A Catalogue of Twists of Supersymmetric Gauge Theories

Chris Elliott Pavel Safronov Brian Williams

January 22, 2020

Abstract

We give a complete classification of supersymmetric twists of super Yang-Mills theories with matter in all dimensions. Super Yang-Mills theories can be modelled classically using the BV formalism; we construct the supersymmetry algebra action using the language of L_∞ algebras, then for each class of square-zero supercharge we give a description of the corresponding twisted theory in terms of partially holomorphic versions of Chern-Simons and BF theory.

Contents

Introduction

d	$\mathcal{N}$	Twist	Description	Invariant Directions
10	$(1,0)$	Rank $(1,0)$	Holomorphic Chern-Simons Theory $\mathrm{Bun}_G(\mathbb{C}^5)$	5 (holomorphic)
9	1	Rank 1	Mixed Chern-Simons Theory $\mathrm{Bun}_G(\mathbb{C}^4 \times \mathbb{R}_{\mathrm{dR}})$	5 (minimal)
8	1	Rank $(1,0)$ Pure	Holomorphic BF Theory $T^*[-1]\,\mathrm{Bun}_G(\mathbb{C}^4, BG)$	4 (holomorphic)
		Rank $(1,1)$	Mixed Chern-Simons Theory $\mathrm{Bun}_G(\mathbb{C}^3 \times \mathbb{R}^2_{\mathrm{dR}})$	5
		Rank $(1,0)$ Impure	Perturbatively trivial (Spin(7) Instanton) $\mathrm{Bun}_G(\mathbb{C}^4, BG)_{\mathrm{dR}}$	8 (topological)
7	1	Rank 1 Pure	Mixed BF Theory $T^*[-1]\,\mathrm{Bun}_G(\mathbb{C}^3 \times \mathbb{R}_{\mathrm{dR}}, BG)$	4 (minimal)
		Rank 2	Mixed Chern-Simons Theory $\mathrm{Bun}_G(\mathbb{C}^2 \times \mathbb{R}^3_{\mathrm{dR}})$	5
		Rank 1 Impure	Perturbatively trivial (G_2 Monopole) $\mathrm{Bun}_G(\mathbb{C}^3 \times \mathbb{R}_{\mathrm{dR}}, BG)_{\mathrm{dR}}$	7 (topological)
6	$(1,1)$	Rank $(1,0)$	Holomorphic BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^3, \mathfrak{g}/\mathfrak{g})$	3 (holomorphic)
		Rank $(1,1)$ special	Mixed BF Theory $T^*[-1]\,\mathrm{Bun}_G(\mathbb{C}^2 \times \mathbb{R}^2_{\mathrm{dR}})$	4
		Rank $(2,2)$	Mixed Chern-Simons Theory $\mathrm{Bun}_G(\mathbb{C} \times \mathbb{R}^2_{\mathrm{dR}})$	5
		Rank $(1,1)$ generic	Perturbatively trivial (6d Hitchin system) $\mathrm{Bun}_G(\mathbb{C}^2 \times \mathbb{R}^2_{\mathrm{dR}})_{\mathrm{dR}}$	6 (topological)
5	2	Rank 1	Mixed BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$	3 (minimal)
		Rank 2 special	Mixed BF Theory $T^*[-1]\,\mathrm{Bun}_G(\mathbb{C} \times \mathbb{R}^3_{\mathrm{dR}})$	4
		Rank 4	5d Chern-Simons Theory $\mathrm{Bun}_G(\mathbb{R}^5_{\mathrm{dR}})$	5 (topological)
		Rank 2 generic	Perturbatively trivial (Haydys-Witten) $\mathrm{Bun}_G(\mathbb{C} \times \mathbb{R}^3_{\mathrm{dR}})_{\mathrm{dR}}$	5 (topological)
4	4	Rank $(1,0)$	Holomorphic BF Theory $T^*[-1]\,\mathrm{Higgs}_G(\mathbb{C}^2)$	2 (holomorphic)
		Rank $(1,1)$	Mixed BF Theory $T^*[-1]\,\mathrm{Bun}_G(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}^2_{\mathrm{dR}})$	3
		Rank $(2,2)$ special	BF Theory (Kapustin-Witten B-model) $T^*[-1]\mathrm{Flat}_G(\mathbb{R}^4)$	4 (topological)
		Rank $(2,1)$	Perturbatively trivial $\mathrm{Bun}_G(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}^2_{\mathrm{dR}})_{\mathrm{dR}}$	4 (topological)
		Rank $(2,0)$	Perturbatively trivial (Kapustin-Witten A-Model) $\mathrm{Higgs}_G(\mathbb{C}^2)_{\mathrm{dR}}$	4 (topological)
		Rank $(2,2)$ generic	Perturbatively trivial (Generic Kapustin-Witten) $\mathrm{Flat}_G(\mathbb{R}^4)_{\mathrm{dR}}$	4 (topological)
3	8	Rank 1	Mixed BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}_{\mathrm{dR}}, T^*[2](\mathfrak{g}/\mathfrak{g}))$	2 (minimal)
		Rank 2 (B)	BF Theory (Rozansky-Witten) $T^*[-1]\mathrm{Map}(\mathbb{R}^3_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$	3 (topological)
		Rank 2 (A)	Perturbatively trivial (dual Rozansky-Witten) $T^*[-1]\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})_{\mathrm{dR}}$	3 (topological)

Table 1: Twists of Maximally Supersymmetric Pure Yang-Mills Theories with gauge group G (16 supercharges).

d	$\mathcal{N}$	Twist	Description	Invariant Directions
6	$(1,0)$	Rank $(1,0)$	Holomorphic BF Theory coupled to a holomorphic symplectic boson $\mathrm{Map}(\mathbb{C}^3, U /\!/ G)$	3 (holomorphic)
5	1	Rank 1	Mixed BF Theory coupled to a mixed symplectic boson $\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}_{\mathrm{dR}}, U /\!/ G)$	3 (minimal)
4	2	Rank $(1,0)$	Holomorphic BF Theory $T^*[-1]\mathrm{Map}(\mathbb{C}^2, U /\!/ G)$	2 (holomorphic)
		Rank $(1,1)$	Mixed BF Theory coupled to a mixed symplectic boson $\mathrm{Map}(\mathbb{C} \times \mathbb{R}^2_{\mathrm{dR}}, U /\!/ G)$	3
		Rank $(2,0)$	Perturbatively trivial (Donaldson theory) $\mathrm{Map}(\mathbb{C}^2, U /\!/ G)_{\mathrm{dR}}$	4 (topological)
3	4	Rank 1	Mixed BF Theory coupled to a mixed symplectic boson $T^*[-1]\mathrm{Map}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, U /\!/ G)$	2 (minimal)
		Rank 2 (B)	BF Theory coupled to a symplectic boson (Rozansky-Witten) $\mathrm{Map}(\mathbb{R}^3_{\mathrm{dR}}, U /\!/ G)$	3 (topological)
		Rank 2 (A)	Perturbatively trivial (dual Rozansky-Witten) $\mathrm{Map}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, U /\!/ G)_{\mathrm{dR}}$	3 (topological)

Table 2: Twists of Supersymmetric Yang-Mills Theories with gauge group G with a hypermultiplet valued in symplectic representation U (8 supercharges).

d	$\mathcal{N}$	Twist	Description	Invariant Directions
4	1	Rank $(1,0)$	Holomorphic BF Theory coupled to R-matter $T^*[-1]\mathrm{Map}(\mathbb{C}^2, R/G)$	2 (holomorphic)
3	2	Rank 1	Mixed BF Theory coupled to R-matter $T^*[-1]\mathrm{Map}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, R/G)$	2 (minimal)

Table 3: Twists of Supersymmetric Yang-Mills Theories with gauge group G with a chiral multiplet valued in representation R (4 supercharges).

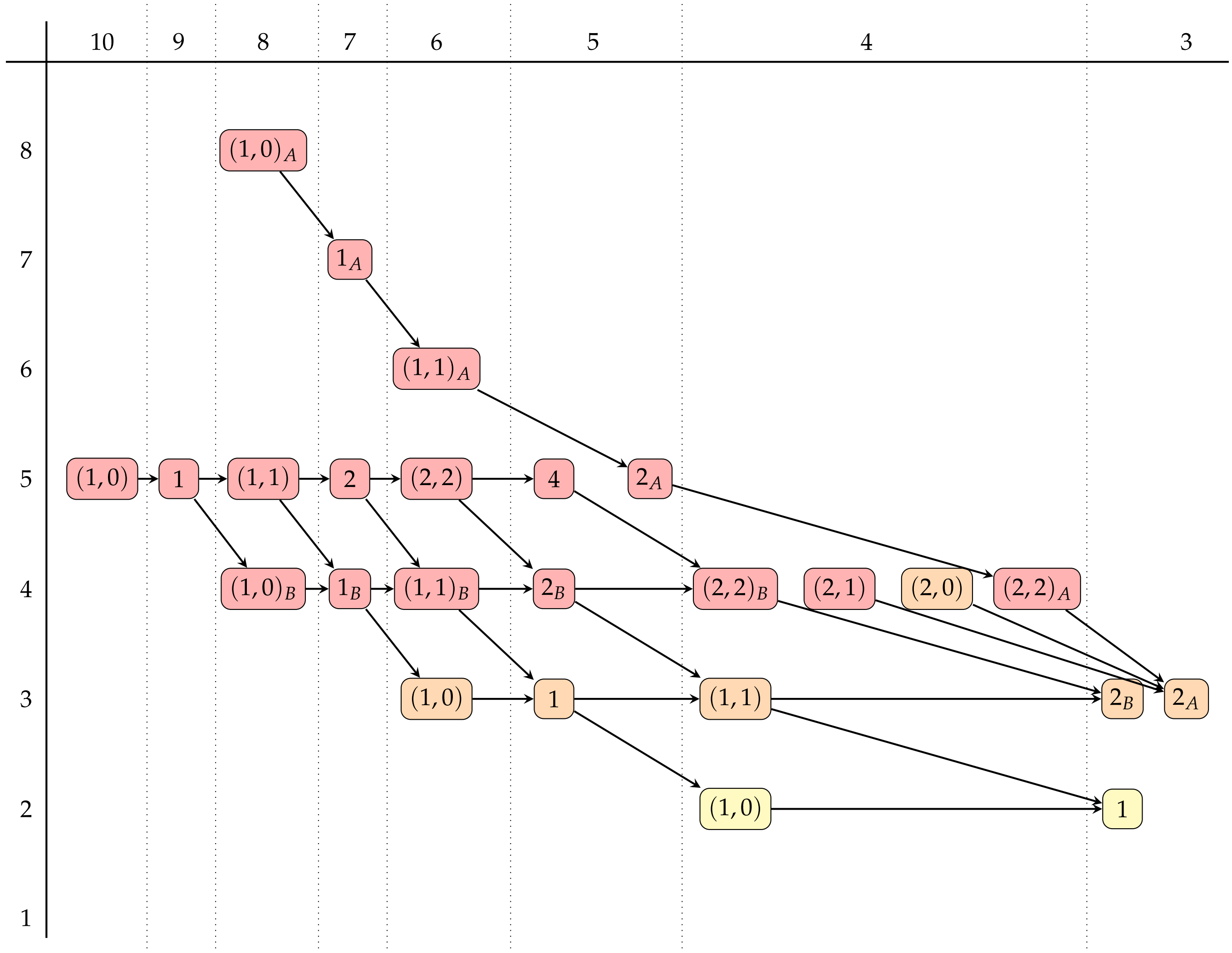

Figure 1: This figure shows the orbits of square-zero supercharges in each dimension. The labels indicate each orbit: the number refers to the rank, and the subscript indicates the situations where the supercharges of a given rank split into multiplet orbits. Each column is labelled by a dimension, and each row by the number of invariant directions of the supercharge. Colours indicate the maximal supersymmetry algebra where the given supercharges live, so red indicates supercharges defined in algebras with 16 supercharges, orange those with 8 supercharges, and yellow those with 4 supercharges.

Part I

Supersymmetric Gauge Theory

1 The BV-BRST Formalism

In this section we will set up the homological formalism in which we study classical field theory: the BV-BRST formalism. Most of the material in this section is not original. We refer the reader to [**CostelloBook**; **Book2**] for more details on this perspective. We will conclude the section by describing a number of fundamental examples of classical field theories that are highly structured: partially holomorphic or topological theories. We will also discuss the concept of *dimensional reduction* of a classical field theory on M along a fibration $M \to N$. We will use the idea of dimensional reduction to construct many of the supersymmetric field theories which we will consider in the next section.

1.1 Conventions

Throughout the paper we will frequently study obects, for instance vector bundles, equipped with a $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-grading. *Degree* will refer to the first (cohomological) grading and *odd* or *even* to the second (fermionic) grading. For an element x we denote by $|x| \in \mathbb{Z}/2\mathbb{Z}$ the total degree.

Given a vector bundle $E \to M$ we denote by $\mathcal{E}$ the topological vector space of smooth sections of E and $\mathcal{E}_c$ the topological vector space of smooth compactly supported sections. We denote by $\mathcal{O}(\mathcal{E})$ (respectively $\mathcal{O}(\mathcal{E}_c)$) the completed algebra of symmetric functions on $\mathcal{E}$ (respectively $\mathcal{E}_c$). We denote by $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ the space of local functionals on $\mathcal{E}$ (see [**Book2**]). An element of $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ will be denoted symbolically by an expression of the form

$$\int_M f(\phi, \phi', \ldots),$$

where f is a density on M depending on infinite jets of sections of E. Note, however, that the integral here is a formal symbol. The space of local functionals can be viewed as a subspace

$$\mathcal{O}_{\mathrm{loc}}(\mathcal{E}) \subset \mathcal{O}(\mathcal{E}_c)$$

where the integral symbol makes sense in earnest when applied to sections which are compactly supported. We denote by $\mathcal{O}^+_{\mathrm{loc}}(\mathcal{E}) \subset \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ the subspace of local functionals which are at least cubic.

Given two vector bundles E, F on M we can also make sense of the space of local functionals from E to F. By definition, this is

$$\mathrm{Fun}_{\mathrm{loc}}(\mathcal{E}, \mathcal{F}) = \prod_{n \geq 0} \mathrm{PolyDiff}(\mathcal{E}^{\times n}, \mathcal{F})_{S_n}$$

where $\mathrm{PolyDiff}(\mathcal{E}^{\times n}, \mathcal{F})$ denotes the space of polydifferential operators, and we take coinvariants for the obvious symmetric group action. When $\mathcal{F} = \mathcal{E}$, we refer to $\mathrm{Fun}_{\mathrm{loc}}(\mathcal{E}, \mathcal{E})$ as the space of local vector fields on E. There is a natural Lie bracket on $\mathrm{Fun}_{\mathrm{loc}}(\mathcal{E}, \mathcal{E})$ and a canonical action of this Lie algebra on local functionals.

1.2 Classical BV Theories

(Pavel: We need to introduce classical field theories linear over a commutative ring (used to talk about symmetries and the Hodge family).)

The classical BV (Batalin-Vilkovisky) formalism [**BatalinVilkovisky**] is a model for classical field theory from the Lagrangian perspective. In brief the classical BV formalism produces a local model for the critical locus of an action functional, but considered in the derived sense. That is, given a space Φ of fields and an action functional with derivative $\mathrm{d}S$, one considers not just the usual locus in Φ of fields with $\mathrm{d}S(\phi) = 0$, but the derived intersection $\mathrm{dCrit}(S) = \Phi \cap^h_{T^*\Phi} \Gamma_{\mathrm{d}S}$ of the zero section in $T^*\Phi$ with the graph of $\mathrm{d}S$. The formalism we describe below can be interpreted as an abstract formalism for modelling the tangent complex at a point to a derived critical locus $\mathrm{dCrit}(S)$.

Definition 1.1. A ***free BV theory*** on a manifold M is the data of:

- a finite rank $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-graded vector bundle $E \to M$ equipped with an even differential operator of cohomological degree $+1$

$$Q_{\mathrm{BV}} \colon \mathcal{E} \to \mathcal{E}[1]$$

such that (1): $Q_{\mathrm{BV}}^2 = 0$ and (2): the pair $(\mathcal{E}, Q_{\mathrm{BV}})$ is an elliptic complex;

- a map of bundles
$$\omega\colon E \otimes E \to \mathrm{Dens}_M[-1]$$
that is
  (1) fiberwise nondegenerate,
  (2) graded skew symmetric, and
  (3) satisfies $\int_M \omega(e_0, Q_{\mathrm{BV}} e_1) = (-1)^{|e_0|} \int_M \omega(Q_{\mathrm{BV}} e_0, e_1)$ where e_i are compactly supported sections of E .

We call $\mathcal{E}$ the ***space of BV fields***. The pairing ω equips the algebra of local functionals on E with a ***BV bracket*** (see [**CostelloBook**])
$$\{-,-\}\colon \mathcal{O}_{\mathrm{loc}}(\mathcal{E}) \times \mathcal{O}_{\mathrm{loc}}(\mathcal{E}) \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E}).$$
This bracket is of cohomological degree $+1$. We will now explain how to define the BV bracket. First note that there is a linear map
$$\mathrm{d}_{\mathrm{dR}} : \mathcal{O}_{\mathrm{loc}}(\mathcal{E}) \to \mathrm{Fun}_{\mathrm{loc}}(\mathcal{E}, \mathcal{E}^!)$$
defined as follows. A local functional $F \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ can be written as an equivalence class of a sum of densities of the form
$$D_1(-) \cdots D_n(-)\Omega$$
where D_i is a differential operator $D_i : \mathcal{E} \to C^\infty_M$ and Ω is a density on M. Without loss of generality, suppose F is of this form. Then, we can view F as a functional in $\mathcal{O}(\mathcal{E}_c)$ by the assignment
$$\phi \mapsto \int_M D_1(\phi) \cdots D_n(\phi)\Omega$$
where ϕ denotes a compactly supported section. Define the symmetric multilinear map
$$\begin{array}{rccl} \mathrm{d}_{\mathrm{dR}}F & : & \mathcal{E}_c^{\times(n-1)} & \to & \mathcal{E}^\vee \\ & & (\phi_1, \ldots, \phi_{n-1}) & \mapsto & D_1(\phi_1) \cdots D_{n-1}(\phi_{n-1}) D_n(-) + \{\text{symmetric terms}\}. \end{array}$$
Integrating by parts, we see that for any $(n-1)$-tuple $\phi_1, \ldots, \phi_{n-1} \in \mathcal{E}_c$ that the linear functional $(\mathrm{d}_{\mathrm{dR}}F)(\phi_1, \ldots, \phi_{n-1})$ is an element of $\mathcal{E}^!$. This implies that $\mathrm{d}_{\mathrm{dR}}F \in \mathrm{Fun}_{\mathrm{loc}}(\mathcal{E}, \mathcal{E}^!)$.

The non-degenerate pairing ω determines a bundle isomorphism $\omega\colon E \cong E^![-1]$ and hence an isomorphism of local functions
$$\omega : \mathrm{Fun}_{\mathrm{loc}}(\mathcal{E}, \mathcal{E}^!) \cong \mathrm{Fun}_{\mathrm{loc}}(\mathcal{E}, \mathcal{E}[1]).$$
We recognize the right hand side as the space of local vector fields placed in a shifted cohomological degree. In total, we see that a local functional F determines a local vector field by applying this isomorphism to $\mathrm{d}_{\mathrm{dR}}F$:
$$X_F := \omega \circ \mathrm{d}_{\mathrm{dR}}(F) \in \mathrm{Fun}_{\mathrm{loc}}(\mathcal{E}, \mathcal{E}[1]).$$
This is the Hamiltonian vector field corresponding to F. Finally, the BV bracket between local functionals F, G is defined by
$$\{F, G\} = X_F(G).$$

The BV bracket enjoys the graded skew symmetry property
$$\{F, G\} = (-1)^{|F||G|}\{G, F\}$$
as well as the graded Jacobi identity. This bracket together wtih Q_{BV} endows $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1]$ with the structure of a dg Lie algebra. Since the space of local functionals is not an algebra, the bracket does not satisfy any type of Leibniz rule.

Now, we will include interactions in the BV picture.

Definition 1.2. A ***classical BV field theory*** (or simply, classical field theory) is a free BV theory (E, Q, ω) equipped with an even functional

$$I \in \mathcal{O}^{+}_{\mathrm{loc}}(\mathcal{E})$$

of cohomological degree zero satisfying the Maurer-Cartan equation

$$Q_{\mathrm{BV}} I + \frac{1}{2}\{I, I\} = 0.$$

Given a classical field theory $(E, Q_{\mathrm{BV}}, \omega, I)$ we denote by

$$S = \frac{1}{2}\int_M \omega(e, Q_{\mathrm{BV}} e) + I \in \mathcal{O}_{\mathrm{loc}}(E)$$

the BV action of the theory.

The local functional S satisfies the *classical master equation*

$$\{S, S\} = 0.$$

In fact, given a degree (-1) nondegenerate pairing ω on E, prescribing the data of a classical field theory (namely a pair Q_{BV}, I satisfying the Maurer-Cartan equation) is equivalent to prescribing a local functional $S \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ that is at least quadratic and satisfies the classical master equation. The operator Q_{BV} is the BV bracket with the quadratic part of the BV action S, and the cubic and higher terms in S coincide with I.

Because of this, we will sometimes refer to the triple (E, S, ω) instead $(E, Q_{\mathrm{BV}}, I, \omega)$ as the data of a classical field theory.

Remark 1.3. We will also consider ***$\mathbb{Z}/2\mathbb{Z}$-graded classical field theories*** which are defined as before, but where E has only a single $\mathbb{Z}/2\mathbb{Z}$-grading and, correspondingly, Q_{BV} is simply an odd operator.

Remark 1.4. The data of a classical BV theory can be equivalently encoded in an elliptic L_∞ algebra $L = E[-1]$ equipped with a cyclic structure, i.e. a symplectic isomorphism $L \cong L^![-3]$ [**Book2**].

Next, we formulate the notion of an equivalence of classical BV theories.

Definition 1.5. A ***morphism*** $F\colon (E, Q_{\mathrm{BV}}, \omega, I) \rightsquigarrow (E', Q'_{\mathrm{BV}}, \omega', I')$ of classical field theories over the same manifold M is a nonlinear map of vector bundles $E \rightsquigarrow E'$, i.e. a collection $F = \sum_{n\geq 1}^{\infty} F$ of poly-differential operators $F_n\colon \mathrm{Sym}^n(E) \to E'$, that intertwines the differentials $Q_{\mathrm{BV}}, Q'_{\mathrm{BV}}$, the pairings ω, ω', and the interactions I, I'. A morphism is a ***perturbative equivalence*** if the map $F_1\colon (\mathcal{E}, Q_{\mathrm{BV}}) \to (\mathcal{E}', Q'_{\mathrm{BV}})$ is a quasi-isomorphism. A classical field theory is ***perturbatively trivial*** if it is perturbatively equivalent to the zero theory ($E = 0$).

We will now describe two primitive examples of equivalences of classical field theories which will be useful in simplifying twisted theories. First, we consider the process of integrating out an auxiliary field.

Proposition 1.6. Fix a volume form dvol_M on M. Suppose (E, Q, ω, I) is a classical field theory, where $E \cong E_0 \oplus (\mathcal{O}_M \oplus \mathrm{Dens}_M[-1])$ with the symplectic pairing ω given by a sum of a symplectic pairing ω_0 on E_0 and the standard symplectic pairing on the second summand. Denote by ϕ a section of $\mathcal{O}_M$ and by ϕ^* a section of $\mathrm{Dens}_M[-1]$. Suppose the BV action is

$$S = S_0 + \frac{1}{2}\int \mathrm{dvol}_M(\phi^2 - 2\phi S_1),$$

where S_0 and S_1 are independent of ϕ. Then the theory (E, Q, ω, I) is perturbatively equivalent to the theory (E_0, Q_0, ω_0, I') with the BV action $S_0 - S_1^2/2$, where we set $\phi^* = 0$ and $\phi = S_1$.

Proof. Concretely, suppose that the linear part of S_1 is given by an operator Q_1, and the interacting part of Q_1 is given by a functional $I_1 = \sum_{n=1}^{\infty} I_1^n$. Then we define the required perturbative equivalence as follows. First $F_1(e) = (e, Q_1(e), 0) \in E$. Then, for $n > 1$, define

$$\begin{aligned} F_n \colon \operatorname{Sym}^n(E_0) &\to E \\ e_1 \otimes \cdots \otimes e_n &\mapsto (0, I_1^n(e_1, \ldots, e_n), 0). \end{aligned}$$

These F_n manifestly intertwine the pairings ω and ω'. To see that they intertwine the action functionals, we observe that

$$\begin{aligned} S(F(e)) &= S(e, S_1(e), 0) \\ &= S_0(e) + \frac{1}{2}\,\mathrm{dvol}_M \int (S_1(e)^2 - 2S_1(e)^2) \\ &= S_0(e) - \frac{1}{2} S_1(e)^2 \\ &= S'(e). \end{aligned}$$

Finally, F_1 is a quasi-isomorphism with quasi-inverse given by the projection $(E, Q) \to (E_0, Q_0)$, therefore our specified morphism is an equivalence. ☐

Remark 1.7. In terms of the classical BV complex, this proposition tells us that given a classical BV complex of the form

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{E}_0^0 \arrow[r, "Q_0"] \arrow[dr] & \mathcal{E}_0^1 \arrow[r] & \cdots \\ & \mathcal{O}_M \arrow[r, "i"] \arrow[ur] & \mathrm{Dens}_M & \end{tikzcd}$$

where i is the canonical inclusion, we can replace it with a quasi-isomorphic complex consisting of only the first line, as long as we suitably modify the differential Q_0 and the Lie structure.

We may also remove a trivial BRST doublet.

Proposition 1.8. Suppose $(E, Q_{\mathrm{BV}}, \omega, I)$ is a classical field theory, $F \to M$ is a graded vector bundle, such that $E \cong E_0 \oplus E_1 \oplus E_1^![-1] \oplus E_1^! \oplus E_1[-1]$. Denote by ϕ, ϕ^* sections of $E_1, E_1^![-1]$ and by ψ, ψ^* sections of $E_1^!, E_1[-1]$. Suppose

$$Q_{\mathrm{BV}}\phi + \{I, \phi\} = \phi - f_\psi \in E_1[-1], \qquad Q\psi + \{I, \psi\} = \psi - f_\phi \in E_1^![-1],$$

where f_ϕ is independent of ψ and f_ψ is independent of ϕ. Then the theory $(E, Q_{\mathrm{BV}}, \omega, I)$ is perturbatively equivalent to the theory $(E_0, Q_0, \omega_0, I_0)$ with the BV action obtained by setting $\phi = f_\psi, \phi^* = 0, \psi = f_\phi, \psi^* = 0$ in the original BV action.

Proof. Again, concretely, we'll write $\sum_{n\geq 1} f_\phi^{(n)}$ and $\sum_{n\geq 1} f_\psi^{(n)}$ for the Taylor expansions of f_ϕ and f_ψ respectively. We can write out a morphism of classical BV theories $F \colon E_0 \to E$ by defining $F_1(e) = (e, f_\phi^{(1)}(e), 0, f_\psi^{(1)}(e), 0)$, and for $n > 1$ defining

$$F_n(e_1 \otimes e_n) = (0, f_\phi^{(n)}(e_1, \ldots, e_n), 0, f_\psi^{(n)}(e_1, \ldots, e_n)).$$

Again, these F_n manifestly intertwine the pairings on E_0 and E, since the image of F_n lands in an isotropic summand of the $E_1 \oplus E_1^![-1] \oplus E_1^! \oplus E_1[-1]$ part of E. Also, by construction, the F_n intertwine the action functionals, since

$$\begin{aligned} S(F(e)) &= S_0(e) + \frac{1}{2} \int_M \omega(f_\psi(e), f_\phi(e) - f_\phi(e)) + \omega(f_\phi(e), f_\psi(e) - f_\psi(e)) \\ &= S_0(e). \end{aligned}$$

Again, F_1 is a quasi-isomorphism between the classical BV complexes with quasi-inverse given by the projection onto the summand E_0. ☐

Remark 1.9. Again, in terms of the classical BV complex, this proposition tells us that given a classical BV complex of the form

$$\begin{tikzcd}
\cdots \arrow[r] & E_0^{-1} \arrow[r] \arrow[dr] \arrow[ddr] & E_0^0 \arrow[r] & E_0^1 \arrow[r] & E_0^2 \arrow[r] & \cdots \\
& & E_1 \arrow[dr] & E_1^! \arrow[ur] & & \\
& & E_1^! \arrow[ur] & E_1 \arrow[uur] & &
\end{tikzcd}$$

with the differentials $E_1 \to E_1$ and $E_1^! \to E_1^!$ given by the identity, we can replace it with a quasi-isomorphic complex consisting of only the first line, as long as we suitably modify its differential and Lie structure. (Chris: Pavel, is this, and the other remark above, what you wanted me to add?)

1.3 Symmetries in the Classical BV Formalism

In this section we define what it means for a (super) Lie algebra to act on a classical field theory (see also **[Book2]** for a related discussion). Let $(E, Q_{\mathrm{BV}}, \omega, I)$ be a classical field theory and $\mathfrak{g}$ a super Lie algebra. We will define $\mathfrak{g}$-equivariant local observables in the classical field theory by introducing $\mathfrak{g}$-valued background fields into our classical field theory, and extending the action functional to a functional that involves these background fields, but still satisfies the classical master equation. We begin by defining an appropriate version of the Chevalley-Eilenberg cochain complex.

Definition 1.10. The *Chevalley-Eilenberg complex* for the Lie algebra $\mathfrak{g}$, with coefficients in $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$, will be defined as follows. Consider the graded vector space

$$C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E})) = \bigoplus_n \mathrm{Hom}(\wedge^n \mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))[-n]$$

(Brian: not crazy about the notation $C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))$ for just the graded vector space before knowing that $\mathcal{O}_{\mathrm{loc}}$ is a module.) parametrizing multilinear maps $f \colon \mathfrak{g}^{\otimes n} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ which satisfy the antisymmetry property

$$f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = (-1)^{|x_1||x_2|+1} f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$$

where $x_j \in \mathfrak{g}$. The Chevalley-Eilenberg differential is given, following the sign conventions of **[SafronovCoisoInt]**, by the formula

$$(\mathrm{d}_{\mathrm{CE}} f)(x_1, \ldots, x_n) = \sum_{i<j} (-1)^{|x_i| \sum_{p=1}^{i-1} |x_p| + |x_j| \sum_{p=1, p\neq i}^{j-1} |x_p| + i + j + |f|} f([x_i, x_j], x_1, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_n).$$

The complex is additionally equipped with a degree $+1$ BV bracket via the formula

$$\{f, g\}(x_1, \ldots, x_{k+l}) = \sum_{\sigma \in S_{k,l}} \mathrm{sgn}(\sigma)(-1)^{\epsilon + \epsilon_1} \{f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), g(x_{\sigma(k+1)}, \ldots, x_{\sigma(k+l)})\},$$

where $S_{k,l}$ is the set of (k, l)-shuffles, ϵ is the usual Koszul sign and

$$\epsilon_1 = |g|k + \sum_{i=1}^{k} |x_{\sigma(i)}|(l + |g|).$$

The operator Q_{BV} on $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ extends $C^\bullet(\mathfrak{g})$-linearly to an operator on $C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))$ by the rule

$$(Q_{\mathrm{BV}} f)(x_1, \ldots, x_n) = Q_{\mathrm{BV}} f(x_1, \ldots, x_n)$$

where $f\colon \mathfrak{g}^{\otimes n} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$. The differentials d_{CE} and Q_{BV} are compatible in the sense that $(\mathrm{d}_{\mathrm{CE}} + Q_{\mathrm{BV}})^2 = 0$ making $C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))$ into a cochain complex with total differential $\mathrm{d}_{\mathrm{CE}} + Q_{\mathrm{BV}}$. Via the BV bracket, the shift of this cochain complex $C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))[-1]$ is a dg Lie algebra. This shifted cotangent complex will model equivariant local observables in our classical field theory, but to finish defining the $\mathfrak{g}$ action we must define the equivariant version of the classical interaction. This is defined as follows.

Definition 1.11. Let $(E, Q_{\mathrm{BV}}, \omega, I)$ be a classical field theory.

(1) An ***action*** of a super Lie algebra $\mathfrak{g}$ on $(E, Q_{\mathrm{BV}}, \omega, I)$ is an element of cohomological degree zero

$$I_{\mathfrak{g}} = \sum_{k \geq 0} I_{\mathfrak{g}}^{(k)} \text{ in } C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E})),$$

where $I_{\mathfrak{g}}^{(k)}$ is a multilinear map $\mathfrak{g}^{\otimes k} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$, that satisfies the following three conditions:

(a) $I_{\mathfrak{g}}^{(0)} = I$.

