CLASSIFICATION OF MASSLESS SUPERMULTIPLETS

1.

Let V be a complex vector space of dimension d with a chosen spin structure. Let Σ be a spinorial representation of $\mathrm{Spin}(V)$, $\Gamma\colon \mathrm{Sym}^2(\Sigma)\to V$ the associated vector-valued pairing, and T_Σ the supertranslation group with Lie algebra $\Pi\Sigma\oplus V$. The superPoincaré group is the semi-direct product $\mathrm{Spin}(V)\ltimes T_\Sigma$.

Recall that massless representations of the Poincaré group are parametrized in the following way. Split $V = V_0\oplus \mathbf{C}e_+\oplus\mathbf{C}e_-$, where e_+ and e_- are null vectors satisfying $(e_+, e_-) = 1$. Then massless representations correspond to finite-dimensional representations of $\mathrm{Spin}(V_0)$.

Remark 1.1. We ignore the continuous-spin representations. Alternatively, we only consider finite-dimensional representations of the little group.

We label irreducible representations of $\mathrm{Spin}(2)$ by ***helicity*** which is a half-integer number.

Definition 1.2. Suppose $\dim(V_0)\geq 2$. A representation ρ of $\mathrm{Spin}(V_0)$ is ***admissible*** if for some (equivalently, any) nondegenerate subspace $\mathbf{C}^2\subset V_0$ the helicities appearing in ρ are at most 1 by absolute value.

In a similar way, massless representations of the superPoincaré group are parametrized as follows (see [Freed, Lecture 6] for details). Consider a quadratic form on Σ defined by $q(Q,Q) = (e_+, \Gamma(Q,Q))$ for $Q\in\Sigma$. Let Σ_0 be the radical of q and $\overline{\Sigma} = \Sigma/\Sigma_0$. Let $\mathrm{Cl}(\overline{\Sigma}, q)$ be the associated Clifford algebra. Then massless representations correspond to finite-dimensional representations of $\mathrm{Cl}(\overline{\Sigma}, q)$ with a compatible action of $\mathrm{Spin}(V_0)$.

1.1. $d = 3$.
We have $\mathrm{Spin}(1)\cong \mathbf{Z}/2$. We denote its irreducible representations by $\mathbf{C}$ and S which correspond to the massless scalar and massless spinor.

The spinorial representation is

$$\Sigma = (S\oplus S)\otimes W,$$

where W carries a nondegenerate symmetric bilinear form. We have

$$\overline{\Sigma} = S\otimes W.$$

There is a unique supermultiplet which contains $\dim(W)$ scalars and $\dim(W)$ spinors.

1.2. $d = 4$.
The admissible irreducible representations of $\mathrm{Spin}(2)$ are the trivial one-dimensional representation $\mathbf{C}$, the semi-spin representations $S_\pm$ of helicity $\pm\frac{1}{2}$ and the half-vector representations $V_\pm$ of helicity ± 1.

The spinorial representation is

$$\Sigma = (S_+\oplus S_-)\otimes W\oplus (S_+\oplus S_-)\otimes W^*$$

for a complex vector space W.

The radical is $\Sigma_0 = S_+ \otimes W \oplus S_- \otimes W^*$, so

$$\overline{\Sigma} = S_- \otimes W \oplus S_+ \otimes W^*$$

with q the obvious quadratic form. In particular, it contains a Lagrangian subspace $S_+ \otimes W^*$, so irreducible massless representations of the superPoincaré group are parametrized by irreducible Spin(2)-representations $\mathbf{C}_h$ labeled by a half-integer h, so that the corresponding representation of the little group is

$$\mathbf{C}_h \otimes \wedge^\bullet(S_+ \otimes W^*).$$

This representation is admissible if $-1 \leq h \leq 1$ and $h + \mathcal{N}/2 \leq 1$. In particular, we must have $\mathcal{N} \leq 4$.

