin}(2, \mathbb{R}) \times \mathrm{Spin}(N_\mathbb{R}) \subset G_R$ given by embedding into the first factor. We consider the twisting homomorphism given by $\mathrm{Spin}(N_\mathbb{R}) \hookrightarrow \mathrm{Spin}(2, \mathbb{R}) \times \mathrm{Spin}(N_\mathbb{R}) \subset G_R$ given by embedding into the second factor.

Theorem 7.3. The twist of $(4+\dim(L))$-dimensional super Yang–Mills on $L \times N_\mathbb{R}$ by Q_t is perturbatively equivalent to the generalized Hodge theory with the space of fields $\mathrm{Map}(L \times (N_\mathbb{R})_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$. Moreover, the equivalence is $\mathrm{SU}(L) \times \mathrm{Spin}(N_\mathbb{R})$-equivariant.

Proof. By Theorem 6.3 the twist of 8d $\mathcal{N} = 1$ super Yang–Mills on $L \times N$ by Q_t is perturbatively equivalent to the holomorphic Hodge theory. Moreover, the equivalence is $\mathrm{SU}(L) \times \mathrm{SU}(N)$-equivariant. The claim then follows from Corollary 1.58. $\square$

Now let $L = \mathbb{C}^3$ and $N_\mathbb{R} = \mathbb{R}$. Dimensionally reducing the family Q_t along $L \times N \to L \times N_\mathbb{R}$ we obtain a family of supercharges which are topological for $t \neq 0$ and has 4 invariant directions for $t = 0$. In other words, at $t = 0$ we get the minimal twist and at $t \neq 0$ the topological twist.

Theorem 7.4. The twist of 7d $\mathcal{N} = 1$ super Yang–Mills with respect to the family Q_t of square-zero supercharges is perturbatively equivalent to the generalized Hodge theory $\mathrm{Map}(\mathbb{C}^3 \times \mathbb{R}_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$. Moreover, this equivalence is SU(3)-equivariant.

Corollary 7.5. The topological twist of 7d $\mathcal{N} = 1$ super Yang–Mills is perturbatively trivial.

7.1.3 Partially Topological Twist

Finally, we discuss the case of a partially topological twist. We consider the twisting homomorphism $\phi\colon \mathrm{SU}(3) \times \mathrm{Spin}(3, \mathbb{R}) \to G_R = \mathrm{Spin}(3, \mathbb{C})$ given by projection on the second factor.

Theorem 7.6. The partially topological twist of 7d $\mathcal{N} = 1$ super Yang–Mills is perturbatively equivalent to the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}^3_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{SU}(2) \times \mathrm{Spin}(3, \mathbb{R})$-equivariant.

Proof. Any partially topological supercharge in the 7-dimensional supersymmetry algebra lifts to a square-zero supercharge in the 10-dimensional supersymmetry algebra. The claim follows from Theorem 5.1 applied to $L = \mathbb{C}^2$. □

8 Dimension 6

The 6-dimensional supersymmetry algebra has odd part $\Sigma \cong S_+ \otimes W_+ \oplus S_- \otimes W_-$, where S_+, S_- are the 4-dimensional semi-spin representations of $\mathrm{Spin}(6, \mathbb{C}) \cong \mathrm{SL}(4, \mathbb{C})$ and W_+, W_- are complex symplectic vector spaces. We have isomorphisms $S_+ \cong S_-^*$.

There are Yang–Mills theories with $\mathcal{N} = (1, 0)$ or $\mathcal{N} = (1, 1)$ supersymmetry, which we consider separately.

8.1 $\mathcal{N} = (1, 0)$ Super Yang–Mills

The general setup for $\mathcal{N} = (1, 0)$ super Yang–Mills is described in Section 3 which we now recall. Let U be a complex symplectic $\mathfrak{g}$-representation. We consider $\mathcal{N} = (1, 0)$ super Yang–Mills theory on $M = \mathbb{R}^6$ with the Euclidean metric. We fix $W_- = 0$ and $W_+ = \mathbb{C}^2$ equipped with a symplectic structure. The R-symmetry group depends on the type of the representation U:

- In general, the theory admits an R-symmetry group $G_R = \mathrm{SL}(2, \mathbb{C})$ with W_+ the two-dimensional defining representation.
- If $U = T^*R = R \oplus R^*$ for a $\mathfrak{g}$-representation R, then the theory admits an R-symmetry group $G_R = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$, where $\mathrm{GL}(1, \mathbb{C})$ acts trivially on W_+, with weight 1 on R and with weight -1 on R^*.

This theory admits a unique twist:

- A square-zero supercharge $Q \neq 0 \in \Sigma$ has 3 invariant directions, so it gives rise to a holomorphic theory. If the representation U is of cotangent type, we have a compatible homomorphism $\alpha\colon \mathrm{U}(1) \to G_R = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$ given by a diagonal embedding, so in this case we get a $\mathbb{Z}$-grading. Such a supercharge is stabilized by $G = \mathrm{SU}(3) \subset \mathrm{Spin}(6, \mathbb{C})$. We have a twisting homomorphism $\phi\colon \mathrm{MU}(3) \xrightarrow{\det^{1/2}} \mathrm{U}(1) \hookrightarrow \mathrm{SL}(2, \mathbb{C})$, so the twisted theory carries an MU(3)-action.

8.1.1 Holomorphic Twist

Consider a nonzero $Q \in S_+ \otimes W_+$. Since $\wedge^2(S_+) \cong V$, the square-zero condition is equivalent to the condition that Q has rank 1, i.e. $Q = q_+ \otimes w_1 \in S_+ \otimes W_+$. We will also fix $w_2 \in W_+$ such that $(w_1, w_2) = 1$.

As in Section 4.1.1, the data of q_+ is equivalent to the data of a Kähler structure L on $V_\mathbb{R}$ together with a complex half-density on L.

Under the embedding $\mathrm{MU}(L) \subset \mathrm{Spin}(V_\mathbb{R})$, the semi-spin representations S_+, S_- decompose as

$$S_+ = \det(L)^{1/2} \oplus L \otimes \det(L)^{-1/2}, \qquad S_- = \det(L)^{-1/2} \oplus L^* \otimes \det(L)^{1/2},$$

where $q_+ \in S_+$ lies in the first summand.

We fix an embedding $\mathrm{U}(1) \subset \mathrm{SL}(2, \mathbb{C})$ under which $w_1 \in W_+$ has weight 1. Under the composite

$$\phi\colon \mathrm{MU}(3) \xrightarrow{\det^{1/2}} \mathrm{U}(1) \subset \mathrm{SL}(2, \mathbb{C})$$

we obtain that $W_+ \cong \det(L)^{-1/2} w_1 \oplus \det(L)^{1/2} w_2$.

We will now rewrite the fields using the twisting homomorphism ϕ from MU(3), where we denote by K the canonical bundle of $L = \mathbb{C}^3$.

Fields: The BRST fields are given by:

- Gauge fields $A_{1,0} \in \Omega^{1,0}(M; \mathfrak{g})$, $A_{0,1} \in \Omega^{0,1}(M; \mathfrak{g})$.
- Gauge fermions $\chi \in \Omega^0(M; \Pi\mathfrak{g})$, $\xi \in \Omega^{3,0}(M; \Pi\mathfrak{g})$, $B \in \Omega^{0,2}(M; \Pi\mathfrak{g})$, $\rho \in \Omega^{1,0}(M; \Pi\mathfrak{g})$.
- Matter bosons $\nu \in \Omega^0(M; U \otimes K^{-1/2})$, $\phi \in \Omega^0(M; U \otimes K^{1/2})$.
- Matter fermions $\psi \in \Omega^{0,1}(M; \Pi U \otimes K^{1/2})$, $\widetilde{\nu} \in \Omega^0(M; \Pi U \otimes K^{-1/2})$.
- A ghost field $c \in \Omega^0(M; \mathfrak{g})[1]$.

Let $\omega \in \Omega^{1,1}(M)$ be the Kähler form. We denote the real volume form on M by

$$\mathrm{dvol} = \frac{\omega^3}{6}.$$

Using Corollary 4.3, the BV action S_{BV} of the Q-twisted theory consists of the sum of the following terms:

$$
\begin{aligned}
S_{\text{gauge}} &= \int \mathrm{dvol} \left(-(F_{2,0}, F_{0,2}) - \frac{1}{4}(\Lambda F_{1,1})^2 \right) + \\
&\quad + \frac{1}{2} \left(\omega(B \wedge \partial_{A_{1,0}}\rho) + \omega^2 \chi \Lambda(\overline{\partial}_{A_{0,1}}\rho) - \omega(\rho \wedge \partial_{A_{1,0}} B) + \xi \overline{\partial}_{A_{0,1}} B \right) \\
S_{\text{matter}} &= \int \Bigg(\mathrm{dvol}((\partial_{A_{1,0}}\nu, \overline{\partial}_{A_{0,1}}\phi) + (\partial_{A_{1,0}}\phi, \overline{\partial}_{A_{0,1}}\nu)) + 2\omega^2 \wedge (\widetilde{\nu}\overline{\partial}_{A_{1,0}}\psi) + \psi \wedge \overline{\partial}_{A_{0,1}}\psi + \\
&\quad + 2\,\mathrm{dvol}(([\xi, \nu], \widetilde{\nu}) + ([\chi, \phi], \widetilde{\nu})) \Bigg) \\
S_{\text{anti}} &= \int \partial_{A_{1,0}} c \wedge A^*_{1,0} + \overline{\partial}_{A_{0,1}} c \wedge A^*_{0,1} + [B, c] \wedge B^* + [\xi, c] \wedge \xi^* + [\chi, c]\chi^* + [\rho, c] \wedge \rho^* + \\
&\quad + \frac{1}{2}[c, c]c^* + [\nu, c] \wedge \nu^* + [\phi, c] \wedge \phi^* + [\psi, c] \wedge \psi^* + [\widetilde{\nu}, c] \wedge \widetilde{\nu}^* \\
I^{(1)}_{\text{gauge}} &= \int \mathrm{dvol} \left(-(\rho, A^*_{1,0}) + \frac{1}{2}(F_{0,2}, B^*) + \frac{1}{2}(F_{1,1}, \chi^*) \right) \\
I^{(1)}_{\text{matter}} &= \int \mathrm{dvol} \left((\widetilde{\nu}, \nu^*) + \frac{1}{2}(\overline{\partial}_{A_{0,1}}\phi, \psi^*) \right) \\
I^{(2)}_{\text{gauge}} &= -\frac{1}{4} \int \mathrm{dvol}(\chi^*)^2.
\end{aligned}
$$

Theorem 8.1. The holomorphic twist of 6d $\mathcal{N} = (1,0)$ super Yang–Mills on $M = \mathbb{R}^6$ with matter valued in a symplectic $\mathfrak{g}$-representation U is perturbatively equivalent to the holomorphic Chern–Simons theory on $M \cong \mathbb{C}^3$ with the space of fields $\mathrm{Sect}(M, (U \otimes K_M^{1/2}) /\!\!/ \mathfrak{g})$. Moreover, the equivalence is MU(3)-equivariant.

Proof. The proof of this theorem is very similar to the proof of Theorem 4.4. First, we eliminate the fields χ and χ^* using Proposition 1.8. We then observe that the action includes the terms $\int \mathrm{dvol}(\rho, A^*_{1,0})$ and $\int \mathrm{dvol}(\widetilde{\nu}, \nu^*)$. In other words, the pairs $(\rho, A_{1,0})$ and $(\nu, \widetilde{\nu})$ form trivial BRST doublets, which can be eliminated using Proposition 1.10. The twisted theory is therefore perturbatively equivalent to the theory with the BV action

$$
\begin{aligned}
S_{\text{BV}} = &\int \xi \overline{\partial}_{A_{0,1}} B + \psi \wedge \overline{\partial}_{A_{0,1}} \psi \\
&+ \mathrm{dvol} \left(\frac{1}{2}(F_{0,2}, B^*) + \frac{1}{2}(\overline{\partial}_{A_{0,1}}\phi, \psi^*) + (\overline{\partial}_{A_{0,1}} c, A^*_{0,1}) + ([B, c], B^*) + ([\xi, c], \xi^*) + \frac{1}{2}[c, c]c^* + [\phi, c]\phi^* + ([\psi, c], \psi^*) \right).
\end{aligned}
$$

Up to rescaling of the antifields, this is the action functional of the required theory. □

If $U = T^*R = R \oplus R^*$, the R-symmetry group is enhanced to $G_R = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$. We have a homomorphism $\alpha\colon \mathrm{U}(1) \hookrightarrow G_R = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{U}(1)$ given by the diagonal embedding which is compatible with the holomorphic supercharge. We may also use a new twisting homomorphism

$$
\widetilde{\phi}\colon \mathrm{MU}(3) \xrightarrow{\det^{1/2}} \mathrm{U}(1) \xrightarrow{\alpha} G_R.
$$

With these modifications the BRST fields are given by:

- Gauge fields $A_{1,0} \in \Omega^{1,0}(M; \mathfrak{g})$, $A_{0,1} \in \Omega^{0,1}(M; \mathfrak{g})$.
- Gauge fermions $\chi \in \Omega^0(M; \mathfrak{g})[-1]$, $\xi \in \Omega^{3,0}(M; \Pi\mathfrak{g})[1]$, $B \in \Omega^{0,2}(M; \mathfrak{g})[-1]$, $\rho \in \Omega^{1,0}(M; \Pi\mathfrak{g})[1]$.
- Matter bosons $\nu \in \Omega^0(M; R^* \oplus R \otimes K^{-1}[-2])$, $\phi \in \Omega^0(M; R \oplus R^* \otimes K[2])$.
- Matter fermions $\psi \in \Omega^{0,1}(M; R[-1] \oplus R^* \otimes K[1])$, $\widetilde{\nu} \in \Omega^0(M; R^*[1] \oplus R \otimes K^{-1}[-1])$.

- A ghost field $c \in \Omega^0(M; \mathfrak{g})[1]$.

Note that the MU(3)-action on the fields factors through U(3) since the square roots of K have canceled out. By comparing the degrees and the transformation rules of the fields in Theorem 8.1 we obtain the following statement.

Theorem 8.2. The holomorphic twist of 6d $\mathcal{N} = (1,0)$ super Yang–Mills on $M = \mathbb{R}^6$ with matter valued in a $\mathfrak{g}$-representation $U = T^*R = R \oplus R^*$ is perturbatively equivalent to the holomorphic BF theory on $M \cong \mathbb{C}^3$ with the space of fields $T^*[-1]\,\mathrm{Map}(M, R/\mathfrak{g})$. Moreover, the equivalence is U(3)-equivariant.

8.2 $\mathcal{N} = (1,1)$ Super Yang–Mills

The 6d $\mathcal{N} = (1,1)$ super Yang–Mills theory is obtained by dimensional reduction from the 10d $\mathcal{N} = (1,0)$ super Yang–Mills. It admits R-symmetry group $G_R = \mathrm{Spin}(4, \mathbb{C})$ under which W_+, W_- are the two semi-spin representations.

Given an element $Q \in S_+ \otimes W_+ \oplus S_- \otimes W_-$ we denote by $W^*_{Q_\pm} \subset S_\pm$ the images of Q. We classify square-zero supercharges according to the ranks of these spaces:

- Rank $(1,0)$ and $(0,1)$. These automatically square to zero and are holomorphic. Such supercharges factor through a copy of the $\mathcal{N} = (1,0)$ (respectively, $\mathcal{N} = (0,1)$) supersymmetry algebra. They admit a twisting homomorphism from MU(3) and a $\mathbb{Z}$-grading $\alpha\colon \mathrm{U}(1) \to G_R$.
- Rank $(1,1)$ and $\langle W^*_{Q_+}, W^*_{Q_-}\rangle = 0$. These automatically square to zero and have 4 invariant directions. There is a $\mathbb{Z}$-grading $\alpha\colon \mathrm{U}(1) \hookrightarrow G_R = \mathrm{SL}(2, CC) \times \mathrm{SL}(2, \mathbb{C})$ given by the diagonal embedding and these admit a twisting homomorphism $\phi\colon \mathrm{MU}(2) \times \mathrm{Spin}(2, \mathbb{R}) \xrightarrow{\det^{1/2} \times \mathrm{id}} \mathrm{U}(1) \xrightarrow{\alpha} G_R$.
- Rank $(1,1)$ and $\langle W^*_{Q_+}, W^*_{Q_-}\rangle \neq 0$. These automatically square to zero and are topological. Such supercharges are stabilized by $\mathrm{SU}(3) \subset \mathrm{Spin}(6, \mathbb{C})$ and have a $\mathbb{Z}$-grading $\alpha\colon \mathrm{U}(1) \to G_R = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ given by the diagonal embedding.
- Rank $(2,2)$. The square-zero supercharges have 5 invariant directions and give rise to a $\mathbb{Z}/2\mathbb{Z}$-graded theory. The twisting homomorphism is given by the obvious embedding $\phi\colon \mathrm{Spin}(4, \mathbb{R}) \to G_R = \mathrm{Spin}(4, \mathbb{C})$, so the twisted theory carries a $\mathrm{Spin}(4, \mathbb{R})$-action.

8.2.1 Holomorphic Twist

The 6d $\mathcal{N} = (1,1)$ super Yang–Mills theory viewed as a $\mathcal{N} = (1,0)$ supersymmetric theory coincides with the 6d $\mathcal{N} = (1,0)$ super Yang–Mills with matter in the representation $U = T^*\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$. Under this isomorphism the R-symmetry group $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$ of 6d $\mathcal{N} = (1,0)$ super Yang–Mills is a subgroup of the R-symmetry group $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ of 6d $\mathcal{N} = (1,1)$ super Yang–Mills. In particular, from Theorem 8.2 we obtain the following statement.

Theorem 8.3. The holomorphic twist of 6d $\mathcal{N} = (1,1)$ super Yang–Mills on $M = \mathbb{R}^6$ is perturbatively equivalent to the holomorphic BF theory on $M \cong \mathbb{C}^3$ with the space of fields $T^*[-1]\,\mathrm{Map}(M, \mathfrak{g}/\mathfrak{g})$. Moreover, the equivalence is U(3)-equivariant.

8.2.2 Rank $(1,1)$ Partially Topological Twist

Let $L = \mathbb{C}^2$ equipped with a Hermitian structure, $N_\mathbb{R} = \mathbb{R}^2$ equipped with a Euclidean structure and $N = N_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ its complexification. Consider the 8-dimensional spacetime $V^8_\mathbb{R} = L \times N$ and the 6-dimensional spacetime $V^6_\mathbb{R} = L \times N_\mathbb{R}$. Under the projection $V^8_\mathbb{R} \to V^6_\mathbb{R}$ a holomorphic square-zero supercharge Q in 8 dimensions dimensionally reduces to a rank $(1,1)$ partially topological square-zero supercharge in 6 dimensions. Therefore, from Theorem 7.1 we obtain the following statement.

Theorem 8.4. The rank $(1,1)$ partially topological twist of 6d $\mathcal{N}=(1,1)$ super Yang–Mills is perturbatively equivalent to the generalized BF theory with the space of fields $T^*[-1]\,\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{MU}(2) \times \mathrm{Spin}(2,\mathbb{R})$-equivariant.

8.2.3 Topological Twist

Let $L=\mathbb{C}^2$ equipped with a Hermitian structure and a complex half-density, $N_{\mathbb{R}}=\mathbb{R}^2$ equipped with a Euclidean structure and $N = N_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. Consider the 8-dimensional spacetime $V^8_{\mathbb{R}} = L \times N$ and the 6-dimensional spacetime $V^6_{\mathbb{R}} = L \times N_{\mathbb{R}}$. Under the projection $V^8_{\mathbb{R}} \to V^6_{\mathbb{R}}$ the family Q_t of 8-dimensional square-zero supercharges given by equation (20) dimensionally reduces to a family of square-zero supercharges which are topological for $t \neq 0$ and have 4 invariant directions at $t=0$. So, we get a rank $(1,1)$ partially topological twist at $t=0$ and a rank $(1,1)$ topological twist at $t\neq 0$. Therefore, from Theorem 7.3 we obtain the following statement.

