A Human-Checkable Four-Color Theorem Proof

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***Abstract**. This paper presents a short and simple proof of the Four-Color Theorem, which can be utterly checkable by human mathematicians, without computer assistance. The new key idea that has permitted it is presented in the Introduction.*

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Keywords. Four-Color Theorem, Three-Edge Color Theorem, WDN, DSCC, 2-DSCC_M, 4-CM, CBG, 3-ECC.

Contents

1. Introduction

The Four-Color Theorem (4CT) [1, 16] is a very beautiful discovery, embraces very deep math with innocent appearance, and furthermore has an epical and passionate history.

Circa a century and two decades since it was conjectured, finally its computer program-based proof was amazing (and even a technological feat in the ancient times of 1970s [13, 14]). However, as this proof – even though it was highly developed later [2, 15] – is yet so very long and composed with separated pieces through excruciating details, it is still no amenable to complete human verification. [2]

This little paper presents a very shorter and simpler proof docile to fully human verification and total understanding [12] (an overview of the proof is presented in Section 3).

A reviewer has asked me: "– What is the new key idea that allows us to get around looking into all these many different configurations that the existing proofs test for?"

The new key idea here is to see a planar map not as formed with possibly *so many different configurations*, but simply see it as formed by *essentially only two closed curves* overlapping on a plane: then, all the regions from the map can only be in exactly a single position with respect to these two curves, from four possible options: out of them, inside of them, out of one and inside the other, or out of the other and inside the one, which naturally generates the four colors that they can be colored with, without adjacent regions having the same color, since crossing a boundary from the map must necessarily change one, or the two, of the out/inside positions. Consequently, adjacent regions cannot have the same position – equivalently, the same color – with respect to the two original (set of) closed curves.

Hence, as only four different positions are possible with respect to these two (set of) closed curves (out-out, out-in, in-out, in-in), if we prove that any hypothetical minimal counter-example to 4CT can always be formed by means of this overlapping construction (hence, by contradiction, that *hypothetical* minimal counter-example cannot exist as a *real* counter-example), then the 4CT stands proved, by *Reductio ad Absurdum*.

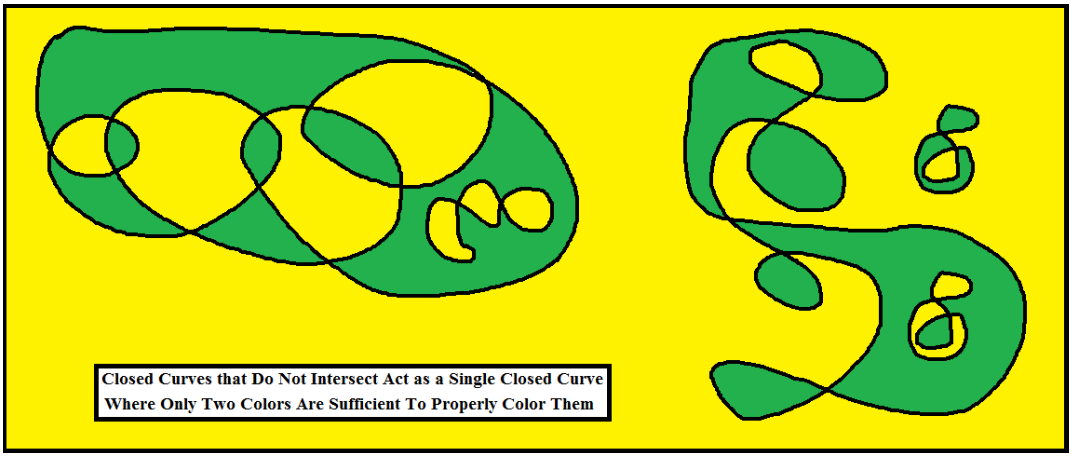

Figure 1.1 Example 1 of Closed Curves Representing the New Key Idea Used in the Proof

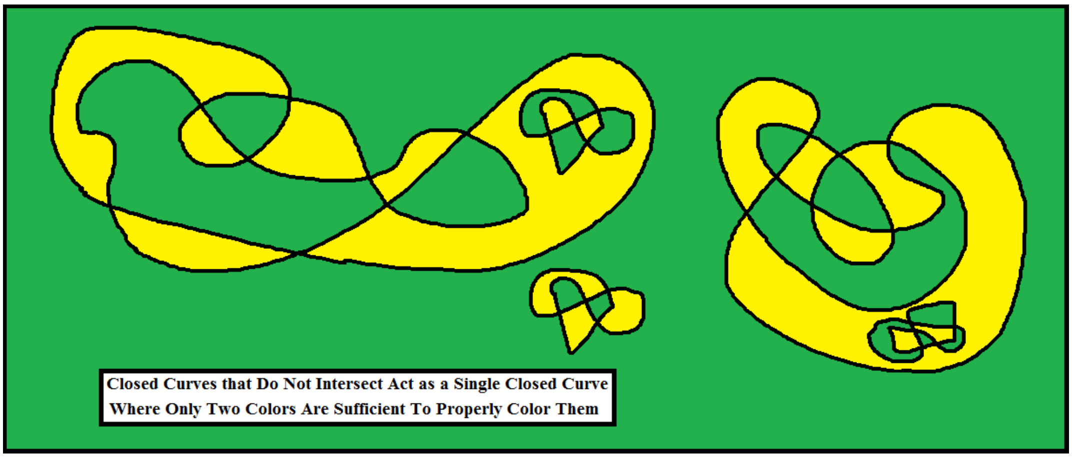

Figure 1.2 Example 2 of Closed Curves Representing the New Key Idea Used in the Proof

With the two figures above, we can intuitively see that a resulting map from overlapping two [no intersecting or disjoint set of] closed curves can always be properly colored with only four different colors: All its regions can be colored with one from the four colors: green-green (1), green-yellow (2), yellow-green (3) or yellow-yellow (4) according to arrangement of the colors where the regions are from on the 2 original (set of) closed curves.

2. A Human-Checkable Four-Color Theorem Proof

Definition 2.1. Disjoint Set of Closed Curves (*DSCC*). A ***DSCC*** is a finite set of disjoint oriented closed curves [either simple (*Jordan curve*) or non-simple ones] on a Euclidean plane [3], where there is no intersection point between any pair of them – that is, they do neither cross, intersect nor "*touch*" themselves in that set. The empty set and a unitary set of a simple closed curve are trivially *DSCC*.

More formally: Let I_1, ..., I_n be nonempty intervals of reals $\mathbf{R}$, and C_1: $I_1 \rightarrow \mathbf{R}^2$, ..., C_n: $I_n \rightarrow \mathbf{R}^2$ be continuous mapping, where $I_i = [a_i, b_i]$, $b_i > a_i$, $(C_i(x) = C_i(y)) \wedge (x \neq y)$ only on finitely many x, $C_i(a_i) = C_i(b_i)$, $\forall i$, $1 \leq i \leq n$, $\{C_1, ..., C_n\}$ is a *DSCC* iff $C_i(x) = C_j(y) \Rightarrow i = j$, $\forall i, j$, $1 \leq i, j \leq n$.

Definition 2.2. *DSCC* Winding Number (DWN). The ***DWN*** of every point (x, y) inside a Euclidean plane where the respective *DSCC* is drawn is the algebraic sum of all the absolute values of the *winding numbers* (as defined in [4, 8]) of that point with respect to all the curves C_i of that *DSCC* (so, the orientations of the curves of the *DSCC* do not matter at all). Note that every arbitrary point of a Euclidean plane has a determined, fixed and effectively calculable winding number with respect to a determined oriented closed curve, so a DWN with respect to a given *DSCC*: DWN$(x, y) = \sum |\text{winding number}(x, y, C_i)|$.

Definition 2.3. Points inside and outside with respect to a *DSCC*. Every point (x, y) inside a Euclidean plane with respect to a *DSCC* (where it is drawn) can be exactly and exclusively in only one of the following position:

(i) *Outside from the DSCC*, iff its respective DWN is even [$2 \mid \text{DWN}(x, y)$];
(ii) *Inside from the DSCC*, iff its respective DWN is odd [$2 \nmid \text{DWN}(x, y)$]; or
(iii) Inside the image of some curve of the *DSCC* (where DWN(x, y) is undefined).

Definition 2.4. Regions inside and outside from a *DSCC*. An arbitrary region R (a contiguous portion of surface that does not contain any point from any curve) in a Euclidean plane where a *DSCC* is drawn can be exactly and exclusively in only one of the following position: (Obs.: Some few small parts of the proof are visual – using geometric intuition –, but their truth are so obvious that they do not need formal combinatorics demonstrations, at all.)

(i) *Outside from the DSCC*, iff some point $(x, y) \in R$ is outside from that *DSCC*; or
(ii) *Inside from the DSCC*, iff some point $(x, y) \in R$ is inside from that *DSCC*.

Note: All the points from a specified region must have the same DWN, since the winding number of a continuously moving point with respect to some closed curve changes only if that point crosses (intersects) that closed curve. [4, 8] Thus, as in a region there is no point from the curves, that DWN of every point inside that region must be equal to one of any other point into that same region. Furthermore, by Def. 2.2, every region into a Euclidean plane has a determined, unique, fixed and effectively calculable answer whether it is either *inside* or *outside* of a given *DSCC*.

Verify that the definitions of *outside* and *inside* above can be swapped without essentially altering the gist of the proof in this paper.

Definition 2.5. 2-*DSCC* Map (2-*DSCC*_M). A ***2-DSCC_M*** is a connected finite simple planar graph that can be represented (drawn) by two *DSCC*, where we can call them a *Blue DSCC* and a *Yellow DSCC* (which can be considered formed by *blue* and *yellow* closed curves, where the blue curves can *touch* (have common points with) the yellow ones and vice versa, whereas the same color curves cannot touch themselves, of course, by Def. 2.1).

Note yet that a *DSCC* in this definition can be the empty one, and, in a 2-*DSCC*_M, regions (countries), boundaries (borders, sides), and vertices (points where different boundaries *touch* themselves) can be represented by faces, edges (set E) and vertices (set V) on a finite simple planar graph $M = (V, E)$, respectively (where all the edges from M can be represented by $e_{i\text{-}j} = \{v_i, v_j\}$, where $v_i, v_j \in V$ and $\{v_i, v_j\} \in E$).

Definition 2.6. Blue, yellow and green edges (boundaries). In a 2-*DSCC*_M represented by a finite simple planar graph $M = (V, E)$, every edge $\mathbf{e_{i\text{-}j}}$ representing a boundary formed only by a blue (respect., yellow) curve is a *blue* (respect., *yellow) edge*, and every edge $e_{k\text{-}l}$ representing boundary formed by an intersection of a blue and a yellow curve (upon infinitely many points) is a *green edge*. These particulars can be represented by the predicates $B(e_{i\text{-}j})$ (respect., $Y(e_{i\text{-}j})$) and $G(e_{k\text{-}l})$, respectively.

In order to formally include these colors to the maps, we shall introduce two new planar graphs $B = (V, E_b)$, and $Y = (V, E_y)$, where a blue (respect., yellow) edge will be represented by $e_{bi\text{-}bj} = \{v_{bi}, v_{bj}\} \in E_b$ (respect., $e_{yk\text{-}yl} = \{v_{yk}, v_{yl}\} \in E_y$), and a green edge is defined as one that is in these two graphs at same time ($e_{bi\text{-}bj}$ and $e_{yi\text{-}yj}$), linking the same points v_i and v_j (where $v_{bi} = v_{yi} = v_i$, $\forall v_i \in V$, and $E_b \cup E_y = E$).

Lemma 2.2. Every edge of a ***2-DSCC_M*** is either a blue, yellow or green edge.

