4d $\mathcal{N} = 1$

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0.1 Theories of matter

In this subsection, S_+/S_- denote the positive/negative irreducible spin representations of $\mathfrak{so}(4;\mathbb{C})$. To define the chiral multiplet we fix a complex vector space R. The spinorial pairing $(-,-) : S_\pm \otimes S_\pm \to \mathbb{C}$ induces a pairing between $S_\pm \otimes R$ and $S_\pm \otimes R^*$ that we also denote by $(-,-)_R$.

The field content for the 4-dimensional chiral multiplet with values in a complex vector space R is:

- a pair of scalars $\phi \in C^\infty(\mathbb{R}^4; R)$ and $\overline{\phi} \in C^\infty(\mathbb{R}^4; R^*)$.
- a positive Weyl spinor $\psi_+ \in C^\infty(\mathbb{R}^4; R \otimes S_+)$ and a negative Weyl spinor $\psi_- \in C^\infty(\mathbb{R}^4; R^* \otimes S_-)$.

The space of BRST fields is

$$F = C^\infty(\mathbb{R}^4; R \oplus R^* \oplus R \otimes S_+ \oplus R^* \otimes S_-).$$

The BRST action is

$$S(\phi, \overline{\phi}, \psi_\pm) = \int_{\mathbb{R}^4} -(\mathrm{d}\phi \wedge *\mathrm{d}\overline{\phi})_R + (\psi_+, \not{\partial}\psi_-)_R.$$

0.2 Matter multiplets

In some dimensions $n \leq 6$ there exists the following *matter* representations of supersymmetry.

- Dimension $n = 4$, with $\Sigma = S_+ \otimes R \oplus S_- \otimes R^*$, with R a complex vector space. This is called the $\mathcal{N} = 1$ chiral multiplet.
- Dimension $n = 6$, with $\Sigma = S_+ \oplus W_+$ (or $\Sigma = S_- \otimes W_-$), with W_+ (or W_-) a complex, symplectic representation. This is called the $\mathcal{N} = (1,0)$ (or $\mathcal{N} = (0,1)$) hyper multiplet.

Lemma 0.1. Let $v \in V$, $\psi \in C^\infty(V, \Sigma)$, and $Q \in \Sigma$. Then

$$(v, \mathrm{d}(Q, \psi))_V = (\rho(v)Q, \not{\partial}\psi)_\Sigma.$$

Proof. Let V_α denote the vector field corresponding to α. Note that $\mathrm{d}(Q,\psi) \wedge *\alpha = *(Q, V_\alpha.\psi)$. The result then follows from invariance of $(-,-)$ and the formula $\not{\partial}\psi = \rho(\mathrm{d}x_i)(\partial_i\psi)$. ☐

Lemma 0.2. For $Q_1, Q_2 \in \Sigma$ and $\psi_+ \in S_+$ we have

$$\rho(\mathrm{d}(Q_1, \psi_+))Q_2 = \rho(\Gamma(Q_1, \not{\partial}\psi_+))Q_2$$

(Brian: I'd also like the following to be true.

Proposition 0.1. Suppose $Q_+ \in S_+$, $Q_- \in S_-$, and $\psi_+ \in C^\infty(V, S_+)$. Then

$$\not{\partial}(\rho(\Gamma(Q_+, Q_-))\psi_+) = \rho(\mathrm{d}(Q_+, \psi_+))Q_- + (Q_-, \not{\partial}\psi_+)Q_+.$$

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0.2.1 The chiral multiplet

In this subsection, S_+/S_- denote the positive/negative irreducible spin representations of $\mathfrak{so}(4;\mathbb{C})$.

