Notes on Pure Spinors

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1 Pure Spinors

Let S be the Dirac spinor representation of $\mathfrak{so}(1,n-1)$, and let V be the complex n-dimensional vector representation. I'll write (v,w) for the inner product on V, $\langle Q,Q'\rangle$ for the invariant nondegenerate $\mathbb{C}$ valued pairing on S, and $\Gamma(Q,Q')$ for the equivariant vector valued pairing on S.

For a spinor Q we define its *null space* T_Q to be

$$T_Q = \{v \in V : \rho(v)Q = 0\} \subseteq V.$$

Note that T_Q is totally isotropic, since if v,w are in T_Q then

$$(v,w)Q = (\rho(v)\rho(w) - \rho(w)\rho(v))Q = 0.$$

Definition 1.1. A spinor Q is called *pure* if the space T_Q is *maximal*, i.e. of dimension $\lfloor \frac{n}{2} \rfloor$. It is called *square zero* if $\Gamma(Q,Q) = 0$

Now, I'd like to investigate the image

$$S_Q = \mathrm{Im}(\Gamma(Q,-)) \subseteq V$$

of the map induced by pairing with a pure or square zero spinor Q.

Proposition 1.2. Let Q be a pure spinor.

- If $n = 2m$ is even, then $S_Q = T_Q$.
- If $n = 2m-1$ is odd, then $T_Q \subseteq S_Q$ is a codimension 1 subspace.

Proof. First observe that $S_Q^\perp = T_Q$. Indeed, let v be a vector in $S_Q^\perp$, i.e. $(v,w) = 0$ for all $w \in S_Q$. That is

$$\begin{aligned} &(v, \Gamma(Q,Q')) = 0 \text{ for all } Q' \in S \\ \Leftrightarrow\ &\langle \rho(v)Q, Q'\rangle = 0 \text{ for all } Q' \in S \\ \Leftrightarrow\ &\rho(v)Q = 0 \end{aligned}$$

So $v \in S_Q^\perp$ if and only if $v \in T^Q$. Furthermore, taking a second orthogonal complement, this means that $S_Q = T_Q^\perp$, and therefore – since T_Q is totally isotropic – $T_Q \subseteq S_Q$. Looking at dimensions, since Q is pure, $\dim T_Q = \lfloor \frac{n}{2} \rfloor$, so $\dim S_Q = \lceil \frac{n}{2} \rceil$. This completes the proof. □

Now, what's the relationship between pure spinors and square zero spinors? To answer this question, we'll use an interesting decomposition. Every spinor Q induces a matrix element $Q \otimes \overline{Q} \in S \otimes S^* \cong \mathrm{End}(S)$ via the diagonal embedding and the invariant pairing on S. If S is irreducible (i.e. for Dirac spinors in odd dimensions or Weyl spinors in even dimensions) we can identify endomorphisms with elements of the Clifford algebra, and thus decompose the spinor bilinear as

$$Q \otimes \overline{Q} = \sum_{p=0}^{n} F_p$$

where $F_p \in \wedge^p V$. We can explicitly identify these elements F_p as, in index notation, the Clifford algebra element $Q_i \gamma_{ij}^{a_1 \cdots a_p} Q_j$. In this language we can characterise pure spinors [Che97].

Theorem 1.3 (Chevalley)**.** A spinor Q is pure if and only if it is Weyl (in even dimensions), and $F_p = 0$ for all $p < n/2$.

In particular, $F_1 = 0$ whenever $n > 2$, so we have

Corollary 1.4. All pure spinors in dimensions $n > 2$ are square zero.

Now, let's investigate pure and square zero spinors in various dimensions.

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References

[Che97] Claude Chevalley. *The Algebraic Theory of Spinors and Clifford Algebras: Collected Works*, volume 2. Springer, 1997.

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