Theorem 8.5. The twist of 6d $\mathcal{N}=(1,1)$ super Yang–Mills with respect to the family Q_t of square-zero supercharges is perturbatively equivalent to the generalized Hodge theory $\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$. Moreover, this equivalence is $\mathrm{SU}(2) \times \mathrm{Spin}(2,\mathbb{R})$-equivariant.

Corollary 8.6. The topological twist of 6d $\mathcal{N}=(1,1)$ super Yang–Mills is perturbatively trivial.

8.2.4 Rank $(2,2)$ Twist

We consider $L=\mathbb{C}$ equipped with a Hermitian structure and a complex half-density. From Theorem 5.1 we obtain the following statement.

Theorem 8.7. The rank $(2,2)$ twist of 6d $\mathcal{N}=(1,1)$ super Yang–Mills is perturbatively equivalent to the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(\mathbb{C}\times\mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{Spin}(4,\mathbb{R})$-equivariant.

9 Dimension 5

The 5-dimensional supersymmetry algebra has odd part $\Sigma \cong S\otimes W$, where S is the 4-dimensional spin representation of $\mathrm{Spin}(5,\mathbb{C}) \cong \mathrm{Sp}(4,\mathbb{C})$ and W is a complex symplectic vector space. The spin representation carries a symplectic paring $S \otimes S \to \mathbb{C}$.

There are Yang–Mills theories with $\mathcal{N}=1$ or $\mathcal{N}=2$ supersymmetry, which we consider separately.

9.1 $\mathcal{N}=1$ Super Yang–Mills

We fix $W=\mathbb{C}^2$ equipped with a symplectic structure. Let U be a symplectic $\mathfrak{g}$-representation. The 5d $\mathcal{N}=1$ super Yang–Mills theory is obtained by a dimensional reduction from the 6d $\mathcal{N}=(1,0)$ super Yang–Mills theory.

The R-symmetry group coincides with the R-symmetry group in 6 dimensions:

- For a general U the R-symmetry group is $G_R = \mathrm{SL}(2,\mathbb{C})$ with W the two-dimensional defining representation.
- For $U = T^*R$ the R-symmetry group is $G_R = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$, where $\mathrm{GL}(1,\mathbb{C})$ acts trivially on W, with weight 1 on R and with weight -1 on R^*.

The theory admits a unique twist:

- A square-zero supercharge $Q \neq 0 \in \Sigma$ has 3 invariant directions. There is a twisting homomorphism $\phi\colon \mathrm{MU}(2) \xrightarrow{\det^{1/2}} \mathrm{U}(1) \hookrightarrow \mathrm{SL}(2,\mathbb{C})$, so the twisted theory carries an MU(2)-action.

9.1.1 Minimal Twist

A square-zero supercharge Q has rank 1, i.e. $Q = q_+ \otimes w_1$ for some $w_1 \in W$. We also choose $w_2 \in W$ such that $(w_1, w_2) = 1$. We have a twisting homomorphism

$$\phi\colon \mathrm{MU}(2) \xrightarrow{\det^{1/2}} \mathrm{U}(1) \subset \mathrm{SL}(2,\mathbb{C})$$

such that $W \cong \det(L)^{-1/2} w_1 \oplus \det(L)^{1/2} w_2$.

As in Section 5.1.1, the data of q_+ is equivalent to the choice of a one-dimensional subspace $N_\mathbb{R} \subset V_\mathbb{R}$ and a complex structure on $V_\mathbb{R}/N_\mathbb{R}$ together with a complex half-density.

We will perform a computation of the twist in a more general setting which will be useful for lower-dimensional computations.

Suppose L is a complex vector space equipped with a Hermitian structure. Suppose $N_\mathbb{R} = \mathbb{R}^{3-\dim(L)}$ equipped with a Euclidean metric and a spin structure. Denote by $N = N_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ its complexification which carries a complex half-density. Let $V_\mathbb{R} = L \times N$ (a 6-dimensional real vector space). We set $W^6_+ = \det(L)^{-1/2} w_1 \oplus \det(L)^{1/2} w_2$ as a representation of $\mathrm{MU}(L)$. By the results of Section 8.1.1, there is a canonical square-zero supercharge $Q = q_+ \otimes w_1 \in \Sigma$ determined by the complex structure on $L \times N$ and a complex half-density on N.

The dimensional reduction of 6d super Yang–Mills on $L \times N$ along $\mathrm{Re}\colon N \to N_\mathbb{R}$ is by definition the $(3 + \dim(L))$-dimensional super Yang–Mills on $L \times N_\mathbb{R}$ with 8 supercharges. Since $N \cong N_\mathbb{R} \oplus iN_\mathbb{R}$, this theory carries an action of the R-symmetry group $G_R = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{Spin}(N_\mathbb{R})$. We consider a twisting homomorphism

$$\phi\colon \mathrm{MU}(L) \times \mathrm{Spin}(N_\mathbb{R}) \to G_R = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{Spin}(N_\mathbb{R})$$

whose first component is $\mathrm{MU}(L) \to \mathrm{SU}(2)$ as before and the second component is the identity.

Theorem 9.1. The twist by Q of $(3 + \dim(L))$-dimensional super Yang–Mills on $L \times N_\mathbb{R}$ with 8 supercharges and matter valued in a symplectic $\mathfrak{g}$-representation U is perturbatively equivalent to the generalized Chern–Simons theory with the space of fields $\mathrm{Sect}(L \times (N_\mathbb{R})_{\mathrm{dR}}, (U \otimes K_L^{1/2}) /\!\!/ \mathfrak{g})$. Moreover, the equivalence is $\mathrm{MU}(L) \times \mathrm{Spin}(N_\mathbb{R})$-equivariant.

Proof. By Theorem 8.1 the twist of 6d $\mathcal{N} = (1,0)$ super Yang–Mills on $L \times N$ by Q is perturbatively equivalent to the holomorphic Chern–Simons theory with the space of fields $\mathrm{Sect}(L \times N, (U \otimes K_L^{1/2}) /\!\!/ \mathfrak{g})$. By Proposition 1.57 we get that the dimensional reduction of holomorphic Chern–Simons on $L \times N$ along $\mathrm{Re}\colon N \to N_\mathbb{R}$ is isomorphic to the generalized Chern–Simons theory with the space of fields $\mathrm{Sect}(L \times N_\mathbb{R}, (U \otimes K_L^{1/2}) /\!\!/ \mathfrak{g})$ and this isomorphism is $\mathrm{MU}(L) \times \mathrm{SO}(N_\mathbb{R})$-equivariant, where $\mathrm{SO}(N_\mathbb{R})$ acts on N via the homomorphism (2). ∎

We will now concentrate on the 5-dimensional case.

Theorem 9.2. The minimal twist of 5d $\mathcal{N} = 1$ super Yang–Mills on $M = \mathbb{C}^2 \times \mathbb{R}$ with matter valued in a symplectic $\mathfrak{g}$-representation U is perturbatively equivalent to the generalized Chern–Simons theory with the space of fields $\mathrm{Sect}(\mathbb{C}^2 \times \mathbb{R}_{\mathrm{dR}}, (U \otimes K_{\mathbb{C}^2}^{1/2}) /\!\!/ \mathfrak{g})$. Moreover, the equivalence is MU(2)-equivariant.

If U is of cotangent type, we may enhance the R-symmetry group to $G_R = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}) \times \mathrm{Spin}(N_\mathbb{R})$. We have a homomorphism $\alpha\colon \mathrm{U}(1) \to \mathrm{SL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}) \times \mathrm{Spin}(N_\mathbb{R})$ given by the diagonal embedding into the first two components. We also use a new twisting homomorphism

$$\widetilde{\phi}\colon \mathrm{MU}(L) \times \mathrm{Spin}(N_\mathbb{R}) \xrightarrow{\det^{1/2} \times \mathrm{id}} \mathrm{U}(1) \times \mathrm{Spin}(N_\mathbb{R}) \xrightarrow{\alpha \times \mathrm{id}} G_R.$$

Theorem 9.3. The minimal twist of 5d $\mathcal{N} = 1$ super Yang–Mills on $M = \mathbb{C}^2 \times \mathbb{R}$ with matter valued in the $\mathfrak{g}$-representation $U = T^*R = R \oplus R^*$ is perturbatively equivalent to the generalized BF theory with the space of fields $T^*[-1]\,\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}_{\mathrm{dR}}, R/\mathfrak{g})$. Moreover, the equivalence is U(2)-equivariant.

9.2 $\mathcal{N} = 2$ Super Yang–Mills

The 5d $\mathcal{N} = 2$ super Yang–Mills theory is obtained by dimensional reduction from the 10d $\mathcal{N} = (1, 0)$ super Yang–Mills. It admits R-symmetry group $G_R = \mathrm{Spin}(5, \mathbb{C})$ under which W is the 4-dimensional spin representation which carries a symplectic pairing.

An element $Q \in S \otimes W$ gives rise to maps $S^* \to W$ and $W^* \to S$. Both S and W are 4-dimensional symplectic vector spaces and the classification of supercharges will use their relative position.

This theory admits four twists:

- Rank 1. These automatically square to zero and have 3 invariant directions. Such supercharges come from the $\mathcal{N} = 1$ supersymmetry algebra. They admit a twisting homomorphism from MU(2) and a $\mathbb{Z}$-grading $\alpha\colon \mathrm{U}(1) \to G_R$.
- Rank 2, where the image of $W^* \to S$ is Lagrangian. These automatically square to zero and have 4 invariant directions. There is a $\mathbb{Z}$-grading $\alpha\colon \mathrm{U}(1) \to G_R$ and a twisting homomorphism $\phi\colon \mathrm{Spin}(2, \mathbb{R}) \times \mathrm{Spin}(3, \mathbb{R}) \hookrightarrow G_R = \mathrm{Spin}(5, \mathbb{C})$.
- Rank 2, where the image of $W^* \to S$ is symplectic. These automatically square to zero and are topological. There is a $\mathbb{Z}$-grading $\alpha\colon \mathrm{U}(1) \to G_R$ and a twisting homomorphism $\phi\colon \mathrm{Spin}(4, \mathbb{R}) \hookrightarrow G_R = \mathrm{Spin}(5, \mathbb{C})$.
- Rank 4. These are topological and do not admit a compatible homomorphism α. There is the obvious twisting homomorphism $\phi\colon \mathrm{Spin}(5, \mathbb{R}) \to \mathrm{Spin}(5, \mathbb{C})$.

9.2.1 Minimal Twist

The 5d $\mathcal{N} = 2$ super Yang–Mills viewed as a $\mathcal{N} = 1$ supersymmetry theory coincides with the 5d $\mathcal{N} = 1$ super Yang–Mills with matter valued in the representation $U = T^*\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$. From Theorem 9.3 we obtain the following statement.

Theorem 9.4. The minimal twist of 5d $\mathcal{N} = 2$ super Yang–Mills on $M = \mathbb{C}^2 \times \mathbb{R}$ is perturbatively equivalent to the generalized BF theory with the space of fields $T^*[-1]\,\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$. Moreover, the equivalence is U(2)-equivariant.

9.2.2 Rank 2 Partially Topological Twist

Let $L = \mathbb{C}$ equipped with a Hermitian structure, $N_{\mathbb{R}} = \mathbb{R}^3$ equipped with a Euclidean structure and $N = N_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. Consider the 8-dimensional spacetime $V^8_{\mathbb{R}} = L \times N$ and the 5-dimensional spacetime $V^5_{\mathbb{R}} = L \times N_{\mathbb{R}}$. Under the projection $V^8_{\mathbb{R}} \to V^5_{\mathbb{R}}$ a holomorphic square-zero supercharge Q in 8 dimensions dimensionally reduces to a rank 2 partially topological square-zero supercharge in 5 dimensions. Therefore, from Theorem 7.1 we obtain the following statement.

Theorem 9.5. The rank 2 partially topological twist of 5d $\mathcal{N} = 2$ super Yang–Mills is perturbatively equivalent to the generalized BF theory with the space of fields $T^*[-1]\,\mathrm{Map}(\mathbb{C} \times \mathbb{R}^3_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{Spin}(2, \mathbb{R}) \times \mathrm{Spin}(3, \mathbb{R})$-equivariant.

9.2.3 Rank 2 Topological Twist

Let $L = \mathbb{C}$ equipped with a Hermitian structure and a complex half density, $N_{\mathbb{R}} = \mathbb{R}^3$ equipped with a Euclidean structure and $N = N_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. Consider the 8-dimensional spacetime $V_{\mathbb{R}} = L \times N$. Under the projection $\mathrm{Re}\colon N \to N_{\mathbb{R}}$ the family Q_t of 8-dimensional square-zero supercharges given by (20) dimensionally reduces to a family of 5-dimensional square-zero supercharges which at $t \neq 0$ are topological at and $t = 0$ have 4 invariant directions. Since they admit a compatible $\mathbb{Z}$-grading, at $t = 0$ we obtain the rank 2 partially topological twist and at $t \neq 0$ we obtain the rank 2 topological twist. Therefore, from Theorem 7.3 we obtain the following statement.

Theorem 9.6. The twist of 5d $\mathcal{N} = 2$ super Yang–Mills with respect to the family Q_t of square-zero supercharges is perturbatively equivalent to the generalized Hodge theory $\mathrm{Map}(\mathbb{C} \times \mathbb{R}^3_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$. Moreover, this equivalence is $\mathrm{Spin}(3)$-equivariant.

Corollary 9.7. The rank 2 topological twist of 5d $\mathcal{N} = 2$ super Yang–Mills is perturbatively trivial.

9.2.4 Rank 4 Twist

We consider $V_{\mathbb{R}} = \mathbb{R}^5$ equipped with a Euclidean structure and as before let $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. V carries a Hermitian structure and a half-density, so by the results of Section 4.1.1 we obtain a square-zero supercharge Q. Under the projection $\mathrm{Re}\colon V \to V_{\mathbb{R}}$ the supercharge Q dimensionally reduces to a rank 4 supercharge in 5 dimensions. Therefore, from Theorem 5.1 we obtain the following statement.

Theorem 9.8. The rank 4 twist of 5d $\mathcal{N} = 2$ super Yang–Mills is perturbatively equivalent to the topological Chern–Simons theory with the space of fields $\mathrm{Map}(\mathbb{R}^5_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{Spin}(5, \mathbb{R})$-equivariant.

10 Dimension 4

The 4-dimensional supersymmetry algebra has odd part $\Sigma \cong S_+ \otimes W \oplus S_- \otimes W^*$, where S_+, S_- are the 2-dimensional semi-spin representations of $\mathrm{Spin}(4, \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ and W is a complex vector space. The semi-spin representations carry symplectic pairings $S_\pm \otimes S_\pm \to \mathbb{C}$.

There are Yang–Mills theories with $\mathcal{N} = 1, 2, 4$ supersymmetry, which we consider separately.

10.1 $\mathcal{N} = 1$ Super Yang–Mills

The general setup for $\mathcal{N} = 1$ super Yang–Mills is described in Section 3 which we now recall. Let R be a complex $\mathfrak{g}$-representation. We consider $\mathcal{N} = 1$ super Yang–Mills theory on $M = \mathbb{R}^4$ with the Euclidean metric. The theory admits an R-symmetry group $G_R = \mathrm{GL}(1, \mathbb{C})$ which acts on $W = \mathbb{C}$ with weight 1.

Fields: The BRST fields are given by:

- Gauge field $A \in \Omega^1(M; \mathfrak{g})$.
- Gauge fermions $(\lambda_+, \lambda_-) \in \Omega^0(M; \Pi S_+ \otimes \mathfrak{g} \oplus \Pi S_- \otimes \mathfrak{g})$.
- Matter bosons $(\overline{\phi}, \phi) \in \Omega^0(M; R \oplus R^*)$.
- Matter fermions $(\psi_-, \psi_+) \in \Omega^0(M; \Pi S_+ \otimes R^* \oplus \Pi S_- \otimes R)$.
- A ghost field $A_0 \in \Omega^0(M; \mathfrak{g})[1]$.

The R-symmetry acts with weight ± 1 on $\lambda_\pm, \psi_\pm$.

The theory admits a unique twist:

- Elements $Q \in S_+ \oplus S_-$ of rank $(1,0)$ or rank $(0,1)$. Such supercharges are automatically square-zero and are holomorphic. We have a compatible twisting homomorphism
$$\phi\colon \mathrm{MU}(2) \xrightarrow{\det^{1/2}} \mathrm{U}(1) \hookrightarrow G_R$$
with the second arrow the natural embedding. The twist is $\mathbb{Z}$-graded with homomorphism $\alpha\colon \mathrm{U}(1) \hookrightarrow G_R$ given by the natural embedding.

10.1.1 Holomorphic Twist

Choose a complex structure L on $V_\mathbb{R}$. Under the embedding $\mathrm{MU}(L) = \mathrm{MU}(2) \subset \mathrm{Spin}(V_\mathbb{R})$, the semi-spin representations decompose as
$$S_+ = \det(L)^{-1/2} \oplus \det(L)^{1/2}, \qquad S_- = L \otimes \det(L)^{-1/2}.$$
Consider the twisting homomorphism $\phi\colon \mathrm{MU}(2) \to G_R$ under which $W = \det(L)^{-1/2}$. Then the spinorial representation becomes
$$\Sigma = (\mathbb{C}Q \oplus \det(L)^{-1}) \oplus L.$$

The embedding $\alpha\colon \mathrm{U}(1) \hookrightarrow G_R$ makes Q weight 1.

We first decompose the fields of the 4-dimensional $\mathcal{N} = 1$ theory with respect to $\mathrm{MU}(2)$.

Fields: The BRST fields are given by:

- Gauge fields $A_{1,0} \in \Omega^{1,0}(M; \mathfrak{g})$, $A_{0,1} \in \Omega^{0,1}(M; \mathfrak{g})$.
- Gauge fermions $\lambda_0 \in \Omega^0(M; \mathfrak{g})[-1]$, $A_{0,2} \in \Omega^{0,2}(M; \mathfrak{g})[-1]$, $\lambda_{1,0} \in \Omega^{1,0}(M; \mathfrak{g})[1]$.
- Matter bosons $\phi \in \Omega^0(M; R^*)$, $\gamma_0 \in \Omega^0(M; R)$.
- Matter fermions $\psi_0 \in \Omega^0(M; R^*)[1]$, $\beta_{2,0} \in \Omega^{2,0}(M; R^*)[1]$, $\gamma_{0,1} \in \Omega^{0,1}(M; R)[-1]$.
- A ghost field $A_0 \in \Omega^0(M; \mathfrak{g})[1]$.

Let $\omega \in \Omega^{1,1}(M)$ be the Kähler form. We denote the real volume form on M by
$$\mathrm{dvol} = \frac{\omega^2}{2}.$$

Using Corollary 4.3, the BV action S_{BV} of the Q-twisted theory consists of the sum of the following terms:

$$
\begin{aligned}
S_{\text{gauge}} &= \int \mathrm{dvol}\left(-(F_{2,0}, F_{0,2}) - \frac{1}{4}(\Lambda F_{1,1})^2\right) + \frac{1}{2}\left((\lambda_{1,0} \wedge \partial_{A_{1,0}} A_{0,2}) + \omega(\lambda_{1,0} \wedge \overline{\partial}_{A_{0,1}} \lambda_0)\right) \\
S_{\text{matter}} &= \int \mathrm{dvol}\left((\partial_{A_{1,0}}\phi, \overline{\partial}_{A_{0,1}}\gamma_0) + (\partial_{A_{1,0}}\gamma_0, \overline{\partial}_{A_{0,1}}\phi)\right) + \omega(\psi_0 \partial_{A_{1,0}}\gamma_{0,1}) + \beta_{2,0}\overline{\partial}_{A_{0,1}}\gamma_{0,1} + \\
&\quad + 2(\omega \wedge ([\lambda_{1,0}, \gamma_{0,1}], \gamma_0) + ([A_{0,2}, \beta_{2,0}], \phi)) \\
S_{\text{anti}} &= \int \Big(\partial_{A_{1,0}} A_0 \wedge A^*_{1,0} + \overline{\partial}_{A_{0,1}} A_0 \wedge B_{2,1} + [\lambda_{1,0}, A_0] \wedge \lambda^*_{1,0} + \mathrm{dvol}([\lambda_0, A_0], \lambda^*_0) + [A_0, A_{0,1}] \wedge B_{2,1} + [A_0, A_{1,0}] \wedge A^*_{1,0} \\
&\quad + \frac{1}{2}[A_0, A_0] B_{2,2} + [\phi, A_0]\phi^* + [\gamma_0, A_0]\beta_{2,2} + [\gamma_{0,1}, A_0] \wedge \beta_{2,1} + [\psi_0, A_0]\psi^*_0 + [\beta_{2,0}, A_0] \wedge \gamma_{0,2}\Big) \\
I^{(1)}_{\text{gauge}} &= \int \left(-\lambda_{1,0} \wedge A^*_{1,0} + \frac{1}{2} F_{0,2} \wedge B_{2,0} + \frac{1}{2}\omega \wedge F_{1,1}\lambda^*_0\right) \\
I^{(1)}_{\text{matter}} &= \int \left(\psi_0 \phi^* + \frac{1}{2}\overline{\partial}_{A_{0,1}}\gamma_0 \wedge \beta_{2,1}\right) \\
I^{(2)}_{\text{gauge}} &= -\frac{1}{4}\int \mathrm{dvol}(\lambda^*_0)^2.
\end{aligned}
$$

Notice that a priori the theory is only MU(2)-equivariant, but manifestly descends to a U(2)-equivariant theory.