(b) For each $k \geq 1$ and $x_1, \ldots, x_k \in \mathfrak{g}$ the local functional $I_{\mathfrak{g}}^{(k)}(x_1, \ldots, x_k)$ is at least quadratic in the fields.

(c) $I_{\mathfrak{g}}$ satisfies the Maurer–Cartan equation:

$$(\mathrm{d}_{\mathrm{CE}} + Q_{\mathrm{BV}}) I_{\mathfrak{g}} + \frac{1}{2}\{I_{\mathfrak{g}}, I_{\mathfrak{g}}\} = 0.$$

(2) For each $k \geq 1$ there is a decomposition $I_{\mathfrak{g}}^{(k)} = \sum_\ell I_{\mathfrak{g}}^{(k),\ell}$ where $I_{\mathfrak{g}}^{(k),\ell}(x_1, \ldots, x_k) \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ is ℓ-linear in the fields $\mathcal{E}$ for all $x_1, \ldots, x_k \in \mathfrak{g}$. An action $I_{\mathfrak{g}}$ is called ***elliptic*** if the induced linear differential operator

$$Q_{\mathrm{BV}} + \sum_{k \geq 1} \{I_{\mathfrak{g}}^{(k),2}, -\}$$

is elliptic as a $C^\bullet(\mathfrak{g})$-linear differential operator.

Remark 1.1. (Brian: Explain $C^\bullet(\mathfrak{g})$-linear elliptic)

Remark 1.2. We have seen that a classical BV theory can also be presented in terms of a BV action $S \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ satisfying the classical master equation $\{S, S\} = 0$. One can also formulate actions of a Lie algebra on a classical theory in these terms. The data of an action of a Lie algebra $\mathfrak{g}$ on a classical field theory (E, ω, S) is equivalent to the choice of a local functional $\mathfrak{S}_{\mathfrak{g}} = \sum_k \mathfrak{S}_{\mathfrak{g}}^{(k)} \in C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))$, at least quadratic in the fields, such that $\mathfrak{S}_{\mathfrak{g}}^{(0)} = S$ and such that the classical master equation

$$\mathrm{d}_{\mathrm{CE}} \mathfrak{S}_{\mathfrak{g}} + \frac{1}{2}\{\mathfrak{S}_{\mathfrak{g}}, \mathfrak{S}_{\mathfrak{g}}\} = 0$$

is satisfied.

Remark 1.3. We have defined an action of a Lie algebra on a classical field theory in terms of a Noether current $I_{\mathfrak{g}}$. Such data gives rise to an L_∞ action of $\mathfrak{g}$ on the space of fields $\mathcal{E}$ in the following way. By the Maurer-Cartan equation, the operator $\mathrm{d}_{\mathrm{CE}} + Q_{\mathrm{BV}} + \{I_{\mathfrak{g}}, -\}$ defines a differential on the graded vector space $\mathcal{O}(\mathfrak{g}[1] \oplus \mathcal{E})$. By assumption that the Noether current is at least quadratic in the fields, we see that this differential defines a family of maps

$$\mathfrak{g}^{\otimes k} \otimes \mathcal{E}^{\otimes \ell} \to \mathcal{E}$$

combining to give $\mathcal{E}$ the structure of an L_∞-module for $\mathfrak{g}$.

We may also define actions of supergroups on classical field theories. The action of a supergroup G is more data than the action of a super Lie algebra $\mathfrak{g}$: it includes the infinitesimal action of the Lie algebra $\mathfrak{g}$, along with an action of G on the fields exponentiating this infinitesimal action. That is, we make the following definition.

Definition 1.12. Let $(E, Q_{\mathrm{BV}}, \omega, I)$ be a classical field theory, and let G be a supergroup acting on spacetime M. An ***action*** of G on $(E, Q_{\mathrm{BV}}, \omega, I)$ is given by the following data:

- An action of G on $\mathcal{E}$ compatible with the G-action on M.
- An (elliptic) action $I_{\mathfrak{g}}$ of its super Lie algebra $\mathfrak{g}$ with $I_{\mathfrak{g}}^{(k)} = 0$ for $k \geq 2$

These are required to satisfy the following conditions:

- The G-action on $\mathcal{E}$ preserves the symplectic pairing ω, the differential Q and the interaction term I.
- For every $x \in \mathfrak{g}$, the vector field $X_{I_{\mathfrak{g}}^{(1)}(x)}$ on $\mathcal{E}$ coincides with the infinitesimal action of $\mathfrak{g}$ on $\mathcal{E}$.

Remark 1.13. While we allow for L_∞ actions of Lie algebras, we will only consider strict actions of Lie groups in the present work.

1.4 From BRST to BV

We will now explain how to build classical BV theories from more traditional data: that of the *usual* fields of a classical field theory, together with the usual action functional and the action of gauge transformations. These data can be packaged into what's known as a BRST theory, where fermionic fields (ghosts) are introduced to generate the infinitesimal gauge transformations, in the following way.

Definition 1.14. A ***classical BRST theory*** on a manifold M consists of the following data:

- a $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-graded vector bundle F together with the structure of a local L_∞ algebra on the shift $F[-1]$;
- A local functional $S_{\mathrm{BRST}} \in \mathcal{O}_{\mathrm{loc}}(\mathcal{F})$ of polynomial degree ≥ 2.

Together, these data must satisfy the equation

$$Q_{\mathrm{BRST}} S_{\mathrm{BRST}} = 0,$$

where Q_{BRST} is the Chevalley–Eilenberg differential defined by the L_∞ structure on $F[-1]$.

We call $\mathcal{F}$ the ***space of BRST fields***.

Remark 1.15. In the most typical examples, the bundle F is concentrated in $\mathbb{Z}$-degrees -1 and 0. In this case, sections in degree 0 are thought of as physical fields, and ghosts – sections in degree -1 – are thought of as generators of the infinitesimal gauge symmetry. The action of gauge transformations on fields is then encoded by the Lie structure.

From a classical BRST theory $(\mathcal{F}, S_{\mathrm{BRST}})$, one can construct a classical BV theory as follows. Let $\{\ell_k\}_{k\geq 1}$ be the L_∞ structure maps underlying the local Lie algebra $F[-1]$.

First, we define the free BV theory. Split $S_{\mathrm{BRST}} = S_{\mathrm{BRST}}^{\mathrm{free}} + I_{\mathrm{BRST}}$, where $I_{\mathrm{BRST}} \in \mathcal{O}_{\mathrm{loc}}^{+}(\mathcal{F})$ and $S_{\mathrm{BRST}}^{\mathrm{free}}$ is a quadratic local functional which we may view as defining a map

$$S_{\mathrm{BRST}}^{\mathrm{free}} \colon F \to F^!.$$

The underlying bundle of the BV theory is

$$E = F \oplus F^![-1].$$

The BV pairing ω on E is defined in terms of the natural pairing between F and $F^!$. The differential of the free BV theory is

$$Q_{\mathrm{BV}} = \ell_1 + S_{\mathrm{BRST}}^{free}.$$

The interacting theory is constructed as follows. First, note that for $k \geq 2$ the L_∞ structure maps $\{\ell_k\}_{k\geq 2}$ on $\mathcal{F}$ pull back to multilinear maps on $\mathcal{E}$ via the obvious projection $p\colon \mathcal{E} \to \mathcal{F}$. These structure maps assemble into a local functional $I_F \in \mathcal{O}^+_{\mathrm{loc}}(\mathcal{E})$ defined by

$$I_F(e) = \sum_{k\geq 2} \frac{1}{(k+1)!} \int_M \omega_F(e, (p^*\ell_k)(e, \ldots, e))$$

which is linear along $\mathcal{F}^!$. Likewise, the BRST action I_{BRST} pulls back to $\mathcal{E}$, and we define the BV interaction as the sum

$$I_{\mathrm{BV}} = I_F + p^* I_{\mathrm{BRST}} \in \mathcal{O}^+_{\mathrm{loc}}(\mathcal{E}).$$

Lemma 1.16. Suppose (F, S_{BRST}) is a classical BRST theory such that $(\mathcal{E}, Q)$ defined above is an elliptic complex. Then $(E, Q_{\mathrm{BV}}, \omega, I)$ is a classical BV theory.

We refer to the classical BV theory $(E, Q_{\mathrm{BV}}, \omega, I)$ from the above statement as the (-1)***-shifted cotangent bundle*** of $\mathcal{F}$ and by abuse of notation we often denote it simply by $\mathcal{E} = T^*[-1]\mathcal{F}$. In the case $S_{\mathrm{BRST}} = 0$ we refer to the theory $T^*[-1]\mathcal{F}$ as a theory of ***cotangent type***.

Remark 1.17. In general, multiple BRST theories can give rise to the same BV theory. A BV theory $(E, Q_{\mathrm{BV}}, \omega, I)$ is of cotangent type as long as there is *some* F with $S_{\mathrm{BRST}} = 0$ producing the given theory using the construction above. Theories of cotangent type can still have interesting, non-trivial action functionals, encoded by the L_∞ structure on F.

If the fields of the classical BRST theory are denoted by ϕ, we denote their antifields in the classical BV theory by ϕ^*, so that

$$\{\phi(x), \phi^*(y)\} = \{\phi^*(y), \phi(x)\} = \delta(x-y).$$

(Brian: Looks like you're using a bracket we haven't defined. We've only defined brackets on local functionals, but I think when you write $\phi(x)$ you mean a point-like observable, which is far from local.) (Chris: I don't think we need to say this so precisely. I think it's enough to say that a homogeneous splitting $F = \bigoplus_i \Phi_i$ induces a splitting $F^! = \bigoplus_i \Phi_i^!$, and if we denote a general element of the summand Φ_i by ϕ then we will denote a general element of the summand $\Phi_i^!$ by ϕ^*.)

1.5 Examples of Classical Field Theories

In this section we give some examples of classical field theories we will use in our classification of twisted supersymmetric field theories. All theories we consider in this section are $\mathbb{Z}$-graded, i.e. the space E of fields will be completely even with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading.

1.5.1 Generalized BF Theory

Our first example will generalize the fundamental example of *BF theory* to a not entirely topological context. Ordinarily, BF theory describes the classical BRST theory on a d-manifold M with fields given by a G-gauge field A and a $\mathfrak{g}$-valued $(d-2)$-form B, with action functional

$$S(A, B) = \int_M \langle F_A \wedge B \rangle.$$

This theory is, in fact, of cotangent type, where the base of the cotangent includes A and its antifield, and the fiber includes B and its antifield. This basic setup can be generalized to a setting where M need not be entirely topological, and where $\mathfrak{g}$ may be a more general L_∞ algebra, in the following way.

Definition 1.18. Let X and Y be complex manifolds and M a smooth oriented manifold. Fix an L_∞ algebra $\mathfrak{g}$. The ***generalized** BF **theory*** is the (-1)-shifted cotangent bundle of the following classical BRST theory:

- The spacetime is the smooth manifold $X \times Y \times M$.
- The bundle of BRST fields is the $\mathbb{Z}$-graded bundle $F = \Omega_X^{0,\bullet} \otimes \Omega_Y^{\bullet,\bullet} \otimes \Omega_M^\bullet \otimes \mathfrak{g}[1]$. $F[-1]$ is equipped with a natural local L_∞ algebra structure from $\mathfrak{g}$.
- The BRST action is $S_{\mathrm{BRST}} = 0$.

We denote the space of BV fields by $\mathcal{E} = \mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, T^*[d]B\mathfrak{g})$, where $d = \dim_{\mathbb{C}}(X) + 2\dim_{\mathbb{C}}(Y) + \dim(M) - 1$.

Remark 1.19. The mapping stack above has a precise meaning in the language of derived algebraic geometry (Chris: Let's say a few words about where the terminology comes from at least, even if for us it's just shorthand. I can add this later. I should also remark on the fact that we can pull the $T^*[d]$ out to a $T^*[k]$, with $k = -1$ if X, Y, M are vector spaces.) For the purposes of this paper, while we think of the derived stack as the global moduli space of solutions the equations of motion in generalized BF theory, we will only use the perturbative datum of the classical BV complex. The language of the mapping stack will simply be used as a compact way of referring to the theory above.

Let us unpack the definition. Let $d = \dim_{\mathbb{C}}(X) + 2\dim_{\mathbb{C}}(Y) + \dim(M)$. Then the bundle of BV fields is

$$E = \Omega_X^{0,\bullet} \otimes \Omega_Y^{\bullet,\bullet} \otimes \Omega_M^\bullet \otimes \mathfrak{g}[1] \oplus \Omega_X^{\dim(X),\bullet} \otimes \Omega_Y^{\bullet,\bullet} \otimes \Omega_M^\bullet \otimes \mathfrak{g}^*,$$

(Chris: I changed a shift here which I don't think was correct, but someone should double check.) where we denote the two fields by A and B. The BV action is

$$S = \int_{X\times Y\times M} \langle B \wedge (\bar\partial_X + \bar\partial_Y + \mathrm{d}_{\mathrm{dR},M})A\rangle + \sum_{k\geq 1} \frac{1}{k!}\int_{X\times Y\times M} \langle B \wedge \ell_k(A, \dots, A)\rangle,$$

where $\langle -, - \rangle$ is the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ and ℓ_k denote the components of the L_∞ structure on $\mathfrak{g}$.

Example 1.20. For $X = Y = \mathrm{pt}$ and $\mathfrak{g}$ an ordinary Lie algebra we recover the usual topological *BF* theory with the BV action

$$S = \int_M \left\langle B \wedge \left(\mathrm{d}_{\mathrm{dR}} A + \frac{1}{2}[A \wedge A]\right)\right\rangle.$$

We will see many *BF* theories as the output when we twists a supersymmetric gauge theories. In fact, a special case of the definition above also arises when twisting theories of matter. We will refer to as a generalized $\beta\gamma$ system, extending the usual 2d $\beta\gamma$ system.

Definition 1.21. Let X and Y be complex manifolds and M a smooth manifold. Fix a complex vector space V. The ***generalized** $\beta\gamma$ **system*** is the BV theory given by the (-1)-shifted cotangent bundle of the following classical BRST theory:

- The spacetime is the smooth manifold $X \times Y \times M$.
- The bundle of BRST fields is the $\mathbb{Z}$-graded bundle $F = \Omega_X^{0,\bullet} \otimes \Omega_Y^{\bullet,\bullet} \otimes \Omega_M^\bullet \otimes V$. $F[-1]$ is equipped with an L_∞ structure with $\ell_k = 0$ for $k \geq 2$.
- The BRST action is $S_{\mathrm{BRST}} = 0$.

Remark 1.4. This is indeed a special case of generalized BF theory: the generalized $\beta\gamma$ system appears as generalized *BF* theory with the L_∞ algebra given by $\mathfrak{g} = V[-1]$, with trivial L_∞ structure.

We may also couple a $\beta\gamma$ system to a more general generalized BF theory as follows.

Example 1.22. Let X, Y, M be as before. Suppose $\mathfrak{g}$ is a dg Lie algebra and V is a dg representation of $\mathfrak{g}$. Consider the dg Lie algebra

$$\mathcal{L} = \mathfrak{g} \oplus V[-1]$$

with the only nontrivial brackets $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ given by the Lie bracket on $\mathfrak{g}$ and $\mathfrak{g} \otimes V \to V$ given by the $\mathfrak{g}$-action on V. The space of fields in the corresponding generalized BF theory will be denoted using the mapping stack notation (as in Remark 1.19) as

$$\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, T^*[d](V/\mathfrak{g})) := \mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, T^*[d]\mathcal{L}),$$

where $d = \dim_{\mathbb{C}}(X) + 2\dim_{\mathbb{C}}(Y) + \dim(M) - 1$.

The following is obvious from the definition.

Proposition 1.23. Consider the generalized BF theory on $\mathbb{R}^{2n_1+2n_2+n_3}$ with the space of BV fields

$$\mathcal{E} = \mathrm{Map}(\mathbb{C}^{n_1} \times (\mathbb{C}^{n_2})_{\mathrm{Dol}} \times (\mathbb{R}^{n_3})_{\mathrm{dR}}, T^*[d]B\mathfrak{g}).$$

Then it carries an action of $\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{SO}(n_3)$ given by the pullback action on differential forms on $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{R}^{n_3}$.

Remark 1.24. In fact, the $\mathrm{SO}(n_3)$-action given by the previous proposition extends to a homotopically trivial action in the sense of **[ElliottSafronov]**.

Generalized BF theories can be naturally deformed to theories which are perturbatively trivial, but which arise as shadows of non-trivial non-perturbative theories. These will often appear as topological twists of supersymmetric field theories, the most famous example being the 2d A-model (Chris: refer also to the introduction). In order to remain within the world of perturbation theory, we will study these theories as deformations families parameterized by a complex parameter t. So at $t = 0$ we will recover generalized BF theory.

We will refer to these families as Hodge deformations of generalized BF theory. The terminology here comes from Simpson's Hodge stack **[Simpson]**: if X is a smooth scheme, or more generally a derived Artin stack, one can define a derived stack X_{Hod} over $\mathbb{A}^1$, where the fiber at $0 \in \mathbb{A}^1$ is the Dolbeault stack X_{Dol} of X, essentially the 1-shifted tangent bundle, and the fiber at a non-zero point is equivalent to the de Rham stack X_{dR} of X, which has contractible tangent complex.

Definition 1.25. Consider the generalized BF theory on $X \times Y \times M$ associated to the L_∞ algebra $\mathfrak{g}$, and supposed $\mathfrak{g}$ is equipped with a non-degenerate pairing. The *Hodge deformation* of this generalized BF theory is the $\mathbb{C}[t]$ family of classical BV theories on $X \times Y \times M$, with $(E_t, \omega_t, I_t) = (E_{\mathrm{BF}}, \omega_{\mathrm{BF}}, I_{\mathrm{BF}})$ the same data as for generalized BF theory, so

$$E_t = \left(\Omega_X^{0,\bullet} \otimes \Omega_Y^{\bullet,\bullet} \otimes \Omega_M^{\bullet} \otimes \mathfrak{g}[1]\right) \oplus \left(\Omega_X^{\dim(X),\bullet} \otimes \Omega_Y^{\bullet,\bullet} \otimes \Omega_M^{\bullet} \otimes \mathfrak{g}^*\right),$$

but $Q_{\mathrm{BV},t} = Q_{\mathrm{BV,BF}} + tQ'$, where Q' is the isomorphism

$$Q' \colon \Omega_X^{0,\bullet} \otimes \Omega_Y^{\bullet,\bullet} \otimes \Omega_M^{\bullet} \otimes \mathfrak{g}[1] \to \Omega_X^{\dim(X),\bullet} \otimes \Omega_Y^{\bullet,\bullet} \otimes \Omega_M^{\bullet} \otimes \mathfrak{g}^*$$

induced by the Calabi-Yau structure on X and the non-degenerate pairing on $\mathfrak{g}$.

Using our mapping stack terminology, we will refer to the Hodge deformation of generalized BF theory as

$$\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}}).$$

1.5.2 Generalized Chern–Simons Theory

The next class of examples of classical BV theories we give are generalizations of Chern-Simons theory. Unlike the example of the generalized BF theory, these theories are not of cotangent type.

Definition 1.26. Let X and Y be complex manifolds and M a smooth oriented manifold. Fix an L_∞ algebra $\mathfrak{g}$. We assume X is equipped with a holomorphic volume form $\Omega_X \in \Omega^{\dim(X),0}(X)$ and $\mathfrak{g}$ is equipped with a nondegenerate invariant symmetric pairing $\langle -,-\rangle\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{C}[\dim_{\mathbb{C}}(X)+2\dim_{\mathbb{C}}(Y)+\dim(M)-3]$. The ***generalized Chern–Simons theory*** is the following classical BV theory:

- The spacetime is the smooth manifold $X\times Y\times M$.
- The bundle of BV fields is the $\mathbb{Z}$-graded bundle $E=\Omega^{0,\bullet}_X\otimes\Omega^{\bullet,\bullet}_Y\otimes\Omega^\bullet_M\otimes\mathfrak{g}[1]$.
- $Q=\overline{\partial}_X+\overline{\partial}_Y+\mathrm{d}_{\mathrm{dR},M}+\ell_1$.
- The pairing $\omega\colon E\otimes E\to\mathrm{Dens}_M[-1]$ is given by the combination of the wedge product of differential forms, integration $\int_{X\times Y\times M}\Omega_X\wedge(-)$ and the pairing $\langle -,-\rangle$ on $\mathfrak{g}$.
- The interaction term is
$$I=\sum_{k\geq 2}\frac{1}{(k+1)!}\int_{X\times Y\times M}\Omega_X\wedge\langle A\wedge\ell_k(A,\dots,A)\rangle.$$

We denote the space of BV fields by $\mathcal{E}=\mathrm{Map}(X\times Y_{\mathrm{Dol}}\times M_{\mathrm{dR}},B\mathfrak{g})$.

We may also consider a $\mathbb{Z}/2\mathbb{Z}$-graded version of the above theory where $\mathfrak{g}$ is merely $\mathbb{Z}/2\mathbb{Z}$-graded.

Example 1.27. For $X=Y=\mathrm{pt}$, M a 3-manifold and $\mathfrak{g}$ an ordinary Lie algebra we recover the usual 3-dimensional Chern-Simons theory with the BV action
$$S=\int_M\left(\frac{1}{2}\langle A\wedge\mathrm{d}_{\mathrm{dR}}A\rangle+\frac{1}{6}\langle A\wedge[A\wedge A]\rangle\right).$$
More generally, if $X=Y=\mathrm{pt}$ and M is any d-dimensional manifold where d is odd, we recover d-dimensional Chern-Simons theory. This has the same BV action, where now A is a (not necessarily homogeneous) differential form on M. If d is not 3 this theory is only $\mathbb{Z}/2\mathbb{Z}$-graded.

Example 1.28. For $Y=M=\mathrm{pt}$, X a Calabi-Yau 3-fold and $\mathfrak{g}$ an ordinary Lie algebra we recover the holomorphic Chern–Simons theory with the BV action
$$S=\int_X\Omega_X\wedge\left(\frac{1}{2}\langle A\wedge\overline{\partial}A\rangle+\frac{1}{6}\langle A\wedge[A\wedge A]\rangle\right).$$
As in the previous example, this still makes sense if X is a Calabi-Yau d-fold with d odd, as a $\mathbb{Z}/2\mathbb{Z}$-graded theory.

Example 1.29. If $\mathfrak{h}$ is an L_∞ algebra, $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{h}^*[d-3]$ carries a natural L_∞ structure given by combining the original L_∞ structure on the first term and the coadjoint action of the first term on the second term. The L_∞ algebra $\mathfrak{g}$ carries a natural symmetric pairing of degree $d-3$ given by the obvious pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$. Generalized Chern–Simons theory for $\mathfrak{g}$ in this case recovers the generalized BF theory from Definition 1.18.

Example 1.30. Let X,Y,M be as before and denote $d=\dim_{\mathbb{C}}(X)+2\dim_{\mathbb{C}}(Y)+\dim(M)$. Suppose $\mathfrak{g}$ is a dg Lie algebra and V is a $\mathfrak{g}$-representation equipped with a $(d-1)$-shifted symplectic structure $V\otimes V\to\mathbb{C}[d-1]$. Consider the dg Lie algebra
$$\mathcal{L}=\mathfrak{g}\oplus V[-1]\oplus\mathfrak{g}^*[d-3]$$
with the brackets $\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ given by the Lie bracket on $\mathfrak{g}$, $\mathfrak{g}\otimes V\to V$ given by the $\mathfrak{g}$-action on V, $\mathfrak{g}\otimes\mathfrak{g}^*\to\mathfrak{g}^*$ given by the coadjoint action and $\mu\colon V\otimes V\to\mathfrak{g}^*[d-1]$ defined by $(\mu(v,w),x)_{\mathfrak{g}}=([x,v],w)_V$. The dg Lie algebra

$\mathcal{L}$ carries nondegenerate invariant symmetric pairing of cohomological degree $d-3$ given by pairing $\mathfrak{g}$ and $\mathfrak{g}^*$ and pairing V with itself. The space of fields in the corresponding Chern–Simons theory will be denoted by

$$\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, V /\!/ \mathfrak{g}) := \mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathcal{L}).$$

Note that if X is merely a complex manifold equipped with a square root $K_X^{1/2}$ of the canonical bundle, we may still define the Chern–Simons theory with the bundle of BV fields

$$\begin{aligned} E =&\Omega_X^{0,\bullet} \otimes \Omega_Y^{\bullet,\bullet} \otimes \Omega_M^\bullet \otimes \mathfrak{g}[1] \\ &\oplus \Omega_X^{0,\bullet} \otimes K_X^{1/2} \otimes \Omega_Y^{\bullet,\bullet} \otimes \Omega_M^\bullet \otimes V \\ &\oplus \Omega_X^{\dim_{\mathbb{C}} X,\bullet} \otimes \Omega_Y^{\bullet,\bullet} \otimes \Omega_M^\bullet \otimes \mathfrak{g}^*[d-2] \end{aligned}$$

and the action functional as before.

As for BF theories, the Chern–Simons theory on $\mathbb{R}^n$ carries a natural rotation action.

Proposition 1.31. Suppose $\mathfrak{g}$ is an L_∞ algebra equipped with a nondegenerate invariant symmetric pairing of degree $n_1 + 2n_2 + n_3 - 3$. Consider the generalized Chern–Simons theory on $\mathbb{R}^{2n_1+2n_2+n_3}$ with the space of BV fields

$$\mathcal{E} = \mathrm{Map}(\mathbb{C}^{n_1} \times (\mathbb{C}^{n_2})_{\mathrm{Dol}} \times (\mathbb{R}^{n_3})_{\mathrm{dR}}, B\mathfrak{g}).$$

Then it carries an action of $\mathrm{SU}(n_1) \times \mathrm{U}(n_2) \times \mathrm{SO}(n_3)$ given by the pullback action on differential forms on $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{R}^{n_3}$.

We may slightly enhance the previous proposition if we are in the setting of Example 1.30 and we choose a square root of the canonical bundle. We define the unitary metalinear group to be

$$\mathrm{MU}(n) = \mathrm{U}(n) \times_{\mathrm{U}(1)} \mathrm{U}(1),$$

where $\mathrm{U}(n) \to \mathrm{U}(1)$ is the determinant map and $\mathrm{U}(1) \to \mathrm{U}(1)$ is the map $z \mapsto z^2$. We denote by $\det^{1/2}\colon \mathrm{MU}(n) \to \mathrm{U}(1)$ the projection on the second factor; this may be thought of as a square root of the determinant representation of $\mathrm{U}(n)$. The natural $\mathrm{U}(n)$-action on $\mathbb{C}^n$ lifts to an $\mathrm{MU}(n)$-action on the bundle of half-densities $K_{\mathbb{C}^n}^{1/2} \to \mathbb{C}^n$.

Proposition 1.32. Suppose $\mathfrak{g}$ is a dg Lie algebra and V a $\mathfrak{g}$-representation equipped with a $(n_1 + 2n_2 + n_3 - 1)$-shifted symplectic structure. Consider the generalized Chern–Simons theory on $\mathbb{R}^{2n_1+2n_2+n_3}$ with the space of BV fields

$$\mathcal{E} = \mathrm{Map}(\mathbb{C}^{n_1} \times (\mathbb{C}^{n_2})_{\mathrm{Dol}} \times (\mathbb{R}^{n_3})_{\mathrm{dR}}, V /\!/ \mathfrak{g}).$$

Then it carries an action of $\mathrm{MU}(n_1) \times \mathrm{U}(n_2) \times \mathrm{SO}(n_3)$ given by the pullback action on differential forms on $\mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \times \mathbb{R}^{n_3}$.

1.5.3 Generalized Hodge theory

We will also consider deformations of the generalized BF theory to a perturbatively trivial theory. By a deformation we will mean a $\mathbb{C}[t]$-family of classical BV theories which reduce to the given theory at $t = 0$.

Given an L_∞ algebra $\mathfrak{g}$ we denote by $\mathfrak{g}_{\mathrm{Hod}}$ the $\mathbb{C}[t]$-linear L_∞ algebra

$$\mathfrak{g}_{\mathrm{Hod}} = \mathbb{C}[t] \otimes (\mathfrak{g} \oplus \mathfrak{g}[1])$$

with the L_∞ brackets coming from the L_∞ brackets on $\mathfrak{g}$ in the first term, where we consider $\mathfrak{g}[1]$ as the adjoint representation of $\mathfrak{g}$. The differential is given by the original differential on $\mathfrak{g}$ plus the term $t\,\mathrm{id}$ from the second summand to the first summand.

If $\mathfrak{g}$ carries a nondegenerate invariant symmetric paiirng of degree d, so does $\mathfrak{g}_{\mathrm{Hod}}$.

Definition 1.33. Let X and Y be complex manifolds and M a smooth oriented manifold. Fix an L_∞ algebra $\mathfrak{g}$. We assume X is equipped with a holomorphic volume form $\Omega_X \in \Omega^{\dim(X),0}(X)$ and $\mathfrak{g}$ is equipped with a nondegenerate invariant symmetric pairing $\langle -,- \rangle \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}[\dim_{\mathbb{C}}(X) + 2\dim_{\mathbb{C}}(Y) + \dim(M) - 3]$. The ***generalized Hodge theory*** is the $\mathbb{C}[t]$-linear classical BV theory given by the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$.

Proposition 1.34. The $t = 0$ specialization of the generalized Hodge theory with the space of fields $\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$ is isomorphic to the generalized BF theory with the space of fields $\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, T^*[d]B\mathfrak{g})$, where $d = \dim_{\mathbb{C}}(X) + 2\dim_{\mathbb{C}}(Y) + \dim(M) - 1$.

The specialization of the generalized Hodge theory at $t \neq 0$ is perturbatively trivial.

Proof. At $t = 0$ we get
$$\mathfrak{g}_{\mathrm{Hod}}|_{t=0} \cong \mathfrak{g} \oplus \mathfrak{g}[1] \cong \mathfrak{g} \oplus \mathfrak{g}^*[d-1],$$
where we use the symmetric bilinear pairing on $\mathfrak{g}$ in the second isomorphism. The first claim then follows from Example 1.29.

At $t \neq 0$ the L_∞ algebra $\mathfrak{g}_{\mathrm{Hod}}$ becomes acyclic which proves the second claim. □

1.6 Dimensional Reduction

In this section we formulate the procedure of dimensional reduction of a classical field theory. Fix a submersion $p \colon M \to N$ equipped with a fiberwise volume form, i.e. an isomorphism $p^*\mathrm{Dens}_N \cong \mathrm{Dens}_M$. The idea is that the *dimensional reduction* of a classical field theory on M along the submersion p is the theory obtained by restricting to those fields which are constant along the fibers of p. We will begin with an abstract definition of dimensional reduction, then prove that if $M = N \times \mathbb{R}^k$, and we consider field theories which are translation invariant along the fiber, then this procedure is well-defined.

Definition 1.35. We say that a classical field theory $(E_N, Q_N, \omega_N, I_N)$ on N is a ***dimensional reduction*** along p of the classical field theory $(E_M, Q_M, \omega_M, I_M)$ on a manifold M if one is given the data of an isomorphism $p^*E_N \cong E_M$ of the bundles of BV fields satisfying the following conditions:

- The diagram
$$\begin{tikzcd} p^*E_N \otimes p^*E_N \arrow[r, "\omega_N"] \arrow[d, "\sim"] & p^*\mathrm{Dens}_N[-1] \arrow[d, "\sim"] \\ E_M \otimes E_M \arrow[r, "\omega_M"] & \mathrm{Dens}_M[-1] \end{tikzcd}$$
is commutative.
- The diagram
$$\begin{tikzcd} \mathcal{E}_N \arrow[r, "Q_N"] \arrow[d, "p^*"] & \mathcal{E}_N[1] \arrow[d, "p^*"] \\ \mathcal{E}_M \arrow[r, "Q_M"] & \mathcal{E}_M[1] \end{tikzcd}$$
is commutative.
- Under the map $p^* \colon \mathcal{E}_N \to \mathcal{E}_M$ we have $p^*I_M = I_N$.

We have an obvious notion of isomorphisms of dimensional reductions: these are isomorphisms of classical field theories on N which are compatible with the isomorphisms $p^*E_N \cong E_M$. Thus, the collection of dimensional reductions of a given classical field theory on M forms a groupoid. In fact, when dimensional reduction makes sense, this groupoid is always contractible.

Proposition 1.36. Suppose $(E_M, Q_M, \omega_M, I_M)$ is a classical field theory on M and $p\colon M \to N$ is a homotopy equivalence. Then the groupoid of dimensional reductions of $(E_M, Q_M, \omega_M, I_M)$ is either contractible or empty.

Suppose $M = N \times \mathbb{R}$ and choose a translation-invariant density along the $\mathbb{R}$ direction. If the original classical field theory is translation-invariant along the $\mathbb{R}$ direction, dimensional reductions exist.

