1 Some BRST Twisting Calculations

Example 1.1 (10d $\mathcal{N}=1$ Super Yang-Mills)**.** I'm going to write this following the calculation in Baulieu [**Baulieu**]. We choose a complex structure on $\mathbb{R}^{10}$ and decompose the (BRST) fields into irreducible components for the action of SU(5). The fields one obtains are

$$\begin{aligned} A &= A_{0,1} + A_{1,0} \in \Omega^{0,1}(\mathbb{C}^5;\mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C}^5;\mathfrak{g}) \\ \lambda &= \chi + \psi_{1,0} + B_{0,2} \in \Omega^0(\mathbb{C}^5; S_+), \end{aligned}$$

where the subscripts indicate the form type of the component fields. There's also a ghost c, which is a $\mathfrak{g}$-valued scalar in degree -1. Finally we'll introduce an auxiliary scalar field h (bosonic, in degree 0). The BRST differential is given by $c \mapsto (\partial c, \overline{\partial} c)$. Denote this complex by $(\Phi^{\mathrm{YM}}, \mathrm{d}^{\mathrm{YM}}_{\mathrm{BRST}})$. The dg Lie bracket is given by the action of c on the other component fields. The holomorphic twist is only $\mathbb{Z}/2\mathbb{Z}$-graded, and is given by the identity map from ψ to $A_{1,0}$, the identity map from h to χ, along with the map $\overline{\partial}$ from $A_{0,1}$ to B [1].

We expect the holomorphically twisted theory to be equivalent to 5d holomorphic Chern-Simons theory. The BRST fields here are given by $c, A_{0,1}$ and $B_{0,2}$ in even, odd and even degrees respectively (this is only a $\mathbb{Z}/2\mathbb{Z}$-graded theory). Denote this complex by $(\Phi^{\mathrm{hCS}}, \mathrm{d}^{\mathrm{hCS}}_{\mathrm{BRST}})$. It's clear that the projection $\Phi^{\mathrm{YM}} \to \Phi^{\mathrm{hCS}}$ is a quasi-isomorphism with respect to the Q-twisted BRST differential: its fiber is the contractible complex $h \mapsto \chi, \psi \mapsto A_{1,0}$. It remains for us to verify that this projection is compatible with the Lagrangian densities of the two theories. In other words, if we take the Lagrangian density of 10d $\mathcal{N}=1$ Yang-Mills theory and subtract the Lagrangian density of holomorphic Chern-Simons theory applied to the fields $c, A_{0,1}$ and B then the result is Q-exact.

So, let's describe the Lagrangian density of 10d $\mathcal{N}=1$ Yang-Mills theory in terms of our component fields. We have

$$\begin{aligned} \mathcal{L}_{\mathrm{YM}} &= (B \wedge \overline{\partial}_{A_{0,1}} B) + J^2(F_{2,0} \wedge F_{0,2})\Omega + \|h\|^2\Omega + hJ(F_{1,1})\Omega + J(\chi \wedge (\overline{\partial}_{A_{0,1}}\psi))\Omega + J^2(B \wedge (\partial_{A_{1,0}}\psi))\Omega \\ &= (B \wedge \overline{\partial}_{A_{0,1}} B) + \frac{\delta}{\delta Q}\Big(2J^2(F_{2,0} \wedge B)\Omega + h\chi\Omega + \chi J(F_{1,1})\Omega + J(\chi \wedge (F_{1,1}))\Omega\Big). \end{aligned}$$

Therefore, after twisting the Lagrangian density is equivalent to just $B \wedge \overline{\partial}_{A_{0,1}} B$, which is the Lagrangian density in 5d holomorphic Chern-Simons theory, as required.

Example 1.2 (9d $\mathcal{N}=1$ Super Yang-Mills)**.** Let's decompose our fields further into irreducible summands with respect to the action of SU(4). We also restrict to 10d fields which are constant in the x^{10}-direction. The only fields we'll consider are those which survived the 10d holomorphic twist. These fields are

$$\begin{aligned} B^{10} &= B + A \wedge \mathrm{d}\overline{z}^5 \in (\Omega^{0,2}(\mathbb{C}^4;\mathfrak{g}) \oplus \Omega^{0,1}(\mathbb{C}^4;\mathfrak{g})) \otimes \Omega^0(\mathbb{R}_9) \\ A^{10}_{0,1} &= A' + \phi \mathrm{d}\overline{z}^5 \in (\Omega^{0,1}(\mathbb{C}^4;\mathfrak{g}) \oplus \Omega^{0}(\mathbb{C}^4;\mathfrak{g})) \otimes \Omega^0(\mathbb{R}_9) \end{aligned}$$

along with the ghost c.