Theorem 10.1 (See also [**SWchar**]). The holomorphic twist of 4d $\mathcal{N} = 1$ super Yang–Mills with matter valued in a $\mathfrak{g}$representation-representation R is perturbatively equivalent to holomorphic BF theory with the space of fields $T^*[-1]\,\mathrm{Map}(\mathbb{C}^2, R/\mathfrak{g})$. Moreover, the equivalence is U(2)-equivariant.

Proof. The proof of this theorem is very similar to the proof of Theorem 4.4. First, we eliminate the fields λ_0 and λ_0^* using Proposition 1.8. We then observe that the action includes the terms $\int \lambda_{1,0} \wedge A^*_{1,0}$ and $\int \psi_0 \phi^*$. Thus, the two pairs $(\lambda_{1,0}, A_{1,0})$ and (ϕ, ψ_0) form trivial BRST doublets, which can be eliminated using Proposition 1.10.

The twisted theory is therefore perturbatively equivalent to the theory with BV action

$$
\begin{aligned}
S_{\text{BV}} = \int \Big(&(B_{2,1}\overline{\partial}_{A_{0,1}} A_0) + B_{2,0}F_{0,2} + \beta_{2,0}\overline{\partial}_{A_{0,1}}\gamma_{0,1} + \beta_{2,1}\overline{\partial}_{A_{0,1}}\gamma_0 \\
&+ [A_{0,2}, A_0] \wedge B_{2,0} + \frac{1}{2}[A_0, A_0]B_{2,2} + [\gamma_0, A_0] \wedge \beta_{2,2} + [\gamma_{0,1}, A_0] \wedge \beta_{2,1}\Big).
\end{aligned}
$$

This is indeed the action functional of the required theory, where $A_{0,\bullet}, B_{2,\bullet}$ comprise the fields of holomorphic BF theory and $\gamma_{0,\bullet}, \beta_{2,\bullet}$ comprise the fields of the $\beta\gamma$ system. □

10.2 $\mathcal{N} = 2$ Super Yang–Mills

The 4d $\mathcal{N} = 2$ super Yang–Mills theory is obtained by a dimensional reduction from the 6d $\mathcal{N} = (1, 0)$ super Yang–Mills theory with matter valued in a symplectic $\mathfrak{g}$-representation U. Let W be a two-dimensional complex symplectic vector space. The theory admits the R-symmetry group $G_R = \mathrm{SL}(2; \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$, where $\mathrm{GL}(1, \mathbb{C})$ acts on W with weight 1.

Fields: The BRST fields are given by:

- Gauge field $A \in \Omega^1(M; \mathfrak{g})$.
- Scalar fields $a, \widetilde{a} \in \Omega^0(M; \mathfrak{g})$.

- Gauge fermions $(\lambda_+, \lambda_-) \in \Omega^0(M; \Pi S_+ \otimes W \otimes \mathfrak{g} \oplus \Pi S_- \otimes W^* \otimes \mathfrak{g})$.
- Matter boson $\phi \in \Omega^0(M; U \otimes W)$.
- Matter fermions $(\psi_-, \psi_+) \in \Omega^0(M; \Pi S_+ \otimes U \oplus \Pi S_- \otimes U)$.
- A ghost field $c \in \Omega^0(M; \mathfrak{g})[1]$.

The subgroup $\mathrm{GL}(1, \mathbb{C}) \subset G_R$ has the following action on fields: weight 2 on a, weight -2 on $\widetilde{a}$ and weight ± 1 on $\lambda_\pm, \psi_\pm$.

If the representation U is $T^*R = R \oplus R^*$, the R-symmetry group is enhanced to $G_R = \mathrm{SL}(2) \times \mathrm{GL}(1, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$, where the last $\mathrm{GL}(1, \mathbb{C})$ acts with weight 1 on R and weight -1 on R^*.

There are three classes of square-zero supercharge in the 4d $\mathcal{N} = 2$ supersymmetry algebra, distinguished by the ranks of the two summands $(Q_+, Q_-) \in S_+ \otimes W \oplus S_- \otimes W^*$:

- Rank $(1, 0)$ and $(0, 1)$ supercharges automatically square to zero. The corresponding twists are holomorphic. Such twists factor through a copy of the $\mathcal{N} = 1$ supersymmetry algebra. As before, they admit a $\mathbb{Z}$-grading and a twisting homomorphism from $\mathrm{MU}(2)$.
- Rank $(2, 0)$ and $(0, 2)$ supercharges also automatically square to zero. The corresponding twists are topological (the *Donaldson twist*). There is a twisting homomorphism from $\mathrm{MU}(2)$ and a compatible homomorphism $\alpha \colon \mathrm{U}(1) \to G_R$.
- Rank $(1, 1)$ square-zero supercharges have three invariant directions. There is a twisting homomorphism from $\mathrm{Spin}(2, \mathbb{R}) \times \mathrm{Spin}(2, \mathbb{R}) \subset \mathrm{Spin}(4, \mathbb{R})$. For a general U there is no compatible homomorphism $\alpha \colon \mathrm{U}(1) \to G_R$.

10.2.1 Holomorphic Twist

Choose a basis for W given by $\{w_1, w_2\}$, where $(w_1, w_2) = 1$, and for concreteness we take $Q = q_+ \otimes w_1$ for some nonzero vector $q_+ \in S_+$. Denote by $L \subset V$ the image of $\Gamma(Q, -) \colon S_- \to V$. Under the embedding $\mathrm{MU}(L) \subset \mathrm{Spin}(V_\mathbb{R})$, the semi-spin representations decompose as

$$S_+ = \det(L)^{-1/2} \oplus \det(L)^{1/2}, \qquad S_- = L \otimes \det(L)^{-1/2}.$$

Recall that the R-symmetry group is $G_R = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$. For any integer $n \in \mathbb{Z}$ consider the homomorphism

$$\begin{array}{rccc} \alpha_n \colon & \mathrm{U}(1) & \to & \mathrm{SL}(2, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C}) \\ & z & \mapsto & \big(\mathrm{diag}(z^{2n}, z^{-2n}), z^{-2n+1}\big) \end{array}$$

under which w_1 has weight 1 and w_2 has weight $-4n + 1$.

We consider the twisting homomorphism

$$\phi \colon \mathrm{MU}(2) \xrightarrow{\det^{1/2}} \mathrm{U}(1) \to G_R$$

under which we have an $\mathrm{MU}(2)$-identification $W = \det(L)^{-1/2} w_1 \oplus \det(L)^{1/2} w_2$, so that

$$S_+ \otimes W \cong \mathbb{C}Q \oplus \det(L)^{-1} \oplus \mathbb{C} \oplus \det(L).$$

Fields: The BRST fields are given by:

- Gauge fields $A_{1,0} \in \Omega^{1,0}(M;\mathfrak{g})$ and $A_{0,1} \in \Omega^{0,1}(M;\mathfrak{g})$.
- Scalar fields $\widetilde{a} \in \Omega^0(M;\mathfrak{g})[4n-2]$ and $a \in \Omega^0(M;\mathfrak{g})[-4n+2]$;
- Gauge fermions $\chi \in \Omega^0(M;\mathfrak{g})[-1]$, $\xi \in \Omega^{2,0}(M;\mathfrak{g})[4n-1]$, $B \in \Omega^{0,2}(M;\mathfrak{g})[-1]$, $b \in \Omega^{0,1}(M;\mathfrak{g})[-4n+1]$, $\rho \in \Omega^{1,0}(M;\mathfrak{g})[1]$, $\widetilde{\chi} \in \Omega^0(M;\mathfrak{g})[4n-1]$.
- Matter bosons $\nu \in \Omega^0(M;U \otimes K_M^{-1/2})[-2n]$, $\phi \in \Omega^0(M;U \otimes K_M^{1/2})[2n]$.
- Matter fermions $\psi \in \Omega^{0,1}(M;U \otimes K_M^{1/2})[2n-1]$, $\varsigma \in \Omega^{2,0}(M;U \otimes K_M^{-1/2})[-2n+1]$, $\widetilde{\nu} \in \Omega^{0,2}(M;U \otimes K_M^{-1/2})[-2n+1]$.
- A ghost field $c \in \Omega^0(M;\mathfrak{g})[1]$.

Theorem 10.2. Fix the twisting datum $\alpha = \alpha_n$. The holomorphic twist of 4d $\mathcal{N} = 2$ super Yang–Mills is perturbatively equivalent to the holomorphic BF theory with the space of fields $T^*[-1]\mathrm{Sect}(M,(U \otimes K_M^{1/2}[2n]) /\!\!/ \mathfrak{g})$. Moreover, the equivalence is MU(2)-equivariant.

Proof. 4d $\mathcal{N} = 2$ super Yang–Mills theory is obtained by dimensionally reducing 6d $\mathcal{N} = (1,0)$ super Yang–Mills theory. Under dimensional reduction the 6d fields from Section 8.1.1 decompose as follows:

$$\begin{aligned} A_{1,0} &\rightsquigarrow A_{1,0}, \widetilde{a} \\ A_{0,1} &\rightsquigarrow A_{0,1}, a \\ B &\rightsquigarrow B, b \\ \rho &\rightsquigarrow \rho, \widetilde{\chi} \\ \psi &\rightsquigarrow \psi, \varsigma. \end{aligned}$$

The claim about the underlying $\mathbb{Z}/2\mathbb{Z}$-graded MU(2)-equivariant theories follows by applying dimensional reduction (Proposition 1.56) to the minimal twist of 5d $\mathcal{N} = 1$ super Yang–Mills theory (Theorem 9.2).

Next, we check that the equivalence respects the gradings. Indeed, the equivalence given by Theorem 9.2 eliminates fields $A_{1,0}, \widetilde{a}, \rho, \chi, \widetilde{\chi}$. The rest of the fields organize into the following collections:

$$\begin{aligned} c + A_{0,1} + B &\in \Omega^{0,\bullet}(M;\mathfrak{g})[1] \\ B^* + A_{0,1}^* + c^* &\in \Omega^{2,\bullet}(M;\mathfrak{g}) \end{aligned}$$

$$\begin{aligned} a + b + \xi^* &\in \Omega^{0,\bullet}(M;\mathfrak{g})[2-4n] \\ \xi + b^* + a^* &\in \Omega^{2,\bullet}(M;\mathfrak{g})[4n-1] \end{aligned}$$

$$\begin{aligned} \phi + \psi + \varsigma^* &\in \Omega^{0,\bullet}(M;U \otimes K_M^{1/2})[2n] \\ \varsigma + \psi^* + \phi^* &\in \Omega^{2,\bullet}(M;U \otimes K_M^{-1/2})[1-2n] \end{aligned}$$

These fields have the same degrees as in the holomorphic BF theory.

$\square$

10.2.2 Rank $(2,0)$ Topological Twist

Next we discuss the case of the topological twist. As in Section 6.1.2 it will be useful to consider a family of topological supercharges degenerating to a rank $(1,0)$ holomorphic supercharge.

Consider the same twisting homomorphism $\phi\colon \mathrm{MU}(2) \to G_R$ as in Section 10.2.1 and $\alpha = \alpha_0 \colon \mathrm{U}(1) \to G_R$. With respect to the MU(2)-action we have a decomposition

$$S_+ \otimes W \cong \mathbb{C}Q_0 \oplus \det(L)^{-1} \oplus \mathbb{C}\overline{Q}_0 \oplus \det(L).$$

Consider a family of supercharges

$$Q_t = Q_0 + t\overline{Q}_0 \tag{21}$$

where $t \in \mathbb{C}$. When $t \neq 0$, this supercharge is of rank $(2,0)$, while at $t=0$ it reduces to the holomorphic supercharge from the previous section.

Remark 10.3. With respect to $\alpha_n\colon \mathrm{U}(1) \to G_R$ the supercharge Q_0 has weight 1, while $\overline{Q}_0$ has weight $-4n+1$. So, requiring Q_t to have weight 1 forces us to choose $n=0$.

We will use the notation for fields from Section 10.2.1. First, we are going to write the functionals (6), (7), (10), (11) in terms of these fields.

Proposition 10.4. Suppose Q_t is the rank $(2,0)$ supercharge of 21. The MU(2) decomposition of the functionals $I^{(1)}_{\mathrm{gauge}}, I^{(1)}_{\mathrm{matter}}, I^{(2)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{matter}}$ (see (6), (7), (10), (11)) in terms of the fields of 4d $\mathcal{N}=2$ super Yang–Mills theory are

$$\begin{aligned}
I^{(1)}_{\mathrm{gauge}}(Q_t) &= \int \mathrm{dvol}\left(-(\rho, A^*_{1,0}) - t(b, A^*_{0,1}) - (\widetilde{\chi} + t\chi)\widetilde{a}^*\right) \\
&\quad + \int \mathrm{dvol}\left((F_{0,2}, B^*) + (\overline{\partial}_{A_{0,1}} a, b^*) + \frac{1}{2}\Lambda(F_{1,1} + [a,\widetilde{a}])\chi^* + [\phi,\phi]\xi^*\right) \\
&\quad + \int \mathrm{dvol}\left(t\Omega^{-1}F_{2,0}\wedge B^* + (t\partial_{A_{1,0}} a, \rho^*)\right) \\
I^{(2)}_{\mathrm{gauge}}(Q_t, Q_t) &= \int \mathrm{dvol}\left(t\chi^*\widetilde{\chi}^* + \frac{t}{2}\xi^* B^* - \frac{1}{4}(\chi^* + t\widetilde{\chi}^*)^2 + tac^*\right) \\
I^{(1)}_{\mathrm{matter}}(Q_t) &= \int \mathrm{dvol}\left((\widetilde{\nu}, \nu^*) + t(\varsigma, \phi^*) + [\phi, \nu] + \frac{1}{2}(\overline{\partial}_{A_{0,1}}\phi, \psi^*) + [\nu, a]\varsigma^*\right) \\
I^{(2)}_{\mathrm{matter}}(Q_t) &= \int \mathrm{dvol}\,\frac{t}{4}(\psi^*, \psi^*)
\end{aligned}$$

Theorem 10.5. The twist of 4d $\mathcal{N}=2$ super Yang–Mills with respect to the family Q_t of square-zero supercharges is perturbatively equivalent to the holomorphic Hodge theory $\mathrm{Sect}\left(\mathbb{C}^2, \left((U\otimes K^{1/2}_{\mathbb{C}^2})/\!\!/\mathfrak{g}\right)_{\mathrm{Hod}}\right)$. Moreover, this equivalence is MU(2)-equivariant.

Proof. The proof proceeds as in the proof of Theorem 4.4 with slight modifications.

Observe that the quadruple of fields $\{\chi^*, \chi, \widetilde{\chi}^*, \widetilde{\chi}\}$ has the same Poisson brackets as the quadruple $\{\chi^* - t\widetilde{\chi}^*, \chi, \widetilde{\chi}^*, \widetilde{\chi} + t\chi\}$. Therefore, we may eliminate the fields $\chi^* - t\widetilde{\chi}^*, \chi$ using Proposition 1.8. We then have trivial BRST doublets $\{\widetilde{\chi} + t\chi, \widetilde{a}\}$, $\{\nu, \widetilde{\nu}\}$ and $\{\rho, A_{1,0}\}$ which may be eliminated using Proposition 1.10. We are left with the action

$$S_{BF} + t\int \mathrm{dvol}\left(-(b, A^*_{0,1}) + ac^* + \frac{1}{2}\xi^* B^* + (\varsigma, \phi^*) + \frac{1}{4}(\psi^*, \psi^*)\right),$$

where S_{BF} is the action functional of the holomorphic twist at $t=0$ found in the previous section. Since the extra terms are quadratic in the fields, the claim is reduced to a comparison of the underlying local L_∞ algebra of the twisted theory and that of the holomorphic Hodge theory. The former is given by (cf. the proof of Theorem 10.2)

$$
\begin{tikzcd}
\underline{-1} & \underline{0} & \underline{1} & \underline{2} & \underline{3} & \underline{4} \\
& \Omega^0(\mathbb{C}^2;\mathfrak{g})_c \arrow[r] & \Omega^{0,1}(\mathbb{C}^2;\mathfrak{g})_{A_{0,1}} \arrow[r] & \Omega^{0,2}(\mathbb{C}^2;\mathfrak{g})_B & & \\
\Omega^0(\mathbb{C}^2;\mathfrak{g})_a \arrow[r] \arrow[ur, dashed, "t\,\mathrm{id}"] & \Omega^{0,1}(\mathbb{C}^2;\mathfrak{g})_b \arrow[r] \arrow[ur, dashed, "t\,\mathrm{id}"] & \Omega^{0,2}(\mathbb{C}^2;\mathfrak{g})_{\xi^*} \arrow[ur, dashed, "t\,\mathrm{id}"] & & & \\
& & \Omega^{2,0}(\mathbb{C}^2;\mathfrak{g})_{B^*} \arrow[r] \arrow[dr, dashed, "t\,\mathrm{id}"] & \Omega^{2,1}(\mathbb{C}^2;\mathfrak{g})_{A^*_{0,1}} \arrow[r] \arrow[dr, dashed, "t\,\mathrm{id}"] & \Omega^{2,2}(\mathbb{C}^2;\mathfrak{g})_{c^*} \arrow[dr, dashed, "t\,\mathrm{id}"] & \\
& & & \Omega^{2,0}(\mathbb{C}^2;\mathfrak{g})_{\xi} \arrow[r] & \Omega^{2,1}(\mathbb{C}^2;\mathfrak{g})_{b^*} \arrow[r] & \Omega^{2,2}(\mathbb{C}^2;\mathfrak{g})_{a^*} \\
& & \Omega^0(\mathbb{C}^2;U\otimes K^{1/2})_\phi \arrow[r] & \Omega^{0,1}(\mathbb{C}^2;U\otimes K^{1/2})_\psi \arrow[r] & \Omega^{0,2}(\mathbb{C}^2;U\otimes K^{1/2})_{\varsigma^*} & \\
& \Omega^{2,0}(\mathbb{C}^2;U\otimes K^{-1/2})_\varsigma \arrow[r] \arrow[ur, dashed, "t\,\mathrm{id}"] & \Omega^{2,1}(\mathbb{C}^2;U\otimes K^{-1/2})_{\psi^*} \arrow[r] \arrow[ur, dashed, "t\,\mathrm{id}"] & \Omega^{2,2}(\mathbb{C}^2;U\otimes K^{-1/2})_{\phi^*} \arrow[ur, dashed, "t\,\mathrm{id}"] & &
\end{tikzcd}
$$

which is exactly the local L_∞ algebra of the holomorphic Hodge theory. □

Corollary 10.6. The rank $(2,0)$ topological twist of 4d $\mathcal{N}=2$ super Yang–Mills is perturbatively trivial.