Proof. In a ***2-DSCC_M*** there are only blue and yellow curves, and every segment where they intersect is (must be) green (representing a overlap of the two curves). Hence, can there be only blue, yellow and green edges (boundaries) in a ***2-DSCC_M***. ☐

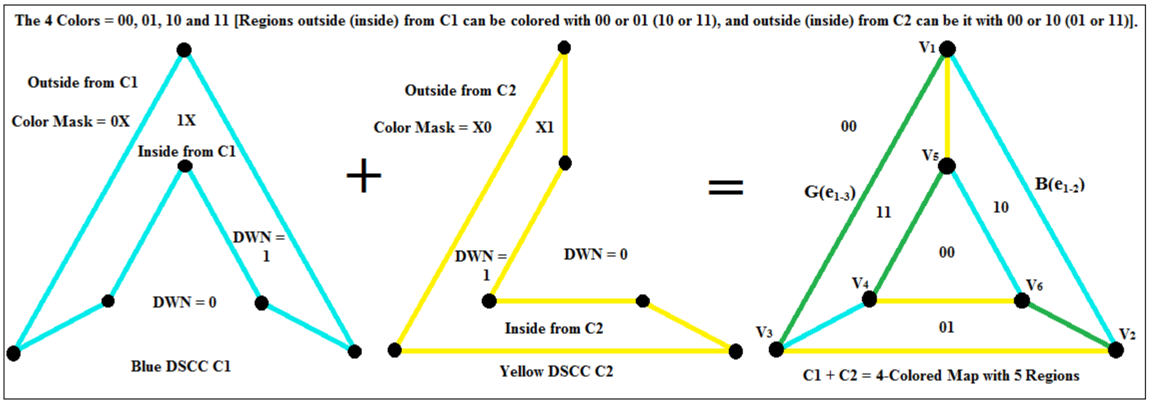

Figure 2.1 Example of How 2 Closed Curves form a 4-Coloring (or 3-Edge-Coloring) 2-*DSCC*_M

More formally, the 2-*DSCC*_M in the Fig. 2.1 above can be represented by $M = (\{v_1, v_2, v_3, v_4, v_5, v_6\}, \{e_{b1\text{-}b2}, e_{b3\text{-}b4}, e_{b5\text{-}b6}, e_{y1\text{-}y5}, e_{y2\text{-}y3}, e_{y4\text{-}y6}, e_{g1\text{-}g3}, e_{g4\text{-}g5}, e_{g2\text{-}g6}\})$.

Lemma 2.3. All regions R_i are exclusively either inside or outside from each one of the two *DSCC* of a ***2-DSCC_M*** (more formally, R_i(*Blue DSCC*) = *Inside* exclusive-or *Outside*, and R_i(*Yellow DSCC*) = *Inside* exclusive-or *Outside*).

Proof. All the points from a specified region must have the same DWN, since the winding number of a continuously moving point with respect to some closed curve changes only if that point crosses that closed curve. [4, 8] Since the intersection of a region with the curves from a ***2-DSCC_M*** is an empty set, this region is exclusively either inside or outside each curve from that ***2-DSCC_M***, of course, by Def. 2.4:

$$(x_1, y_1), (x_2, y_2) \in R_i \Rightarrow \mathrm{DWN}(x_1, y_1) = \mathrm{DWN}(x_2, y_2). \ \square$$

Definition 2.7. Four colorable map (4-CM). A ***4-CM*** is a *4-colorable* connected finite simple planar graph, that is, a map where at most four colors suffice to color it without adjacent regions having the same color. [1, 16]

Lemma 2.4. Every ***2-DSCC_M*** is also a ***4-CM***.

Proof. We can color the regions in any *2-DSCC_M* as indicated in the table below:

Is the Region Inside the *Blue DSCC*?	Is the Region Inside the *Yellow DSCC*?	Color the Region With the Color:
No	No	00 (■)
No	Yes	01 (■)
Yes	No	10 (■)
Yes	Yes	11 (■)

Table 2.1 How to 4-color a *2-DSCC_M* accordingly regions' positioning

Thus, within this Table 2.1, since there are in the *2-DSCC_M* two *DSCC*, and all the regions of the map are inside exclusive-or outside each curve, without third option, by Lemma 2.3, four colors are sufficient in order to color it, in general, as if each region inside (respect., outside) the *Blue DSCC* had a mask color ***1X*** (respect., ***0X***), and each region inside (respect., outside) the *Yellow DSCC* had a mask color ***X1*** (respect., ***X0***), where the color of that region is like the composition of these two masks, as shown in the Table 2.1. But with only four colors, couldn't there be two adjacent regions with the same color? We shall see below that not:

If two adjacent regions have the same color, then the position of those regions is the same with respect to two *DSCC*, but this is impossible, at all, since in order to cross the edges of the map we can only cross either exactly one blue or exactly one yellow curve, or exactly two ones at same time (when crossing a green edge), exactly one of each color, by Def. 2.1 (remember that the blue (respect., yellow) curves of a *DSCC* cannot have common edge (infinitely many points) – only finitely many [isolated] points – with themselves); that is, when we cross (intersect) an edge, moving us from a region to another one, we must change at least one of the answers above (from *No* to *Yes* or from *Yes* to *No*) with respect to the arriving region, by Def. 2.4; thus, it is impossible that two adjacent regions have the same set of those two positioning answers in the table above, so it is impossible that they have the same color.

Hence, the coloring of the map by means of the table above guarantee that there is no adjacent region with the same color, so at most four color suffice to properly color every *2-DSCC_M*; thus, every *2-DSCC_M* is also a *4-CM*. ☐

Lemma 2.5. Every ***4-CM*** is also a ***2-DSCC_M***.

Proof. This proof is constructive, since we shall demonstrate that we can construct a *2-DSCC_M* from any proper 4-coloring of every *4-CM*.

See, from any proper 4-coloring of every *4-CM M*, we can decide which regions in it are either inside or outside each curve of a supposed *2-DSCC_M*, as indicated in the table below:

Color of the Region In *M*	Is the Region Inside the *Blue DSCC*?	Is the Region Inside the *Yellow DSCC*?
00 (■)	No	No
01 (■)	No	Yes
10 (■)	Yes	No
11 (■)	Yes	Yes

Table 2.2 How to position regions in a *2-DSCC_M* accordingly their coloring

Thus, the *Blue DSCC* is formed by all edges adjacent to regions colored with color 10 (■) or 11 (■), when they are also adjacent to regions colored with some from the other two colors, 00 (■) or 01 (■), and the *Yellow DSCC* is formed by all edges adjacent to regions colored with color 01 (■) or 11 (■), when they are also adjacent to regions colored with some from the other two colors, 00 (■) or 10 (■).

Then, from the table above, we can decide which edges in every *4-CM* form each curve of that supposed *2-DSCC_M*, as indicated in the table below:

Edge Adjacent to Colored Regions	Is the Edge in the *Blue DSCC*?	Is the Edge in the *Yellow DSCC*?
00 (■) | 01 (■)	No	Yes
00 (■) | 10 (■)	Yes	No
00 (■) | 11 (■)	Yes	Yes
01 (■) | 10 (■)	Yes	Yes
01 (■) | 11 (■)	Yes	No
10 (■) | 11 (■)	No	Yes

Table 2.3 How to decide pertinacity of edges in *DSCCs* accordingly their adjacency

Proposition 2.1. In order to construct the *Blue DSCC* of the map, we can make a new map *M'* simply excluding from the original *4-CM* all the edges that are not in this *DSCC*, that is, those that are adjacent to regions colored with 00 (■) and 01 (■), and those that are adjacent to regions colored with 10 (■) and 11 (■), coloring the resulting regions of *M'* with 00 (■) and 10 (■), respectively, and also coloring with 00 (■) all possible remaining regions colored with 01 (■). Therefore, *M'* turns out to be a proper *2-colored* map, since it is impossible that two new regions with the same new color are adjacent in *M'*.

Proof. Suppose, by absurdity, that *M'* after new coloring turns out to have two adjacent regions colored with 00 (■). That is, we have in *M'* a region formed with two adjacent ones from *M* originally colored with 00 (■) and 01 (■), either adjacent to other region formed with two adjacent ones from *M* originally colored with 00 (■) and 01 (■) too, or formed with only one from *M* originally colored with 01 (■). See that the unique possibility in order to this fact to happen is that the edge adjacent to those regions in *M'* is adjacent to regions colored with 00 (■) and 01 (■) in *M*, but this is impossible, since this kind of edge must be excluded in the formation of *M'*, accordingly table above.

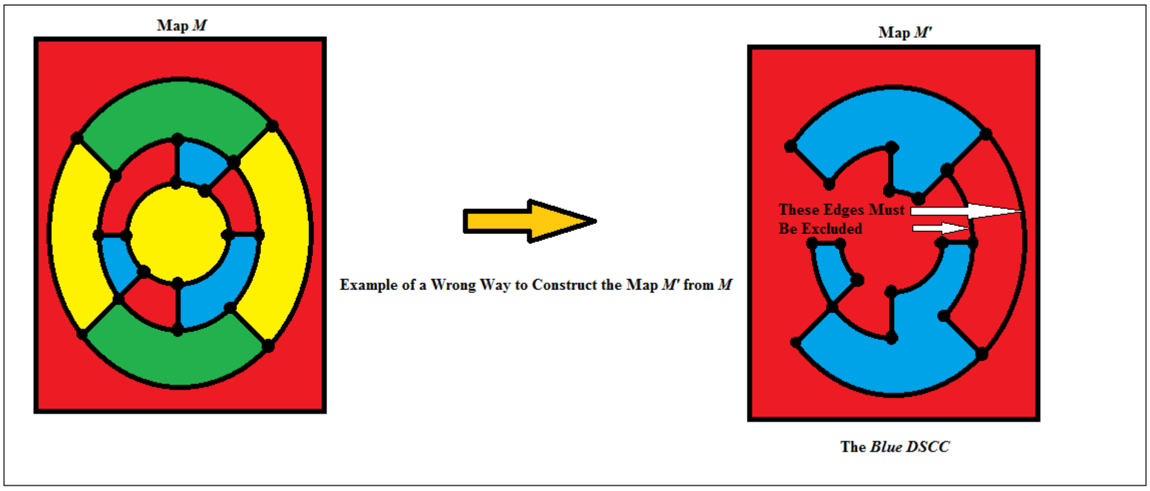

Figure 2.2 Example of How *NOT* to Construct a *Blue and* a *Yellow DSCC* from a 4-CM

On the other hand, suppose, by absurdity too, that *M'* after new coloring turns out to have two adjacent regions colored with 10 (■). That is, we have in *M'* a region formed with two adjacent ones from *M* originally colored with 10 (■) and 11 (■), adjacent to other region formed with two adjacent ones from *M* originally colored with 10 (■) and 11 (■) too. The unique possibility in order to this fact to happen is that the edge adjacent to those regions in *M'* is adjacent to regions colored with 10 (■) and 11 (■) in *M*; but this is impossible, since this kind of edge must be excluded in the formation of *M'*, accordingly table above. ☐

Now, those remaining edges in *M'* form at least one closed curve, since all the vertices from a *2-colorable* map must have an even degree (otherwise, it could not be *2-colorable*, of course), therefore they form one or more *Eulerian cycles*, [5, 16] which can be represented by a set of disjoint closed curves (since *Eulerian cycles* neither pass by any edge of the map more than once, nor cross any another one at all, so those closed curves do not also cross another ones, at all, obeying the Def. 2.1). Then, we can call this set *the Blue DSCC* of the map. Similarly by symmetry, reversing the roles of the colors 10 (■) and 01 (■) in the above argument, we see the same process shall build *the Yellow DSCC* of the *2-DSCC_M*.

Notice that all the edges of the *4-CM* must be in some *DSCC*, since the conditions of the edges forming each one of them exhaust all the edges in the map (the edges excluded in the formation of a *DSCC* are not excluded in the formation of the other one, thus all the edges of the map shall eventually participate (must do it) in the formation of some *DSCC*, after all).

