A CLASSIFICATION OF SPINORS UP TO DIMENSION TWELVE.[1]

By Jun-ichi Igusa.

A spinor is an element of the vector space in which the spin representation takes place. The group G that is being represented is the two-sheeted covering group of the special orthogonal group called the spin group. We regard two spinors to be "congruent" or G-equivalent if there exists an element of G which transforms one to another via the spin representation. By a *classification of spinors*, we understand (1) *the decomposition of the space of spinors into equivalence classes or "orbits"* and (2) *the determining of the structure of the stabilizer subgroup of G for each orbit.*

The objective of this paper is to classify spinors up to dimension twelve. Although we can certainly make a similar investigation for slightly higher dimensions, it appears that the classification of spinors for an arbitrary dimension is a difficult problem. In this connection, we would like to mention that, as far as we know, the only G-orbit that has been closely examined (by E. Cartan and Chevalley) is the (projectively closed) orbit consisting of "simple" or "pure" spinors. We shall investigate some such special orbits in general (enough to settle the problem up to dimension twelve). We shall fix our ground field k of characteristic different from 2 and classify spinors relative to k. This is more precise than the absolute classification that we have formulated. Instead, we shall assume that G "splits over k." In other words, we shall assume that the quadratic form for the spin group G has a maximal index over k. The restriction on the dimension comes from the fact that we are primarily interested in what we call "absolutely admissible representations." We recall that a spin representation (half-spin representation for even dimension) is absolutely admissible if and only if the dimension is at most twelve (cf. 6). Actually, the spin representations up to dimension twelve can best be described in connection with a certain representation of the simple group of type E_7 via the theory of Jordan algebras. In writing up the theory of absolutely admissible representations, however, we find it convenient to separate this part from the general theory. We have also treated a similar classification problem for a certain (absolutely admis-

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sible) representation which is contained in the spin representation. For the convenience of the reader, we have tried to make this paper as elementary (and self-contained) as possible.

1. Preliminaries. We shall denote by Ω a universal domain and by k, K its subfields. Let G denote an algebraic group, X a vector space, and ρ a (rational) representation of G in X all defined over a field k. If x is an arbitrary point of X, the subset $\rho(G)\cdot x$ of X, i. e., the set of points of X of the form $\rho(g)\cdot x$ for all g in G, is called the G-orbit of x with x as its represntative. The stabilizer subgroup of G at x consists of all g satisfying $\rho(g)\cdot x = x$. It is an algebraic group defined over $k(x)$. The space X decomposes into G-orbits. We say that a G-orbit U satisfies the *Witt condition* if G_K operates transitively on U_K for any extension K of k over which U is defined. We are denoting by U_K the set of K-rational points of U; similarly for G_K. We shall consider the special case when G is the spin group of a quadratic form in m variables and ρ the spin representation (half-spin representation for even m). We shall assume that the characteristic of Ω is different from 2. We shall start by recalling basic definitions and results. We refer to (2), (3), (5) for the details.

Let V denote a vector space of dimension $m \geqq 3$ and f a non-degenerate quadratic form on V both defined over k. We shall denote by (u, v) the associated Ω-bilinear form, i. e., we put $(u, v) = f(u+v) - f(u) - f(v)$ for every u, v in V. We shall denote by $\mathrm{Aut}(V, f)$ the orthogonal group of (V, f). It consists of invertible Ω-linear transformations σ in V satisfying $f(\sigma\cdot v) = f(v)$ for every v in V. We shall denote by $\mathrm{Aut}(V, f)^{+}$ the connected component of the identity in $\mathrm{Aut}(V, f)$. It is defined by $\det(\sigma) = 1$. Let C denote the *Clifford algebra* of (V, f). This is an associative algebra over Ω generated by V in which $v^2 = f(v)$ for every v in V and which is "universal" with respect to this property. It has the structure of a graded module over Ω, which is not intrinsic. It has the so-called canonical anti-automorphism $x \to x'$, which is an involution of C, defined over k, keeping every element of V invariant. If we denote by $C^{\pm}$ the sums of homogeneous submodules of C of even and odd degrees respectively, we get the structure of a semi-graded algebra, which is intrinsic. In other words, the sum $C = C^{+} + C^{-}$ is Ω-direct and $(C^{\pm})^2 \subset C^{+}$, $C^{+}C^{-}$, $C^{-}C^{+} \subset C^{-}$. In particular, C^{+} is a subalgebra of C invariant by the canonical anti-automorphism of C. For any subset S of C, we put $S^{\pm} = S \cap C^{\pm}$. Also, for any subspace W of V, we shall denote by C_W the subalgebra of C generated by W. If W is totally isotropic, i.e., if $f = 0$ on W, C_W is isomorphic to the exterior algebra of W. On the other hand, if V decomposes into a direct sum of orthogonal subspaces as

$V = V_1 \oplus V_2$ and if C_i for each i is the Clifford algebra of V_i with respect to the restriction of f to V_i, we can identify C with $C_1 \otimes C_2$ as semi-graded algebras. We recall that, if x_1, y_1 and x_2, y_2 are elements of C_1 and C_2, each one of which is even or odd, the product $(x_1 \otimes x_2)(y_1 \otimes y_2)$ is $\pm (x_1y_1 \otimes x_2y_2)$. We take the minus-sign if and only if both x_2 and y_1 are odd.

The Clifford algebra C and its subalgebra C^+ are of dimension 2^m and 2^{m-1} respectively, and they are both semi-simple. Their centers are of dimension 1, 2 when m is even and 2, 1 when m is odd. We shall define the spin group. Let G^* denote the subgroup of the unit group of C consisting of those s with the property that $sVs^{-1} = V$. Then G^* forms an algebraic group defined over k. This is the *Clifford group* of (V, f). If s is in G^*, $s's$ is in the center of C, which is Ω when m is even. The correspondence $s \rightarrow s's$ gives rise to a homomorphism $G^* \rightarrow G_m$ $(= \Omega^\times)$ defined over k. In general, for any associative algebra A, we shall denote by $A^\times$ the unit group of A. In the case when m is odd, if s is in $(G^*)^+$, $s's$ is in the center of C^+, which is Ω. Therefore, we get a homomorphism $(G^*)^+ \rightarrow G_m$ defined over k. The *spin group* G is defined, in each case, as the kernel of the homomorphism $(G^*)^+ \rightarrow G_m$. It is a connected, simply connected, semi-simple group defined over k. In the case when $m = 4$, G is isomorphic to $SL_2 \times SL_2$. Except for this case, G is simple. If we put $\phi(s) \cdot v = svs^{-1}$ for every s in G^* and v in V, we get a homomorphism $\phi\colon G^* \rightarrow \mathrm{Aut}(V, f)$ defined over k. This homomorphism is surjective, and it gives rise to an epimorphism $G \rightarrow \mathrm{Aut}(V, f)^+$ with ± 1 as its kernel. This is called the *vector representation* of G. On the other hand, we observe that C^+ is the enveloping algebra of G. Furthermore, C^+ is isomorphic to the direct sum of two total matric algebras of degree 2^{n-1} when $m = 2n$ and to the total matric algebra of degree 2^n when $m = 2n + 1$. Therefore, in each case, we get a representation of G of degree 2^n. This is called the *spin representation* of G. The spin representation is a monomorphism. An element of the representation space is called a *spinor*. We observe that, unlike the vector representation, the spin representation is not necessarily defined over k. The spin representation for $m = 2n$ is not irreducible, but it is a sum of *half-spin representations*. They are irreducible and of degree 2^{n-1}.

We shall impose a condition on f relative to k so that the spin representation will be defined over k. The condition is that f is of *maximal index* over k, i.e., of index n over k for $m = 2n$, $2n + 1$. We shall explicitly construct the spin representation and the space of spinors. The construction is rational over k. We take a maximal totally isotropic subspace of V which is defined over k. By the Witt theorem, any two such subspaces are conjugate by an operation of $\mathrm{Aut}(V,f)_k$. We fix a maximal totally isotropic subspace L of V defined over k and an Ω-base $e_1,\cdots,e_n$ of L rational over k. Then, there exists a maximal totally isotropic subspace M of V also defined over k and an Ω-base $e_{n+1},\cdots,e_{2n}$ of M rational over k satisfying $(e_i,e_{n+i})=1$ for $1\leqq i\leqq n$ and $(e_i,e_j)=0$ for other pairs (i,j), $i\leqq j$. If W is a totally isotropic subspace of V, we shall denote by e_W the product of an Ω-base of W in C_W. We observe that $\Omega^\times e_W$ is uniquely determined by W. We take $e_L=e_1\cdots e_n$ and $e_M=e_{n+1}\cdots e_{2n}$. In the case when $m=2n$, Ce_M is a minimal left ideal of C and the correspondence $x\to xe_M$ gives an isomorphism $C_L\to Ce_M$ of these vector spaces defined over k. Therefore, if s is in C and x in C_L, there exists one and only one element y of C_L satisfying $sxe_M=ye_M$. Moreover, if we put $\rho(s)\cdot x=y$, we will get a representation ρ of C in C_L defined over k. We put

$$X=(C_L)^+=C_L\cap C^+.$$

This is a vector space of dimension 2^{n-1} defined over k. By restricting ρ to G, we get a half-spin representation of G in X. Another half-spin representation of G is obtained by using $(C_L)^-$ instead of $(C_L)^+$. In the case when n is even and $n\neq 2$, the kernel of the half-spin representation consists of 1 and

$$s_0=(e_1+e_{n+1})(e_1-e_{n+1})\cdots(e_n+e_{2n})(e_n-e_{2n}).$$

We observe that $\pm s_0$ are the two elements of the center of C^+ whose squares are 1 and which anticommute with every element of V. In the case when $m=2n+1$, let V_0 denote the subspace of V generated by $e_1,e_2,\cdots,e_{2n}$. Then, the subspace of V orthogonal to V_0 is of the form $\Omega\epsilon_0$ for some ϵ_0 rational over k and satisfying $f(\epsilon_0)\neq 0$. By replacing f by λf for $\lambda=f(\epsilon_0)^{-1}$, at the same time replacing $e_{n+1},\cdots,e_{2n}$ by $\lambda^{-1}e_{n+1},\cdots,\lambda^{-1}e_{2n}$, we may assume that $f(\epsilon_0)=1$. This affects the spin group G etc. only up to an isomorphism over k. Consider a vector space V_1 of dimension $m+1$ defined over k and represent it as $V+\Omega\epsilon_1$. We define a quadratic form f_1 on V_1 as $f_1(v+\lambda\epsilon_1)=f(v)-\lambda^2$ for every v in V and λ in Ω. Then f_1 is a non-degenerate quadratic form on V_1 of index $n+1$ over k. In fact, if we put

$$e_0=(\tfrac{1}{2})(\epsilon_0+\epsilon_1),\quad e_{2n+1}=(\tfrac{1}{2})(\epsilon_0-\epsilon_1),$$

we have $(e_i,e_{n+i})=(e_0,e_{2n+1})=1$ for $1\leqq i\leqq n$ and $(e_i,e_j)=0$ for other pairs (i,j), $i\leqq j$. We are denoting by (u,v) the associated Ω-bilinear form of f_1. Therefore $L_1=\Omega e_0+L$, $M_1=M+\Omega e_{2n+1}$ are maximal totally isotropic subspaces of V_1 defined over k, and they are similar to L, M for V. Let G_1 denote the spin group of (V_1, f_1) and put

$$X = X_1 = (C_{L_1})^+.$$

Then, the spin group G of (V, f) becomes the stabilizer subgroup of G_1 at ϵ_1 via its vector representation ϕ_1 in V_1. In other words, an element s of G_1 belongs to G if and only if $\phi_1(s) \cdot \epsilon_1 = \epsilon_1$. By restricting the half-spin representation of G_1 in X_1 to G, we get the spin representation ρ of G in X.

We shall recall less standardized facts. Suppose first that u, v are elements of V satisfying $f(u)f(v) = 0$, $(u, v) = 0$. Then, for $x = uv$ in C and λ in Ω, the exponential

$$\exp(\lambda x) = 1 + \lambda x + (2!)^{-1}(\lambda x)^2 + \cdots$$

represents a well-defined element of C. This is because we have

$$x^2 = -f(u)f(v) = 0.$$

Furthermore, $\exp(\lambda x) = 1 + \lambda x$ is contained in G. In particular, if W is a totally isotropic subspace of V and if $(C_W)_2$ denotes the homogeneous part of degree two in the exterior algebra C_W, we get a well-defined homomorphism $\exp\colon (C_W)_2 \to G$ such that $\exp(x) = 1 + x +$ higher terms for every x in $(C_W)_2$. In this connection, we remark that, if u, v are elements of V satisfying $f(u) = f(v) = 0$, $(u, v) = 1$, for any λ in $\Omega^\times$, the element of C of the form

$$\lambda^{-1} + (\lambda - \lambda^{-1})uv = (u + \lambda^{-1}v)(u + \lambda v)$$

is contained in G. Furthermore, this element depends multiplicatively on λ Suppose next that m is even and N a maximal totally isotropic subspace of V. Then $Ce_M \cap e_N C$ is of dimension 1. Hence, it is of the form $\Omega x e_M$ for some $x \neq 0$ in C_L such that $\Omega^\times x$ is uniquely determined by N. The fact is that x determines N uniquely and the set of all such x forms the G^*-orbit of 1. Moreover, an element $x \neq 0$ of C_L is contained in this G^*-orbit if and only if it is of the form $\exp(y)e_W$ for some y in $(C_L)_2$ and W in L. It is understood that e_W for $W = 0$ means an element of $\Omega^\times$. This is the *parametric representation of pure spinors* due to Chevalley (cf. 3, p. 76). Finally, still in the case when m is even, we take x, y from C_L and put

$$x'y = g(x, y)e_L + \text{lower terms},$$

in which $g(x, y)$ is in Ω. The Ω-bilinear form g on $C_L \times C_L$ is defined over k, and it is a G-invariant in the sense that $g(\rho(s) \cdot x, \rho(s) \cdot y) = g(x, y)$ for every s in G and x, y in C_L. It is called the *spinor invariant*. The spinor invariant gives a non-degenerate quadratic form on $(C_L)^{\pm}$ if and only if m is a multiple of 8.