All the square-zero supercharges lie in the same Spin(9)-orbit, so choose a square-zero Q which is still square-zero in 10 dimensions. The twist with respect to Q is then just the dimensional reduction of the 10d twisted theory calculated above. So, the twisted action functional is just calculated by decomposing the 10d twisted action

$$\mathcal{L}^Q = B \wedge (\partial_{A'} A + [\phi, B])\mathrm{d}x^9.$$

Example 1.3 (8d $\mathcal{N}=1$ Super Yang-Mills)**.** There are two ways of dimensionally reducing a second time: we can either reduce in the direction x^9 to obtain another holomorphic twist, or we can reduce in a non-invariant direction to obtain a partially topological twist.

[1] More precisely, if we're doing perturbation theory around a background connection A' we take the covariant derivative with respect to this A'.

1. The latter – the partially topological twist – is acted upon by SU(3). Again, let's decompose our twisted component fields with respect to the action of SU(3), and restrict to fields which are constant in the x^{10} and x^9 directions. The result is

$$\begin{aligned} B^{10} &= B + A_2 + A_3 + \phi_1 \in (\Omega^{0,2}(\mathbb{C}^3;\mathfrak{g}) \oplus \Omega^{0,1}(\mathbb{C}^3;\mathfrak{g})^2 \oplus \Omega^0(\mathbb{C}^3;\mathfrak{g})) \otimes \Omega^0(\mathbb{R}_7 \times \mathbb{R}_8) \\ A^{10}_{0,1} &= A_1 + \phi_2 + \phi_3 \in (\Omega^{0,1}(\mathbb{C}^3;\mathfrak{g}) \oplus \Omega^0(\mathbb{C}^3;\mathfrak{g})^2) \otimes \Omega^0(\mathbb{R}_7 \times \mathbb{R}_8) \end{aligned}$$

again, along with the ghost c (I've skipped writing the $\mathrm{d}\overline{z}^4$ and $\mathrm{d}\overline{z}^5$ factors in here).

As in dimension 9 we choose a square-zero Q of rank $(1,1)$ which is still square-zero in 10 dimensions, and decompose the 10d twisted theory calculated above into these component fields. The twisted action functional looks like

$$\mathcal{L}^Q = B \wedge (\overline{\partial}\phi_1 + [A_1, \phi_1] - [A_3, \phi_2] - [A_2, \phi_3])\mathrm{d}x^7 \wedge \mathrm{d}x^8.$$

2. The former – the holomorphic twist – is essentially equivalent to the theory discussed in dimension 9, but without the $\Omega^0(\mathbb{R}_9)$ factor. It is still acted upon by SU(4). This theory should be a cotangent theory where the base – in terms of BV fields – is spanned by $c, A', B, {}^*A, {}^*\phi$, and the fiber is spanned by $\phi, A, {}^*B, {}^*A', {}^*c$. We can see this by thinking about which fields are bosonic and which are fermionic. This example should, for the first time, be promotable into a $\mathbb{Z}$-graded theory. For this to produce the correct degrees, we need the following R-charges under the 8d $\mathbb{C}^\times$ R-symmetry group:

$$\begin{aligned} B&\colon \text{charge } +1 \\ A&\colon \text{charge } -1 \\ A'&\colon \text{charge } 0 \\ \phi&\colon \text{charge } -2. \end{aligned}$$

In 10-dimensions, A^{10}, and hence $A^{10}_{0,1}$, has R-charge 0, and λ, and hence B^{10}, has R-charge $+1$. So writing $A^{10}_{0,1} = A' + \phi\,\mathrm{d}\overline{z}^5$ and $B^{10} = B + A \wedge \mathrm{d}\overline{z}^5$ we get the correct R-charges if $\mathrm{d}\overline{z}^5$ has R-charge 2.

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