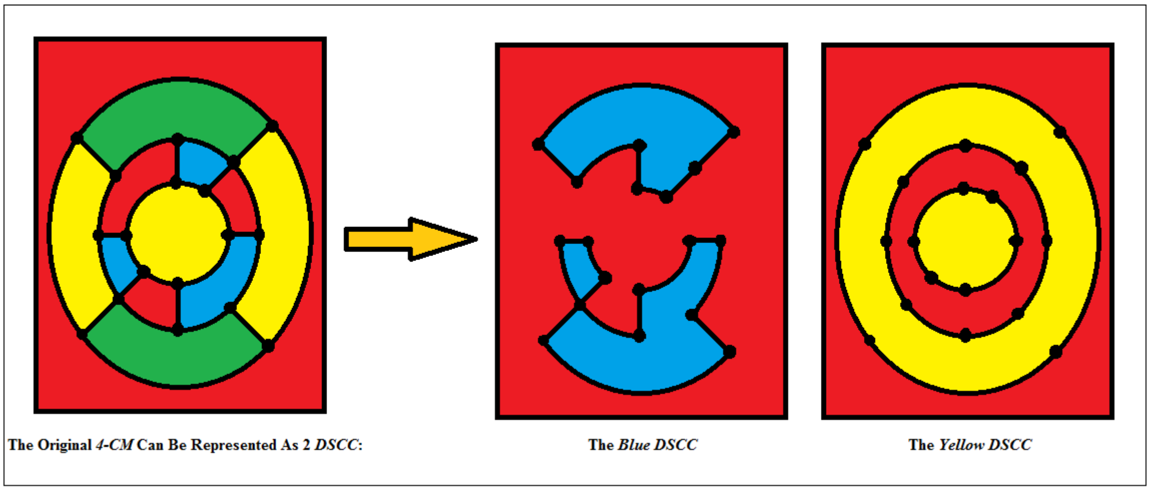

Figure 2.3 Example 1 of How to Construct a *Blue and* a *Yellow DSCC* from a 4-CM

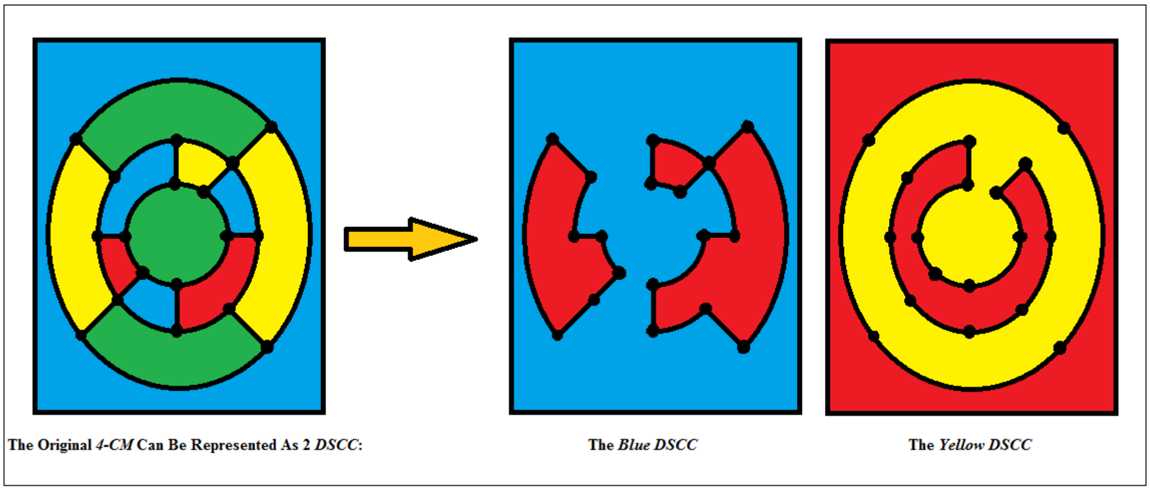

Figure 2.4 Example 2 of How to Construct a *Blue and* a *Yellow DSCC* from a 4-CM

Hence, by the construction above, every *4-CM* is also a *2-DSCC_M*. ☐

Lemma 2.6. The quantity of blue (respect., yellow) plus green incident edges to every vertex from a *2-DSCC_M* must be even. A *2-DSCC_M* colored in this way is called *properly edge-colored*.

Proof. As all curves from an arbitrary *DSCC* are closed curves, by Def. 2.1, and all green edges represent overlap of two edges, one blue and the other yellow, the quantity of incident blue (respect., yellow) plus green edges on every vertex of an arbitrary 4-CM is (must be) even, for the edges in each *DSCC* always occur in pairs at the vertices, one coming and other going out of each vertex of the *2-DSCC_M*. ☐

Corollary 2.1. Every 3-degree vertex from a properly edge-colored *2-DSCC_M* can only have the three incident edges on it colored with blue, yellow and green (where the order does not matter), that is, it is a properly 3-edge-colored vertex.

Definition 2.8. CBG. A ***CBG*** is a connected finite simple planar cubic bridgeless graph (or a cubic polyhedral map). [6, 16]

Definition 2.9. 3-ECC. A ***3-ECC*** is a *CBG* that admits a *Tait coloring* (a proper 3-edge coloring). [6, 16]

Definition 2.10. Blue-green, yellow-green and blue-yellow chain. A *blue-green* (respect., *yellow-green* or *blue-yellow*) *chain* is a cycle in a *2-DSCC_M* that contains only blue (respect., yellow) and green (respect., yellow) edges. Note that it is allowed that there are adjacent edges with the same color in that cycle (when the map is <u>not</u> cubic, of course).

Definition 2.11. Local inversion of colors. A *local inversion of colors* is the exchanging or swapping a color for the other one on all the edges in a [blue-green, yellow-green, or blue-yellow] chain.

Lemma 2.7. If two edges are in a blue (respect., yellow) simple cycle (closed walk without repetitions of vertices) [9] of a curve of a *DSCC* from a *2-DSCC_M* and are incident to the 5-edged vertex ***A***, where all the other vertices of this *2-DSCC_M* are 3-edged, then there is a blue-green (respect., yellow-green) chain containing them in that *2-DSCC_M*.

Proof. As all curves from an arbitrary *DSCC* are closed curves, and all the green edges represent overlap of a blue and a yellow edges, when we walk at that blue (respect., yellow) simple cycle, all the edges that we can pass by (only once, for all others vertices are 3-edged, but ***A***) are either blue (respect., yellow) or green edge (when that blue (respect., yellow) curve intersects a yellow (respect., blue) one at infinitely many points), that is, we are walking only upon blue (respect., yellow) and green edges, where this closed walking forms a blue-green (respect., yellow-green) chain. ☐

Notice that in general the blue-green (respect., yellow-green) chains herein can have adjacent edges with the same color, by Def. 2.4, but in a 3-edge properly colored *3-ECC* this is <u>not</u> possible at all, where every adjacent edges must have alternating color, since otherwise would be two edges with the same color focusing (abutting) on a same properly colored 3-edged vertex, which is not possible at all, by Corollary 2.1.

Lemma 2.8. Every ***3-ECC*** is also a ***4-CM*.**

Proof. By *CBG*'s Tait coloring, every *3-ECC* is also a *4-CM* [10, 16]. □

See that from a *3-ECC* we can generate a *2-DSCC_M* by ascribing all its blue-green (respect., yellow-green) chains to blue (respect., yellow) curves, generating so a *2-DSCC_M*.

Corollary 2.2. Every ***3-ECC*** is also a ***2-DSCC_M*** (***3-ECC*** ⇒ ***4-CM*** ⇔ ***2-DSCC_M***), by Lemmas 2.4, 2.5 and 2.8.

Theorem 2.1. Four-Color Theorem (4CT). Every connected finite simple planar graph is a ***4-CM***. [1, 16]

Proof. As very well known, 4CT is equivalent to Theorem 2.2 below [7, 11, 16]:

Theorem 2.2. Three Edge-Coloring Theorem (3-ECT). Every *CBG* is a *3-ECC*.

Therefore, we shall prove the Theorem 2.2, so the Theorem 2.1. So, as it is very well known too, since every finite map that has no region completely surrounded by another region can be converted into a cubic map, a hypothetical minimal counter-example to Theorem 2.1 (and to Theorem 2.2 too) would be a *CBG* ***N*** that has a pentagon inside it. [2, 16]

Thus, the resulting map ***C*** when that pentagon is removed from ***N*** is a *4-CM*, since ***N*** is hypothetically *minimal*, as demonstrated by the sequence of Figs. 2.5, 2.6 and 2.7 below (for if ***C'*** in the Figure 2.5 below [***C'*** = ***N*** without an edge from that pentagon] was not four-colorable, then it would be a counter-example smaller than ***N***, which is impossible, as ***N*** is minimal, by hypothesis); in consequence, ***C*** is also a *2-DSCC_M*, by Lemma 2.5:

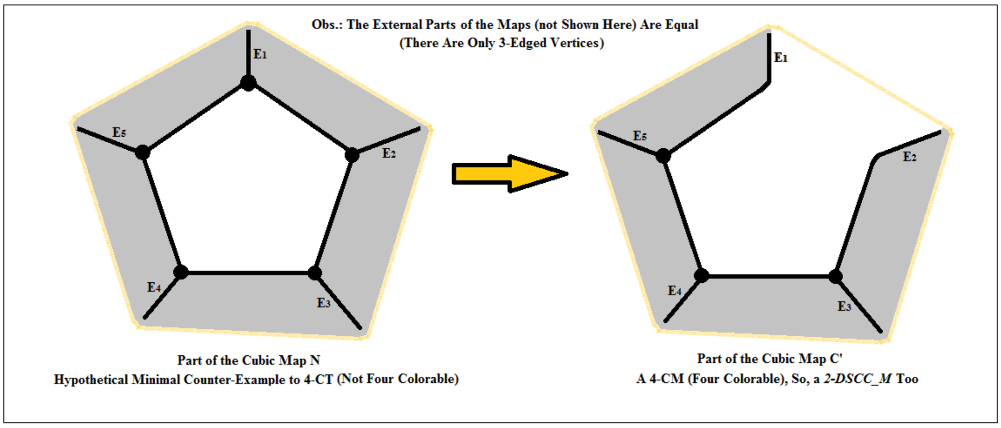

Figure 2.5 Demonstration that from a hypothetical minimal counter-example we can create a smaller *3-ECC*

Verify above that, as ***N*** is a cubic map, then ***C'*** is a cubic map too:

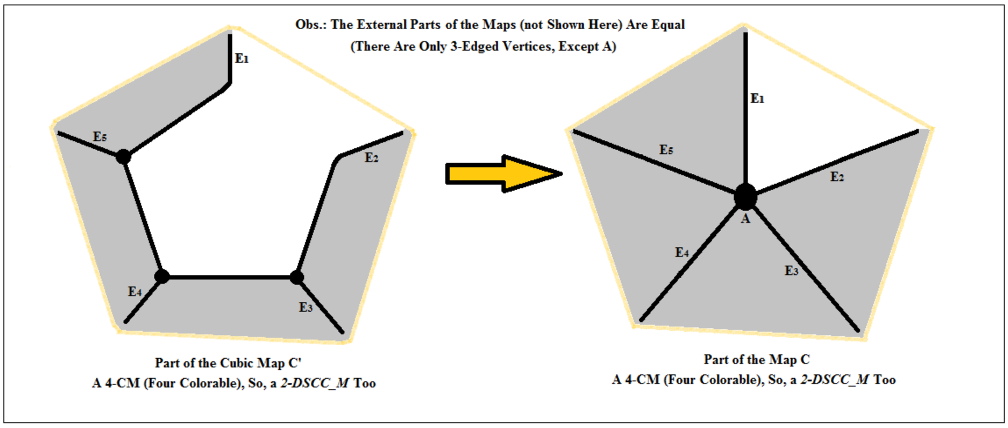

Figure 2.6 Demonstration that from a *3-ECC* we can create a *2-DSCC_M*

Then, mixing the two previous figures, we reach the result seen in the Fig. 2.7 below, where from ***N*** we construct ***C*** (that is not a cubic map, unlike ***C'***), when that pentagonal face is contracted:

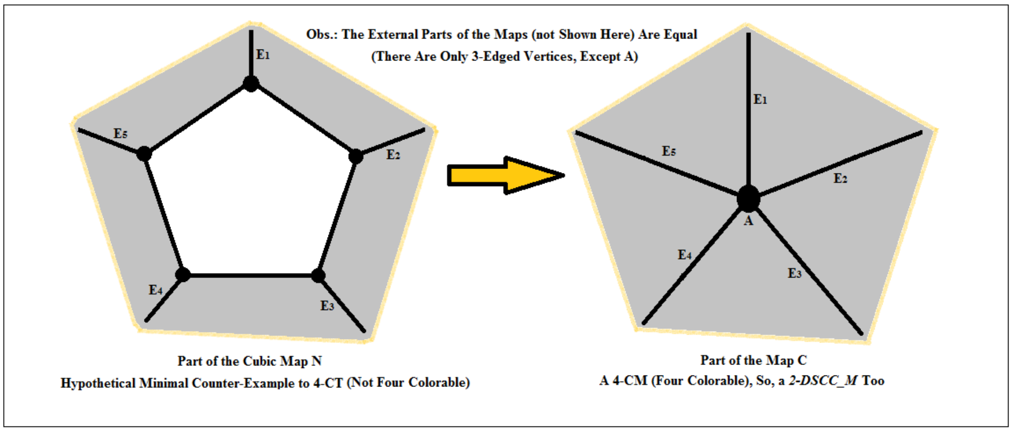

Figure 2.7 Demonstration that from a hypothetical minimal counter-example we can create a *2-DSCC_M*