The action of the odd part of the $4d$ $\mathcal{N} = 1$ supersymmetry algebra on the 4-dimensional matter theory is encoded by the following linear and quadratic functionals

$$I^{(1)}(Q) = \int \langle \phi^*, (Q, \psi_+) \rangle_R + \langle \overline{\phi}^*, (Q, \psi_-) \rangle_R + \langle \psi_+^*, \rho(\mathrm{d}\phi)Q \rangle_R + \langle \psi_-^*, \rho(\mathrm{d}\overline{\phi})Q \rangle_R$$

$$I^{(2)}(Q_1 \otimes Q_2) = \int \frac{1}{2}(\Gamma(Q_1, Q_2), \Gamma(\psi_+^*, \psi_-^*)_R) - ((Q_1, \psi_+^*), (Q_2, \psi_-^*))_R - ((Q_2, \psi_+^*), (Q_1, \psi_-^*))_R$$

for $Q, Q_1, Q_2 \in S_+ \oplus S_-$ A word on the notation used in the definition of $I^{(2)}$. The V-valued pairing $\Gamma(\psi_+^*, \psi_-^*)_R$ denotes the image of $\psi_+^* \otimes \psi_-^*$ under the composition

$$(S_+ \otimes R^*) \otimes (S_- \otimes R) \cong (S_+ \otimes S_-) \otimes (R^* \otimes R) \xrightarrow{\Gamma \otimes (-,-)_R} V.$$

Additionally, we view (Q_1, ψ_+^*) as a scalar valued in R^* and (Q_2, ψ_-^*) as a scalar valued in R. The pairing $((Q_1, \psi_+^*), (Q_2, \psi_-^*))_R$ is the obvious one between R and R^*.

Theorem 0.3. The functional $\mathfrak{S} = S + I^{(1)} + I^{(2)}$ satisfies the classical master equation

$$\mathrm{d}_{\mathrm{Lie}}\mathfrak{S} + \frac{1}{2}\{\mathfrak{S}, \mathfrak{S}\} = 0. \tag{1}$$

Before proving the theorem, we decompose the classical master equation (1) into the following equations:

$$\begin{array}{rcl} \{S, I^{(1)}\} & = & 0 \\ \{S, I^{(2)}\} + \mathrm{d}_{CE} I^{(1)} + \frac{1}{2}\{I^{(1)}, I^{(1)}\} & = & 0 \\ \mathrm{d}_{CE} I^{(2)} + \{I^{(1)}, I^{(2)}\} & = & 0 \\ \{I^{(2)}, I^{(2)}\} & = & 0 \end{array} \tag{2}$$

The last equation is automatically satisfied since $I^{(2)}$ is independent of $\phi, \overline{\phi}$, and $\psi_\pm$.

The first equation in (2) states that the classical action for the chiral multiplet is supersymmetric.

Lemma 0.2. One has $\{S, I^{(1)}\}(Q) = 0$ for all $Q \in S_+ + S_-$.

Proof. The BV bracket involving terms in S depending on $\phi, \overline{\phi}$ is:

$$-\left\{(\mathrm{d}\phi, \mathrm{d}\overline{\phi}) \ , \ I^{(1)}\right\}(Q) = (\mathrm{d}(Q, \psi_+), \mathrm{d}\overline{\phi}) + (\mathrm{d}\phi, \mathrm{d}(Q_-, \psi_-))$$

The BV bracket involving terms in S depending on $\psi_\pm$ is:

$$\begin{aligned} \left\{(\psi_+, \not{\partial}\psi_-) \ , \ I^{(1)}\right\}(Q) &= (\rho(\mathrm{d}\phi)Q, \not{\partial}\psi_-) + (\psi_+, \not{\partial}(\rho(\mathrm{d}\overline{\phi})Q)) \\ &= (\rho(\mathrm{d}\phi)Q, \not{\partial}\psi_-) - (\not{\partial}\psi_+, (\rho(\mathrm{d}\overline{\phi})Q) \\ &= (\mathrm{d}\phi, \Gamma(\not{\partial}\psi_-, Q)) - (\mathrm{d}\overline{\phi}, \Gamma(\not{\partial}\psi_+, Q)) \end{aligned}$$

Adding the two terms up, we see that $\{S, I^{(1)}\}(Q) = 0$ by Lemma 0.1 as desired. $\square$

Next, we move on to the second equation in (2).