Proof. The topological twist of 4d $\mathcal{N}=2$ super Yang–Mills is the twist by Q_t with $t\neq 0$. By Theorem 10.5 it is equivalent to the $t\neq 0$ specialization of the holomorphic Hodge theory which by Proposition 1.52 is perturbatively trivial. □

10.2.3 Rank $(1,1)$ Twist

We finally consider the twist with respect to a rank $(1,1)$ supercharge. In this case, the twist is compatible with the group $G=\mathrm{Spin}(2,\mathbb{R})\times\mathrm{Spin}(2,\mathbb{R})$. We denote each factor by $\mathrm{Spin}(2,\mathbb{R})_i$, $i=1,2$. The twisting homomorphism is

$$\phi\colon \mathrm{Spin}(2,\mathbb{R})_1\times\mathrm{Spin}(2,\mathbb{R})_2\to G_R=\mathrm{SL}(W)\times\mathrm{GL}(1,\mathbb{C}),$$

where on the first factor $\mathrm{Spin}(2,\mathbb{R})_1\cong\mathrm{U}(1)\hookrightarrow\mathrm{SL}(2,\mathbb{C})$ is given by the diagonal embedding and on the second factor $\mathrm{Spin}(2,\mathbb{R})_2\hookrightarrow\mathrm{GL}(1,\mathbb{C})$ is the obvious inclusion. If we denote by $S_{\pm,i}$, $i=1,2$ the semi-spin representations of the factor $\mathrm{Spin}(2,\mathbb{R})_i$, we have

$$W\cong S_{+,2}\otimes(S_{+,1}\oplus S_{-,1}).$$

The semi-spin representations of $\mathrm{Spin}(4,\mathbb{R})$ decompose with respect to $\mathrm{Spin}(2,\mathbb{R})_1\times\mathrm{Spin}(2,\mathbb{R})_2\subset\mathrm{Spin}(4,\mathbb{R})$ as

$$S_+\cong S_{+,1}\otimes S_{+,2}\oplus S_{-,1}\otimes S_{-,2},\qquad S_-\cong S_{+,1}\otimes S_{-,2}\oplus S_{-,1}\otimes S_{+,2}.$$

So, both $S_+\otimes W$ and $S_-\otimes W^*$ contain a trivial one-dimensional subspace and hence we obtain a rank $(1,1)$ square-zero supercharge.

Theorem 10.7. The rank $(1,1)$ partially topological twist of 4d $\mathcal{N}=2$ super Yang-Mills theory is perturbatively equivalent to the generalized Chern-Simons theory with space of fields $\mathrm{Map}(\mathbb{C}\times\mathbb{R}^2_{\mathrm{dR}},U/\!\!/\mathfrak{g})$. Moreover, this equivalence is $\mathrm{U}(1)\times\mathrm{Spin}(2,\mathbb{R})$-equivariant.

Proof. Any square-zero supercharge of rank $(1,1)$ lifts to a rank 1 supercharge in the 5d $\mathcal{N}=1$ supersymmetry algebra. The result then follows from Theorem 9.1 applied to $L=\mathbb{C}$. □

If $U = T^*R$ is of cotangent type, we may enhance the R-symmetry group to $G_R = \mathrm{SL}(W) \times \mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$. The last $\mathrm{GL}(1,\mathbb{C})$ acts trivially on W, by weight $+1$ on R and weight -1 on R^*.

We have a homomorphism $\alpha\colon \mathrm{U}(1) \to \mathrm{SL}(W) \times \mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ given by the diagonal embedding into the first and the third components. We also use a new twisting homomorphism $\widetilde{\phi}\colon \mathrm{Spin}(2,\mathbb{R})_1 \times \mathrm{Spin}(2,\mathbb{R})_2 \to G_R$ given by composing ϕ with the obvious homomorphism from the first factor $\mathrm{Spin}(2,\mathbb{R})_1$ to the last $\mathrm{GL}(1,\mathbb{C})$ factor in G_R (cf. the definition of α and $\widetilde{\phi}$ in Section 8.1.1).

Theorem 10.8. The minimal twist of 4d $\mathcal{N} = 2$ super Yang–Mills on $M = \mathbb{C} \times \mathbb{R}^2$ with matter valued in the $\mathfrak{g}$-representation $U = T^*R = R \oplus R^*$ is perturbatively equivalent to the generalized BF theory with the space of fields $T^*[-1]\,\mathrm{Map}(\mathbb{C} \times \mathbb{R}^2_{\mathrm{dR}}, R/\mathfrak{g})$. Moreover, the equivalence is $\mathrm{U}(1) \times \mathrm{Spin}(2,\mathbb{R})$-equivariant.

10.3 $\mathcal{N} = 4$ Super Yang-Mills

The 4d $\mathcal{N} = 4$ super Yang–Mills theory is obtained by dimensional reduction from the 10d $\mathcal{N} = (1,0)$ super Yang–Mills. It admits R-symmetry group $G_R = \mathrm{Spin}(6;\mathbb{C}) \cong \mathrm{SL}(4,\mathbb{C})$ under which $W \cong S^6_+$ is the positive six-dimensional semi-spin representation and $W^* \cong S^6_-$.

Let us decompose the rotation group as $\mathrm{Spin}(4,\mathbb{R}) \cong \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$. The classification of orbits of square-zero supercharges in the 4d $\mathcal{N} = 4$ supersymmetry algebra is the most interesting among the examples we consider in this paper. We have the following classes.

- Rank $(1,0)$ and $(0,1)$ supercharges automatically square to zero. The corresponding twists are holomorphic. Such twists factor through a copy of the $\mathcal{N} = 1$ supersymmetry algebra. As before, they admit a $\mathbb{Z}$-grading and a twisting homomorphism from $\mathrm{MU}(2)$.
- Rank $(2,0)$ and $(0,2)$ supercharges automatically square to zero. The corresponding twists are topological. Such twists factor through a copy of the $\mathcal{N} = 2$ supersymmetry algebra. They admit the following twisting homomorphisms:
  1. The half twisting homomorphism $\phi_{1/2}\colon \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \to \mathrm{SL}(4,\mathbb{C})$ given by $(A,B) \mapsto \mathrm{diag}(A,1,1)$. This is the twisting homomorphism that comes from the $\mathcal{N} = 2$ supersymmetry algebra.
  2. The Kapustin–Witten twisting homomorphism $\phi_{\mathrm{KW}}\colon \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \to \mathrm{SL}(4,\mathbb{C})$ given by $(A,B) \mapsto \mathrm{diag}(A,B)$.
  3. The Vafa–Witten twisting homomorphism $\phi_{\mathrm{VW}}\colon \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \to \mathrm{SL}(4,\mathbb{C})$ given by $(A,B) \mapsto \mathrm{diag}(A,A)$.
- Rank $(1,1)$ supercharges. Such supercharges have three invariant directions. They factor through a copy of the $\mathcal{N} = 2$ supersymmetry algebra. As before, they admit a twisting homomorphism from $\mathrm{Spin}(2,\mathbb{R}) \times \mathrm{Spin}(2,\mathbb{R})$ and admit a $\mathbb{Z}$-grading.
- Rank $(2,1)$ and $(1,2)$ supercharges. Such supercharges are topological, compatible with a twisting homomorphism from $\mathrm{MU}(2)$ and admit a $\mathbb{Z}$-grading.
- Rank $(2,2)$ square-zero supercharges are topological. They correspond to a choice of an exact sequence

  $$0 \to S_+^* \to W \to S_- \to 0.$$

  Since S_+, S_- and W all carry volume forms, such a square-zero supercharge is parametrized by a continuous parameter $s \in \mathbb{C}^\times$ given by the ratio of the isomorphism $\det(W) \cong \det(S_+)^* \otimes \det(S_-)$ induced by Q and the isomorphism induced by the volume forms. These supercharges admit a $\mathbb{Z}$-grading and are compatible with the twisting homomorphism $\phi_{\mathrm{KW}}\colon \mathrm{Spin}(4,\mathbb{R}) \to G_R$.

10.3.1 Holomorphic Twist

Let L be a complex structure on $V_{\mathbb{R}}$. Consider a twisting homomorphism $\mathrm{MU}(L) \to G_R = \mathrm{Spin}(6;\mathbb{C})$ under which W decomposes as

$$W = L \otimes \det(L)^{-1/2} \oplus \det(L)^{-1/2} w_1 \oplus \det(L)^{1/2} w_2.$$

In particular,

$$S_+ \otimes W \cong L \oplus L^* \oplus \det(L) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \det(L)^{-1}$$

and we consider the supercharge $Q \in S_+ \otimes W$ of rank $(1,0)$ contained in the scalar summand which spans the subspace $\mathbb{C}w_1 \subset W$.

We consider a homomorphism $\alpha \colon \mathrm{U}(1) \to G_R$ under which $L \otimes \det(L)^{-1/2} \subset W$ has weight -1 and w_1, w_2 have weight 1. In particular, Q has α-weight 1.

Fields: In the notation of Section 10.2.1, the BRST fields are given by:

- Gauge fields $A_{1,0} \in \Omega^{1,0}(M;\mathfrak{g})$ and $A_{0,1} \in \Omega^{0,1}(M;\mathfrak{g})$.
- Scalar fields $\widetilde{a} \in \Omega^0(M;\mathfrak{g})[-2]$ and $a \in \Omega^0(M;\mathfrak{g})[2]$;
- Gauge fermions $\chi \in \Omega^0(M;\mathfrak{g})[-1]$, $\xi \in \Omega^{2,0}(M;\mathfrak{g})[-1]$, $B \in \Omega^{0,2}(M;\mathfrak{g})[-1]$, $b \in \Omega^{0,1}(M;\mathfrak{g})[1]$, $\rho \in \Omega^{1,0}(M;\mathfrak{g})[1]$, $\widetilde{\chi} \in \Omega^0(M;\mathfrak{g})[-1]$.
- Matter bosons $\nu \in \Omega^{0,1}(M;\mathfrak{g})$, $\phi \in \Omega^{1,0}(M;\mathfrak{g})$.
- Matter fermions $\psi \in \Omega^{1,1}(M;\mathfrak{g})[-1]$, $\varsigma \in \Omega^{1,0}(M;\mathfrak{g})[1]$, $\widetilde{\nu} \in \Omega^{0,1}(M;\mathfrak{g})[1]$.
- A ghost field $c \in \Omega^0(M;\mathfrak{g})[1]$.

Note that the $\mathrm{MU}(2)$-action on fields factors through a $\mathrm{U}(2)$-action.

Theorem 10.9. The holomorphic twist of 4d $\mathcal{N} = 4$ super Yang–Mills on $M = \mathbb{R}^4$ is perturbatively equivalent to the BF theory with the space of fields $T^*[-1]\,\mathrm{Map}(\mathbb{C}^2_{\mathrm{Dol}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{U}(2)$-equivariant.

Proof. The 4d $\mathcal{N} = 4$ super Yang–Mills theory viewed as a $\mathcal{N} = 2$ theory is 4d $\mathcal{N} = 2$ Yang–Mills theory with matter valued in $U = T^*\mathfrak{g}$. Under this correspondence $\alpha \colon \mathrm{U}(1) \to G_R$ defined above coincides with α_0 from Section 10.2.1. From Theorem 10.2 we obtain that the twist is given by $T^*[-1]\,\mathrm{Map}(\mathbb{C}^2, (T^*\mathfrak{g} \otimes K_{\mathbb{C}^2}^{1/2})/\!\!/\mathfrak{g})$ as a $\mathbb{Z}$-graded theory.

Note, however, that the twisting homomorphism used in Section 10.2.1 differs from the twisting homomorphism defined above. In particular, this equivalence is not $\mathrm{U}(2)$-equivariant. In the present case the fields organize into the following collections:

$$\begin{aligned} c + A_{0,1} + B &\in \Omega^{0,\bullet}(M;\mathfrak{g})[1] \\ \phi + \psi + \varsigma^* &\in \Omega^{1,\bullet}(M;\mathfrak{g}) \\ \xi + b^* + a^* &\in \Omega^{2,\bullet}(M;\mathfrak{g})[-1] \end{aligned}$$

$$\begin{aligned} a + b + \xi^* &\in \Omega^{0,\bullet}(M;\mathfrak{g})[2] \\ \varsigma + \psi^* + \phi^* &\in \Omega^{1,\bullet}(M;\mathfrak{g})[1] \\ B^* + A_{0,1}^* + c^* &\in \Omega^{2,\bullet}(M;\mathfrak{g}) \end{aligned}$$

These are exactly the fields in $T^*[-1]\,\mathrm{Map}(\mathbb{C}^2_{\mathrm{Dol}}, B\mathfrak{g})$. $\square$

10.3.2 Rank $(2,0)$ Topological Twist

Next we look at the case of the twist by a rank $(2,0)$ supercharge. As in Section 10.2.2, it will be useful to consider a family of topological supercharges which degenerate to the rank $(1,0)$ supercharge we just discussed.

We use the same twisting homomorphism $\phi\colon \mathrm{MU}(2) \to \mathrm{Spin}(6;\mathbb{C})$ and twisting data $\alpha\colon U(1) \to G_R$ as in Section 10.3.1. Then, $S_+ \otimes W$ decomposes under MU(2) as

$$S_+ \otimes W \cong L \oplus L^* \oplus \det(L) \oplus \mathbb{C} \cdot Q_0 \oplus \mathbb{C} \cdot \overline{Q}_0 \oplus \det(L)^{-1}.$$

Consider the family of supercharges $Q_t = Q_0 + t\overline{Q}_0 \in S_+ \otimes W$ of rank $(1,0)$ contained in the scalar summands above.

Theorem 10.10. The twist of 4d $\mathcal{N} = 4$ super Yang–Mills with respect to the family Q_t of square-zero supercharges is perturbatively equivalent to the holomorphic Hodge theory $\mathrm{Map}(\mathbb{C}^2_{\mathrm{Dol}}, B\mathfrak{g}_{\mathrm{Hod}})$. Moreover, this equivalence is MU(2)-equivariant.

Proof. The 4d $\mathcal{N} = 4$ super Yang–Mills theory viewed as a $\mathcal{N} = 2$ theory is 4d $\mathcal{N} = 2$ Yang–Mills theory with matter valued in $U = T^*\mathfrak{g}$. Under this correspondence $\alpha\colon \mathrm{U}(1) \to G_R$ defined above coincides with α_0 from Section 10.2.1. From Theorem 10.5 we obtain that the twist is $\mathrm{Sect}\left(\mathbb{C}^2, \left((T^*\mathfrak{g} \otimes K^{1/2}_{\mathbb{C}^2}) /\!\!/ \mathfrak{g}\right)_{\mathrm{Hod}}\right)$ as a $\mathbb{Z}$-graded theory.

The twisting homomorphism used in Section 10.2.1 differs from the twisting homomorphism defined above. In particular, this equivalence is not U(2)-equivariant. In the present case the fields decompose in the same fashion as in the proof of Theorem 10.9 which are precisely the fields of $\mathrm{Map}(\mathbb{C}^2_{\mathrm{Dol}}, B\mathfrak{g}_{\mathrm{Hod}})$. □

10.3.3 Rank (1, 1) Partially Topological Twist

Next we consider the twist with respect to a rank $(1,1)$ supercharge. As in Section 10.2.3 the twist is compatible with the group $G = \mathrm{Spin}(2,\mathbb{R}) \times \mathrm{Spin}(2,\mathbb{R})$. However, we will use a different twisting homomorphism. We denote each factor by $\mathrm{Spin}(2,\mathbb{R})_i$, $i = 1,2$, and by $S_{\pm,i}$ the semi-spin representations of the factor $\mathrm{Spin}(2,\mathbb{R})_i$.

The twisting homomorphism is

$$\phi\colon \mathrm{Spin}(2,\mathbb{R})_1 \times \mathrm{Spin}(2,\mathbb{R})_2 \to G_R = \mathrm{SL}(W),$$

under which W splits as

$$W \cong (S_{+,1} \otimes S_{+,2} \oplus S_{-,1} \otimes S_{-,2}) \oplus (S_{+,1} \otimes S_{-,2} \oplus S_{-,1} \otimes S_{+,2}). \tag{22}$$

In this case $S_+ \otimes W$ and $S_- \otimes W^*$ have two-dimensional trivial G-subrepresentations. Any scalar rank $(1,1)$ supercharge is square-zero. We choose a homomorphism $\alpha\colon \mathrm{U}(1) \to G_R$ under which the first two summands in (22) have weight 1 and the last two summands have weight -1. This makes the chosen rank $(1,1)$ supercharge have weight 1.

Theorem 10.11. The rank (1, 1) twist of 4d $\mathcal{N} = 4$ super Yang–Mills on $M = \mathbb{R}^4$ is perturbatively equivalent to the generalized BF theory with the space of fields $T^*[-1]\,\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{Spin}(2,\mathbb{R}) \times \mathrm{Spin}(2,\mathbb{R})$-equivariant.

Proof. The 4d $\mathcal{N} = 4$ super Yang–Mills theory viewed as a $\mathcal{N} = 2$ theory is 4d $\mathcal{N} = 2$ Yang–Mills theory with matter valued in $U = T^*\mathfrak{g}$. By Theorem 10.7 we obtain that the twist is equivalent to $T^*[-1]\,\mathrm{Map}(\mathbb{C} \times \mathbb{R}^2_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$ as a $\mathbb{Z}$-graded theory. Let us now analyze the $\mathrm{Spin}(2,\mathbb{R}) \times \mathrm{Spin}(2,\mathbb{R})$-action.

By construction the twisting homomorphism $\mathrm{Spin}(2,\mathbb{R}) \times \mathrm{Spin}(2,\mathbb{R}) \to G_R$ defined by (22) factors as $\mathrm{Spin}(2,\mathbb{R}) \times \mathrm{Spin}(2,\mathbb{R}) \subset \mathrm{MU}(2) \to G_R$, where the latter map is the twisting homomorphism used in Section 10.3.1. Therefore, we have to restrict the fields used in that section to $\mathrm{Spin}(2,\mathbb{R}) \times \mathrm{Spin}(2,\mathbb{R})$. But by Theorem 10.9 the fields belong to $T^*[-1]\,\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{C}_{\mathrm{Dol}}, B\mathfrak{g})$ whose underlying graded $\mathrm{Spin}(2,\mathbb{R}) \times \mathrm{Spin}(2,\mathbb{R})$-equivariant bundle coincides with that of $T^*[-1]\,\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})$. □

10.3.4 Rank $(1,2)$ Topological Twist

Next we look at the case of the twist by a rank $(1,2)$ supercharge. As in many cases so far, it will be useful to consider a family of supercharges which are generically of rank $(1,2)$.

We use the same twisting homomorphism $\phi\colon \mathrm{MU}(2) \to \mathrm{Spin}(6;\mathbb{C})$ and twisting data $\alpha\colon U(1) \to G_R$ as in Section 10.3.1. Then, $S_+ \otimes W$ decomposes under $\mathrm{MU}(2)$ as

$$S_+ \otimes W \cong L \oplus L^* \oplus \det(L) \oplus \mathbb{C}\cdot Q_0 \oplus \mathbb{C} \oplus \det(L)^{-1}.$$

Further, the representation $S_- \otimes W^*$ decomposes as

$$S_- \otimes W^* = L \oplus L^* \oplus \mathbb{C}Q' \oplus \mathrm{End}_0(L),$$

where $\mathrm{End}_0(L) \subset \mathrm{End}(L)$ is the subspace of traceless endomorphisms. The supercharge Q_0 has rank $(1,0)$ and the supercharge Q' has rank $(0,2)$. Moreover, both are square-zero and commute with each other. Therefore, we may consider a family of square-zero supercharges

$$Q_t = Q_0 + tQ' \tag{23}$$

which is of rank $(1,0)$ at $t=0$ and of rank $(1,2)$ at $t \neq 0$.

We are going to use the notation of the fields from Section 10.3.1.