2. Even dimensional cases. We shall consider the case when $\dim(V) = 2n \geqq 4$. Since the quadratic form f on V is assumed to be of maximal index over k, we may denote the corresponding spin group G without ambiguity by Spin_{2n}. Similarly, we may use the notation Spin_{2n+1} in the case when $\dim(V) = 2n+1$. We shall fix a maximal k-split torus T of G and make the roots and the corresponding unipotent groups explicit. We recall that, for each root α of G relative to T there exists a unique connected unipotent subgroup P_α of G, normalized by T, and n isomorphism θ_α of G_a $(=\Omega)$ to P_α defined over k such that $t\cdot\theta_\alpha(\lambda)\cdot t^{-1} = \theta_\alpha(\alpha(t)\cdot\lambda)$ for every t in T and λ in G_a (cf. 1, 4).

If λ is in G_m, for any index i satisfying $1 \leqq i \leqq n$, we put

$$s_i(\lambda) = \lambda^{-1} + (\lambda - \lambda^{-1})\, e_i e_{n+i}.$$

Then $s_i(\lambda)$ is an element of G satisfying

$$\phi(s_i(\lambda))\cdot(\sum_{j=1}^{2n}\lambda_j e_j) = \lambda^2\lambda_i e_i + \lambda^{-2}\lambda_{n+i}e_{n+i} + \sum_{j\neq i,n+i}\lambda_j e_j.$$

Moreover, if we associate $s_1(t_1)\cdots s_n(t_n)$ to $t = (t_1,\cdots,t_n)$ for t_i in G_m, we get a homomorphism over k of $(G_m)^n$ to its image group, say T, in G such that the kernel consists of $(\pm 1,\cdots,\pm 1)$ with even numbers of minus-signs. Its order is 2^{n-1}. We observe that T is a maximal k-split torus of G. We shall sometimes denote the image of t also by t. On the other hand, if λ is in G_a, for any two distinct indices i, j satisfying $1 \leqq i,\ j \leqq 2n$, $(e_i, e_j) = 0$, we put

$$s_{ij}(\lambda) = 1 + \lambda e_i e_j = \exp(\lambda e_i e_j).$$

Then $s_{ij}(\lambda)$ is an element of G satisfying $s_{ij}(\lambda)s_{ji}(\lambda) = 1$ and

$$\phi(s_{ij}(\lambda))\cdot v = v + \lambda(e_j, v)\, e_i - \lambda\,(e_i, v)\, e_j$$

for every v in V. Moreover, the correspondence $\lambda \rightarrow s_{ij}(\lambda)$ defines an isomorphism over k of G_a to its image group, say P_{ij}, in G. These are the unipotent subgroups of G corresponding to the $2n(n-1)$ roots of G relative to T.

We say that a point x of X is of rank r if the stabilizer subgroup of G at x is of rank r (in the sense that its maximal torus is of dimension r). We shall determine points of rank $n-1$ up to G-equivalence. Since every maximal torus of G is conjugate to T, we have only to consider fixed points of subtori of T. Suppose that

$$x = \sum x_{i_1 \cdots i_h} e_{i_1} \cdots e_{i_h}$$

is an arbitrary point of X. Then h is even and $1 \leqq i_1, \cdots, i_h \leqq n$. If we apply $s_1(t_1) \cdots s_n(t_n) = t$ to x, we get

$$\rho(t) \cdot x = (t_1 \cdots t_n)^{-1} \sum (t_{i_1} \cdots t_{i_h})^2 x_{i_1 \cdots i_h} e_{i_1} \cdots e_{i_h}.$$

Therefore, we have $\rho(t) \cdot x = x$ if and only if $t_1 \cdots t_n = (t_{i_1} \cdots t_{i_h})^2$ for $x_{i_1 \cdots i_h} \neq 0$. In particular, there is no point of rank n except 0. Suppose that x is of rank $n-1$. Then, either one or two coefficients of x are different from 0. In either case, let $x_{i_1 \cdots i_h}$ denote the non-zero coefficient of x with smaller h. In the case when $h > 0$, we pick two indices i, j from $i_1, \cdots, i_h$ and apply $s = s_{n+i, n+j}(1) s_{ij}(1) s_{n+i, n+j}(1)$ to x. Then $\rho(s) \cdot x$ has the same form as x with h replaced by $h-2$. By a repeated application of this procedure, we will get a G-equivalent point with $h = 0$. If we denote this point by x and apply $s = s_1(x_0)$ to x, we get $\rho(s) \cdot x = 1 + \cdots$. Therefore, a point x of rank $n-1$ is G-equivalent to 1. In the case when n is even, it can be G-equivalent to $1 + q e_L$ with $q \neq 0$.

After this remark, we shall examine stabilizer subgroups of G at various points of X. We shall prove five general lemmas using three different methods. Actually, we shall be interested only in the cases when $n \leqq 6$. However, the restriction to these cases will give minor simplication of the proofs (with sacrifice of clarity). At any rate, we first observe that, if H is the stabilizer subgroup of G at any point $x \neq 0$ of X, the vector representation ϕ of G gives rise to an isomorphism of H to $\phi(H)$ defined over $k(x)$. This follows from the fact that ϕ is a separable epimorphism defined over k with ± 1 as its kernel and -1 is not contained in H. In particular, if σ is an arbitrary element of $\phi(H)$, the corresponding element of H is rational over $k(x, \sigma)$. This remark will be used later in the case when $x = 1$.

If s is an arbitrary element of G^*, we shall define four square matrices α, β, γ, δ of degree n by

$$\phi(s) \cdot (e_1 \cdots e_{2n}) = (e_1 \cdots e_{2n}) \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

Then, $\alpha {}^t\beta, \gamma {}^t\delta$ are alternating and $\alpha {}^t\delta + \beta {}^t\gamma = 1_n$ or, equivalently, ${}^t\alpha\gamma$, ${}^t\beta\delta$ are alternating and ${}^t\alpha\delta + {}^t\gamma\beta = 1_n$. We shall denote homogeneous components of X and of its elements by putting subscripts. We observe that X, in fact any $(C_W)^+$ for a totally isotropic subspace W of V, is a graded, commutative local ring over Ω.

LEMMA 1. *Every element* $x \neq 0$ *of* X *is* G*-equivalent over* $K = k(x)$ *to an element of* $1 + X_4 + \cdots$. *An element* x *of* $1 + X_4 + \cdots$ *is* G*-equivalent to* 1 *if and only if* $x = 1$. *Moreover, if* H *is the stabilizer subgroup of* G *at* 1, $\phi(H)$ *is the subgroup of* $\mathrm{Aut}(V, f)^+$ *defined by* $\beta = 0$, $\det(\delta) = 1$. *In particular,* H *is isomorphic over* k *to a semidirect product* $(SL_n)(G_a)^{\frac{1}{2}n(n-1)}$.

Proof. First of all, every element $x \neq 0$ of X is G_K-equivalent to an element of the form $y = 1 + y_2 + \cdots$. The proof is similar to the one that we have used in determining points of rank $n - 1$: If $x_0 \neq 0$ in x, we apply $s_1(x_0)$ to x. If $x_0 = 0$ in x, we write x in the form $a + be_i + ce_j + de_ie_j$ with a, b, c, d free from e_i, e_j for some i, j satisfying $1 \leqq i < j \leqq n$, and apply $s = s_{n+i,n+j}(1)$ to x. Then, we get $\rho(s) \cdot x = x - d$. A repeated application of this procedure will produce an element with non-zero homogeneous component of degree 0. Now, for $y = 1 + y_2 + \cdots, \rho(\exp(-y_2)) \cdot y$ is in $1 + X_4 + \cdots$. This proves the first part. In order to prove the second part, suppose more generally that $x_0 \neq 0$, $x_2 = 0$. Let s denote an element of G^* for which $\rho(s) \cdot 1 = x$. Then, there exists an element y of X_2 and a subspace W of L satisfying $x = \exp(y) e_W$. This is the parametric representation of pure spinors that we recalled in the previous section. We observe that we have an identity in the graded ring C_L. We compare the degrees of lowest terms on both sides. Since we have $x_0 \neq 0$ by assumption, we get $0 = \dim(W)$, hence $W = 0$. We compare the homogeneous components of degree 2 on both sides. Since we have $x_2 = 0$ by assumption, we get $y = 0$, hence $\rho(s) \cdot 1 = x_0$. This proves the second part. Since the spinor $\rho(s) \cdot 1$ represents the maximal totally isotropic subspace sMs^{-1} of V for every s in G^*, we have $\rho(s) \cdot 1 = x_0$ if and only if $sMs^{-1} = M$, i.e., if and only if $\beta = 0$. Consider the subgroup H^* of G^* defined by $\beta = 0$. Then $\chi(s) = \rho(s) \cdot 1$ defines a rational character χ of H^* satisfying $\chi(-1) = -1$. On the other hand, if we apply the canonical anti-automorphism to $se_M = \chi(s) e_M$, we get $e_M s' = \chi(s) e_M$. This implies $se_M s' = \chi(s)^2 e_M = \det(\delta)(ss') e_M$, and hence $\chi(s)^2 = \det(\delta)(ss')$. Since we have $H = G \cap \mathrm{Ker}(\chi)$, we see that $\phi(H)$ is the subgroup of $\mathrm{Aut}(V, f)^+$ defined by $\beta = 0$, $\det(\delta) = 1$.

As we remarked before, if $\beta = 0$, $\det(\delta) = 1$ in an element σ of $\mathrm{Aut}(V, f)^+$, the element s of the stabilizer subgroup of G at 1 satisfying $\phi(s) = \sigma$ is rational over $k(\sigma)$. In the case when $n = 2$ and $\gamma = 0$, this element is given by

$$\begin{aligned} s = 1 &+ ((\alpha_1 - 1) e_1 + \alpha_{21} e_2) e_3 + (\alpha_{12} e_1 + (\alpha_2 - 1) e_2) e_4 \\ &+ (\alpha_1 + \alpha_2 - 2) e_1 e_2 e_3 e_4, \end{aligned}$$

in which $\alpha_1, \alpha_{12}, \alpha_{21}, \alpha_2$ are the coefficients of α. An element like s will appear later in Lemma 4.

In order to prove Lemma 2 without assuming that the characteristic is 0, we shall make a general observation. Suppose that G is a connected reductive group, T a maximal torus and S a subtorus of T. We shall assume that T is the centralizer of S. Let P denote a connected unipotent subgrounp of G which is normalized by S. Then, the product SP is semidirect, hence a connected solvable subgroup. Therefore, it is contained in a Borel subgroup B of G. Since T is the unique maximal torus of G containing S, B is the semidirect product TB_u and B_u contains P. Since B_u is "directly spanned" by those P_α which are contained in B_u, we get a set of morphisms $P \to P_\alpha$ compatible with the operation of S. Therefore, if P is one dimensional and $\theta\colon G_a \to P$ the corresponding isomorphism, each morphism $P \to P_\alpha$ gives rise to a polynomial f_α satisfying $f_\alpha(0)=0$,

$$f_\alpha(\chi(t)\cdot\lambda)=\alpha(t)\cdot f_\alpha(\lambda),$$

in which $t\cdot\theta(\lambda)\cdot t^{-1}=\theta(\chi(t)\cdot\lambda)$, for every t in S and λ in G_a. We recall that we have $t\cdot\theta_\alpha(\lambda)\cdot t^{-1}=\theta_\alpha(\alpha(t)\cdot\lambda)$. In some cases, this fact can be used to eliminate infinitesimal argument. For instance, if the restriction to S is injective on the set of roots of G relative to T and if no restricted root is a power (with an exponent $\geqq 2$) of an element of $\mathrm{Hom}(S, G_m)$, we will get $P=P_\alpha$ for some α.

LEMMA 2. *Suppose that* $n\equiv 2 \bmod 4$ *and* $n\neq 2$. *Then, if H is the stabilizer subgroup of G at* $x=1+qe_L$ *for* $q\neq 0$, $\phi(H)$ *is the subgroup of* $\mathrm{Aut}(V,f)^+$ *defined by* $\beta=\gamma=0$, $\det(\delta)=1$. *In particular, H is isomorphic over k to* SL_n.