Lemma 0.3. One has

$$\{S, I^{(2)}\} + \mathrm{d}_{CE} I^{(1)} + \frac{1}{2}\{I^{(1)}, I^{(1)}\} = 0. \tag{3}$$

Proof. Evaluating the equation (3) on $v_1, v_2 \mathfrak{iso}(V)$ reduces to the claim that (??) defines a strict Lie action. Evaluating on $v \in \mathfrak{iso}(V)$ and $Q \in S_+ \oplus S_-$, the claim reduces to the fact that $I^{(1)}$ is Poincaré invariant. So, the only nontrivial term to check is the evaluation on $Q_1, Q_2 \in S_+ \oplus S_-$.

The individual terms are:

$$\begin{aligned}\{I^{(1)}, I^{(1)}\}(Q_1, Q_2) = & - \phi^*(Q_1, \rho(\mathrm{d}\phi)Q_2) - \phi^*(Q_2, \rho(\mathrm{d}\phi)Q_1) \\ & - \overline{\phi}^*(Q_1, \rho(\mathrm{d}\overline{\phi})Q_2) - \overline{\phi}^*(Q_2, \rho(\mathrm{d}\overline{\phi})Q_1) \\ & - (\psi_+^*, \rho(\mathrm{d}(Q_1, \psi_+))Q_2) - (\psi_+^*, \rho(\mathrm{d}(Q_2, \psi_+))Q_1) \\ & - (\psi_-^*, \rho(\mathrm{d}(Q_1, \psi_-))Q_2) - (\psi_-^*, \rho(\mathrm{d}(Q_2, \psi_-))Q_1)\end{aligned}$$

$$\begin{aligned}(\mathrm{d}_{CE} I^{(1)})(Q_1, Q_2) = & - \phi^* L_{\Gamma(Q_1,Q_2)}(\phi) - \overline{\phi}^* L_{\Gamma(Q_1,Q_2)}\overline{\phi} \\ & - (\psi_+^*, \Gamma(Q_1, Q_2).\psi_+) - (\psi_-^*, \Gamma(Q_1, Q_2).\psi_-)\end{aligned}$$

and

$$\begin{aligned}\{S, I^{(2)}(Q_1, Q_2)\} = \left\{(\psi_+, \partial\!\!\!/\psi_-), I^{(2)}\right\} = & \Gamma(Q_1, Q_2)\Gamma(\psi_+^*, \partial\!\!\!/\psi_+) - (Q_1, \psi_+^*)(Q_2, \partial\!\!\!/\psi_+) - (Q_2, \psi_+^*)(Q_1, \partial\!\!\!/\psi_+) \\ & + \Gamma(Q_1, Q_2)\Gamma(\psi_-^*, \partial\!\!\!/\psi_-) - (Q_1, \psi_-^*)(Q_2, \partial\!\!\!/\psi_-) - (Q_2, \psi_-^*)(Q_1, \partial\!\!\!/\psi_-) \\ = & (\psi_+^*, \rho(\Gamma(Q_1, Q_2))\partial\!\!\!/\psi_+) - (Q_1, \psi_+^*)(Q_2, \partial\!\!\!/\psi_+) - (Q_2, \psi_+^*)(Q_1, \partial\!\!\!/\psi_+) \\ & + (\psi_-^*, \rho(\Gamma(Q_1, Q_2))\partial\!\!\!/\psi_-) - (Q_1, \psi_-^*)(Q_2, \partial\!\!\!/\psi_-) - (Q_2, \psi_-^*)(Q_1, \partial\!\!\!/\psi_-).\end{aligned}$$

We first collect all terms in Equation (3) proportional to ϕ^*:

$$-\frac{1}{2}(Q_1, \rho(\mathrm{d}\phi)Q_2) - \frac{1}{2}(Q_2, \rho(\mathrm{d}\phi)Q_1) - L_{\Gamma(Q_1,Q_2)}\phi.$$

By the Clifford identity (Brian: $v \wedge \Gamma(Q_1, Q_2) = (Q_1, \rho(v)Q_2)$) we observe that the first two terms cancel with the third term. Similarly, all terms proportional to $\overline{\phi}^*$ also vanish.

