1.

1.1. Sign convention. Each field has two gradings: the ghost grading and the fermionic number. The braiding is defined so that $xy = (-1)^{(F(x)+|x|)(F(y)+|y|)} yx$.

1.2. Useful formulas. The Clifford relation is

$$vw + wv = (v, w).$$

The relationship between Clifford multiplication and Γ is

$$(v, \Gamma(Q_1, Q_2)) = \langle Q_1, \rho(v) Q_2 \rangle.$$

For a one-form β we have

$$\alpha \wedge \beta = (*\alpha, \beta)\mathrm{dVol}.$$

The map $\wedge^2 V \to \mathrm{Cl}(V)$ is given by

$$v_1 \wedge v_2 \mapsto v_1 v_2 - \frac{1}{2}(v_1, v_2).$$

If X is a two-form and Q_1, Q_2 are two spinors, we have

$$\Gamma(Q_1, \rho(X) Q_2) + \Gamma(Q_2, \rho(X) Q_1) = \iota_{\Gamma(Q_1, Q_2)} X.$$

Since we are in the Euclidean signature, we have

$$**\alpha = (-1)^{k(n-k)} \alpha,$$

where α is a k-form.

Theorem 1.1 (3-ψ rule)**.**

$$\rho(\Gamma(Q_1, Q_2)) Q_3 + \rho(\Gamma(Q_2, Q_3)) Q_1 + \rho(\Gamma(Q_3, Q_1)) Q_2 = 0.$$

Equivalently,

$$(\Gamma(-, Q_1), \Gamma(Q_2, Q_3)) + (\Gamma(-, Q_2), \Gamma(Q_3, Q_1)) + (\Gamma(-, Q_3), \Gamma(Q_1, Q_2)) = 0.$$

1.3. Computation. To simplify the notation, we drop the integral, the pairing on $\mathfrak{g}$ and the volume form. We denote by $\langle -, - \rangle$ the spinorial pairing.

The BRST action is

$$S_{BRST} = \frac{1}{2} F_A \wedge *F_A - \langle \lambda, \not{D}_A \lambda \rangle.$$

The BV action is

$$S_{BV} = \frac{1}{2} F_A \wedge *F_A - \langle \lambda, \not{D}_A \lambda \rangle + d_A c \wedge A^* - \langle [\lambda, c], \lambda^* \rangle - \frac{1}{2}[c, c] c^*.$$

The supersymmetry action is given by

$$I^{(1)}(Q) = -2\Gamma(Q, \lambda) \wedge A^* - \langle \rho(F_A) Q, \lambda^* \rangle$$
$$I^{(2)}(Q_1, Q_2) = (\Gamma(Q_1, Q_2), \Gamma(\lambda^*, \lambda^*)) - \langle Q_1, \lambda^* \rangle \langle Q_2, \lambda^* \rangle - 2(\iota_{\Gamma(Q_1, Q_2)} A) c^*.$$

We need to check

$$\{S_{BV}, I^{(1)}\} = 0 \tag{1}$$

$$\{S_{BV}, I^{(2)}\} + d_{CE}I^{(1)} + \frac{1}{2}\{I^{(1)}, I^{(1)}\} = 0 \tag{2}$$

$$d_{CE}I^{(2)} + \{I^{(1)}, I^{(2)}\} = 0. \tag{3}$$

1.4. **Checking (1).**

Lemma 1.2 (Snygg)**.** *Suppose X is an adjoint-valued p-form and Q a constant spinor. Then*

$$\not{D}_A(\rho(X)Q) = \rho(d_A X)Q + (-1)^{n(p+1)}\rho(*d_A * X)Q.$$

Lemma 1.3.

$$\{I^{(1)}(Q), S_{BV}\} = (\Gamma(Q,\lambda), \Gamma(\lambda,\lambda)).$$

Proof. Let us split $S_{BV} = \sum_{i=1}^{5} S_{BV}^i$ into individual summands.

The first term gives

$$\begin{aligned}
-\frac{1}{2}\{I^{(1)}(Q), S_{BV}^1\} &= d_A\Gamma(Q,\lambda) \wedge *F_A \\
&= \Gamma(Q,\lambda) \wedge d_A * F_A \\
&= (-1)^{n-1} d_A * F_A \wedge \Gamma(Q,\lambda) \\
&= (-1)^{n-1}(*d_A * F_A, \Gamma(Q,\lambda)).
\end{aligned}$$

The second term gives

$$\begin{aligned}
-\frac{1}{2}\{I^{(1)}(Q), S_{BV}^2\} &= (\lambda, \rho(\Gamma(Q,\lambda))\lambda) - (\rho(F_A)Q, \not{D}_A\lambda) \\
&= (\Gamma(Q,\lambda), \Gamma(\lambda,\lambda)) + \langle\lambda, \not{D}_A(\rho(F_A)Q)\rangle \\
&= (\Gamma(Q,\lambda), \Gamma(\lambda,\lambda)) + (-1)^n\langle\lambda, \rho(*d_A * F_A)Q\rangle,
\end{aligned}$$

where we have used Lemma lemma 1.1 and the Bianchi identity in the last line.