Proof. **Uniqueness**. We begin by showing that any two dimensional reductions are isomorphic and moreover such an isomorphism is unique if it exists. Since $p\colon M \to N$ is a homotopy equivalence, the functor p^* establishes an isomorphism between the category of graded vector bundles on N and on M. In a similar way, p^* establishes an equivalence between the category of graded vector bundles E_N on N equipped with a nondegenerate pairing $E_N \otimes E_N \to \mathrm{Dens}_N[-1]$ and a similar category for M.

Since $\mathcal{E}_N \to \mathcal{E}_M$ is injective, the diagram

$$\begin{tikzcd} \mathcal{E}_N \arrow[r, "Q_N"] \arrow[d, "p^*"] & \mathcal{E}_N[1] \arrow[d, "p^*"] \\ \mathcal{E}_M \arrow[r, "Q_M"] & \mathcal{E}_M[1] \end{tikzcd}$$

uniquely determines Q_N from Q_M. Moreover, the condition $p^* I_M = I_N$ uniquely determines N.

Existence. Now suppose $(E_M, Q_M, \omega_M, I_M)$ is translation-invariant along the $\mathbb{R}$ direction. Translation invariance provides the descent datum to construct the bundle of fields E_N on N equipped with a nondegenerate pairing ω_N. Moreover, it shows that the differential Q_M preserves the subspace $\mathcal{E}_N \hookrightarrow \mathcal{E}_M$. The restriction of I_M under the same embedding is independent of the $\mathbb{R}$ factor by translation invariance, so $I_N = p^* I_M$ is again a local functional. ☐

Remark 1.37. Therefore, it makes sense to talk about "the" dimensional reduction of a classical field theory along the projection $p\colon N \times \mathbb{R} \to N$: there exists a dimensional reduction which is unique up to a canonical isomorphism.

We will now describe dimensional reductions of the classical Chern–Simons theory.

Proposition 1.38. Let X and Y be complex manifolds and M a smooth manifold. Fix an L_∞ algebra $\mathfrak{g}$ equipped with a nondegenerate invariant pairing and consider the generalized Chern–Simons theory

$$\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times (M \times \mathbb{R})_{\mathrm{dR}}, B\mathfrak{g}).$$

Its dimensional reduction along the projection $p\colon X \times Y \times (M \times \mathbb{R}) \to X \times Y \times M$ is isomorphic to the generalized BF theory

$$\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, T^*[\dim_{\mathbb{C}}(X) + 2\dim_{\mathbb{C}}(Y) + \dim(M) - 1]B\mathfrak{g}).$$

Proof. To simplify the proof, we assume $X = Y = M = \mathrm{pt}$. Then $\mathfrak{g}$ carries a (-2)-shifted pairing $\langle -, - \rangle$. In particular, the generalized BF theory

$$\mathrm{Map}(\mathrm{pt}, T^*[-1]B\mathfrak{g})$$

has the bundle of BV fields $\mathfrak{g}[1] \oplus \mathfrak{g}^*[-2]$. We may identify it with $\mathfrak{g}[1] \oplus \mathfrak{g}$, where the pairing ω_N pairs the two factors using $\langle -, - \rangle$.

We may identify $p^*(\mathfrak{g}[1] \oplus \mathfrak{g}) \cong \Omega^\bullet_{\mathbb{R}} \otimes \mathfrak{g}[1]$. Under this identification the integration pairing ω_M on differential forms reduces to the pairing ω_N. The de Rham differential vanishes on translation-invariant forms which shows a compatibility of dimensional reduction with the differentials Q_{BV}. Finally, in both cases the interaction term comes from the L_∞ structure on $\mathfrak{g}$. ☐

Corollary 1.5. Let $X, Y, M, \mathfrak{g}$ be as before. Consider the generalized Hodge theory

$$\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times (M \times \mathbb{R})_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}}).$$

Its dimensional reduction along the projection $p\colon X \times Y \times (M \times \mathbb{R}) \to X \times Y \times M$ is isomorphic to the generalized Hodge theory

$$\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, T^*[\dim_{\mathbb{C}}(X) + 2\dim_{\mathbb{C}}(Y) + \dim(M) - 1]B\mathfrak{g}_{\mathrm{Hod}}).$$

Let $V_{\mathbb{R}}$ be a real vector space equipped with a nondegenerate symmetric bilinear pairing and an orientation. Recall that the symmetric bilinear pairing trivializes $\det(V_{\mathbb{R}})^{\otimes 2}$ and the orientation allows us to obtain a trivialization of $\det(V_{\mathbb{R}})$, i.e. a real volume form. We denote by $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. V inherits a nondegenerate Hermitian form from the symmetric bilinear pairing on $V_{\mathbb{R}}$. Moreover, V carries a complex volume form Ω_V:

$$\det(V) \cong \det(V_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$$

and we use the real volume form on $V_{\mathbb{R}}$.

Functoriality of this construction gives a group homomorphism

$$\mathrm{SO}(V_{\mathbb{R}}) \longrightarrow \mathrm{SU}(V) \tag{1}$$

such that the real projection $\mathrm{Re}\colon V \to V_{\mathbb{R}}$ is $\mathrm{SO}(V_{\mathbb{R}})$-equivariant.

Proposition 1.39. Let $X, Y, M, \mathfrak{g}$ be as before and $V_{\mathbb{R}}, V$ as above. Consider the generalized Chern–Simons theory

$$\mathrm{Map}((X \times V) \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g}).$$

Its dimensional reduction along the projection $p\colon (X \times V) \times Y \times M \to X \times Y \times (M \times V_{\mathbb{R}})$ induced by the projection $\mathrm{Re}\colon V \to V_{\mathbb{R}}$ is isomorphic to the generalized Chern–Simons theory

$$\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times (M \times V_{\mathbb{R}})_{\mathrm{dR}}, B\mathfrak{g}).$$

The isomorphism is moreover $\mathrm{SO}(V_{\mathbb{R}})$-equivariant.

Proof. We may assume $X, Y, M = \mathrm{pt}$ as in the previous proof.

We have

$$\Omega_V^{0,\bullet} \cong \mathrm{Sym}^\bullet_{\mathbb{C}}(\underline{V}[-1])$$

and

$$\Omega^\bullet_{V_{\mathbb{R}}} \cong \mathrm{Sym}^\bullet_{\mathbb{R}}(\underline{V}_{\mathbb{R}}[-1]) \otimes_{\mathbb{R}} \mathbb{C}.$$

Under the pullback map

$$\Omega^\bullet(V_{\mathbb{R}}; \mathbb{C}) \to \Omega^\bullet(V) \to \Omega^{0,\bullet}(V)$$

the integration map $\int_V \Omega_V \wedge (-)\colon \Omega^{0,\bullet}(V) \to \mathbb{C}$ reduces to the integration map $\int_{V_{\mathbb{R}}}(-)\colon \Omega^\bullet(V_{\mathbb{R}}; \mathbb{C}) \to \mathbb{C}$. (Brian: This is not defined) (Chris: is compactly supported enough? We want to compare pairings.) □

Corollary 1.6. Let $X, Y, M, V_{\mathbb{R}}, V$ be as before. Consider the generalized BF theory

$$\mathrm{Map}((X \times V) \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, T^*[d]B\mathfrak{g}).$$

Its dimensional reduction along the projection $p\colon (X \times V) \times Y \times M \to X \times Y \times (M \times V_{\mathbb{R}})$ induced by the projection $\mathrm{Re}\colon V \to V_{\mathbb{R}}$ is isomorphic to the generalized BF theory

$$\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times (M \times V_{\mathbb{R}})_{\mathrm{dR}}, T^*[d]B\mathfrak{g}).$$

The isomorphism is moreover $\mathrm{SO}(V_{\mathbb{R}})$-equivariant.

Corollary 1.7. Let $X, Y, M, V_\mathbb{R}, V$ be as before. Consider the generalized Hodge theory

$$\mathrm{Map}((X \times V) \times Y_{\mathrm{Dol}} \times M_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}}).$$

Its dimensional reduction along the projection $p\colon (X \times V) \times Y \times M \to X \times Y \times (M \times V_\mathbb{R})$ induced by the projection $\mathrm{Re}\colon V \to V_\mathbb{R}$ is isomorphic to the generalized Hodge theory

$$\mathrm{Map}(X \times Y_{\mathrm{Dol}} \times (M \times V_\mathbb{R})_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}}).$$

The isomorphism is moreover $\mathrm{SO}(V_\mathbb{R})$-equivariant.

2 Supersymmetry

Having set up the formalism behind classical field theories in the BV and BRST formalisms, we will introduce the other main formal ingredient of this paper: the supersymmetry action. So, we will discuss the classification of supersymmetry algebras, the notion of a supersymmetric field theory, and the idea of a *twist* of a supersymmetric field theory, extending work of the first two authors in [**ElliottSafronov**]. We will introduce the classification of supersymmetry algebras using the division algebra perspective of Baez and Huerta [**BaezHuerta**], which will be useful for the classification of super Yang-Mills theories in Section 3 below.

2.1 Supersymmetry Algebras

In this section we will recall the framework for supersymmetry algebras and their classification following Deligne [**DeligneSpinors**] and our previous work [**ElliottSafronov**], we refer there for more details.

Let $V_\mathbb{R}$ be a finite-dimensional real vector space of dimension $n = \dim_\mathbb{R}(V_\mathbb{R})$ equipped with an orientation and a nondegenerate symmetric bilinear form. Denote by $V = V_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ its complexification. Consider the Lie algebra $\mathfrak{so}(V)$. Let us recall the following facts:

- If n is odd, $\mathfrak{so}(V)$ has a distinguished fundamental representation called the ***spin*** representation S.
- If n is even, $\mathfrak{so}(V)$ has a pair of distinguished fundamental representations called the ***semi-spin*** representations S_+ and S_-.

Definition 2.1. A ***spinorial representation*** Σ is a finite sum of spin or semi-spin representations of $\mathfrak{so}(V)$.

So, in odd dimensions we have $\Sigma = S \otimes W$ and in even dimensions we have $\Sigma = S_+ \otimes W_+ \oplus S_- \otimes W_-$, where W denotes a multiplicity space.

We have an embedding $\mathrm{U}(n) \subset \mathrm{SO}(2n, \mathbb{R})$ which lifts to an embedding $\mathrm{MU}(n) \subset \mathrm{Spin}(2n, \mathbb{R})$. If we denote by L the standard n-dimensional representation of $\mathrm{U}(n)$, then the semi-spin representations of $\mathrm{Spin}(2n, \mathbb{R})$ restrict to $\mathrm{MU}(n)$ as

$$S_+ \cong \det(L)^{-1/2} \otimes \wedge^{\mathrm{even}} L, \qquad S_- \cong \det(L)^{-1/2} \otimes \wedge^{\mathrm{odd}} L.$$

Definition 2.2. Fix a spinorial representation Σ and a nondegenerate $\mathfrak{so}(V)$-equivariant pairing $\Gamma\colon \mathrm{Sym}^2(\Sigma) \to V$. The ***supertranslation Lie algebra*** is the $\mathfrak{so}(V)$-equivariant super Lie algebra $\mathfrak{A} = \Pi\Sigma \oplus V$ whose only nontrivial bracket is given by Γ.

For a given spinorial representation, the pairing Γ is typically unique up to a scale, so a supertranslation Lie algebra is specified by fixing a spinorial representation. In turn, a spinorial representation is determined by the dimension of the multiplicity space, so we will talk about $\mathcal{N}$ or $(\mathcal{N}_+, \mathcal{N}_-)$ supertranslation Lie algebras, where the numbers are specified as follows.

- If $n \equiv 0, 1, 3, 4 \pmod 8$, we let $\mathcal{N} = \dim(W)$.
- If $n \equiv 2 \pmod 8$, we let $\mathcal{N}_\pm = \dim(W_\pm)$.
- If $n \equiv 5, 7 \pmod 8$, we let $2\mathcal{N} = \dim(W)$.
- If $n \equiv 6 \pmod 8$, we let $2\mathcal{N}_\pm = \dim(W_\pm)$.

Fix the following data:

- A spinorial representation Σ of $\mathfrak{so}(V)$.
- An $\mathfrak{so}(V)$-equivariant nondegenerate pairing $\Gamma\colon \operatorname{Sym}^2(\Sigma) \to V$.
- A Lie group G_R, the ***group of*** R***-symmetries***, which acts on Σ by $\mathfrak{so}(V)$-equivariant automorphisms preserving Γ.

Note that the supertranslation Lie algebra $\mathfrak{A}$ is a $\operatorname{Spin}(V_\mathbb{R}) \times G_R$-equivariant super Lie algebra. We will sometimes want to refer to the infinitesimal version of this action.

Definition 2.3. Let $\mathfrak{A}$ be a supertranslation algebra. The corresponding *supersymmetry algebra* is the Lie algebra $(\mathfrak{so}(V) \oplus \mathfrak{g}_R) \ltimes \mathfrak{A}$.

We will now define the fundamental notion of a supersymmetric field theory. Consider a spacetime manifold $M = V_\mathbb{R}$. Let $\operatorname{ISO}(V_\mathbb{R}) = \operatorname{Spin}(V_\mathbb{R}) \ltimes V_\mathbb{R}$ be the ***Poincaré group*** which acts by affine transformations on M.

Definition 2.4. A classical field theory (E, S, ω) is ***supersymmetric*** if $E \to M$ is an $\operatorname{ISO}(V_\mathbb{R}) \times G_R$-equivariant vector bundle and the infinitesimal strict action of the translation Lie algebra V on the classical theory is extended to a $\operatorname{Spin}(V_\mathbb{R}) \times G_R$-equivariant L_∞ action of the supertranslation Lie algebra $\mathfrak{A}$ on the classical theory.

2.2 Composition Algebras and Minimal Supersymmetry

We will now recall a relationship between certain "minimal" supersymmetry algebras and composition algebras. Our treatment will essentially follow that of Baez and Huerta [**BaezHuerta**].

Let A be a unital (possibly non-associative) complex algebra equipped with an antiinvolution $\sigma\colon A \to A$. We make the following assumptions:

1. The map $\operatorname{Re}(x) = x \mapsto (x + \sigma(x))/2$ defines a projector onto the subspace of A spanned by the unit.
2. By the previous assumption we have a quadratic form $x\sigma(x)\colon A \to \mathbb{C}$. We assume that it is nondegenerate.

In fact, the data of the antiinvolution σ may equivalently be encoded in the data of a non-degenerate multiplicative norm $x \mapsto x\sigma(x)$, i.e. A is a real composition algebra [**SpringerVeldkamp**].

For a 2×2-matrix M with entries in A we define its hermitian adjoint $M^\dagger$ by transposing the matrix and applying σ to the entries. We define V to be the space of 2×2 Hermitian matrices with values in A, so that $\dim(V) = \dim(A) + 2$. V carries a nondegenerate quadratic form given by $M \mapsto -\det(M)$. Moreover, it carries an orthogonal involution $\widetilde{M} = M - \operatorname{Tr}(M) \cdot 1$.

Given a left A-module, we may turn it into a right A-module via the antiinvolution $\sigma\colon A \to A$. Since A is a Frobenius algebra, we have the following basic construction

Lemma 2.5. Suppose M is a left A-module and N a right A-module equipped with a nondegenerate pairing $(-,-)\colon M \otimes N \to \mathbb{C}$ satisfying $(am, n) = (m, na)$ for every $a \in A$, $m \in M$ and $n \in N$. Then there is a unique map $(-,-)^A\colon M \otimes N \to A$ of (A, A)-bimodules whose real part is $(-,-)$.

Consider two isomorphic spinorial representations $\Sigma = A \oplus A$ and $\Sigma^* = A \oplus A$; we will equip the sum of these representations with an action of the Clifford algebra. The action on the two summands Σ and Σ^* will be different, hence the different notation. We define action maps

$$\rho\colon V \otimes \Sigma \to \Sigma^*, \qquad \rho\colon V \otimes \Sigma^* \to \Sigma$$

given, respectively, by

$$\rho(M)Q = MQ, \qquad \rho(M)Q = \widetilde{M}Q.$$

The following is proved in [**BaezHuerta**].

Proposition 2.6. The action maps $V \otimes \Sigma \to \Sigma^*$ and $V \otimes \Sigma^* \to \Sigma$ satisfy the Clifford relation

$$\rho(M)\rho(M) = -\det(M) \cdot 1.$$

So, $\Sigma \oplus \Sigma^*$ forms a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford module.

We have a nondegenerate scalar spinorial pairing

$$\Sigma \otimes \Sigma^* \longrightarrow \mathbb{C}$$

given by

$$(Q_1, Q_2) = \mathrm{Re}(Q_1^\dagger Q_2).$$

It obviously satisfies $(Q_1\sigma(a), Q_2) = (Q_1, Q_2 a)$. The extension to an A-valued pairing provided by Lemma 2.5 is given by

$$(Q_1, Q_2)^A = Q_1^\dagger Q_2.$$

By duality we obtain maps $\Gamma\colon \mathrm{Sym}^2(\Sigma) \to V$ and $\Gamma\colon \mathrm{Sym}^2(\Sigma^*) \to V$ given, respectively, by

$$\Gamma(Q_1, Q_2) = Q_1\widetilde{Q_2^\dagger + Q}_2 Q_1^\dagger, \qquad \Gamma(Q_1, Q_2) = Q_1 Q_2^\dagger + Q_2 Q_1^\dagger.$$

We will now state two important properties of Γ and the spinorial pairing. The following statement was proved in [**BaezHuerta**] (see also [**Schray**] for the case $\dim(V) = 10$).

Theorem 2.1. Suppose A is alternative, i.e. $a \otimes b \otimes c \mapsto (ab)c - a(bc)$ is completely antisymmetric. For $Q_1, Q_2, Q_3 \in \Sigma$ we have

$$\rho(\Gamma(Q_1, Q_2))Q_3 + \rho(\Gamma(Q_2, Q_3))Q_1 + \rho(\Gamma(Q_2, Q_3))Q_1 = 0.$$

If we moreover assume A is associative, there is a relationship between the scalar spinorial pairing and Γ.

Theorem 2.2. Suppose A is associative. For $Q_1, Q_2 \in \Sigma$ and $Q_3 \in \Sigma^*$ we have

$$Q_1(Q_2, Q_3)^A + Q_2(Q_1, Q_3)^A = \rho(\Gamma(Q_1, Q_2))Q_3.$$

Proof. The right-hand side is
$$(Q_1 Q_2^\dagger + Q_2 Q_1^\dagger) Q_3$$
which by associativity can be rewritten as
$$Q_1(Q_2^\dagger Q_3) + Q_2(Q_1^\dagger Q_3)$$
which is the left-hand side. $\square$

We will be interested in the following examples:

1. (**3d** $\mathcal{N} = 1$ **supersymmetry**) $A = \mathbb{C}$. We have $\dim(V) = 3$ and Σ is the spin representation of $\mathrm{Spin}(3, \mathbb{C})$.
2. (**4d** $\mathcal{N} = 1$ **supersymmetry**) $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}[x]/(x^2 + 1)$ with $\sigma(x) = -x$. Moreover, $\dim(V) = 4$ and $\Sigma = S_+ \oplus S_-$ is the sum of semi-spin representations.
3. (**6d** $\mathcal{N} = (1, 0)$ **supersymmetry**) $A = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{End}(W_+)$, where W_+ is a two-dimensional symplectic vector space with σ given by the dual operator. Moreover, $\dim(V) = 6$ and $\Sigma = S_+ \otimes W_+$.
4. (**10d** $\mathcal{N} = (1, 0)$ **supersymmetry**) $A = \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$. We have $\dim(V) = 10$ and $\Sigma = S_+$ is a semi-spin representation.

All four examples are alternative, while the first three examples are moreover associative.

2.3 Two-dimensional Chiral Supersymmetry

In the previous section we have related composition algebras to minimal supersymmetry algebras in dimensions 3, 4, 6 and 10. In this section we explain a different relationship between composition algebras and supersymmetry algebras, this time in the case of 2d $\mathcal{N} = (\mathcal{N}_+, 0)$ supersymmetry.

Recall that in the case $\dim(V) = 2$ we have two one-dimensional semi-spin representations S_+, S_-. Moreover, we have an isomorphism
$$V \cong S_+^{\otimes 2} \oplus S_-^{\otimes 2}$$
and a pairing $(-,-)\colon S_+ \otimes S_- \to \mathbb{C}$, both of which are $\mathfrak{so}(V)$-equivariant. We denote the embeddings $S_\pm^{\otimes 2} \subset V$ by $\Gamma_\pm$, so that
$$(\Gamma_+(s, s), \Gamma_-(f, f)) = 2(s, f)^2. \tag{2}$$

Let W be a complex vector space of dimension $\mathcal{N}_+$ equipped with a nondegenerate symmetric bilinear pairing. We consider the spinorial representation
$$\Sigma = S_+ \otimes W$$
and its dual
$$\Sigma^* = S_- \otimes W.$$
The Clifford action $V \otimes S_+ \to S_-$ is defined so that
$$\rho(\Gamma_-(f, f))s = 2(s, f)f$$
and similarly for $V \otimes S_- \to S_+$.

Proposition 2.7. For $v, w \in V$ and $s \in S_+ \oplus S_-$ we have
$$\rho(v)\rho(w)s + \rho(w)\rho(v)s = 2(v, w)s.$$

Proof. It is enough to prove the claim for $s \in S_+$, $w \in S_+^{\otimes 2}$ and $v \in S_-^{\otimes 2}$. Assume $w = \Gamma_+(s,s)$ and $v = \Gamma_-(f,f)$ for $f \in S_-$. Then we have

$$\begin{aligned}\rho(\Gamma_+(s,s))\rho(\Gamma_-(f,f))s &= 2(s,f)\rho(\Gamma_+(s,s))f \\ &= 4(s,f)^2 s.\end{aligned}$$

But by (2) we have

$$(\Gamma_+(s,s),\Gamma_-(f,f)) = 2(s,f)^2$$

which proves the claim. □

The Clifford action $V \otimes S_\pm \to S_\mp$ extends in an obvious way to a Clifford action $V \otimes \Sigma \to \Sigma^*$ and $V \otimes \Sigma^* \to \Sigma$ given by the identity on the W component. Thus, $\Sigma \oplus \Sigma^*$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford module.

The spaces Σ, Σ^* are equipped with $\mathfrak{so}(V)$-equivariant nondegenerate pairings $\Gamma\colon \mathrm{Sym}^2(\Sigma) \to V$ defined by

$$\Gamma(s \otimes q_1, s \otimes q_2) = \Gamma_+(s,s)(q_1,q_2)$$

and $\Gamma\colon \mathrm{Sym}^2(\Sigma^*) \to V$ defined similarly.

Proposition 2.8. For any $v \in V$ and $Q_1, Q_2 \in \Sigma$ or $Q_1, Q_2 \in \Sigma^*$ we have

$$(v, \Gamma(Q_1,Q_2)) = (\rho(v)Q_1, Q_2).$$

Proof. It is enough to prove the claim with $Q_1, Q_2 \in \Sigma$. Assume $v = \Gamma_-(f,f)$ for some $f \in S_-$, $Q_1 = s \otimes q_1$ and $Q_2 = s \otimes q_2$. Then the left-hand side is

$$(\Gamma_-(f,f),\Gamma_+(s,s))(q_1,q_2) = 2(s,f)^2(q_1,q_2).$$

The right-hand side is

$$\begin{aligned}(\rho(\Gamma_-(f,f)s \otimes q_1, s \otimes q_2) &= (2(s,f)f \otimes q_1, s \otimes q_2) \\ &= 2(s,f)^2(q_1,q_2).\end{aligned}$$

□

An important property of two-dimensional chiral supersymmetry is the following analog of Theorem 2.1.

Theorem 2.3. For $Q_1, Q_2, Q_3 \in \Sigma$ we have

$$\rho(\Gamma(Q_1,Q_2))Q_3 = 0.$$

Proof. Indeed, $\Gamma(Q_1,Q_2)$ lies in $S_+^{\otimes 2} \subset V$, but the nonzero Clifford action is given by

$$S_-^{\otimes 2} \otimes (S_+ \otimes W) \longrightarrow S_- \otimes W.$$

□

We will now fix a composition algebra A with an antiinvolution σ as in Section 2.2 and set $W = A$. The nondegenerate symmetric bilinear pairing $a_1, a_2 \mapsto \mathrm{Re}(a_1\sigma(a_2))$ on A endows W with a pairing. Both Σ and Σ^* are right A-modules and the Clifford actions $V \otimes \Sigma \to \Sigma^*$ and $V \otimes \Sigma^* \to \Sigma$ are maps of right A-modules.

Since Σ and Σ^* are right A-modules, by Lemma 2.5 we may extend the scalar spinorial pairing to an A-valued pairing $\Sigma \otimes \Sigma^* \to A$ by

$$(s_1 \otimes q_1, s_2 \otimes q_2)^A = (s_1,s_2)\sigma(q_1)q_2.$$

Theorem 2.4. For $Q_1, Q_2 \in \Sigma$ and $Q_3 \in \Sigma^*$ we have

$$Q_1(Q_2, Q_3)^A + Q_2(Q_1, Q_3)^A = \rho(\Gamma(Q_1, Q_2))Q_3.$$

Proof. Pick basis elements $s \in S_+$ and $f \in S_-$, so that

$$Q_1 = s \otimes q_1, \qquad Q_2 = s \otimes q_2, \qquad Q_3 = f \otimes q_3.$$

The right-hand side is

$$\begin{aligned}(q_1, q_2)\rho(\Gamma_+(s, s)) f \otimes q_3 &= 2(s, f)s \otimes (q_1, q_2)q_3 \\ &= (s, f)s \otimes (q_1\sigma(q_2) + q_2\sigma(q_1))q_3.\end{aligned}$$

We have

$$Q_1(Q_2, Q_3)^A = s \otimes q_1(s, f)(\sigma(q_2)q_3),$$

so the left-hand side is

$$s(s, f) \otimes (q_1(\sigma(q_2)q_3) + q_2(\sigma(q_1)q_3)).$$

By associativity of A the two expressions are equal. □

2.4 Supersymmetric Twisting

The idea of twisting, originally developed by Witten [**WittenTQFT**], is to modify a classical BV theory by deforming the differential Q_{BV} by the action of a square-zero fermionic symmetry.

Definition 2.9. A ***square-zero supercharge*** is a nonzero element $Q \in \Sigma$ such that $\Gamma(Q, Q) = 0$. Its ***number of invariant directions*** is the dimension of the image of $\Gamma(Q, -)\colon \Sigma \to V$.

It is shown in [**ElliottSafronov**] that the number d of invariant directions is at least $n/2$. We will use the following adjectives for square-zero supercharges depending on d:

- A supercharge Q is ***topological*** if $d = n$.
- A supercharge Q is ***holomorphic*** if n is even and $d = n/2$.
- A supercharge Q is ***minimal*** if n is odd and $d = (n+1)/2$.

In the intermediate case we refer to Q as a ***holomorphic-topological*** (alternatively, partially topological) supercharge. The collection of all square-zero supercharges in dimensions 2 through 10 (where one restricts to supersymmetries with at most 16 supercharges) was studied in [**ElliottSafronov**] and [**EagerSaberiWalcher**]. In particular, the orbits of square-zero supercharges under the R-symmetry group, $\mathrm{Spin}(V)$ and the obvious scaling action of $\mathbb{C}^\times$ are shown in Figure 1.

Let (E, S, ω) be a supersymmetric classical field theory. Recall, this means we have a Maurer-Cartan element

$$I_{\mathfrak{A}} = \sum_{k \geq 0} I_{\mathfrak{A}}^{(k)} \in C^\bullet(\mathfrak{A}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))$$

where $I_{\mathfrak{A}}^{(k)}\colon \mathfrak{A}^{\otimes k} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ as in Definition 1.11 and the classical field theory has an action of the R-symmetry group G_R.

Definition 2.10. Suppose (E, S, ω) is a supersymmetric classical field theory and Q a square-zero supercharge. The ***Q-twisted classical field theory*** is the $\mathbb{Z}/2$-graded classical field theory with the same bundle of BV fields and the symplectic pairing ω and the BV action

$$S^Q = S + \sum_{k \geq 0} I^{(k)}_{\mathfrak{A}}(Q, \ldots, Q).$$

Given additional data, we may enhance the classical field theory.

Definition 2.11. Let $Q \in \Sigma$ be a square-zero supercharge. A homomorphism $\alpha\colon U(1) \to G_R$ is ***compatible*** with Q if Q has weight 1 and the α-grading on mod 2 on E coincides with the fermionic grading.

Given such an α we may consider a new $\mathbb{Z}$-grading on E given by the sum of the cohomological grading and the grading given by α. The map $I^{(k)}_{\mathfrak{A}} : \mathfrak{A}^{\otimes k} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ is G_R-equivariant, so the element $I^{(k)}_{\mathfrak{A}}(Q, \ldots, Q)$ has α-grading k. But it also has cohomological degree $-k$. In other words, the twisted action S^Q has total degree zero, so (E, S^Q, ω) is a $\mathbb{Z}$-graded classical field theory.

Definition 2.12. Let $Q \in \Sigma$ be a square-zero supercharge and suppose $G \to \mathrm{Spin}(V_{\mathbb{R}})$ is a fixed group homomorphism. A ***twisting homomorphism*** is a homomorphism $\phi\colon G \to G_R$ such that Q is preserved under the composite $G \to \mathrm{Spin}(V_{\mathbb{R}}) \times G_R$.

The classical field theory (E, S, ω) carries a $\mathrm{Spin}(V_{\mathbb{R}}) \times G_R$-action. However, the Q-twisted theory (E, S^Q, ω) does not in general carry a $\mathrm{Spin}(V_{\mathbb{R}}) \times G_R$-action since the elements $I^{(k)}_{\mathfrak{A}}(Q, \ldots, Q)$ are not in general invariant under $\mathrm{Spin}(V_{\mathbb{R}}) \times G_R$. However, given a twisting homomorphism ϕ we see that $I^{(k)}_{\mathfrak{A}}(Q, \ldots, Q)$ is preserved under G, so (E, S^Q, ω) carries a G-action.

2.5 Dimensional Reduction of Supersymmetric Theories

Suppose $V_{\mathbb{R}} = \mathbb{R}^n$ as before and choose a subspace $W_{\mathbb{R}} \subset V_{\mathbb{R}}$, so that $V_{\mathbb{R}} = W_{\mathbb{R}} \oplus W_{\mathbb{R}}^{\perp}$. We denote $W = W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

Fix a spinorial representation Σ of $\mathfrak{so}(V)$, a nondegenerate pairing $\Gamma_V\colon\ \mathrm{Sym}^2(\Sigma) \to V$ and a group of R-symmetries G_V. This datum generates a supersymmetry algebra, which we'll denote by $\mathfrak{A}$. We have a natural embedding

$$\mathfrak{so}(W) \oplus \mathfrak{so}(W^{\perp}) \subset \mathfrak{so}(V),$$

so Σ restricts to a spinorial $\mathfrak{so}(W)$ representation. We define the dimensionally reduced Γ-pairing as the composite

$$\Gamma_W\colon\ \mathrm{Sym}^2(\Sigma) \xrightarrow{\Gamma_V} V \to W,$$

where the last map is the orthogonal projection onto W. Finally, we have a new R-symmetry group

$$G_W = G_V \times \mathrm{Spin}(W_{\mathbb{R}}^{\perp}).$$

This datum generates a supersymmetry algebra $\mathfrak{A}'$ in dimension $\dim(W_{\mathbb{R}})$ as defined in Section 2.1.

Recall from Proposition 1.36 that the dimensional reduction of a classical field theory along the projection $p\colon V_{\mathbb{R}} \to W_{\mathbb{R}}$ exists and is unique. We have the following generalization of this statement to supersymmetric theories.

Proposition 2.13. Suppose (E, Q, ω, I) is an $\mathfrak{A}$-supersymmetric classical field theory on $V_{\mathbb{R}}$. Then its dimensional reduction along the projection $p\colon V_{\mathbb{R}} \to W_{\mathbb{R}}$ has a unique $\mathfrak{A}'$-supersymmetric structure, compatible with the supersymmetry on $V_{\mathbb{R}}$ in the sense that $p^* I^{(i)}_{V_{\mathbb{R}}} = I^{(i)}_{W_{\mathbb{R}}}$.

Proof. This follows from the proof of Proposition 1.36 by coupling the theory (E, Q, ω, I) to auxiliary fields generated by the representation Σ. $\square$

The following proposition is an immediate consequence of Proposition 2.13 and Definition 2.10.

Proposition 2.14. Fix a twisting datum (Q, α), where $\Gamma_V(Q, Q) = 0$. Then the dimensional reduction of the twist of the classical field theory E is isomorphic to the twist of the dimensional reduction of E.