Next, we collect all terms in Equation (3) proportional to ψ_+^*:

$$\begin{aligned}& -\frac{1}{2}\rho(\mathrm{d}(Q_1, \psi_+))Q_2 - \frac{1}{2}\rho(\mathrm{d}(Q_2, \psi_+))Q_1 - \Gamma(Q_1, Q_2).\psi_+ \\ & \pm \frac{1}{2}\rho(\Gamma(Q_1, Q_2))\partial\!\!\!/\psi_+ \mp \frac{1}{2}(Q_2, \partial\!\!\!/\psi_+)Q_1 \mp \frac{1}{2}(Q_1, \partial\!\!\!/\psi_+)Q_2\end{aligned}$$

By Lemma 0.2 the first, second terms, and fourth terms are equal to

$$-\frac{1}{2}\rho(\Gamma(Q_1, \partial\!\!\!/\psi_+))Q_2 - \frac{1}{2}\rho(\Gamma(Q_2, \partial\!\!\!/\psi_+))Q_1 + \frac{1}{2}\rho(\Gamma(Q_1, Q_2))\partial\!\!\!/\psi_+$$

This is zero by Proposition (Brian: 3-psi).

The remaining terms are

$$-\Gamma(Q_1, Q_2).\psi_+ + \frac{1}{2}(Q_2, \partial\!\!\!/\psi_+)Q_1 + \frac{1}{2}(Q_1, \partial\!\!\!/\psi_+)Q_2.$$

$\square$

Lemma 0.4.

$$\{I^{(1)}, I^{(2)}\}(Q_1, Q_2, Q_3) = 0$$

for every $Q_1, Q_2, Q_3 \in S_+ \oplus S_-$.

Proof. We have

$$\{I^{(1)}(Q_1), I^{(2)}(Q_2, Q_3)\} =$$

$\{I^{(1)}, I^{(2)}\}$ is obtained by cyclically symmetrizing the above expression. By Proposition **??** the cyclic symmetrization of the term with c^* is zero. The Clifford relation implies that □

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The 4-dimensional $\mathcal{N} = 1$ pure super Yang-Mills theory has BRST fields given by a ghost c, a 4d gauge field A, and a pair of opposite chirality Lie algebra valued Weyl spinor fields $\lambda_\pm$: In addition, 4-dimensional supersymmetry supports a chiral (or anti-chiral) matter multiplet, which comprises a complex scalar ϕ and a left (right) Weyl spinor ψ_+ (ψ_-).

The most general theory 4-dimensional $\mathcal{N} = 1$ theory we will consider is super Yang-Mills theory valued in a Lie algebra $\mathfrak{g}$ minimally coupled to a chiral multiplet with values in a complex $\mathfrak{g}$-representation R. The field content for the BRST fields in the gauge sector is:

$$(c, A, \lambda_\pm) \in (\Omega^0(\mathbb{R}^4) \oplus \Omega^1(\mathbb{R}^4) \oplus \Omega^0(\mathbb{R}^4) \otimes \Pi S_\pm) \otimes \mathfrak{g}.$$

For the chiral multiplet, the fields ϕ, ψ_+ take values in R and $\overline{\phi}, \psi_-$ take values in R^*:

$$(\phi, \psi_+) \in \Omega^0(\mathbb{R}^4) \otimes (R \oplus R \otimes \Pi S_+) \quad , \quad (\overline{\phi}, \psi_-) \in \Omega^0(\mathbb{R}^4) \otimes (R^* \oplus R^* \otimes \Pi S_-).$$

Th full BV action for $\mathcal{N} = 1$ supersymmetric pure Yang-Mills theory on $\mathbb{R}^4$ was defined in Section **??**, which we call $S_{\text{BV,gauge}}$ in this section.

