Assignment #10

Due Friday, 9 December 2022, at the start of class

Please read sections 14.1–14.7 and 15.1–15.4 from the textbook.[1]

DO THE FOLLOWING EXERCISES from section 14.2, pages 489–491:

- Exercise 2.7 *You may use techniques from* either *section 14.2 or 14.3.*

Problem P21. Suppose $c \in \mathbb{R}^n$ is a nonzero vector and consider the problem

$$\begin{aligned} \text{minimize} \quad & z = c^\top x \\ \text{subject to} \quad & \sum_{i=1}^n x_i^2 = 1 \end{aligned}$$

where $x \in \mathbb{R}^n$. Note that the single equality constraint can be written as $\|x\|^2 = 1$.

(a) By arguing informally explain why the solution is

$$x_* = -\frac{c}{\|c\|}.$$

Use a sketch of the $n = 2$ case to explain.

(b) The necessary optimality conditions for this problem are addressed by Theorem 14.15 on page 504 of the textbook. Compute the Lagrangian and state the first-order necessary conditions in detail. (*Note that you do* not *need to compute a null-space matrix in order to do this.*)

(c) Solve the conditions in **(b)** algebraically to confirm the solution in part **(a)**. How many points (x_*, λ_*) are there which satisfy the first-order necessary conditions?

Problem P22. *Before doing this problem read Example 14.20 on pages 506–507. This problem asks for a similar analysis.*

Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x) = (x_1 - 1)^2 + (x_2 + 1)^2 \\ \text{subject to} \quad & x_1^2 + x_2^2 \le 9 \\ & x_2 \ge 0 \end{aligned}$$

(a) Sketch the feasible set and explain informally, perhaps using contours of f, why $x_* = (1, 0)^\top$ is the solution.

(b) Write the constraints in the form $g_i(x) \ge 0$. Compute the Lagrangian and its gradient. For each of the points $A = (0, 0)^\top$, $B = (0, 3)^\top$, and $C = (1, 0)^\top$ compute

[1] Griva, Nash, and Sofer, *Linear and Nonlinear Optimization*, 2nd ed., SIAM Press 2009.

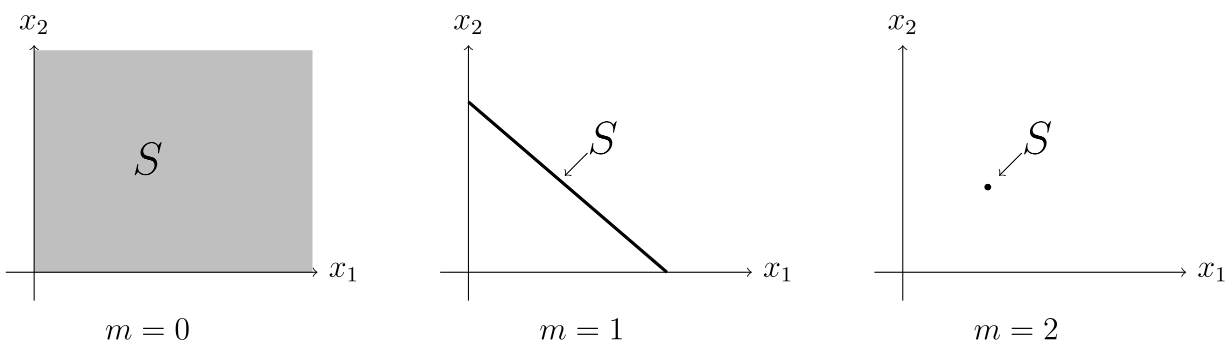
the values of λ_i satisfying the zero-gradient condition. Address whether these points satisfy the first-order optimality conditions, that is, whether they are candidates for a local minimizer. Show in particular that C satisfies all the first-order conditions in Theorem 14.18. (*You do* not *need to find null-space matrices to answer this question.*)

Problem P23. Consider nonlinear optimization problems on $x \in \mathbb{R}^n$ which have standard-form linear constraints:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Assume that there are m scalar constraint equations and that A has full row rank. Thus $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $m \leq n$ (as usual).

We want to visualize the possible feasible sets for such problems. In 2D ($n = 2$) there are exactly three possibilities for the dimension of the feasible set. The cartoons below illustrate these possibilities when the feasible set S is non-empty and bounded. (Except that for $m = 0$ the set is unbounded.)



For 3D ($n = 3$) there are four nonempty, and bounded if $m > 0$, possibilities. Sketch the four corresponding cartoons. These cartoons should have the same annotations as the 2D versions above.

Problem P24. *This problem asks you to write the general KKT conditions.*

Consider the general nonlinear constrained optimization problem over $x \in \mathbb{R}^n$:

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & g_i(x) = 0, & i = 1, \dots, \ell \\ & h_i(x) \geq 0, & i = 1, \dots, m \end{array}$$

(a) In section 14.5 the meaning of the phrase "*x_* is a regular point of the constraints*" is given. State this definition precisely for the above problem.

(b) State the Lagrangian for this problem.

(c) Suppose x_* is a local minimizer of the above problem which is also a regular point. State the first-order necessary conditions for the above problem. (*Again, you do* not *need to find null-space matrices to answer this question.*)