Proof. Let H_0 denote the subgroup of the stabilizer subgroup of G at 1 defined by the additional condition that $\gamma=0$. Then, we have $se_Ls^{-1}=e_L$ for every s in H_0. Therefore H_0 is contained in H, and it is isomorphic to SL_n by $s\to\alpha$. Consider the subtorus, say S, of T defined by $t_1\cdots t_n=1$. Then S is a maximal torus of H contained in H_0 and T is its centralizer in G. Furthermore, the restriction to S is injective on the set of roots of G relative to T and no restricted root is a power of an element of $\mathrm{Hom}(S, G_m)$. Therefore, a subgroup P of G ismorphic to G_a is normalized by S if and only if $P=P_{ij}$ for some P_{ij}. On the other hand, we can verify easily that P_{ij} is contained in H if and only if it is contained in H_0. Let H^0 denote the connected component of the identity in H. Then H^0 is reductive. Otherwise, its unipotent radical and H_0 will contain at least one P_{ij}. This contradicts the fact that H_0 is semi-simple. Since H^0 is reductive, it is generated by its maximal torus S and by certain subgroups P which are isomorphic to G_a

and normalized by S. Therefore, by what we have said, we get $H^0 = H_0$. We shall show that $H = H_0$. At any rate, H_0 is normal in H. Now, for an arbitrary $n \neq 2$, the normalizer in $\mathrm{Aut}(V, f)$ of its subgroup defined by $\beta = \gamma = 0$, $\det(\delta) = 1$ consists of the subgroup defined by $\beta = \gamma = 0$ and of its coset with the matrix defined by $\alpha = \delta = 0$, $\beta = \gamma = -1_n$ as its representative. In the case when n is even, therefore, the normalizer is contained in $\mathrm{Aut}(V, f)^+$. Moreover, if we put $s_1 = (e_1 + e_{n+1}) \cdots (e_n + e_{2n})$, this is an element of G with $\phi(s_1)$ equal to the above representative. Furthermore, we have

$$\rho(s_1) \cdot (1 + qe_L) = (-1)^{\frac{1}{2}n(n-1)} q + e_L.$$

Now, suppose that s is in H for $n \equiv 2 \bmod 4$ and $n \neq 2$. If $\beta = \gamma = 0$ for s, using the notations in the proof of Lemma 1, we have $\rho(s) \cdot (1 + qe_L) = (1 + q \cdot \det(\alpha) e_L) \chi(s)$. This implies $\chi(s) = 1$ and $\det(\alpha) = 1$, hence $\phi(s)$ is in $\phi(H_0)$. Since ϕ is injective on H, therefore, s is in H_0. If $\alpha = \delta = 0$ for s, we can write s in the form $s_0 s_1$ such that $\beta = \gamma = 0$ for s_0. We shall use α, β, γ, δ for s_0. Then, we have

$$\rho(s) \cdot (1 + qe_L) = (-q + \det(\alpha) e_L) \chi(s_0),$$

and hence $-q\chi(s_0) = 1$ and $\det(\alpha)\chi(s_0) = q$. On the other hand, we have seen in the proof of Lemma 1 that $\chi(s_0)^2 = \det(\delta)$. This and the second identity imply $q\chi(s_0) = 1$. This contradicts the first identity. We have thus shown that $H = H_0$.

The above proof shows that, in the case when $n \equiv 0 \bmod 4$ and $n \neq 4$, the stabilizer subgroup of G at $1 + qe_L$ for $q \neq 0$ has two connected components. Also, Lemma 1 shows that the stabilizer subgroup at $1 + qe_L$ for $q \neq 0$ is connected but not isomorphic to SL_n for $n = 2$. We shall see that we have the same situation for $n = 4$. A similar remark applies to the lemmas that will follow.

We shall denote by L_0 and M_0 the subspaces of V generated by $e_1, \cdots, e_{n-1}$ and $e_{n+1}, \cdots, e_{2n-1}$. Also, we shall denote by f_0 the restriction of f to $V_0 = L_0 + M_0$ and by C_0, G_0 the Clifford algebra, the spin group of (V_0, f_0). They are, respectively, a subalgebra of C and a subgroup of G. With these notations, we shall prove the following lemma:

LEMMA 3. *Suppose that* $n \equiv 1 \bmod 4$. *Then, every element* s *of the stabilizer subgroup* H *of* G *at* $x = 1 + qe_{L_0}$ *for* $q \neq 0$ *can be written uniquely in the form* $s_0(1 + v_0 e_{2n})$, *in which* s_0 *is an element of the stabilizer subgroup of* G_0 *at* x *and* v_0 *is in* V_0. *The converse is also true. Moreover, elements like* $1 + v_0 e_{2n}$ *form a connected unipotent normal subgroup of* H.

Proof. Every element s of C^+ can be written uniquely in the form $a + be_n + ce_{2n} + de_ne_{2n}$ with a, b, c, d in C_0, and we have

$$s' = (a' + d') - b'e_n - c'e_{2n} - d'e_ne_{2n}.$$

Therefore, we have $ss' = 1$ if and only if

$$aa' + ad' + cb' = 1, \quad ab' - ba' = bd' - db',$$
$$ac' - ca' = 0, \quad ad' - da' = bc' - cb'.$$

Also, we have

$$se_ns' = c(a' + d') + (a + d)(a' + d')e_n - cc'e_{2n}$$
$$+ ((a + d)c' - c(a' + d'))e_ne_{2n},$$
$$se_{2n}s' = ab' - bb'e_n + aa'e_{2n} + (ba' - ab')e_ne_{2n}.$$

Now, if s is in the stabilizer subgroup of G at x, we have $sxe_M = xe_M$, hence $axe_{M_0} = xe_{M_0}$ and $bxe_{M_0} = 0$. Since $s' = s^{-1}$ is in the stabilizer subgroup of G at x, we also have $b'xe_{M_0} = 0$. The expression for $se_{2n}s'$ shows that ab' is in V_0. This and $ab'xe_{M_0} = 0$ imply that $ab' = 0$. The expressions for se_ns', $se_{2n}s'$ show that aa', bb' and cc' are in Ω. Suppose that $aa' = 0$. Then, we have $(se_ns', se_{2n}s') = (bb')(cc') = 1$, hence bb' is in $\Omega^\times$. Therefore b and b' are units of C_0. Consequently, $ab' = 0$ implies $a = 0$. This contradicts $axe_{M_0} = xe_{M_0} \neq 0$. Therefore aa' is in $\Omega^\times$, hence a is a unit of C_0. This implies $aa' = a'a$ and $b = 0$. On the other hand, the assumption that $n \equiv 1 \bmod 4$ is equivalent with the condition that $x' = x$. Therefore $axe_{M_0} = xe_{M_0}$ implies $e_{M_0}xa' = e_{M_0}x$, and hence $(a'a)e_{M_0}x^2e_{M_0} = e_{M_0}x^2e_{M_0}$. Since this implies $(a'a)e_{M_0} = e_{M_0}$, we get $a'a = 1$. Therefore, by the condition for $ss' = 1$, we get $ad' = 0$, hence $d = 0$. The expression for se_ns' shows that ca' is in V_0. If u_0 is an element of V_0, we have $su_0s^{-1} = v_0 +$ a linear combination of e_n, e_{2n} with v_0 in V_0. This implies $au_0 = v_0a$ for every u_0 in V_0, and hence $s_0 = a$ is in the stabilizer subgroup of G_0 at x. Moreover, we have $s_0{}^{-1}s = 1 + v_0e_{2n}$ with $v_0 = a'(ca')a$ in V_0. Conversely, if v_0 is in V_0, $1 + v_0e_{2n}$ is in the stabilizer subgroup of G at x. The rest is clear.

We shall denote by L_0 and M_0 the subspaces of V generated by $e_1, \cdots, e_{n-2}$ and $e_{n+1}, \cdots, e_{2n-2}$. Also, we shall denote by f_0 the restriction of f to $V_0 = L_0 + M_0$ and by C_0, G_0 the Clifford algebra, the spin group of (V_0, f_0), respectively. With these notations, we shall prove the following lemma:

LEMMA 4. *Suppose that $n \equiv 2 \bmod 4$. Then, every element s of the stabilizer subgroup H of G at $x = 1 + qe_{L_0}$ for $q \neq 0$ can be written uniquely* *in the form $s_0s_1s_2$, in which s_0 is an element of the stabilizer subgroup of G_0 at x and*

$$s_1 = 1 + (\lambda_1 e_{n-1} + \lambda_{21} e_n) e_{2n-1} + (\lambda_{12} e_{n-1} + \lambda_2 e_n) e_{2n} + (\lambda_1 + \lambda_2) e_{n-1} e_n e_{2n-1} e_{2n}$$

with $1+\lambda_1, \lambda_{12}, \lambda_{21}, 1+\lambda_2$ forming an element of SL_2 and

$$s_2 = (1 + u_0 e_{2n-1})(1 + (v_0 + \lambda e_{2n-1}) e_{2n}) = (1 + (v_0 + (\lambda - (u_0, v_0)) e_{2n-1}) e_{2n})(1 + u_0 e_{2n-1})$$

with u_0, v_0 in V_0 and λ in Ω. The converse is also true. Moreover, we have $s_0s_1 = s_1s_0$ and elements like s_2 form a connected unipotent normal subgroup of H.

Proof. We shall prove only that s can be written in the form $s_0s_1s_2$ (because the remaining parts are straightforward). We proceed as in the proof of Lemma 3. We shall use the same notations except that we replace V_0, C_0, G_0 etc. by other notations, say by V_1, C_1, G_1 etc. We have $s = a + be_n + ce_{2n} + de_ne_{2n}$ with a, b, c, d in C_1, and $axe_{M_1} = xe_{M_1}$, $bxe_{M_1} = 0$. This time, however, ab' in V_1 and $ab'xe_{M_1} = 0$ imply that ab' is in Ωe_{2n-1}. Therefore, we have to make an adjustment. If we multiply s_1^{-1} to s from the left, we get $s_1^{-1}s = a^* + b^*e_n + c^*e_{2n} + d^*e_ne_{2n}$ with

$$a^* = (1 + \lambda_2 e_{n-1} e_{2n-1}) a + \lambda_{12} e_{n-1} b$$
$$b^* = \lambda_{21} e_{2n-1} a + ((1 + \lambda_1) - \lambda_1 e_{n-1} e_{2n-1}) b.$$

Therefore, if we put $ab' = \lambda e_{2n-1}$, we have

$$a^*(a^*)' = \lambda\lambda_{12} + (aa')(1 + \lambda_2)$$
$$a^*(b^*)' = (bb')\lambda_{12} e_{n-1} + (\lambda(1 + \lambda_1) + (aa')\lambda_{21}) e_{2n-1}.$$

Since s_1 and s, hence also $s_1^{-1}s$ are in the stabilizer subgroup of G at x, $a^*(b^*)'$ has to be in Ωe_{2n-1}. Therefore, we have $(bb')\lambda_{12} = 0$ for every λ_{12}, hence $bb' = 0$. Since $se_{2n}s' = \lambda e_{2n-1} + (aa')e_{2n}$, either λ or aa' is different from 0. Using this fact, we can find s_1 such that $a^*(a^*)' = 1$, $a^*(b^*)' = 0$. In fact, not only this is possible, but there still remains the freedom of multiplying $1 - \lambda_{12} e_{n-1} e_{2n}$ to $s_1^{-1}s$ from the left. At any rate, this implies $b^* = 0$ and, by the condition for $ss' = 1$, $d^* = 0$. We change our notation and assume that s instead of $s_1^{-1}s$ has this property. Then, we have $s = a(1 + ve_{2n})$ with a in $(C_1)^+$ satisfying $aa' = 1$, $axe_{M_1} = xe_{M_1}$ and v in V_1. As in the proof of Lemma 3, we also have $au_1 = v_1a$ for every u_1 in V_1. Therefore a is in the stobilizer subgroup of G_1 at x. According to Lemma 3, we have $a = s_0(1 + u_0 e_{2n-1})$, in which s_0 is in the stabilizer subgroup of G_0

at x and u_0 in V_0. Let λ_{12} denote the coefficient of e_{n-1} in v. Then, by multiplying $1-\lambda_{12}e_{n-1}e_{2n}$ to s from the left, we get

$$(1-\lambda_{12}e_{n-1}e_{2n})s = s_0(1+u_0e_{2n-1})(1+(v_0+\lambda e_{2n-1})e_{2n}),$$

in which v_0 is in V_0 and λ in Ω.

We need another lemma of a slightly different nature. If r is a positive integer, we shall denote by h_r the "standard alternating matrix" of degree $2r$, i.e., the alternating matrix composed of 0, 1_r, -1_r, 0. The proof of the following lemma is valid only in the case when the characteristic is 0, a restriction that will be removed later for $n=6$:

LEMMA 5. *Suppose that $n=2r$ with $r\geqq 3$. Let H denote the stabilizer subgroup of G at $x=e_1e_{r+1}+\cdots+e_re_{2r}$ and H_0 the connected component of the identity in H. Then H/H_0 is isomorphic to the multiplicative group of $(r-2)$-th roots of unity. As for H_0 itself, every element σ of $\phi(H_0)$ can be written uniquely in the form $\sigma_0\sigma_1$, in which $\beta=\gamma=0$, δ in Sp_n for σ_0 and $\beta=0$, $\delta=1_n$, $\operatorname{tr}(\gamma h_r)=0$ for σ_1. The converse is also true. Moreover, elements like σ_1 form a connected unipotent normal subgroup of $\phi(H_0)$.*