Proposition 10.12. Suppose Q_t is the rank $(1,2)$ supercharge of (23). The $\mathrm{MU}(2)$ decomposition of the functionals $I^{(1)}, I^{(2)}$ (see (6), (7)) in terms of the fields of 4d $\mathcal{N}=4$ super Yang–Mills theory are

$$\begin{aligned}
I^{(1)}(Q_t) = &\int \mathrm{dvol}\left(-(\rho, A^*_{1,0}) - \widetilde{\chi}\widetilde{a}^* - (\widetilde{\nu},\nu^*) - t(\varsigma, A^*_{1,0}) - t(\widetilde{\nu}, A^*_{0,1}) - t(b,\nu^*) - t(\rho,\phi^*) - t(\Lambda\psi)\widetilde{a}^*\right) \\
&+ \int \mathrm{dvol}\left((F_{0,2}, B^*) + (\overline{\partial}_{A_{0,1}} a, b^*) + \frac{1}{2}\Lambda(F_{1,1} + [\phi,\nu] + \omega[a,\widetilde{a}])\chi^* + [\phi,\phi]\xi^* + [\nu,a]\varsigma^* + \frac{1}{2}(\overline{\partial}_{A_{0,1}}\phi, \psi^*)\right) \\
&+ t\int \mathrm{dvol}\left(\frac{1}{2}(F_{1,1} + [\phi,\nu] + \omega[a,\widetilde{a}], \psi^*) + \Lambda\overline{\partial}_{A_{0,1}}\phi\widetilde{\chi}^* + \Lambda\partial_{A_{1,0}}\nu\chi^* + \partial_{A_{1,0}} a\varsigma^* + \overline{\partial}_{A_{0,1}} a\widetilde{\nu}^* + [\phi,a]\rho^* + [\nu,a]b^*\right) \\
I^{(2)}(Q_t,Q_t) = &\int \mathrm{dvol}\left(t^2(\xi^* B^* + \chi^*\widetilde{\chi}^* + \frac{1}{2}(\psi^*,\psi^*)) - \frac{1}{4}(\chi^* + t\Lambda\psi^*)^2 + t^2 a c^*\right)
\end{aligned}$$

Theorem 10.13. The twist of 4d $\mathcal{N}=4$ super Yang–Mills with respect to the family Q_t of square-zero supercharges of Equation 23 is perturbatively equivalent to $\mathrm{Map}\left(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}}\right)$. Moreover, this equivalence is $\mathrm{Spin}(2;\mathbb{R}) \times \mathrm{Spin}(2;\mathbb{R})$-equivariant.

Proof. Fix a rank $(2,1)$ supercharge of the form $Q_{s,t} = Q_0 + tQ_1 + sQ_2$, where $Q_t = Q_0 + tQ_1$ is a rank $(2,0)$ supercharge lying within a 4d $\mathcal{N}=2$ subalgebra of the 4d $\mathcal{N}=4$ supertranslation algebra. Consider the supersymmetry actions

$$\begin{aligned}
&I^{(1)}(Q_0 + tQ_1 + sQ_2) \\
&I^{(2)}(Q_0 + tQ_1 + sQ_2, Q_0 + tQ_1 + sQ_2).
\end{aligned}$$

We will compute the corresponding $\mathbb{Z}$-graded Q_{st}-twisted theory. Let us first recall the steps of the proof of Theorem 10.5: the twist of 4d $\mathcal{N}=2$ super Yang-Mills with matter by Q_t. We will do the same thing to eliminate fields in the Q_{st}-twisted theory. The key fact is that the part of the deformation of the action proportional to s, namely

$$I^{(1)}(sQ_2) + 2I^{(2)}(Q_0, sQ_2) + 2I^{(2)}(tQ_1, sQ_2) + I^{(2)}(sQ_2, sQ_2)$$

only includes terms involving both $\mathcal{N}=4$ fields in the set $\{c, A_{1,0}, A_{0,1}, a, \widetilde{a}, \chi\widetilde{\chi}, b, B, \xi, \rho\}$ coming from the $\mathcal{N}=2$ vector multiplet (plus their antifields) *and* $\mathcal{N}=4$ fields in the set $\{\nu, \widetilde{\nu}, \phi, \psi, \varsigma\}$ coming from the $\mathcal{N}=2$ hypermultiplet (and their antifields). More precisely, the action involves the images of the fields in these sets after restricting from $\mathrm{U}(2)$-representations to $\mathrm{Spin}(2;\mathbb{R})\times\mathrm{Spin}(2;\mathbb{R})$-representations. In what follows we will write $\mathrm{U}(2)$-expressions, with the understanding that the further deformation by sQ_2 will further break these component fields.

So, we first performed a chance of variables sending $\{\chi^*, \chi, \widetilde{\chi}^*, \widetilde{\chi}\}$ to $\{\chi^* - t\widetilde{\chi}^*, \chi, \widetilde{\chi}^*, \widetilde{\chi} + t\chi\}$: this only used the fact that the two quadruples had the same Poisson brackets, which still applies here. We then used Proposition 1.8 to eliminate the fields $\chi^* - t\widetilde{\chi}^*, \chi$. The proposition still applies here, because the further twist by sQ_2 does not introduce a χ^2 or $(\chi^* - t\widetilde{\chi}^*)^2$ term. Finally, we used Proposition 1.10 to eliminate the trivial BRST doublets $\{\widetilde{\chi} + t\chi, \widetilde{a}\}$, $\{\nu, \widetilde{\nu}\}$ and $\{\rho, A_{1,0}\}$. Again, the proposition still applies here: each time we apply the proposition, we must only ensure that there are no terms in the twisted action proportional to a pair of fields (X, Y), we need only observe that the deformation could not introduce terms to the action proportional to XX^* or YY^*.

As a result of this argument, we are free, as in the proof of Theorem 10.5, to set the eliminated fields $\chi, \widetilde{\chi}^*, \widetilde{\chi} + t\chi, \widetilde{a}, \widetilde{a}^*, \nu, \nu^*, \widetilde{\nu}, \widetilde{\nu}^*\rho, \rho^*, A_{1,0}$ and $A_{1,0}^*$ to 0 in the twisted action functional. Finally, we set the remaining eliminated field $\chi^* - t\widetilde{\chi}^*$ equal to the expression S_1, as in Proposition 1.8. This twisted action coincides with the sum of the $Q_0 + sQ_2$-twisted action functional from Theorem 10.7, with the deformation by tQ_1, namely

$$\left(I^{(1)}(tQ_1) + 2I^{(2)}(Q_0, tQ_1) + I^{(2)}(tQ_1, tQ_1)\right) + 2I^{(2)}(tQ_1, sQ_2). \tag{24}$$

To see this, we must verify that the expression S_1 occuring in the Q_{st}-twisted action functional agrees with the corresponding expression in the $Q_0 + sQ_2$-twisted action functional. This follows from the fact that the only term proportional to $t(\chi^* - t\widetilde{\chi}^*)$ in the $Q_0 + tQ_1$-twisted action functional from equation 10.4 is proportional to $\widetilde{\chi}^*$, and so vanishes after setting $\widetilde{\chi}^* = 0$.

The deformation given by the first three terms in 24 was computed in the proof of Theorem 10.5 to be

$$t\int \mathrm{dvol}\left(-(b, A_{0,1}^*) + ac^* + \frac{1}{2}\xi^* B^* + t(\varsigma, \phi^*) + \frac{1}{4}(\psi^*, \psi^*)\right). \tag{25}$$

The fourth and final term, proportional to st, vanishes after the field elimination we have performed, since $\Gamma_{10}(Q_1, Q_2) = 0$, and the remaining term from equation 7, of form $st(Q_1, \lambda^*)(Q_2, \lambda^*)$, is proportional to (Q_1, λ^*), which is proportional to χ^*.

Therefore, the Q_{st} twisted action theory is equivalent to the rank $(1,1)$-twisted theory from Theorem 10.7, with action deformed by the expression 25. This deformation gives the desired theory. The rank $(1,1)$-twisted theory was $\mathbb{Z}$-graded, and the deformation here by tQ_1 is a deformation by a degree 1 operator, so the twisted theory is also $\mathbb{Z}$-graded. Finally, the rank $(1,1)$-twisted theory was shown to be $\mathrm{Spin}(2;\mathbb{R}) \times \mathrm{Spin}(2;\mathbb{R})$-invariant, and the deformation 25 was $\mathrm{MU}(2)$-equivariant, so the rank $(1,2)$-twisted theory is still $\mathrm{Spin}(2;\mathbb{R}) \times \mathrm{Spin}(2;\mathbb{R})$-invariant.

□

10.3.5 Rank $(2,2)$ Topological Twist

Consider a rank $(2,2)$ supercharge $Q \in S_+ \otimes W \oplus S_- \otimes W^*$. It defines embeddings $S_+^* \hookrightarrow W$ and $S_-^* \hookrightarrow W^*$ and the square-zero condition is that their images pair to zero. In other words, we have a short exact sequence

$$0 \longrightarrow S_+^* \longrightarrow W \longrightarrow S_- \longrightarrow 0.$$

The semi-spin representations $S_\pm$ carry volume forms induced by scalar spinorial pairings. Moreover, W has a canonical volume form since it is the semi-spin representation of $\mathrm{Spin}(6,\mathbb{C}) \cong \mathrm{SL}(4,\mathbb{C})$. Comparing these volume forms under the above exact sequence gives an invariant $s \in \mathbb{C}^\times$ of a rank $(2,2)$ square-zero supercharge. Moreover, $\mathrm{Spin}(6,\mathbb{C})$-orbits of rank $(2,2)$ square-zero supercharges are parametrized by this invariant.

Let $N_\mathbb{R} = \mathbb{R}^4$ equipped with a Euclidean structure and $N = N_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ its complexification. We consider the 8-dimensional Euclidean vector space N which carries a complex half-density. By the results of Section 6.1.2 we obtain a family Q_t of 8d square-zero supercharges. Its dimensional reduction to 4 dimensions also gives a family of 4d square-zero supercharges. Then from Theorem 7.3 we obtain the following statement.

Theorem 10.14. The twist of 4d $\mathcal{N} = 4$ super Yang–Mills with respect to the family Q_t of square-zero supercharges is perturbatively equivalent to the topological Hodge theory $\mathrm{Map}(\mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$.

Let us now rewrite the family Q_t in 4-dimensional terms. Consider the Kapustin–Witten twisting homomorphism $\phi_{\mathrm{KW}}\colon \mathrm{Spin}(4,\mathbb{R}) \subset \mathrm{Spin}(4,\mathbb{R}) \times \mathrm{Spin}(2,\mathbb{R}) \subset \mathrm{Spin}(6,\mathbb{C})$ under which W decomposes as

$$W \cong S_+ \oplus S_-.$$

In this case the spinorial representation becomes

$$\Sigma \cong S_+ \otimes S_+ \oplus S_+ \otimes S_- \oplus S_- \otimes S_+ \oplus S_- \otimes S_-.$$

In particular, there are two scalar supercharges Q_+ and Q_- given by the volume forms on S_+ and S_- respectively. We may then consider a family of supercharges

$$Q = uQ_+ + ivQ_-$$

for $u, v \in \mathbb{C}$. If $u, v \neq 0$ we obtain a rank $(2,2)$ supercharge. In this case the map $Q\colon S_+^* \cong S_+ \to W$ is given by multiplication by u and the map $Q\colon W \to S_-$ is given by multiplication by iv. Therefore, its s-invariant is

$$s = -\frac{u^2}{v^2}.$$

Remark 10.15. The family $uQ_+ + ivQ_-$ of square-zero supercharges is the same family as studied by Kapustin and Witten, see [**KapustinWitten**].

These supercharges are related to Q_t as follows. Let S^8_+ be the semi-spin representation of $\mathrm{Spin}(8,\mathbb{C})$ and $S_\pm$ the semi-spin representations of $\mathrm{Spin}(4,\mathbb{C})$ as before. Under the embedding

$$\mathrm{Spin}(4,\mathbb{C}) \subset \mathrm{Spin}(4,\mathbb{C}) \times \mathrm{Spin}(4,\mathbb{C}) \subset \mathrm{Spin}(8,\mathbb{C})$$

S^8_+ splits as

$$S^8_+ \cong S_+ \otimes S_+ \oplus S_- \otimes S_-,$$

so $Q_+, Q_- \in S^8_+$. We then obtain

$$Q_0 = Q_+ + Q_-, \qquad \overline{Q}_0 = Q_+ - Q_-.$$

Therefore, the s-invariant of the family Q_t is

$$s = \frac{(1+t)^2}{(1-t)^2}. \tag{26}$$

Corollary 10.16. The rank $(2,2)$ twist of 4d $\mathcal{N} = 4$ super Yang–Mills for $s = 1$ is perturbatively equivalent to the topological BF theory $T^*[-1]\,\mathrm{Map}(\mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g})$.

Proof. The supercharge Q_0 has s-invariant $s = 1$. By Theorem 10.14 the twist by Q_0 is perturbatively equivalent to the specialization of the theory $\mathrm{Map}(\mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$ at $t = 0$. By Proposition 1.52 the latter is isomorphic to the topological BF theory $T^*[-1]\,\mathrm{Map}(\mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g})$. ☐

Corollary 10.17. The rank $(2,2)$ twist of 4d $\mathcal{N} = 4$ super Yang–Mills for $s \neq 1$ is perturbatively trivial.

Proof. For any $s \neq 1$ we may find $t \neq 0$ solving (26). But by Proposition 1.52 the specialization of the topological Hodge theory $\mathrm{Map}(\mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$ at $t \neq 0$ is perturbatively trivial. ☐

11 Dimension 3

The 3-dimensional $\mathcal{N} = k$ supersymmetry algebra has odd part $\Sigma \cong S \otimes W$, where S is the 2-dimensional spinor representation of $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, and where W is a k-dimensional vector space equipped with a bilinear pairing. The maximal supersymmetric gauge theory has $\mathcal{N} = 8$. However, there are $\mathcal{N} = 4$ super Yang-Mills theories for every choice U of symplectic representation of the gauge group, and $\mathcal{N} = 2$ super Yang-Mills theories for every choice R of arbitrary representation of the gauge group. In dimension 3, much like we saw in dimensions 5 and 7, all twisted theories can be obtained by dimensional reduction from theories one dimension higher.

11.1 $\mathcal{N} = 2$ Super Yang-Mills with Chiral Matter

We'll begin with the minimal super Yang-Mills theory that admits non-trivial twists (if $\mathcal{N} = 1$ there are no square-zero supercharges). So, fix a gauge group G, and a representation R. The 3d $\mathcal{N} = 2$ super Yang-Mills theory arises by dimensional reduction from $\mathcal{N} = 1$ super Yang-Mills theory on $\mathbb{R}^4$ with an R-valued chiral multiplet. In this case, $W = \mathbb{C}^2$ equipped with a nondegenerate symmetric bilinear pairing.

The R-symmetry group is $\mathrm{Spin}(2; \mathbb{C}) = \mathbb{C}^\times$, acting on W by the defining representation.

Fields: We can describe the BRST field content of $\mathcal{N} = 2$ super Yang-Mills by restricting the 4d fields from Section 10.1 to representations of the group $\mathrm{Spin}(3; \mathbb{C})$. The fields we obtain are

- $\mathfrak{g}$-valued bosons: $A \in \Omega^1(\mathbb{R}^3; \mathfrak{g})$, and a scalar $\phi \in \Omega^0(\mathbb{R}^3; \mathfrak{g})$.
- R-valued bosons: $(\overline{\phi}, \phi) \in \Omega^0(\mathbb{R}^3; R \oplus R^*)$.
- $\mathfrak{g}$-valued fermions: $\lambda \otimes u \in \Omega^0(\mathbb{R}^3; S \otimes W \otimes \mathfrak{g})$.
- R-valued fermions: $(\psi_-, \psi_+) \in \Omega^0(\mathbb{R}^3; S \otimes R \oplus S \otimes R^*)$.
- Ghost: $c \in \Omega^0(\mathbb{R}^3; \mathfrak{g})[1]$.

This theory admits a unique twist up to equivalence:

- A square zero supercharge $Q \neq 0 \in \Sigma$ has two invariant directions. There is a twisting homomorphism $\phi = (-)^{1/2} \colon \mathrm{U}(1) \hookrightarrow G_R$, so the twisted theory carries a $U(1)$-action. The twist is $\mathbb{Z}$-graded with $\alpha : \mathrm{U}(1) \hookrightarrow G_R$ given by the natural embedding.

11.1.1 Minimal Twist

A square-zero supercharge Q has rank 1, i.e. $Q = q \otimes w$ for some $w \in W$. We use the twisting homomorphism $\phi = (-)^{1/2} \colon \mathrm{U}(1) \hookrightarrow G_R$.

As in Section 5.1.1, the data of q_+ is equivalent to the choice of a one-dimensional subspace $N_\mathbb{R} \subset V_\mathbb{R}$ and a complex structure on $V_\mathbb{R}/N_\mathbb{R}$ together with a complex half-density. Note that in one dimension the choice of a complex half-density is equivalent to a choice of spin structure.

Theorem 11.1. The minimal twist of 3d $\mathcal{N} = 2$ super Yang-Mills with Lie algebra $\mathfrak{g}$ coupled to the $\mathcal{N} = 2$ chiral multiplet valued in a representation R is perturbatively equivalent to generalized BF theory for the Lie algebra $\mathfrak{g}$ coupled to the generalized $\beta\gamma$ system with values in the representation R, with moduli space $T^*[-1]\mathrm{Map}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, R/\mathfrak{g})$. Moreover this equivalence is U(1)-equivariant.

Proof. By Theorem 10.1 the twist of 4d $\mathcal{N}=1$ super Yang–Mills on $L \times N$ by Q is perturbatively equivalent to the holomorphic Chern–Simons theory with the space of fields $T^*[-1]\,\mathrm{Map}(L \times N, R/\mathfrak{g})$. By Proposition 1.57 we get that the dimensional reduction of holomorphic Chern–Simons on $L \times N$ along $\mathrm{Re}\colon N \to N_{\mathbb{R}}$ is isomorphic to the generalized Chern–Simons theory with the space of fields $T^*[-1]\,\mathrm{Map}(L \times N_{\mathbb{R}}, R/\mathfrak{g})$ and this isomorphism is U(1)-equivariant by our choice of a twisting homomorphism. □

11.2 $\mathcal{N}=4$ Super Yang-Mills with Hypermultiplet Matter

We will now consider 3d $\mathcal{N}=4$ theories, which arise by dimensional reduction from 4d $\mathcal{N}=2$ super Yang-Mills with gauge group G, coupled to a hypermultiplet valued in a symplectic representation U, or equivalently by dimensional reduction from 6d $\mathcal{N}=(1,0)$ super Yang-Mills coupled to a hypermultiplet. The R-symmetry group is $G_R = \mathrm{Spin}(4;\mathbb{C})$, and W is its defining representation.

Fields: We can describe the BRST fields of $\mathcal{N}=4$ super Yang-Mills by restricting the 4d fields from Section 10.2 to representations of the group $\mathrm{Spin}(3;\mathbb{C})$. The fields we obtain are

- $\mathfrak{g}$-valued bosons: $A \in \Omega^1(\mathbb{R}^3;\mathfrak{g})$, and three scalar fields $(\phi_1,\phi_2,\phi_3) \in \Omega^0(\mathbb{R}^3;\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g})$.
- U-valued bosons: a W-valued scalar field $(\phi \otimes v) \in \Omega^0(\mathbb{R}^3; W \otimes U)$.
- $\mathfrak{g}$-valued fermions: a pair of W-valued spinors $(\lambda_1 \otimes u_1, \lambda_2 \otimes u_2) \in \Omega^0(\mathbb{R}^3; S \otimes (W \oplus W) \otimes \mathfrak{g})$.
- U-valued fermions: a pair of spinors $(\psi_1,\psi_2) \in \Omega^0(\mathbb{R}^3; (S \oplus S) \otimes U)$.
- Ghost field: $c \in \Omega^0(\mathbb{R}^3;\mathfrak{g})[1]$.

In the $\mathcal{N}=4$ supersymmetry algebra there are now three non-trivial orbits of square-zero supercharges. An element $Q \in S \otimes W$ gives rise to a map $S^* \to W$; Q squares to zero if its image is totally isotropic. The classification of orbits includes the rank of this map.

