Coupling

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0.1 Supersymmetric coupling

0.1.1 The 6d vector multiplet coupled to the hypermultiplet

(Brian: Some conventions

- For $x \in \mathfrak{g}$ and $a \in U$ I will write $[x, a]$ for the action of x on a.

)

We choose a quadratic moment map $\mu : U^{\otimes 2} \to \mathfrak{g}^*$ which satisfying

$$(x, \mu(a, b))_{\mathfrak{g}} = ([x, a], b)_U$$

for all $x \in \mathfrak{g}$, and $a, b \in U$.

Let S_{gauge} and S_{matter} be the BV actions for the $6d$ $\mathcal{N} = (1, 0)$ vector and hyper multiplets, respectively. The coupled theory has action $S_{\text{gauge}} + S_{\text{matter}} + I_{\text{couple}}$ where

$$\begin{aligned} I_{\text{couple}} = & g(A, \mu \otimes \Gamma(\psi, \psi))_{\mathfrak{g}} - g(A, \mu(\mathrm{d}\phi, \phi)) - g(A, \mu(\phi, [A, \phi]))_{\mathfrak{g}} \\ & - g(\lambda, \mu(\phi, \psi))_{\mathfrak{g}} \\ & + g(\psi^*, [c, \psi])_U + g(\phi^*, [c, \phi])_U \\ & + g^2 \left(\mu(\phi, \phi), \mu(\phi, \phi)\right)_{\mathfrak{g}} . \end{aligned}$$

The action of supersymmetry on the pure vector multiplet was constructed in Section **??** and was shown to be encoded by linear and quadratic functionals that we denote $I^{(1)}_{\text{gauge}}$ and $I^{(2)}_{\text{gauge}}$. The action of supersymmetry on the pure hyper multiplet was constructed in Section **??** and was shown to be encoded by linear and quadratic functionals that we denote $I^{(1)}_{\text{matter}}$ and $I^{(2)}_{\text{matter}}$. Additionally, in the coupled theory, there are the following new terms in the action of supersymmetry:

$$\begin{aligned} I^{(1)}_{\text{couple}}(Q) &= \int g(\psi^*, \rho([A, \phi])Q)_U + \frac{1}{2} g(\lambda^*, Q\mu(\phi))_{\mathfrak{g}} \\ I^{(2)}_{\text{couple}}(Q_1, Q_2) &= 0 \end{aligned}$$

(Brian: It appears there are no additional L_∞ corrections)

Theorem 0.1. Let $S_{\mathrm{BV}} = S_{\mathrm{gauge}} + S_{\mathrm{matter}} + I_{\mathrm{couple}}$ be the full BV action for the coupled vector to hyper multiplet and let

$$I^{(1)} = I^{(1)}_{\mathrm{gauge}} + I^{(1)}_{\mathrm{matter}} + I^{(1)}_{\mathrm{couple}}$$
$$I^{(2)} = I^{(2)}_{\mathrm{gauge}} + I^{(2)}_{\mathrm{matter}}.$$

Then, $\mathfrak{S} = S_{\mathrm{BV}} + I^{(1)} + I^{(2)}$ satisfies the classical master equation

$$\mathrm{d}_{\mathrm{Lie}}\mathfrak{S} + \frac{1}{2}\{\mathfrak{S}, \mathfrak{S}\} = 0. \tag{1}$$

We have already seen that S_{BV} is consistent, so satisifies the ordinary classical master equation $\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\}$. We decompose the remaining terms in the classical master equation 2 into components based on the cardinality of the number of odd supersymmetry inputs, many of which have already been studied in Sections **??** and **??**. The new equations we must verify are the following:

$$\begin{aligned} \{S_{\mathrm{BV}}, I^{(1)}_{\mathrm{couple}}\} + \{I_{\mathrm{couple}}, I^{(1)}_{\mathrm{gauge}} + I^{(1)}_{\mathrm{matter}}\} + \{I_{\mathrm{couple}}, I^{(1)}_{\mathrm{couple}}\} &= 0 \\ \{I_{\mathrm{couple}}, I^{(2)}_{\mathrm{gauge}} + I^{(2)}_{\mathrm{matter}}\} + \mathrm{d}_{CE} I^{(1)}_{\mathrm{couple}} + \{I^{(1)}_{\mathrm{gauge}} + I^{(1)}_{\mathrm{matter}}, I^{(1)}_{\mathrm{couple}}\} + \tfrac{1}{2}\{I^{(1)}_{\mathrm{couple}}, I^{(1)}_{\mathrm{couple}}\} &= 0 \\ \{I^{(1)}_{\mathrm{couple}}, I^{(2)}\} &= 0 \end{aligned} \tag{2}$$

The third equation is zero by type reasons.

Lemma 0.2. We have $\{I^{(1)}_{\mathrm{couple}}, I^{(2)}\} = 0$.

Proof. This follows from the simple observation that $I^{(2)}$ only involves the antifields λ^*, ψ^* whereas $I^{(1)}_{\mathrm{couple}}$ only involves the fields A and ϕ. So by type reasons, all terms in the bracket are zero. □

This leaves us to verify the first two equations of (2). We begin with the first equation.

Lemma 0.3. One has

$$\{S_{\mathrm{BV}}, I^{(1)}_{\mathrm{couple}}\} + \{I_{\mathrm{couple}}, I^{(1)}_{\mathrm{gauge}} + I^{(1)}_{\mathrm{matter}}\} + \{I_{\mathrm{couple}}, I^{(1)}_{\mathrm{couple}}\} = 0$$

Proof. First, we compute

$$\{S_{\mathrm{BV}}, I^{(1)}_{\mathrm{couple}}\} = g(\rho([A,\phi])Q, \partial\!\!\!/\psi) + g(\psi^*, \rho([\mathrm{d}_A c, \phi])Q) + \frac{1}{2} g(\partial\!\!\!/_A\lambda, \mu(\phi,\phi)Q) + g(\lambda^*, [c, \mu(\phi)Q])$$

Next,

$$\begin{aligned} \{I_{\mathrm{couple}}, I^{(1)}_{\mathrm{gauge}} + I^{(1)}_{\mathrm{matter}}\}(Q) = &- g(\Gamma(Q,\lambda), (\mu\otimes\Gamma)(\psi,\psi)) - g(\Gamma(Q,\lambda), \mu(\phi, \mathrm{d}_A\phi)) + g(\rho(F_A)Q, \mu(\phi,\psi)) \\ &+ g(A, \mu((Q,\psi), \mathrm{d}_A\phi)) + g(A, (\mu\otimes\Gamma)(\psi, \rho(\mathrm{d}\phi)Q)) + g(\lambda, \mu((Q,\psi),\psi)) + g(\lambda, \mu(\phi, \rho(\mathrm{d}\phi)Q)) \\ &+ g(\phi^*, [c, (Q,\psi)]) + g(\psi^*, \rho([c, \mathrm{d}\phi])Q) + g^2(\mu(\phi,\phi), \mu(\phi, (Q,\psi))). \end{aligned}$$

Finally,

$$\begin{aligned} \{I_{\mathrm{couple}}, I^{(1)}_{\mathrm{couple}}\}(Q) = &- g^2(A, (\mu\otimes\Gamma)(\psi, \rho([A,\phi])Q)) - g^2(\lambda, \mu(\phi, \rho([A,\phi])Q)) - g^2(\mu(\phi,\phi)Q, \mu(\phi,\psi)) \\ &- g^2(\psi^*, \rho([A,[c,\phi]])Q) - g^2(\psi^*, [c, \rho([A,\phi])Q]) + g^2(\lambda^*, \mu([c,\phi],\phi)Q) \end{aligned}$$

First, we collect all terms involving only A, ϕ and ψ. This is obtained by extracting the A-dependent terms in the expression

$$\pm(\mathrm{d}_A(Q,\psi), \mathrm{d}_A\psi) \pm (\rho(\mathrm{d}_A\phi)Q, \not{\partial}_A\psi).$$

The proof that this expression vanishes is identical to the proof of Lemma **??**.