Proof. Consider the subgroup of the stabilizer subgroup, say H_1, of G at 1 defined by the additional condition that δ is in Sp_n and $\gamma=0$. Then, this subgroup is certainly contained in H. Also, the subgroup of H_1 defined by the additional condition that $\delta=1_n$ and $\operatorname{tr}(\gamma h_r)=0$ is connected unipotent, and it is contained in H. In fact, this subgroup is generated by $s_{n+i,n+j}(\lambda)$ with $(i,j)\neq(1,r+1),\cdots,(r,2r)$ for $1\leqq i<j\leqq n$ and by

$$s_{n+1,n+r+1}(\lambda_1)\cdots s_{n+r,n+2r}(\lambda_r),$$

in which $\lambda_1+\cdots+\lambda_r=0$. Moreover, each one of these elements keeps x invariant. We observe that the subgroup of H which is isomorphic to Sp_n normalizes this unipotent group, and hence their product is semidirect. We shall denote this product by H_0. In order to idenify H_0 with the connected component of the identity in H, we shall use an infinitesimal argument (in a slightly disguised form). Consider the subtorus, say S, of T defined by $t_1t_{r+1}=\cdots=t_rt_{2r}=1$. Consider the product of P_{ij}, in a suitable order, which correspond to one same restricted root (of G relative) to S different from 0. Also, consider the product of T and those P_{ij} which correspond to the restricted root 0. We determine its element which keeps x invariant. Then, using the assumption that $r\geqq 3$, we can show that every such element is contained in H_0. For instance, if the restricted root is 0, we examine the condition that an element of the form

$$s = t \cdot s_{1,r+1}(\lambda_1) \cdot \cdot \cdot s_{r,2r}(\lambda_r) \cdot s_{n+1,n+r+1}(\lambda_1') \cdot \cdot \cdot s_{n+r,n+2r}(\lambda_r')$$

keeps x invariant. If we apply the third factor to x, we get $a_0 + x$ with $-a_0 = \lambda_1' + \cdot \cdot \cdot + \lambda_r'$. Therefore, we have $\rho(s) \cdot x = \rho(t) \cdot (a_0 + \text{higher})$, hence $a_0 = 0$. Consequently, we get

$$\rho(s) \cdot x = \rho(t) \cdot (x - \sum_{i<j} (\lambda_i + \lambda_j) e_i e_j e_{r+i} e_{r+j} + \text{higher}),$$

hence $\lambda_i + \lambda_j = 0$ for all $1 \leqq i < j \leqq r$. Because of the assumption that $r \geqq 3$, we get $\lambda_i = 0$ for all i. The rest is clear. We have thus shown that H_0 is normal in H. (This is the only part which depends on the assumption that the characteristic is 0.) Now, let σ with submatrices α, β, γ, δ denote an element of the normalizer of $\phi(H_0)$ in $\mathrm{Aut}(V, f)$. Then σ normalizes the unipotent radical of $\phi(H_0)$. In particular, we have $\beta v = 0$ for every alternating matrix v of degree n satisfying $\mathrm{tr}(v h_r) = 0$. This implies $\beta = 0$ when $r \neq 1$, hence certainly for $r \geqq 3$. Therefore α is in the normalizer of Sp_n in GL_n. This implies that α is a scalar multiple of an element of Sp_n. On the other hand, if $\delta = 1_n$, $\beta = 0$ and γ alternating in σ, it is in the normalizer. Consequently, every coset of $\phi(H_0)$ in its normalizer has a representative of the form

$$\begin{bmatrix} \lambda 1_n & 0 \\ 0 & \lambda^{-1} 1_n \end{bmatrix} \begin{bmatrix} 1_n & 0 \\ \lambda' h_r & 1_n \end{bmatrix},$$

in which λ is in $\Omega^{\times}$ and λ' in Ω. Furthermore, the representative is unique up to $\lambda \to \pm \lambda$. Now, the element s of G such that $\phi(s)$ is of the above form is

$$s_1(t_1) \cdot \cdot \cdot s_n(t_n) s_{n+1,n+r+1}(\lambda') \cdot \cdot \cdot s_{n+r,n+2r}(\lambda'),$$

in which $(t_1)^2 = \cdot \cdot \cdot = (t_n)^2 = \lambda$. Since we have

$$\rho(s) \cdot x = (t_1 \cdot \cdot \cdot t_n)^{-1} (\lambda^2 x - r\lambda'),$$

we see that s keeps x invariant if and only if $\lambda^2 = t_1 \cdot \cdot \cdot t_n$ and $\lambda' = 0$. The only condition imposed on λ is $\lambda^{n-4} = 1$. The rest is clear.

3. Even dimensional cases (continued). We recall that we have $G = \mathrm{Spin}_{2n}$ $(n \geqq 2)$, $X = (C_L)^+$ and ρ the half-spin representation of G in X defined over k. We shall determine G-orbits in X and also the corresponding stabilizer subgroups for $n \leqq 6$. In the first two cases when $n = 2, 3$, $X - 0$ is a G-orbit by Lemma 1. Moreover, the Witt condition is satisfied, and the stabilizer subgroups of G at 1 are isomorphic over k to $(SL_2) G_a$, $(SL_3)(G_a)^3$ by Lemma 1. These are trivial cases.

PROPOSITION 1. *In the case when* $n=4$, *put*

$$g(x)=x_0x_1-x_{12}x_{34}+x_{13}x_{24}-x_{14}x_{23}$$

for

$$x=x_0+\sum_{i<j}x_{ij}e_ie_j+x_1e_L.$$

Then, the G-orbits in $X-0$ *are* $g^{-1}(i)$ *for* $i\neq 0$ *and* $g^{-1}(0)-0$. *The Witt condition is satisfied. Moreover, the stabilizer subgroup of G at* $x\neq 0$ *is isomorphic over* $k(x)$ *to* Spin_7 *when* $g(x)\neq 0$ *and to* $(SL_4)(G_a)^6$ *when* $g(x)=0$.

Proof. We recall that we have an Ω-bilinear form $g(x,y)$ on $C_L\times C_L$ called the spinor invariant. We observe that the restriction of $g(x,y)$ to $X\times X$ is precisely the associated Ω-bilinear form of $g(x)$. In particular, $g(x)$ is a G-invariant. On the other hand, every element $x\neq 0$ of X is G_K-equivalent for $K=k(x)$ to an element of the form $1+qe_L$ with q in Ω by Lemma 1. If we compute $g(x)$ at x and $1+qe_L$, we get $q=g(x)$. Therefore, the stabilizer subgroup of G at $x\neq 0$, $g(x)=0$ is isomorphic over K to $(SL_4)(G_a)^6$ by Lemma 1. According to the "principle of triality," we can interchange ϕ and ρ. In other words, we may consider G as the spin group of (X,g), ρ as the vector representation and ϕ as the half-spin representation of G (cf. 3, 7). In particular, the stabilizer subgroup of G at x, $g(x)\neq 0$ is isomorphic over K to Spin_7.

PROPOSITION 2. *In the case when* $n=5$, *there are two G-orbits in* $X-0$ *with* 1, $1+e_1e_2e_3e_4$ *as their representatives. The Witt condition is satisfied. Moreover, the stabilizer subgroups of G at the above representatives are respectively isomorphic over k to* $(SL_5)(G_a)^{10}$, $(\mathrm{Spin}_7)(G_a)^8$.

Proof. Every element $x\neq 0$ of X is G_K-equivalent for $K=k(x)$ to an element of $1+X_4$ by Lemma 1. If we consider the subgroup of the stabilizer subgroup of G at 1 defined by the additional condition that $\gamma=0$, it is isomorphic over k to SL_5 by $s\to\alpha$. This group operates transitively on X_4-0 satisfying the Witt condition. The rest follows from Lemma 1, Lemma 3, and Proposition 1.

We shall denote by $e_i^*, e_{ij}^*, \cdots$ the partial products of $e_1,\cdots,e_n$ satisfying $e_ie_i^*=e_ie_je_{ij}^*=\cdots=e_L$. We put, e.g.,

$$e_{ij}^*=(-1)^{i+j-1}e_1\cdots e_{i-1}e_{i+1}\cdots e_{j-1}e_{j+1}\cdots e_n$$

for $1\leqq i<j\leqq n$. Also, if a is an alternating matrix of degree $2r$, we shall

denote by $Pf(a)$ the Pfaffian of a normalized by $Pf(h_r)=1$, in which h_r is the standard alternating matrix of degree $2r$.

PROPOSITION 3. *In the case when* $n=6$, *put*

$$J(x)=x_0Pf((y_{ij}))+y_0Pf((x_{ij}))+\sum_{i<j}Pf(X_{ij})Pf(Y_{ij}) \\ -(\tfrac{1}{4})(x_0y_0-\sum_{i<j}x_{ij}y_{ij})^2$$

for

$$x=x_0+\sum_{i<j}x_{ij}e_ie_j+\sum_{i<j}y_{ij}e_{ij}{}^*+y_0e_L,$$

in which, e.g., (x_{ij}) *is the alternating matrix determined by* x_{ij} *and* X_{ij} *the alternating matrix obtained from* (x_{ij}) *by crossing out its i-th and j-th lines and columns. Then, the G-orbits in* $X-0$ *are* $J^{-1}(i)$ *for* $i\neq 0$ *and three orbits in* $J^{-1}(0)$ *with* $1, 1+e_{14}{}^*$, $1+e_{14}{}^*+e_{25}{}^*$ *as their representatives. The Witt condition is satisfied. Moreover, the stabilizer subgroup of* G *at the representative* $1+e_{14}{}^*+e_{25}{}^*+ie_{36}{}^*$ *of* $J^{-1}(i)$ *for* $i\neq 0$ *is isomorphic over* $k((-i)^{\frac{1}{2}})$ *to* SL_6. *The stabilizer subgroups at the above three representatives are respectively isomorphic over* k *to* $(SL_6)(G_a)^{15}$, $(\mathrm{Spin}_7\times SL_2)$-*times a connected unipotent group of dimension* 17, $(Sp_6)(G_a)^{14}$.

Proof. We know by Lemma 1 that every element $x\neq 0$ of X is G_K-equivalent for $K=k(x)$ to an element, say y, of $1+X_4+\Omega e_L$. We can write y in the form

$$1+\sum_{i<j}y_{ij}e_{ij}{}^*+y_0e_L$$

with y_0 in Ω and with an alternating matrix (y_{ij}) with coefficients in Ω. If we consider, as in the proof of Proposition 2, the subgroup of the stabilizer subgroup of G at 1 defined by $\gamma=0$, with respect to this subgroup, the element y is equivalent over $k(y)$ to

$$1+qe_L,\ 1+e_{14}{}^*+qe_L,\ \text{or}\ 1+e_{14}{}^*+e_{25}{}^*+pe_{36}{}^*+qe_L$$

according as $\operatorname{rank}((y_{ij}))=0$, 2, or $\geqq 4$. If we apply

$$s_{1,4}(-\lambda)s_{2,5}(-\lambda)s_{9,12}(\lambda)$$

to the third element, we get

$$1+e_{14}{}^*+e_{25}{}^*+(p-q\lambda+\lambda^2)e_{36}{}^*+(q-2\lambda)e_L.$$

Consequently, by taking $\lambda=(\tfrac{1}{2})q$, we can replace (p,q) by $(p-(\tfrac{1}{4})q^2,0)$. Now, if the given element x is expressed as in the proposition, we can make each step of the above "normalization-process" explicit and we get $p-(\tfrac{1}{4})q^2=J(x)$. However, this is not necessary. We have only to verify directly that $J(x)$ is a G-invariant. Then, the transformability of x into $1+e_{14}{}^*+e_{25}{}^*+ie_{36}{}^*$ implies $i=J(x)$.

We shall examine G-orbits in $X-0$. Suppose first that $J(x)=0$. Then, by what we have said, x is G_K-equivalent for $K=k(x)$ to 1, $1+e_{14}{}^*$, or $1+e_{14}{}^*+e_{25}{}^*$. We observe that the first polar δJ vanishes at 1, $1+e_{14}{}^*$ but does not vanish at $1+e_{14}{}^*+e_{25}{}^*$. Since 1 and $1+e_{14}{}^*$ are not G-equivalent by Lemma 1, no two of these three elements are G-equivalent. We shall examine the case when $J(x)=i\neq 0$. In this case, x is G_K-equivalent for $K=k(x)$ to $1+qe_L$, $1+e_{14}{}^*+qe_L$ with $q^2=-4i$, or to $1+e_{14}{}^*+e_{25}{}^*+ie_{36}{}^*$. We shall show that they are G-equivalent over $k(q)$. If we apply $\exp(\lambda_1e_7e_{10}+\lambda_2e_8e_{11}+\lambda_3e_9e_{12})$ to $1+qe_L$, we get

$$(1+q\lambda_1\lambda_2\lambda_3)-q\lambda_2\lambda_3e_1e_4-q\lambda_3\lambda_1e_2e_5-q\lambda_1\lambda_2e_3e_6$$
$$-q\lambda_1e_{14}{}^*-q\lambda_2e_{25}{}^*-q\lambda_3e_{36}{}^*+qe_L.$$

This becomes $1+e_{14}{}^*+qe_L$ for $\lambda_1=-q^{-1}$, $\lambda_2=\lambda_3=0$. Therefore, $1+qe_L$ and $1+e_{14}{}^*+qe_L$ are G-equivalent over $k(q)$. On the other hand, we can transform the above element into an element of $1+X_4+\Omega e_L$ over $k(q,\lambda_1,\lambda_2,\lambda_3)$. If we further normalize (over the same field) the component in X_4, we will get $1+e_{14}{}^*+e_{25}{}^*+p_1e_{36}{}^*+q_1e_L$. In the case when $1+q\lambda_1\lambda_2\lambda_3\neq 0$, the normalization-process that we have explained in the proof of Lemma 1 is unique, and we get

$$p_1=-q^3\lambda_1\lambda_2\lambda_3(1+q\lambda_1\lambda_2\lambda_3)^{-2}$$
$$q_1=q(1-q\lambda_1\lambda_2\lambda_3)(1+q\lambda_1\lambda_2\lambda_3)^{-1}.$$

If we take $\lambda_1=q$, $\lambda_2=\lambda_3=q^{-1}$, therefore, $1+qe_L$ is transformed over $k(q)$ into $1+e_{14}{}^*+e_{25}{}^*+ie_{36}{}^*$. This proves the assertion. The above proof shows also that the Witt condition is satisfied.

We shall determine the stabilizer subgroups of G at 1, $1+e_{14}{}^*$, $1+e_{14}{}^*+e_{25}{}^*+ie_{36}{}^*$. We observe that Lemma 1 and Lemma 4 settle the first two cases. We shall examine the third case with $i=0$. If we apply $s_{9,12}(1)s_{3,6}(1)s_{9,12}(1)$ to $1+e_{14}{}^*+e_{25}{}^*$, we get $e_1e_4+e_2e_5+e_3e_6$. We recall that the G-orbit of $1+e_{14}{}^*+e_{25}{}^*$ is precisely the set of simple points of the hypersurface $J(x)=0$. Therefore, the dimension of this G-orbit is equal to $\dim(X)-1=31$. Consequently, the dimension of the stabilizer subgroup H of G at $e_1e_4+e_2e_5+e_3e_6$ is of dimension $\dim(G)-31=35$. This information enables us to conclude (in the notation of the proof of Lemma 5) that H_0 is the connected component of H. Therefore H is isomorphic over k to a semidirect product $(Sp_6)(G_a)^{14}$ by Lemma 5. We shall examine the third case with $i\neq 0$. We have seen that this representative is G-equivalent over $k(q)$ to $1+qe_L$ for $q^2=-4i$. Therefore, the stabilizer subgroup of G at $1+e_{14}^*+e_{25}^*+ie_{36}^*$ for $i\neq 0$ is isomorphic over $k((-i)^{\frac{1}{2}})$ to SL_6 by Lemma 2. This completes the proof.