- Rank 1. In this case Q is minimal, with 2 invariant directions. Such supercharges lie in a subalgebra isomorphic to the $\mathcal{N}=2$ supersymmetry algebra and are unique up to equivalence. They admit a twisting homomorphism $\phi\colon \mathrm{U}(1) \to G_R$ and a $\mathbb{Z}$-grading defined by a homomorphism $\alpha\colon \mathrm{U}(1) \to G_R$.
- Rank 2. Such supercharges are topological. A rank 2 supercharge defines a Lagrangian subspace of W, and therefore an orientation. The $G_R = \mathrm{Spin}(4;\mathbb{C})$-action factors through an SO(4) action on W, and so preserves orientation, so there are two G_R orbits corresponding to the two choices of orientation. We refer to these as the A-twist and the B-twist, distinguished by whether or not they can be promoted to a twisting datum.
  1. An A-twist supercharge admits a twisting homomorphism $\phi\colon \mathrm{U}(1) \to G_R$ and a $\mathbb{Z}$-grading $\alpha\colon \mathrm{U}(1) \to G_R$.
  2. A B-twist supercharge admits the diagonal twisting homomorphism $\phi'\colon \mathrm{SU}(2) \to \mathrm{SU}(2)\times\mathrm{SU}(2) \to G_R$. This twist is only $\mathbb{Z}/2\mathbb{Z}$-graded

The distinction via twisting homomorphisms and $\mathbb{Z}$-gradings follows by identifying the twists as dimensional reductions from 4d $\mathcal{N}=2$.

Lemma 11.2. A rank $(2,0)$ square 0 supercharge in the 4d $\mathcal{N}=2$ supersymmetry algebra restricts to an A-twisting supercharge in 3d $\mathcal{N}=4$. Likewise a rank $(1,1)$ square zero supercharge in 4d $\mathcal{N}=2$ restricts to a B-twisting supercharge in 3d $\mathcal{N}=4$.

Proof. Let W_4 be the complex two-dimensional auxiliary space of the 4d $\mathcal{N}=2$ supesymmetry algebra. The projection from the 4d $\mathcal{N}=2$ supertranslation algebra to the 3d $\mathcal{N}=4$ supertranslation algebra induces an isomorphism $W_4 \oplus W_4^* \to W$ of representations of the group $\mathrm{Spin}(3;\mathbb{C})$. This splits the fundamental representation W into the sum of two Lagrangians, defining an orientation on W. A rank $(2,0)$ supercharge induces the Lagrangian subspace $W_4 \subseteq W$, which is oriented. The corresponding twist can be promoted to a twisting datum using a copy of $\mathrm{U}(1)$ acting with weight 1 on W_4; explicitly, this is the homomorphism $\alpha : \mathrm{U}(1) \to G_R = \mathrm{SL}(2;\mathbb{C}) \times \mathrm{SL}(2;\mathbb{C})$ given by the embedding into the second factor. This is the A-twist in our classification above.

A rank $(1,1)$ supercharges induces a Lagrangian subspace of the form $L \oplus L^* \subseteq W$, where L is a 1-dimensional subspace of W_4. This subspace has the opposite orientation, so corresponds to the B-twist in our classification above. $\square$

11.2.1 Minimal Twist

The 3d $\mathcal{N} = 4$ super Yang–Mills viewed as a $\mathcal{N} = 2$ supersymmetric theory coincides with the 3d $\mathcal{N} = 2$ super Yang–Mills with matter valued in the representation $R = U \oplus \mathfrak{g}$.

There is a unique twisting homomorphism $\phi : \mathrm{U}(1) \to \mathrm{Spin}(4;\mathbb{C})$ given by the restriction of the 4d $\mathcal{N} = 2$ twisting homomorphism for the minimal twist as in Theorem 12.16 to the subgroup $\mathrm{U}(1) \subset \mathrm{MU}(2)$.

This twist also admits the following twisting datum. To incorporate the $\mathbb{Z}$-grading, we use the twisting datum $\alpha\colon \mathrm{U}(1) \to \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \cong G_R$. The action of this α on W coincides with the action of α_0 in Section 10.2.1 on the sum of the 4d auxiliary space with its dual $W_4 \oplus W_4^*$, composed with the isomorphism $W_4 \oplus W_4^* \to W$: indeed, both have weights $(1, 1, -1, -1)$. Therefore our equivalence is compatible with the $\mathbb{Z}$-grading in Theorem 12.16 induced by α_0.

Theorem 11.3. The minimal twist of 3d $\mathcal{N} = 4$ super Yang–Mills on $N \times L = \mathbb{C} \times \mathbb{R}$ is perturbatively equivalent to the generalized BF theory coupled to a higher holomorphic symplectic U-valued boson, with space of fields $T^*[-1]\mathrm{Sect}(N \times L_{\mathrm{dR}}, (U \otimes K_N^{1/2}) /\!\!/ \mathfrak{g}))$. Moreover, the equivalence is $\mathrm{U}(1)$-equivariant.

Proof. The statement about the equivalence of the underlying $\mathbb{Z}/2\mathbb{Z}$-graded theories follows from applying dimensional reduction (Theorem 1.57) to Theorem 12.16 calculating the holomorphic twist of 4d $\mathcal{N} = 2$ super Yang-Mills on $N \times L_{\mathbb{C}}$. This equivalence is equivariant for the action of $\mathrm{U}(1) \cong \mathrm{MU}(N) \subseteq \mathrm{MU}(N \times L_{\mathbb{C}})$. $\square$

11.2.2 Topological A-Twist

Let $L = \mathbb{C}$ equipped with a Hermitian structure and a complex half density, $N_{\mathbb{R}} = \mathbb{R}$ equipped with a Euclidean structure and $N = N_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. Consider the 4-dimensional spacetime $V_{\mathbb{R}} = L \times N$. Under the projection $\mathrm{Re}\colon N \to N_{\mathbb{R}}$ the family Q_t of 4-dimensional square-zero supercharges given by (21) dimensionally reduces to a family of 3-dimensional square-zero supercharges which at $t \neq 0$ are topological at and $t = 0$ have 2 invariant directions. Since they admit a compatible $\mathbb{Z}$-grading, at $t = 0$ we obtain the holomorphic twist and at $t \neq 0$ we obtain the topological B twist. Therefore, from Theorem 10.5 we obtain the following statement.

Theorem 11.4. The twist of 3d $\mathcal{N} = 4$ super Yang–Mills with respect to the family Q_t of square-zero supercharges is perturbatively equivalent to the generalized Hodge theory $\mathrm{Map}(N \times L_{\mathrm{dR}}, ((U \otimes K_N^{1/2}) /\!\!/ \mathfrak{g})_{\mathrm{Hod}})$. Moreover, this equivalence is $\mathrm{U}(1)$-equivariant.

Corollary 11.5. The topological A twist of 3d $\mathcal{N} = 4$ super Yang–Mills is perturbatively trivial.

11.2.3 Topological B-Twist

We consider $V_{\mathbb{R}} = \mathbb{R}^3$ equipped with a Euclidean structure and as before let $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. V carries a Hermitian structure and a half-density, so by the results of Section 8.1.1 we obtain a square-zero supercharge Q. Under the projection $\mathrm{Re}\colon V \to V_{\mathbb{R}}$ the supercharge Q dimensionally reduces to the topological B supercharge in 3 dimensions. Therefore, from Theorem 9.1 we obtain the following statement.

Theorem 11.6. The rank 2 B-twist of 3d $\mathcal{N} = 4$ super Yang-Mills theory with symplectic matter representation U is perturbatively equivalent to the generalized BF theory coupled to a U-valued higher holomorphic boson with space of fields $\mathrm{Map}(\mathbb{R}^3_{\mathrm{dR}}, U/\!\!/\mathfrak{g})$. Moreover, the equivalence is $\mathrm{Spin}(3;\mathbb{R})$-equivariant.

11.3 $\mathcal{N} = 8$ Super Yang-Mills Theory

Finally, let's consider maximal super Yang-Mills theory with gauge group G in dimension 3, with $\mathcal{N} = 8$ supersymmetry. This is the theory obtained by compactifying $\mathcal{N} = (1,0)$ super Yang-Mills on $\mathbb{R}^{10}$ down to $\mathbb{R}^3$. The R-symmetry group is $G_R = \mathrm{Spin}(7;\mathbb{C})$, and W is its 8-dimensional (Dirac) spin representation.

Fields: We can describe the BRST fields of $\mathcal{N} = 4$ super Yang-Mills by restricting the 4d fields from Section 10.3 to representations of the group $\mathrm{Spin}(3;\mathbb{C})$. The fields we obtain are

- Bosons: $A \in \Omega^1(\mathbb{R}^3;\mathfrak{g})$, and seven scalar fields $\phi_i \in \Omega^0(\mathbb{R}^3;\mathfrak{g})$.
- Fermions: eight spinor fields $\lambda_i \in \Omega^0(\mathbb{R}^3; S \otimes \mathfrak{g})$.
- Ghost Field: $c \in \Omega^0(\mathbb{R}^3;\mathfrak{g})[1]$.

In the $\mathcal{N} = 8$ supersymmetry algebra the classification of twists is the same as we saw in $\mathcal{N} = 4$. That is, there are three orbits: one consisting of rank 1 supercharges, and two orbits of rank 2 supercharges. We can see this in the following way.

Lemma 11.7. There are two distinct $\mathrm{Spin}(3;\mathbb{C}) \times G_R$-orbits of square-zero supercharges of rank 2 in the 3d $\mathcal{N} = 8$ supersymmetry algebra.

Proof. Choose a symplectic basis $\langle s, s'\rangle$ for S, and let $Q = s \otimes w + s' \otimes w'$ be a square-zero rank 2 element of $S \otimes W$. Let V_7 denote the fundamental representation of G_R. As in Section 5, the element $w \in W$ is equivalent to the data of a maximal isotropic subspace $L \subseteq V_7$, together with a choice of a half-density. This element w is stabilized, in particular, by a copy of the metalinear group $\mathrm{ML}(L)$. Under the group $\mathrm{ML}(L)$ the auxiliary space W decomposes as

$$W \cong \left(\mathbb{C} \oplus L \oplus \wedge^2 L \oplus \wedge^3 L\right) \otimes \det(L)^{-1/2},$$

(see also [**ElliottSafronov**]), with w lying in the last summand. Under this decomposition, split the remaining element $w' = (v_0, v_1, v_2, v_3)$. Then

- $v_0 = 0$, because the square-zero condition implies, in particular, that $(w, w') = 0$ with respect to the scalar spinor pairing on W, here given by the wedge pairing.
- Without loss of generality $v_3 = 0$, since under the action of $\mathrm{Spin}(3;\mathbb{C})$, $s \otimes w + s' \otimes w' \sim s \otimes w + s' \otimes (w' - w)$.
- If $v_1 = 0$ then $v_2 \neq 0$, and all choices of non-zero v_2 are in the same orbit under $\mathrm{SL}(L) \subseteq \mathrm{Stab}(w) \subseteq G_R$.
- If $v_1 \neq 0$ then without loss of generality $v_2 = 0$, using the action by wedge product of $L \subseteq \mathrm{Stab}(w) \subseteq G_R$. Finally all choices of non-zero v_1 are likewise in the same orbit under $\mathrm{SL}(L) \subseteq \mathrm{Stab}(w) \subseteq G_R$. The stabilizer of w acts on the space $v_1 \neq 0$, so these latter two cases comprise two inequivalent orbits.

□

The classification of twists therefore takes the following form. We identify how the two orbits of rank 2 supercharges arise by dimensional reduction from 4d $\mathcal{N}=4$ by forgetting the supertranslation action down to 3d $\mathcal{N}=4$ (this may partially break the R-symmetry, but this is enough to distinguish the two classes of twisted theory).

- Rank 1. In this case Q is minimal, with 2 invariant directions. Such supercharges come from the $\mathcal{N}=2$ supersymmetry algebra. They admit a twisting homomorphism from $\mathrm{U}(1)$ and a $\mathbb{Z}$-grading $\alpha\colon \mathrm{U}(1) \to G_R$.
- Rank 2 A-twist. These twists are topological, and come from the $\mathcal{N}=4$ supersymmetry algebra. The A-twist arises by dimensional reduction from the rank $(2,0)$ twist in 4d. It admits a twisting homomorphism from $\mathrm{U}(1)$ and a $\mathbb{Z}$-grading $\alpha\colon \mathrm{U}(1) \to G_R$.
- Rank 2 B-twist. These twists are topological, and come from the $\mathcal{N}=4$ supersymmetry algebra. The B twist arises by dimensional reduction from the rank $(1,1)$ twist in 4d. It admits a twisting homomorphism $\phi\colon \mathrm{Spin}(3;\mathbb{R}) \to G_R$.

11.3.1 Minimal Twist

The 3d $\mathcal{N}=8$ super Yang–Mills viewed as an $\mathcal{N}=4$ supersymmetric theory coincides with the 3d $\mathcal{N}=4$ super Yang–Mills with matter valued in the representation $U = T^*\mathfrak{g}$. Alternatively, it is obtained by dimensional reduction from the holomorphic twist of 4d $\mathcal{N}=4$ super Yang-Mills.

Theorem 11.8. The minimal twist of 3d $\mathcal{N}=8$ super Yang–Mills on $M = \mathbb{C}\times\mathbb{R}$ is perturbatively equivalent to the generalized BF theory with space of fields $T^*[-1]\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$. Moreover, the equivalence is $\mathrm{U}(1)$-equivariant.

Proof. By Proposition 10.9 and Proposition 1.60 the dimensional reduction of the minimal twist of 4d $\mathcal{N}=4$ super Yang-Mills along $\mathbb{C}^2 \to \mathbb{C}\times\mathbb{R}$ is equivalent to the desired generalized BF theory. □

11.3.2 Topological A-Twist

From Theorem 11.4, using the same argument, we obtain the following statement.

Theorem 11.9. The twist of 3d $\mathcal{N}=8$ super Yang–Mills with respect to the family Q_t of square-zero supercharges is perturbatively equivalent to the generalized Hodge theory $T^*[-1]\mathrm{Map}(\mathbb{C}\times\mathbb{R}_{\mathrm{dR}}, (\mathfrak{g}/\mathfrak{g})_{\mathrm{Hod}})$. Moreover, this equivalence is $\mathrm{U}(1)$-equivariant.

Corollary 11.10. The topological A-twist of 3d $\mathcal{N}=8$ super Yang-Mills is perturbatively trivial.

11.3.3 Topological B-Twist

From Theorem 11.6 we obtain the following statement.

Theorem 11.11. The rank 2 B-twist of 3d $\mathcal{N}=8$ super Yang-Mills theory with gauge group G is perturbatively equivalent to the BF theory with space of fields $T^*[-1]\mathrm{Map}(\mathbb{R}^3_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$.

12 Dimension 2

In dimension 2, the semispin representations $S_\pm$ are 1-dimensional, where $\mathrm{Spin}(2;\mathbb{C}) \cong \mathbb{C}^\times$ acts with weight ± 1. There is an independent pairing $\Gamma_\pm\colon S_\pm^{\otimes 2} \to V_2 \cong \mathbb{C}_1 \oplus \mathbb{C}_{-1}$ for each chirality. There are two classes of twisted supersymmetric gauge theory that we will consider in 2d. First we have theories with $(\mathcal{N},\mathcal{N})$ supersymmetry; these arise via dimensional reduction from supersymmetric gauge theories in higher dimensions. We additionally have gauge theories with chiral, i.e. $(0,\mathcal{N})$ supersymmetry, which we saw using the observations in Section 2.3. We'll address twists for these two classes of theory in turn.

12.1 $\mathcal{N}=(2,2)$ Super Yang-Mills with Matter

First, let's consider the $\mathcal{N}=(2,2)$ super Yang-Mills theory, which arises via dimensional reduction from 3d $\mathcal{N}=2$ super Yang-Mills. This theory includes a gauge multiplet with gauge group G and a chiral multiplet valued in a representation R of G. The R-symmetry group is $G_R = \mathbb{Z}/2\mathbb{Z} \ltimes (\mathrm{Spin}(2;\mathbb{C}) \times \mathrm{Spin}(2;\mathbb{C})) \cong \mathbb{C}^\times \times \mathbb{C}^\times$, with the factors acting by their vector representation on W_+ and W_- repectively, and with $\mathbb{Z}/2\mathbb{Z}$ acting on both W_+ and W_- by $(a,b) \mapsto (a^{-1}, b^{-1})$.

Fields: We can describe the BRST fields of $\mathcal{N}=(2,2)$ super Yang-Mills by restricting the fields in dimension 3 from Section 11.1, or equivalently the 3d fields from Section 10.1 to representations of the group $\mathrm{Spin}(2;\mathbb{C})$. The fields we obtain are

- $\mathfrak{g}$-valued bosons: $A \in \Omega^1(\mathbb{R}^2;\mathfrak{g})$, and a pair of scalars $(\phi_1,\phi_2) \in \Omega^0(\mathbb{R}^2;\mathfrak{g}\oplus\mathfrak{g})$.
- R-valued bosons: $(\overline{\phi},\phi) \in \Omega^0(\mathbb{R}^2; R\oplus R^*)$.
- $\mathfrak{g}$-valued fermions: $(\lambda_+ \otimes u_+, \lambda_- \otimes u_-) \in \Omega^0(\mathbb{R}^2; (S_+ \otimes W_+ \oplus S_- \otimes W_-)\otimes \mathfrak{g})$.
- R-valued fermions: $(\psi_+^+, \psi_-^+, \psi_+^-, \psi_-^-) \in \Omega^0(\mathbb{R}^2; (S_+ \oplus S_-)\otimes R \oplus (S_+\oplus S_-)\otimes R^*)$.
- Ghost field: $c \in \Omega^0(\mathbb{R}^2;\mathfrak{g})[1]$.

In the $\mathcal{N}=(2,2)$ supersymmetry algebra there is are three classes of non-trivial orbits of square-zero supercharges.

- Square-zero supercharges of rank $(1,0)$ or $(0,1)$, which are holomorphic. There is a compatible twisting homomorphism from $\mathrm{U}(1)$, and a compatible twisting datum $\alpha\colon \mathrm{U}(1) \to \mathrm{Spin}(2;\mathbb{R})$ acting with weight 1 on S_+ and weight -1 acting on S_-.
- Square-zero supercharges of rank $(1,1)$ are topological, and split into four orbits under the action of $\mathrm{Spin}(2;\mathbb{C})\times(\mathbb{C}^\times \times \mathbb{C}^\times)$. Indeed, we can identify a square-zero supercharge of rank $(1,1)$ as a pair of vectors

  $((\lambda,\pm i\lambda),(\mu,\pm i\mu)) \in W_+ \oplus W_-$ with $\lambda,\mu \neq 0$. By acting by G_R we can set $\lambda=\mu=1$, leaving four orbits, represented by the supercharges $Q_A = ((1,i),(1,i)), Q_A^\dagger = ((1,-i),(1,-i)), Q_B = ((1,i),(1,-i))$ and $Q_B^\dagger = ((1,-i),(1,i))$. The $\mathbb{Z}/2\mathbb{Z}$-action swaps the two A supercharges and the two B supercharges, leaving two orbits under $\mathrm{Spin}(2;\mathbb{C})\times G_R$.

  1. The A-supercharges are compatible with the twisting homomorphism $\phi_A\colon \mathrm{Spin}(2;\mathbb{C}) \to \mathrm{Spin}(2;\mathbb{C}) \times \mathrm{Spin}(2;\mathbb{C})$ with weights $(1,1)$.
  2. The B-supercharges are compatible with the twisting homomorphism $\phi_B\colon \mathrm{Spin}(2;\mathbb{C}) \to \mathrm{Spin}(2;\mathbb{C}) \times \mathrm{Spin}(2;\mathbb{C})$ with weights $(1,-1)$.

  Moreover, the A-supercharges can be promoted to twisting data using the homomorphism $\alpha_A = \phi_B$, and the B-supercharges can be promoted to twisting data using $\alpha_B = \phi_A$.

The calculation of the twists here is similar to what we saw in 4d $\mathcal{N}=2$ supersymmetry. The minimal twist and the B-twist arise by dimensional reduction from twists of $3d$ $\mathcal{N}=2$ supersymmetric Yang-Mills theory. On the contrary, the A-twist as a deformation of the holomorphic twist does not arise as a dimensional reduction.