3 Supersymmetric Yang–Mills theories

In this section we construct supersymmetry action on super Yang–Mills theories. We have the following variants of the super Yang–Mills theory depending on $\dim(\Sigma)$:

- (16 supercharges). This theory exists in dimensions 2 through 10 and it depends on a Lie algebra $\mathfrak{g}$.
- (8 supercharges). This theory exists in dimensions 2 through 6 and it depends on a Lie algebra $\mathfrak{g}$ together with a symplectic $\mathfrak{g}$-representation U.
- (4 supercharges). This theory exists in dimensions 2 through 4 and it depends on a Lie algebra $\mathfrak{g}$ together with a $\mathfrak{g}$-representation R.
- (2 supercharges). This theory exists in dimensions 2 through 3 and it depends on a Lie algebra $\mathfrak{g}$ together with an orthogonal $\mathfrak{g}$-representation P.

There are a few additional possibilities that occur in dimension 2.

- ($\mathcal{N}_+$ supercharges, chiral supersymmetry). This theory exists in dimension 2 and depends on a Lie algebra $\mathfrak{g}$.
- (4 supercharges, chiral supersymmetry). This theory exists in dimension 2 and depends on a Lie algebra $\mathfrak{g}$ together with a symplectic $\mathfrak{g}$-representation U.
- (2 supercharges, chiral supersymmetry). This theory exists in dimension 2 and depends on a Lie algebra $\mathfrak{g}$ together with a $\mathfrak{g}$-representation R.
- (1 supercharge, chiral supersymmetry). This theory exists in dimension 2 and depends on a Lie algebra $\mathfrak{g}$ together with an orthogonal $\mathfrak{g}$-representation P.

In each case the lower-dimensional theories are obtained by dimensional reduction from the theory in the highest dimension: for instance, 7d $\mathcal{N} = 1$ super Yang–Mills (16 supercharges) is obtained by dimensional reduction from 10d $\mathcal{N} = (1, 0)$ super Yang–Mills. So, it will be enough to construct the supersymmetry action in these highest-dimensional theories.

3.1 Pure Super Yang–Mills Theory

We begin with a description of certain pure supersymmetric Yang–Mills theories. Let $V_{\mathbb{R}}$ be a real vector space equipped with a nondegenerate symmetric bilinear pairing and V its complexification. Fix a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford module $\Sigma \oplus \Sigma^* \to \mathbb{C}$ with the associated Γ-pairings

$$\Gamma\colon \mathrm{Sym}^2(\Sigma) \to V, \qquad \Gamma\colon \mathrm{Sym}^2(\Sigma^*) \to V$$

defined as in Appendix A. We make the following assumption on this setup.

Assumption 3.1. For $Q_1, Q_2, Q_3 \in \Sigma$ we have

$$\rho(\Gamma(Q_1, Q_2))Q_3 + \rho(\Gamma(Q_2, Q_3))Q_1 + \rho(\Gamma(Q_3, Q_1))Q_2 = 0.$$

Explicitly, we consider the following cases:

- (**2d** $\mathcal{N} = (\mathcal{N}_+, 0)$ **supersymmetry**) We have $\dim(V) = 2$ and $\Sigma = S_+ \otimes W$ for some complex vector space W equipped with a nondegenerate symmetric bilinear pairing. Assumption 3.1 is satisfied by Theorem 2.3.
- (**3d** $\mathcal{N} = 1$ **supersymmetry**) We have $\dim(V) = 3$ and $\Sigma = S$. Assumption 3.1 is satisfied by Theorem 2.1.
- (**4d** $\mathcal{N} = 1$ **supersymmetry**) We have $\dim(V) = 4$ and $\Sigma = S_+ \oplus S_-$. Assumption 3.1 is satisfied by Theorem 2.1.
- (**6d** $\mathcal{N} = (1, 0)$ **supersymmetry**) We have $\dim(V) = 6$ and $\Sigma = S_+ \otimes W_+$ for a two-dimensional complex symplectic vector space W_+. Assumption 3.1 is satisfied by Theorem 2.1.
- (**10d** $\mathcal{N} = (1, 0)$ **supersymmetry**) We have $\dim(V) = 10$ and $\Sigma = S_+$. Assumption 3.1 is satisfied by Theorem 2.1.

Let $\mathfrak{g}$ be a Lie algebra equipped with a nondegenerate symmetric bilinear pairing. The fields of the Yang–Mills theory are as follows:

- A connection $A \in \Omega^1(V_{\mathbb{R}}; \mathfrak{g})$ on the trivial bundle.
- A spinor $\lambda \in \Gamma(V_{\mathbb{R}}; \Pi\Sigma \otimes \mathfrak{g})$.
- A ghost field $c \in \Gamma(V_{\mathbb{R}}; \mathfrak{g}[1])$.

We also introduce their antifields $A^* \in \Omega^1(V_{\mathbb{R}}; \mathfrak{g}[-1])$, $\lambda^* \in \Gamma(V_{\mathbb{R}}; \Pi\Sigma^* \otimes \mathfrak{g}[-1])$ and $c^* \in \Gamma(V_{\mathbb{R}}; \mathfrak{g}[-2])$. Let $\mathcal{E}_{\mathrm{gauge}}$ denote the space of BV fields.

Denote by $F_A = \mathrm{d}A + \frac{1}{2}[A \wedge A]$ the curvature of A and let $\not{d}_A$ be the twisted Dirac operator obtained from Γ (see Appendix A).

The BRST action is given by

$$S_{\mathrm{BRST}}(A, \lambda) = \int_{V_{\mathbb{R}}} \mathrm{dvol} \left(-\frac{1}{4}(F_A, F_A) + \frac{1}{2}(\lambda, \not{d}_A \lambda) \right).$$

We introduce gauge transformations

$$\delta A = -\mathrm{d}_A c, \qquad \delta\lambda = [c, \lambda], \qquad \delta c = \frac{1}{2}[c, c],$$

so that the BV action is given by

$$S_{\mathrm{gauge}} = \int_{V_{\mathbb{R}}} \mathrm{dvol} \left(-\frac{1}{4}(F_A, F_A) + \frac{1}{2}(\lambda, \not{d}_A \lambda) + (\mathrm{d}_A c, A^*) + ([c, \lambda], \lambda^*) + \frac{1}{2}([c, c], c^*) \right). \tag{3}$$

To simplify the notation, the pairing on $\mathfrak{g}$ from now on will be implicit.

The Poincaré group acts, in the sense of Definition 1.12, on Yang–Mills theory on $\mathbb{R}^n$. Indeed, there is an obvious Poincaré action on fields where we use that Σ is a representation of $\mathrm{Spin}(V_{\mathbb{R}})$. The corresponding Hamiltonian is given by

$$I^{(1)}_{\mathrm{gauge}}(v) = \int_{V_{\mathbb{R}}} \mathrm{dvol}\left((L_v A, A^*) - (v.\lambda, \lambda^*) - (v.c)c^*\right), \tag{4}$$

for $v \in \mathfrak{iso}(V)$, where $v.\lambda$ contains both a derivative and the $\mathfrak{so}(V)$ action on Σ.

We will now construct an elliptic L_∞ action of the super Lie algebra $\mathfrak{A}$ on the theory. Following Definition 1.11, we have to prescribe a collection of functionals $I^{(1)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}, \dots$, where $I^{(k)}_{\mathrm{gauge}}\colon \mathfrak{A}^{\otimes k} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$, together satisfying the classical master equation. The supersymmetry action we construct will extend the Poincaré action from (4), so we just have to specify the values of $I^{(k)}_{\mathrm{gauge}}$ on the supersymmetry generators in Σ. The action of supersymmetry is given by a linear and a quadratic functional

$$I^{(1)}_{\mathrm{gauge}}(Q) = \int_{V_{\mathbb{R}}} \mathrm{dvol}\left(-(\Gamma(Q,\lambda), A^*) + \frac{1}{2}(\rho(F_A)Q, \lambda^*)\right) \tag{5}$$

$$I^{(2)}_{\mathrm{gauge}}(Q_1, Q_2) = \int_{V_{\mathbb{R}}} \mathrm{dvol}\left(\frac{1}{4}(\Gamma(Q_1,Q_2), \Gamma(\lambda^*,\lambda^*)) - \frac{1}{2}(Q_1,\lambda^*)(Q_2,\lambda^*) - \iota_{\Gamma(Q_1,Q_2)}A c^*\right). \tag{6}$$

The following theorem summarizes the fact that super Yang–Mills theory is indeed supersymmetric in the sense of Definition 2.4.

Theorem 3.1. The functional $\mathfrak{S}_{\mathrm{gauge}} = S_{\mathrm{gauge}} + I^{(1)}_{\mathrm{gauge}} + I^{(2)}_{\mathrm{gauge}} \in C^\bullet(\mathfrak{A}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_{\mathrm{gauge}}))$ satisfies the classical master equation

$$\mathrm{d}_{\mathrm{CE}}\mathfrak{S}_{\mathrm{gauge}} + \frac{1}{2}\{\mathfrak{S}_{\mathrm{gauge}}, \mathfrak{S}_{\mathrm{gauge}}\} = 0.$$

Thus, $\mathfrak{S}_{\mathrm{gauge}}$ defines an elliptic L_∞ action of the super Lie algebra $\mathfrak{A}$ on super Yang–Mills theory and so super Yang–Mills theory is supersymmetric.

The rest of the section will be devoted to the proof of the above theorem. The classical master equation decomposes into the following equations:

$$\begin{aligned}
\{S_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}\} &= 0 \\
\{S_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\} + \mathrm{d}_{\mathrm{CE}} I^{(1)}_{\mathrm{gauge}} + \frac{1}{2}\{I^{(1)}_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}\} &= 0 \\
\mathrm{d}_{\mathrm{CE}} I^{(2)}_{\mathrm{gauge}} + \{I^{(1)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\} &= 0 \\
\{I^{(2)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\} &= 0.
\end{aligned}$$

Note that the last equation is automatically satisfied since $I^{(2)}_{\mathrm{gauge}}$ is independent of λ and c. The rest of the claims will be proved in a sequence of Lemmas. To simplify the expressions, we drop the integrals.

Lemma 3.2. One has $\{S_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}(Q)\} = 0$.

Proof. Let us decompose $S_{\mathrm{gauge}} = \sum_{i=1}^5 S^i_{\mathrm{gauge}}$ into individual summands.

The first term gives

$$\begin{aligned}
\{S^1_{\mathrm{gauge}}, I^{(1)}_{\mathrm{gauge}}(Q)\} &= -\frac{1}{2}(\mathrm{d}_A\Gamma(Q,\lambda), F_A) \\
&= -\frac{1}{2}(-1)^{n-1}(*\mathrm{d}_A * F_A, \Gamma(Q,\lambda)).
\end{aligned}$$

The second term gives

$$\begin{aligned}\{S^2_{\text{gauge}}, I^{(1)}_{\text{gauge}}(Q)\} &= -\frac{1}{2}(\lambda, \rho(\Gamma(Q,\lambda))\lambda) + \frac{1}{2}(\rho(F_A)Q, \not{d}_A\lambda) \\ &= -\frac{1}{2}(\Gamma(Q,\lambda), \Gamma(\lambda,\lambda)) - \frac{1}{2}(\lambda, \not{d}_A(\rho(F_A)Q)) \\ &= -\frac{1}{2}(\Gamma(Q,\lambda), \Gamma(\lambda,\lambda)) - \frac{1}{2}(-1)^n(\lambda, \rho(*\mathrm{d}_A * F_A)Q),\end{aligned}$$

where we have used Proposition A.3 and the Bianchi identity in the last line.

By (22) and Assumption 3.1 we have $(\Gamma(Q,\lambda), \Gamma(\lambda,\lambda)) = 0$, so $\{S^1_{\text{gauge}} + S^2_{\text{gauge}}, I^{(1)}_{\text{gauge}}(Q)\} = 0$.

Finally, $\{S^3_{\text{gauge}} + S^4_{\text{gauge}} + S^5_{\text{gauge}}, I^{(1)}_{\text{gauge}}(Q)\} = 0$ due to gauge-invariance of $I^{(1)}(Q)$. □

Remark 3.3. The previous Lemma expresses the fact that the pure super Yang–Mills action is supersymmetric; this was proven by Baez and Huerta in [**BaezHuerta**], and our proof essentially follows the proof in loc. cit.

Lemma 3.4. One has

$$\{S_{\text{gauge}}, I^{(2)}_{\text{gauge}}\} + \mathrm{d}_{\text{CE}} I^{(1)}_{\text{gauge}} + \frac{1}{2}\{I^{(1)}_{\text{gauge}}, I^{(1)}_{\text{gauge}}\} = 0.$$

Proof. Evaluating the equation

$$\{S_{\text{gauge}}, I^{(2)}_{\text{gauge}}\} + \mathrm{d}_{\text{CE}} I^{(1)}_{\text{gauge}} + \frac{1}{2}\{I^{(1)}_{\text{gauge}}, I^{(1)}_{\text{gauge}}\} = 0$$

on $v_1, v_2 \in \mathfrak{iso}(V)$, the claim reduces to the fact that (4) defines a strict Lie action. Evaluating it on $v \in \mathfrak{iso}(V)$ and $Q \in \Sigma$, the claim reduces to the fact that $I^{(1)}_{\text{gauge}}$ is Poincaré-invariant. So, the only nontrivial check is for $Q_1, Q_2 \in \Sigma$.

The individual terms are

$$\begin{aligned}\frac{1}{2}\{I^{(1)}_{\text{gauge}}, I^{(1)}_{\text{gauge}}\}(Q_1, Q_2) &= -\{I^{(1)}_{\text{gauge}}(Q_1), I^{(1)}_{\text{gauge}}(Q_2)\} \\ &= -\frac{1}{2}(\rho(\mathrm{d}_A\Gamma(Q_1,\lambda))Q_2, \lambda^*) + \frac{1}{2}(\Gamma(Q_2, \rho(F_A)Q_1), A^*) \\ &\quad - \frac{1}{2}(\rho(\mathrm{d}_A\Gamma(Q_2,\lambda))Q_1, \lambda^*) + \frac{1}{2}(\Gamma(Q_1, \rho(F_A)Q_2), A^*),\end{aligned}$$

$$\begin{aligned}(\mathrm{d}_{\text{CE}} I^{(1)}_{\text{gauge}})(Q_1, Q_2) &= I^{(1)}_{\text{gauge}}(\Gamma(Q_1, Q_2)) \\ &= (L_{\Gamma(Q_1,Q_2)}(A), A^*) - (\Gamma(Q_1,Q_2).\lambda, \lambda^*) - (\Gamma(Q_1,Q_2).c)c^*\end{aligned}$$

and

$$\begin{aligned}\{S_{\text{gauge}}, I^{(2)}_{\text{gauge}}(Q_1, Q_2)\} = &-\frac{1}{2}(Q_2, \lambda^*)(Q_1, \not{d}_A\lambda + [c, \lambda^*]) - \frac{1}{2}(Q_1, \lambda^*)(Q_2, \not{d}_A\lambda + [c, \lambda^*]) \\ &+ \frac{1}{2}(\Gamma(Q_1, Q_2), \Gamma(\lambda^*, \not{d}_A\lambda + [c, \lambda^*])) + \iota_{\Gamma(Q_1,Q_2)}(\mathrm{d}_A c)c^* - (\mathrm{d}_A \iota_{\Gamma(Q_1,Q_2)}A, A^*) \\ &+ ([\lambda, \iota_{\Gamma(Q_1,Q_2)}A], \lambda^*) - [\iota_{\Gamma(Q_1,Q_2)}A, c]c^*\end{aligned}$$

The total coefficient in front of A^* is

$$\frac{1}{2}\Gamma(Q_1, \rho(F_A)Q_2) + \frac{1}{2}\Gamma(Q_2, \rho(F_A)Q_1) + L_{\Gamma(Q_1,Q_2)}A - \mathrm{d}_A \iota_{\Gamma(Q_1,Q_2)}A.$$

Using Proposition A.1 we get that the sum of the first two terms is $-\iota_{\Gamma(Q_1,Q_2)}F_A$ which cancels the last two terms.

The total coefficient in front of c^* is

$$-\Gamma(Q_1, Q_2).c + \iota_{\Gamma(Q_1,Q_2)}(\mathrm{d}_A c) - [\iota_{\Gamma(Q_1,Q_2)} A, c] = 0.$$

The total coefficient in front of λ^* is

$$\begin{aligned}
&-\frac{1}{2}\rho(\mathrm{d}_A\Gamma(Q_1,\lambda))Q_2 - \frac{1}{2}\rho(\mathrm{d}_A\Gamma(Q_2,\lambda))Q_1 - \Gamma(Q_1,Q_2).\lambda \\
&+\frac{1}{2}\rho(\Gamma(Q_1,Q_2))\not{\mathrm{d}}_A\lambda - \frac{1}{2}(Q_2, \not{\mathrm{d}}_A\lambda)Q_1 - \frac{1}{2}(Q_1, \not{\mathrm{d}}_A\lambda)Q_2 + [\lambda, (\Gamma(Q_1,Q_2), A)]
\end{aligned}$$

Using Proposition A.2 the first, second, fifth and sixth terms combine to

$$-\frac{1}{2}\not{\mathrm{d}}_A\rho(\Gamma(Q_1,\lambda))Q_2 - \frac{1}{2}\not{\mathrm{d}}_A\rho(\Gamma(Q_2,\lambda))Q_1$$

which is equal to $\frac{1}{2}\not{\mathrm{d}}_A\rho(\Gamma(Q_1,Q_2))\lambda$ by Assumption 3.1. Using the Clifford relation this term cancels the rest of the terms. $\square$

Evaluating the equation

$$\mathrm{d}_{\mathrm{CE}} I^{(2)}_{\mathrm{gauge}} + \{I^{(1)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\} = 0$$

on $v_1, v_2, v_3 \in \mathfrak{iso}(V)$ or on $v_1, v_2 \in \mathfrak{iso}(V)$ and $Q \in \Sigma$ we automatically get zero. Evaluating it on $v \in \mathfrak{iso}(V)$ and $Q_1, Q_2 \in \Sigma$ we get Poincaré-invariance of $I^{(2)}_{\mathrm{gauge}}$.

Lemma 3.5.

$$\{I^{(1)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\}(Q_1, Q_2, Q_3) = 0$$

for every $Q_1, Q_2, Q_3 \in \Sigma$.

Proof. We have

$$\begin{aligned}
\{I^{(1)}_{\mathrm{gauge}}(Q_1), I^{(2)}_{\mathrm{gauge}}(Q_2,Q_3)\} = &-\iota_{\Gamma(Q_2,Q_3)}\Gamma(Q_1,\lambda)c^* - \frac{1}{2}(\Gamma(Q_2,Q_3), \Gamma(\rho(A^*)Q_1, \lambda^*)) \\
&+\frac{1}{2}(Q_2, \rho(A^*)Q_1)(Q_3,\lambda^*) + \frac{1}{2}(Q_3, \rho(A^*)Q_1)(Q_2,\lambda^*).
\end{aligned}$$

$\{I^{(1)}_{\mathrm{gauge}}, I^{(2)}_{\mathrm{gauge}}\}(Q_1, Q_2, Q_3)$ is obtained by cyclically symmetrizing the above expression. By Assumption 3.1 the cyclic symmetrization of the term with c^* is zero. The Clifford relation implies that

$$\begin{aligned}
\frac{1}{2}(\Gamma(Q_2,Q_3), \Gamma(\rho(A^*)Q_1,\lambda^*)) &= -\frac{1}{2}(\Gamma(Q_2,Q_3), \Gamma(\rho(A^*)\lambda^*, Q_1)) + (\Gamma(Q_2,Q_3), A^*)(Q_1,\lambda^*) \\
&= -\frac{1}{2}(\rho(\Gamma(Q_2,Q_3))Q_1, \rho(A^*)\lambda^*) + (\Gamma(Q_2,Q_3), A^*)(Q_1,\lambda^*).
\end{aligned}$$

Therefore, again using Assumption 3.1 we see that the cyclic symmetrization of the terms with A^* vanishes. $\square$

3.2 Coupling to Matter Multiplets

(Brian: How do we see the various flavors of matter multiplets? For instance, in $2d(0,2)$ there is the Fermi multiplet and the chiral multiplet. Your description only sees the chiral. In $2d(0,4)$ there are even more: the hypermultiplet (two $(0,2)$ chirals), the twisted hypermultiplet (the same as the hyper, different R-symmetry), the fermi multiplet (two $(0,2)$ fermi multiplets). I'm not sure which your description sees, but I have a hunch it's the $(0,4)$ hypermultiplet. What is the reason for only seeing certain multiplets via this general setup?) (Chris: Presumably pure

$\mathcal{N} = (0, N)$ means a $(0, 2)$ vector coupled to $N - 2$ Fermi multiplets, so some of these are already covered by the classification? Perhaps your concern is that you want a formalism that allows Fermis that are not adjoint valued? I don't think the difference between this hyper and twisted hyper is germane here: the proof of supersymmetry is that there is an action of super Poincaré, it doesn't involve the action of R-symmetry.)

In this section we describe the coupling of super Yang–Mills theory to matter valued in a $\mathfrak{g}$-representation P, i.e. the supersymmetric gauged linear σ-models. Our description of the supersymmetry of the matter multiplet is inspired by the presentation of the supersymmetric nonlinear σ-models by Deligne and Freed in **[DeligneFreed]**.

Consider as before $V_\mathbb{R}$ and a Clifford module $\Sigma \oplus \Sigma^*$ satisfying Assumption 3.1. In addition, fix a complex associative composition algebra A equipped with an antiinvolution σ as in Section 2.2. Suppose $\Sigma \oplus \Sigma^*$ carries a compatible right A-module structure. Let $(-,-)^A \colon \Sigma \otimes \Sigma^* \to A$ be the correspoding A-valued pairing given by Lemma 2.5. We make the following additional assumption.

Assumption 3.6. For $Q_1, Q_2 \in \Sigma$ and $Q_3 \in \Sigma^*$ we have

$$Q_1(Q_2, Q_3)^A + Q_2(Q_1, Q_3)^A = \rho(\Gamma(Q_1, Q_2))Q_3.$$

Explicitly, we consider the following cases:

- (**2d** $\mathcal{N} = (1, 0)$ **supersymmetry**) $A = \mathbb{C}$. Assumption 3.6 is satisfied by Theorem 2.4.
- (**2d** $\mathcal{N} = (2, 0)$ **supersymmetry**) $A = \mathbb{C}[x]/(x^2 + 1)$. Assumption 3.6 is satisfied by Theorem 2.4.
- (**2d** $\mathcal{N} = (4, 0)$ **supersymmetry**) $A = \mathrm{End}(Z)$. Assumption 3.6 is satisfied by Theorem 2.4.
- (**3d** $\mathcal{N} = 1$ **supersymmetry**) $A = \mathbb{C}$. Assumption 3.6 is satisfied by Theorem 2.2.
- (**4d** $\mathcal{N} = 1$ **supersymmetry**) $A = \mathbb{C}[x]/(x^2 + 1)$. Assumption 3.6 is satisfied by Theorem 2.2.
- (**6d** $\mathcal{N} = (1, 0)$ **supersymmetry**) $A = \mathrm{End}(Z)$. Assumption 3.6 is satisfied by Theorem 2.2.

Let P be a left A-module equipped with a $\mathbb{C}$-valued nondegenerate symmetric bilinear pairing such that

$$(av, w) = (v, \sigma(a)w).$$

Moreover, assume P carries a $\mathfrak{g}$-action commuting with the A-module structure and preserving the bilinear pairing. Explicitly, for $A = \mathbb{C}, \mathbb{C} \otimes_\mathbb{R} \mathbb{C}, \mathbb{H} \otimes_\mathbb{R} \mathbb{C}$ we get the following data:

- $A = \mathbb{C}$. We are looking for a $\mathfrak{g}$-representation P equipped with a nondegenerate symmetric bilinear pairing.
- $A = \mathbb{C}[x]/(x^2 + 1)$. A left A-module P splits as $P = P_+ \oplus P_-$, where x acts as $\pm i$ on $P_\pm$. Note that with respect to the right A-action x acts as $\mp i$ on $P_\pm$. So, the symmetric bilinear pairing identifies $P_+ \cong P_-^*$. In other words, the data boils down to a $\mathfrak{g}$-representation R, so that $P = R \oplus R^*$.
- $A = \mathrm{End}(Z)$. A left A-module is necessarily of the form $P \cong Z \otimes U$, where A just acts on Z. Compatibility of the orthogonal pairing on P with the A-action implies that it is given by a product of the symplectic pairing on Z and a symplectic pairing on U. So, the data boils down a symplectic $\mathfrak{g}$-representation U.

We are going to construct a theory on $V_\mathbb{R}$ describing a matter multiplet valued in P. The BRST fields are given as follows:

- a scalar $\phi \in \Gamma(V_\mathbb{R}; P)$;

- a spinor $\psi \in \Gamma(V_{\mathbb{R}}; \Pi\Sigma^* \otimes_A P)$.

As usual, we denote the antifields by $\phi^* \in \Gamma(V_{\mathbb{R}}; \Pi P)$ and $\psi^* \in \Gamma(V_{\mathbb{R}}; \Sigma \otimes_A P)$.

We extend the pairings on P and between Σ and Σ^* to a pairing between $\Sigma \otimes_A P$ and $\Sigma^* \otimes_A P$ in the following way. Given $\sum_i \tilde{s}_i \otimes v_i \in \Sigma^* \otimes_A P$ and $\sum_j s_j \otimes w_j \in \Sigma \otimes_A P$, their pairing is

$$\sum_{i,j} \mathrm{Re}((v_i, w_j)^A (s_j, \tilde{s}_i)^A), \tag{7}$$

where we extend both pairings to A-valued pairings using Lemma 2.5. We may also extend the Γ-pairing to a map

$$\Gamma\colon \mathrm{Sym}^2(\Sigma^* \otimes_A P) \to V$$

defined by the property

$$(v, \Gamma(\psi_1, \psi_2)) = (\psi_1, \rho(v)\psi_2), \qquad v \in V,\ \psi_i \in \Sigma^* \otimes_A P.$$

The BV action for the matter multiplet is

$$S_{\mathrm{matter}} = \int_{V_{\mathbb{R}}} \mathrm{dvol} \left(\frac{1}{2}(\mathrm{d}_A\phi, \mathrm{d}_A\phi) + (\psi, \not{\mathrm{d}}_A\psi) + 2(\lambda\phi, \psi) + (c\psi, \psi^*) - (c\phi, \phi^*) \right),$$

where we use the pairing (7) in the second term.

It is Poincaré-invariant with the corresponding Hamiltonian

$$I^{(1)}_{\mathrm{matter}}(v) = \int_{V_{\mathbb{R}}} \mathrm{dvol}\left((L_v A, \phi^*) - (v.\psi, \psi^*)\right), \tag{8}$$

for $v \in \mathfrak{iso}(V)$.

The action of supersymmetry is given by a linear and quadratic functional:

$$I^{(1)}_{\mathrm{matter}}(Q) = \int_{V_{\mathbb{R}}} \mathrm{dvol} \left(((Q, \psi), \phi^*) + \frac{1}{2}(\rho(\mathrm{d}_A\phi)Q, \psi^*) \right) \tag{9}$$

$$I^{(2)}_{\mathrm{matter}}(Q_1, Q_2) = \frac{1}{4}\int_{V_{\mathbb{R}}} \mathrm{dvol}(\Gamma(Q_1, Q_2), \Gamma(\psi^*, \psi^*)) \tag{10}$$

where $Q, Q_1, Q_2 \in \Sigma$.

We consider the full action of the super Yang–Mills theory

$$S_{\mathrm{BV}} = S_{\mathrm{gauge}} + S_{\mathrm{matter}}$$

together with a supersymmetry action functionals

$$I^{(1)} = I^{(1)}_{\mathrm{gauge}} + I^{(1)}_{\mathrm{matter}}, \qquad I^{(2)} = I^{(2)}_{\mathrm{gauge}} + I^{(2)}_{\mathrm{matter}}.$$

The following result states that these functionals encode an off-shell action of the supersymmetry algebra.

Theorem 3.2. The functional $\mathfrak{S} = S_{\mathrm{BV}} + I^{(1)}_{\mathrm{BV}} + I^{(2)}_{\mathrm{BV}}$ satisfies the classical master equation

$$\mathrm{d}_{\mathrm{CE}}\mathfrak{S} + \frac{1}{2}\{\mathfrak{S}, \mathfrak{S}\} = 0. \tag{11}$$

The rest of this section is devoted to the proof of Theorem 3.2. The classical master equation (11) decomposes into a sequence of equations

$$\begin{aligned} \{S_{\mathrm{BV}}, I^{(1)}\} &= 0 \\ \{S_{\mathrm{matter}}, I^{(2)}\} + \mathrm{d}_{\mathrm{CE}} I^{(1)}_{\mathrm{matter}} + \{I^{(1)}_{\mathrm{gauge}}, I^{(1)}_{\mathrm{matter}}\} + \tfrac{1}{2}\{I^{(1)}_{\mathrm{matter}}, I^{(1)}_{\mathrm{matter}}\} &= 0 \\ \mathrm{d}_{\mathrm{CE}} I^{(2)}_{\mathrm{matter}} + \{I^{(1)}_{\mathrm{matter}}, I^{(2)}_{\mathrm{matter}}\} &= 0 \\ \{I^{(2)}_{\mathrm{matter}}, I^{(2)}_{\mathrm{matter}}\} &= 0 \end{aligned} \tag{12}$$

The last equation is automatically satisfied since $I^{(2)}$ is independent of the fields ϕ, ψ, A, λ.

The first equation in (12) states that the classical action is supersymmetric.

Lemma 3.7. One has $\{S_{\mathrm{BV}}, I^{(1)}\}(Q) = 0$ for all $Q \in \Sigma$.

Proof. Let us decompose $S_{\mathrm{matter}} = \sum_{i=1}^{5} S^i_{\mathrm{matter}}$ into individual summands.

The first term gives

$$\begin{aligned} \{S^1_{\mathrm{matter}}, I^{(1)}(Q)\} &= -(\mathrm{d}_A\phi, \mathrm{d}_A(Q,\psi)) + (\Gamma(Q,\lambda)\phi, \mathrm{d}_A\phi) \\ &= \mathrm{d}_A^*\mathrm{d}_A\phi(Q,\psi) + (\Gamma(Q,\lambda)\phi, \mathrm{d}_A\phi). \end{aligned}$$

The second term gives

$$\begin{aligned} \{S^2_{\mathrm{matter}}, I^{(1)}(Q)\} &= -(\psi, \not{\mathrm{d}}_A\rho(\mathrm{d}_A\phi)Q) - (\psi, \rho(\Gamma(Q,\lambda))\psi) \\ &= -(\psi, \rho(F_A)Q)\phi - \mathrm{d}_A^*\mathrm{d}_A\phi(\psi, Q) - (\psi, \rho(\Gamma(Q,\lambda))\psi), \end{aligned}$$

where we have used Proposition A.3 in the second line.

The third term gives

$$\begin{aligned} \{S^3_{\mathrm{matter}}, I^{(1)}(Q)\} &= ((\rho(F_A)Q)\phi, \psi) + 2(\lambda(Q,\psi), \psi) - (\lambda\phi, \rho(\mathrm{d}_A\phi)Q) \\ &= ((\rho(F_A)Q)\phi, \psi) + (\rho(\Gamma(Q,\lambda))\psi, \psi) - (\Gamma(\lambda\phi, Q), \mathrm{d}_A\phi), \end{aligned}$$

where we have used Assumption 3.6 in the middle term and (22) in the last term. It is then obvious that

$$\{S^1_{\mathrm{matter}} + S^2_{\mathrm{matter}} + S^3_{\mathrm{matter}}, I^{(1)}(Q)\} = 0.$$

Finally, the terms $\{S^4_{\mathrm{matter}} + S^5_{\mathrm{matter}} + S^3_{\mathrm{gauge}}, I^{(1)}(Q)\}$ are zero due to gauge-invariance of $I^{(1)}(Q)$ while the rest of the terms are zero by Lemma 3.2. ☐

Next, we move on to the second equation in (12).

Lemma 3.8. One has

$$\{S_{\mathrm{matter}}, I^{(2)}\} + \mathrm{d}_{\mathrm{CE}} I^{(1)}_{\mathrm{matter}} + \{I^{(1)}_{\mathrm{gauge}}, I^{(1)}_{\mathrm{matter}}\} + \frac{1}{2}\{I^{(1)}_{\mathrm{matter}}, I^{(1)}_{\mathrm{matter}}\} = 0. \tag{13}$$

Proof. Evaluating expression (13) on $v_1, v_2 \in \mathfrak{iso}(V)$ reduces to the claim that (8) defines a strict Lie action. Evaluating on $v \in \mathfrak{iso}(V)$ and $Q \in \Sigma$, the claim reduces to the fact that $I^{(1)}$ is Poincaré-invariant. So, the only nontrivial term to check is the evaluation on $Q_1, Q_2 \in \Sigma$.

