PURE SPINOR FORMALISM

PAVEL SAFRONOV

1. Pure spinors

Let $V = \mathbf{C}^{10}$ and S_+, S_- be the semi-spin representations of $\mathrm{Spin}(V)$. We have a nondegenerate $\mathrm{Spin}(V)$-equivariant pairing $\Gamma\colon \mathrm{Sym}^2(S_+) \to V$.

For a vector space L we denote by $\mathrm{ML}(L)$ the metalinear group, i.e. the $2:1$ cover of $\mathrm{GL}(L)$ given by the pullback

$$\begin{tikzcd} \mathrm{ML}(L) \arrow[r] \arrow[d] & \mathrm{GL}(1) \arrow[d, "z\mapsto z^2"] \\ \mathrm{GL}(L) \arrow[r, "\det"] & \mathrm{GL}(1) \end{tikzcd}$$

Fact: the choice of a spin structure on V endows any Lagrangian subspace $L \subset V$ with a metalinear structure, i.e. a choice of $\det(L)^{1/2}$.

Proposition 1.1.

(1) *The group* $\mathrm{Spin}(V)$ *acts transitively on the set* $\mathrm{LGr}(V)$ *of Lagrangian subspaces* $L \subset V$.

(2) *The stabilizer of a Lagrangian subspace* $L \subset V$ *is a parabolic subgroup* $G_L \subset \mathrm{Spin}(V)$ *which fits into an exact sequence*

$$1 \longrightarrow \wedge^2 L \longrightarrow G_L \longrightarrow \mathrm{ML}(L) \longrightarrow 1.$$

The choice of a Lagrangian complement $L^* \subset V$ *to* $L \subset V$ *determines a splitting of this exact sequence, i.e. it gives an identification* $G_L \cong \mathrm{ML}(L) \ltimes \wedge^2 L$.

(3) *Under the restriction* $G_L \subset \mathrm{Spin}(V)$ *the semi-spin representations split as*

$$S_+ = (\mathbf{C} \oplus \wedge^2(L^*) \oplus \wedge^4(L^*)) \otimes \det(L)^{1/2}, \qquad S_- = (\mathbf{C} \oplus \wedge^2 L \oplus \wedge^4 L) \otimes \det(L)^{-1/2}.$$

The tangent bundle $\mathrm{T}_{\mathrm{LGr}(V)}$ to $\mathrm{LGr}(V)$ is naturally $\mathrm{Spin}(V)$-equivariant. Its fiber at $L \in \mathrm{LGr}(V)$ is isomorphic to

$$\wedge^2(L \oplus L^*)/(\mathrm{End}(L) \oplus \wedge^2 L) \cong \wedge^2 L^*$$

as a G_L-representation (here $\wedge^2 L$ acts trivially). In particular, $\dim(\mathrm{LGr}(V)) = 10$.

We have $\det(\wedge^2 L^*) \cong \deg(L)^{-4}$. This representation is not G_L-invariant, so $\mathrm{LGr}(V)$ does not have a $\mathrm{Spin}(V)$-invariant Calabi–Yau structure.

Proposition 1.2. *Let* P *be the set of nonzero elements* $Q \in S_+$ *satisfying* $\Gamma(Q, Q) = 0$ *and* $\tilde{\mathrm{P}} = \mathrm{P} \cup \{0\}$.

- *For* $Q \in \mathrm{P}$ *the image of* $\Gamma(Q, -)\colon S_+ \to V$ *is a Lagrangian subspace. In particular, we have a projection* $\mathrm{P} \to \mathrm{LGr}(V)$.

- *The natural action of $\mathbf{C}^\times$ on P by scaling gives $\mathrm{P} \to \mathrm{LGr}(V)$ the structure of a $\mathbf{C}^\times$-torsor. The fiber of $\mathrm{P} \to \mathrm{LGr}(V)$ at $L \subset V$ may be identified with nonzero elements $Q \in \det(L)^{1/2}$.*

The tangent bundle T_P to P is naturally $\mathrm{Spin}(V)$-equivariant. Its fiber at $Q \in \mathrm{P}$ is isomorphic to

$$\wedge^2(L \oplus L^*)/(\mathrm{End}_0(L) \oplus \wedge^2 L) \cong \wedge^2 L^* \oplus \mathbf{C}$$

as a $\mathrm{SL}(L) \ltimes \wedge^2 L$-representation ($\wedge^2 L$ acts from the first to the second summand). In particular, $\det(\mathrm{T}_{\mathrm{P},Q}) \cong \det(L)^{-4}$ which is trivial as a $\mathrm{SL}(L) \ltimes \wedge^2 L$-representation. In particular, there is a unique $\mathrm{Spin}(V)$-invariant Calabi–Yau structure on P.

Choose a point $Q \in P$. We can introduce a coordinate chart near Q in the following way. We split

$$S_+ = (\mathbf{C} \oplus \wedge^2(L^*) \oplus \wedge^4(L^*)) \otimes \det(L)^{1/2}.$$

Let $(\ell, A, M) \in S_+$ be components of a spinor with respect to this splitting. The pure spinor constraint is

$$\ell M + \Lambda \wedge \Lambda = 0,$$
$$\langle \Lambda, M \rangle = 0,$$

where in the last line the pairing is $\wedge^2 L^* \otimes \wedge^4 L^* \to \det(L)^* \otimes L^*$.

In particular, in a neighborhood of Q (i.e. in a neighborhood of $\Lambda = 0$, $M = 0$ and $\ell \neq 0$) the pair (ℓ, Λ) gives a coordinate chart. We may identify

$$\det(\wedge^2(L^*) \otimes \det(L)^{1/2}) \cong \det(L),$$

so in this chart the unique $\mathrm{Spin}(V)$-invariant Calabi–Yau structure has the form

$$\Omega = \ell^{-3} \mathrm{d}\ell \mathrm{d}^{10}\Lambda.$$

2. Pure spinor formulation of 10d SYM

Let

$$T = \Pi\Sigma_+ \oplus V$$

be the supertranslation Lie algebra and G_T the supertranslation group. Then $C^\infty(G_T)$ carries two commuting T-actions given by left and right translations. For $\sigma \in S_+$ denote by Q_σ and $\mathcal{D}_\sigma$ the corresponding vector fields.

We assign the ghost number number 1 and odd fermionic degree to coordinates on P. The fields in our theory are

$$\mathcal{F} = C^\infty(G_T) \otimes \mathcal{O}(\mathrm{P}) \otimes \mathfrak{g}[1].$$

The differential at $\sigma \in \mathrm{P}$ is given by $\mathcal{D}_\sigma$. There is a residual supersymmetry action on $\mathcal{F}$ given by Q_σ.

The differential $\mathcal{D}$ can be split as $\mathcal{D}^0 + \mathcal{D}^1$, where $\mathcal{D}^0$ is $\mathcal{D}$ with $\Gamma = 0$. The differential $\mathcal{D}^0$ does not act on $C^\infty(V)$, so it just becomes an overall factor.

Definition 2.1. The ***zero-mode cohomology*** is the cohomology of $C^\infty(\Pi S) \otimes \mathcal{O}(\mathrm{P})$ with respect to $\mathcal{D}^0$.