Here, we collect all terms involving only ψ and λ:

$$-(\psi, \rho(\Gamma(Q,\lambda))\psi) + (\lambda, \mu((Q,\psi),\psi))$$

By Proposition **??** we have

$$(\psi, \rho(\Gamma(Q,\lambda))\psi) = (\psi, [\lambda, (Q,\psi)]).$$

So, by the moment map condition the total expression involving ψ, λ vanishes.

Here, we collect all terms involving only λ and ϕ:

$$-(\lambda, \mu(\phi, \rho(\mathrm{d}\phi)Q)) + ([\Gamma(\lambda, Q), \phi], \mathrm{d}\phi).$$

By the moment map condition the first term is $([\lambda,\phi], \rho(\mathrm{d}\phi)Q)$. So, by the Clifford relation, these terms cancel.

Here, we collect all terms involving only A, λ and ϕ:

Here, we collect all terms involving only ϕ and ψ:

$$\pm(\mu(\phi,\phi), \mu(\phi, (Q,\psi)) \pm (\mu(\phi,\psi), \mu(\phi,\phi)Q)$$

Since $(\mu(\phi,\psi), \mu(\phi,\phi)Q) = (\mu(\phi,\phi), \mu(\phi,(Q,\psi))$ this expression is identically zero. (Brian: fix signs)

Here, we collect all terms involving only ϕ^*, c and ψ:

$$(\phi^*, [c, (Q,\psi)]) \pm (\phi^*, (Q, [c,\psi]))$$

which clearly vanishes.

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$$(\psi^*, [c, \rho(\mathrm{d}\phi)Q]) \pm (\psi^*, \rho([c, \mathrm{d}\phi])Q)$$

which clearly vanishes.

Here, we collect all terms involving only ψ^*, c, A, and ϕ:

$$\pm(\psi^*, [c, \rho([A,\phi])Q]) \pm (\psi^*, \rho([A,[c,\phi]])Q)$$

which clearly vanishes.

Here, we collect all terms depending only on λ^*, ϕ and c:

$$-(\lambda^*, \mu([c,\phi],\phi)Q) + (\lambda^*, [c, Q\mu(\phi,\phi)]).$$

Since the moment map μ is $\mathfrak{g}$-invariant, this expression vanishes. □

Finally, we verify the second equation in (2).

Lemma 0.4. One has

$$\{I_{\text{couple}}, I^{(2)}_{\text{gauge}} + I^{(2)}_{\text{matter}}\} + \mathrm{d}_{CE} I^{(1)}_{\text{couple}} + \{I^{(1)}_{\text{gauge}} + I^{(1)}_{\text{matter}}, I^{(1)}_{\text{couple}}\} + \frac{1}{2}\{I^{(1)}_{\text{couple}}, I^{(1)}_{\text{couple}}\} = 0.$$

Proof. The second and fourth terms are zero by type reasons. Thus, we must show

$$\{I_{\text{couple}}, I^{(2)}_{\text{gauge}} + I^{(2)}_{\text{matter}}\} + \{I^{(1)}_{\text{gauge}} + I^{(1)}_{\text{matter}}, I^{(1)}_{\text{couple}}\} = 0 \tag{3}$$

We begin by expanding $\{I_{\text{couple}}, I^{(2)}_{\text{gauge}}\}$:

$$\begin{aligned}\{I_{\text{couple}}, I^{(2)}_{\text{gauge}}\}(Q_1, Q_2) = &-\frac{1}{2}(\rho(\Gamma(Q_1, Q_2))\lambda^*, \mu(\phi, \psi)) - \frac{1}{2}(Q_1, \mu(\phi, \psi))(Q_2, \lambda^*) - \frac{1}{2}(Q_1, \lambda^*)(Q_2\mu(\phi, \psi)) \\ &+(\psi^*, [i_{\Gamma(Q_1,Q_2)}A, \psi]) + (\phi^*, [i_{\Gamma(Q_1,Q_2)}A, \phi])\end{aligned}$$