The individual terms are:

$$\begin{aligned}\frac{1}{2}\{I^{(1)}_{\text{matter}}, I^{(1)}_{\text{matter}}\}(Q_1,Q_2) =& -\{I^{(1)}_{\text{matter}}(Q_1), I^{(1)}_{\text{matter}}(Q_2)\}\\ =& -\frac{1}{2}(Q_1,\rho(\mathrm{d}_A\phi)Q_2)\phi^* - \frac{1}{2}(Q_2,\rho(\mathrm{d}_A\phi)Q_1)\phi^*\\ &+\frac{1}{2}(\rho(\mathrm{d}(Q_1,\psi))Q_2,\psi^*) + \frac{1}{2}(\rho(\mathrm{d}(Q_2,\psi))Q_1,\psi^*),\end{aligned}$$

$$\begin{aligned}\{I^{(1)}_{\text{gauge}}, I^{(1)}_{\text{matter}}\}(Q_1,Q_2) =& -\{I^{(1)}_{\text{gauge}}(Q_1), I^{(1)}_{\text{matter}}(Q_2)\} - \{I^{(1)}_{\text{gauge}}(Q_2), I^{(1)}_{\text{matter}}(Q_1)\}\\ =& -\frac{1}{2}(\rho(\Gamma(Q_1,\lambda))(\phi Q_2),\psi^*) - \frac{1}{2}(\rho(\Gamma(Q_2,\lambda))(\phi Q_1),\psi^*),\end{aligned}$$

$$(\mathrm{d}_{\mathrm{CE}} I^{(1)}_{\text{matter}})(Q_1,Q_2) = L_{\Gamma(Q_1,Q_2)}(\phi)\phi^* - (\Gamma(Q_1,Q_2).\psi,\psi^*),$$

and

$$\{S_{\text{matter}}, I^{(2)}(Q_1,Q_2)\} = \frac{1}{2}\Gamma(Q_1,Q_2)\Gamma(\psi^*, \not{d}_A\psi - 2\lambda\phi + c\psi^*) - ((\iota_{\Gamma(Q_1,Q_2)}A)\psi,\psi^*) + ((\iota_{\Gamma(Q_1,Q_2)}A)\phi,\phi^*)$$

We first collect all terms in equation (13) proportional to ϕ^*:

$$-\frac{1}{2}(Q_1,\rho(\mathrm{d}_A\phi)Q_2) - \frac{1}{2}(Q_2,\rho(\mathrm{d}_A\phi)Q_1) + L_{\Gamma(Q_1,Q_2)}\phi + (\iota_{\Gamma(Q_1,Q_2)}A)\phi.$$

By (22) we observe that the first two terms cancel with the last two terms.

Next, we collect all terms in equation (13) containing ψ^* and ψ:

$$\frac{1}{2}\not{d}_A Q_2(Q_1\psi)^A + \frac{1}{2}\not{d}_A Q_1(Q_2,\partial_i\psi)^A - \Gamma(Q_1,Q_2).\psi + \frac{1}{2}\rho(\Gamma(Q_1,Q_2))\not{d}_A\psi - (\iota_{\Gamma(Q_1,Q_2)}A)\psi. \tag{14}$$

Applying Assumption 3.6 to $Q_3 = \psi$, the first two terms become $\frac{1}{2}\not{d}_A\rho(\Gamma(Q_1,Q_2)\psi)$. Finally, by the Clifford identity the sum of this term with the fourth term in (14) is precisely $\iota_{\Gamma(Q_1,Q_2)}\mathrm{d}_A\psi$ which cancels the remaining terms. □

Lemma 3.9.

$$\{I^{(1)}_{\text{matter}}, I^{(2)}_{\text{matter}}\}(Q_1,Q_2,Q_3) = 0$$

for every $Q_1,Q_2,Q_3 \in \Sigma$.

Proof. We have

$$\begin{aligned}\{I^{(1)}_{\text{matter}}(Q_1), I^{(2)}_{\text{matter}}(Q_2,Q_3)\} &= \frac{1}{2}(\Gamma(Q_2,Q_3),\Gamma(\psi^*,\phi^*Q_1))\\ &= (\psi^*,\phi^*\rho(\Gamma(Q_2,Q_3))Q_1).\end{aligned}$$

The expression $\{I^{(1)}_{\text{matter}}, I^{(2)}_{\text{matter}}\}(Q_1,Q_2,Q_3)$ is obtained by cyclically symmetrizing the above expression. By Assumption 3.1 the cyclic symmetrization is identically zero. □

Part II

Classification of Twists

4 Dimension 10

The 10-dimensional supersymmetry algebra has odd part $\Sigma \cong S_+ \otimes W_+ \oplus S_- \otimes W_-$, where S_+, S_- are the 16-dimensional semi-spin representations of $\mathrm{Spin}(10, \mathbb{C})$, and where W_+ and W_- are complex vector spaces equipped with nondegenerate symmetric bilinear pairings. There are Yang–Mills theories with $\mathcal{N} = (1,0)$ or $\mathcal{N} = (0,1)$ supersymmetries. We concentrate on the first case, the second case being identical. So, we fix $W_+ = \mathbb{C}$ and $W_- = 0$.

4.1 $\mathcal{N} = (1,0)$ Super Yang–Mills

Let $\mathfrak{g}$ be a complex Lie algebra equipped with a symmetric bilinear invariant nondegenerate pairing. We consider $\mathcal{N} = (1,0)$ super Yang–Mills theory on $M = \mathbb{R}^{10}$ with the Euclidean metric, where $\mathfrak{g}$ is the complexified Lie algebra of the gauge group.

This theory admits a unique twist:

- A square-zero supercharge $Q \neq 0 \in \Sigma$ has 5 invariant directions and does not admit a compatible homomorphism α. So, it gives rise to a $\mathbb{Z}/2\mathbb{Z}$-graded holomorphic theory. Such a supercharge is stabilized by $G = \mathrm{SU}(5) \subset \mathrm{Spin}(10, \mathbb{C})$.

4.1.1 Holomorphic twist

Let $Q \in \Sigma$ be a square-zero supercharge. The image of $\Gamma(Q, -)\colon \Sigma \to V$ is a complex Lagrangian subspace $L \subset V$. Denote by $\sigma\colon V \to V$ the complex conjugation induced by the real structure $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Since the bilinear form on $V_{\mathbb{R}}$ is positive-definite, $L \cap \sigma(L) = 0$. In other words, L defines a (linear) complex structure on $V_{\mathbb{R}}$. Moreover, we may canonically identify $\sigma(L) \cong L^*$.

Let $\mathrm{ML}(L)$ be the metalinear group of L. Under the embedding $\mathrm{ML}(L) \subset \mathrm{Spin}(V)$, the semi-spin representation $\Sigma = S_+$ decomposes as

$$\Sigma = \det(L)^{1/2} \oplus \wedge^2 L^* \otimes \det(L)^{1/2} \oplus L \otimes \det(L)^{-1/2}.$$

$Q \in \Sigma$ lies in the first summand, so the choice of Q is equivalent to the choice of a (linear) Kähler structure L on $V_{\mathbb{R}}$ together with a complex half-density on L. The square of this half-density defines a Calabi–Yau structure on M.

We will now rewrite the fields and the action in terms of the Calabi–Yau structure. Let $\omega \in \Omega^{1,1}(M)$ be the Kähler form, $\Omega \in \Omega^{5,0}(M)$ the holomorphic volume form and $\Lambda\colon \Omega^{p+1,q+1}(M) \to \Omega^{p,q}(M)$ the dual Lefschetz operator. We denote the real volume form on M by

$$\mathrm{dvol} = \frac{\omega^5}{5!}.$$

The vector representation decomposes as

$$\Omega^1(M) \cong \Omega^{1,0}(M) \oplus \Omega^{0,1}(M),$$

the semi-spin representation S_+ decomposes as

$$\Omega^0(M;S_+) \cong \Omega^{1,0}(M) \oplus \Omega^{0,2}(M) \oplus \Omega^0(M)$$

and the semi-spin representation S_- decomposes as

$$\Omega^0(M;S_-) \cong \Omega^{0,1}(M) \oplus \Omega^{2,0}(M) \oplus \Omega^0(M).$$

Under this decomposition the scalar pairing $S_+ \otimes S_- \to \mathbb{C}$ corresponds to the wedge product of individual components post-composed with Λ. Under the above decompositions the Clifford multiplication of a vector $A = A_{1,0} + A_{0,1}$ and a spinor $\lambda = \rho + B + \chi \in S_+$ is given by

$$\rho(A)\lambda = (A_{0,1}\chi + \Lambda(A_{1,0} \wedge B), A_{1,0} \wedge \rho + *(A_{0,1} \wedge B \wedge \Omega), \Lambda(A_{0,1} \wedge \rho)) \in S_-.$$

Fields: The BRST fields are given by:

- Gauge fields $A_{1,0} \in \Omega^{1,0}(M;\mathfrak{g})$, $A_{0,1} \in \Omega^{0,1}(M;\mathfrak{g})$.
- Fermions $\rho \in \Omega^{1,0}(M;\Pi\mathfrak{g})$, $B \in \Omega^{0,2}(M;\Pi\mathfrak{g})$, $\chi \in \Omega^0(M;\Pi\mathfrak{g})$.
- A ghost field $c \in \Omega^0(M;\mathfrak{g})[1]$.

We denote their antifields by $A^*_{1,0}, A^*_{0,1}, \rho^*, B^*, \chi^*, c^*$.

The BV action of the theory is obtained from (3) by decomposing it in terms of the above fields. To write it out we will need an expression for the Hodge star operator on Kähler manifolds, see [**Huybrechts**].

Proposition 4.1. Let (M, ω) be a Kähler d-fold and decompose

$$\Omega^2(M) = \Omega^{2,0}(M) \oplus \Omega^{0,2}(M) \oplus (\mathbb{C}\omega \oplus \Omega^{1,1}_\perp(M)).$$

Then

1. The spaces $\Omega^{2,0}(M) \oplus \Omega^{0,2}(M)$, $\mathbb{C}\omega$ and $\Omega^{1,1}_\perp(M)$ are mutually orthogonal.
2. For a form $\alpha \in \Omega^{2,0}(M) \oplus \Omega^{0,2}(M)$ we have
$$*\alpha = \frac{1}{(d-2)!}\alpha \wedge \omega^{d-2}.$$
3. For $\alpha \in \Omega^{1,1}_\perp(M)$ we have
$$*\alpha = -\frac{1}{(d-2)!}\alpha \wedge \omega^{d-2}.$$
4. For $\alpha \in \mathbb{C}\omega$ we have
$$*\alpha = \frac{1}{(d-1)!}\alpha \wedge \omega^{d-2}.$$

Corollary 4.2. Let M be a Kähler d-fold and $F = F_{2,0} + F_{1,1} + F_{0,2}$ a two-form. Then

$$F \wedge *F + \frac{1}{(d-2)!} F \wedge F \wedge \omega^{d-2} = \left(4(F_{2,0}, F_{0,2}) + (\Lambda F_{1,1})^2\right) \frac{\omega^d}{d!}.$$

Since we are working near the trivial connection, the topological term $\int F \wedge F \wedge \omega^3$ is exact, so we will drop it. The BV action of the twisted theory S_{BV} is then the sum of the following terms:

$$S_{\mathrm{BRST}} = \int \mathrm{dvol}\left(-(F_{2,0}, F_{0,2}) - \frac{1}{4}(\Lambda F_{1,1})^2 + \chi\Lambda(\bar{\partial}_{A_{0,1}}\rho) + (B, \partial_{A_{1,0}}\rho)\right) + \frac{1}{2}B \wedge \bar{\partial}_{A_{0,1}} B \wedge \Omega \tag{15}$$

$$S_{\mathrm{anti}} = \int \mathrm{dvol}\left((\partial_{A_{1,0}}c, A^*_{1,0}) + (\bar{\partial}_{A_{0,1}}c, A^*_{0,1}) + ([\rho, c], \rho^*) + [\chi, c]\chi^* + ([B, c], B^*) + \frac{1}{2}[c, c]c^*\right) \tag{16}$$

$$I^{(1)} = \int \mathrm{dvol}\left(-(\rho, A^*_{1,0}) + (F_{0,2}, B^*) + \frac{1}{2}\Lambda F_{1,1}\chi^*\right) \tag{17}$$

$$I^{(2)} = -\frac{1}{4}\int \mathrm{dvol}(\chi^*)^2. \tag{18}$$

Theorem 4.1. The holomorphic twist of 10d $\mathcal{N} = (1,0)$ super Yang–Mills on $M = \mathbb{R}^{10}$ is perturbatively equivalent to the holomorphic Chern–Simons theory on $M \cong \mathbb{C}^5$ with the space of fields $\mathrm{Map}(M, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{SU}(5)$-equivariant.

Proof. First, we may eliminate χ and χ^* using Proposition 1.6. So, the above theory is perturbatively equivalent to the theory without χ and χ^* with the BV action

$$\begin{aligned} S_{\mathrm{no}\,\chi} = &\int \mathrm{dvol}\left(-(F_{2,0}, F_{0,2}) + (B, \partial_{A_{1,0}}\psi)\right) + \frac{1}{2}B \wedge \bar{\partial}_{A_{0,1}} B \wedge \Omega \\ &+ \int \mathrm{dvol}\left((\partial_{A_{1,0}}c, A^*_{1,0}) + (\bar{\partial}_{A_{0,1}}c, A^*_{0,1}) + ([\rho, c], \rho^*) + ([B, c], B^*) + \frac{1}{2}[c, c]c^*\right) \\ &+ \int \mathrm{dvol}(-(\psi, A^*_{1,0}) + (F_{0,2}, B^*)) \end{aligned}$$

Next, we have a term $\int \mathrm{dvol}\, \rho \wedge A^*_{1,0}$ in the action, i.e. $(\rho, A_{1,0})$ is a trivial BRST doublet, so by Proposition 1.8 we may remove it. The above theory then becomes perturbatively equivalent to the theory without fields $\rho, \rho^*, A_{1,0}, A^*_{1,0}$ and with the BV action

$$S_0 = \int \frac{1}{2}B \wedge \bar{\partial}_{A_{0,1}} B \wedge \Omega + \mathrm{dvol}\left((\bar{\partial}_{A_{0,1}}c, A^*_{0,1}) + ([B, c], B^*) + \frac{1}{2}[c, c]c^* + (F_{0,2}, B^*)\right)$$

Up to rescaling of the antifields, it coincides with the BV action for holomorphic Chern–Simons (see Section 1.5.2). ☐

Remark 4.3. A similar claim was previously proved by Baulieu **[Baulieu]** by adding an auxiliary field to 10d $\mathcal{N} = (1,0)$ super Yang–Mills.

5 Dimension 9

The 9-dimensional supersymmetry algebra has odd part $\Sigma \cong S \otimes W$, where S is the 16-dimensional spin representation of $\mathrm{Spin}(9, \mathbb{C})$ and W is a complex vector space equipped with a nondegenerate symmetric bilinear pairing. There is a Yang–Mills theory with $\mathcal{N} = 1$ supersymmetry, so we fix $W = \mathbb{C}$.

5.1 $\mathcal{N} = 1$ Super Yang–Mills

Let $\mathfrak{g}$ be a complex Lie algebra equipped with a symmetric bilinear invariant nondegenerate pairing. We consider $\mathcal{N} = 1$ super Yang–Mills theory on $M = \mathbb{R}^9$ equipped with the Euclidean metric.

This theory admits a unique twist:

- A square-zero supercharge $Q \neq 0 \in \Sigma$ has 5 invariant directions and does not admit a compatible homomorphism α. So, it gives rise to a $\mathbb{Z}/2\mathbb{Z}$-graded holomorphic theory. Such a supercharge is stabilized by $G = \mathrm{SU}(4) \subset \mathrm{Spin}(9, \mathbb{C})$.

We may identify the odd part of the 9d $\mathcal{N} = 1$ supersymmetry algebra with the odd part of the 10d $\mathcal{N} = (1, 0)$ supersymmetry algebra. Under this identification a supercharge Q squares to zero in 9d iff it squares to zero in 10d.

5.1.1 Minimal twist

Let $Q \in \Sigma$ be a square-zero supercharge. Denote the image of $\Gamma(Q, -)\colon \Sigma \to V$ by $L^\perp \subset V$. Its orthogonal complement L is maximal isotropic and $L^\perp/L$ is one-dimensional. Since the bilinear form on $V_\mathbb{R}$ is positive-definite, $L \cap \sigma(L) = 0$. Moreover, $N = L^\perp \cap \sigma(L^\perp) \subset V$ is a σ-stable one-dimensional subspace, we let $N_\mathbb{R}$ be the σ-invariants of N. Therefore, we get a decomposition

$$V = L \oplus \sigma(L) \oplus N,$$

where $L^\perp = L \oplus N$.

Under the embedding $\mathrm{ML}(L) \subset \mathrm{Spin}(V)$ the spin representation $\Sigma = S$ decomposes as

$$\Sigma = \wedge^\bullet L \otimes \det(L)^{-1/2}$$

and the supercharge Q lies in the one-dimensional subspace $\det(L)^{1/2} \subset \Sigma$. Therefore, the choice of Q is equivalent to the choice of a one-dimensional subspace $N_\mathbb{R} \subset V_\mathbb{R}$ and a complex structure on $V_\mathbb{R}/N_\mathbb{R}$ together with a complex half-density.

It will be convenient to perform a computation of the twist in a slightly more general setting which will be useful for lower-dimensional computations.

Suppose L is a complex vector space equipped with a Kähler structure and a complex half-density. Suppose $N_\mathbb{R} = \mathbb{R}^{5-\dim(L)}$ equipped with a Euclidean metric and a spin structure. Denote by $N = N_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ its complexification which carries a complex half-density. Let $V_\mathbb{R} = L \times N$ (a 10-dimensional real vector space). By the results of Section 4.1.1, there is a canonical square-zero supercharge $Q \in \Sigma$ determined by the complex structure on $L \times N$ and a complex half-density.

The dimensional reduction of 10d super Yang–Mills on $L \times N$ alond $\mathrm{Re}\colon N \to N_\mathbb{R}$ is by definition the $(5 + \dim(L))$-dimensional super Yang–Mills on $L \times N_\mathbb{R}$. Since $N \cong N_\mathbb{R} \oplus iN_\mathbb{R}$, this theory carries an action of the R-symmetry group $G_R = \mathrm{Spin}(N_\mathbb{R})$. We consider a twisting homomorphism $\phi\colon \mathrm{SU}(L) \times \mathrm{Spin}(N_\mathbb{R}) \to G_R = \mathrm{Spin}(N_\mathbb{R})$ given by the projection onto the second factor under which Q is preserved.

Theorem 5.1. The twist of $(5 + \dim(L))$-dimensional super Yang–Mills on $L \times N_\mathbb{R}$ by Q is perturbatively equivalent to the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(L \times (N_\mathbb{R})_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{SU}(L) \times \mathrm{Spin}(N_\mathbb{R})$-equivariant.

Proof. By Theorem 4.1 the twist of 10d $\mathcal{N} = 1$ super Yang–Mills on $L \times N$ by Q is perturbatively equivalent to the holomorphic Chern–Simons theory. Moreover, the equivalence is $\mathrm{SU}(L) \times \mathrm{SU}(N)$-equivariant. By Proposition 1.39 we get that the dimensional reduction of holomorphic Chern–Simons on $L \times N$ along $\mathrm{Re}\colon N \to N_\mathbb{R}$ is isomorphic to the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(L \times N_\mathbb{R}, B\mathfrak{g})$ and this isomorphism is $\mathrm{SU}(L) \times \mathrm{SO}(N_\mathbb{R})$-equivariant, where $\mathrm{SO}(N_\mathbb{R})$ acts on N via the homomorphism (1).

Therefore, we just need to establish that the $\mathrm{Spin}(N_\mathbb{R})$-action on the twisted $(5 + \dim(L))$-dimensional super Yang–Mills obtained using the twisting homomorphism coincides with the $\mathrm{Spin}(N_\mathbb{R})$-action on the generalized Chern–Simons theory. The $\mathrm{Spin}(N_\mathbb{R})$-action on the fields of $(5 + \dim(L))$-dimensional super Yang–Mills is obtained via the homomorphism

$$\mathrm{Spin}(N_{\mathbb{R}}) \xrightarrow{\text{diagonal}} \mathrm{Spin}(N_{\mathbb{R}}) \times \mathrm{Spin}(N_{\mathbb{R}}) \hookrightarrow \mathrm{Spin}(N_{\mathbb{R}} \oplus N_{\mathbb{R}}),$$

where the diagonal embedding comes from the identity map to the partial Lorentz group $\mathrm{Spin}(N_{\mathbb{R}})$ and the twisting homomorphism, i.e. the identity map, to the R-symmetry group $G_R = \mathrm{Spin}(N_{\mathbb{R}})$. The $\mathrm{SO}(N_{\mathbb{R}})$-action on the fields of the generalized Chern–Simons theory is given by the composite

$$\mathrm{SO}(N_{\mathbb{R}}) \xrightarrow{(1)} \mathrm{SU}(N) \longrightarrow \mathrm{SO}(N_{\mathbb{R}} \oplus N_{\mathbb{R}})$$

The claim then follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{SO}(N_{\mathbb{R}}) \arrow[r, "\text{diagonal}"] \arrow[d, "(1)"] & \mathrm{SO}(N_{\mathbb{R}}) \times \mathrm{SO}(N_{\mathbb{R}}) \arrow[d] \\
\mathrm{SU}(N) \arrow[r] & \mathrm{SO}(N_{\mathbb{R}} \oplus N_{\mathbb{R}}).
\end{tikzcd}$$

□

We will now concentrate on the 9-dimensional case.

Theorem 5.2. The minimal twist of 9d $\mathcal{N} = 1$ super Yang–Mills on $M = \mathbb{C}^4 \times \mathbb{R}$ is perturbatively equivalent to the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(\mathbb{C}^4 \times \mathbb{R}_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{SU}(4)$-equivariant.

Proof. Any square-zero supercharge in the 9-dimensional supersymmetry algebra is square-zero in the 10-dimensional supersymmetry algebra. The claim follows from Theorem 5.1 applied to $L = \mathbb{C}^4$. □

6 Dimension 8

The 8-dimensional supersymmetry algebra has odd part $\Sigma \cong S_+ \otimes W \oplus S_- \otimes W^*$, where S_+, S_- are the 8-dimensional semi-spin representations of $\mathrm{Spin}(8, \mathbb{C})$ and W is a complex vector space. The semi-spin representations carry non-degenerate symmetric bilinear pairings $S_\pm \otimes S_\pm \to \mathbb{C}$. There is a Yang–Mills theory with $\mathcal{N} = 1$ supersymmetry, so we fix $W = \mathbb{C}$.

6.1 $\mathcal{N} = 1$ Super Yang–Mills

Let $\mathfrak{g}$ be a complex Lie algebra equipped with a symmetric bilinear invariant nondegenerate pairing. We consider $\mathcal{N} = 1$ super Yang–Mills theory on $M = \mathbb{R}^8$ with the Euclidean metric. It admits R-symmetry group $G_R = \mathrm{Spin}(2, \mathbb{R})$ which acts with weight $1/2$ on W and weight $-1/2$ on W^*.

This theory admits three twists:

- Supercharges $(Q, 0)$ and $(0, Q)$ with $(Q, Q)_{S_\pm} = 0$. These are holomorphic. Moreover, we have an isomorphism $\alpha\colon \mathrm{U}(1) \xrightarrow{\sim} \mathrm{Spin}(2, \mathbb{R})$ under which they have weight 1, so they give rise to a $\mathbb{Z}$-graded holomorphic theory. Such a supercharge is stabilized by $G = \mathrm{SU}(4) \subset \mathrm{Spin}(8, \mathbb{C})$. We have a twisting homomorphism $\phi\colon \mathrm{MU}(4) \xrightarrow{\det^{1/2}} \mathrm{U}(1) \cong G_R$, so the twisted theory carries an action of $\mathrm{MU}(4)$.

- Supercharges $(Q,0)$ and $(0,Q)$ with $(Q,Q)_{S_\pm} \neq 0$. These are topological. As before, we may choose a compatible homomorphism α, so they give rise to a $\mathbb{Z}$-graded topological theory. Such a supercharge is stabilized by $\mathrm{Spin}(7,\mathbb{R}) \subset \mathrm{Spin}(8,\mathbb{C})$.
- Square-zero supercharges (Q_+, Q_-) where both $Q_\pm$ are nonzero. These have 5 invariant directions and do not admit a compatible homomorphism α, so they give rise to a $\mathbb{Z}/2\mathbb{Z}$-graded theory. We have $(Q_\pm, Q_\pm)_{S_\pm} = 0$. The supercharge Q_+ is stabilized by $\mathrm{SU}(4) \subset \mathrm{Spin}(8,\mathbb{R})$. The supercharge Q_- is stabilized by $\mathrm{SU}(3) \subset \mathrm{SU}(4) \subset \mathrm{Spin}(8,\mathbb{R})$. We have a twisting homomorphism $\phi\colon \mathrm{SU}(3) \times \mathrm{Spin}(2,\mathbb{R}) \to G_R = \mathrm{Spin}(2,\mathbb{R})$ given by projection onto the second factor, so the twisted theory in fact carries an action of $\mathrm{SU}(3) \times \mathrm{Spin}(2,\mathbb{R})$.

6.1.1 Holomorphic twist

Suppose $Q \in S_+$ such that $(Q,Q)_{S_+} = 0$. As in Section 4.1.1, the data of such Q is equivalent to the data of a Kähler structure L on $V_\mathbb{R}$ together with a complex half-density on L.

We consider the twisting homomorphism $\det^{1/2}\colon \mathrm{MU}(4) \to \mathrm{Spin}(2,\mathbb{R})$ under which Q becomes scalar. Moreover, we have an isomorphism $\alpha\colon \mathrm{U}(1) \cong \mathrm{Spin}(2,\mathbb{R})$, so the theory is $\mathbb{Z}$-graded and carries an $\mathrm{MU}(4)$-action. In fact, this action will factor through $\mathrm{U}(4)$ by a direct observation.

Fields: The BRST fields are given by:

- Gauge fields $A_{1,0} \in \Omega^{1,0}(M;\mathfrak{g})$ and $A_{0,1} \in \Omega^{0,1}(M;\mathfrak{g})$.
- Scalar fields $\phi \in \Omega^{4,0}(M;\mathfrak{g})[2]$ and $\widetilde{\phi} \in \Omega^{0,4}(M;\mathfrak{g})[-2]$.
- Fermions $\chi \in \Omega^0(M;\mathfrak{g})[-1]$, $B \in \Omega^{0,2}(M;\mathfrak{g})[-1]$, $\widetilde{\chi} \in \Omega^{0,4}(M;\mathfrak{g})[-1]$, $\rho \in \Omega^{1,0}(M;\mathfrak{g})[1]$ and $C \in \Omega^{3,0}(M;\mathfrak{g})[1]$.
- A ghost field $c \in \Omega^0(M;\mathfrak{g})[1]$.

Theorem 6.1. The holomorphic twist of 8d $\mathcal{N}=1$ super Yang–Mills on $M = \mathbb{R}^8$ is perturbatively equivalent to the holomorphic BF theory on $M \cong \mathbb{C}^4$ with the space of fields $\mathrm{Map}(M, T^*[3]B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{U}(4)$-equivariant.

Proof. 8d $\mathcal{N}=1$ super Yang–Mills theory is obtained by dimensionally reducing 10d $\mathcal{N}=1$ super Yang–Mills theory. Under dimensional reduction the 10d fields from Section 4.1.1 decompose as follows:

$$\begin{aligned} A_{1,0} &\rightsquigarrow A_{1,0}, \widetilde{\phi} \\ A_{0,1} &\rightsquigarrow A_{0,1}, \phi \\ \rho &\rightsquigarrow \rho, \widetilde{\chi} \\ B &\rightsquigarrow B, C. \end{aligned}$$

The claim about the underlying $\mathbb{Z}/2$-graded theories follows by applying dimensional reduction (Proposition 1.38) to the computation of the minimal twist of 9d $\mathcal{N}=1$ super Yang–Mills (Theorem 5.2). We are left to check that the equivalence respects the gradings and the $\mathrm{U}(4)$-action. Indeed, the equivalence given by Theorem 5.2 eliminates fields $A_{1,0}, \widetilde{\phi}, \rho, \chi, \widetilde{\chi}$ and hence the underlying local L_∞ algebra after the twist becomes

$$\begin{gathered} \Omega^0(\mathbb{C}^4;\mathfrak{g})_c \longrightarrow \Omega^{0,1}(\mathbb{C}^4;\mathfrak{g})_{A_{0,1}} \longrightarrow \Omega^{0,2}(\mathbb{C}^4;\mathfrak{g})_B \longrightarrow \Omega^{0,3}(\mathbb{C}^4;\mathfrak{g})_{C^*} \longrightarrow \Omega^{0,4}(\mathbb{C}^4;\mathfrak{g})_{\phi^*} \\ \oplus \\ \Omega^{4,0}(\mathbb{C}^4;\mathfrak{g})_\phi \longrightarrow \Omega^{4,1}(\mathbb{C}^4;\mathfrak{g})_C \longrightarrow \Omega^{4,2}(\mathbb{C}^4;\mathfrak{g})_{B^*} \longrightarrow \Omega^{4,3}(\mathbb{C}^4;\mathfrak{g})_{A_{0,1}^*} \longrightarrow \Omega^{4,4}(\mathbb{C}^4;\mathfrak{g})_{c^*} \end{gathered}$$

concentrated in cohomological degrees $-2,\ldots,3$. These fields have the same degrees as in the holomorphic BF theory. $\square$

6.1.2 Topological twist

Next we discuss the case of the topological twist. We are going to prove that it is perturbatively trivial. In fact, it will be useful to study a degeneration of the topological twist to a holomorphic twist and decribe the corresponding family of twisted theories.

Let $V_{\mathbb{R}} = \mathbb{R}^8$ and $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Fix a Kähler structure on $\mathbb{R}^8$ and denote by $L \subset V$ the i-eigenspace of the complex structure. Moreover, fix a complex volume form on L. Under $\mathrm{SU}(L) \subset \mathrm{Spin}(V)$ the semi-spin representation S_+ decomposes as

$$S_+ \cong \mathbb{C}\alpha \oplus \wedge^2 L \oplus \mathbb{C}\beta.$$

The scalar spinorial pairing $S_+ \otimes S_+ \to \mathbb{C}$ is given by pairing the outer terms with each other and $\wedge^2 L$ with itself using the complex volume form on L. Consider a family of square-zero supercharges

$$Q_t = \alpha + t\beta \in S_+ \tag{19}$$

for $t \in \mathbb{C}$. We have

$$(Q_t, Q_t) = t,$$

so at $t = 0$ we have a holomorphic supercharge and and $t \neq 0$ we have a topological supercharge.

We will use the notation for fields of 8d $\mathcal{N} = 1$ super Yang–Mills from Section 6.1.1. Using the Calabi–Yau structure we will regard $C \in \Omega^{0,1}(\mathbb{C}^4; \mathfrak{g})[1]$ and $\widetilde{\chi} \in \Omega^0(\mathbb{C}^4; \mathfrak{g})[-1]$. First, we are going to write the functionals (5) and (6) in terms of these fields.

Proposition 6.1. The functionals $I^{(1)}$ and $I^{(2)}$ (see (5) and (6)) in terms of the fields of 8d $\mathcal{N} = 1$ super Yang–Mills are

$$\begin{aligned} I^{(1)}(Q_t) &= \int \mathrm{dvol}\, \left(-(\rho, A^*_{1,0}) - t(C, A^*_{0,1}) - (\widetilde{\chi} + t\chi)\widetilde{\phi}^*\right) \\ &\quad + \int \mathrm{dvol} \left((F_{0,2}, B^*) + t\Omega^{-1} F_{2,0} \wedge B^* + (\overline{\partial}_{A_{0,1}}\phi, C^*) + (t\partial_{A_{1,0}}\phi, \rho^*) + \frac{1}{2}\Lambda F_{1,1}(\chi^* - t\widetilde{\chi}^*) \right) \\ I^{(2)}(Q_t, Q_t) &= \int \mathrm{dvol} \left(t\chi^*\widetilde{\chi}^* + \frac{t}{2}\Omega^{-1} B^* \wedge B^* - \frac{1}{4}(\chi^* + t\widetilde{\chi}^*)^2 + t\phi c^* \right). \end{aligned}$$

The action of the twisted 8d super Yang–Mills theory is given by

$$S_{Q_t} = S_{\mathrm{BRST}} + S_{\mathrm{anti}} + I^{(1)}(Q_t) + I^{(2)}(Q_t),$$

where S_{BRST} and S_{anti} are given by (15) and (16) respectively.

We have a homomorphism $\alpha \colon \mathrm{U}(1) \to G_R = \mathrm{Spin}(2, \mathbb{R})$ with respect to which Q_t has weight 1, so the Q_t-twisted theory will have a $\mathbb{Z}$-grading.

