BCOV THEORY ON A COMPLEX SURFACE

Suppose X is a complex manifold of complex dimension $\dim_{\mathbb{C}}(X) = 2$ equipped with a holomorphic symplectic form $\omega \in \Omega^{2,hol}(X)$. We study BCOV theory on X, which should be thought of as the theory of "divergence-free holomorphic polyvector fields" with respect to the volume form determined by the fixed symplectic structure.

Let $\mathrm{PV}^{k,hol}(X)$ be the space of holomorphic k-polyvector fields; these are holomorphic sections of the kth exterior power of the holomorphic tangent bundle $\wedge^k T^{1,0}X$. The holomorphic volume form identifies

$$\mathrm{PV}^{0,hol}(X) \cong_\omega \Omega^{2,hol}(X) \;\; , \;\; \mathrm{PV}^{1,hol}(X) \cong_\omega \Omega^{1,hol}(X) \;\; , \;\; \mathrm{PV}^{2,hol}(X) \cong_\omega \Omega^{0,hol}(X) = \mathcal{O}^{hol}(X).$$

In particular, the holomorphic de Rham operator $\partial : \Omega^{i,hol} \to \Omega^{i+1,hol}$ defines an operator

$$\partial : \mathrm{PV}^{k,hol}(X) \to \mathrm{PV}^{k-1,hol}(X). \tag{1}$$

At the crudest level, the fields consist of those holomorphic polyvector fields in the kernel of the operator ∂.

The problem with this definition is that it does not fit the usual definition of the space of fields for a classical field theory. In particular, the space of fields is not equal to the smooth sections of a vector bundle. The solution to this problem consists of two steps:

- Resolve the space of holomorphic polyvector fields via its Dolbeualt resolution. This amounts to replacing $\mathrm{PV}^{k,hol}(X)$ by the complex $\mathrm{PV}^{k,*}(X)$ equipped with its Dolbeualt differential

  $$\overline{\partial} : \mathrm{PV}^{k,l}(X) \to \mathrm{PV}^{k,l+1}(X).$$

  As sheaves one has $\mathrm{PV}^{k,*} \simeq \mathrm{PV}^{k,hol}$. We equip the double complex $\mathrm{PV}^{*,*}(X)$ with the grading coming from totalization. Notice that since the operator ∂ in (1) is a holomorphic differential operator it extends in a natural way to $\mathrm{PV}^{*,*}(X)$.
- The next step is to resolve the kernel

  $$\ker \partial \subset \mathrm{PV}^{*,hol}(X),$$

  or more accurately its Dolbeualt version

  $$(\ker \partial, \overline{\partial}) \subset \left(\mathrm{PV}^{*,*}(X), \overline{\partial}\right)$$

  One introduces the formal parameter t of cohomological degree 2 ...

The non-local form of the action functional is

$$S(\alpha) = \frac{1}{2}\int_X \alpha\overline{\partial}(\partial^{-1}\alpha) + \frac{1}{3}\int_X \alpha^3.$$

Notice that in this form only elements in $\mathrm{PV}^{\geq 1,*}$ appear in the action functional. There is a change of coordinates that will put this action functional in a more familiar form.

Suppose $\alpha^{1,*}$ is in the kernel of $\partial : \mathrm{PV}^{1,*}(X) \to \mathrm{PV}^{0,*}(X)$. Then, locally, we can assume that $\alpha^{1,*} = \partial(\alpha^{2,*})$ for some $\alpha^{2,*} \in \mathrm{PV}^{2,*}(X)$. Under the isomorphisms between polyvector fields and Dolbeualt forms, we redefine our fields via

$$\begin{aligned} \mathrm{PV}^{2,*}(X) &\cong_\omega \Omega^{0,*}(X) \\ \alpha^{2,*} &\leftrightarrow A^{0,*} \end{aligned}$$

and

$$\begin{aligned} PV^{0,*}(X) &\cong_\omega \Omega^{2,*}(X) = \omega \cdot \Omega^{0,*}(X) \\ \alpha^{0,*} &\leftrightarrow \omega \cdot B^{0,*} \end{aligned}$$

In particular, the new fields A, B are now Dolbeault forms. Keeping track of the cohomological degree, we find that the our new space of fields is

$$A + \epsilon B \in \Omega^{0,*}(X)[\epsilon][1]$$

where ϵ is a formal parameter of degree $+1$. In particular, the fields in degree zero are $A^{0,1} \in \Omega^{0,1}(X)$ and $B^0 \in \Omega^0(X)$. In terms of these new fields, the action functional takes the form

$$S(A, B) = \int_X \omega B \overline{\partial} A + \frac{1}{2} \int_X B \partial A \partial A. \tag{2}$$

There is yet another way we can recast this classical theory that more clearly reflects the dependence of the theory on the symplectic form. As above, suppose X is a symplectic surface with holomorphic symplectic form ω. We will consider the Lie algebra of holomorphic vector fields $\mathcal{X}^{hol}(X)$, or more accurately, its Dobleault resolution $\mathcal{X}^*(X) = \Omega^{0,*}(X, T^{1,0}X)$.

There is a sub Lie algebra of $\mathcal{X}^{hol}(X)$ consisting of holomorphic *symplectic* vector fields

$$\{X \in \mathcal{X}^{hol}(X) \mid L_X \omega = 0\} \subset \mathcal{X}^{hol}(X).$$

Since the symplectic form is holomorphic, there is a resolution via the Dolbeualt complex. Indeed, there is a sub dg Lie algebra

$$\mathcal{X}^*_{symp}(X) = \{\xi \in \Omega^{0,*}(X, T^{1,0}X) \mid L_\xi \omega = 0\} \subset \mathcal{X}^*(X).$$

When one equips the left hand side with the $\overline{\partial}$ differential, this is a resolution for holomorphic symplectic vector fields.

There is a related Lie algebra of holomorphic *Hamiltonian* vector fields. This Lie algebra has underlying vector space the space of holomorphic functions $\mathcal{O}^{hol}(X)$ and is equipped with Lie bracket given by the holomorphic Poisson bracket $\{-,-\}_\omega$ determined by the symplectic form ω. As we've done several times already, this Lie algebra admits a Dolbeualt resolution as a dg Lie algebra

$$\left(\Omega^{0,*}(X), \overline{\partial}, \{-,-\}_\omega\right). \tag{3}$$

Locally, every holomorphic symplectic vector field is Hamiltonian. Thus, as sheaves of dg Lie algebras we might as well replace $\mathcal{X}^*_{symp}(X)$ with the dg Lie algebra (3).

Given any dg Lie algebra, there is a standard construction to define a classical BV theory. BW: cotangent theory

The fields are

$$(A, B) \in \Omega^{0,*}(X)[1] \oplus \Omega^{2,*}(X)$$

The action functional is

$$\int_X B\overline{\partial} A + \frac{1}{2}\int_X B\{A, A\}. \tag{4}$$

The following lemma is an immediate consequence of our discussion.

Lemma 0.1. *The action functional (2) is equivalent to the action functional (4). In other words, BCOV theory on a complex surface is equivalent to the cotangent theory associated to the Lie algebra of holomorphic symplectic vector fields.*