We observe that the set of points of $X-0$ where the first polar δJ vanishes is the union of the two G-orbits with 1 and $1+e_{14}^*$ as their representatives. On the other hand, in the case when $(-i)^{\frac{1}{2}}$ is not contained in $K=k(i)$, we can show that the stabilizer subgroup H of G at $1+e_{14}^*+e_{25}^*+ie_{36}^*$ is the special unitary group relative to $K((-i)^{\frac{1}{2}})/K$ for a hermitian matrix of degree 6 and of index 3. In particular, the K-rank of H is 3.

4. Odd dimensional cases. We shall pass to the case when $\dim(V)=2n+1\geqq 3$. We recall that $L_1=L+\Omega e_0$, $M_1=M+\Omega e_{2n+1}$, $V_0=L+M$, $V_1=L_1+M_1$ and

$$\epsilon_0=e_0+e_{2n+1},\qquad \epsilon_1=e_0-e_{2n+1}.$$

The Clifford algebras etc. of (V_0,f_0), (V_1,f_1) are denoted by C_0, C_1 etc. We recall that G has been identified with the subgroup of $G_1=\mathrm{Spin}_{2n+2}$ consisting of those s with the property that $\phi_1(s)\cdot\epsilon_1=\epsilon_1$. Moreover, the spin representation is defined as the restriction to G of the half-spin representation of G_1 in $X_1=X$. If we decompose α etc. into submatrices α_1, α_{12}, α_{21}, α_2, in which α_{12} for instance is of type $(n,1)$, the condition $\phi_1(s)\cdot\epsilon_1=\epsilon_1$ simply means that $\alpha_{12}=\beta_{12}$, $\alpha_2=\beta_2+1$, $\gamma_{12}=\delta_{12}$, $\gamma_2=\delta_2-1$. This is a trivial (but useful) restatement. We shall prove the following lemma:

LEMMA 6. *Every element s of $G=\mathrm{Spin}_{2n+1}$ is either of the form $s_0(1+v_0\epsilon_0)$ with s_0 in the even Clifford group $(G_0^*)^+$ and v_0 in V_0 satisfying $(s_0s_0')(1+f_0(v_0))=1$, or of the form $s_1(v_0+\epsilon_0)$ with s_1 in $(G_0^*)^-$ and v_0 in V_0 satisfying $s_1s_1'=1$, $f_0(v_0)=0$. The converse is also true.*

Proof. Every element s of C_1 can be written uniquely in the form $a+b\epsilon_0+c\epsilon_1+d\epsilon_0\epsilon_1$ with a, b, c, d in C_0. Suppose that s is even. Then, we have $s\epsilon_1=\epsilon_1 s$ if and only if $c=d=0$. In this case, we have

$$ss'=(aa'+bb')+(ba'-ab')\epsilon_0$$
$$s\epsilon_0 s'=(ab'+ba')+(aa'-bb')\epsilon_0.$$

Moreover, for every u_0 in V_0, we have

$$su_0s'=(au_0a'-bu_0b')-(au_0b'+bu_0a')\epsilon_0.$$

Therefore s is in G if and only if $s=a+b\epsilon_0$ with a in $(C_0)^+$, b in $(C_0)^-$ such that $aa'+bb'=1$, ab' in V_0, aa' in Ω, $au_0a'-bu_0b'$ in V_0, $au_0'+bu_0a'$ in

Ω for every u_0 in V_0. Suppose first that aa' is in $\Omega^{\times}$. . Then a is a unit of $(C_0)^+$ and we have $b = w_0 a$ with w_0 in V_0. Moreover $au_0a'w_0 + w_0au_0a'$ is in Ω, hence $u_0a'w_0a + a'w_0au_0$ is also in Ω for every u_0 in V_0. Now, if we express $a'w_0a$ as a linear combination of 1, e_n, e_{2n}, e_ne_{2n} and write down the above condition for $u_0 = e_n$, e_{2n}, we see that the coefficients of e_n, e_{2n} are in Ω and the coefficient of e_ne_{2n} is 0. Therefore, we see by induction that $a'w_0a$ is in V_0. In particular, we have $b = w_0a = av_0$ with v_0 in V_0. Moreover, the condition for s to be in G becomes equivalent to the condition that a is in $(G_0{}^*)^+$ and $(aa')(1 + f_0(v_0)) = 1$. The case when $aa' = 0$, hence $bb' = 1$ can be treated similarly. This completes the proof.

We observe that the maximal torus T of $G_0 = \mathrm{Spin}_{2n}$ consisting of $t = s_1(t_1) \cdots s_n(t_n)$ is a maximal torus of G. Moreover, if we put

$$s_{0i}(\lambda) = 1 + \lambda e_i\epsilon_0 = \exp(\lambda e_i\epsilon_0)$$

for $1 \leqq i \leqq 2n$, the correspondence $\lambda \to s_{0i}(\lambda)$ gives an isomorphism over k of G_a to its image group, say P_i, in G. These are the unipotent subgroups of G which correspond to the additional $2n$ roots of G with respect to T, and G is generated by T, the P_{ij} in G_0 and by these P_i. We shall consider G for $n \leqq 5$ excluding the trivial cases when $n = 1, 2$.

Proposition 4. *In the case when $n = 3$, let $g(x)$ denote the G_1-invariant in Proposition 1 evaluated at*

$$x = x_0 + \sum_{i<j} x_{ij}e_ie_j + \Big(\sum_i x_ie_i + y_0e_1e_2e_3\Big)e_0$$

after replacing e_0 by e_4. Then, the G-orbits in $X - 0$ are $g^{-1}(i)$ for $i \neq 0$ and $g^{-1}(0)$. The Witt condition is satisfied. Moreover, the stabilizer subgroup of G at x, $g(x) \neq 0$ is connected and it is a $k(x)$-split simple group of type G_2. The stabilizer subgroup at $x \neq 0$, $g(x) = 0$ is isomorphic over $k(x)$ to a semidirect product of SL_3 and a connected unipotent group of dimension 6.

Proof. We shall show that every element $x \neq 0$ of X is G_K-equivalent for $K = k(x)$ to $1 + g(x)e_1e_2e_3e_0$. We observe that $g(x)$ is of the form $x_0y_0 - x_1x_{23} + x_{213} - x_3x_{12}$ when x is expressed as in the proposition. Suppose first that $x_0 \neq 0$ or $(x_{ij}) \neq 0$. Then x is G_0-equivalent over K to another element with $x_0 = 1$, $(x_{ij}) = 0$ (cf. the proof of Lemma 1). This element is G_0-equivalent over K to $1 + (pe_1 + g(x)e_1e_2e_3)e_0$ for some p in K. If we apply $s_{01}(-p)$ to this element, we get $1 + g(x)e_1e_2e_3e_0$. Suppose next that $x_0 = 0$, $(x_{ij}) = 0$ but $x_i \neq 0$ for some i. Then, the application of $s_{0j}(1)$ to x for any $j \neq i$ will reduce this case to the previous case. Finally, if y_0 is the only non-zero coefficient of x, the application of $s_{4,5}(1)$ to x will reduce this case to the previous case. This proves the assertion. In particular, the Witt condition is satisfied.

We shall determine the stabilizer subgroup H of G at $1+qe_1e_2e_3e_0$. In the case when $q=0$, we apply Lemma 6. We see that H consists of $s=s_0(1+v_0\epsilon_0)$ with s_0 in the stabilizer subgroup of G_0 at 1 and v_0 in M. Therefore, it is isomorphic over k to a semidirect product of SL_3 and a connected unipotent group of dimension 6. In the case when $q\neq 0$, we observe that the intersection S of the maximal torus T and H consists of those $t=s_1(t_1)s_2(t_2)s_3(t_3)$ in which $t_1t_2t_3=1$. Moreover, (either directly or by the general theory) we see that T is the centralizer of S in G. Therefore S is a maximal torus of H. Let H_0 denote the connected component of the identity in H. Then, by examining the restricted root system to S, we can convince ourselves easily that H_0 is a simple group of type G_2, which clearly splits over k. In particular, every automorphism of H_0 is inner (cf. 4). Therefore H is contained in the product of H_0 and the centralizer of H_0 in G. However, the centralizer of H_0 is contained in the centralizer of S (in G), which is T. Hence H is contained in $H_0T\cap H=H_0S=H_0$ (cf. also 7). Since the Witt condition is satisfied, we can state our results as in the proposition.

Proposition 5. *In the case when* $n=4$, *put*

$$g(x)=x_0y_0+Pf((x_{ij}))+\sum_i x_iy_i$$

for

$$x=x_0+\sum_{i<j}x_{ij}e_ie_j+y_0e_1e_2e_3e_4+(\sum_i x_ie_i+\sum_i y_ie_i{}^*)e_0.$$

Then, the G-orbits in $X-0$ are $g^{-1}(i)$ for $i\neq 0$ and two orbits in $g^{-1}(0)$ with $1, 1+e_1{}^*e_0$ as their representatives. The Witt condition is satisfied. Moreover, the stabilizer subgroup of G at x, $g(x)\neq 0$ is isomorphic over $k(x)$ to Spin_7. *The stabilizer subgroups at the above two representatives are respectively isomorphic over k to semidirect products of SL_4 and a connected unipotent group of dimension* 10, *of a connected k-split simple group of type G_2 and $(G_a)^7$.*

Proof. In general, if we restrict the spin representation of $G=\mathrm{Spin}_{2n+1}$ in X to $G_0=\mathrm{Spin}_{2n}$, we get the spin representation of G_0 provided that we identify X with C_L. In the case when $n=8$, we can identify the spinor invariant $g(x,y)$ with the associated Ω-bilinear form of the $g(x)$ in the proposition. The fact is that $g(x)$ is not only a G_0-invariant but it is a G-invariant. After this remark, we take an arbitrary element $x \neq 0$ of X and express it as in the proposition. Suppose first that $x_0 \neq 0$, $y_0 \neq 0$, or $(x_{ij}) \neq 0$. Then x is G_0-equivalent over $k(x)$ to another element with $x_0 = 1$, $(x_{ij}) = 0$. We can further assume that $x_i = 0$ for $i \neq 1$. If we apply $s_{01}(-x_1)$ to this element, we will get an element in which the coefficient of e_1e_0 is also 0. Such an element is clearly G_0-equivalent to $1 + qe_1e_2e_3e_4 + pe_1{}^*e_0$, and the above normalization-process is rational over $k(x)$. Suppose next that $x_0 = y_0 = 0$, $(x_{ij}) = 0$. Then, the application of $s_{0j}(1)$ to x for a suitable j will reduce this case to the previous case. Now, if we apply $s_{05}(\lambda)$ to $1 + qe_1e_2e_3e_4 + pe_1{}^*e_0$, we get $1 + qe_1e_2e_3e_4 + (p + q\lambda)e_1{}^*e_0$. Therefore, we have only two types of elements $1 + ge_1e_2e_3e_4$, $1 + pe_1{}^*e_0$ to be examined. We observe that $1 + pe_1{}^*e_0{}^*$ for $p \neq 0$ is G_0-equivalent over $k(p)$ to $1 + e_1{}^*e_0$. We have only to apply, e. g.,

$$1 + (p-1)e_1e_5 + (p^{-1}-1)e_2e_6 + (p + p^{-1} - 2)e_1e_2e_5e_6$$

to $1 + pe_1{}^*e_0$. On the other hand, 1 and $1 + e_1{}^*e_0$ are not even G_1-equivalent by Lemma 1. We have thus shown that $g^{-1}(i)$ is a G-orbit for $i \neq 0$ and $g^{-1}(0) - 0$ consists of two G-orbits with 1, $1 + e_1{}^*e_0$ as their representatives. Moreover, the Witt condition is satisfied.

We shall examine the stabilizer subgroup H of G at $x = 1 + e_1{}^*e_0$, $1 + qe_1e_2e_3e_4$. Suppose first that $x = 1 + qe_1e_2e_3e_4$ with $q \neq 0$. According to Lemma 3, every element of the stabilizer subgroup of G_1 at x is of the form $s_0(1 + v_0e_9)$, in which s_0 is an element of the stabilizer subgroup of G_0 at x and v_0 is in V_0. An element of this form is contained in G if and only if $v_0 = 0$. Therefore H is the stabilizer subgroup of G_0 at x, and it is isomorphic over $k(q)$ to Spin_7 by Proposition 1. Suppose next that $q = 0$. Then, Lemma 6 shows that every element of H is of the form $s_0(1 + v_0\epsilon_0)$ with s_0 in the stabilizer subgroup of G_0 at 1 and v_0 in M. The converse is also true. Therefore H is isomorphic over k to a semidirect product of SL_4 and a connected unipotent group of dimension 10 by Proposition 1. Finally, suppose that $x = 1 + e_1{}^*e_0$. According to Lemma 3, every element of the stabilizer subgroup of G_1 at x is of the form $s_0(1 + v_0e_5)$, in which s_0 is an element of the stabilizer subgroup at x of the spin group of the restriction of f_1 to the eight dimensional vector space generated over Ω by those e's different from e_1, e_5, and v_0 is in the same vector space. An element of this form commutes with ϵ_1 if and only if s_0 commutes with ϵ_1 and $(v_0, \epsilon_1) = 0$. Therefore, by Proposition 4 we see that H is isomorphic over k to a semidirect product of a connected k-split simple group of type G_2 and $(G_a)^7$.