In the resulting *2-DSCC_M* at the right in the Fig. 2.7 above (note that a *2-DSCC_M* does not need to be cubic, by Def. 2.5), the quantity of blue (respect., yellow) plus green incident edges to the vertex **A** must be even (by the Lemma 2.6), where the only quantities allowed are 3-1-1 (one color appears three times and the other ones do it once), as demonstrated by exhaustion in the table below, where **σ** represents an arbitrary permutation of those five edges:

$E_{\sigma(1)}$	$E_{\sigma(2)}$	$E_{\sigma(3)}$	$E_{\sigma(4)}$	$E_{\sigma(5)}$	Qty. Blue	Qty. Yellow	Allowed?
Blue	Blue	Blue	Blue	Blue	Odd (5)	Even (0)	No
Blue	Blue	Blue	Blue	Yellow	Even (4)	Odd (1)	No
…	…	…	…	…	…	…	… No …
…	…	…	…	…	…	…	… No …
Blue	Blue	Blue	Yellow	Green	Even (4)	Even (2)	Yes
Blue	Yellow	Yellow	Yellow	Green	Even (2)	Even (4)	Yes
Blue	Yellow	Green	Green	Green	Even (4)	Even (4)	Yes

Table 2.4 Quantities Allowed of Colored-Edges Incidents to the Vertex ***A*** of the Map ***C*** in Fig. 2.8

We shall analyze only the case with three blue edges, 1 green and 1 yellow one, since the other cases are only permutations of those colors, leading to the same results, by symmetry (rotation or reflection of the map, or global inversion of some pair of colors).

If those three blue edges are contiguously incident to the vertex ***A*** (that is, we can draw a continuous line crossing all these three edges without crossing any other edge from the map), as shown in the Fig. 2.8 below (see that this Fig. 2.8 embraces all the possible cases where three edges with the same color are contiguously incident to the vertex ***A***, up to the colors and their positioning), we can create a new region over this vertex, returning to the original map ***N***, but now proving that it is really a *3-ECC* at all, as demonstrated in that Fig. 2.8, since ***C*** is a *2-DSCC_M*, so ***N*** is really a *4-CM* too, by Corollary 2.2. See why into more details below:

Well, as all vertices from that map ***C***, except the vertex ***A***, are properly 3-edge colored, by Corollary 2.1, when we include that new region over the vertex ***A***, we demonstrate that that map ***N*** is also a *3-ECC*, as the five new vertices replacing the vertex ***A*** are also all properly 3-edge colored (see it at Fig. 2.8 below), and all the remaining vertices of ***N*** are properly 3-edge colored too, because the remaining external part of ***N*** is equal to the remaining external part of ***C*** (not explicitly shown in that Fig. 2.8):

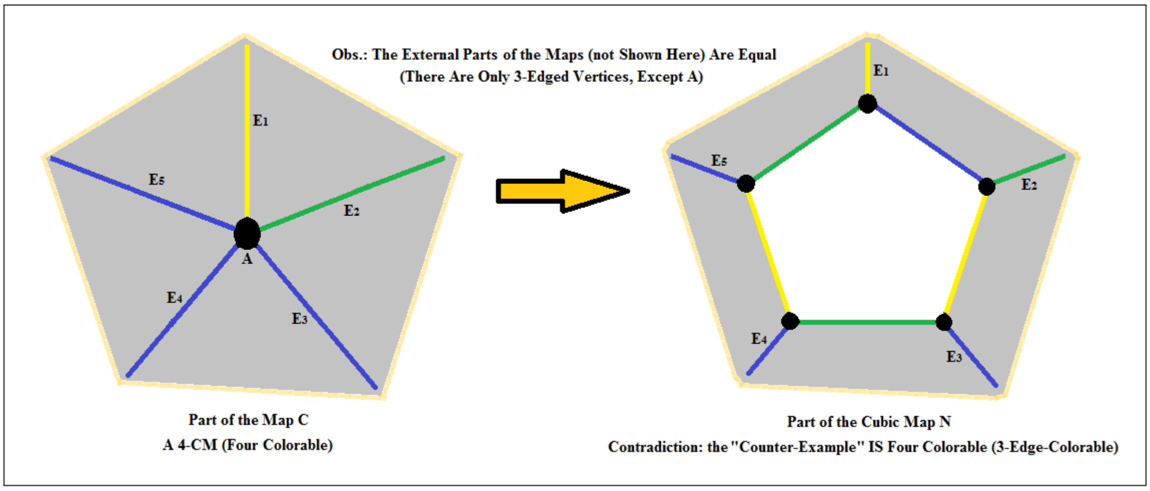

Figure 2.8 Demonstration that a supposed minimal counter-example (***N***) is NOT really a counter-example

So, because we can delete a region from the map ***N***, then properly color that resulting map ***C*** with only four colors, and then return that region, generating a *3-ECC*, so also a *4-CM*, by Corollary 2.2, this process proves that the original map ***N*** cannot be a true minimal counter-example to the 4CT, which generates a contradiction, for our thesis is that ***N*** is so. So, the initial assumption that three blue (or yellow/green) edges are contiguously incident to the vertex ***A*** is (must be) wrong, in order to maintain our thesis (even though temporarily, as we'll see).

Consequently, the three blue (or yellow/green) edges are not (cannot be) contiguously incident to the vertex ***A*** in the map ***C***; hence it must be like shown in the Fig. 2.9 below:

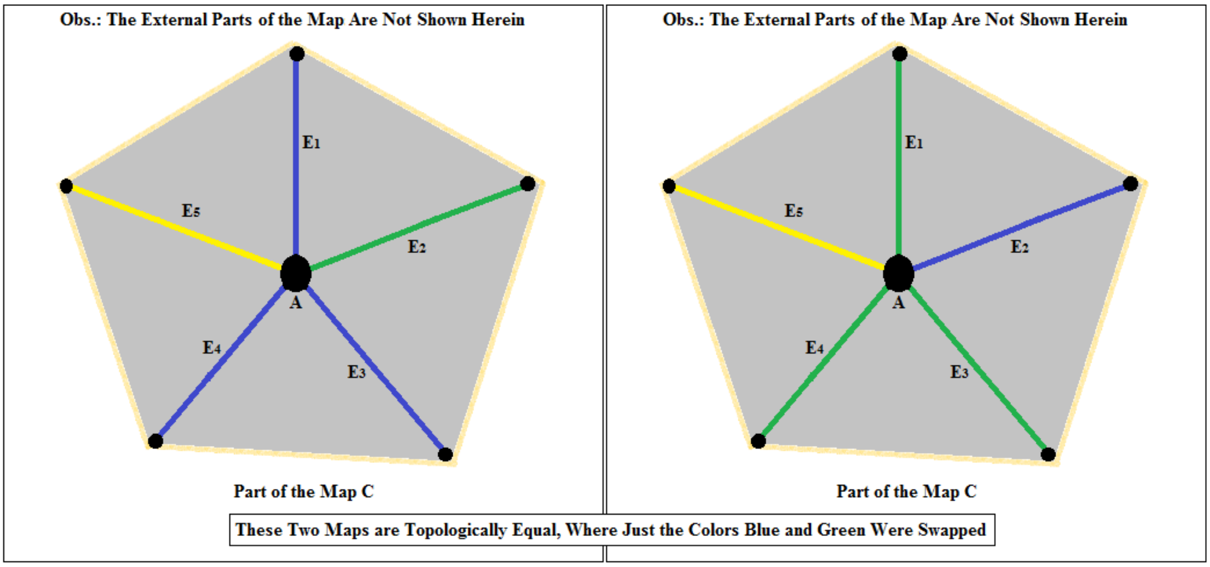

Figure 2.9 In order to ***N*** can be a minimal counter-example to the 4CT, ***C*** must have this kind of coloring

So, two possible topologies (structures or conformations) of the blue curve from that map ***C*** (where the edges E_1, E_3 and E_4 are blue, E_2 is green, and E_5 is yellow [where alternatively, if E_1, E_3 and E_4 are green (respect., yellow), E_2 is blue (respect., green), and E_5 is yellow (respect., blue), we swap the colors blue and green (respect., yellow), generating a topologically identical properly 4-colorable map]) are represented below, since those blue edges must belong to the blue curve(s), and that (those) blue curve(s) must pass by the vertex ***A***, by all the other five vertices surrounding it, and by those four edges, three blue (E_1, E_3 and E_4) and one green ones (E_2) (remembering that a green edge is an overlapped edge composed by a piece from a blue curve and another one from a yellow one):

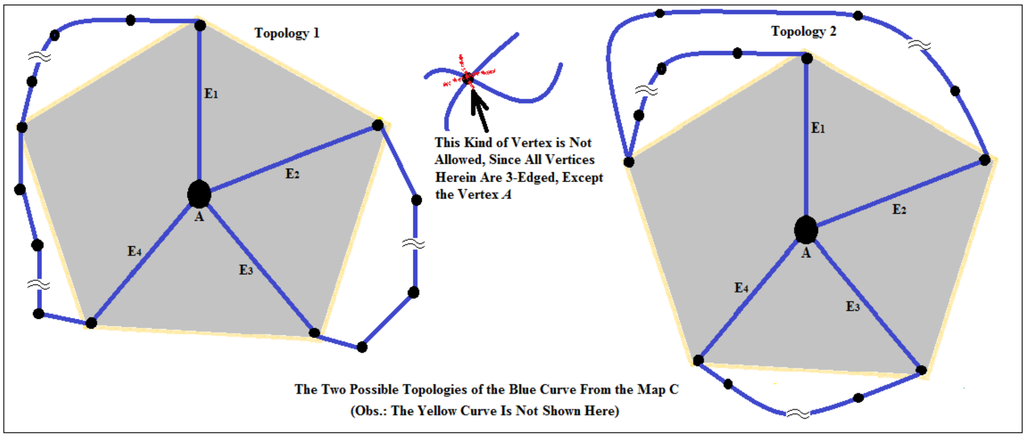

Figure 2.10 Two possible topologies of the blue curve that pass at vertex ***A*** from the map ***C***

Note that there are also two other possible topologies, 1’ and 2’, but they are essentially the same as those 1 and 2 represented in the Fig. 2.5 above respectively, since all following conclusions are completely applicable when we replace the topology 1 by the 1’, or the topology 2 by the 2’ (obs.: verify by exhaustion that there is no other possible ones, at all):

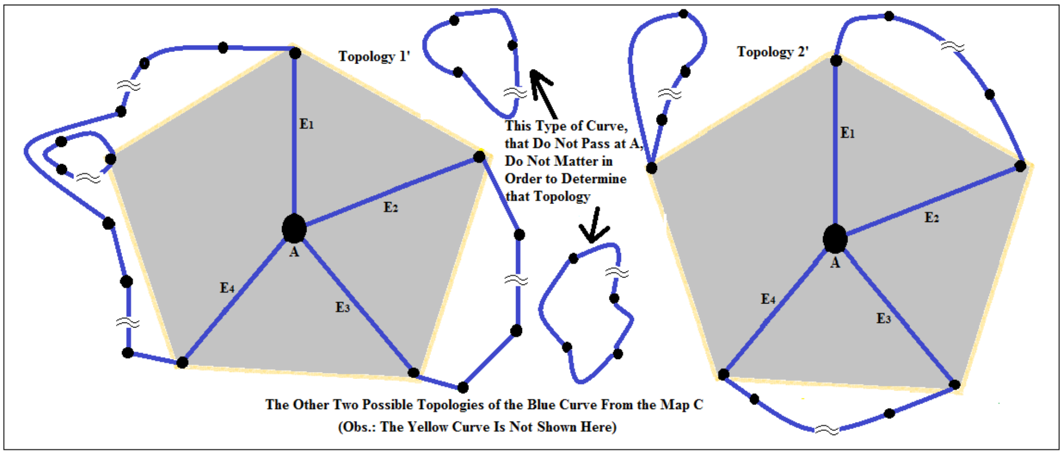

Figure 2.11 Other two possible topologies of the blue curves that pass by vertex ***A*** of the map ***C***