12.1.1 Minimal Twist

Theorem 12.1. The minimal twist of 2d $\mathcal{N} = (2,2)$ super Yang-Mills theory with gauge group G coupled to the $\mathcal{N} = (2,2)$ chiral multiplet valued in a representation R is perturbatively equivalent to holomorphic BF theory for the Lie algebra $\mathfrak{g}$ coupled to the $\beta\gamma$ system with values in the representation R, with moduli space $T^*[-1]\mathrm{Map}(\mathbb{C}, (R/\mathfrak{g})_{\mathrm{Dol}})$. Moreover, this equivalence is U(1)-equivariant.

Proof. The statement about the equivalence of the underlying $\mathbb{Z}/2\mathbb{Z}$-graded theories follows from applying dimensional reduction (Theorem 1.56) to Theorem 11.1 calculating the minimal twist of 3d $\mathcal{N} = 2$ super Yang-Mills on $\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}$. This equivalence is U(1)-equivariant. It is also $\mathbb{Z}$-graded, using the twisting datum $\alpha = \phi_A$. □

12.1.2 Topological A Twist

To deform the holomorphic twist to the topological A-twist, obtaining a Hodge deformation, we use similar techniques to those of Section 10.2.2. We first analyze how the supersymmetry action decomposes. Consider the 1-parameter family of supercharges

$$Q_t = Q_0 + tQ', \tag{27}$$

where $Q_0 = (1, i) \in S_+$ is a holomorphic supercharge, and $Q' = (1, i) \in S_-$. This family of supercharges are all compatible with the twisting homomorphism ϕ_A – the map $\mathrm{U}(1) \to \mathrm{Spin}(2;\mathbb{C}) \times \mathrm{Spin}(2;\mathbb{C})$ with weights $(1,1)$. They are promoted to twisting data using the homomorphism ϕ_B with weights $(1,-1)$. Let us first decompose our fields according to the twisting homomorphism ϕ_A. The fields we obtain are given as follows, where the subscripts denote the U(1)-weight.

- $\mathfrak{g}$-valued bosons: $A_{1,0} \in \Omega^{1,0}(\mathbb{C};\mathfrak{g})$, $A_{0,1} \in \Omega^{0,1}(\mathbb{C};\mathfrak{g})$, $\chi_1 \in \Omega^0(\mathbb{C};\mathfrak{g}[2])$, $\chi_{-1} \in \Omega^0(\mathbb{C};\mathfrak{g}[2])$.
- R-valued bosons: $\phi_0 \in \Omega^0(\mathbb{C};R)$, $\overline{\phi}_0 \in \Omega^0(\mathbb{C};R^*)$.
- $\mathfrak{g}$-valued fermions: $(\lambda_2, \lambda_0, \lambda_0', \lambda_{-2}) \in \Omega^0(\mathbb{R}^2;\mathfrak{g}[-1] \oplus \mathfrak{g}[-1] \oplus \mathfrak{g}[1] \oplus \mathfrak{g}[1])$.
- R-valued fermions: $(\psi_1, \psi_{-1}, \overline{\psi}_1, \overline{\psi}_{-1}) \in \Omega^0(\mathbb{R}^2; R[-1] \oplus R[-1] \oplus R^*[1] \oplus R^*[1])$.
- Ghost field: $c_0 \in \Omega^0(\mathbb{R}^2;\mathfrak{g}[1])$.

We write the antifields similarly: the antifield of a field x_n is written x^*_{-n}, with opposite U(1)-weight.

Proposition 12.2. Suppose Q_t is the rank $(1,1)$ supercharge of 27. The U(1) decomposition of the functionals $I^{(1)}_{\mathrm{gauge}}, I^{(1)}_{\mathrm{matter}}, I^{(2)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{matter}}$ (see (6), (7), (10), (11)) in terms of the fields of 2d $\mathcal{N} = (2,2)$ super Yang–Mills theory are

$$\begin{aligned}
I^{(1)}_{\mathrm{gauge}}(Q_t) &= \int \mathrm{dvol}\left(-(\lambda_{-2}, A_1^*) - (\lambda_0, \chi^*_{-1})) + \frac{1}{2}((\rho(\overline{\partial}_{A_1} A_1), \lambda^*_{-2}) + (\rho(\overline{\partial}_{A_1}\chi_{-1}), \lambda_0^*))\right) \\
&\quad + \int \mathrm{dvol}\, t\left(-(\lambda_2, A_1^*) - (\lambda_0', \chi_1^*)) + \frac{1}{2}((\rho(\partial_{A_{-1}} A_{-1}), \lambda_2^*) + (\rho(\partial_{A_{-1}}\chi_1), \lambda_0'^*))\right) \\
I^{(2)}_{\mathrm{gauge}}(Q_t, Q_t) &= \int \mathrm{dvol}\left(\frac{t}{2}(\lambda^*_{-2}, \lambda_0'^*) - \frac{1}{2}(\lambda^*_{-2} + t\lambda_0'^*)^2 - t(\chi_{-1}, c_0^*)\right) \\
I^{(1)}_{\mathrm{matter}}(Q_t) &= \int \mathrm{dvol}\left(\left((\overline{\psi}_{-1}, \phi_0^*) + \frac{1}{2}(\overline{\partial}_{A_1}\overline{\phi}_0, \psi^*_{-1})\right) + t\left((\overline{\psi}_1, \overline{\phi}_0^*) + \frac{1}{2}(\partial_{A_{-1}}\phi_0, \psi_1^*)\right)\right) \\
I^{(2)}_{\mathrm{matter}}(Q_t) &= \int \mathrm{dvol}\,\frac{t}{4}\left((\psi^*_{-1}, \psi^*_{-1}) + (\psi_1^*, \psi_1^*)\right).
\end{aligned}$$

Theorem 12.3. The topological A-twist of 2d $\mathcal{N} = (2,2)$ super Yang-Mills theory with gauge group G coupled to the $\mathcal{N} = (2,2)$ chiral multiplet valued in a representation R is perturbatively equivalent, as a one-parameter deformation of the holomorphic twist, to the Hodge family with moduli space $T^*[-1]\mathrm{Map}(\mathbb{C}, (R/\mathfrak{g})_{\mathrm{Hod}})$. This equivalence is U(1)-invariant.

Proof. We first perform a change of variables sending λ_2 to $\lambda_2 + t\lambda_0'$, and its antifield λ_{-2}^* to $\lambda_{-2}^* - t\lambda_0^{\prime *}$, integrating out the field $\lambda_2 + t\lambda_0'$ and its antifield using Proposition 1.8. We can then eliminate the trivial BRST doublets $(\lambda_{-2}, A_{-1}), (\overline{\psi}_{-1}, \phi_0)$ and (λ_0, χ_1) using Proposition 1.10. At $t = 0$, we are left with the holomorphic twist from Theorem 12.1, i.e, the theory $T^*[-1]\mathrm{Map}(\mathbb{C}, (R/\mathfrak{g})_{\mathrm{Dol}})$. The action functional is further deformed by the remaining t-linear terms, which together take the form

$$\int \mathrm{dvol}\, t \left(-(\lambda_0', \chi_1^*) - (\chi_{-1}, c_0^*) + (\overline{\psi}_1, \overline{\phi}_0^*) + \frac{1}{4}(\psi_{-1}^*, \psi_{-1}^*) + \frac{1}{4}(\psi_1^*, \psi_1^*) \right).$$

In terms of the classical BV complex we can identify these terms as introducing the following dotted isomorphism between two summands:

$$\begin{tikzcd}
\underline{-2} & \underline{-1} & \underline{0} & \underline{1} & \underline{2} & \underline{3} \\
& \Omega^0(\mathbb{R}^2;\mathfrak{g})_{c_0} \arrow[r] & \Omega^{0,1}(\mathbb{R}^2;\mathfrak{g})_{A_1} & & \Omega^0(\mathbb{R}^2;\mathfrak{g}^*)_{\lambda_0^{\prime *}} \arrow[r] & \Omega^{0,1}(\mathbb{R}^2;\mathfrak{g}^*)_{\chi_1^*} \\
\Omega^0(\mathbb{R}^2;\mathfrak{g})_{\chi_{-1}} \arrow[r] \arrow[ur, dashed] & \Omega^{0,1}(\mathbb{R}^2;\mathfrak{g})_{\lambda_0'} \arrow[ur, dashed] & & \Omega^0(\mathbb{R}^2;\mathfrak{g}^*)_{A_1^*} \arrow[r] \arrow[ur, dashed] & \Omega^{0,1}(\mathbb{R}^2;\mathfrak{g}^*)_{c_0^*} \arrow[ur, dashed] & \\
& \Omega^0(\mathbb{R}^2;R^*)_{\overline{\psi}_{-1}} \arrow[r] \arrow[dr, dashed] & \Omega^{0,1}(\mathbb{R}^2;R^*)_{\psi_1^*} \arrow[dr, dashed] & & & \\
& & \Omega^0(\mathbb{R}^2;R^*)_{\overline{\phi}_0} \arrow[r] & \Omega^{0,1}(\mathbb{R}^2;R^*)_{\psi_{-1}} & & \\
& & \Omega^0(\mathbb{R}^2;R)_{\psi_{-1}^*} \arrow[r] \arrow[dr, dashed] & \Omega^{0,1}(\mathbb{R}^2;R)_{\overline{\phi}_0^*} \arrow[dr, dashed] & & \\
& & & \Omega^0(\mathbb{R}^2;R)_{\psi_1} \arrow[r] & \Omega^{0,1}(\mathbb{R}^2;R)_{\overline{\psi}_1^*}. &
\end{tikzcd}$$

This is exactly the deformation to $T^*[-1]\mathrm{Map}(\mathbb{C}, (R/\mathfrak{g})_{\mathrm{Hod}})$. ☐

12.1.3 Topological B Twist

Finally, there is the 2d B-twist. This is much easier, since it arises directly from 3d via dimensional reduction in a non-invariant direction. Indeed, let W_3 be the 2d auxiliary space in the 3d $\mathcal{N} = 2$ supertranslation algebra. The 3d spin representation $S \otimes W_3$ splits into $S_+ \otimes W_3 \oplus S_- \otimes W_3$ as a $\mathrm{Spin}(2;\mathbb{C})$-module. Rank one square zero elements in 3d $\mathcal{N} = 2$ generically become rank $(1,1)$ square zero elements in the 2d $\mathcal{N} = (2,2)$ supersymmetry algebra. There is a square-zero supercharge in 3d $\mathcal{N} = 2$ compatible with the identity twisting homomorphism $\mathrm{Spin}(2;\mathbb{C}) \to \mathrm{Spin}(2;\mathbb{C})$. This becomes the B-twisting homomorphism of $\mathrm{Spin}(2;\mathbb{C}) \to \mathrm{Spin}(2;\mathbb{C}) \times \mathrm{Spin}(2;\mathbb{C})$ of weight $(1,-1)$ in the 2d $\mathcal{N} = (2,2)$-algebra.

Theorem 12.4. The topological B-twist of 2d $\mathcal{N} = (2,2)$ super Yang-Mills theory with gauge group G coupled to the $\mathcal{N} = (2,2)$ chiral multiplet valued in a representation R is perturbatively equivalent to the topological BF theory with moduli space $T^*[-1]\mathrm{Map}(\mathbb{R}^2_{\mathrm{dR}}, (R/\mathfrak{g}))$. This equivalence is U(1)-invariant.

Proof. We again dimensionally reduce the holomorphic twist of 3d $\mathcal{N} = 2$ super Yang-Mills, but this time in one of the non-invariant directions. That is, we start with the result from Theorem 11.1, then apply Proposition 1.57 to turn the complex direction into a second real de Rham direction. ☐

12.2 $\mathcal{N} = (4,4)$ Super Yang-Mills with Matter

We can likewise consider $\mathcal{N} = (4,4)$ super Yang-Mills theory with gauge group G, with a hypermultiplet valued in a symplectic representation U, by dimensionally reducing 3d $\mathcal{N} = 4$ super Yang-Mills. No new twists arise, i.e. every square-zero supercharge sits inside an $\mathcal{N} = (2,2)$ subalgebra.

The R-symmetry group in the $\mathcal{N}=(4,4)$ case is $G_R = \mathrm{SU}(2) \times \mathrm{Spin}(4;\mathbb{C})$, where $\mathrm{Spin}(4;\mathbb{C})$ acts on $S_+ \otimes W_+ \oplus S_- \otimes W_- \cong S_+^6 \otimes W_+^6$ – where W_+^6 is the 6d $\mathcal{N}=(1,0)$ auxiliary space – by its fundamental representation on both W_+ and W_-, and where SU(2) acts by permuting the two factors, i.e. by its fundamental representation on W^6.

In addition to rank $(1,0)$ supercharges, which are holomorphic, there are square-zero rank $(1,1)$ supercharges. As in Section 12.1, these supercharges split into two orbits.

Proposition 12.5. There are two G_R-orbits in the space of rank $(1,1)$ square-zero supercharges in the $\mathcal{N}=(4,4)$ supersymmetry algebra, which we will refer to as the A- and B-supercharges. The B-supercharge occurs as a degeneration of A-type supercharges.

Proof. First, let us decompose the 6d semispinor representation S_+^6 into $S_+^4 \oplus S_-^4$ as a representation of $\mathrm{Spin}(4;\mathbb{C})$. A rank $(1,1)$ square-zero supercharge can be identified with the data of a null vector in S_+^4, a null vector in S_-^4 and a pair of null vectors in W_+^6, all non-zero. There are only two G_R-orbits in this space of quadruples of null vectors: either the two null vectors in W_+^6 are colinear, or they are distinct. We will refer to these as the B- and A-supercharges respectively. □

Upon dimensional reduction from 3d $\mathcal{N}=4$, the B-supercharge reduced to the 2d $\mathcal{N}=(4,4)$ B-supercharge. Indeed, the 3d B-supercharge squares to zero as an element of the 6d $\mathcal{N}=(1,0)$ supersymmetry algebra, which means that in this 6d algebra it has rank 1. Therefore the two vectors in W_+^6 discussed above must be colinear.

12.2.1 Minimal Twist

We obtain the following from the 3d $\mathcal{N}=4$ minimal twist of Theorem 11.3 by dimensional reduction, i.e. by applying Proposition 1.56.

Theorem 12.6. The holomorphic twist of 2d $\mathcal{N}=(4,4)$ super Yang-Mills with gauge group G and symplectic matter representation U is perturbatively equivalent to a holomorphic BF theory, with moduli space given by $T^*[-1]\mathrm{Map}(\mathbb{C}, (U/\!\!/\mathfrak{g})_{\mathrm{Dol}})$. This equivalence is U(1)-invariant.

12.2.2 Topological A Twist

We likewise obtain the following from the 3d $\mathcal{N}=4$ A-twist of Theorem 11.4 by dimensional reduction, i.e. by applying Proposition 1.56.

Theorem 12.7. The holomorphic twist of 2d $\mathcal{N}=(4,4)$ super Yang-Mills with gauge group G and symplectic matter representation U admits a one-parameter deformation to the A-twist. This family is perturbatively equivalent to a Hodge theory, with moduli space given by $T^*[-1]\mathrm{Map}(\mathbb{C}, (U/\!\!/\mathfrak{g})_{\mathrm{Hod}})$. This equivalence is U(1)-invariant.

12.2.3 Topological B Twist

Finally, we obtain the following from the 3d $\mathcal{N}=4$ A-twist of Theorem 11.6 by dimensional reduction, i.e. by applying Proposition 1.56.

Theorem 12.8. The B-twist of 2d $\mathcal{N}=(4,4)$ super Yang-Mills with gauge group G and symplectic matter representation U is perturbatively equivalent to the topological BF theory with moduli space $T^*[-1]\mathrm{Map}(\mathbb{R}^2_{\mathrm{dR}}, U/\!\!/\mathfrak{g})$. This equivalence is U(1)-invariant.

12.3 $\mathcal{N} = (8,8)$ Super Yang-Mills with Matter

Now, let us consider $\mathcal{N} = (8,8)$ super Yang-Mills theory by dimensionally reducing 3d $\mathcal{N} = 8$ super Yang-Mills, or we can consider $\mathcal{N} = (8,8)$ super Yang-Mills. No new twists arise, i.e. every square-zero supercharge sits inside an $\mathcal{N} = (2,2)$ subalgebra.

The R-symmetry group in the $\mathcal{N} = (4,4)$ case is $G_R = \mathrm{Spin}(8;\mathbb{C})$, acting on W_+ and W_- by its two semispin representations. As in the $\mathcal{N} = (4,4)$ case, the classification of square-zero supercharge is identical to the classification we say in the $\mathcal{N} = (4,4)$ case. Rank $(1,0)$ square zero supercharges, and rank $(1,1)$ square zero supercharges split into two orbits in the following way.

Proposition 12.9. There are two G_R-orbits in the space of rank $(1,1)$ square-zero supercharges in the $\mathcal{N} = (8,8)$ supersymmetry algebra, which we will refer to as the A- and B-supercharges. The B-supercharge occurs as a degeneration of A-type supercharges.

Proof. Our aim is to understand the classification of $\mathrm{Spin}(8;\mathbb{C})$-orbits in the space of pairs of non-zero null-vectors $w_+ \in W_+$ and $w_- \in W_-$. Since $\mathrm{Spin}(8;\mathbb{C})$ acts transitively on the possible choices of w_+, it remains for us to understand the action of the stabilizer $\mathrm{Stab}(w_+) \subseteq \mathrm{Spin}(8;\mathbb{C})$ on the space of null vectors w_-. The element w_+ is equivalent to the data of a Lagrangian subspace $L \subseteq V_8$, along with a half-density. As a representation of the subgroup $\mathrm{ML}(L)$ of $\mathrm{Stab}(w_+)$, the two semispin representations decompose as

$$\begin{aligned} W_+ &\cong (\mathbb{C} \oplus \wedge^2 L \oplus \wedge^4 L) \otimes \det(L)^{-1/2} \\ W_- &\cong (L \oplus \wedge^3 L) \otimes \det(L)^{-1/2}, \end{aligned}$$

with $w_+ \in \wedge^4 L$. With respect to this decomposition, say $w_- = (v_1, v_3)$. If $v_1 \neq 0$, then we can act by $\wedge^2 L \subseteq \mathrm{Stab}(w_+)$ to make $v_3 = 0$. Under the action of $\mathrm{SL}(L)$ all such non-zero v_1 are in the same orbit. Likewise if $v_1 = 0$ then $v_3 \neq 0$ and we can act by $\mathrm{SL}(L)$ to see that all such non-zero v_3 are in the same orbit. There are, therefore, two orbits once again, with one degenerating to the other. ☐

Upon dimensional reduction from 3d $\mathcal{N} = 8$, the B-supercharge reduces to the 2d $\mathcal{N} = (8,8)$ B-supercharge. Indeed, if we forget down to $\mathcal{N} = 4$ supersymmetry, then the B-twist should correspond to the B-twist. In both $\mathcal{N} = (4,4)$ and $\mathcal{N} = (8,8)$ the two orbits are distinguished, in that the orbit of B-twists lies in the closure of the orbit of A-twists.

12.3.1 Minimal Twist

We obtain the following from the 3d $\mathcal{N} = 8$ minimal twist of Theorem 11.8 by dimensional reduction, i.e. by applying Proposition 1.56.

Theorem 12.10. The holomorphic twist of 2d $\mathcal{N} = (8,8)$ super Yang-Mills is perturbatively equivalent to a holomorphic BF theory, with moduli space given by $T^*[-1]\mathrm{Map}(\mathbb{C}, (T^*[2](\mathfrak{g}/\mathfrak{g}))_{\mathrm{Dol}})$. This equivalence is $\mathrm{U}(1)$-invariant.

12.3.2 Topological A Twist

We obtain the following from the 3d $\mathcal{N} = 8$ A-twist of Theorem 11.9 by dimensional reduction, i.e. by applying Proposition 1.56.