Proposition 6. *In the case when* $n=5$, *let* $J(x)$ *denote the* G_1-*invariant in Proposition* 3 *evaluated at*

$$x = x_0 + \sum_{i<j} x_{ij}e_ie_j + \sum_i y_ie_i^* + (\sum_i x_ie_i + \sum_{i<j} y_{ij}e_{ij}^* + y_0e_L)e_0$$

after replacing e_0 *by* e_6. *Then, the* G-*orbits in* $X-0$ *are* $J^{-1}(i)$ *for* $i\neq 0$ *and four orbits in* $J^{-1}(0)$ *with* $1, 1+e_{13}^*e_0, 1+e_1^*, 1+(e_{24}^*+e_{35}^*)e_0$ *as their representatives. The Witt condition is satisfied. Moreover, the stabilizer subgroup of* G *at the representative* $1+ie_1^*+(e_{24}^*+e_{35}^*)e_0$ *of* $J^{-1}(i)$ *for* $i\neq 0$ *is isomorphic over* $k((-i)^{\frac{1}{2}})$ *to* SL_5. *The stabilizer subgroups at the above four representatives are respectively isomorphic over* k *to semidirect products of* SL_5 *and a connected unipotent group of dimension* 15, *of (connected* k-*split simple group of type* $G_2) \times SL_2$ *and a connected unipotent group of dimension* 15, *of* Spin_7 *and* $(G_a)^9$, *of* Sp_4 *and a connected unipotent group of dimension* 14.

Proof. Let $x\neq 0$ denote an arbitrary element of X expressed as in the proposition. Suppose that $x_0\neq 0$, $(x_{ij})\neq 0$, or $y_i\neq 0$ for some i. Then, by the same argument as in the proof of Proposition 5, we see that x is G-equivalent over $k(x)$ to another element in which $x_0=1$, $(x_{ij})=0$, $x_i=0$ for all i. We may further assume that $y_i=0$ for $i\neq 1$. Moreover, the case when $x_0=0$, $(x_{ij})=0$, $x_i=0$ for all i can be reduced to the previous case by applying $s_{0j}(1)$ for a suitable j. Therefore, we have only to examine the G-equivalence of elements of the following form:

$$y = 1 + y_1e_1^* + (\sum_{i<j} y_{ij}e_{ij}^* + y_0e_L)e_0.$$

If we evaluate J at y, we get

$$J(y) = y_1(-y_{23}y_{45}+y_{24}y_{35}-y_{25}y_{34}) - (\tfrac{1}{4})(y_0)^2.$$

Suppose first that $y_1=0$. Then y is G_0-equivalent over $k(y)$ to

$$1+qe_{L_1} \quad (e_{L_1}=e_Le_0), \quad 1+(e_{13}^*+pe_{24}^*+qe_L)e_0$$

according as $\mathrm{rank}(y_{ij})=0$, or $\geqq 2$. In the case when $p\neq 0$, if we apply

$$1+(p-1)e_4e_9+(p^{-1}-1)e_5e_{10}+(p+p^{-1}-2)e_4e_5e_9e_{10}$$

to $1+(e_{13}^*+pe_{24}^*+qe_L)e_0$, we get $1+(e_{13}^*+e_{24}^*+qe_L)e_0$. In the case when $q\neq 0$, if we apply $s_{6,8}(q^{-1})$ to $1+(e_{13}^*+qe_L)e_0$, we get $1+qe_L$. Moreover, still in the case when $q\neq 0$, if we apply $s_{6,8}(q^{-1})s_{05}(-q^{-1})s_{7,9}(q^{-1})$ to $1+(e_{13}^*+e_{24}^*+qe_L)e_0$, we get $1+qe_{L_1}$. Therefore, every y with $y_1=0$

is G-equivalent over $k(y)$ to $1+qe_{L_1}$, $1+e_{13}{}^*e_0$, or $1+(e_{13}{}^*+e_{24}{}^*)e_0$. We know by Proposition 3 that they are not even G_1-equivalent.

Suppose next that $y_1\neq 0$. If we apply $s_{06}(\lambda)$ to y and then apply $\exp^{-1}$ of its homogeneous component of degree 2, we get

$$1+z_1e_1{}^*+(\sum y_{ij}e_{ij}{}^*+z_0e_L)e_0$$

with $z_0=y_0-2p\lambda$ and $z_1=y_1-y_0\lambda+p\lambda^2$, in which

$$p=-y_{23}y_{45}+y_{24}y_{35}-y_{25}y_{34}.$$

Therefore, if $y_1-y_0\lambda+p\lambda^2=0$ has a solution, y is G-equivalent over $k(y,\lambda)$ to another element in which the coefficient of $e_1{}^*$ is 0. Since the discriminant of the quadratic equation is $-4J(y)$, if this is a square in $k(y)$, the root λ is rational over $k(y)$. In the case when $-4J(y)$ is not a square in $k(y)$, we necessarily have $p\neq 0$. We shall examine this case. Consider the subgroup of the stabilizer subgroup of G_0 at 1 defined by the additional condition that $\gamma=0$ and δ is of the form

$$\delta=\begin{bmatrix}\delta_1 & \delta_{12}\\ 0 & \delta_2\end{bmatrix},$$

in which δ_{12} for instance is of type $(1,4)$. An element of this subgroup will map the matrix (y_{ij}) to $\delta(y_{ij})^t\delta$. Therefore, if we take δ suitably, we see that y is G_0-equivalent over $k(y)$ to $z=1+z_1e_1{}^*+(e_{24}{}^*+e_{35}{}^*+z_0e_L)e_0$, and we have $J(z)=z_1-(\frac{1}{4})(z_0)^2$. If we apply $s_{06}((\frac{1}{2})z_0)$ to z and then apply $\exp^{-1}$ of its homogeneous component of degree 2, we get $1+ie_1{}^*+(e_{24}{}^*+e_{35}{}^*)e_0$ with $i=J(z)$. We have seen that this element is G-equivalent over $k((-i)^{\frac{1}{2}})$ to another element in which the coefficitnt of $e_1{}^*$ is 0. Then, it is necessarily G-equivalent over $k(q)$ to $1+qe_{L_1}$ for $q^2=-4i$. Now, the above normalization-process breaks down when $y_0=p=0$. In this case, we shall again use the subgroup of the stabilizer subgroup of G_0 at 1 with $\gamma=0$ and δ as above. By taking δ suitably, we see that y is G_0-equivalent over $k(y)$ to $1+e_1{}^*$, $1+e_1{}^*+e_{13}{}^*e_0$,

$$1+e_1{}^*+e_{24}{}^*e_0,\ \text{or}\ 1+e_1{}^*+(e_{13}{}^*+e_{24}{}^*)e_0.$$

If we apply $s_{08}(1)$ to the first element, we get the second element. Similarly, if we apply $s_{1,5}(1)s_{08}(1)$ to the third element, we get the fourth element. On the other hand, $1+(e_{13}{}^*+e_{24}{}^*)e_0$ and $1+(e_{24}{}^*+e_{35}{}^*)e_0$ are certainly G_0-equivalent over k. We have thus shown that every element $x\neq 0$ of X is G-equivalent over $k(x)$ to $1+ie_1{}^*+(e_{24}{}^*+e_{35}{}^*)e_0$ when $i=J(x)\neq 0$ and to 1, $1+e_{13}{}^*e_0$, $1+(e_{24}{}^*+e_{35}{}^*)e_0$, $1+e_1{}^*$, or $1+e_1{}^*+e_{24}{}^*e_0$ when $J(x)=0$. Now, if we apply $s_{8,10}(1)s_{3,5}(1)s_{8,10}(1)$ to $1+e_1{}^*+e_{24}{}^*e_0$, we get $e_2e_4+e_3e_5$

$+ e_1 e_0$. We observe that, if we apply $s_{01}(1)s_{06}(1)s_{8,10}(1)$ to $1 + (e_{24}{}^* + e_{35}{}^*)e_0$, we get the same element. Therefore $1 + e_1{}^* + e_{24}{}^* e_0$ and $1 + (e_{24}{}^* + e_{35}{}^*) e_0$ are G-equivalent over k. We shall see that the remaining elements are not G-equivalent.

We shall examine the stabilizer subgroups of G at various representatives. If x is one of them, we shall denote the subgroup in question by H. Suppose first that $x = 1 + ie_1{}^* + (e_{24}{}^* + e_{35}{}^*) e_0$ with $i \neq 0$. We know that x is G-equivalent over $k((-i)^{\frac{1}{2}})$ to $1 + qe_{L_1}$ for $q^2 = -4i$. Furthermore, the stabilizer subgroup of G at $1 + qe_{L_1}$ for $q \neq 0$ is isomorphic over $k(q)$ to SL_5. This follows from Lemma 3 applied to G_1 (and from the "trivial restatement" before Lemma 6). In the case when $i = 0$ in x, we have seen that it is G-equivalent over k to $e_2 e_4 + e_3 e_5 + e_1 e_0$. The stabilizer subgroup of G at this element can be determined by Lemma 5 applied to G_1. It is isomorphic over k to a semidirect product of Sp_4 and a connected unipotent group of dimension 14. Suppose that $x = 1 + e_1{}^*$. Then, Lemma 6 shows that every element of H is of the form $s_0(1 + v_0 \epsilon_0)$, in which s_0 is an element of the stabilizer subgroup of G_0 at x and v_0 is in Ωe_6. Therefore, by Lemma 3 and Proposition 1, we see that H is isomorphic over k to a semidirect product of Spin_7 and $(G_a)^9$. Suppose that $x = 1 + e_{13}{}^* e_0$. According to Lemma 4, every element of the stabilizer subgroup of G_1 at x can be written uniquely in the form $s_0 s_1 s_2$. This element is contained in H if and only if it is contained in G. We shall examine this condition. Since s_1 is in G, we may assume that $s_1 = 1$. For a similar reason, we may assume that s_2 is of the form $1 + v_0 \epsilon_1$ with v_0 in $\Omega e_6 + \Omega e_8$. Then, we see that $s_0 s_2$ is in G, i.e., commutes with ϵ_1, if and only if $v_0 = 0$ and s_0 is in G. Therefore H is isomorphic over k to a semidirect product of (connected k-split simple group of type G_2) $\times SL_2$ and a connected unipotent group of dimension 15. In particular, we see that $1 + e_1{}^*$ and $1 + e_{13}{}^* e_0$ are not G-equivalent (although they are G_1-equivalent). Finally, suppose that $x = 1$. Then, by Lemma 1 applied to G_1 we see that H is isomorphic over k to a semidirect product of SL_5 and a connected unipotent group of dimension 15. This completes the proof.

We observe that the set of points of $X - 0$ where the first polar δJ vanishes is the union of three G-orbits with 1, $1 + e_{13}{}^* e_0$, $1 + e_1{}^*$ as their representatives. On the other hand, in the case when $(-i)^{\frac{1}{2}}$ is not contained in $K = k(i)$, we can show that the stabilizer subgroup H of G at $1 + ie_1{}^* + (e_{24}{}^* + e_{35}{}^*) e_0$ is the special unitary group relative to $K((-i)^{\frac{1}{2}})/K$ for a hermitian matrix of degree 5 and of index 2. In particular, the K-rank of H is 2.

5. A singular case. We take the spin representation of Spin_{4r} in the exterior algebra $E = C_L$ as our starting point. We have seen in Lemma 5 that $G = Sp_{2r}$ is embedded in the stabilizer subgroup of Spin_{4r} at

$$a = e_1 e_{r+1} + \cdots + e_r e_{2r}.$$

We recall that $e_1, e_2, \cdots, e_{2r}$ form an Ω-base of L rational over k. We shall identify $\mathrm{Aut}(L)$ with GL_{2r} as

$$g \cdot (e_1 \cdots e_{2r}) = (e_1 \cdots e_{2r}) \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

Then g is contained in G if and only if $\alpha^t\beta$, $\gamma^t\delta$ are symmetric and $\alpha^t\delta - \beta^t\gamma = 1_r$ or, equivalently, ${}^t\alpha\gamma$, ${}^t\beta\delta$ are symmetric and ${}^t\alpha\delta - {}^t\gamma\beta = 1_r$. We observe that $\mathrm{Aut}(L)$ operates on E as a group of degree-preserving algebra automorphisms. Consequently, we have $g \cdot \exp(x) = \exp(g \cdot x)$ for every g in $\mathrm{Aut}(L)$ and x in E. Let X denote the kernel of the Ω-linear mapping $E_p \to E_{2r-p+2}$ defined by $x \to x \cdot \exp(a)_{2(r-p+1)}$, in which $a = e_1 e_{r+1} + \cdots + e_r e_{2r}$. Then X is a G-invariant subspace of E_p defined over k for $1 \leqq p \leqq r$.

On the other hand, the half-spin representations of Spin_{4r} in $(C_L)^{\pm}$ are conjugate (with respect to an outer automorphism). We recall that in the special case when $r = 3$, there exists a quartic invariant J for the half-spin representation of Spin_{12} in $(C_L)^+$ (cf. Proposition 3). Therefore, the half-spin representation of Spin_{12} in $(C_L)^-$ also has a quartic invariant. If we restrict this invariant to X for $p = 3$, we will get a G-invariant. In the following, we shall make this G-invariant explicit and classify G-orbits in $X - 0$.