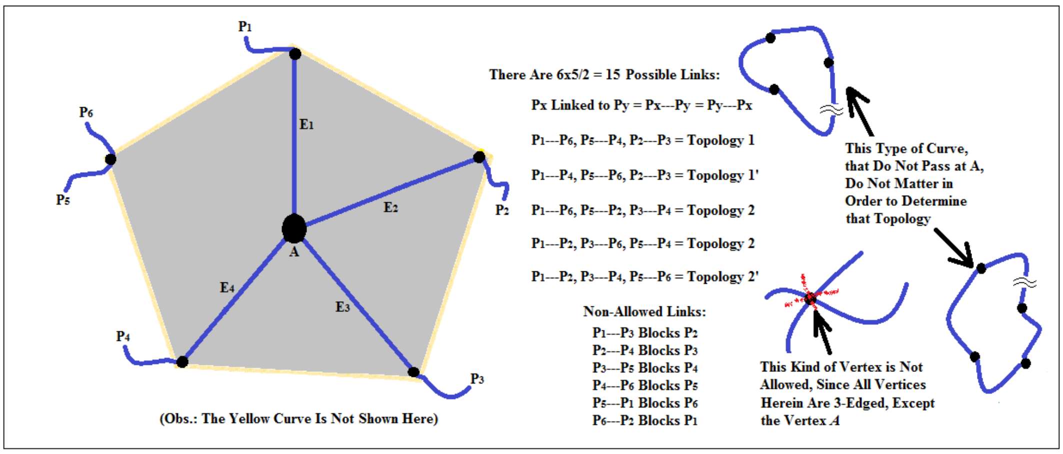

Figure 2.12 Demonstration that there are herein only those four possible topologies (1, 1’, 2 and 2’)

Note: Furthermore, it can be other blue curves that do not pass at any of these six vertices, but these curves, even though they do exist, do not matter at all with respect to the topologies above and the proof in this paper, for the arguments herein utilized shall be demonstrated true ones independently of the existence or nonexistence of such curves. Verify yet that there is no other possible topologies for any curve (or curves) that pass at all those six vertices and four edges, at all: it is enough to think of a torn blue curve with loose ends at those six vertices and four edges and exhaustively try to link those loose ends (P_1, P_2, P_3, P_4, P_5 and P_6 in the Fig. 2.12 above) of the curve in order to fix the entire curve into a closed one, with neither intersection nor crossing of it with itself (nor with another one with the same color), except at vertex ***A***, as demonstrated in the Fig. 2.12 above.

If it was topology 2 (or 2'), however, then would exist a blue-green chain including E_1 and E_2 (by Lemma 2.7, since these edges are contained in a simple cycle, as demonstrated in the Fig. 2.10 (and in the Fig. 2.11, w.r.t. topology 2')), which would permit a local inversion of these colors in this chain (as in Def. 2.11), generating a 3-edge-coloring map with three blue edges contiguously incident to the vertex ***A***, allowing for creation of more one region and continuing so, properly 3-edge-coloring (as in the Figure 2.8), which also would imply that that map ***N*** could not be at all a true minimal counter-example to the 4CT:

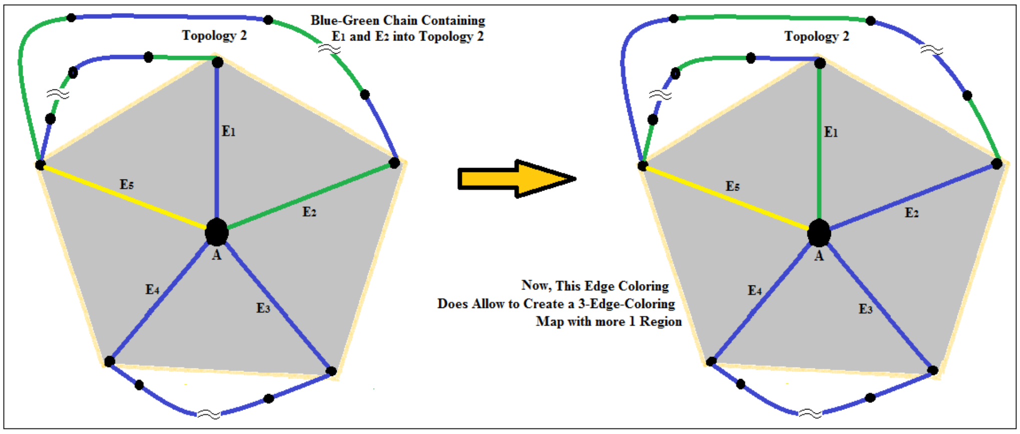

Figure 2.13 Locally inverting colors in a blue-green chain resulting a map as one in the Fig. 2.8

Hence, as that map ***N*** is a minimal counter-example to the 4CT, by hypothesis, that topology is (must be) the 1 (or the 1').

So, as the three blue edges are not contiguously incident to the vertex ***A***, and they are positioned according to topology 1 (or 1'), we can locally invert the colors blue and yellow in a blue-yellow chain containing E_1 following exactly one from the three ways below:

1. The blue-yellow chain contains the edges E_1 and E_5, as demonstrated in the Fig. 2.14 below, which generates a map ***C*** coloring identical to the one in the Fig. 2.8, which implies that same conclusion (that map ***N*** cannot be a real minimal counter-example to the 4CT), then generating that same kind of contradiction with our thesis, that that map ***N*** is a true minimal counter-example to the 4CT:

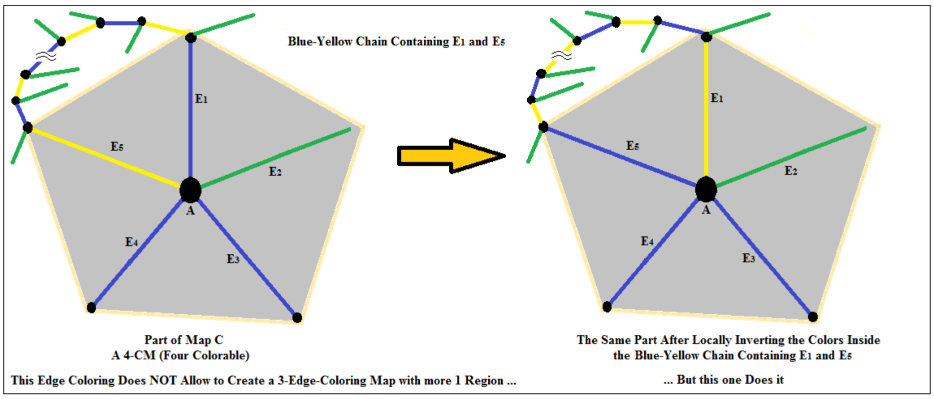

Figure 2.14 Locally inverting colors in a blue-green chain resulting a map as one in the Fig. 2.8

2. The blue-yellow chain contains the edges E_1 and E_4, as demonstrated in the Fig. 2.15 below, which also generates a map C coloring identical to the one in the Fig. 2.8 (up to the colors and positioning of those figures), which implies that same conclusion (that map ***N*** cannot be a minimal counter-example to the 4CT), then generating that same kind of contradiction with our thesis, that that map ***N*** is a true minimal counter-example to the 4CT:

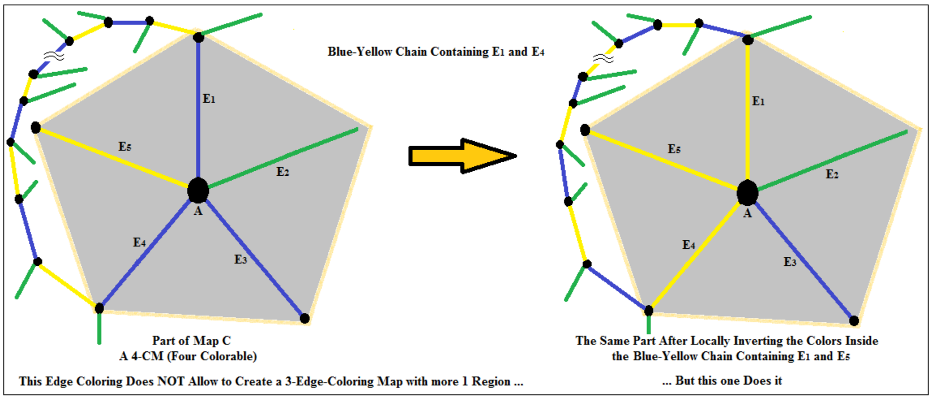

Figure 2.15 Locally inverting colors in a blue-yellow chain resulting a map as one in the Fig. 2.8

3. The blue-yellow chain contains the edges E_1 and E_3, as demonstrated in the Fig. 2.16 below. Now, however, the resulting map ***C*** coloring is not as to one in the Fig. 2.8, whereby that conclusion is not valid herein (that map ***N*** now seems right, can still be a minimal counter-example to the 4CT). We call this *local inversion* $\boldsymbol{L_1}$:

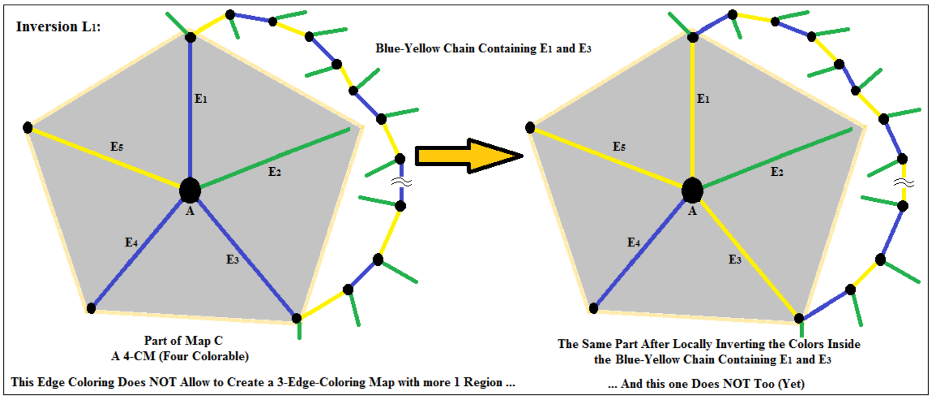

Figure 2.16 Locally inverting colors in a yellow-green chain resulting a map like one in the Fig. 2.9

Proposition 2.2. A local inversion of colors as that in Fig. 2.16 leaves the property of being *2-DSCC_M* intact on map ***C*** above, that is, after it the map *C* continues being so.

Proof. After that local inversion of colors, all the 3-edged vertices of that map *C* continues properly colored, because there was only a swap of the colors on two incident edges in each 3-edged vertex. And the 5-edged vertex ***A*** can be expanded into three properly colored 3-edged vertices, generating after all a *3-ECC*; thereby the resulting map is a *2-DSCC_M* one too, by Corollary 2.2.