The kinetic part of the action functional for 4-dimensional $\mathcal{N} = 1$ chiral multiplet is of the form

$$S_{\text{matter}}(\phi, \overline{\phi}, \psi_\pm) = \int d^4x \, \langle d\phi, *d\overline{\phi} \rangle_R + \int d^4x \, \langle \psi_+, \partial\!\!\!/ \psi_- \rangle_R.$$

Here $\langle -, - \rangle_R$ denotes the duality pairing between the representation R and its dual R^*.

There is also a term S_{couple} describing the coupling between Yang-Mills and the matter fields, which we record here:

$$\begin{aligned} S_{\text{couple}} = &\pm g \int \langle \psi_+, [\lambda_+, \phi] \rangle_R \pm g \int \langle \psi_-, [\lambda_-, \overline{\phi}] \rangle_{R^*} \\ &\pm g \int [A, \phi] \wedge *d_A \overline{\phi} \pm g \int [A, \overline{\phi}] \wedge *d_A \phi \end{aligned}$$

(Brian: add terms involving antifields) (Brian: There is a term of the form $\pm g^2 \int \langle (\phi\overline{\phi}), *(\phi\overline{\phi}) \rangle_R$ that is usually written down. This indeed changes the EOM, and is needed to compensate an extra term in the definition of $I^{(1)}_{\text{gauge}}$ when we couple to matter. My claim is that if we disregard both of these terms we will get a consistent supersymmetric system still.)

The action

$$S_{\mathrm{BV}} = S_{BV}(\phi, \overline{\phi}, \psi_\pm, A, \lambda_\pm, \text{ a.f's}) = S_{\mathrm{BV,gauge}}(A, \lambda_\pm, \text{ a.f's}) + S_{\mathrm{matter}}(\phi, \overline{\phi}, \psi_\pm) + S_{\mathrm{couple}}(\phi, \overline{\phi}, \psi_\pm, A, \lambda_\pm, \text{ a.f's})$$

is the full BV action of $\mathcal{N} = 1$ super Yang-Mills coupled to matter.

The full BV action S_{BV} is clearly Poincaré invariant, so there is a functional I_{Poin} encoding the action by the 4-dimensional Poincaré algebra.

The action of supersymmetry on the 4-dimensional matter theory is encoded by a linear and quadratic functional:

$$\begin{aligned} I^{(1)}_{\mathrm{matter}}(Q_+ + Q_-) &= \int \langle \phi^*, (Q_+, \psi_+)\rangle_R + \int \langle \overline{\phi}^*, (Q_-, \psi_-)\rangle_R + \int \langle \psi_+^*, \rho(\mathrm{d}\phi) Q_-\rangle_R + \int \langle \psi_-^*, \rho(\mathrm{d}\overline{\phi}) Q_+\rangle_R \\ I^{(2)}_{\mathrm{matter}}(Q_+ \otimes Q_-) &= \int \langle \psi_-^*, \rho(\Gamma(Q_+, Q_-))\psi_+^*\rangle_R + (Brian: writethisdifferently?). \end{aligned}$$

where $Q_\pm \in S_\pm$. As usual, we will use the notation δ_Q to denote the endomorphism on fields satisfying

$$I^{(1)}_{\mathrm{matter}} = \int \langle \phi^* + \overline{\phi}^* + \psi_+^* + \psi_-^*, \delta_Q(\phi + \overline{\phi} + \psi_+ + \psi_-)\rangle.$$