Theorem 6.2. The twist of 8d $\mathcal{N} = 1$ super Yang–Mills with respect to the family Q_t of square-zero supercharges is perturbatively equivalent to the holomorphic Hodge theory $\mathrm{Map}(\mathbb{C}^4, B\mathfrak{g}_{\mathrm{Hod}})$. Moreover, this equivalence is $\mathrm{SU}(4)$-equivariant.

Proof. The proof proceeds as in the proof of Theorem 4.1 with slight modifications.

Observe that the quadruple of fields $\{\chi^*, \chi, \widetilde{\chi}^*, \widetilde{\chi}\}$ has the same Poisson brackets as the quadruple $\{\chi^* - t\widetilde{\chi}^*, \chi, \widetilde{\chi}^*, \widetilde{\chi} + t\chi\}$. Therefore, we may eliminate the fields $\chi^* - t\widetilde{\chi}^*, \chi$ using Proposition 1.6. We then have trivial BRST doublets $\{\widetilde{\chi} + t\chi, \widetilde{\phi}\}$ and $\{\rho, A_{1,0}\}$ which may be eliminated using Proposition 1.8. We are left with the action

$$S_{BF} + \int \mathrm{dvol} \left(-t(C, A^*_{0,1}) + t\phi c^* + \frac{t}{2}\Omega^{-1} B^* \wedge B^* \right),$$

where S_{BF} is the action functional of the holomorphic twist at $t = 0$. Since the extra terms are quadratic in the fields, the claim is reduced to a comparison of the underlying local L_∞ algebra of the twisted theory and that of the holomorphic Hodge theory. The former is given by (cf. the proof of Theorem 6.1)

$$\begin{tikzcd}[column sep=tiny]
 & \Omega^0(\mathbb{C}^4;\mathfrak{g})_c \arrow[rr] & & \Omega^{0,1}(\mathbb{C}^4;\mathfrak{g})_{A_{0,1}} \arrow[rr] & & \Omega^{0,2}(\mathbb{C}^4;\mathfrak{g})_B \arrow[rr] & & \Omega^{0,3}(\mathbb{C}^4;\mathfrak{g})_{C^*} \arrow[rr] & & \Omega^{0,4}(\mathbb{C}^4;\mathfrak{g})_{\phi^*} \\
\Omega^0(\mathbb{C}^4;\mathfrak{g})_\phi \arrow[rr] \arrow[ur, dashed, "t\,\mathrm{id}"] & & \Omega^{0,1}(\mathbb{C}^4;\mathfrak{g})_C \arrow[rr] \arrow[ur, dashed, "t\,\mathrm{id}"] & & \Omega^{0,2}(\mathbb{C}^4;\mathfrak{g})_{B^*} \arrow[rr] \arrow[ur, dashed, "t\,\mathrm{id}"] & & \Omega^{0,3}(\mathbb{C}^4;\mathfrak{g})_{A^*_{0,1}} \arrow[rr] \arrow[ur, dashed, "t\,\mathrm{id}"] & & \Omega^{0,4}(\mathbb{C}^4;\mathfrak{g})_{c^*} \arrow[ur, dashed, "t\,\mathrm{id}"] &
\end{tikzcd}$$

which is exactly the local L_∞ algebra of the holomorphic Hodge theory. □

Corollary 6.3. The topological twist of 8d $\mathcal{N} = 1$ super Yang–Mills is perturbatively trivial.

Proof. The topological twist of 8d $\mathcal{N} = 1$ super Yang–Mills is the twist by Q_t with $t \neq 0$. By Theorem 6.2 it is equivalent to the $t \neq 0$ specialization of the holomorphic Hodge theory which by Proposition 1.34 is perturbatively trivial. □

6.1.3 Partially topological twist

Finally we discuss the case of the partially topological supercharge $(Q_+, Q_-) \in \Sigma$. We consider the twisting homomorphism $\phi\colon \mathrm{SU}(3) \times \mathrm{Spin}(2, \mathbb{R}) \to G_R = \mathrm{Spin}(2, \mathbb{R})$ given by projection on the second factor.

Theorem 6.4. The partially topological twist of 8d $\mathcal{N} = 1$ super Yang–Mills is perturbatively equivalent to the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(\mathbb{C}^3 \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{SU}(3) \times \mathrm{Spin}(2, \mathbb{R})$-equivariant.

Proof. Since Q_+ and Q_- satisfy $(Q_\pm, Q_\pm)_{S_\pm} = 0$, they lift to a square-zero supercharge in the 10-dimensional supersymmetry algebra. The claim follows from Theorem 5.1 applied to $L = \mathbb{C}^3$. □

7 Dimension 7

The 7-dimensional supersymmetry algebra has odd part $\Sigma \cong S \otimes W$, where S is the 8-dimensional spin representation of $\mathrm{Spin}(7, \mathbb{C})$ and W is a complex symplectic vector space. The spin representation carries a nondegenerate symmetric bilinear pairing $S \otimes S \to \mathbb{C}$. There is a Yang–Mills theory with $\mathcal{N} = 1$ supersymmetry, so we fix $W = \mathbb{C}^2$.

7.1 $\mathcal{N} = 1$ Super Yang–Mills

Let $\mathfrak{g}$ be a complex Lie algebra equipped with a symmetric bilinear invariant nondegenerate pairing. We consider $\mathcal{N} = 1$ super Yang–Mills theory on $M = \mathbb{R}^7$ with the Euclidean metric. It admits R-symmetry group $G_R = \mathrm{Spin}(3, \mathbb{R})$ with W the two-dimensional spin representation.

This theory admits three twists:

- Rank 1 supercharges $Q = \alpha \otimes w \in S \otimes W$, where $(\alpha, \alpha)_S = 0$. These are minimal, i.e. the number of invariant directions is 4. We have a homomorphism $\alpha\colon \mathrm{U}(1) \to G_R = \mathrm{Spin}(3, \mathbb{R})$ under which they have weight 1. We also have a twisting homomorphism $\phi\colon \mathrm{MU}(3) \xrightarrow{\det^{1/2}} \mathrm{U}(1) \to \mathrm{Spin}(3, \mathbb{R})$, so the twisted theory is $\mathbb{Z}$-graded and carries an action of $\mathrm{MU}(3)$.

- Rank 1 supercharges $Q = \alpha \otimes w \in S \otimes W$, where $(\alpha, \alpha)_S \neq 0$. These are topological and stabilized by $G_2 \subset \mathrm{Spin}(7, \mathbb{C})$. We have a homomorphism $\alpha \colon \mathrm{U}(1) \to G_R = \mathrm{Spin}(3, \mathbb{R})$ under which they have weight 1.
- Square-zero supercharges Q of rank 2. These have 5 invariant directions and do not admit a compatible homomorphism α, so they give rise to a $\mathbb{Z}/2\mathbb{Z}$-graded theory. We have a twisting homomorphism $\phi \colon \mathrm{SU}(2) \times \mathrm{Spin}(3, \mathbb{R}) \to G_R = \mathrm{Spin}(3, \mathbb{R})$ given by projection on the second factor, so tht theory carries an action of $\mathrm{SU}(2) \times \mathrm{Spin}(3, \mathbb{R})$.

7.1.1 Minimal twist

Denote the image of $\Gamma(Q, -) \colon \Sigma \to V$ by $L^\perp$, so that its orthogonal complement $L \subset V$ is a 3-dimensional isotropic subspace. As in Section 5.1.1, the data of a partially topological supercharge is equivalent to the choice of a one-dimensional subspace $N_\mathbb{R} \subset V_\mathbb{R}$ and a complex structure on $V_\mathbb{R}/N_\mathbb{R}$ together with a half-density.

We have $Q = \alpha \otimes w \in S \otimes W$ and denote by $\alpha \colon \mathrm{U}(1) \hookrightarrow G_R = \mathrm{Spin}(3, \mathbb{R})$ the subgroup under which w has weight 1. We consider the twisting homomorphism given by the composite

$$\phi \colon \mathrm{MU}(3) \xrightarrow{\det^{1/2}} \mathrm{U}(1) \xrightarrow{\alpha} G_R.$$

As before, the $\mathrm{MU}(3)$-action on the theory factors through a $\mathrm{U}(3)$-action.

Theorem 7.1. The minimal twist of 7d $\mathcal{N} = 1$ super Yang–Mills on $M = \mathbb{R}^7$ is perturbatively equivalent to the generalized BF theory on $\mathbb{C}^3 \times \mathbb{R}$ with the space of fields $\mathrm{Map}(\mathbb{C}^3 \times \mathbb{R}_{\mathrm{dR}}, T^*[3]B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{U}(3)$-equivariant.

Proof. The minimal supercharge $Q = \alpha \otimes w$ may be obtained by dimensional reduction of the holomorphic supercharge in the 8-dimensional supersymmetry algebra. Therefore, the minimal twist of 7d $\mathcal{N} = 1$ super Yang–Mills on $\mathbb{C}^3 \times \mathbb{R}$ is obtained by dimensional reduction from the holomorphic twist of 8d $\mathcal{N} = 1$ super Yang–Mills on $\mathbb{C}^4$. The claim then follows from Theorem 6.1 and Corollary 1.6. ☐

7.1.2 Topological twist

Next we study the topological twist. Let $L = \mathbb{C}^3$ equipped with a Hermitian structure and consider the spacetime $V_\mathbb{R} = L \times \mathbb{R}$. Under $\mathrm{SU}(L) \subset \mathrm{Spin}(V_\mathbb{R})$ the spin representation decomposes as

$$S \cong \wedge^\bullet L.$$

In particular, it contains two trivial summands: $\mathbb{C}\alpha$ in the lowest degree and $\mathbb{C}\beta$ in the top degree.

Consider a family of square-zero supercharges

$$Q_t = (\alpha + t\beta) \otimes w \tag{20}$$

for some nonzero $w \in W$. We have $(\alpha + t\beta, \alpha + t\beta) = t$, so at $t = 0$ we have a minimal supercharge and at $t \neq 0$ we have a topological supercharge.

Theorem 7.2. The twist of 7d $\mathcal{N} = 1$ super Yang–Mills with respect to the family Q_t of square-zero supercharges is perturbatively equivalent to the generalized Hodge theory $\mathrm{Map}(\mathbb{C}^3 \times \mathbb{R}_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$. Moreover, this equivalence is $\mathrm{SU}(3)$-equivariant.

Proof. The family of square-zero supercharges given by equation (20) may be obtained by dimensional reduction of the corresponding family Q_t^8 (see equation (19)) in 8 dimensions. The twist of 7d $\mathcal{N} = 1$ super Yang–Mills with respect to Q_t is obtained by dimensional reduction of the twist of 8d $\mathcal{N} = 1$ super Yang–Mills with respect to Q_t^8. The claim then follows from Theorem 6.2 and Corollary 1.7. ☐

Corollary 7.3. The topological twist of 7d $\mathcal{N} = 1$ super Yang–Mills is perturbatively trivial.

7.1.3 Partially topological twist

Finally, we discuss the case of a partially topological twist. We consider the twisting homomorphism $\phi\colon \mathrm{SU}(3) \times \mathrm{Spin}(3,\mathbb{R}) \to G_R = \mathrm{Spin}(3,\mathbb{R})$ given by projection on the second factor.

Theorem 7.4. The partially topological twist of 7d $\mathcal{N} = 1$ super Yang–Mills is perturbatively equivalent to the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}^3_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{SU}(2) \times \mathrm{Spin}(3,\mathbb{R})$-equivariant.

Proof. Any partially topological supercharge in the 7-dimensional supersymmetry algebra lifts to a square-zero supercharge in the 10-dimensional supersymmetry algebra. The claim follows from Theorem 5.1 applied to $L = \mathbb{C}^2$. □

8 Dimension 6

The 6-dimensional supersymmetry algebra has odd part $\Sigma \cong S_+ \otimes W_+ \oplus S_- \otimes W_-$, where S_+, S_- are the 4-dimensional semi-spin representations of $\mathrm{Spin}(6,\mathbb{C}) \cong \mathrm{SL}(4,\mathbb{C})$ and W_+, W_- are complex symplectic vector spaces. We have isomorphisms $S_+ \cong S_-^*$.

There are Yang–Mills theories with $\mathcal{N} = (1,0)$ or $\mathcal{N} = (1,1)$ supersymmetry, which we consider separately.

8.1 $\mathcal{N} = (1,0)$ Super Yang–Mills

The general setup for $\mathcal{N} = (1,0)$ super Yang–Mills is described in Section 3 which we now recall. Let $\mathfrak{g}$ be a complex Lie algebra equipped with a symmetric bilinear invariant nondegenerate pairing and U a complex symplectic $\mathfrak{g}$-representation. We consider $\mathcal{N} = (1,0)$ super Yang–Mills theory on $M = \mathbb{R}^6$ with the Euclidean metric. We fix $W_- = 0$ and $W_+ = \mathbb{C}^2$ equipped with a symplectic structure. The R-symmetry group depends on the type of the representation U:

- In general, the theory admits an R-symmetry group $G_R = \mathrm{SU}(2)$ with W_+ the two-dimensional defining representation.
- If $U = T^*R = R \oplus R^*$ for a $\mathfrak{g}$-representation R, then the theory admits an R-symmetry group $G_R = \mathrm{SU}(2) \times \mathrm{U}(1)$, where $\mathrm{U}(1)$ acts trivially on W_+, with weight 1 on R and with weight -1 on R^*.

This theory admits a unique twist:

- A square-zero supercharge $Q \neq 0 \in \Sigma$ has 3 invariant directions, so it gives rise to a holomorphic theory. If the representation U is of cotangent type, we have a compatible homomorphism $\alpha\colon \mathrm{U}(1) \to G_R = \mathrm{SU}(2) \times \mathrm{U}(1)$ given by a diagonal embedding, so in this case we get a $\mathbb{Z}$-grading. Such a supercharge is stabilized by $G = \mathrm{SU}(3) \subset \mathrm{Spin}(6,\mathbb{C})$. We have a twisting homomorphism $\phi\colon \mathrm{MU}(3) \xrightarrow{\det^{1/2}} \mathrm{U}(1) \hookrightarrow \mathrm{SU}(2)$, so the twisted theory carries an $\mathrm{MU}(3)$-action.

Fields: The BRST fields are given by:

- A gauge field $A \in \Omega^1(M; \mathfrak{g})$.

- Gauge fermion $\lambda \in \Omega^0(M; \Pi S_+ \otimes W_+ \otimes \mathfrak{g})$.
- Matter boson $\phi \in \Omega^0(M; W_+ \otimes U)$.
- Matter fermion $\psi \in \Omega^0(M; \Pi S_- \otimes U)$.
- A ghost field $c \in \Omega^0(M; \mathfrak{g})[1]$.

8.1.1 Holomorphic twist

Consider a nonzero $Q \in S_+ \otimes W_+$. Since $\wedge^2(S_+) \cong V$, the square-zero condition is equivalent to the condition that Q has rank 1, i.e. $Q = q_+ \otimes w_1 \in S_+ \otimes W_+$. We will also fix $w_2 \in W_+$ such that $(w_1, w_2) = 1$.

As in Section 4.1.1, the data of q_+ is equivalent to the data of a Kähler structure L on $V_\mathbb{R}$ together with a complex half-density on L.

Under the embedding $\mathrm{MU}(L) \subset \mathrm{Spin}(V_\mathbb{R})$, the semi-spin representations S_+, S_- decompose as

$$S_+ = \det(L)^{1/2} \oplus L \otimes \det(L)^{-1/2}, \qquad S_- = \det(L)^{-1/2} \oplus L^* \otimes \det(L)^{1/2},$$

where $q_+ \in S_+$ lies in the first summand.

We fix an embedding $\mathrm{U}(1) \subset \mathrm{SU}(2)$ under which $w_1 \in W_+$ has weight 1. Under the composite

$$\phi \colon \mathrm{MU}(3) \xrightarrow{\det^{1/2}} \mathrm{U}(1) \subset \mathrm{SU}(2)$$

we obtain that $W_+ \cong \det(L)^{-1/2} w_1 \oplus \det(L)^{1/2} w_2$.

We will now rewrite the fields using the twisting homomorphism ϕ from $\mathrm{MU}(3)$, where we denote by K the canonical bundle of $L = \mathbb{C}^3$.

Fields: The BRST fields are given by:

- Gauge fields $A_{1,0} \in \Omega^{1,0}(M; \mathfrak{g})$, $A_{0,1} \in \Omega^{0,1}(M; \mathfrak{g})$.
- Gauge fermions $\chi \in \Omega^0(M; \Pi\mathfrak{g})$, $\xi \in \Omega^{3,0}(M; \Pi\mathfrak{g})$, $B \in \Omega^{0,2}(M; \Pi\mathfrak{g})$, $\rho \in \Omega^{1,0}(M; \Pi\mathfrak{g})$.
- Matter bosons $\nu \in \Omega^0(M; U \otimes K^{-1/2})$, $\phi \in \Omega^0(M; U \otimes K^{1/2})$.
- Matter fermions $\psi \in \Omega^{1,0}(M; \Pi U \otimes K^{1/2})$, $\widetilde{\nu} \in \Omega^0(M; \Pi U \otimes K^{-1/2})$.
- A ghost field $c \in \Omega^0(M; \mathfrak{g})[1]$.

Let $\omega \in \Omega^{1,1}(M)$ be the Kähler form. We denote the real volume form on M by

$$\mathrm{dvol} = \frac{\omega^3}{6}.$$

Using Corollary 4.2, the BV action S_{BV} of the Q-twisted theory consists of the sum of the following terms:

$$\begin{aligned}
S_{\text{gauge}} &= \int \mathrm{dvol}\left(-(F^{2,0},F^{0,2}) - \frac{1}{4}(\Lambda F_{1,1})^2\right) + \\
&\quad + \frac{1}{2}\left(\omega(B\wedge\partial_{A_{1,0}}\rho) + \omega^2\chi\Lambda(\overline{\partial}_{A_{0,1}}\rho) - \omega(\rho\wedge\partial_{A_{1,0}}B) + \xi\overline{\partial}_{A_{0,1}}B\right) \\
S_{\text{matter}} &= \int\Bigg(\mathrm{dvol}((\partial_{A_{1,0}}\nu,\overline{\partial}_{A_{0,1}}\phi) + (\partial_{A_{1,0}}\phi,\overline{\partial}_{A_{0,1}}\nu)) + 2\omega^2\wedge(\widetilde{\nu}\overline{\partial}_{A_{1,0}}\psi) + \psi\wedge\overline{\partial}_{A_{0,1}}\psi + \\
&\quad + 2\,\mathrm{dvol}(([\xi,\nu],\widetilde{\nu}) + ([\chi,\phi],\widetilde{\nu}))\Bigg) \\
S_{\text{anti}} &= \int \partial_{A_{1,0}}c\wedge A^*_{1,0} + \overline{\partial}_{A_{0,1}}c\wedge A^*_{0,1} + [B,c]\wedge B^* + [\xi,c]\wedge\xi^* + [\chi,c]\chi^* + [\rho,c]\wedge\rho^* + \\
&\quad + \frac{1}{2}[c,c]c^* + [\nu,c]\wedge\nu^* + [\phi,c]\wedge\phi^* + [\psi,c]\wedge\psi^* + [\widetilde{\nu},c]\wedge\widetilde{\nu}^* \\
I^{(1)}_{\text{gauge}} &= \int \mathrm{dvol}\left(-(\rho,A^*_{1,0}) + \frac{1}{2}(F_{0,2},B^*) + \frac{1}{2}(F_{1,1},\chi^*)\right) \\
I^{(1)}_{\text{matter}} &= \int \mathrm{dvol}\left((\widetilde{\nu},\nu^*) + \frac{1}{2}(\overline{\partial}_{A_{0,1}}\phi,\psi^*)\right) \\
I^{(2)}_{\text{gauge}} &= -\frac{1}{4}\int \mathrm{dvol}(\chi^*)^2.
\end{aligned}$$

Theorem 8.1. The holomorphic twist of 6d $\mathcal{N}=(1,0)$ super Yang–Mills on $M=\mathbb{R}^6$ with matter valued in a $\mathfrak{g}$-representation U is perturbatively equivalent to the holomorphic Chern–Simons theory on $M\cong\mathbb{C}^3$ with the space of fields $\mathrm{Map}(M, U/\!/\mathfrak{g})$. Moreover, the equivalence is $\mathrm{MU}(3)$-equivariant.

Proof. The proof of this theorem is very similar to the proof of Theorem 4.1. First, we eliminate the fields χ and χ^* using Proposition 1.6. We then observe that the action includes the terms $\int \mathrm{dvol}(\rho, A^*_{1,0})$ and $\int\mathrm{dvol}(\widetilde{\nu},\nu^*)$. In other words, the pairs $(\rho, A_{1,0})$ and $(\nu,\widetilde{\nu})$ form trivial BRST doublets, which can be eliminated using Proposition 1.8. The twisted theory is therefore perturbatively equivalent to the theory with the BV action

$$\begin{aligned}
S_{\mathrm{BV}} = \int & \xi\overline{\partial}_{A_{0,1}}B + \psi\wedge\overline{\partial}_{A_{0,1}}\psi \\
&+ \mathrm{dvol}\left(\frac{1}{2}(F_{0,2},B^*) + \frac{1}{2}(\overline{\partial}_{A_{0,1}}\phi,\psi^*) + (\overline{\partial}_{A_{0,1}}c,A^*_{0,1}) + ([B,c],B^*) + ([\xi,c],\xi^*) + \frac{1}{2}[c,c]c^* + [\phi,c]\phi^* + ([\psi,c],\psi^*)\right).
\end{aligned}$$

Up to rescaling of the antifields, this is the action functional of the required theory. □

If $U = T^*R = R\oplus R^*$, the R-symmetry group is enhanced to $G_R = \mathrm{SU}(2)\times\mathrm{U}(1)$. We have a homomorphism $\alpha\colon \mathrm{U}(1)\hookrightarrow G_R = \mathrm{SU}(2)\times\mathrm{U}(1)$ given by the diagonal embedding which is compatible with the holomorphic supercharge. We may also use a new twisting homomorphism

$$\phi\colon \mathrm{MU}(3)\xrightarrow{\det^{1/2}}\mathrm{U}(1)\xrightarrow{\alpha} G_R.$$

With these modifications the BRST fields are given by:

- Gauge fields $A_{1,0}\in\Omega^{1,0}(M;\mathfrak{g})$, $A_{0,1}\in\Omega^{0,1}(M;\mathfrak{g})$.
- Gauge fermions $\chi\in\Omega^0(M;\mathfrak{g})[-1]$, $\xi\in\Omega^{3,0}(M;\Pi\mathfrak{g})[1]$, $B\in\Omega^{0,2}(M;\mathfrak{g})[-1]$, $\rho\in\Omega^{1,0}(M;\Pi\mathfrak{g})[1]$.
- Matter bosons $\nu\in\Omega^0(M;R^*\oplus R\otimes K^{-1}[-2])$, $\phi\in\Omega^0(M;R\oplus R^*\otimes K[2])$.
- Matter fermions $\psi\in\Omega^{1,0}(M;R[-1]\oplus R^*\otimes K[1])$, $\widetilde{\nu}\in\Omega^0(M;R^*[1]\oplus R\otimes K^{-1}[-1])$.

- A ghost field $c \in \Omega^0(M;\mathfrak{g})[1]$.

Note that the MU(3)-action on the fields factors through U(3) since the square roots of K have canceled out. By comparing the degrees and the transformation rules of the fields in Theorem 8.1 we obtain the following statement.

Theorem 8.2. The holomorphic twist of 6d $\mathcal{N} = (1,0)$ super Yang–Mills on $M = \mathbb{R}^6$ with matter valued in a $\mathfrak{g}$-representation $U = T^*R = R \oplus R^*$ is perturbatively equivalent to the holomorphic BF theory on $M \cong \mathbb{C}^3$ with the space of fields $\mathrm{Map}(M, T^*[2](R/\mathfrak{g}))$. Moreover, the equivalence is U(3)-equivariant.

8.2 $\mathcal{N} = (1,1)$ Super Yang–Mills

The 6d $\mathcal{N} = (1,1)$ super Yang–Mills theory is obtained by dimensional reduction from the 10d $\mathcal{N} = (1,0)$ super Yang–Mills. It admits R-symmetry group $G_R = \mathrm{Spin}(4,\mathbb{R})$ under which W_+, W_- are the two semi-spin representations.

Given an element $Q \in S_+ \otimes W_+ \oplus S_- \otimes W_-$ we denote by $W^*_{Q_\pm} \subset S_\pm$ the images of Q. We classify square-zero supercharges according to the ranks of these spaces:

- Rank $(1,0)$ and $(0,1)$. These automatically square to zero and are holomorphic. Such supercharges factor through a copy of the $\mathcal{N} = (1,0)$ (respectively, $\mathcal{N} = (0,1)$) supersymmetry algebra. They admit a twisting homomorphism from MU(3) and a $\mathbb{Z}$-grading $\alpha\colon \mathrm{U}(1) \to G_R$.
- Rank $(1,1)$ and $\langle W^*_{Q_+}, W^*_{Q_-}\rangle = 0$. These automatically square to zero and have 4 invariant directions. There is a $\mathbb{Z}$-grading $\alpha\colon \mathrm{U}(1) \hookrightarrow G_R = \mathrm{SU}(2) \times \mathrm{SU}(2)$ given by the diagonal embedding and these admit a twisting homomorphism $\phi\colon \mathrm{MU}(2) \times \mathrm{Spin}(2,\mathbb{R}) \xrightarrow{\det^{1/2} \times \mathrm{id}} \mathrm{U}(1) \xrightarrow{\alpha} G_R$.
- Rank $(1,1)$ and $\langle W^*_{Q_+}, W^*_{Q_-}\rangle \neq 0$. These automatically square to zero and are topological. Such supercharges are stabilized by $\mathrm{SU}(3) \subset \mathrm{Spin}(6,\mathbb{C})$ and have a $\mathbb{Z}$-grading $\alpha\colon \mathrm{U}(1) \to G_R = \mathrm{SU}(2) \times \mathrm{SU}(2)$ given by the diagonal embedding.
- Rank $(2,2)$. The square-zero supercharges have 5 invariant directions and give rise to a $\mathbb{Z}/2$-graded theory. There is the identity twisting homomorphism $\phi\colon \mathrm{Spin}(4,\mathbb{R}) \to G_R = \mathrm{Spin}(4,\mathbb{R})$, so the twisted theory carries a $\mathrm{Spin}(4,\mathbb{R})$-action.

8.2.1 Holomorphic twist

The 6d $\mathcal{N} = (1,1)$ super Yang–Mills theory viewed as a $\mathcal{N} = (1,0)$ supersymmetric theory coincides with the 6d $\mathcal{N} = (1,0)$ super Yang–Mills with matter in the representation $U = T^*\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$. Under this isomorphism the R-symmetry group $\mathrm{SU}(2) \times \mathrm{U}(1)$ of 6d $\mathcal{N} = (1,0)$ super Yang–Mills is a subgroup of the R-symmetry group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ of 6d $\mathcal{N} = (1,1)$ super Yang–Mills. In particular, from Theorem 8.2 we obtain the following statement.

Theorem 8.3. The holomorphic twist of 6d $\mathcal{N} = (1,1)$ super Yang–Mills on $M = \mathbb{R}^6$ is perturbatively equivalent to the holomorphic BF theory on $M \cong \mathbb{C}^3$ with the space of fields $\mathrm{Map}(M, T^*[2](\mathfrak{g}/\mathfrak{g}))$. Moreover, the equivalence is U(3)-equivariant.

8.2.2 Rank $(1,1)$ partially topological twist

We define α to be the diagonal embedding $\mathrm{U}(1) \hookrightarrow G_R = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and the twisting homomorphism ϕ to be given by the composite

$$\phi\colon \mathrm{MU}(2) \times \mathrm{Spin}(2) \xrightarrow{\det^{1/2} \times \mathrm{id}} \mathrm{U}(1) \xrightarrow{\alpha} G_R.$$

Let $L = \mathbb{C}^2$ equipped with a Hermitian structure and consider the 7-dimensional spacetime $V^7_{\mathbb{R}} = L \times \mathbb{R}^3$ and the 6-dimensional spacetime $V^6_{\mathbb{R}} = L \times \mathbb{R}^2$. Under the projection $V^7_{\mathbb{R}} \to V^6_{\mathbb{R}}$ a rank 2 square-zero supercharge Q in 7 dimensions dimensionally reduces to a rank $(1,1)$ partially topological square-zero supercharge in 6 dimensions.

Theorem 8.4. The rank $(1,1)$ partially topological twist of 6d $\mathcal{N} = (1,1)$ super Yang–Mills is perturbatively equivalent to the generalized BF theory with the space of fields $\mathrm{Map}(\mathbb{C}^2 \times \mathbb{R}^2_{\mathrm{dR}}, T^*[4]B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{MU}(2) \times \mathrm{Spin}(2,\mathbb{R})$-equivariant.

Proof. The claim about the underlying $\mathbb{Z}/2$-graded theories follows by dimensional reduction (Proposition 1.39) along $V^7_{\mathbb{R}} \to V^6_{\mathbb{R}}$ from the corresponding statement for the 7-dimensional super Yang–Mills given by Theorem 7.4.

To check that the $\mathbb{Z}$-gradings are compatible, we observe that α defined above coincides with α defined for 6d $\mathcal{N} = (1,0)$ super Yang–Mills with matter valued in a $\mathfrak{g}$-representation $U = T^*\mathfrak{g}$. In particular, the $\mathbb{Z}$-grading of fields in this twist coincides with the $\mathbb{Z}$-grading of fields in the holomorphic twist given in Theorem 8.2. The claim then follows from a direct comparison. □

8.2.3 Topological twist

Next we study the rank $(1,1)$ topological twist. Let $L = \mathbb{C}^3$ equipped with a Hermitian structure and consider the spacetime $V_{\mathbb{R}} = L$. Under $\mathrm{SU}(L) \subset \mathrm{Spin}(V_{\mathbb{R}})$ the semi-spin representations decompose as

$$S_+ \cong \mathbb{C}\alpha \oplus L, \qquad S_- \cong \mathbb{C}\beta \oplus L^*.$$

Consider a family of square-zero supercharges

$$Q_t = \alpha + t\beta \in \Sigma. \tag{21}$$

At $t = 0$ we obtain a rank $(1,0)$ holomorphic supercharge and at $t \neq 0$ we obtain a rank $(1,1)$ topological supercharge.

Theorem 8.5. The twist of 6d $\mathcal{N} = (1,1)$ super Yang–Mills with respect to the family Q_t of square-zero supercharges is perturbatively equivalent to the generalized Hodge theory $\mathrm{Map}(\mathbb{C}^3, T^*[2]B\mathfrak{g}_{\mathrm{Hod}})$. Moreover, this equivalence is $\mathrm{SU}(3)$-equivariant.

Proof. The family of square-zero supercharges given by equation (21) lifts to the family of square-zero supercharges in 7 dimensions given by equation (20). So, the twist of 6d $\mathcal{N} = (1,1)$ super Yang–Mills is obtained by a dimensional reduction of the twist of 7d $\mathcal{N} = 1$ super Yang–Mills. Therefore, the claim follows from Theorem 7.2 and Corollary 1.5. □

Corollary 8.6. The topological twist of 6d $\mathcal{N} = (1,1)$ super Yang–Mills is perturbatively trivial.

8.2.4 Rank $(2,2)$ twist

We finally consider twist by a rank $(2,2)$ square-zero supercharge $Q \in \Sigma$.

Theorem 8.7. The rank $(2,2)$ twist of 6d $\mathcal{N} = (1,1)$ super Yang–Mills is perturbatively equivalent to the generalized Chern–Simons theory with the space of fields $\mathrm{Map}(\mathbb{C} \times \mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g})$. Moreover, the equivalence is $\mathrm{Spin}(4,\mathbb{R})$-equivariant.

Proof. Consider the $\mathrm{Spin}(8,\mathbb{C})$ semi-spin representations $S^8_{\pm}$. Under the embedding $\mathrm{Spin}(6,\mathbb{C}) \times \mathrm{Spin}(2,\mathbb{C}) \subset \mathrm{Spin}(8,\mathbb{C})$, these decompose as

$$S^8_+ = S^6_+ \otimes S^2_+ \oplus S^6_- \otimes S^2_-, \qquad S^8_- = S^6_+ \otimes S^2_- \oplus S^6_- \otimes S^2_+,$$

where $S^6_\pm$ are the semi-spin representations of $\mathrm{Spin}(6,\mathbb{C})$ and $S^2_\pm$ are the semi-spin representations of $\mathrm{Spin}(2,\mathbb{C})$. The odd part of the supersymmetry algebra is

$$\Sigma = S^8_+ \oplus S^8_- \cong S^6_+ \otimes W_+ \oplus S^6_- \otimes W_-,$$

where $W_\pm \cong S^2_+ \oplus S^2_-$. Let $\alpha_\pm$ be basis vectors of $S^2_\pm$ and consider an arbitrary supercharge

$$Q = s^+_+ \otimes \alpha_+ + s^-_+ \otimes \alpha_- + s^+_- \otimes \alpha_- + s^-_- \otimes \alpha_+ \in \Sigma,$$

where the lower index on s refers to $S^8_\pm$ and the upper index on s refers to $S^6_\pm$.