Then, contracting those three vertices and returning them into the vertex ***A***, the simple closed yellow curve that passes by those three vertices intersects itself at a single point (the vertex ***A***) and transforms itself into a non-simple closed yellow curve (or two simple closed yellow curves that pass by those three vertices join themselves into only one non-simple closed yellow curve), and the simple closed blue curve that passes by those three vertices continues unchanged passing by the vertex ***A***, thereby leaving intact that property of the map *C* (being *2-DSCC_M*), as shown in the Figs. 2.17, 2.18 and 2.19 below: □

Obs.: The left map below is the same as that one at right in the Fig. 2.16 above, just drawn 144° (0.8 π rad) counter-clockwise rotated:

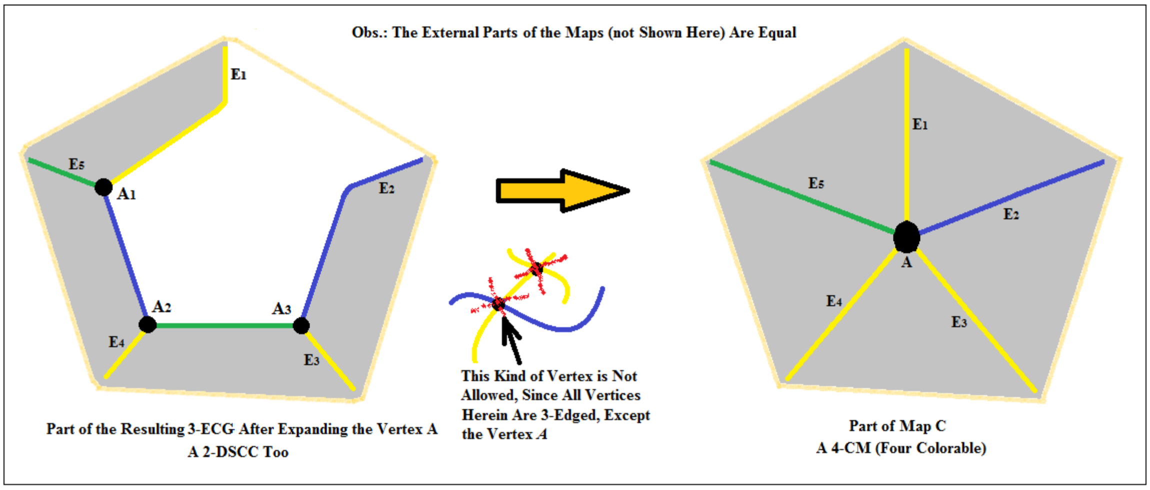

Figure 2.17 Expanding the vertex ***A*** and generating a *2-DSCC*

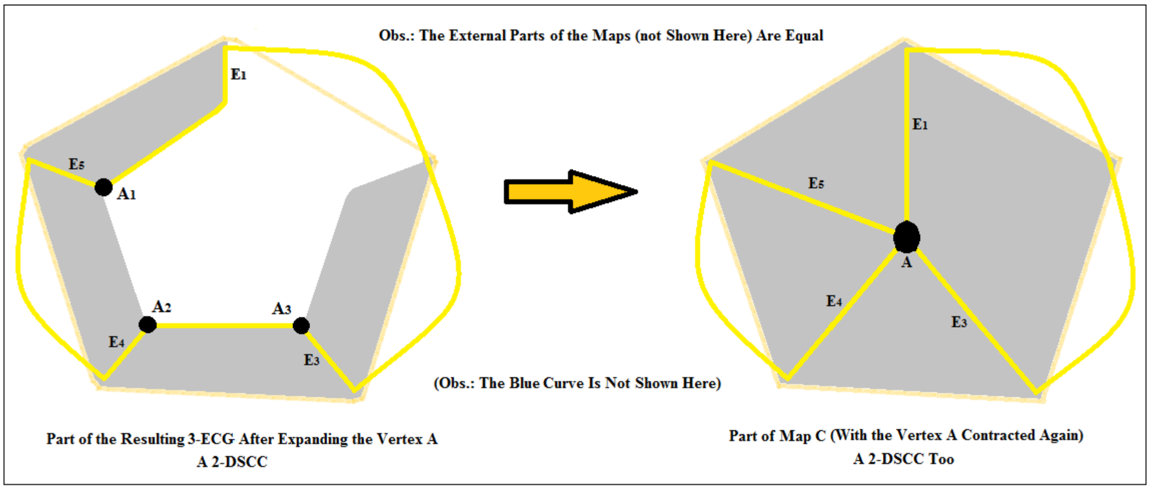

Figure 2.18 Contracting the vertex ***A*** again and showing that the map continues being a *2-DSCC*

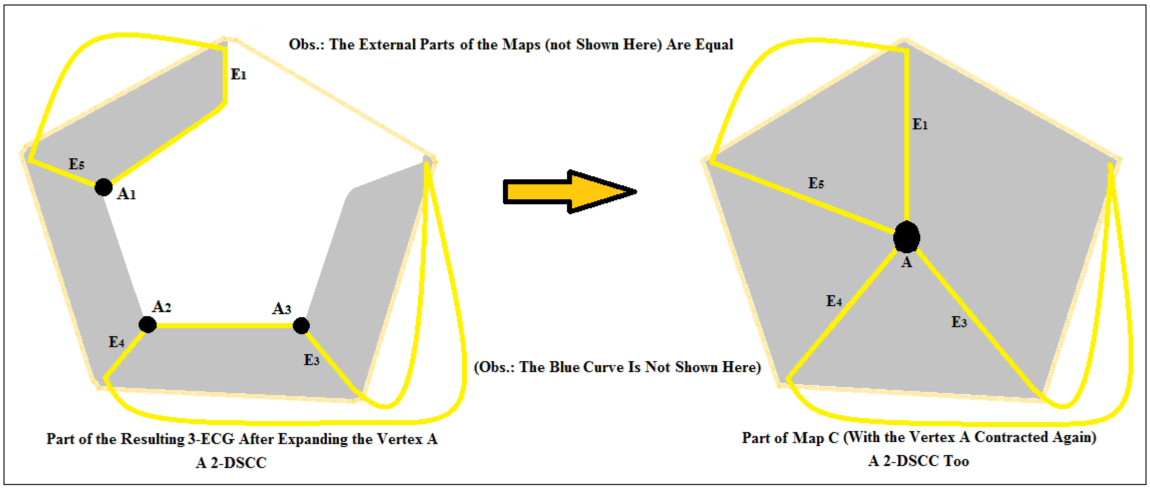

Figure 2.19 Contracting the vertex ***A*** again and showing that the map continues being a *2-DSCC*

Consequently, the two only possible topologies of the yellow curve from the resulting map ***C*** above, the only ones acceptable (where the edges E_1, E_3 and E_5 are yellow, and E_2 is green), are (must be):

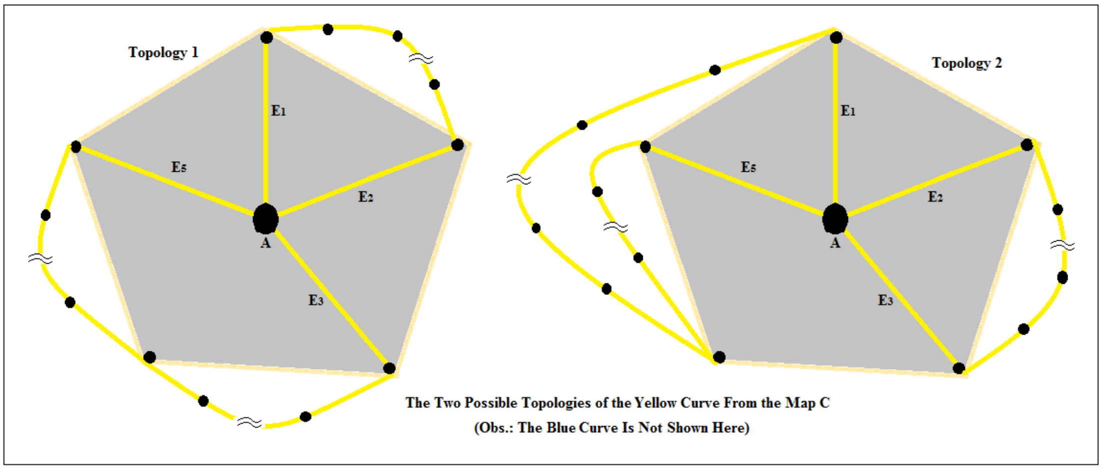

Figure 2.20 Two possible topologies of the yellow curve of the map ***C*** that pass at vertex ***A***, after $\boldsymbol{L_1}$

Note yet that, as before with that blue curves in the Fig. 2.10, here also there are two other possible topologies, 1’ and 2’, but they are essentially the same as those topologies 1 and 2 represented in the Fig. 2.20 above respectively, for the same reasons:

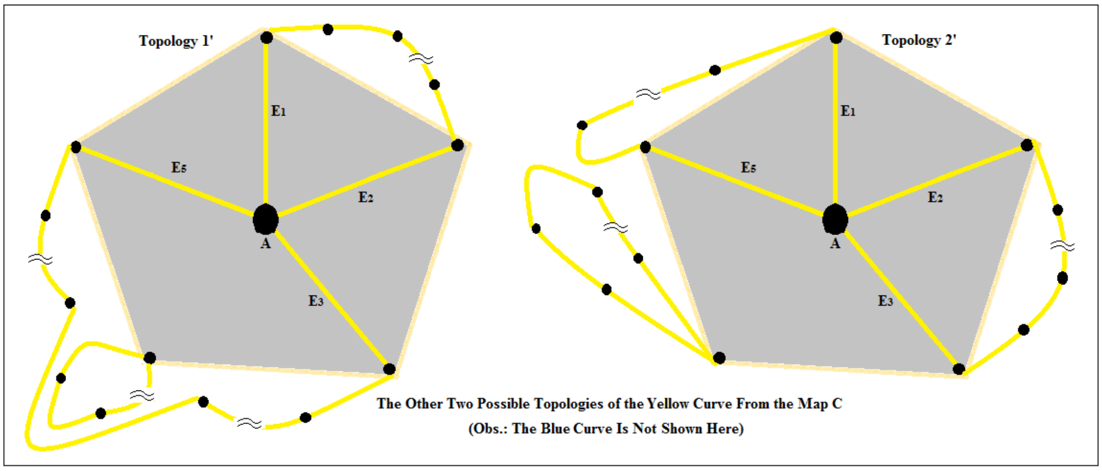

Figure 2.21 Other two possible topologies of the yellow curve of the map ***C*** that pass at vertex ***A***, after $\boldsymbol{L_1}$

If it was topology 2 (or 2’), however, then, as in Fig. 2.13, it will be a flaw too:

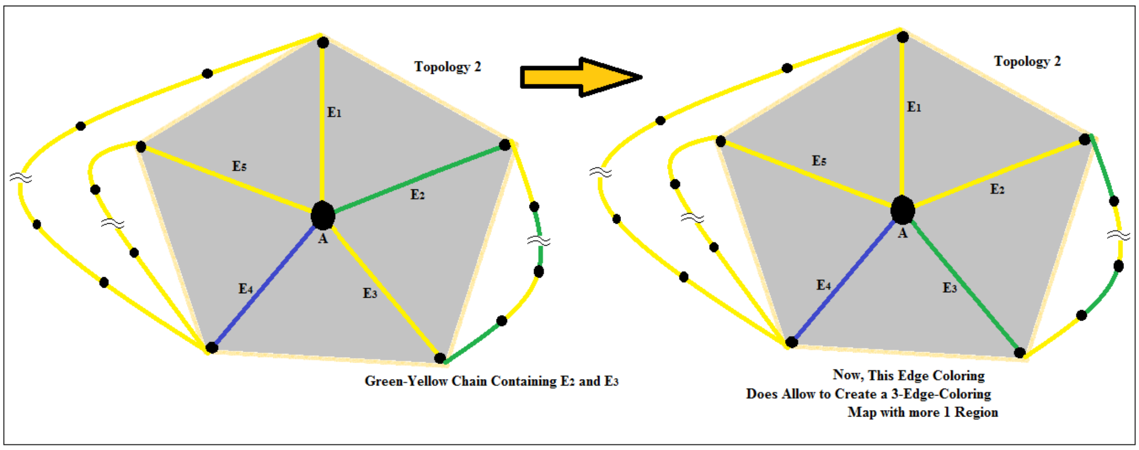

Figure 2.22 Locally inverting colors in a yellow-green chain resulting a map as one in the Fig. 2.8

Hence, as ***N*** is a minimal counter-example to the 4CT, by thesis, in order to avoid that flaw, that topology is (must be) also the 1 (or 1’). Now, upon the resulting ***C*** colored in way 3, we can locally invert the colors on the blue-yellow chain holding E_4 and E_5 (as inversion $\boldsymbol{L_2}$):

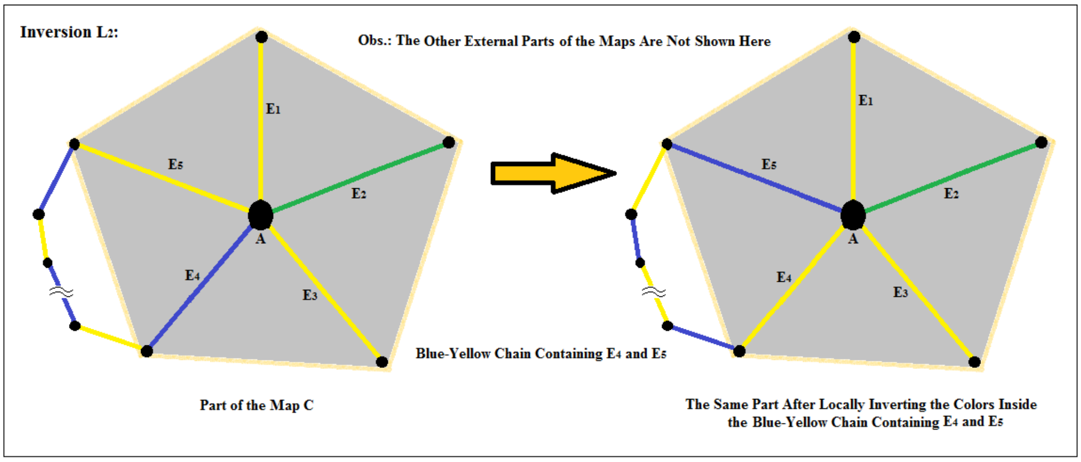

Figure 2.23 Locally inverting the colors in a blue-yellow chain, then resulting a map as one in the Fig. 2.9

Note that if the blue-yellow chain containing E_4 also contained E_1 (or E_3), then we could locally invert these colors in this chain, generating a properly 3-edge-coloring map allowing for more one region and continuing so (as in the Fig. 2.8), which also would imply that that map ***N*** could not be a minimal counter-example to the 4CT. Thus, by maintaining our thesis, this blue-yellow chain contains neither E_1 nor E_3 (it cannot contain any of them, at all).