Next, we expand $\{I_{\text{couple}}, I^{(2)}_{\text{matter}}\}$:

$$\begin{aligned}\{I_{\text{couple}}, I^{(2)}_{\text{matter}}\}(Q_1, Q_2) =&\frac{1}{2}(\rho(\Gamma(Q_1, Q_2))\psi^*, [A, \psi]) - \frac{1}{2}(\lambda, \mu(\phi, \rho(\Gamma(Q_1, Q_2))\psi^*)) \\ &+ (\Gamma(Q_1, Q_2), \Gamma([c, \psi^*], \psi^*)) + (\Gamma(Q_1, Q_2), \Gamma(\psi^*, [c, \psi^*])).\end{aligned}$$

The last line vanishes by the $\mathfrak{g}$-invariance of the symplectic pairing for U.

Expanding $\{I^{(1)}_{\text{gauge}}, I^{(1)}_{\text{couple}}\}$:

$$\begin{aligned}\{I^{(1)}_{\text{gauge}}, I^{(1)}_{\text{couple}}\}(Q_1, Q_2) = &- (\psi^*, \rho([\Gamma(Q_1, \lambda), \phi])Q_2) - (\psi^*, \rho([\Gamma(Q_2, \lambda), \phi])Q_1) \\ &+(A^*, \Gamma(Q_1, \mu(\phi, \phi)Q_2)) - (A^*, \Gamma(Q_2, \mu(\phi, \phi)Q_1)).\end{aligned}$$

Notice the term proportional to A^* vanishes.

$$\begin{aligned}\{I^{(1)}_{\text{matter}}, I^{(1)}_{\text{couple}}\}(Q_1, Q_2) = &\pm (\phi^* Q_1, \rho([A, \phi])Q_2) \pm (\phi^* Q_2, \rho([A, \phi])Q_1) \\ &-(\psi^*, \rho([A, (Q_1, \psi)])Q_2) - (\psi^*, \rho([A, (Q_2, \psi)])Q_1) \\ &-(\lambda^*, \mu((Q_1, \psi), \phi)Q_2) - (\lambda^*, \mu((Q_2, \psi), \phi)Q_1)\end{aligned}$$

We collect terms involving ϕ, A and ϕ^*:

$$[i_{\Gamma(Q_1,Q_2)}A, \phi] \pm (Q_1, \rho([A, \phi])Q_2) \pm (Q_2, \rho([A, \phi])Q_1)$$

By Proposition **??** the last two terms cancel with the first term. (Brian: Pavel, what is the version of that identity we need to use here, that proposition isn't quite what we need. This is the only sign I'm unsure of in this proposition.)

Next, we collect terms involving ϕ, ψ and λ^*. This term is obtained by taking the $\mathfrak{g}$-pairing of $\mu(\phi, \psi)$ with

$$-\frac{1}{2}\rho(\Gamma(Q_1, Q_2))\lambda^* + \frac{1}{2}(Q_2, \lambda^*)Q_1 + \frac{1}{2}(Q_1, \lambda^*)Q_2$$

which vanishes by Proposition **??**.

Next, we collect terms involving λ, ϕ, ψ^*. This term can be written as pairing the antifield ψ^* with the field

$$-\frac{1}{2}\rho(\Gamma(Q_1, Q_2))[\lambda, \phi] - \frac{1}{2}\rho([\Gamma(Q_1, \lambda), \phi])Q_2 - \frac{1}{2}\rho([\Gamma(Q_2, \lambda), \phi])Q_1.$$

This whole expression is equal to applying the $\mathfrak{g}$-valued field

$$-\frac{1}{2}\rho(\Gamma(Q_1, Q_2))\lambda - \frac{1}{2}\rho(\Gamma(Q_1, \lambda))Q_2 - \frac{1}{2}\rho(\Gamma(Q_2, \lambda))Q_1$$

to the field ϕ by the representation. This expression vanished by Proposition **??**.

