Homework 7: Car Tracking

Course: CS 221 Spring 2019
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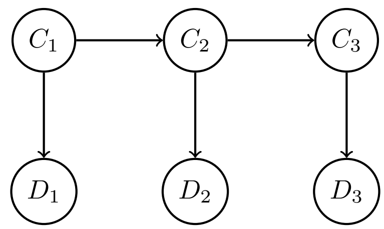
Problem 1: Bayesian Network Basics

First, let us look at a simplified version of the car tracking problem. For this problem only, let $C_t \in \{0, 1\}$ be the actual location of the car we wish to observe at time step $t \in \{1, 2, 3\}$. Let $D_t \in \{0, 1\}$ be a sensor reading for the location of that car measured at time t. Here's what the Bayesian network (it's an HMM, in fact) looks like:



The distribution over the initial car distribution is uniform; that is, for each value $c_1 \in \{0, 1\}$:

$$p(c_1) = 0.5$$

The following local conditional distribution governs the movement of the car (with probability ϵ, the car moves). For each $t \in \{2, 3\}$:

$$p(c_t \mid c_{t-1}) = \begin{cases} \epsilon & \text{if } c_t \neq c_{t-1} \\ 1 - \epsilon & \text{if } c_t = c_{t-1} \end{cases}$$

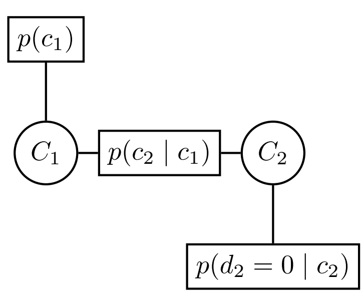
The following local conditional distribution governs the noise in the sensor reading (with probability η, the sensor reports the wrong position). For each $t \in \{1, 2, 3\}$:

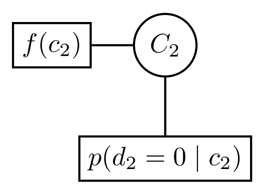
$$p(d_t \mid c_t) = \begin{cases} \eta & \text{if } d_t \neq c_t \\ 1 - \eta & \text{if } d_t = c_t \end{cases}$$

Below, you will be asked to find the posterior distribution for the car's position at the second time step (C_2) given different sensor readings.

Important: For the following computations, try to follow the general strategy described in lecture (marginalize non-ancestral variables, condition, and perform variable elimination). Try to delay normalization until the very end. You'll get more insight than trying to chug through lots of equations.

(a) Suppose we have a sensor reading for the second timestep, $D_2 = 0$. Compute the posterior distribution $\mathbb{P}(C_2 = 1 \mid D_2 = 0)$. We encourage you to draw out the (factor) graph.



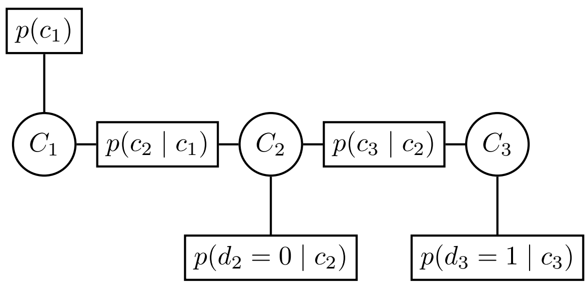


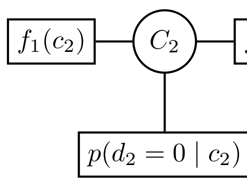
$$f(c_2) = \sum_{c_1} p(c_1)p(c_2 \mid c_1) = \begin{cases} .5(1-\epsilon) + .5\epsilon = .5 & \text{if } c_2 = 0 \\ .5\epsilon + .5(1-\epsilon) = .5 & \text{if } c_2 = 1 \end{cases}$$

$$\mathbb{P}(C_2 = c_2 \mid D_2 = 0) \propto f(c_2)p(d_2 = 0 \mid c_2) = \begin{cases} .5(1-\eta) & \text{if } c_2 = 0 \\ .5\eta & \text{if } c_2 = 1 \end{cases}$$

$$\mathbb{P}(C_2 = 1 \mid D_2 = 0) = \eta$$

(b) Suppose a time step has elapsed and we got another sensor reading, $D_3 = 1$, but we are still interested in C_2. Compute the posterior distribution $\mathbb{P}(C_2 = 1 \mid D_2 = 0, D_3 = 1)$. The resulting expression might be moderately complex. We encourage you to draw out the (factor) graph.





$$f_1(c_2) = \sum_{c_1} p(c_1)p(c_2 \mid c_1) = \begin{cases} .5(1-\epsilon) + .5\epsilon = .5 & \text{if } c_2 = 0 \\ .5\epsilon + .5(1-\epsilon) = .5 & \text{if } c_2 = 1 \end{cases}$$

$$f_2(c_2) = \sum_{c_3} p(c_3 \mid c_2)p(d_3 = 1 \mid c_3) = \begin{cases} (1-\epsilon)\eta + \epsilon(1-\eta) & \text{if } c_2 = 0 \\ \epsilon\eta + (1-\epsilon)(1-\eta) & \text{if } c_2 = 1 \end{cases}$$

$$\mathbb{P}(C_2 = c_2 \mid D_2 = 0, D_3 = 1) \propto f_1(c_2)p(d_2 = 0 \mid c_2)f_2(c_2) = \begin{cases} .5(1-\eta)((1-\epsilon)\eta + \epsilon(1-\eta)) & \text{if } c_2 = 0 \\ .5\eta(\epsilon\eta + (1-\epsilon)(1-\eta)) & \text{if } c_2 = 1 \end{cases}$$

$$\mathbb{P}(C_2 = 1 \mid D_2 = 0, D_3 = 1) = \frac{\epsilon\eta^2 + (1-\epsilon)(1-\eta)\eta}{\epsilon\eta^2 + 2(1-\epsilon)(1-\eta)\eta + \epsilon(1-\eta)^2}$$

(c) Suppose $\epsilon = 0.1$ and $\eta = 0.2$.