Observe yet that the yellow-blue chain beginning in E_5 must end up in E_4 in this case, since if it did it in E_1 or E_3, then a map as in the Fig. 2.8 would be generated, when the colors from that yellow-blue chain (beginning in E_5 and ending up in either E_1 or E_3) were locally inverted (which also would imply that that map ***N*** could not be a true minimal counter-example to the 4CT).

So, as in the Fig. 2.10 above, the two possible topologies of the yellow curve from the resulting map ***C*** colored as at right in the Fig. 2.23 (where the edges E_1, E_3 and E_4 are yellow, E_2 is green, and E_5 is blue) are represented below:

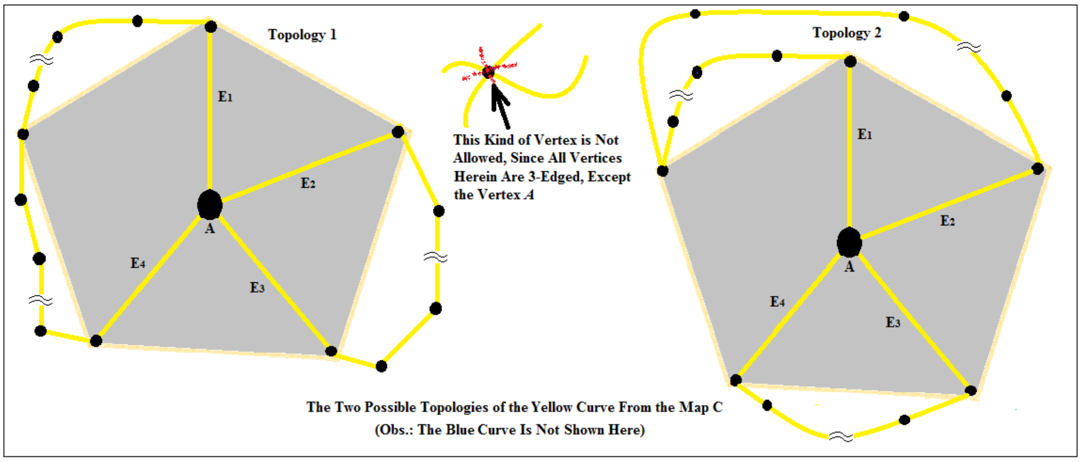

Figure 2.24 Two possible topologies of a yellow curve of the map ***C*** that pass at vertex ***A***, after ***L₂***

Again, as in the Fig. 2.11, if it was topology 2 (or 2'), then would exist a yellow-green chain including E_1 and E_2, which would permit a local inversion of these colors in this chain, generating a 3-edge-coloring map allowing for more one region and continuing so (as in the Fig. 2.8), which also would imply that that map ***N*** could not be a real minimal counter-example to the 4CT:

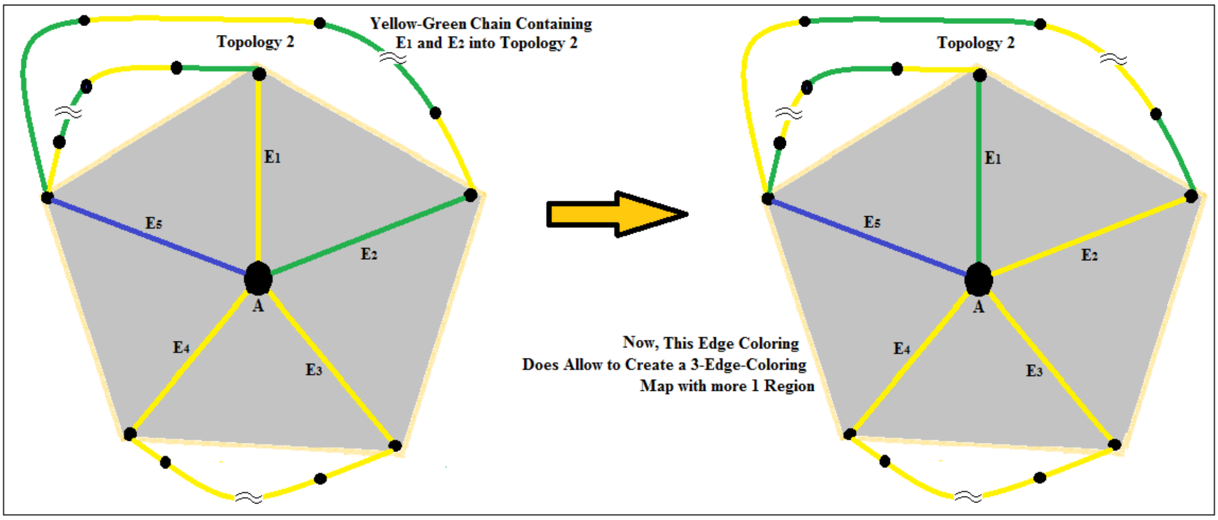

Figure 2.25 Locally inverting colors in a yellow-green chain, then resulting a map as one in the Fig. 2.8

Hence, as that map ***N*** is a minimal counter-example to the 4CT, by thesis, that topology is (must be) the 1 (or 1'). Note that these two local inversions (L_1 and L_2) must be disjoint (they cannot have any edge in common, because a blue-yellow chain cannot cross another one on a 3-edged vertex, at all, for there is only one pair of blue-yellow edges in each vertex of this type), which implies that that blue-yellow chain containing the edges E_4 and E_5, shown in the Fig. 2.23, cannot contain any edge in that blue-yellow chain containing the edges E_1 and E_3, shown in the Fig. 2.16.

Moreover, as demonstrated in the Fig. 2.29 below, even though the blue-yellow chain containing the edges E_4 and E_5 crosses the green-yellow chain containing the edges E_1 and E_2 in the map, the local inversion L_2 cannot cut the part of that yellow curve linking the edges E_1 and E_2 in that original map, in order to generate the topology 1 (or 1') into another position in this map. This part of that yellow curve is just changed in order to pass at other vertices.

However, in order to hold our thesis that map ***N*** is a minimal counter-example to the 4CT, this fact could not happen at all, since if it was so, then the link of the yellow curve *from* E_1 to E_2 could not change *to* E_2 to E_3, neither *from* E_3 to E_5 *to* E_1 to E_4, as it is mandatory in our obligatory sequence of topologies of the yellow curve after those local inversions L_1 and L_2 (when we insist holding that thesis):

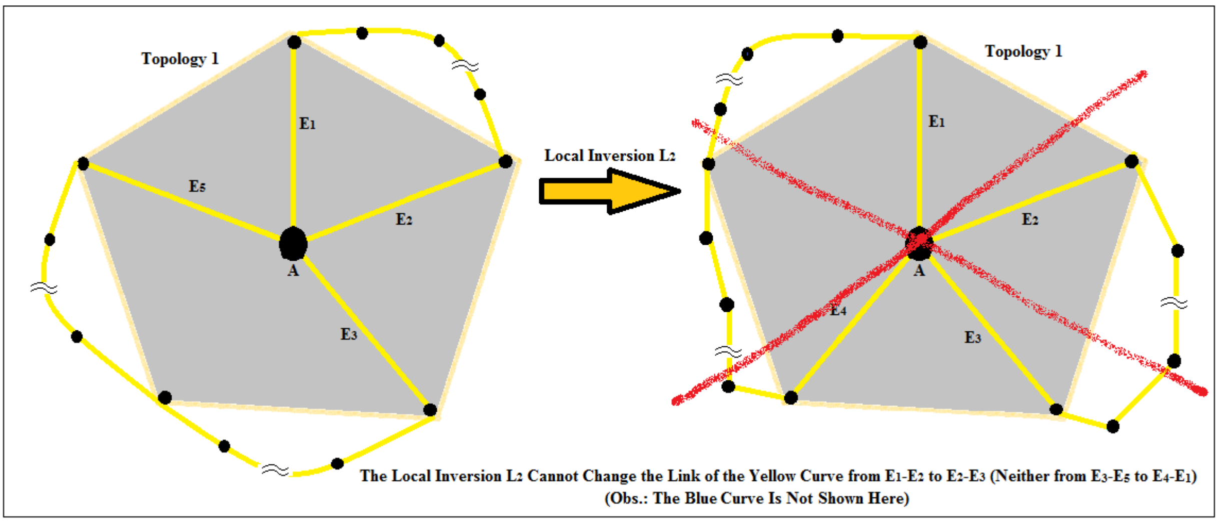

Figure 2.26 The inversion L_2 upon a map with topology 1 cannot generate a map with topology 1 (or 1')

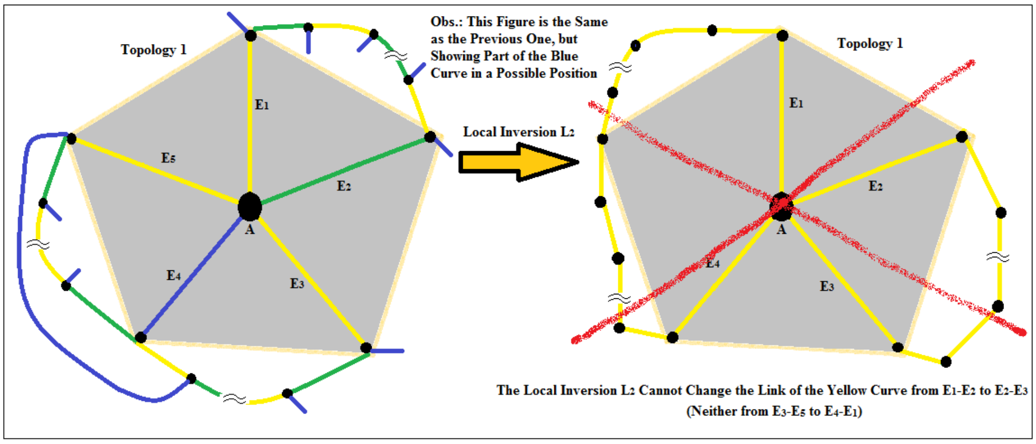

Figure 2.27 The inversion L_2 upon a map with topology 1 cannot generate a map with topology 1 (or 1')

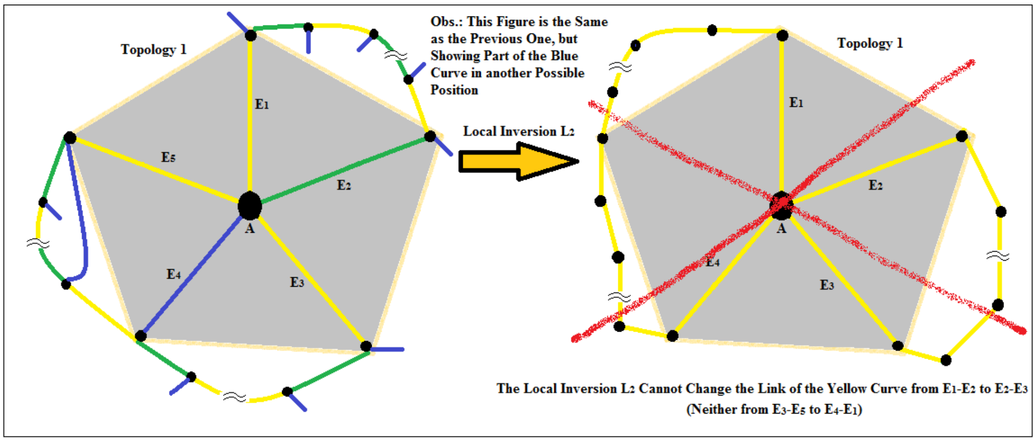

Figure 2.28 The inversion L_2 upon a map with topology 1 cannot generate a map with topology 1 (or 1')