The action of supersymmetry on the gauge sector is encoded by the functionals:

$$\begin{aligned} I^{(1)}_{\mathrm{gauge}}(Q_+ + Q_-) &= \int \langle A^*, \Gamma(Q_+ + Q_-, \lambda_+ + \lambda_-)\rangle + \langle \lambda_+^* + \lambda_-^*, \not{F}_A(Q_+ + Q_-)\rangle \\ I^{(2)}_{\mathrm{gauge}}(Q_+ \otimes Q_-) &= \int \left\langle \lambda^*, \rho(\Gamma(Q_+, Q_-))\lambda^* + \frac{1}{2}((Q_+, \lambda_+^* + \lambda_-^*)Q_- + (Q_-, \lambda_+^* + \lambda_-^*)Q_+)\right\rangle. \end{aligned}$$

(Brian: C and P claim there should be some extra terms in $I^{(2)}_{\mathrm{gauge}}$ coming from reduction. If a few of those terms are of the form $\langle \lambda_-^*, \lambda_-^*\rangle\langle Q_-, Q_-\rangle$ and $\langle \lambda_+^*, \lambda_+^*\rangle\langle Q_+, Q_+\rangle$ we'd be in business. The issue is that in the twist calculation below has an extra copy of a complex involving components of λ_-^* in degree zero and λ_- in degree $+1$. Such terms would turn this acyclic. Can we see why such terms are *necessary* to preserve off-shell SUSY?)

These functionals prescribe an off-shell action of 4-dimensional $\mathcal{N} = 1$ super Yang-Mills coupled to matter, as we summarize in the following proposition.

Proposition 1.1 ([SWchar]). Let $I^{(1)} = I^{(1)}_{\mathrm{matter}} + I^{(1)}_{\mathrm{gauge}}$ and $I^{(2)} = I^{(2)}_{\mathrm{matter}} + I^{(2)}_{\mathrm{gauge}}$. Then, the functional

$$\mathfrak{S} = S_{\mathrm{BV}} + I_{\mathrm{Poin}} + I^{(1)} + I^{(2)} \in \mathrm{C}^\bullet_{\mathrm{Lie}}(\mathfrak{A}) \otimes \mathrm{C}^\bullet_{\mathrm{loc}}(\mathfrak{L})[-1]$$

satisfies the Maurer-Cartan equation

$$\left(\mathrm{d}_{\mathrm{Lie}}\mathfrak{S} + \frac{1}{2}\{\mathfrak{S}, \mathfrak{S}\}\right) = 0.$$

Notice that we have not introduced any auxiliary fields, which is at the cost of us formulating the action of supersymmetry as an L_∞ action by the super Lie algebra of supertranslations.

Proof. Throughout this proof we drop the pairing $\langle -, -\rangle_R$ from the notation. Also, we continue to denote the spinorial pairing by $\langle -, -\rangle$ and the standard inner product by $(-, -)$.

First, consider the pure matter sector $\mathfrak{S}_{\mathrm{matter}} = S_{\mathrm{matter}} + I_{\mathrm{Poin}} + I^{(1)}_{\mathrm{matter}} + I^{(2)}_{\mathrm{matter}}$. We will show that $\mathfrak{S}_{\mathrm{matter}}$ satisfies the classical master equation. From the form of S_{matter} and by Poincaré invariance of the matter action, it is clear that

$$\mathrm{d}_{\mathrm{Lie}} S_{\mathrm{matter}} + \mathrm{d}_{\mathrm{Lie}}(I^{(1)}_{\mathrm{matter}} + I^{(2)}_{\mathrm{matter}}) + \frac{1}{2}\{S_{\mathrm{matter}} + I_{\mathrm{Poin}}, S_{\mathrm{matter}} + I_{\mathrm{Poin}}\} = 0.$$

So, we only need to consider the terms involving $\{S_{\text{matter}}, I^{(1)}_{\text{matter}}\}$, $\{S_{\text{matter}}, I^{(2)}_{\text{matter}}\}$, $\mathrm{d}_{\text{Lie}} I_{\text{Poin}}$, and $\{I^{(1)}_{\text{matter}}, I^{(1)}_{\text{matter}}\}$.