Let us assume that Q is a rank $(2,2)$ supercharge, i.e. $s^\pm_+, s^\pm_-$ are all nonzero. The supercharge Q squares to zero in 6d iff $s^+_+ \wedge s^+_-$ under the isomorphism $\wedge^2 S^6_+ \cong V$ is equal to $s^-_+ \wedge s^-_-$ under the isomorphism $\wedge^2 S^6_- \cong V$.

Q squares to zero in 8d iff it squares to zero in 6d and $(s^+_+, s^-_-) = (s^+_-, s^-_+) = 0$, where $(-,-)$ denotes the natural pairing between S^6_+ and $S^6_- \cong (S^6_+)^*$. But by **[ElliottSafronov]** the condition that Q squares to zero in 6d already implies the vanishing of these pairings.

So, any square-zero rank $(2,2)$ supercharge in 6d lifts to a square-zero rank $(1,1)$ supercharge in 8d and hence by Theorem 6.4 they are obtained by a dimensional reduction of a square-zero supercharge in 10d along $(\mathrm{id} \times \mathrm{Re})\colon \mathbb{R}^2 \times \mathbb{C}^4 \to \mathbb{R}^2 \times \mathbb{R}^4$. The statement therefore follows from Theorem 5.1. □

9 Dimension 5

The 5-dimensional $\mathcal{N} = k$ supersymmetry algebra has odd part $\Sigma \cong S \otimes W$, where S is the 4-dimensional spinor representation of $\mathrm{Spin}(5) \cong \mathrm{Sp}(4)$, and where W is a $2k$-dimensional symplectic vector space. The maximal supersymmetric gauge theory has $\mathcal{N} = 2$. However, there are $\mathcal{N} = 1$ super Yang-Mills theories for every choice U of symplectic representation of the gauge group.

9.1 $\mathcal{N} = 1$ Super Yang-Mills Theory with Matter

We begin by treating $\mathcal{N} = 1$ Super Yang-Mills theory with gauge group G, with a matter (half) hypermultiplet valued in a symplectic representation U. This is the theory obtained from 6d $\mathcal{N} = (1,0)$ super Yang-Mills theory with matter, after dimensional reduction to 5 dimensions.

Fields: We can describe the BRST fields of $\mathcal{N} = 1$ super Yang-Mills by restricting the 6d fields from Section **??** to representations of the group $\mathrm{O}(5)$. In addition to the ghost c, the fields we obtain are

- $\mathfrak{g}$-valued bosons: a gauge field $A \in \Omega^1(\mathbb{R}^5; \mathfrak{g})$, and a scalar $\phi' \in \Omega^0(\mathbb{R}^5; \mathfrak{g})$.
- U-valued bosons: a (symplectic) scalar field $\phi \in \Omega^0(\mathbb{R}^5; W \otimes U)$.
- $\mathfrak{g}$-valued fermions: a spinor $\lambda \in \Omega^0(\mathbb{R}^5; S \otimes W \otimes \mathfrak{g})$.
- U-valued fermions: a spinor $\psi \in \Omega^0(\mathbb{R}^5; S \otimes U)$.

Twisting data: There is a unique class of square-zero supercharges in the 5d $\mathcal{N} = 1$ supersymmetry algebra. Twists by such supercharges are minimal: they are invariant in three directions. The stabilizer of a minimal supercharge is isomorphic to $\mathrm{SU}(2) \subseteq \mathrm{Spin}(5)$.

Theorem 9.1. The minimal twist of 5d $\mathcal{N} = 1$ super Yang-Mills theory with gauge group G and symplectic matter representation U, on the product of a Calabi-Yau surface S and an oriented 1-manifold L is equivalent to the generalized BF theory coupled to a higher holomorphic symplectic U-valued boson, with space of fields $\mathrm{Map}(S \times L_{\mathrm{dR}}, U/\!/G)$.

Proof. This follows from the calculation of the minimal twist in 6-dimensional $\mathcal{N} = (1,0)$ theory in Theorem 8.1 above, by applying Proposition 1.39 to dimensionally reduce along one of the non-invariant Calabi-Yau directions. □

9.2 $\mathcal{N} = 2$ Super Yang-Mills Theory

Fields: We can describe the BRST fields of $\mathcal{N} = 2$ super Yang-Mills by restricting the fields of maximal super Yang-Mills from 10d, or alternatively by restricting the 7d $\mathcal{N} = 1$ fields from Section 7, to representations of the group $\mathrm{O}(5)$. In addition to the ghost c, the fields we obtain are

- Bosons: A gauge field $A \in \Omega^1(\mathbb{R}^5; \mathfrak{g})$, and five scalars $\phi_1, \ldots, \phi_5 \in \Omega^1(\mathbb{R}^5; \mathfrak{g})$.
- Fermions: A pair of spinors λ_1 and λ_2 in $\Omega^0(\mathbb{R}^5; S \otimes W \otimes \mathfrak{g})$.

We could alternatively have obtained these fields by considering the $\mathcal{N} = 1$ 5d super Yang-Mills theory, with matter valued in the symplectic representation $T^*\mathfrak{g}$.

Twisting data: Exactly as in the case of the 6d $\mathcal{N} = (1,1)$ supersymmetry algebra, there are 4 inequivalent classes of twists (see [**ElliottSafronov**]). These supercharges all occur as square-zero supercharges in the 6-dimensional $\mathcal{N} = (1,1)$ supersymmetry algebra from the previous section.

1. Rank 1 spinors, $Q \otimes w$. These automatically square to zero, and admit three invariant directions. Twists by such supercharges were already described in the previous section.
2. Rank 2 square-zero spinors. Like in dimension 6, these split into two inequivalent classes:
   (a) The open locus, consisting of topological supercharges.
   (b) There is a special locus consisting of supercharges with 4 invariant directions. The former class can be realised as a degeneration of this case.
3. Rank 4 square-zero spinors. These supercharges are also topological. The corresponding twists are only $\mathbb{Z}/2\mathbb{Z}$-graded.

Theorem 9.2. The twist of 5d $\mathcal{N} = 2$ super Yang-Mills theory by a rank 2 supercharge with 4 invariant directions on the product of a Calabi-Yau curve C and an oriented 3-manifold M_3 is equivalent to mixed BF theory on the product $C \times (M_3)_{\mathrm{dR}}$. This twist degenerates to a one-parameter family of topological twists of the form $\mathrm{Map}(C \times (M_3)_{\mathrm{dR}}, B\mathfrak{g})_{\mathrm{Hod}}$.

Proof. As we have done in higher dimensions, we start with with the 1-parameter family of twisted theories from Theorem **??**, and dimensionally reduce along one of the Calabi-Yau directions by applying Proposition 1.39 to the family of twists in 6-dimensions. □

Let us describe the partition function of the A-type topological twist appearing at generic points of this family.

Theorem 9.3. (Chris: Something about the Haydys-Witten equations...)

Proof. □

Theorem 9.4. The twist by a rank 4 supercharge of 5d $\mathcal{N}=2$ super Yang-Mills theory on an oriented 5-manifold M_5 is equivalent to 5d Chern-Simons theory on M_5: the $\mathbb{Z}/2\mathbb{Z}$-graded theory with space of fields $\mathrm{Map}((M_5)_{\mathrm{dR}}, B\mathfrak{g})$.

Proof. We now take the twist of 6d $\mathcal{N}=(1,1)$ super Yang-Mills with 5 invariant directions, as calculated in Theorem **??**, and dimensionally reduce in the unique non-invariant direction. That is, we apply Proposition 1.39 to the flat curve C to obtain the required topological theory. □

10 Dimension 4

10.1 $\mathcal{N}=1$ with chiral multiplet matter

We consider $\mathcal{N}=1$ supersymmetric Yang-Mills for a Lie algebra $\mathfrak{g}$ coupled to supersymmetric matter. The matter consists of the $\mathcal{N}=1$ chiral multiplet valued in a complex representation R.

The field content for the 4-dimensional $\mathcal{N}=1$ super Yang-Mills theory coupled to chiral matter is given by:

- $\mathfrak{g}$-valued bosons: a gauge field $A \in \Omega^1(\mathbb{R}^4;\mathfrak{g})$.
- R-valued bosons: $(\overline{\phi},\phi) \in C^\infty(\mathbb{R}^4; R \oplus R^*)$.
- $\mathfrak{g}$-valued fermions: $(\lambda_+,\lambda_-) \in \Omega^0(\mathbb{R}^4; S_+ \otimes \mathfrak{g} \oplus S_- \otimes \mathfrak{g})$.
- R-valued fermions: $(\psi_-,\psi_+) \in C^\infty(\mathbb{R}^4; S_+ \otimes R \oplus \otimes S_- \otimes R^*)$.

The R-symmetry group is $U(1)$ which acts by ± 1 on $\lambda_\pm$ and ∓ 1 on $\psi_\pm$.

10.1.1 Holomorphic twist

There is a single non-zero equivalence class of square-zero supercharges in 4-dimensional $\mathcal{N}=1$, which we fix from here on. Such supercharges are necessarily of rank 1, and for concreteness we will fix a square-zero supercharge of negative chirality

$$Q = Q_+ \in S_+$$

to twist by. Such supercharges are minimal (i.e. have two dimensional image). The twisted theory by such a supercharge can always be made $\mathbb{Z}$-graded where the twisting datum is given by the identity map $\alpha = \mathrm{id} : U(1) \to U(1)$. Such a Q is preserved under a twisting homomorphism

$$\mathrm{MU}(2) \to \mathrm{Spin}(4) \times U(1)$$

where on the first component is the map $\mathrm{MU}(2) \to \mathrm{Spin}(4)$ lifting $\mathrm{U}(2) \to \mathrm{SO}(4)$ and on the second component $\mathrm{MU}(2) \to U(1)$ is the square root of determinant.

We first decompose the fields of the 4-dimensional $\mathcal{N}=1$ theory with respect to MU(2). The fields decompose as:

			Untwisted	R	Twisted
c	$\mapsto$	$A_0 \in \Omega^0(\mathbb{C}^2;\mathfrak{g})$	-1	0	-1
A	$\mapsto$	$A_{0,1}+A_{1,0} \in \Omega^{0,1}(\mathbb{C}^2;\mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C}^2;\mathfrak{g})$	0	0	0
λ_-	$\mapsto$	$\lambda_{1,0} \in \Omega^{1,0}(\mathbb{C}^2;\mathfrak{g})$	0	-1	-1
λ_+	$\mapsto$	$\lambda_0 + A_{0,2} \in \Omega^0(\mathbb{C}^2;\mathfrak{g}) \oplus \Omega^{0,2}(\mathbb{C}^2;\mathfrak{g})$	0	$+1$	$+1$
ϕ	$\mapsto$	$\phi \in \Omega^0(\mathbb{C}^2;R^*)$	0	0	0
$\overline{\phi}$	$\mapsto$	$\gamma_0 \in \Omega^{0,0}(\mathbb{C}^2;R)$	0	0	0
ψ_+	$\mapsto$	$\psi_0 + \beta_{2,0} \in \Omega^0(\mathbb{C}^2;R^*) \oplus \Omega^{2,0}(\mathbb{C}^2;R^*)$	0	-1	-1
ψ_-	$\mapsto$	$\gamma_{0,1} \in \Omega^{0,1}(\mathbb{C}^2;R)$	0	$+1$	$+1$

(Brian: Is it necessary to put in the antifields?) and their antifields:

$$
\begin{aligned}
c^* &\mapsto B_{2,2} \in \Omega^{2,2}(\mathbb{C}^2;\mathfrak{g})\\
A^* &\mapsto B_{2,1} + A^*_{1,0} \in \Omega^{2,1}(\mathbb{C}^2;\mathfrak{g}) \oplus \Omega^{1,2}(\mathbb{C}^2;\mathfrak{g})\\
\lambda_-^* &\mapsto \lambda^*_{1,0} \in \Omega^{1,2}(\mathbb{C}^2;\mathfrak{g})\\
\lambda_+^* &\mapsto \lambda^*_0 + B_{2,0} \in \Omega^{0}(\mathbb{C}^2;\mathfrak{g}) \oplus \Omega^{2,0}(\mathbb{C}^2;\mathfrak{g})\\
\phi^* &\mapsto \phi^* \in \Omega^{2,2}(\mathbb{C}^2;R^*)\\
\overline{\phi}^* &\mapsto \beta_{2,2} \in \Omega^{2,2}(\mathbb{C}^2;R)\\
\psi_+^* &\mapsto \psi^*_{2,2} + \gamma_{0,2} \in \Omega^{2,2}(\mathbb{C}^2;R) \oplus \Omega^{0,2}(\mathbb{C}^2;R)\\
\psi_-^* &\mapsto \beta_{2,1} \in \Omega^{2,1}(\mathbb{C}^2;R)
\end{aligned}
$$

The untwisted action functionals decompose as follows.

$$
\begin{aligned}
S_{\text{gauge}} &= \int \left(-(F^{2,0},F^{0,2}) - \frac{1}{4}(\Lambda F_{1,1})^2\right) + \frac{1}{2}\left(\lambda_{1,0}\wedge \partial_{A_{1,0}} A_{0,2}) + \omega(\lambda_{1,0}\wedge \overline{\partial}_{A_{0,1}}\lambda_0)\right)\\
S_{\text{matter}} &= \int \Big(((\partial_{A_{1,0}}\phi, \overline{\partial}_{A_{0,1}}\gamma_0) + (\partial_{A_{1,0}}\gamma_0, \overline{\partial}_{A_{0,1}}\phi)) + \omega(\psi_0 \partial_{A_{1,0}}\gamma_{0,1}) + \beta_{2,0}\overline{\partial}_{A_{0,1}}\gamma_{0,1} +\\
&\qquad + 2(\omega\wedge([\lambda_{1,0},\gamma_{0,1}],\gamma_0) + ([A_{0,2},\beta_{2,0}],\phi))\Big)\\
S_{\text{anti}} &= \int \partial_{A_{1,0}} A_0 \wedge A^*_{1,0} + \overline{\partial}_{A_{0,1}} A_0 \wedge B_{2,1} + [\lambda_{1,0}, A_0]\wedge \lambda^*_{1,0} + [\lambda_0, A_0]\wedge \lambda_0^* + B_{2,1}\wedge [A_0, A_{0,1}] + A^*_{1,0}\wedge [A_0, A_{1,0}]\\
&\qquad + \frac{1}{2}[A_0,A_0]A_0^* + [\phi, A_0]\phi^* + [\gamma_0, A_0]\beta_{2,2} + [\gamma_{0,1}, A_0]\wedge \beta_{2,1} + [\psi_0, A_0]\psi_0^* + [\beta_{2,0}, A_0]\wedge \gamma_{0,2}\\
I^{(1)}_{\text{gauge}} &= \int \left(-(\lambda_{1,0}, A^*_{0,1}) + \frac{1}{2}(F_{0,2}, B_{2,0}) + \frac{1}{2}(F_{1,1},\lambda_0^*)\right)\\
I^{(1)}_{\text{matter}} &= \int \left((\phi,\psi_0^*) + \frac{1}{2}(\overline{\partial}_{A_{0,1}}\gamma_0, \beta_{2,1})\right)\\
I^{(2)}_{\text{gauge}} &= -\frac{1}{4}\int (\lambda_0^*)^2.
\end{aligned}
$$

Here, in terms of the Kähler form ω, the volume form is identified as $\mathrm{dvol} = \frac{\omega^2}{2}$.

Theorem 10.1 (See also **[SWchar]**)**.** The minimal twist of 4d $\mathcal{N}=1$ super Yang-Mills with Lie algebra $\mathfrak{g}$ coupled to the $\mathcal{N}=1$ chiral multiplet valued in a representation R is MU(2)-equivariantly equivalent to holomorphic BF theory for the Lie algebra $\mathfrak{g}$ coupled to the holomorphic $\beta\gamma$ system with values in the representation R, with classical moduli space $T^*[-1]\mathrm{Map}(\mathbb{C}^2, R/\mathfrak{g})$.

Proof. The proof of this theorem is very similar to the proof of Theorem 4.1. First, we integrate out the fields λ_0 and λ_0^* using Proposition 1.6. We then observe that the action includes the terms $\int(\lambda_{1,0}, A^*_{0,1})$ and $\int(\phi, \psi^*_{1,1})$. Thus, the two pairs $(\lambda_{1,0}, A_{1,0})$ and (ϕ, ψ_0) form BRST doublets, which can be integrated out using Proposition 1.8.

The twisted theory is therefore perturbatively equivalent to the theory with BV action

$$S_{\mathrm{BV}} = \int \Big((B_{2,1}\bar\partial_{A_{0,1}} A_0) + B_{2,0}F_{0,2} + \beta_{2,0}\bar\partial_{A_{0,1}}\gamma_{0,1} + \beta_{2,1}\bar\partial_{A_{0,1}}\gamma_0 \\ + [A_{0,2}, A_0] \wedge B_{2,0} + \frac{1}{2}[A_0, A_0]B_{2,2} + [\gamma_0, A_0] \wedge \beta_{2,2} + [\gamma_{0,1}, A_0] \wedge \beta_{2,1} \Big).$$

This is indeed the action functional of the required theory, where $A_{0,\bullet}, B_{2,\bullet}$ comprise the fields of holomorphic BF theory and $\gamma_{0,\bullet}, \beta_{2,\bullet}$ comprise the fields of the $\beta\gamma$ system. □

10.2 $\mathcal{N} = 2$ Super Yang-Mills with Matter

We obtain $\mathcal{N} = 2$ super Yang-Mills theory on $\mathbb{R}^4$ by dimensionally reducing $\mathcal{N} = (1, 0)$ super Yang-Mills from $\mathbb{R}^6$, or $\mathcal{N} = 1$ super Yang-Mills from $\mathbb{R}^5$, with a hypermultiplet valued in a symplectic representation U. In these terms we can describe the BRST fields.

Fields: We can describe the BRST fields of $\mathcal{N} = 2$ super Yang-Mills with matter valued in the symplectic representation U by restricting the 5d $\mathcal{N} = 2$ fields from Section 9.1 to representations of the group $\mathrm{O}(4)$. In addition to the ghost c, the fields we obtain are

- $\mathfrak{g}$-valued Bosons: a gauge field $A \in \Omega^1(\mathbb{R}^4; \mathfrak{g})$, and a pair of scalar fields $(\phi_1, \phi_2) \in \Omega^1(\mathbb{R}^4; \mathfrak{g} \otimes \wedge^2(W))^2$.
- U-valued bosons: a W-valued scalar field $\phi \otimes v \in \Omega^0(\mathbb{R}^4; W \otimes U)$.
- $\mathfrak{g}$-valued fermions: a W-valued Dirac spinor $\lambda = (\lambda_+ \otimes u_+, \lambda_- \otimes u_-) \in \Omega^0(\mathbb{R}^4; (S_+ \otimes W \oplus S_- \otimes W^*) \otimes \mathfrak{g})$.
- U-valued fermions: a Dirac spinor $\psi = (\psi_+, \psi_-) \in \Omega^0(\mathbb{R}^4; (S_+ \oplus S_-) \otimes U)$.

Supersymmetry action: It will be enough for our purposes to describe only the action of a chiral supercharge.

Proposition 10.1. After reduction to $\mathrm{O}(4)$, the 6d $\mathcal{N} = (1, 0)$ interaction terms $I^{(1)}$ and $I^{(2)}$ become

$$I^{(1)}_{\mathrm{gauge}}(Q) = \int \mathrm{dvol} \Big(-((\Gamma(Q_+, \lambda_-)(w_+, u_-), A^*) + \langle Q_+, \lambda_+ \rangle w_+ \wedge u_+, \phi_2^*) + \\ + \frac{1}{2}((\rho(F_A)Q_+, \lambda_+^*)(u_+^*, w_+) + (\rho(\mathrm{d}_A\phi_2(w_+ \wedge u_-^*))Q_+, \lambda_-^*) \Big)$$

$$I^{(1)}_{\mathrm{matter}}(Q) = \int \mathrm{dvol}(((Q_+, \psi_+), \phi^*)(v^*, w_+) + \frac{1}{2}(\rho(\mathrm{d}_A\phi)Q_+, \psi_+^*)(v, w_+))$$

$$I^{(2)}_{\mathrm{gauge}}(Q, Q) = \int \mathrm{dvol}\,(\frac{1}{4}(Q_+, Q_+)(\lambda_-, \lambda_-)(u_-, w_+)^2 - \frac{1}{2}(Q_+, \lambda_-^*)^2(w_+ \wedge u_-^*)^2 - (Q_+, Q_+)\phi_1(w_+ \wedge w_+)c^*)$$

$$I^{(2)}_{\mathrm{matter}}(Q, Q) = \int \frac{1}{4}\mathrm{dvol}(Q_+, Q_+)(\psi_-^*, \psi_-^*)(w_-^* \wedge w_-^*)$$

if $Q = Q_+ \otimes w_+$ is a non-zero element of $S_+ \otimes W$. (Chris: missing thing here: trivialising those wedge pairs, or equivalently restricting $\Gamma(-, -)$ from 6d to 4d.)

Twisting data: There are three classes of square-zero supercharge in the 3d $\mathcal{N} = 2$ supersymmetry algebra, distinguished by the ranks of the two summands $(Q_+, Q_-) \in S_+ \otimes W \oplus S_- \otimes W^*$.

- Rank $(1,0)$ and $(0,1)$ supercharges automatically square to zero. The corresponding twists are holomorphic, and coincide with the 4d $\mathcal{N} = 1$ twists discussed above.
- Rank $(2,0)$ and $(0,2)$ supercharges also automatically square to zero. The corresponding twists are topological (the *Donaldson twist*).
- Rank $(1,1)$ square-zero supercharges have three invariant directions.

We will identify the latter two twists as further deformations of the minimal twist, which we can describe similarly to what we saw for $\mathcal{N} = 1$.

Theorem 10.2. The minimal twist of 4d $\mathcal{N} = 2$ super Yang-Mills theory with gauge group G and symplectic matter representation U on a Calabi-Yau surface S is perturbatively equivalent to the generalized BF theory coupled to a higher holomorphic symplectic U-valued boson, with space of fields $\mathrm{Map}(S, T^*[1](U /\!/ \mathfrak{g}))$.

Proof. To show this, we start with the 5d $\mathcal{N} = 1$ minimal twist from Theorem 9.1 and dimensionally reduce in the de Rham direction L_{dR}. That is, we apply Proposition 1.38 to the theorem obtained in Theorem 9.1. □

We'll first consider the deformation of the holomorphic twist to the topological twist corresponding to a rank $(2,0)$ supercharge.

Theorem 10.3. The deformation of the holomorphic twist to the Donaldson twist of 4d $\mathcal{N} = 2$ super Yang-Mills theory with gauge group G and symplectic matter representation U is perturbatively equivalent, as a 1-parameter family of theories, to the Hodge family $\mathrm{Map}(S, (U /\!/ \mathfrak{g})_{\mathrm{Hod}})$.

Proof. Suppose the family of generically rank $(2,0)$ supercharge splits as a sum of two rank 1 supercharges as $Q_{\mathrm{hol}} + tQ$, where $t \in \mathbb{C}$. It is enough to understand the deformation of the holomorphically twisted action functional, which we wrote down in the proof of Theorem 10.1, by $I^{(1)}(tQ) + I^{(2)}(tQ, tQ)$ using Proposition 10.1.

(Chris: that term should correspond to an isomorphism between the two copies of $\mathfrak{g} \oplus \mathfrak{g}[-1] \oplus U[-1]$ in the BV complex. Fields that survive will be $c, A_{0,1}, \phi_1, \phi_2, \psi_+$, then a single scalar from ϕ, a single $(0,1)$-form from λ_- and a single scalar from λ_+, along with anti-fields. The terms that will contribute to the deformation of the differential will be an $A^*\lambda_-$ and a $\phi_2^*\lambda_+$ term from $I^{(1)}_{\mathrm{gauge}}$, a $\phi^*\psi_+$ term from $I^{(1)}_{\mathrm{matter}}$, a $c^*\phi_1$ term from $I^{(2)}_{\mathrm{gauge}}$, and a $\psi_-^*\psi_-^*$ term from $I^{(2)}_{\mathrm{matter}}$.) □

We now address the rank $(1,1)$ twist. This is more straightforward: we can understand it by dimensional reduction from the 5d holomorphic twist, this time in a non-invariant direction.

Theorem 10.4. The rank $(1,1)$ twist of 4d $\mathcal{N} = 2$ super Yang-Mills theory with gauge group G and symplectic matter representation U on a product $C_1 \times C_2$ of two Riemann surfaces, is perturbatively equivalent to the generalized BF theory coupled to a U-valued higher holomorphic boson with space of fields $\mathrm{Map}(C_1 \times C_{2\mathrm{dR}}, U /\!/ \mathfrak{g})$. This theory is generally only $\mathbb{Z}/2\mathbb{Z}$-graded.

Proof. We start with the 5d $\mathcal{N} = 1$ holomorphic twist, as in Theorem 9.1, and dimensionally reduce in one of the two non-invariant directions. That is, we apply Proposition 1.39. □

10.3 $\mathcal{N} = 4$ Super Yang-Mills

Fields: We can describe the BRST fields of $\mathcal{N} = 4$ super Yang-Mills by restricting the 5d $\mathcal{N} = 2$ fields from Section 9.2 to representations of the group $O(4)$. In addition to the ghost c, the fields we obtain are

- Bosons: a gauge field $A \in \Omega^1(\mathbb{R}^4; \mathfrak{g})$, and six scalars, which we'll denote by $\phi \in \Omega^1(\mathbb{R}^4; \mathfrak{g} \otimes \wedge^2(W))$.
- Fermions: four spinors, which we'll denote by $(\lambda_+ \otimes u_+, \lambda_- \otimes u_u) \in \Omega^0(\mathbb{R}^5; (S_+ \otimes W \oplus S_- \otimes W^*) \otimes \mathfrak{g})$.

Twisting data: The classification of orbits of square-zero supercharges in the 4d $\mathcal{N} = 4$ supersymmetry algebra is the most interesting among the examples we consider in this paper. We have the following classes. (Chris: some comments claiming we'll discuss things above, to be added. Also this is to revisit after writing the below.)

- Rank $(1,0)$ (and $(0,1)$) supercharges. Such supercharges automatically square to zero, and are holomorphic: indeed they sit inside an $\mathcal{N} = 1$ subalgebra. They are compatible with $SU(2)_- \subseteq Spin(4; \mathbb{R})$, which can be promoted to a twisting homomorphism from the group $U(2)$ as we say in the 4d $\mathcal{N} = 1$ section above.
- Rank $(2,0)$ (and $(0,2)$) supercharges. Such supercharges automatically square to zero, and are topological: indeed they sit inside an $\mathcal{N} = 2$ subalgebra. They are compatible with any of three possible twisting homomorphisms:
  1. The half twisting homomorphism $\phi_{1/2}\colon SU(2)_+ \oplus SU(2)_- \to SL(4; \mathbb{C})$ given by $(A, B) \mapsto \mathrm{diag}(A, 1, 1)$.
  2. The Kapustin-Witten twisting homomorphism $\phi_{KW}\colon SU(2)_+ \oplus SU(2)_- \to SL(4; \mathbb{C})$ given by $(A, B) \mapsto \mathrm{diag}(A, B)$.
  3. The Vafa-Witten twisting homomorphism $\phi_{VW}\colon SU(2)_+ \oplus SU(2)_- \to SL(4; \mathbb{C})$ given by $(A, B) \mapsto \mathrm{diag}(A, A)$.
- Rank $(1,1)$ supercharges. Such supercharges have three invariant directions, and also sit inside an $\mathcal{N} = 2$ subalgebra. Twists by such supercharges are compatible with a twisting homomorphism from $Spin(2; \mathbb{R}) \times Spin(2; \mathbb{R})$, but are only $\mathbb{Z}/2\mathbb{Z}$-graded, as we discussed in the $\mathcal{N} = 2$ section above.
- Rank $(2,1)$ (and $(1,2)$) supercharges. Such supercharges are topological, and compatible with a twisting homomorphism from $Spin(2; \mathbb{R}) \times Spin(2; \mathbb{R})$.
- Rank $(2,2)$ square-zero supercharges are topological, and split into two classes. Generically, such supercharges correspond to perturbatively trivial twisted theories, and are only compatible with $Spin(2; \mathbb{R}) \times Spin(2; \mathbb{R})$. There is a special point of this type which is compatible with Kapustin-Witten twisting homomorphism.
- There is also a special rank $(2,2)$ point (the *B-twisting supercharge*) arising by dimensional reduction from the 5d $\mathcal{N} = 2$ rank 4 supercharge, which is also compatible with the Kapustin-Witten twisting homomorphism.

We can study all the possible twists using what we say in the $\mathcal{N} = 2$ subsection, as well as by dimensional reduction from 5d $\mathcal{N} = 2$ super Yang-Mills theory.

Theorem 10.5. (Chris: holomorphic deforming to rank $(2,0)$, proven above) $\mathrm{Map}(\mathbb{C}^2_{\mathrm{Dol}}, B\mathfrak{g}_{\mathrm{Hod}})$.

Theorem 10.6. (Chris: Rank $(1,1)$ deforming to rank $(2,1)$, the former is done above, the deformation maybe doesn't come from dimensional reduction?) $\mathrm{Map}(\mathbb{C}_{\mathrm{Dol}} \times \mathbb{R}^2_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$.

Theorem 10.7. (Chris: Special rank $(2,2)$ deforming to generic rank $(2,2)$, by dimensional reduction from 5d 2_A deforming to 2_B) $\mathrm{Map}(\mathbb{R}^4_{\mathrm{dR}}, B\mathfrak{g}_{\mathrm{Hod}})$.

(Chris: todo: $(1,0)$, $(2,0)$, $(1,1)$ as above, with $U = T^*\mathfrak{g}$. Then $(2,1)$ and generic $(2,2)$ arise as deformations of $(1,1)$ by dimensional reduction from 5d. Finally the special $(2,2)$ arises by dimensional reduction from the 5d B-twist.)

11 Dimension 3

The 3-dimensional $\mathcal{N} = k$ supersymmetry algebra has odd part $\Sigma \cong S \otimes W$, where S is the 2-dimensional spinor representation of $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, and where W is a k-dimensional vector space equipped with a bilinear pairing. The maximal supersymmetric gauge theory has $\mathcal{N} = 8$. However, there are $\mathcal{N} = 4$ super Yang-Mills theories for every choice U of symplectic representation of the gauge group, and $\mathcal{N} = 2$ super Yang-Mills theories for every choice R of arbitrary representation of the gauge group. In dimension 3, much like we saw in dimensions 5 and 7, all twisted theories can be obtained by dimensional reduction from theories one dimension higher.

11.1 $\mathcal{N} = 2$ Super Yang-Mills with Chiral Matter

We'll begin with the minimal super Yang-Mills theory that admits non-trivial twists (if $\mathcal{N} = 1$ there are no square-zero supercharges). So, fix a gauge group G, and a representation R. The 3d $\mathcal{N} = 2$ super Yang-Mills theory arises by dimensional reduction from $\mathcal{N} = 1$ super Yang-Mills theory on $\mathbb{R}^4$ with an R-valued chiral multiplet.

Fields: We can describe the BRST fields of $\mathcal{N} = 2$ super Yang-Mills by restricting the 4d fields from Section 10.1 to representations of the group $\mathrm{O}(3)$. In addition to the ghost c, the fields we obtain are

- $\mathfrak{g}$-valued bosons: $A \in \Omega^1(\mathbb{R}^3; \mathfrak{g})$, and a scalar $\phi \in \Omega^0(\mathbb{R}^3; \mathfrak{g})$.
- R-valued bosons: $(\overline{\phi}, \phi) \in \Omega^0(\mathbb{R}^3; R \oplus R^*)$.
- $\mathfrak{g}$-valued fermions: $\lambda \otimes u \in \Omega^0(\mathbb{R}^3; S \otimes \mathfrak{g} \otimes W)$.
- R-valued fermions: $(\psi_-, \psi_+) \in \Omega^0(\mathbb{R}^3; S \otimes R \oplus \otimes S \otimes R^*)$.

Twisting data: In the $\mathcal{N} = 2$ supersymmetry algebra there is a unique non-trivial orbit of square-zero supercharges, with rank one. (Chris: todo, say something about $\mathbb{Z}$-gradings and twisting homomorphisms.)

Theorem 11.1. The minimal twist of 3d $\mathcal{N} = 2$ super Yang-Mills with Lie algebra $\mathfrak{g}$ coupled to the $\mathcal{N} = 2$ chiral multiplet valued in a representation R is perturbatively equivalent to mixed BF theory for the Lie algebra $\mathfrak{g}$ coupled to the mixed $\beta\gamma$ system with values in the representation R, with moduli space $T^*[-1]\mathrm{Map}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, R/\mathfrak{g})$.