We first observe that every element x of E_3 can be written uniquely in the form

$$x = \sum_{i_1 < i_2 < i_3} x_{i_1 i_2 i_3} e_{i_1} e_{i_2} e_{i_3}$$

with $(x_{i_1 i_2 i_3})$ forming an alternating tensor of rank three with coefficients in Ω. Moreover, the element x is contained in X if and only if $x_{i14} + x_{i25} + x_{i36} = 0$ for $1 \leqq i \leqq 6$. Now, suppose that $x_{123} \neq 0$. Then, there exists one and only one g with $\alpha = \delta = 1_3$, $\beta = 0$ such that the transformed element $g \cdot x$ is free from $e_{i_1} e_{i_2} e_{i_3}$ in which exactly two of the i_1, i_2, i_3 are among 1, 2, 3. In fact, the unique choice is given by

$$-x_{123}\gamma = \begin{pmatrix} x_{423} & x_{143} & x_{124} \\ x_{523} & x_{153} & x_{125} \\ x_{623} & x_{163} & x_{126} \end{pmatrix}.$$

Similarly, if $x_{456} \neq 0$, there exists one and only one g with $\alpha = \delta = 1_3$, $\gamma = 0$

such that $g \cdot x$ is free from $e_{i_1}e_{i_2}e_{i_3}$ in which exactly two of the i_1, i_2, i_3 are among 4, 5, 6. The unique choice is given by

$$-x_{456}\beta = \begin{bmatrix} x_{156} & x_{416} & x_{451} \\ x_{256} & x_{426} & x_{452} \\ x_{356} & x_{436} & x_{453} \end{bmatrix}.$$

We shall denote the two symmetric matrices (standing on the right hand sides) by (x_{ij}) and (y_{ij}), respectively. Also, we put $x_0 = -x_{123}$ and $y_0 = -x_{456}$. We observe that, if $\beta = \gamma = 0$ in g, the transformation $x \to g \cdot x$ gives rise to $x_0 \to \det(\alpha)x_0$, $y_0 \to \det(\delta)y_0$, and $(x_{ij}) \to \det(\alpha)\delta(x_{ij})^t\delta$, $(y_{ij}) \to \det(\delta)\alpha(y_{ij})^t\alpha$.

We shall examine G-orbits in X. We shall first show that every element $x \neq 0$ of X is G-equivalent over $k(x)$ to another element in which the coefficient of $e_1e_2e_3$ is different from 0. We observe that the set of all products $e_{i_1}e_{i_2}e_{i_3}$ splits into two equivalence classes with respect to the Weyl group of G. Moreover $e_1e_2e_3$ and $e_1e_2e_5$ are their representatives. If $x_{i_1i_2i_3} \neq 0$ for at least one $e_{i_1}e_{i_2}e_{i_3}$ which is equivalent to $e_1e_2e_3$, therefore, we have only to apply the Weyl group of G. Also, in the other case, we may assume that $x_{125} \neq 0$. If we apply the element of G defined by $e_5 \to e_5 + e_3$, $e_6 \to e_6 + e_2$, $e_i \to e_i$ $(i \neq 5, 6)$ to x, the coefficient of $e_1e_2e_3$ will become $2x_{125}$. This implies the assertion.

Now, if $x_{123} \neq 0$ in x, by applying the element of G defined by $\alpha = \delta = 1_3$, $\beta = 0$, $x_0\gamma = (x_{ij})$, we will get another element, say x', in which

$$\begin{aligned} x'_0 &= x_0, \quad x'_{ij} = 0 \\ y'_0 &= y_0 - (x_0)^{-1}\sum_{i,j} x_{ij}y_{ij} - 2(x_0)^{-2}\det((x_{ij})) \\ y'_{ij} &= y_{ij} + (x_0)^{-1}\cdot(-1)^{i+j}\det(X_{ij}). \end{aligned}$$

We are denoting by X_{ij} the matrix obtained from (x_{ij}) by crossing out its i-th line and j-th column. We can further convert the matrix (y'_{ij}) into a diagonal form. Since the whole process is rational, we may assume that the given element is of the form

$$x = -x_0e_1e_2e_3 - y_0e_4e_5e_6 + y_1e_1e_5e_6 + y_2e_4e_2e_6 + y_3e_4e_5e_3,$$

in which $x_0 \neq 0$. We observe that, if we express $J = x_0y_1y_2y_3 - (\frac{1}{4})(x_0y_0)^2$ by the original x_0, y_0, (x_{ij}), (y_{ij}), we will get

$$\begin{aligned} J = x_0\det((y_{ij})) + y_0\det((x_{ij})) + \sum_{i,j}\det(X_{ij})\det(Y_{ij}) \\ - (\tfrac{1}{4})(x_0y_0 - \sum_{i,j} x_{ij}y_{ij})^2. \end{aligned}$$

We shall show that, if we have $J \neq 0$, x is G-equivalent over $k(x, q)$ to

$-e_1e_2e_3-qe_4e_5e_6$ for $q^2=-4J$. We observe that the element of G defined by β_1 $(=\beta_{11})=q^{-1}$, $\beta_2=\beta_3=1$, $\gamma_1=-q$, $\gamma_2=\gamma_3=-1$, all other coefficients $=0$, will transform $-e_1e_2e_3-qe_4e_5e_6$ into $-e_1e_2e_3+qe_4e_5e_6$. Therefore, we have the freedom of choosing the sign in $\pm q$. Furthermore, if we admit the above transformability and recall the existence of a quartic G-invariant, we see that J has to be this G-invariant. We can also verify the G-invariance of J by a direct computation.

In general, if we apply an arbitrary element g of G to $-pe_1e_2e_3-qe_4e_5e_6$, we will get an element of X with

$$x_0=p\det(\alpha)+q\det(\beta),\quad y_0=p\det(\gamma)+q\det(\delta)$$
$$(x_{ij})=-p\gamma\alpha^*-q\delta\beta^*,\quad (y_{ij})=-p\alpha\gamma^*-q\beta\delta^*,$$

in which α^* etc denote $\det(\alpha)\alpha^{-1}$ etc. More precisely, if $\det(\alpha)\neq 0$, α^* is well defined as $\det(\alpha)\alpha^{-1}$ and its coefficients are polynomials in the coefficients of α. Therefore, we may define α^* using this universal expression even when $\det(\alpha)=0$. Now, in order to prove the transformability, we have only to solve the above system of equations over $k(x,q)$ for $p=1$, $(x_{ij})=0$, $(y_{ij})=$ a diagonal matrix, under the assumption that $(x_0y_0)^2-4x_0y_1y_2y_3=q^2\neq 0$. We shall show that this can be done by using diagonal matrices as α, β, γ, δ. In fact, if $y_1y_2y_3\neq 0$, we have $x_0(x_0y_0+q)\neq 0$. Therefore, we have only to choose α_1, α_2, α_3 from $k(x,q)^\times$ satisfying

$$\alpha_1\alpha_2\alpha_3=(2q)^{-1}x_0(x_0y_0+q),$$

which is certainly possible, and put

$$\beta_i=-q^{-1}x_0y_i(\alpha_i)^{-1},\quad \gamma_i=-(2x_0)^{-1}(x_0y_0-q)(y_i)^{-1}\alpha_i$$
$$\delta_i=(2q)^{-1}(x_0y_0+q)(\alpha_i)^{-1}$$

for $1\leqq i\leqq 3$. If we have $y_1y_2y_3=0$, we may assume that $q=x_0y_0$. Then, we have only to choose α_1, α_2, α_3 from $k(x,q)^\times$ satifying $\alpha_1\alpha_2\alpha_3=x_0$, take $\delta_i=(\alpha_i)^{-1}$ and put

$$\beta_i=-(y_0)^{-1}y_i(\alpha_i)^{-1},\quad \gamma_i=-(x_0y_0y_i)^{-1}y_1y_2y_3\alpha_i$$

for $1\leqq i\leqq 3$. (The notation $(y_3)^{-1}y_1y_2y_3$ stands for y_1y_2, which is meaningful even when $y_3=0$.) This completes the proof. As a special case, we see that $-e_1e_2e_3-qe_4e_5e_6$ is G-equivalent over $k(q)$ to $-e_1e_2e_3+e_1e_5e_6+e_4e_2e_6+ie_4e_5e_3$ for $q^2=-4i\neq 0$. Therefore, if we have $i=J(x)\neq 0$ for an arbitrary element x of X, it is G-equivalent over $k(x,q)$ to $-e_1e_2e_3+e_1e_5e_6+e_4e_2e_6+ie_4e_5e_3$, in which $q^2=-4i$. We shall see later that the G-equivalence may not be defined over $k(x)$.

We shall determine the stabilizer subgroup, say H, of G at $-e_1e_2e_3$

$-qe_4e_5e_6$ for $q \neq 0$. We have determined the effect of an arbitrary element g of G to this element. In particular, we see that g is contained in H if and only if (i) $\det(\alpha) + q\det(\beta) = 1$, (ii) $\det(\gamma) + q\det(\delta) = q$, (iii) $\gamma\alpha^* + q\delta\beta^* = 0$, (iv) $\alpha\gamma^* + q\beta\delta^* = 0$. We shall first show that these conditions imply $\det(\gamma) = 0$. If $\det(\gamma) \neq 0$, (iv) implies $\alpha = -q\det(\gamma)^{-1}\beta\delta^*\gamma$. In view of $\alpha^t\delta - \beta^t\gamma = 1_3$, using (ii), we get $\beta = -q^{-1t}(\gamma^*)$, hence $\alpha = {}^t(\delta^*)$. Putting these in (i) and using (ii) again, we get $-\det(\gamma) + q\det(\delta) = q$. However, this and (ii) imply $\det(\gamma) = 0$, a contradiction. Therefore, we have $\det(\gamma) = 0$ for every g in H. This implies that the rank of γ^* is at most one. On the other hand, (ii) and $\det(\gamma) = 0$ imply $\det(\delta) = 1$, hence (iv) implies that the rank of β is at most one. Consequently, we have $\beta^* = 0$, hence (i) implies $\det(\alpha) = 1$. Therefore (iii) implies $\gamma = 0$. Also (iv) implies $\beta = 0$. Conversely, if $\det(\alpha) = 1$, $\beta = \gamma = 0$, the corresponding element g of G is contained in H. Therefore H consists of such elements, and it is isomorphic over k to SL_3.

We shall consider points of the hypersurface in X defined by $J(x) = 0$. We may assume that all coefficients of x are 0 except x_0, y_0, y_1 y_2, y_3 and that $x_0 \neq 0$. We shall show that, in the case when $y_1y_2y_3 \neq 0$, we can find an element g of G which will transform $-e_1e_2e_3 + r_1e_1e_5e_6 + r_2e_4e_2e_6$ for some $r_1, r_2 \neq 0$ into x. In fact, we can use diagonal matrices as α, β, γ, δ. We observe that, because of $(x_0y_0)^2 = 4x_0y_1y_2y_3 \neq 0$, we have $y_0 \neq 0$. We choose α_1, α_2 so that we have $r_i(\alpha_i)^2 = x_0y_i$ for $i = 1, 2$ and determine β_3 by $2r_1r_2\alpha_1\alpha_2\beta_3 = -(x_0)^2y_0$. Then, we take β_1, β_2 freely and determine α_3 by

$$\alpha_1\alpha_2\alpha_3 - r_1\alpha_1\beta_2\beta_3 - r_2\beta_1\alpha_2\beta_3 = x_0.$$

Finally, we put

$$x_0\gamma_1 = r_2\alpha_2\beta_3, \quad x_0\gamma_2 = r_1\alpha_1\beta_3, \quad x_0\gamma_3 = r_1\alpha_1\beta_2 + r_2\beta_1\alpha_2$$
$$x_0\delta_1 = \alpha_2\alpha_3 - r_1\beta_2\beta_3, \quad x_0\delta_2 = \alpha_1\alpha_3 - r_2\beta_1\beta_3, \quad x_0\delta_3 = \alpha_1\alpha_2.$$

The element g so determined is contained in G, and it gives the required transformation. We may assume that g is rational over $k(x)$. We shall consider the case when $y_1y_2y_3 = 0$ or rather $y_3 = 0$. We may assume that $x_0 = 1$. Then, we have $x = -e_1e_2e_3 + y_1e_1e_5e_6 + y_2e_4e_2e_6$. If $y_1y_2 \neq 0$, clearly x is G-equivalent to $-e_1e_2e_3 + e_1e_5e_6 + e_4e_2e_6$. If $y_1y_2 = 0$, we may assume that $y_2 = 0$. Then x is G-equivalent either to $-e_1e_2e_3 + e_1e_5e_6$ or to $-e_1e_2e_3$. We shall show that they are G-inequivalent. If an element g of G transforms $-e_1e_2e_3$ into $-e_1e_2e_3 + e_1e_5e_6$, we will get $\det(\alpha) = 1$, $\det(\gamma) = 0$, $\gamma\alpha^* = 0$, $\alpha\gamma^* =$ the diagonal matrix with -1, 0, 0 as its diagonal coefficients. But this brings a contradiction. On the other hand, the first polar δJ vanishes at $-x_0e_1e_2e_3 - y_0e_4e_5e_6 + y_1e_1e_5e_6 + y_2e_4e_2e_6 + y_3e_4e_5e_3$ for $x_0 \neq 0$ if and only if y_0 and at least two of the y_1, y_2, y_3 are 0. Therefore δJ does not vanish at x if and only if either $J(x) \neq 0$ or x is G-equivalent to $-e_1e_2e_3 + e_1e_5e_6 + e_4e_2e_6$. In particular, the three G-orbits in $J^{-1}(0)$ are distinct.