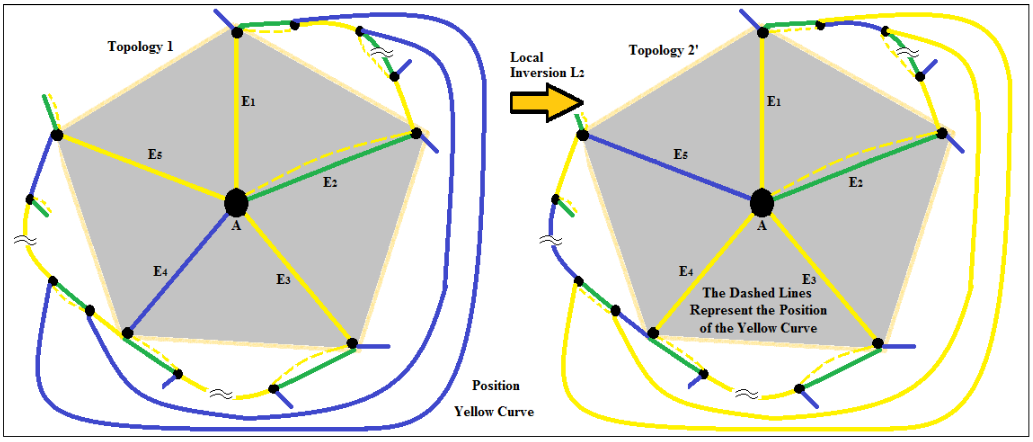

Figure 2.29 The inversion L_2 upon the previous map cannot generate a map with topology 1 (or 1')

Therefore, that thesis (affirming that N is a minimal counter-example) is (must be) false, and that minimal counter-example N cannot exist, at all, since all the allowed colorings in the Table 2.4 above lead to the fact that map N is (must be) a *3-ECC* (so, also a *4-CM*).

So, in synthesis, if the local inversion L_1 generates a map with topology 2 (or 2'), then we can invert the colors in some blue-green chain (by the Lemma 2.7, since in this case the edges E_2 and E_3, e.g., are kept in a simple cycle), generating a 3-edge-coloring map with three blue edges contiguously incident to the vertex A, letting creation of more one region in it and continuing so, properly 3-edge-coloring (as in the Fig. 2.8), which also would imply that that map N could not be at all a minimal counter-example to the 4CT, contradicting our thesis:

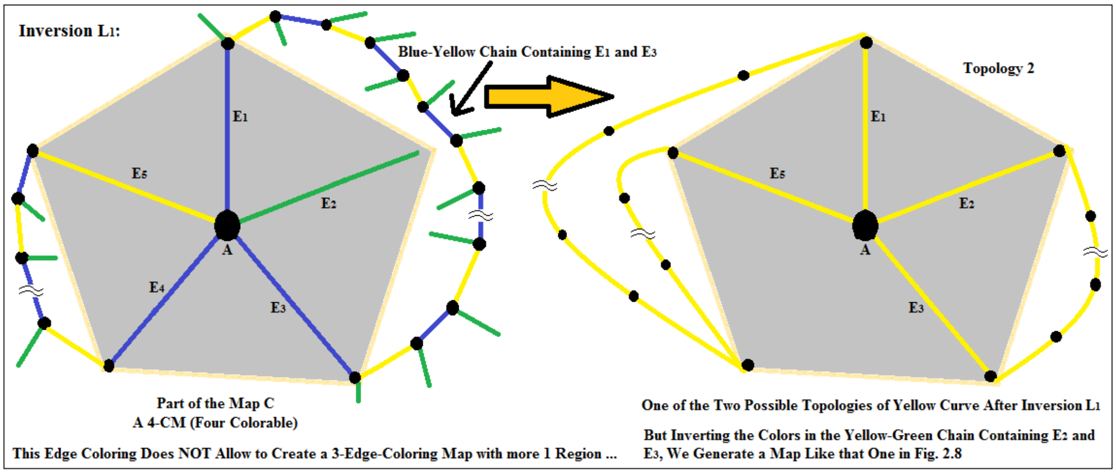

Figure 2.30 If local inversion L_1 leads to topology 2 (or 2'), then N cannot be a minimal counter-example

Hence, as the local inversion L_1 does not (cannot) generate a map with yellow curve with topology 2 (or 2'), it can only generate a map with yellow curve with topology 1 (or 1'):

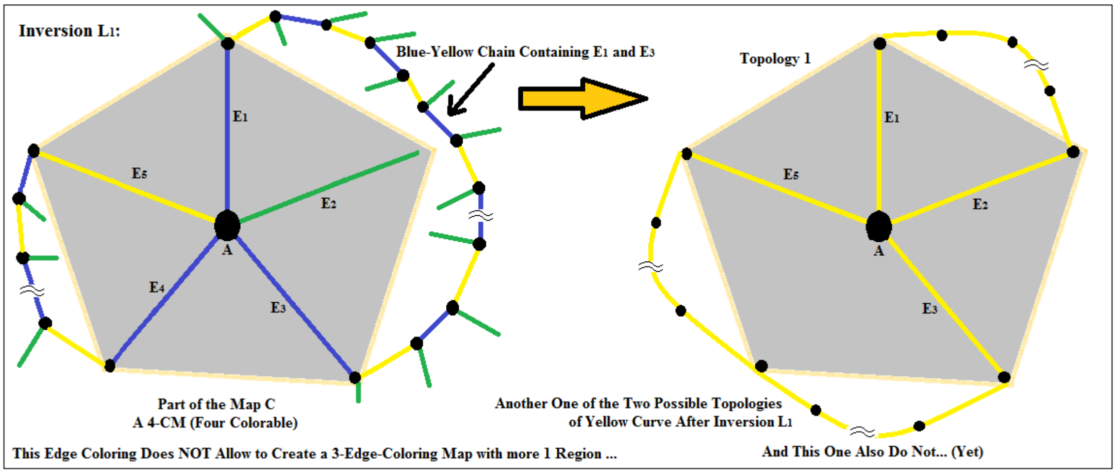

Figure 2.31 If inversion L_1 leads to topology 1, then the map N can be a minimal counter-example (yet)

Now, after the inversion L_1, we must have certainly a map with yellow curve with topology 1 (or 1').

Then, in turn after the inversion L_2, we have certainly a map with yellow curve with topology 2 (or 2'), since – as seen above – the inversion L_2 cannot from a map with yellow curve with topology 1 (or 1') create a map with yellow curve with this same topology 1 or 1':

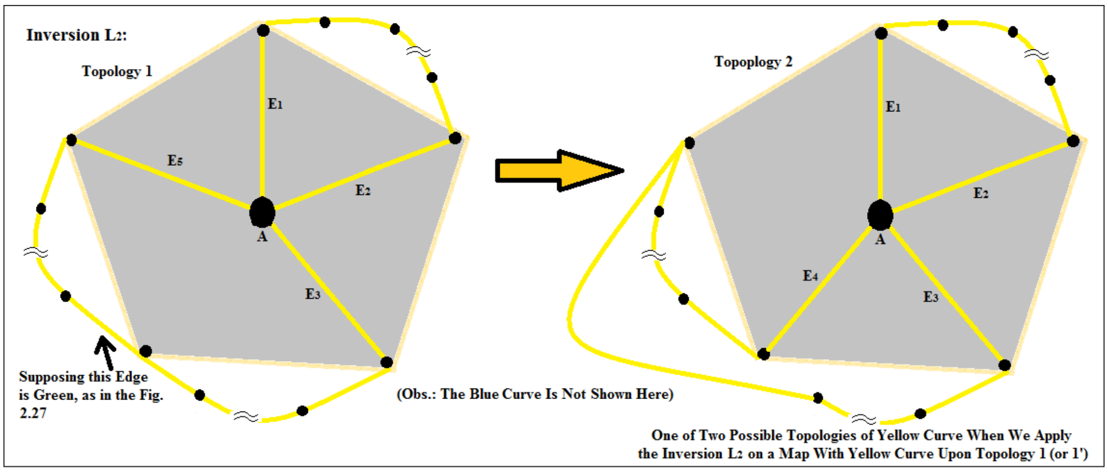

Figure 2.32 The inversion L_2 upon a map with topology 1 can only generate a map with topology 2 (or 2')

Then, with the generated map above, with topology 2, as already shown above we can generate a map as that one in Fig. 2.8, where that map ***N*** is proved cannot be an actual minimal counter-example to the 4CT.

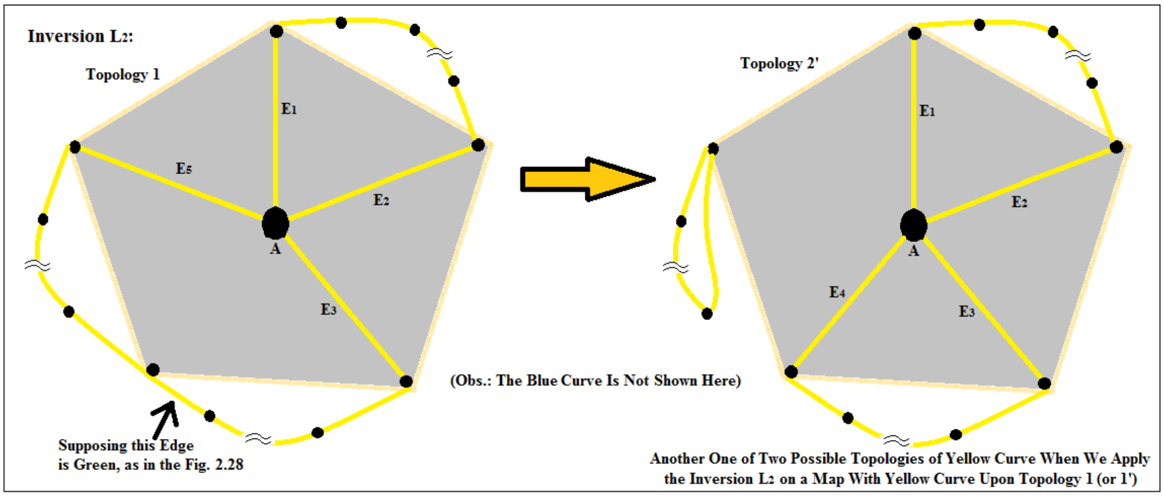

Figure 2.33 The inversion $\boldsymbol{L_2}$ upon a map with topology 1 can only generate a map with topology 2' (or 2)

Finally, with the generated map above, with topology 2', as already shown with topology 2, we can also generate a map as that one in Fig. 2.8, applying the local inversion $\boldsymbol{L_2}$, where that map ***N*** is also proved cannot be a true minimal counter-example to the 4CT.

Hence, every connected finite simple planar graph is a ***3-ECC***, so it is also a ***4-CM***. ☐

3. Conclusion & Understanding

The conclusion is that now we utterly understand [12] why the 4CT is really true: Every finite simple planar graph can be properly four-colorable because all they can be represented by only two sets of closed curves, where all the regions of that map is either inside or outside with respect to each one of these two sets, and when we cross any edge (arriving at some adjacent region) at least one of these positions must change.

So, a properly four-coloring emerges naturally, in every finite map, associating every region of the map to each one of four arbitrary colors biunivocally associated to the four possible positioning of that region:

Color of the Region	Location w.r.t. *DSCC* 1	Location w.r.t. *DSCC* 2
1	Outside	Outside
2	Outside	Inside
3	Inside	Outside
4	Inside	Inside

Table 3.1 Understanding utterly why the 4CT is really true

4. Freedom & Mathematics

"**– The essence of Mathematics is Freedom.**" (Georg Cantor) [11]

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