Finally, we collect terms involving ψ^*, A and ψ. This can be written as the pairing of the antifield ψ^* with the field

$$\frac{1}{2}[A, \rho(\Gamma(Q_1, Q_2))\psi] - \frac{1}{2}\rho([A, (Q_1, \psi)])Q_2 - \frac{1}{2}\rho([A, (Q_2, \psi)])Q_1.$$

This expression is equal to applying the $\mathfrak{g}$-valued field A to the field

$$\frac{1}{2}\rho(\Gamma(Q_1, Q_2))\psi - \frac{1}{2}(Q_1, \psi)Q_2 - \frac{1}{2}(Q_2, \psi)Q_1$$

which vanishes by Proposition **??**.

□

1 The holomorphic twist

The holomorphic twist of the 6d $\mathcal{N} = (1, 0)$ vector multiplet coupled to the hyper multiplet can be described in the BV formalism as a ($\mathbb{Z}/2$-graded) generalized Chern-Simons theory on any Calabi-Yau 3-fold X.

Suppose the vector multiplet takes values in a Lie algebra $\mathfrak{g}$ and the hyper multiplet is valued in a symplectic $\mathfrak{g}$-representation (U, ω_U). Define the following graded Lie algebra

$$\begin{array}{cccc} & \underline{0} & \underline{1} & \underline{2} \\ \mathfrak{g}_U \quad = & \mathfrak{g} & U & \mathfrak{g}^* \end{array}$$

where the Lie brackets are given by:

- the usual Lie bracket $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$;
- the representation $\mathfrak{g} \otimes U[-1] \to U[-1]$;
- the coadjoint action $\mathfrak{g} \otimes \mathfrak{g}^*[-2] \to \mathfrak{g}^*[-2]$;
- the moment map $\mu_k : (U[-1])^{\otimes k} \to \mathfrak{g}^*[-2]$ (Brian: could be L_∞-maps, $k = 2$ is the ordinary situation.)

The Lie algebra has a natural nondegenerate invariant pairing ω of degree $(+2)$ defined by combining the symplectic pairing ω_U on U together with the natural dual pairing between $\mathfrak{g}$ and its dual.

Given $\mathfrak{g}_U$ we can consider generalized Chern-Simons theory on $(X, \Omega^{0,\bullet})$ with values in $\mathfrak{g}_U$. Note that this is only a $\mathbb{Z}/2$-graded theory in the BV formalism, since $\mathfrak{g}_U$ has a pairing of degree $+2$ and $\dim_{\mathbb{C}}(X) = 3$. The underlying space of BV fields is

$$\Omega^{0,\bullet}_X \otimes \mathfrak{g}_U[1]$$

and the action functional is of Chern-Simons type $S(\alpha) = \int_X (\alpha, \bar{\partial}\alpha) + \sum_{k\geq 2} \frac{1}{(k+1)!} \int_X \Omega_X \wedge \langle \alpha \wedge \ell_k(\alpha, \ldots, \alpha)\rangle$.

For the specific graded Lie algebra $\mathfrak{g}_U$ we can decompose $\alpha = A + \varphi + B$ where A is $\mathfrak{g}$-valued, φ is U-valued, and B is $\mathfrak{g}^*$-valued. The action then takes the form

$$S(A + \varphi + B) = \int_X \left(B, \bar{\partial}A + \frac{1}{2}[A, A]\right)_{\mathfrak{g}} + \omega_U\left(\varphi, \bar{\partial}\varphi + [A, \varphi]\right) + \sum_{k\geq 2} (A, \mu(\varphi, \ldots, \varphi))_{\mathfrak{g}}.$$