Proof. This follows by dimensionally reducing the holomorphic twist of 4d $\mathcal{N} = 1$ super Yang-Mills. That is, we start with the result from Theorem 10.1, then apply Proposition 1.39 to turn one of the complex directions into a real de Rham direction. □

11.2 $\mathcal{N} = 4$ Super Yang-Mills with Hypermultiplet Matter

We will now consider 3d $\mathcal{N} = 4$ theories, which arise by dimensional reduction from 4d $\mathcal{N} = 2$ super Yang-Mills with gauge group G, coupled to a hypermultiplet valued in a symplectic representation U, or equivalently by dimensional reduction from 6d $\mathcal{N} = (1, 0)$ super Yang-Mills coupled to a hypermultiplet.

Fields: We can describe the BRST fields of $\mathcal{N} = 4$ super Yang-Mills by restricting the 4d fields from Section 10.2 to representations of the group $\mathrm{O}(3)$. In addition to the ghost c, the fields we obtain are

- $\mathfrak{g}$-valued bosons: $A \in \Omega^1(\mathbb{R}^3; \mathfrak{g})$, and three scalar fields $(\phi_1, \phi_2, \phi_3) \in \Omega^0(\mathbb{R}^3; \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g})$.
- U-valued bosons: a W-valued scalar field $(\phi \otimes v) \in \Omega^0(\mathbb{R}^3; W \otimes U)$.

- $\mathfrak{g}$-valued fermions: a pair of W-valued spinors $(\lambda_1 \otimes u_1, \lambda_2 \otimes u_2) \in \Omega^0(\mathbb{R}^3; S \otimes \mathfrak{g} \otimes (W \oplus W))$.
- U-valued fermions: a pair of spinors $(\psi_1, \psi_2) \in \Omega^0(\mathbb{R}^3; (S \oplus S) \otimes U)$.

Twisting data: In the $\mathcal{N} = 4$ supersymmetry algebra there are now three non-trivial orbits of square-zero supercharges. In addition to rank one square-zero supercharges, which sit inside an $\mathcal{N} = 2$ subalgebra, there are two possible classes of rank two square-zero supercharges. The corresponding twisted theories are obtained by dimensionally reducing the 4d $\mathcal{N} = 2$ theory twisted by a rank $(1,1)$, or a rank $(2,0)$ square-zero supercharge. Let's first discuss the rank 1 twist, and its deformation to a perturbatively trivial rank 2 twist.

Theorem 11.1. The minimal twist of 3d $\mathcal{N} = 4$ super Yang-Mills theory with gauge group G and symplectic matter representation U is perturbatively equivalent to the generalized BF theory coupled to a higher holomorphic symplectic U-valued boson, with space of fields $\mathrm{Map}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, T^*[1](U /\!/ \mathfrak{g}))$. The deformation of this theory to the rank 2 A-twist corresponds to deforming this theory to the 1-parameter family $\mathrm{Map}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, (U /\!/ \mathfrak{g})_{\mathrm{Hod}})$.

Proof. To prove this, we start with the one-parameter family of 4d twisted theories which we computed Theorem 10.3, then apply Proposition 1.39 to reduce to a one-parameter family of 3d twisted theories. □

Theorem 11.2. The rank 2 B-twist of 3d $\mathcal{N} = 4$ super Yang-Mills theory with gauge group G and symplectic matter representation U is perturbatively equivalent to the generalized BF theory coupled to a U-valued higher holomorphic boson with space of fields $\mathrm{Map}(\mathbb{R}^3_{\mathrm{dR}}, U /\!/ \mathfrak{g})$. This theory is generally only $\mathbb{Z}/2\mathbb{Z}$-graded.

Proof. Again, we prove this by dimensional reduction from a 4d $\mathcal{N} = 2$ twisted theory, in this case the rank $(1,1)$ twist from Theorem 10.4, by applying Proposition 1.39. □

11.3 $\mathcal{N} = 8$ Super Yang-Mills Theory

Finally, let's consider maximal super Yang-Mills theory with gauge group G in dimension 3, with $\mathcal{N} = 8$ supersymmetry. This is the theory obtained by compactifying $\mathcal{N} = (1,0)$ super Yang-Mills on $\mathbb{R}^{10}$ down to $\mathbb{R}^3$.

Fields: We can describe the BRST fields of $\mathcal{N} = 4$ super Yang-Mills by restricting the 4d fields from Section 10.3 to representations of the group $\mathrm{O}(3)$. In addition to the ghost c, the fields we obtain are

- Bosons: $A \in \Omega^1(\mathbb{R}^3; \mathfrak{g})$, and seven scalar fields $\phi_i \in \Omega^0(\mathbb{R}^3; \mathfrak{g})$.
- Fermions: eight spinor fields $\lambda_i \in \Omega^0(\mathbb{R}^3; S \otimes \mathfrak{g})$.

Twisting data: The classes of square-zero supercharge in the 3d $\mathcal{N} = 8$ supersymmetry algebra are categorized in the same way as in the $\mathcal{N} = 4$ supersymmetry algebra. There are minimal twisting supercharges of rank 1, and two classes of topological twisting supercharges of rank 2, which we refer to as the A-twist (obtained from a rank $(2,0)$, $(2,1)$ or generic $(2,2)$ supercharge in $4d\mathcal{N} = 4$), and the B-twist (obtained from a rank $(1,1)$ or special rank $(2,2)$ supercharge in 4d $\mathcal{N} = 4$). The twisted theories can be calculated either by dimensional reduction – by applying Proposition 1.39 to the theories from Section 10.3, or from the 3d $\mathcal{N} = 4$ twists by considering the representation $U = T^*\mathfrak{g}$.

Theorem 11.3. The minimal twist of 3d $\mathcal{N} = 8$ super Yang-Mills theory with gauge group G is perturbatively equivalent to the BF theory with space of fields $T^*[-1]\mathrm{Map}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, (\mathfrak{g}/\mathfrak{g})_{\mathrm{Dol}})$. The deformation of this theory to the rank 2 A-twist corresponds to deforming this theory to the 1-parameter family $T^*[-1]\mathrm{Map}(\mathbb{C} \times \mathbb{R}_{\mathrm{dR}}, (\mathfrak{g}/\mathfrak{g})_{\mathrm{Hod}})$.

Theorem 11.4. The rank 2 B-twist of 3d $\mathcal{N} = 8$ super Yang-Mills theory with gauge group G is perturbatively equivalent to the BF theory with space of fields $T^*[-1]\mathrm{Map}(\mathbb{R}^3_{\mathrm{dR}}, \mathfrak{g}/\mathfrak{g})$.

12 Dimension 2

There are two classes of twisted supersymmetric gauge theory that we will consider in 2d. First we have theories with $(\mathcal{N},\mathcal{N})$ supersymmetry; these arise via dimensional reduction from supersymmetric gauge theories in higher dimensions. We additionally have gauge theories with chiral, i.e. $(0,\mathcal{N})$ supersymmetry, which we saw using the observations in Section 2.3. We'll address twists for these two classes of theory in turn.

12.1 $\mathcal{N}=(2,2)$ Super Yang-Mills with matter

First, let's consider the $\mathcal{N}=(2,2)$ super Yang-Mills theory, which arises via dimensional reduction from 3d $\mathcal{N}=2$ super Yang-Mills. This theory includes a gauge multiplet with gauge group G and a chiral multiplet valued in a representation R of G.

Fields: We can describe the BRST fields of $\mathcal{N}=(2,2)$ super Yang-Mills by restricting the fields in dimension 3 from Section 11.1, or equivalently the 3d fields from Section 10.1 to representations of the group $\mathrm{O}(2)$. In addition to the ghost c, the fields we obtain are

- $\mathfrak{g}$-valued bosons: $A\in\Omega^1(\mathbb{R}^2;\mathfrak{g})$, and a pair of scalars $(\phi_1,\phi_2)\in\Omega^0(\mathbb{R}^2;\mathfrak{g}\oplus\mathfrak{g})$.
- R-valued bosons: $(\overline{\phi},\phi)\in\Omega^0(\mathbb{R}^2;R\oplus R^*)$.
- $\mathfrak{g}$-valued fermions: $(\lambda_+\otimes u_+,\lambda_-\otimes u_-)\in\Omega^0(\mathbb{R}^2;S\otimes(W_+\oplus S_-\otimes W_-)\otimes\mathfrak{g})$.
- R-valued fermions: $(\psi_+^+,\psi_-^+,\psi_+^-,\psi_-^-)\in\Omega^0(\mathbb{R}^2;(S_+\oplus S_-)\otimes R\oplus\otimes(S_+\oplus S_-)\otimes R^*)$.

Supersymmetry action: (Chris: todo)

Twisting data: In the $\mathcal{N}=(2,2)$ supersymmetry algebra there is are three classes of non-trivial orbits of square-zero supercharges.

- Square-zero supercharges of rank $(1,0)$ or $(0,1)$, which are holomorphic
- Square-zero supecharges of rank $(1,1)$ are topological, and split into two classes: the B-supercharge arises by dimensional reduction from 3 dimensions, and the A-supercharge does not.

(Chris: todo, say something about $\mathbb{Z}$-gradings and twisting homomorphisms.)

The calculation of the twists here is similar to what we saw in 4d $\mathcal{N}=2$. The holomorphic twist and the B-twist arise by dimensional reduction from 3d, but the A-twist as a deformation of the holomorphic twist is not, and must be calculated directly.

Theorem 12.1. The minimal twist of 2d $\mathcal{N}=(2,2)$ super Yang-Mills theory with gauge group G coupled to the $\mathcal{N}=(2,2)$ chiral multiplet valued in a representation R is perturbatively equivalent to holomorphic BF theory for the Lie algebra $\mathfrak{g}$ coupled to the mixed $\beta\gamma$ system with values in the representation R, with moduli space $T^*[-1]\mathrm{Map}(\mathbb{C},(R/\mathfrak{g})_{\mathrm{Dol}})$.

Proof. This follows, as in 3d, by dimensionally reducing the holomorphic twist of 3d $\mathcal{N}=2$ super Yang-Mills. That is, we start with the result from Theorem 11.1, then apply Proposition 1.38 to dimensionally reduce in the real de Rham direction. ☐

Theorem 12.2. The topological A-twist of 2d $\mathcal{N} = (2,2)$ super Yang-Mills theory with gauge group G coupled to the $\mathcal{N} = (2,2)$ chiral multiplet valued in a representation R is perturbatively equivalent, as a one-parameter deformation of the holomorphic twist, to the Hodge family with moduli space $T^*[-1]\mathrm{Map}(\mathbb{C}, (R/\mathfrak{g})_{\mathrm{Hod}})$.

Proof. □

Finally, there is the 2d B-twist. This is much easier, since it arises directly from 3d via dimensional reduction in a non-invariant direction.

Theorem 12.3. The topological B-twist of 2d $\mathcal{N} = (2,2)$ super Yang-Mills theory with gauge group G coupled to the $\mathcal{N} = (2,2)$ chiral multiplet valued in a representation R is perturbatively equivalent to the topological BF theory with moduli space $T^*[-1]\mathrm{Map}(\mathbb{R}^2, (R/\mathfrak{g}))$.

Proof. We again dimensionally reduce the holomorphic twist of 3d $\mathcal{N} = 2$ super Yang-Mills, but this time in one of the non-invariant directions. That is, we start with the result from Theorem 11.1, then apply Proposition 1.39 to turn the complex direction into a second real de Rham direction. □

12.2 $\mathcal{N} = (4,4)$ and $\mathcal{N} = (8,8)$ Super Yang-Mills with matter

We can likewise consider $\mathcal{N} = (4,4)$ super Yang-Mills theory with gauge group G, with a hypermultiplet valued in a symplectic representation U, by dimensionally reducing 3d $\mathcal{N} = 4$ super Yang-Mills, or we can consider $\mathcal{N} = (8,8)$ super Yang-Mills with gauge group G, by dimensionally reducing 3d $\mathcal{N} = 8$ super Yang-Mills. We will address these two theories together, since in neither case are there any new classes of twist which were not visible already in the $\mathcal{N} = (2,2)$ section above.

The following equivalences follow from the results in the previous section by substituting $R = U \oplus \mathfrak{g}$.

Theorem 12.4. The holomorphic twist of 2d $\mathcal{N} = (4,4)$ super Yang-Mills with gauge group G and symplectic matter representation U admits a one-parameter deformation to the A-twist. Together they are perturbatively equivalent to a holomorphic BF theory and its Hodge deformation, with moduli space given by $T^*[-1]\mathrm{Map}(\mathbb{C}, (U /\!/ \mathfrak{g})_{\mathrm{Hod}})$.

If $U = T^*\mathfrak{g}$ then the $\mathcal{N} = (4,4)$ supersymmetry is promoted to $\mathcal{N} = (8,8)$, and the twisted moduli space can be identified with $T^*[-1]\mathrm{Map}(\mathbb{C}, (T^*[2](\mathfrak{g}/\mathfrak{g}))_{\mathrm{Hod}})$.

Theorem 12.5. The B-twist of 2d $\mathcal{N} = (4,4)$ super Yang-Mills with gauge group G and symplectic matter representation U is perturbatively equivalent to the topological BF theory with moduli space $T^*[-1]\mathrm{Map}(\mathbb{R}^2_{\mathrm{dR}}, U /\!/ \mathfrak{g})$. If $U = T^*\mathfrak{g}$ then the $\mathcal{N} = (4,4)$ supersymmetry is promoted to $\mathcal{N} = (8,8)$, and the twisted moduli space can be identified with $T^*[-1]\mathrm{Map}(\mathbb{R}^2_{\mathrm{dR}}, T^*[2](\mathfrak{g}/\mathfrak{g}))$.

12.3 $\mathcal{N} = (0,2)$ Super Yang-Mills with matter

We consider $\mathcal{N} = (0,2)$ super Yang-Mills for a Lie algebra $\mathfrak{g}$ coupled to $\mathcal{N} = (0,2)$ supersymmetric matter. The supersymmetric matter consists of the $\mathcal{N} = (0,2)$ chiral multiplet with values in a representation R.

(Brian: Should we also compute what happens to the $(0,2)$ Fermi multiplet?)

The untwisted fields are given by:

- $\mathfrak{g}$-valued bosons: a gauge field $A \in \Omega^1(\mathbb{R}^2; \mathfrak{g})$.
- $\mathfrak{g}$-valued fermions: a spinor $\lambda \in C^\infty(\mathbb{R}^2; S_- \otimes W_- \otimes \mathfrak{g})$.

- $R \oplus R^*$-valued scalars: $\phi \in C^\infty(\mathbb{R}^2; R)$ and $\overline{\phi} \in C^\infty(\mathbb{R}^2; R^*)$;
- a pair of $R \oplus R^*$-valued fermions: $\psi \in C^\infty(\mathbb{R}^2; S_+ \otimes R^*)$ and $\overline{\psi} \in C^\infty(\mathbb{R}^2; S_+ \otimes R)$.

Here, W is a complex two-dimensional vector space equipped with a symmetric pairing. Choose the basis $\{u, v\}$ for W_- with pairing given by $\langle u, v\rangle = 1$.

We will twist by an element $Q = q \otimes u \in S_- \otimes W_-$.

The fields decompose as

			Untwisted	R	Twisted
c	$\mapsto$	$A_0 \in \Omega^0(\mathbb{C}; \mathfrak{g})$	-1	0	-1
A	$\mapsto$	$A_{0,1} + A_{1,0} \in \Omega^{0,1}(\mathbb{C}; \mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C}; \mathfrak{g})$	$(0,0)$	$(0,0)$	$(0,0)$
λ	$\mapsto$	$\lambda_0 u + \lambda_{1,0} v \in \Omega^0(\mathbb{C}; \mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C}; \mathfrak{g})$	$(0,0)$	$(1,-1)$	$(1,-1)$
ϕ	$\mapsto$	$\gamma_0 \in \Omega^0(\mathbb{C}; R)$	0	0	0
$\overline{\phi}$	$\mapsto$	$\overline{\phi} \in \Omega^0(\mathbb{C}; R^*)$	0	0	0
ψ	$\mapsto$	$\psi_0 \in \Omega^0(\mathbb{C}; R^*)$	0	-1	-1
$\overline{\psi}$	$\mapsto$	$\gamma_{0,1} \in \Omega^{0,1}(\mathbb{C}; R)$	0	1	1

The anti-fields decompose as

			Untwisted	R	Twisted
c^*	$\mapsto$	$B_{1,1} \in \Omega^{1,1}(\mathbb{C}; \mathfrak{g})$	2	0	2
A^*	$\mapsto$	$A^*_{0,1} + B_{1,0} \in \Omega^{0,1}(\mathbb{C}; \mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C}; \mathfrak{g})$	$(1,1)$	$(0,0)$	$(1,1)$
λ^*	$\mapsto$	$\lambda^*_{1,1} v + \lambda^*_{0,1} u \in \Omega^0(\mathbb{C}; \mathfrak{g}) \oplus \Omega^{0,1}(\mathbb{C}; \mathfrak{g})$	$(1,1)$	$(-1,1)$	$(0,2)$
ϕ^*	$\mapsto$	$\beta_{1,1} \in \Omega^{1,1}(\mathbb{C}; R^*)$	1	0	1
$\overline{\phi}^*$	$\mapsto$	$\overline{\phi}^* \in \Omega^{1,1}(\mathbb{C}; R)$	1	0	1
ψ	$\mapsto$	$\psi^*_{1,1} \in \Omega^{1,1}(\mathbb{C}; R)$	1	1	2
$\overline{\psi}$	$\mapsto$	$\beta_{1,0} \in \Omega^{1,0}(\mathbb{C}; R^*)$	1	-1	0

The action functionals decompose as:

$$S_{\text{gauge}} = \int \left(-(F^{2,0}, F^{0,2}) - \frac{1}{4}(\Lambda F_{1,1})^2 \right) + \frac{1}{2}\left((\lambda_0, \overline{\partial}_{A_{0,1}} \lambda_{1,0}) + (\lambda_{1,0} \wedge \overline{\partial}_{A_{0,1}} \lambda_0) \right)$$

$$S_{\text{matter}} = \int \Big(((\partial_{A_{1,0}} \overline{\phi}, \overline{\partial}_{A_{0,1}} \gamma_0) + (\partial_{A_{1,0}} \gamma_0, \overline{\partial}_{A_{0,1}} \overline{\phi})) + (\psi_0, \partial_{A_{1,0}} \gamma_{0,1}) + (\gamma_{0,1} \wedge \partial_{A_{1,0}} \psi_0) + $$
$$+ (([\lambda_{1,0}, \gamma_{0,1}], \psi_0) + ([\lambda_{1,0}, \psi_0], \gamma_{0,1})) \Big)$$

$$S_{\text{anti}} = \int \partial_{A_{1,0}} A_0 \wedge A^*_{1,0} + \overline{\partial}_{A_{0,1}} A_0 \wedge B_{1,0} + [\lambda_{1,0}, A_0] \wedge \lambda^*_{0,1} + [\lambda_0, A_0] \wedge \lambda^*_{1,1} + B_{1,0} \wedge [A_0, A_{0,1}] + A^*_{0,1} \wedge [A_0, A_{1,0}]$$
$$+ \frac{1}{2}[A_0, A_0] A^*_0 + [\gamma_0, A_0]\beta_{1,1} + [\overline{\phi}, A_0]\overline{\phi}^* + [\gamma_{0,1}, A_0] \wedge \beta_{1,0} + [\psi_0, A_0]\psi^*_{1,1}$$

$$I^{(1)}_{\text{gauge}} = \int \left(-(\lambda_{1,0}, A^*_{0,1}) \right)$$

$$I^{(1)}_{\text{matter}} = \int \left((\psi^*_{1,1}, \overline{\phi}) + \frac{1}{2}(\beta_{1,0} \wedge \overline{\partial} \gamma_0) \right)$$

$$I^{(2)}_{\text{gauge}} = -\frac{1}{4} \int (\lambda^*_{1,1})^2.$$

Theorem 12.1 (See also **[SWchar]**)**.** The minimal twist of 2d $\mathcal{N} = (0,2)$ super Yang-Mills with Lie algebra $\mathfrak{g}$ coupled to the $\mathcal{N} = (0,2)$ chiral multiplet valued in a representation R is $U(1)$-equivariantly equivalent to holomorphic BF theory on $\mathbb{C}$ for the Lie algebra $\mathfrak{g}$ coupled to the holomorphic $\beta\gamma$ system on $\mathbb{C}$ with values in the representation R.

Proof. The proof of this theorem is very similar to the proof of Theorem 4.1. First, we integrate out the fields λ_0 and $\lambda^*_{1,1}$ using Proposition 1.6. We then observe that the action includes the terms $\int(\lambda_{1,0}, A^*_{1,0})$ and $\int(\phi, \psi^*_0)$. Thus, the two pairs $(\lambda_{1,0}, A_{1,0})$ and (ϕ, ψ_0) form BRST doublets, which can be integrated out using Proposition 1.8.

The twisted theory is therefore perturbatively equivalent to the theory with BV action

$$S_{\mathrm{BV}} = \int \Big((B_{1,0}\bar\partial_{A_{0,1}} A_0) + \beta_{1,0}\bar\partial_{A_{0,1}}\gamma_0 \\ + [A_{0,1}, A_0] \wedge B_{1,0} + \frac{1}{2}[A_0, A_0]B_{1,1} + [\gamma_0, A_0] \wedge \beta_{1,1} + [\gamma_{0,1}, A_0] \wedge \beta_{1,0} \Big).$$

This is indeed the action functional of the required theory, where $A_{0,\bullet}, B_{1,\bullet}$ comprise the fields of holomorphic BF theory and $\gamma_{0,\bullet}, \beta_{1,\bullet}$ comprise the fields of the $\beta\gamma$ system. □

12.4 $\mathcal{N} = (0, 4)$ Super Yang-Mills with matter

We consider $\mathcal{N} = (0, 4)$ super Yang-Mills for a Lie algebra $\mathfrak{g}$ coupled to $\mathcal{N} = (0, 4)$ supersymmetric matter. The supersymmetric matter consists of the $\mathcal{N} = (0, 4)$ hypermultiplet with values in a complex symplectic representation U. The spinorial representation is $\Sigma = S_- \otimes W_-$ where $W_- = \mathrm{span}_{\mathbb{C}}\{u_1, u_2, v_1, v_2\}$ is the four-dimensional auxiliary space equipped with the pairing $\langle u_i, v_j\rangle = \delta_{ij}$. Let $W = \mathrm{span}_{\mathbb{C}}\{u_1, u_2\}$.

The field content is:

- $\mathfrak{g}$-valued bosons: a gauge field $A \in \Omega^1(\mathbb{R}^2; \mathfrak{g})$.
- $\mathfrak{g}$-valued fermions: a spinor $\lambda \in C^\infty(\mathbb{R}^2; S_- \otimes W_- \otimes \mathfrak{g})$.
- $W \otimes U$-valued scalars: $\phi \in C^\infty(\mathbb{R}^2; W \otimes U)$;
- $W \otimes U$-valued fermions: $\psi \in C^\infty(\mathbb{R}^2; W \otimes S_+ \otimes U)$.

We twist by the nilpotent supercharge $Q = q \otimes u_1$ where $q \in S_-$ is any nonzero vector.

Theorem 12.2. The minimal twist of 2d $\mathcal{N} = (0, 4)$ super Yang-Mills with Lie algebra $\mathfrak{g}$ coupled to the $\mathcal{N} = (0, 4)$ hypermultiplet is $U(1)$-equivariantly equivalent to holomorphic BF theory on $\mathbb{C}$ for the Lie algebra $\mathfrak{g}$ coupled to the free fermion system on $\mathbb{C}$ with values in the representation $\mathfrak{g} \otimes W'_- \cong \mathfrak{g} \oplus \mathfrak{g}$ where $\mathfrak{g}$ acts by the adjoint action plus the free $\beta\gamma$ system on $\mathbb{C}$ valued in the $\mathfrak{g}$-representation U.

Proof. The supercharge Q lies in a $\mathcal{N} = (0, 2)$ subalgebra of the full $\mathcal{N} = (0, 4)$ algebra. On the odd part of the supersymmetry algebras, this embedding is induced by the algebra map

$$S_- \otimes \mathbb{C}[x]/x^2 \to S_- \otimes \mathrm{End}(W)$$

given by the identity on S_- and the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbb{C}[x]/x^2 \to \mathrm{End}(W)$. Under this embedding we see that the scalar supercharge Q is precisely the same as the scalar supercharge we used to twist in Section 12.3. Furthermore, under this embedding the $(0, 4)$ vector multiplet (A, λ) splits as a $(0, 2)$ vector multiplet plus a $(0, 2)$ Fermi multiplet. The $(0, 2)$ supersymmetry algebra acts trivially on the Fermi multiplet. The $(0, 4)$ hypermultiplet becomes a $(0, 2)$ chiral multiplet valued in the representation U. The claim now follows from Theorem 12.1. □

Remark 12.3. The matter sector in the untwisted theory depends on U as a complex symplectic $\mathfrak{g}$-representation. The matter sector in the twisted theory only depends on U as a complex $\mathfrak{g}$-representation, the dependence on the symplectic form disappears when we twist.

12.5 $\mathcal{N} = (0, \mathcal{N}_-)$ Super Yang-Mills

We consider pure $\mathcal{N} = (0, \mathcal{N}_-)$ super Yang-Mills for a Lie algebra $\mathfrak{g}$, where $\mathcal{N}_- \geq 2$. The spinorial representation is $\Sigma = S_- \otimes W_-$ where W_- is the $\mathcal{N}_-$-dimensional auxiliary equipped with a nondegenerate symmetric bilinear pairing.

- $\mathfrak{g}$-valued bosons: a gauge field $A \in \Omega^1(\mathbb{R}^2; \mathfrak{g})$.
- $\mathfrak{g}$-valued fermions: a spinor $\lambda \in C^\infty(\mathbb{R}^2; S_- \otimes W_- \otimes \mathfrak{g})$.

Choose a splitting $W_- = \mathrm{span}_{\mathbb{C}}\{u, v\} \oplus W'_-$ where $\dim(W'_-) = \mathcal{N}_- - 2$ such that the pairing restricted to $\mathrm{span}_{\mathbb{C}}\{u, v\}$ is $\langle u, v\rangle = 1$.

The field content is:

			Untwisted	R	Twisted
c	$\mapsto$	$A_0 \in \Omega^0(\mathbb{C}; \mathfrak{g})$	-1	0	-1
A	$\mapsto$	$A_{0,1} + A_{1,0} \in \Omega^{0,1}(\mathbb{C}; \mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C}; \mathfrak{g})$	$(0,0)$	$(0,0)$	$(0,0)$
λ	$\mapsto$	$\lambda_0 u + \lambda_{1,0} v + \chi \in \Omega^0(\mathbb{C}; \mathfrak{g}) \oplus \Omega^{1,0}(\mathbb{C}; \mathfrak{g}) \oplus C^\infty(\mathbb{R}^2; S_- \otimes W'_- \otimes \mathfrak{g})$	$(0,0,0)$	$(1,-1,0)$	$(1,-1,0)$

We twist by the nilpotent supercharge $Q = q \otimes u_1$ where $q \in S_-$ is any nonzero vector.

Theorem 12.4 (See also **[SWchar]**). The minimal twist of 2d $\mathcal{N} = (0, \mathcal{N}_-)$ super Yang-Mills with Lie algebra $\mathfrak{g}$ is $U(1)$-equivariantly equivalent to holomorphic BF theory on $\mathbb{C}$ for the Lie algebra $\mathfrak{g}$ coupled to a free fermion system on $\mathbb{C}$ with values in the representation $\mathfrak{g} \otimes W'_- \cong \mathfrak{g} \otimes \mathbb{C}^{\mathcal{N}_- - 2}$ where $\mathfrak{g}$ acts by the adjoint action.

Proof. The proof is similar to the proof of Theorem 12.2. The supercharge Q lies in a $\mathcal{N} = (0, 2)$ subalgebra of the full $\mathcal{N} = (0, \mathcal{N}_-)$ algebra. On the odd part of the supersymmetry algebras, this embedding is induced by the algebra map

$$S_- \otimes \mathbb{C}[x]/x^2 \to S_- \otimes \mathrm{End}(\mathbb{C}^2)$$

given by the identity on S_- and the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbb{C}[x]/x^2 \to \mathrm{End}(\mathbb{C}^2)$. Here, $\mathbb{C}^2$ is spanned by $\{u, v\}$ in the decomposition of W_- given above. Under this embedding we see that the scalar supercharge Q is precisely the same as the scalar supercharge we used to twist in Section 12.3. Furthermore, under this embedding the $(0, \mathcal{N}_-)$ vector multiplet (A, λ) splits as a $(0, 2)$ vector multiplet plus a $(0, 2)$ Fermi multiplet valued in $\mathfrak{g} \otimes W'_-$. The $(0, 2)$ supersymmetry algebra acts trivially on the Fermi multiplet. The claim now follows from a special case of Theorem 12.1 where we take $R = 0$. $\square$

A Spinors

In this paper we will extensively use the theory of spinors. Let V be a complex vector space equipped with a nondegenerate symmetric bilinear pairing. Recall that the Clifford algebra $\mathrm{Cl}(V)$ is defined to be the quotient of the tensor algebra on V by the relation

$$v_1 v_2 + v_2 v_1 = 2(v_1, v_2).$$

Consider a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford module $M = M^+ \oplus M^-$. Denote the Clifford action by $\rho(v) \in \mathrm{End}(M)$. We assume the Clifford module is equipped with a nondegenerate pairing $(-,-)\colon M^+ \otimes M^- \to \mathbb{C}$ such that

$$(\rho(v) Q_1, Q_2) = (Q_1, \rho(v) Q_2)$$

for any $Q_1, Q_2 \in M$ and $v \in V$. From now on we denote $M^+ = \Sigma$ and $M^- = \Sigma^*$. We define the Γ-pairings

$$\Gamma\colon \mathrm{Sym}^2(\Sigma) \longrightarrow V, \qquad \Gamma\colon \mathrm{Sym}^2(\Sigma^*) \longrightarrow V$$

by

$$(\rho(v)Q_1, Q_2) = (v, \Gamma(Q_1, Q_2)) \tag{22}$$

We have a subset $\mathrm{Spin}(V) \subset \mathrm{Cl}(V)$, so Σ and Σ^* are representations of the spin group. Moreover, the Clifford action and the maps Γ are $\mathrm{Spin}(V)$-equivariant.

We may identify $\wedge^2(V) \to \mathfrak{so}(V)$ via

$$\omega \mapsto (w \mapsto -2\iota_{(w,-)})\omega).$$

This gives rise to an action map

$$\wedge^2(V) \otimes \Sigma \longrightarrow \Sigma$$

of two-forms on spinors.

Consider the map $q\colon \wedge^\bullet(V) \to \mathrm{Cl}(V)$ given by antisymmetrization, so that, for instance,

$$q(v_1 \wedge v_2) = v_1 v_2 - (v_1, v_2). \tag{23}$$

The resulting action $\wedge^2(V) \otimes \Sigma \to \Sigma$ then coincides with the original action of $\mathfrak{so}(V)$ on the spinorial representation Σ, so that $\mathfrak{so}(V)$-equivariance of Γ gives the following.

Proposition A.1. For $X \in \wedge^2(V)$ and $Q_1, Q_2 \in \Sigma$ we have

$$\Gamma(Q_1, \rho(X)Q_2) + \Gamma(Q_2, \rho(X)Q_1) = -2\iota_{\Gamma(Q_1,Q_2)}X.$$

We may extend the discussion to the case of Riemannian manifolds N, where we replace V by the vector bundle TN. Given a bundle of Clifford modules $M = \Sigma \oplus \Sigma^*$ as before we have the associated Dirac operator

$$\not{d}\colon \Gamma(N, \Sigma) \to \Gamma(N, \Sigma^*).$$

From (23) we get the following property.

Proposition A.2. Suppose $Q_1, Q_2 \in \Sigma$ and $\lambda \in \Gamma(N, \Sigma)$. Then

$$\not{d}\rho(\Gamma(Q_1, \lambda))Q_2 = \rho(d\Gamma(Q_1, \lambda))Q_2 + (Q_1, \not{d}\lambda)Q_2.$$

Finally, we have the following important compatibility between the Clifford action of differential forms and the Dirac operator proved in **[Snygg]**.

Proposition A.3. Suppose $Q \in \Sigma$ and $X \in \Omega^p(N)$. Then

$$\not{d}(\rho(X)Q) = \rho(\mathrm{d}X)Q + (-1)^{n(1+p)}\rho(*\mathrm{d}*X)Q.$$

Note that both Proposition A.2 and Proposition A.3 extend to the case when λ and X respectively are twisted by a vector bundle and $\not{d}$ is the corresponding twisted Dirac operator.