In this dimension we have to make sure our representations are CPT invariant. We will only deal with PT invariance. This corresponds to extending Spin(2) to $\mathbf{Z}/2 \ltimes \mathrm{Spin}(2)$, where $\mathbf{Z}/2$ acts on Spin(2) by inversion.

The list of admissible supermultiplets:

- $\mathcal{N} = 1$. Chiral multiplet $\mathbf{C}^{\oplus 2} \oplus S_+ \oplus S_-$ and the $\mathcal{N} = 1$ vector multiplet $S_+ \oplus S_- \oplus V_+ \oplus V_-$.
- $\mathcal{N} = 2$. Half-hypermultiplet (aka $\mathcal{N} = 1$ chiral multiplet) and an $\mathcal{N} = 2$ vector multiplet $\mathbf{C}^{\oplus 2} \oplus (S_+ \oplus S_-)^{\oplus 2} \oplus V_+ \oplus V_-$.
- $\mathcal{N} = 4, \mathcal{N} = 3$. $\mathcal{N} = 4$ vector multiplet $\mathbf{C}^{\oplus 6} \oplus (S_+ \oplus S_-)^{\oplus 4} \oplus V_+ \oplus V_-$.

1.3. $d = 5$. The admissible irreducible representations of Spin(3) are the trivial one-dimensional representation $\mathbf{C}$, the spin representation S and the vector representation V_3.

The spinorial representation is

$$\Sigma = (S \oplus S) \otimes W,$$

where W is a complex symplectic vector space. The radical is

$$\Sigma_0 = S \otimes W$$

(say, the first summand), so

$$\overline{\Sigma} = S \otimes W.$$

The quadratic form is given by the product of the symplectic structure on W and the symplectic structure on S. Choose a Lagrangian subspace $L \subset W$. Then the masless irreducible representations of the superPoincaré group are parametrized by irreducible representations M of Spin(3) so that the corresponding representation of the little group is

$$M \otimes \wedge^\bullet(S \otimes L).$$

The list of admissible supermultiplets:

- $\mathcal{N} = 1$. The chiral multiplet (aka half-hypermultiplet) $\mathbf{C}^{\oplus 2} \oplus S$ and the $\mathcal{N} = 1$ vector multiplet $\mathbf{C} \oplus S^{\oplus 2} \oplus V_3$.
- $\mathcal{N} = 2$. $\mathcal{N} = 2$ vector multiplet $\mathbf{C}^{\oplus 5} \oplus S^{\oplus 4} \oplus V_3$.

1.4. $d = 6$. The admissible irreducible representations of Spin(4) are the trivial one-dimensional representation $\mathbf{C}$, the semi-spin representations $S_\pm$, the vector representation V_4 and the (anti) self-dual form representations $\mathrm{Sym}^2(S_\pm)$. Note that we have a decomposition

$$\wedge^2 V_4 \cong \mathrm{Sym}^2(S_+) \oplus \mathrm{Sym}^2(S_-).$$

The spinorial representation is

$$\Sigma = (S_+ \oplus S_-) \otimes W_+ \oplus (S_+ \oplus S_-) \otimes W_-,$$

where $W_\pm$ are symplectic vector spaces.

We have

$$\overline{\Sigma} = S_- \otimes W_+ \oplus S_+ \otimes W_-$$

with the quadratic form being the product of the symplectic form on S_-, S_+ and $W_\pm$. Pick Lagrangians $L_\pm \subset W_\pm$. Then the massless irreducile reprsentations of the superPoincaré group are parametrized by an irreducible Spin(4)-representation M, so that the corresponding representation of the little group is

$$M \otimes \wedge^\bullet (S_- \otimes L_+ \oplus S_+ \otimes L_-).$$

The list of admissible supermultiplets:

- $\mathcal{N} = (1,0)$. The chiral multiplet (aka half-hypermultiplet) $\mathbf{C}^2 \oplus S_-$, the tensor multiplet $\mathbf{C} \oplus S_-^{\oplus 2} \oplus \mathrm{Sym}^2(S_-)$ and the $\mathcal{N} = (1,0)$ vector multiplet $S_+^{\oplus 2} \oplus V_4$.
- $\mathcal{N} = (1,1)$. The $\mathcal{N} = (1,1)$ vector multiplet $\mathbf{C}^{\oplus 4} \oplus (S_+ \oplus S_-)^{\oplus 2} \oplus V_4$.
- $\mathcal{N} = (2,0)$. The $\mathcal{N} = (2,0)$ tensor multiplet $\mathbf{C}^{\oplus 5} \oplus S_-^{\oplus 4} \oplus \mathrm{Sym}^2(S_-)$.

1.5. $d = 7$. The only irreducible admissible representations of Spin(5) are the trivial one-dimensional representation $\mathbf{C}$, the 4d spin representation S and the 5d vector representation V_5.

The spinorial representation is

$$\Sigma = (S \oplus S) \otimes W,$$

where W is a symplectic vector space. We have

$$\overline{\Sigma} = S \otimes W$$

with the quadratic form the product of the symplectic structure on S and the symplectic structure on W. Pick a Lagrangian subspace $L \subset W$. Then the irreducible representation of the little group is

$$M \otimes \wedge^\bullet (S \otimes L),$$

where M is an irreducible representation of Spin(5).

The list of admissible supermultiplets:

- $\mathcal{N} = 1$. The vector multiplet $\mathbf{C}^{\oplus 3} \oplus S^{\oplus 2} \oplus V_5$.

1.6. $d = 8$. The only irreducible admissible representations of Spin(6) are the trivial one-dmensional representation $\mathbf{C}$, the 4d semi-spin representations $S_\pm$ and the 6d vector representation V_6. The spinorial representation is

$$\Sigma = (S_+ \oplus S_-) \otimes (W \oplus W^*)$$

for a complex vector space W. We have

$$\overline{\Sigma} = S_+ \otimes W \oplus S_- \otimes W^*$$

with the quadratic form given by a pairing $S_+ \otimes S_- \to \mathbf{C}$. Then the irreducible representation of the little group is

$$M \otimes \wedge^\bullet(S_+ \otimes W),$$

where M is an irreducible representation of Spin(6).

The list of admissible supermultiplets:

- $\mathcal{N} = 1$. The vector multiplet $\mathbf{C}^{\oplus 2} \oplus S_+ \oplus S_- \oplus V_6$.

1.7. $d = 9$. The only admissible irreducible representations of Spin(7) are the trivial one-dimensional representation $\mathbf{C}$, the 8d spin representation S and the 7d vector representation V_7. The spinorial representation is

$$\Sigma = (S \oplus S) \otimes W,$$

where W carries a nondegenerate symmetric bilinear form. We have

$$\overline{\Sigma} = S \otimes W.$$

Note that in the case $\dim(W)$ odd there is no Spin(7)-invariant Lagrangian subspace in $\overline{\Sigma}$.

The list of admissible supermultiplets:

- $\mathcal{N} = 1$. The vector multiplet $\mathbf{C} \oplus S \oplus V$.

1.8. $d = 10$. The only admissible irreducible representations of Spin(8) are the trivial one-dimensional representation $\mathbf{C}$, the semi-spin representations $S_\pm$ and the 8d vector representation V_8. The spinorial representation is

$$\Sigma = (S_+ \oplus S_-) \otimes W_+ \oplus (S_+ \oplus S_-) \otimes W_-,$$

where $W_\pm$ carry nondegenerate symmetric bilinear forms. We have

$$\overline{\Sigma} = S_- \otimes W_+ \oplus S_+ \otimes W_-.$$

The list of admissible supermultiplets:

- $\mathcal{N} = (1, 0)$. The vector multiplet $S_- \oplus V$.

References

[Freed] D. Freed, Classical field theory and supersymmetry, https://web.ma.utexas.edu/users/dafr/pcmi.pdf.