The equality $\{S_{\text{matter}}, I^{(1)}_{\text{matter}}\} = 0$ simply says that S_{matter} is invariant under the usual linear supersymmetric action. The argument for this is completely standard, see for instance (Brian: refs), but we repeat it here for completeness. The BV bracket involving terms in S_{matter} depending on $\phi, \overline{\phi}$ is:

$$\begin{aligned}\frac{1}{2}\left\{\int \mathrm{d}\phi \wedge *\mathrm{d}\overline{\phi}\,,\ I^{(1)}_{\text{matter}}\right\} &= \int \mathrm{d}(\langle Q_+, \psi_+\rangle) \wedge *\mathrm{d}\overline{\phi} + \int \mathrm{d}\phi \wedge *\mathrm{d}(\langle Q_-, \psi_-\rangle) \\ &= \pm \int (\Gamma(\partial\!\!\!/ \psi_+, Q_+), \mathrm{d}\overline{\phi}) \pm \int (\mathrm{d}\phi, \Gamma(\partial\!\!\!/ \psi_-, Q_-)).\end{aligned}$$

The BV bracket involving terms in S_{matter} depending on $\psi_\pm$ is:

$$\begin{aligned}\frac{1}{2}\left\{\int \langle \psi_+, \partial\!\!\!/ \psi_-\rangle\,,\ I^{(1)}_{\text{matter}}\right\} &= \int \langle \rho(\mathrm{d}\phi) Q_-, \partial\!\!\!/ \psi_-\rangle = \int \langle \psi_+, \partial\!\!\!/ (\rho(\mathrm{d}\overline{\phi}) Q_+)\rangle \\ &= \int \langle \rho(\mathrm{d}\phi) Q_-, \partial\!\!\!/ \psi_-\rangle = \int \langle \partial\!\!\!/ \psi_+, (\rho(\mathrm{d}\overline{\phi}) Q_+)\rangle \\ &= \int (\mathrm{d}\phi, \Gamma(\partial\!\!\!/ \psi_-, Q_-)) + \int (\mathrm{d}\overline{\phi}, \Gamma(\partial\!\!\!/ \psi_+, Q_+))\end{aligned}$$

(Brian: Get the signs right and these terms should cancel.)

The computation of the remaining terms in the matter sector are summarized in the following lemma.

Lemma 1.1.

$$\mathrm{d}_{CE} I_{\text{Poin}} + \{S_{\text{matter}}, I^{(2)}_{\text{matter}}\} + \frac{1}{2}\{I^{(1)}_{\text{matter}}, I^{(1)}_{\text{matter}}\} = 0.$$

(Brian: extra terms in $I^{(2)}$?)

Proof. First, we compute the BV bracket $\{I^{(1)}_{\text{matter}}, I^{(1)}_{\text{matter}}\}$:

$$\{I^{(1)}_{\text{matter}}, I^{(1)}_{\text{matter}}\}(Q_+ + Q_-, Q'_+ + Q'_-) = 2\int \left\langle \phi^* + \overline{\phi}^* + \psi^*_+ + \psi^*_-, [\delta_{Q_+ + Q_-}, \delta_{Q'_+ + Q'_-}]\,(\phi + \overline{\phi} + \psi_+ + \psi_-)\right\rangle.$$