We shall determine the stabilizer subgroup, say H, of G at a suitable representative of each G-orbit in $J^{-1}(0)$. Consider first the G-orbit with $-e_1e_2e_3$ as its representative and suppose that H is defined with respect to this representative. Then, an element g of G is contained in H if and only if $\det(\alpha) = 1$, $\det(\gamma) = 0$, $\gamma\alpha^* = \alpha\gamma^* = 0$. Therefore H consists of those g for which $\det(\alpha) = 1$ and $\gamma = 0$, and it is isomorphic over k to a semidirect product $(SL_3)(G_a)^6$. Consider next the G-orbit with $-e_1e_2e_3 + e_1e_5e_6$ as its representative. We recall that every element of this G-orbit is rationally transformable into $-e_1e_2e_3 + y_1e_1e_5e_6$ for some $y_1 \neq 0$. We shall show that, for any field K containing $k(y_1)$, $-e_1e_2e_3 + y_1e_1e_6e_5$ with y_1 in $\Omega^\times$ is G-equivalent over $K((-y_1)^{\frac{1}{2}})$ to $e_1e_4e_3 + e_5e_2e_3$. (We observe that the absolute transformability is a consequence of the classification of G-orbits in $J^{-1}(0)$. It also follows from the subsequent calculation.) Suppose that g is an arbitrary element of G which transforms $x = e_1e_4e_3 + e_5e_2e_3$ into $-e_1e_2e_3 + y_1e_1e_5e_6$. Then $g \cdot xe_3 = (g \cdot x)(g \cdot e_3) = 0$ implies that $\alpha_{23}, \alpha_3, \gamma_{13}, \gamma_{23}, \gamma_3$ are 0. Also ${}^t\alpha\gamma = {}^t\gamma\alpha$ implies that γ_1, γ_{12} are 0 and ${}^t\alpha\delta - {}^t\gamma\beta = 1_3$ implies that δ_1, δ_{12} are 0 and $\alpha_{13}\delta_{13} = 1$. Consequently, the four matrices of degree two which are obtained from α, β, γ, δ by crossing out their first lines and the third columns form an element of Sp_4. This fact and the condition that $g \cdot x = -e_1e_2e_3 + y_1e_1e_5e_6$ can be put together as follows:

$$-\begin{pmatrix} \alpha_2 & \beta \\ \gamma_2 & \delta_2 \end{pmatrix}\begin{pmatrix} \delta_{32} & -\beta_{32} \\ -\gamma_{32} & \alpha_{32} \end{pmatrix} = \begin{pmatrix} \alpha_{21} & \beta_{21} \\ \gamma_{21} & \delta_{21} \end{pmatrix}\begin{pmatrix} \delta_{31} & -\beta_{31} \\ -\gamma_{31} & \alpha_{31} \end{pmatrix} = (\tfrac{1}{2})\delta_{13}\begin{pmatrix} 0 & 1 \\ y_1 & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} = \det\begin{pmatrix} \delta_{32} & -\beta_{32} \\ -\gamma_{32} & \alpha_{32} \end{pmatrix} = \cdots = \det\begin{pmatrix} \delta_{31} & -\beta_{31} \\ -\gamma_{31} & \alpha_{31} \end{pmatrix} = \tfrac{1}{2}.$$

In particular, we have $(\alpha_{13})^2 = -y_1$, hence $K(g)$ contains $(-y_1)^{\frac{1}{2}}$. On the other hand, we can find an element g of G rational over $K((-y_1)^{\frac{1}{2}})$ which satisfies the above system of equations. We first choose α_{13} so that we have $(\alpha_{13})^2 = -y_1$ and put $\delta_{13} = (\alpha_{13})^{-1}$. Then, we simply solve the above system of equations over $K((-y_1)^{\frac{1}{2}})$. This is certainly possible. The remaining conditions for α, β, γ, δ to form an element of Sp_6 are as follows:

$$\alpha_{13}\begin{bmatrix} \beta_{23} \\ \beta_3 \\ \delta_{23} \\ \delta_3 \end{bmatrix} = \begin{bmatrix} \alpha_{21} & \alpha_2 & \beta_{21} & \beta_2 \\ \alpha_{31} & \alpha_{32} & \beta_{31} & \beta_{32} \\ \gamma_{21} & \gamma_2 & \delta_{21} & \delta_2 \\ \gamma_{31} & \gamma_{32} & \delta_{31} & \delta_{32} \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_{12} \\ -\alpha_1 \\ -\alpha_{12} \end{bmatrix}.$$

We remark that we can take β_1, β_{12}, α_1, α_{12} (and also β_{13}) freely from $K((-y_1)^{\frac{1}{2}})$. This proves the assertion. The above observation implies also that the stabilizer subgroup H of G at x has two components H_0, H_1 and that H_0 is isomorphic over k to a semidirect product of $(SL_2)^2$ and a connected unipotent group of dimension 5. Moreover, the element of G defined by $e_1 \to e_2$, $e_2 \to e_1$, $e_3 \to -e_3$ etc. is a representative of H_1. Finally, we shall consider the G-orbit with $-e_1e_2e_3 + e_1e_5e_6 + e_4e_2e_6$ as its representative. We recall that every element of this G-orbit is rationally transformable into $-e_1e_2e_3 + y_1e_1e_5e_6 + y_2e_4e_2e_6$ for some $y_1, y_2 \neq 0$. We shall show that, for any field K containing $k(y_1, y_2)$, $-e_1e_2e_3 + y_1e_1e_5e_6 + y_2e_4e_2e_6$ with y_1, y_2 in $\Omega^\times$ is G-equivalent over $K((-y_1y_2)^{\frac{1}{2}})$ to $e_1e_4e_3 + e_5e_2e_3 + e_1e_2e_6$. If we apply the element of G defined by $e_3 \to -e_6$, $e_6 \to e_3$, $e_i \to e_i$ $(i \neq 3, 6)$ to $-e_1e_2e_3 + y_1e_1e_5e_6 + y_2e_4e_2e_6$, we get $e_1e_2e_6 + y_1e_1e_5e_3 + y_2e_4e_2e_3$. We shall find an element g_0 of G rational over $K((-y_1y_2)^{\frac{1}{2}})$ which transforms $x = e_1e_4e_3 + e_5e_2e_3 + e_1e_2e_6$ into $y_2e_4e_2e_3 + y_1e_1e_5e_3 + e_1e_2e_6$. We choose α_1, α_2 from $K((-y_1y_2)^{\frac{1}{2}})$ so that we have $(2\alpha_1\alpha_2)^2 = -y_1y_2$ and determine α_{12}, α_{21} by $-2\alpha_1\alpha_{12} = y_1$, $-2\alpha_2\alpha_{21} = y_2$. Then, we put $\alpha_3 = \alpha_1\alpha_2 - \alpha_{12}\alpha_{21}$ and the remaining coefficients of α equal to 0. Also, we put $\beta = \gamma = 0$, $\delta = {}^t\alpha^{-1}$. We have only to take this element of G as g_0. We shall determine the stabilizer subgroup H of G at x. We have seen that the G-orbit of x is precisely the set of simple points of the hypersurface $J(x) = 0$. Therefore, we get $\dim(G/H) = \dim(X) - 1$, hence $\dim(H) = 8$. Let SO_3 denote the special orthogonal group of the quadratic form $2x_1x_2 + (x_3)^2$. If h denotes the symmetric matrix of degree three with $h_{12} = h_{21} = h_3 = 1$, all other coefficients $= 0$, it consists of those δ in SL_3 with the property that $\delta h {}^t\delta = h$. At any rate, an element g of G for which $\gamma = 0$ transforms x into another element, say x', with

$$x'_0 = -\det(\alpha)\operatorname{tr}(\beta h {}^t\delta), \quad (x'_{ij}) = \det(\alpha)\delta h {}^t\delta$$
$$y'_0 = 0, \quad (y'_{ij}) = 0.$$

Consequently, such an element is contained in H if and only if $\delta h {}^t\delta = \det(\delta)h$ and $\operatorname{tr}(h\alpha^{-1}\beta) = 0$. Therefore, if we denote by H_0 the connected component of the identity in H, because of $\dim(H) = 8$, it is isomorphic over k to a semidirect product $(SO_3)(G_a)^5$. We shall show that $H = H_0$. At any rate, H is contained in the normalizer of H_0 in G, and it consists of those elements of G with α in GL_3 satisfying ${}^t\alpha h\alpha = \lambda h$ for some λ in $\Omega^\times$ and with $\gamma = 0$. As we have seen, such an element is contained in H if and only if it is contained in H_0. This proves the assertion.

PROPOSITION 7. *Let X denote the subspace of the exterior algebra over $\Omega e_1 + \cdots + \Omega e_6$ consisting of $x = \sum x_{i_1i_2i_3}e_{i_1}e_{i_2}e_{i_3}$ in which $x_{i14} + x_{i25} + x_{i36} = 0$ for $1 \leqq i \leqq 6$. Then $G = Sp_6$ operates on X. Moreover, we can rearrange*

$x_{i_1i_2i_3}$ into x_0, y_0 and symmetric matrices (x_{ij}), (y_{ij}) of degree 3 such that, if we put

$$J(x) = x_0 \det((y_{ij})) + y_0 \det((x_{ij})) + \sum_{i,j} \det(X_{ij}) \det(Y_{ij}) \\ - (\tfrac{1}{4})(x_0y_0 - \sum_{i,j} x_{ij}y_{ij})^2,$$

in which, e.g., X_{ij} stands for the matrix obtained from (x_{ij}) by crossing out its i-th line and j-th column, this defines a G-invariant. The G-orbits in $X - 0$ are $J^{-1}(i)$ for $i \neq 0$ with $-e_1e_2e_3 - 2(-i)^{\frac{1}{2}}e_4e_5e_6$ as its representative and three orbits in $J^{-1}(0)$ with $-e_1e_2e_3$, $e_1e_4e_3 + e_5e_2e_3$, $e_1e_4e_3 + e_5e_2e_3 + e_1e_2e_6$ as their representatives. The Witt condition is satisfied only for the G-orbit of $-e_1e_2e_3$. The stabilizer subgroup H of G at $e_1e_4e_3 + e_5e_2e_3$ has two components H_0, H_1 and H_0 is isomorphic over k to a semidirect product of $SL_2 \times SL_2$ and a connected unipotent group of dimension 5; H_1 has a rational point over k. The stabilizer subgroups at $-e_1e_2e_3$, $e_1e_4e_3 + e_5e_2e_3 + e_1e_2e_6$, $-e_1e_2e_3 - 2(-i)^{\frac{1}{2}}e_4e_5e_6$ for $i \neq 0$ are respectively isomorphic over k to $(SL_3)(G_a)^6$, $(SO_3)(G_a)^5$, SL_3, in which the quadratic form for SO_3 is of index 1 over k.

We observe that the set of points of $X - 0$ where the first polar δJ does not vanish is the union of G-orbits of $-e_1e_2e_3 + e_1e_5e_6 + e_4e_2e_6 + ie_4e_5e_3$ for all i in Ω. On the other hand, we observe that stabilizer subgroups are not all connected in this case. Far more remarkable is the fact that the Witt condition completely breaks down for two orbits in $J^{-1}(0)$, one of which is maximal. We shall briefly explain this singular situation.

Suppose that U is a G-orbit in $X - 0$ defined over k. We shall be interested in decomposing U_k into G_k-orbits. Since G has trivial Galois cohomology, we can identify U_k/G_k with the cohomology set $H^1(k, H)$ for the stabilizer subgroup H of G at any element of U_k (cf. 8). In the following, we shall explain the decomposition of U_k without explicitly using the Galois cohomology. Since the G-orbit of $-e_1e_2e_3$ satisfies the Witt condition, we shall pass to the next case: Suppose that U is the G-orbit of $e_1e_4e_3 + e_5e_2e_3$. We recall that any element x of U_k is G_k-equivalent to $-e_1e_2e_3 + ae_1e_5e_6$ for some a in $k^\times$. We can show that *the correspondence $x \to a$ gives rise to a bijection $U_k/G_k \to k^\times/(k^\times)^2$.* Suppose next that U is the G-orbit of $e_1e_4e_3 + e_5e_2e_3 + e_1e_2e_6$. We recall that any element x of U_k is G_k-equivalent to another element in which only $e_4e_2e_3$, $e_1e_5e_3$, $e_1e_2e_6$ have non-zero coefficients. In particular, x is G_k-equivalent to an element with $x_0 = y_0 = 0$, $(y_{ij}) = 0$, $\det((x_{ij})) \neq 0$. We let $GL_3(k)$ operate on the set of symmetric, non-degenerate matrices in $M_3(k)$ as $(x_{ij}) \to \det(u)^{-1}u(x_{ij})\,{}^tu$. Then, *the corres-*

pondence $x \to (x_{ij})$ *gives rise to a bijection of* U_k/G_k *to the set of equivalence classes of non-degenerate ternary quadratic forms with coefficients in* k. Finally, in the case when U is the G-orbit $J^{-1}(i)$ for $i \neq 0$ contained in k, the Witt condition is satisfied when $-i$ is a square in k. Therefore, we shall assume that $K = k((-i)^{\frac{1}{2}})$ is a quadratic extension of k. We recall that any element x of U_k is G_k-equivalent to another element in which $x_0 = 1$ and $(x_{ij}) = 0$. This implies that $\det((y_{ij})) = N((\frac{1}{2})y_0 + (-i)^{\frac{1}{2}}) \neq 0$, in which N is the norm from K to k. We let $GL_3(K)$ operate on the set of non-degenerate K/k-hermitian matrices of degree 3 as $(y_{ij}) \to u(y_{ij})\,{}^t u'$, in which u' is the conjugate of u relative to k. Then, *the correspondence* $x \to (y_{ij})$ *gives rise to a bijection of* U_k/G_k *to the set of equivalence classes of non-degenerate ternary* K/k*-hermitian forms.* Furthermore, *the stabilizer subgroup at* x *is the special unitary group of the corresponding ternary hermitian form.*

Since all preparations to prove the above underlined statements have been made before, we shall leave their proofs as exercises. We remark that, if k is an algebraic number field, we have $\mathrm{card}(U_k/G_k) = \infty$ for the first two cases. In the last case, if r is the number of real conjugates of k in which $i > 0$, we have $\mathrm{card}(U_k/G_k) = 2^r$.

THE JOHNS HOPKINS UNIVERSITY.

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