Theorem 12.11. The holomorphic twist of 2d $\mathcal{N} = (8,8)$ super Yang-Mills admits a one-parameter deformation to the A-twist. This family is perturbatively equivalent to a Hodge theory, with moduli space given by $T^*[-1]\mathrm{Map}(\mathbb{C}, (T^*[2](\mathfrak{g}/\mathfrak{g}))_{\mathrm{Hod}})$. This equivalence is $\mathrm{U}(1)$-invariant.

12.3.3 Topological B Twist

We obtain the following from the 3d $\mathcal{N}=8$ A-twist of Theorem 11.11 by dimensional reduction, i.e. by applying Proposition 1.56.

Theorem 12.12. The B-twist of 2d $\mathcal{N}=(8,8)$ super Yang-Mills is perturbatively equivalent to the topological BF theory with moduli space $T^*[-1]\mathrm{Map}(\mathbb{R}^2_{\mathrm{dR}}, T^*[2](\mathfrak{g}/\mathfrak{g}))$. This equivalence is U(1)-invariant.

12.4 $\mathcal{N}=(0,2)$ Super Yang-Mills with Matter

We consider $\mathcal{N}=(0,2)$ super Yang-Mills for a Lie algebra $\mathfrak{g}$ coupled to $\mathcal{N}=(0,2)$ supersymmetric matter. The supersymmetric matter consists of the $\mathcal{N}=(0,2)$ chiral multiplet with values in a representation R.

The untwisted fields are given by:

- $\mathfrak{g}$-valued bosons: a gauge field $A \in \Omega^1(\mathbb{R}^2;\mathfrak{g})$.
- $\mathfrak{g}$-valued fermions: a spinor $\lambda \in C^\infty(\mathbb{R}^2; S_- \otimes W_- \otimes \mathfrak{g})$.
- $R \oplus R^*$-valued scalars: $\phi \in C^\infty(\mathbb{R}^2; R)$ and $\overline{\phi} \in C^\infty(\mathbb{R}^2; R^*)$;
- a pair of $R \oplus R^*$-valued fermions: $\psi \in C^\infty(\mathbb{R}^2; S_+ \otimes R^*)$ and $\overline{\psi} \in C^\infty(\mathbb{R}^2; S_+ \otimes R)$.

Here, W is a complex two-dimensional vector space equipped with a symmetric pairing. Choose the basis $\{u, v\}$ for W_- with pairing given by $\langle u, v\rangle = 1$.

We will twist by an element $Q = q \otimes u \in S_- \otimes W_-$.

12.4.1 Minimal Twist

The fields decompose as

			Untwisted	R	Twisted
c	$\mapsto$	$A_0 \in \Omega^0(\mathbb{C};\mathfrak{g})$	-1	0	-1
A	$\mapsto$	$A_{0,1} + A_{1,0} \in \Omega^{0,1}(\mathbb{C};\mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C};\mathfrak{g})$	$(0,0)$	$(0,0)$	$(0,0)$
λ	$\mapsto$	$\lambda_0 u + \lambda_{1,0} v \in \Omega^0(\mathbb{C};\mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C};\mathfrak{g})$	$(0,0)$	$(1,-1)$	$(1,-1)$
ϕ	$\mapsto$	$\gamma_0 \in \Omega^0(\mathbb{C}; R)$	0	0	0
$\overline{\phi}$	$\mapsto$	$\overline{\phi} \in \Omega^0(\mathbb{C}; R^*)$	0	0	0
ψ	$\mapsto$	$\psi_0 \in \Omega^0(\mathbb{C}; R^*)$	0	-1	-1
$\overline{\psi}$	$\mapsto$	$\gamma_{0,1} \in \Omega^{0,1}(\mathbb{C}; R)$	0	1	1

The anti-fields decompose as

			Untwisted	R	Twisted
c^*	$\mapsto$	$B_{1,1} \in \Omega^{1,1}(\mathbb{C};\mathfrak{g})$	2	0	2
A^*	$\mapsto$	$A^*_{0,1} + B_{1,0} \in \Omega^{0,1}(\mathbb{C};\mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C};\mathfrak{g})$	$(1,1)$	$(0,0)$	$(1,1)$
λ^*	$\mapsto$	$\lambda^*_{1,1} v + \lambda^*_{0,1} u \in \Omega^0(\mathbb{C};\mathfrak{g}) \oplus \Omega^{0,1}(\mathbb{C};\mathfrak{g})$	$(1,1)$	$(-1,1)$	$(0,2)$
ϕ^*	$\mapsto$	$\beta_{1,1} \in \Omega^{1,1}(\mathbb{C}; R^*)$	1	0	1
$\overline{\phi}^*$	$\mapsto$	$\overline{\phi}^* \in \Omega^{1,1}(\mathbb{C}; R)$	1	0	1
ψ	$\mapsto$	$\psi^*_{1,1} \in \Omega^{1,1}(\mathbb{C}; R)$	1	1	2
$\overline{\psi}$	$\mapsto$	$\beta_{1,0} \in \Omega^{1,0}(\mathbb{C}; R^*)$	1	-1	0

The action functionals decompose as:

$$S_{\text{gauge}} = \int \left(-(F^{2,0}, F^{0,2}) - \frac{1}{4}(\Lambda F_{1,1})^2\right) + \frac{1}{2}\left((\lambda_0, \overline{\partial}_{A_{0,1}}\lambda_{1,0}) + (\lambda_{1,0} \wedge \overline{\partial}_{A_{0,1}}\lambda_0)\right)$$

$$S_{\text{matter}} = \int \Big(((\partial_{A_{1,0}}\overline{\phi}, \overline{\partial}_{A_{0,1}}\gamma_0) + (\partial_{A_{1,0}}\gamma_0, \overline{\partial}_{A_{0,1}}\overline{\phi})) + (\psi_0, \partial_{A_{1,0}}\gamma_{0,1}) + (\gamma_{0,1} \wedge \partial_{A_{1,0}}\psi_0) +$$
$$+ (([\lambda_{1,0}, \gamma_{0,1}], \psi_0) + ([\lambda_{1,0}, \psi_0], \gamma_{0,1})) \Big)$$

$$S_{\text{anti}} = \int \partial_{A_{1,0}}A_0 \wedge A^*_{1,0} + \overline{\partial}_{A_{0,1}}A_0 \wedge B_{1,0} + [\lambda_{1,0}, A_0] \wedge \lambda^*_{0,1} + [\lambda_0, A_0] \wedge \lambda^*_{1,1} + B_{1,0} \wedge [A_0, A_{0,1}] + A^*_{0,1} \wedge [A_0, A_{1,0}]$$
$$+ \frac{1}{2}[A_0, A_0]A^*_0 + [\gamma_0, A_0]\beta_{1,1} + [\overline{\phi}, A_0]\overline{\phi}^* + [\gamma_{0,1}, A_0] \wedge \beta_{1,0} + [\psi_0, A_0]\psi^*_{1,1}$$

$$I^{(1)}_{\text{gauge}} = \int \left(-(\lambda_{1,0}, A^*_{0,1})\right)$$

$$I^{(1)}_{\text{matter}} = \int \left((\psi^*_{1,1}, \overline{\phi}) + \frac{1}{2}(\beta_{1,0} \wedge \overline{\partial}\gamma_0)\right)$$

$$I^{(2)}_{\text{gauge}} = -\frac{1}{4}\int (\lambda^*_{1,1})^2.$$

Theorem 12.13 (See also [**SWchar**])**.** The minimal twist of 2d $\mathcal{N} = (0,2)$ super Yang-Mills with Lie algebra $\mathfrak{g}$ coupled to the $\mathcal{N} = (0,2)$ chiral multiplet valued in a representation R is $U(1)$-equivariantly equivalent to holomorphic BF theory on $\mathbb{C}$ for the Lie algebra $\mathfrak{g}$ coupled to the holomorphic $\beta\gamma$ system on $\mathbb{C}$ with values in the representation R.

Proof. The proof of this theorem is very similar to the proof of Theorem 4.4. First, we integrate out the fields λ_0 and $\lambda^*_{1,1}$ using Proposition 1.8. We then observe that the action includes the terms $\int(\lambda_{1,0}, A^*_{1,0})$ and $\int(\phi, \psi^*_0)$. Thus, the two pairs $(\lambda_{1,0}, A_{1,0})$ and (ϕ, ψ_0) form BRST doublets, which can be integrated out using Proposition 1.10.

The twisted theory is therefore perturbatively equivalent to the theory with BV action

$$S_{\text{BV}} = \int \Big((B_{1,0}\overline{\partial}_{A_{0,1}}A_0) + \beta_{1,0}\overline{\partial}_{A_{0,1}}\gamma_0$$
$$+ [A_{0,1}, A_0] \wedge B_{1,0} + \frac{1}{2}[A_0, A_0]B_{1,1} + [\gamma_0, A_0] \wedge \beta_{1,1} + [\gamma_{0,1}, A_0] \wedge \beta_{1,0} \Big).$$

This is indeed the action functional of the required theory, where $A_{0,\bullet}, B_{1,\bullet}$ comprise the fields of holomorphic BF theory and $\gamma_{0,\bullet}, \beta_{1,\bullet}$ comprise the fields of the $\beta\gamma$ system. □

12.5 $\mathcal{N} = (0,4)$ Super Yang-Mills with Matter

We consider $\mathcal{N} = (0,4)$ super Yang-Mills for a Lie algebra $\mathfrak{g}$ coupled to $\mathcal{N} = (0,4)$ supersymmetric matter. The supersymmetric matter consists of the $\mathcal{N} = (0,4)$ hypermultiplet with values in a complex symplectic representation U. The spinorial representation is $\Sigma = S_- \otimes W_-$ where $W_- = \mathrm{span}_{\mathbb{C}}\{u_1, u_2, v_1, v_2\}$ is the four-dimensional auxiliary space equipped with the pairing $\langle u_i, v_j\rangle = \delta_{ij}$. Let $W = \mathrm{span}_{\mathbb{C}}\{u_1, u_2\}$.

The field content is:

- $\mathfrak{g}$-valued bosons: a gauge field $A \in \Omega^1(\mathbb{R}^2; \mathfrak{g})$.
- $\mathfrak{g}$-valued fermions: a spinor $\lambda \in C^\infty(\mathbb{R}^2; S_- \otimes W_- \otimes \mathfrak{g})$.
- $W \otimes U$-valued scalars: $\phi \in C^\infty(\mathbb{R}^2; W \otimes U)$;

- $W \otimes U$-valued fermions: $\psi \in C^\infty(\mathbb{R}^2; W \otimes S_+ \otimes U)$.

We twist by the nilpotent supercharge $Q = q \otimes u_1$ where $q \in S_-$ is any nonzero vector.

12.5.1 Minimal Twist

Theorem 12.14. The minimal twist of 2d $\mathcal{N} = (0,4)$ super Yang-Mills with Lie algebra $\mathfrak{g}$ coupled to the $\mathcal{N} = (0,4)$ hypermultiplet is $U(1)$-equivariantly equivalent to holomorphic BF theory on $\mathbb{C}$ for the Lie algebra $\mathfrak{g}$ coupled to the free fermion system on $\mathbb{C}$ with values in the representation $\mathfrak{g} \otimes W'_- \cong \mathfrak{g} \oplus \mathfrak{g}$ where $\mathfrak{g}$ acts by the adjoint action plus the free $\beta\gamma$ system on $\mathbb{C}$ valued in the $\mathfrak{g}$-representation U.

Proof. The supercharge Q lies in a $\mathcal{N} = (0,2)$ subalgebra of the full $\mathcal{N} = (0,4)$ algebra. On the odd part of the supersymmetry algebras, this embedding is induced by the algebra map

$$S_- \otimes \mathbb{C}[x]/x^2 \to S_- \otimes \mathrm{End}(W)$$

given by the identity on S_- and the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbb{C}[x]/x^2 \to \mathrm{End}(W)$. Under this embedding we see that the scalar supercharge Q is precisely the same as the scalar supercharge we used to twist in Section 12.4. Furthermore, under this embedding the $(0,4)$ vector multiplet (A, λ) splits as a $(0,2)$ vector multiplet plus a $(0,2)$ Fermi multiplet. The $(0,2)$ supersymmetry algebra acts trivially on the Fermi multiplet. The $(0,4)$ hypermultiplet becomes a $(0,2)$ chiral multiplet valued in the representation U. The claim now follows from Theorem 12.13. $\square$

Remark 12.15. The matter sector in the untwisted theory depends on U as a complex symplectic $\mathfrak{g}$-representation. The matter sector in the twisted theory only depends on U as a complex $\mathfrak{g}$-representation, the dependence on the symplectic form disappears when we twist.

12.6 $\mathcal{N} = (0, \mathcal{N}_-)$ Super Yang-Mills

We consider pure $\mathcal{N} = (0, \mathcal{N}_-)$ super Yang-Mills for a Lie algebra $\mathfrak{g}$, where $\mathcal{N}_- \geq 2$. The spinorial representation is $\Sigma = S_- \otimes W_-$ where W_- is the $\mathcal{N}_-$-dimensional auxiliary equipped with a nondegenerate symmetric bilinear pairing.

- $\mathfrak{g}$-valued bosons: a gauge field $A \in \Omega^1(\mathbb{R}^2; \mathfrak{g})$.
- $\mathfrak{g}$-valued fermions: a spinor $\lambda \in C^\infty(\mathbb{R}^2; S_- \otimes W_- \otimes \mathfrak{g})$.

Choose a splitting $W_- = \mathrm{span}_{\mathbb{C}}\{u, v\} \oplus W'_-$ where $\dim(W'_-) = \mathcal{N}_- - 2$ such that the pairing restricted to $\mathrm{span}_{\mathbb{C}}\{u, v\}$ is $\langle u, v \rangle = 1$.

The field content is:

			Untwisted	R	Twisted
c	$\mapsto$	$A_0 \in \Omega^0(\mathbb{C}; \mathfrak{g})$	-1	0	-1
A	$\mapsto$	$A_{0,1} + A_{1,0} \in \Omega^{0,1}(\mathbb{C}; \mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C}; \mathfrak{g})$	$(0,0)$	$(0,0)$	$(0,0)$
λ	$\mapsto$	$\lambda_0 u + \lambda_{1,0} v + \chi \in \Omega^0(\mathbb{C}; \mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C}; \mathfrak{g}) \oplus C^\infty(\mathbb{R}^2; S_- \otimes W'_- \otimes \mathfrak{g})$	$(0,0,0)$	$(1,-1,0)$	$(1,-1,0)$

We twist by the nilpotent supercharge $Q = q \otimes u_1$ where $q \in S_-$ is any nonzero vector.

12.6.1 Minimal Twist

Theorem 12.16 (See also [**SWchar**])**.** The minimal twist of 2d $\mathcal{N} = (0, \mathcal{N}_-)$ super Yang-Mills with Lie algebra $\mathfrak{g}$ is $U(1)$-equivariantly equivalent to holomorphic BF theory on $\mathbb{C}$ for the Lie algebra $\mathfrak{g}$ coupled to a free fermion system on $\mathbb{C}$ with values in the representation $\mathfrak{g} \otimes W'_- \cong \mathfrak{g} \otimes \mathbb{C}^{\mathcal{N}_- - 2}$ where $\mathfrak{g}$ acts by the adjoint action.

Proof. The proof is similar to the proof of Theorem 12.14. The supercharge Q lies in a $\mathcal{N} = (0, 2)$ subalgebra of the full $\mathcal{N} = (0, \mathcal{N}_-)$ algebra. On the odd part of the supersymmetry algebras, this embedding is induced by the algebra map

$$S_- \otimes \mathbb{C}[x]/x^2 \to S_- \otimes \operatorname{End}(\mathbb{C}^2)$$

given by the identity on S_- and the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbb{C}[x]/x^2 \to \operatorname{End}(\mathbb{C}^2)$. Here, $\mathbb{C}^2$ is spanned by $\{u, v\}$ in the decomposition of W_- given above. Under this embedding we see that the scalar supercharge Q is precisely the same as the scalar supercharge we used to twist in Section 12.4. Furthermore, under this embedding the $(0, \mathcal{N}_-)$ vector multiplet (A, λ) splits as a $(0, 2)$ vector multiplet plus a $(0, 2)$ Fermi multiplet valued in $\mathfrak{g} \otimes W'_-$. The $(0, 2)$ supersymmetry algebra acts trivially on the Fermi multiplet. The claim now follows from a special case of Theorem 12.13 where we take $R = 0$. ☐

A Spinors

In this paper we will extensively use the theory of spinors. Let V be a complex vector space equipped with a nondegenerate symmetric bilinear pairing. Recall that the Clifford algebra $\mathrm{Cl}(V)$ is defined to be the quotient of the tensor algebra on V by the relation

$$v_1 v_2 + v_2 v_1 = 2(v_1, v_2).$$

Consider a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford module $M = M^+ \oplus M^-$. Denote the Clifford action by $\rho(v) \in \operatorname{End}(M)$. We assume the Clifford module is equipped with a nondegenerate pairing $(-,-)\colon M^+ \otimes M^- \to \mathbb{C}$ such that

$$(\rho(v) Q_1, Q_2) = (Q_1, \rho(v) Q_2)$$

for any $Q_1, Q_2 \in M$ and $v \in V$. From now on we denote $M^+ = \Sigma$ and $M^- = \Sigma^*$. We define the Γ-pairings

$$\Gamma\colon \mathrm{Sym}^2(\Sigma) \longrightarrow V, \qquad \Gamma\colon \mathrm{Sym}^2(\Sigma^*) \longrightarrow V$$

by

$$(\rho(v) Q_1, Q_2) = (v, \Gamma(Q_1, Q_2)) \tag{28}$$

We have a subset $\mathrm{Spin}(V) \subset \mathrm{Cl}(V)$, so Σ and Σ^* are representations of the spin group. Moreover, the Clifford action and the maps Γ are $\mathrm{Spin}(V)$-equivariant.

We may identify $\wedge^2(V) \to \mathfrak{so}(V)$ via

$$\omega \mapsto (w \mapsto -2\iota_{(w,-))}\omega).$$

This gives rise to an action map

$$\wedge^2(V) \otimes \Sigma \longrightarrow \Sigma$$

of two-forms on spinors.

Consider the map $q\colon \wedge^\bullet (V) \to \mathrm{Cl}(V)$ given by antisymmetrization, so that, for instance,

$$q(v_1 \wedge v_2) = v_1 v_2 - (v_1, v_2). \tag{29}$$

The resulting action $\wedge^2(V) \otimes \Sigma \to \Sigma$ then coincides with the original action of $\mathfrak{so}(V)$ on the spinorial representation Σ, so that $\mathfrak{so}(V)$-equivariance of Γ gives the following.

Proposition A.1. For $X \in \wedge^2(V)$ and $Q_1, Q_2 \in \Sigma$ we have

$$\Gamma(Q_1, \rho(X)Q_2) + \Gamma(Q_2, \rho(X)Q_1) = -2\iota_{\Gamma(Q_1,Q_2)}X.$$

We may extend the discussion to the case of Riemannian manifolds N, where we replace V by the vector bundle TN. Given a bundle of Clifford modules $M = \Sigma \oplus \Sigma^*$ as before we have the associated Dirac operator

$$\not{d} \colon \Gamma(N, \Sigma) \to \Gamma(N, \Sigma^*).$$

From (29) we get the following property.

Proposition A.2. Suppose $Q_1, Q_2 \in \Sigma$ and $\lambda \in \Gamma(N, \Sigma)$. Then

$$\not{d}\rho(\Gamma(Q_1, \lambda))Q_2 = \rho(d\Gamma(Q_1, \lambda))Q_2 + (Q_1, \not{d}\lambda)Q_2.$$

Finally, we have the following important compatibility between the Clifford action of differential forms and the Dirac operator proved in [**Snygg**].

Proposition A.3. Suppose $Q \in \Sigma$ and $X \in \Omega^p(N)$. Then

$$\not{d}(\rho(X)Q) = \rho(\mathrm{d}X)Q + (-1)^{n(1+p)}\rho(*\mathrm{d} * X)Q.$$

Note that both Proposition A.2 and Proposition A.3 extend to the case when λ and X respectively are twisted by a vector bundle and $\not{d}$ is the corresponding twisted Dirac operator.

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