We first focus on terms involving variations of $\phi, \overline{\phi}$:

$$\begin{aligned}[\delta_{Q_+ + Q_-}, \delta_{Q'_+ + Q'_-}](\phi + \overline{\phi}) &= (\delta_{Q_+} + \delta_{Q_-})\left(\langle Q'_+, \psi_+\rangle_+ + \langle Q'_-, \psi_-\rangle_-\right) - (\delta_{Q'_+} + \delta_{Q'_-})\left(\langle Q_+, \psi_+\rangle_+ + \langle Q_-, \psi_-\rangle_-\right) \\ &= \langle Q'_+, \rho(\mathrm{d}\phi) Q_-\rangle_+ + \langle Q'_-, \rho(\mathrm{d}\overline{\phi}) Q_+\rangle_- - \langle Q_+, \rho(\mathrm{d}\phi) Q'_-\rangle_+ - \langle Q_-, \rho(\mathrm{d}\overline{\phi}) Q'_+\rangle_- \\ &= (\mathrm{d}(\phi + \overline{\phi}), \Gamma(Q_+ + Q_-, Q'_+ + Q'_-)) \\ &= L_{[Q_+ + Q_-, Q'_+ + Q'_-]}(\phi + \overline{\phi})\end{aligned}$$

In the third line we have used the identity $(v, \Gamma(Q_1, Q_2)) = \langle \rho(v) Q_1, Q_2\rangle$ where $Q_i \in S$, $v \in V$, $\langle -, -\rangle$ is the spinor pairing, and $(-,-)$ is the inner product on V. The third line follows from the relation $(\mathrm{d}\phi, v) = L_v(\phi)$ where L_v is the Lie derivative.

Let $I_{\text{Poin}}(\phi, \overline{\phi})$ be the piece of I_{Poin} depending on $\phi, \overline{\phi}$ and their antifields. The last line above is simply the Lie derivative with respect to the translation invariant vector field $[Q_+ + Q_-, Q'_+ + Q'_-]$ on the field $\phi + \overline{\phi}$, which is precisely the symmetry encoded by I_{Poin}. Thus, this calculation implies

$$\left(\mathrm{d}_{CE} I_{\text{Poin}}(\phi, \overline{\phi}) + \frac{1}{2}\left\{I^{(1)}_{\text{matter}}(\phi, \overline{\phi}), I^{(1)}_{\text{matter}}(\phi, \overline{\phi})\right\}\right)(Q_+ + Q_-, Q'_+ + Q'_-) = 0$$

for all $Q_\pm, Q'_\pm$.

Next, we focus on the terms in the statement of the lemma involving the functionals which on the fields $\psi_\pm$: $I_{\text{Poin}}(\psi_\pm)$, $I^{(1)}_{\text{matter}}(\psi_\pm)$, and $I^{(2)}_{\text{matter}}$. We must show

$$\left(\mathrm{d}_{CE} I_{\text{Poin}}(\psi_\pm) + \{S_{\text{matter}}, I^{(2)}_{\text{matter}}\} + \frac{1}{2}\{I^{(1)}_{\text{matter}}(\psi_\pm), I^{(1)}_{\text{matter}}(\psi_\pm)\}\right)(Q_+, Q_-, Q'_+ + Q'_-) = 0.$$

for all $Q_\pm, Q'_\pm$.

We start by computing the variation of $\psi_+ + \psi_-$:

$$\begin{aligned}[\delta_{Q_+ + Q_-}, \delta_{Q'_+ + Q'_-}](\psi_+ + \psi_-) &= (\textit{Brian : Fierz}/3\psi) \\ &= \delta_{[Q_+ + Q_-, Q'_+ + Q'_-]}(\psi_+ + \psi_-) \pm \rho(\Gamma(Q_+ + Q_-, Q'_+ + Q'_-)) \cdot \partial\!\!\!/(\psi_+ + \psi_-).\end{aligned}$$

□

Next, we move on to the pure gauge sector $\mathfrak{S}_{\text{gauge}} = S_{\text{BV,gauge}} + I_{\text{Poin}} + I^{(1)}_{\text{gauge}} + I^{(2)}_{\text{gauge}}$. □