The third term gives

$$\begin{aligned}
-\frac{1}{2}\{I^{(1)}(Q), S_{BV}^3\} &= [\Gamma(Q,\lambda), c] \wedge A^* + \frac{1}{2}\langle\rho(d_A d_A c)Q, \lambda^*\rangle \\
&= \Gamma(Q, [\lambda, c]) \wedge A^* + \frac{1}{2}\langle\rho([F_A, c])Q, \lambda^*\rangle.
\end{aligned}$$

The fourth term gives

$$-\frac{1}{2}\{I^{(1)}(Q), S_{BV}^4\} = -\frac{1}{2}([\rho(F_A)Q, c], \lambda^*) - \Gamma(Q, [\lambda, c]) \wedge A^*.$$

The fifth term gives

$$\{I^{(1)}(Q), S_{BV}^5\} = 0.$$

□

By the 3-ψ rule (**??**) we get that S_{BV} is supersymmetric.

1.5. **Checking (2).** We have

$$\begin{aligned}\{I^{(1)}, I^{(1)}\}(Q_1, Q_2) &= -2\{I^{(1)}(Q_1), I^{(1)}(Q_2)\} \\ &= -4\langle \rho(d_A\Gamma(Q_1,\lambda))Q_2, \lambda^*\rangle - 4\Gamma(Q_2, \rho(F_A)Q_1) \wedge A^* + 1 \leftrightarrow 2\end{aligned}$$

$$(d_{CE}I^{(1)})(Q_1, Q_2) = 2L_{\Gamma(Q_1,Q_2)}(A) \wedge A^* + 2\langle \Gamma(Q_1,Q_2).\lambda, \lambda^*\rangle + 2(\Gamma(Q_1,Q_2).c)c^*$$

$$\begin{aligned}\{S_{BV}, I^{(2)}(Q_1,Q_2)\} &= 2(\Gamma(Q_1,Q_2), \Gamma(\lambda^*, \not{D}_A\lambda + [\lambda^*, c])) - \langle Q_1, \lambda^*\rangle\langle Q_2, \not{D}_A\lambda + [\lambda^*, c]\rangle \\ &+ 1 \leftrightarrow 2 - 2\iota_{\Gamma(Q_1,Q_2)}(d_Ac)c^* - 2d_A\iota_{\Gamma(Q_1,Q_2)}A \wedge A^* + 2\langle [\lambda, \iota_{\Gamma(Q_1,Q_2)}A], \lambda^*\rangle \\ &+ 2[\iota_{\Gamma(Q_1,Q_2)}A, c]c^*\end{aligned}$$

We must have

$$\begin{aligned}&-2\iota_{\Gamma(Q_1,Q_2)}F_A + 2L_{\Gamma(Q_1,Q_2)}(A) - 2d_A\iota_{\Gamma(Q_1,Q_2)}A = 0 \\ &\ 2(\Gamma(Q_1,Q_2).c) - 2\iota_{\Gamma(Q_1,Q_2)}(d_Ac) + 2[\iota_{\Gamma(Q_1,Q_2)}A, c] = 0 \\ &-2\rho([A, \Gamma(Q_1,\lambda)])Q_2 - 2\rho([A,\Gamma(Q_2,\lambda)])Q_1 + 2\rho(\Gamma(Q_1,Q_2))\rho(A)\lambda \\ &\quad - Q_2\iota_{\Gamma(Q_1,\lambda)}A - Q_1\iota_{\Gamma(Q_2,\lambda)}A + 2[\lambda, \iota_{\Gamma(Q_1,Q_2)}A] = 0 \\ &-2\rho(d\Gamma(Q_1,\lambda))Q_2 - 2\rho(d\Gamma(Q_2,\lambda))Q_1 + 2\Gamma(Q_1,Q_2).\lambda \\ &\quad + 2\rho(\Gamma(Q_1,Q_2))(\not{D}\lambda) - \langle Q_1, \not{D}\lambda\rangle Q_2 - \langle Q_2, \not{D}\lambda\rangle Q_1 = 0\end{aligned}$$

The first two equations are straightforward to check. The third and fourth equations are checked in the same way, so let's just check equation 3. We have

$$\begin{aligned}&-2\rho([A,\Gamma(Q_1,\lambda)])Q_2 - 2\rho([A,\Gamma(Q_2,\lambda)])Q_1 - Q_2\iota_{\Gamma(Q_1,\lambda)}A - Q_1\iota_{\Gamma(Q_2,\lambda)}A \\ &\quad = -2\rho(A)\rho(\Gamma(Q_1,\lambda))Q_2 - 2\rho(A)\rho(\Gamma(Q_2,\lambda))Q_1 \\ &\quad = 2\rho(A)\rho(\Gamma(Q_1,Q_2))\lambda,\end{aligned}$$

where we have applied the 3-ψ rule in the last line.

The Clifford relation then gives

$$2\rho(A)\rho(\Gamma(Q_1,Q_2))\lambda + 2\rho(\Gamma(Q_1,Q_2))\rho(A)\lambda = 2[\iota_{\Gamma(Q_1,Q_2)}A, \lambda]$$

which cancels the last term.