(i) Compute and compare the probabilities $\mathbb{P}(C_2 = 1 \mid D_2 = 0)$ and $\mathbb{P}(C_2 = 1 \mid D_2 = 0, D_3 = 1)$. Give numbers, round your answer to 4 significant digits.

$$\mathbb{P}(C_2 = 1 \mid D_2 = 0) = .2$$
$$\mathbb{P}(C_2 = 1 \mid D_2 = 0, D_3 = 1) = .4157$$

(ii) How did adding the second sensor reading $D_3 = 1$ change the result? Explain your intuition for why this change makes sense in terms of the car positions and associated sensor observations.

It increased the probability that $C_2 = 1$. Additional data is given that supports $C_2 = 1$. Since the probability that the position changes between timesteps is small, having the sensor reading $D_3 = 1$ makes it more likely that $C_2 = 1$.

(iii) What would you have to set ϵ while keeping $\eta = 0.2$ so that $\mathbb{P}(C_2 = 1 \mid D_2 = 0) = \mathbb{P}(C_2 = 1 \mid D_2 = 0, D_3 = 1)$? Explain your intuition in terms of the car positions with respect to the observations.

$$\epsilon = .5$$

This is effectively making it equally likely for C_3 to be 0 or 1 which makes knowing $D_3 = 1$ irrelevant.

Problem 2: Emission Probabilities

(a) coding

Problem 3: Transition Probabilities

(a) coding

Problem 4: Particle Filtering

(a) coding

(b) coding

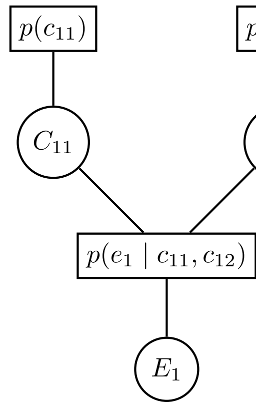
Problem 5: Which Car is It?

So far, we have assumed that we have a distinct noisy distance reading for each car, but in reality, our microphone would just pick up an undistinguished set of these signals, and we wouldn't know which distance reading corresponds to which car. First, let's extend the notation from before: let $C_{ti} \in \mathbb{R}^2$ be the location of the i-th car at the time step t, for $i = 1, \dots, K$ and $t = 1, \dots, T$. Recall that all the cars move independently according to the transition dynamics as before.

Let $D_{ti} \in \mathbb{R}$ be the noisy distance measurement of the i-th car, which is now not observed. Instead, we observe the set of distances $D_t = \{D_{t1}, \dots, D_{tK}\}$ (assume that all distances are all distinct). Alternatively, you can think of $E_t = (E_{t1}, \dots, E_{tK})$ as a list which is a uniformly random permutation of the noisy distances $(D_{t1}, \dots, D_{tK})$. For example, suppose $K = 2$ and $T = 2$. Before, we might have gotten distance readings of 1 and 2 for the first car and 3 and 4 for the second car. Now, our sensor readings would be permutations of $\{1, 3\}$ and $\{2, 4\}$. Thus, even if we knew the second car was distance 3 away at time $t = 1$, we wouldn't know if it moved farther (4 away) or closer (2 away) at time $t = 2$.

(a) Suppose we have $K = 2$ cars and one time step $T = 1$. Write an expression for the conditional distribution $\mathbb{P}(C_{11}, C_{12} \mid E_1 = e_1)$ as a function of the PDF of a Gaussian $p_{\mathcal{N}}(\nu; \mu, \sigma^2)$ and the prior probability $p(c_{11})$ and $p(c_{12})$ over car locations. Your final answer should not contain variables d_{11}, d_{12}.

Remember that $p_{\mathcal{N}}(\nu; \mu, \sigma^2)$ is the probability of a random variable, ν, in a Gaussian distribution with mean μ and standard deviation σ^2.



$$\begin{aligned}
\mathbb{P}(C_{11}, C_{12} \mid E_1 = e_1) &\propto p(c_{11})p(c_{12})(p(e_1 = \{e_{11}, e_{12}\} \mid c_{11}, c_{12}) + p(e_1 = \{e_{12}, e_{11}\} \mid c_{11}, c_{12})) \\
p(e_1 = \{e_{11}, e_{12}\} \mid c_{11}, c_{12}) &\propto p_{\mathcal{N}}(e_{11}; \|a_1 - c_{11}\|, \sigma^2) p_{\mathcal{N}}(e_{12}; \|a_1 - c_{12}\|, \sigma^2) \\
p(e_1 = \{e_{12}, e_{11}\} \mid c_{11}, c_{12}) &\propto p_{\mathcal{N}}(e_{12}; \|a_1 - c_{11}\|, \sigma^2) p_{\mathcal{N}}(e_{11}; \|a_1 - c_{12}\|, \sigma^2) \\
\mathbb{P}(C_{11}, C_{12} \mid E_1 = e_1) &\propto p(c_{11})p(c_{12})(p_{\mathcal{N}}(e_{11}; \|a_1 - c_{11}\|, \sigma^2) p_{\mathcal{N}}(e_{12}; \|a_1 - c_{12}\|, \sigma^2) \\
&\quad + p_{\mathcal{N}}(e_{12}; \|a_1 - c_{11}\|, \sigma^2) p_{\mathcal{N}}(e_{11}; \|a_1 - c_{12}\|, \sigma^2))
\end{aligned}$$