THE FOUR COLOR THEOREM - A NEW PROOF BY INDUCTION

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ABSTRACT. In 1976 Appel and Haken achieved a major break through by thoroughly establishing the Four Color Theorem (4CT). Their proof is based on studying a large number of cases for which a computer-assisted search for hours is required. In 1997 the 4CT was reproved with less need for computer verification by Robertson, Sanders, Seymour and Thomas. In 2000 Ashay Dharwadkar gave an algebraic proof to the 4CT involving Steiner Systems, Eilenberg Modules, Hall Matching and Riemann Surfaces. In this paper, we give a simple proof to the 4CT. The proof is based on the Principle of Mathematical Induction, contraction, and possible colorings of a minimum degree vertex, its adjacent vertices, and adjacent vertices of these adjacent vertices.

1. Introduction

The Four Color Theorem (4CT) plays a predominant role in the currently fashionable field of graph theory. The 4CT states that the vertices of every planar graph can be colored with at most four colors in such a way that any two adjacent vertices have different colors [4, 7, 9, 10, 12].

A coloring of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The set of all vertices with any one color is independent and is called a color class. An n-coloring of a graph G uses n colors; it thereby partitions V, the set of vertices into n color classes.

Heawood [8] showed that the four color problem becomes true when 'four' is replaced by 'five'. Grotzsch [5] proved that any planar graph without triangle can be colored with three colors and Grunbaum [6] extended the result of Grotzsch

In 1976 Appel and Haken [1, 2] achieved a major break through by thoroughly establishing 4CT. Their proof is based on studying a large number

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of cases for which a computer-assisted search for hours is required. In 1997 Robertson, Sanders, Seymour and Thomas reproved the 4CT with less need for computer verification [11]. In 2000 Ashay Dharwadkar gave an algebraic proof to the 4CT involving Steiner Systems, Eilenberg Modules, Hall Matching and Reimann Surfaces [3, 4].

Though Heawood solved the five color theorem using the Principle of Mathematical Induction, over the past 125 years no one could succeed in employing the same technique in proving the 4CT. Also, in the proof of Heawood, there is a need to consider different possible colorings of adjacent vertices of a minimum degree vertex of a graph.

In this paper, we give a simple proof to the 4CT. The proof is based on the Principle of Mathematical Induction, contraction, and possible colorings of a minimum degree vertex, its adjacent vertices, and adjacent vertices of these adjacent vertices. For all basic ideas in graph theory, we follow [7].

2. Main Result

Here we present proof of the four color theorem that is based on the principle of mathematical induction.

Theorem 2.1 (The Four Color Theorem (4CT)). The vertices of every planar graph can be colored with at most four colors in such a way that any two adjacent vertices have different colors.

Proof. The proof is based on the Principle of Mathematical Induction on the number of vertices n of any given simple planar graph. Here graphs are considered in their planar form only.

The theorem is true for graphs of order 1, 2, 3 or 4.

Assume the theorem for simple planar graphs of order less than or equal to n

Let G be a plane triangulation (maximal planar graph) of order n+1. Here plane triangulation is considered since every simple planar graph is a subgraph of a plane triangulation.

Let v be a vertex of G with minimum degree. This implies that $d(v) \leq 5$. If $d(v) \leq 3$, then the theorem can be proved very easily. Otherwise consider the following two cases.

Case 1 G contains at least one cycle of length three, say, $(v_1v_2v_3)$ other than the boundaries of triangular faces (finite or infinite). See Figures 1.1, 1.2 and 1.3.

Let G_i be the induced subgraph of G containing cycle $(v_1v_2v_3)$ but without vertices outside the cycle $(v_1v_2v_3)$.

Similarly let G_o be the induced subgraph of G containing cycle $(v_1v_2v_3)$ but without vertices inside the cycle $(v_1v_2v_3)$. Thus cycle $(v_1v_2v_3)$ is common to both G_i and G_o , $G_i \cup G_o = G$ and $G_i \cap G_o = (v_1v_2v_3)$.

Using the assumption, graphs G_i as well as G_o are four colorable, separately. Now adjust the colors of vertices of G_o so that the colors of vertices of v_1 , v_2 and v_3 of G_o are same as in G_i and thereby the combined graph G of G_i and G_o is four colorable. For example, if v_1 , v_2 and v_3 are assigned with colors 1, 2, 3 in G_i and 3, 4, 1 in G_o respectively, then change the colors of vertices of G_o by $3 \to 1$ (previous color class 3 is replaced by color class 1 in G_o), $4 \to 2$, $1 \to 3$ and $2 \to 4$. Now G is four colorable.

Case 2 G doesn't contain cycle of length three other than the boundaries of triangular faces (finite and infinite).

Let $v_1, v_2, \ldots, v_{d(v)}$ be the vertices adjacent to v in G so that $(v_1 v_2 \ldots v_{d(v)})$ is a cycle in G and are considered in the anti-clockwise direction, d(v) = 4 or 5.

Our aim is to prove that colorings in which $v_1, v_2, \ldots, v_{d(v)}$ are having colors 1, 2, 3 and 4 or colors including 5 are bad colorings of G and can be avoided. (A coloring ζ of a graph G is said to be a bad coloring of G if coloring ζ violates given hypothesis.)

When graph G-v requires at the most four colors, by assumption, and v_1 , $v_2, \ldots, v_{d(v)}$ are taking three colors out of the four colors 1, 2, 3 and 4, then by assigning the fourth color to v the graph G is four colorable.

Let G_1 be the subgraph after removing the d(v)+1 vertices, $v, v_1, v_2, \ldots, v_{d(v)}$ from G. That is $G_1 = G - \{v, v_1, v_2, \ldots, v_{d(v)}\}$.

At first we consider coloring of the subgraph G_1 and from this we try to obtain coloring of G-v. Here we consider all possible colorings of G_1 and for a given coloring of G_1 , we consider different possible colors of $v_1, v_2, \ldots, v_{d(v)}$ in G-v. And in such a coloring if G-v is four colorable and $v_1, v_2, \ldots, v_{d(v)}$ takes less than four colors, then we consider this coloring an acceptable coloring of G-v.

On the other hand, if G-v is four colorable and $v_1, v_2, \ldots, v_{d(v)}$ takes all the four colors, then G is five colorable and so we search for an alternate coloring to G_1 , if possible, so that the alternate coloring admits less than four colors to $v_1, v_2, \ldots, v_{d(v)}$ out of the colors 1, 2, 3 and 4. Here alternate coloring to G_1 is found out by applying contraction on the edge(s) of the cycle $(v_1v_2 \ldots v_{d(v)})$ in G-v.

After applying contraction on the edge(s) of the cycle $(v_1v_2...v_{d(v)})$ in G-v if the contracted graph, say G_c , of G-v is five colorable, then by the assumption the coloring of G_1 is an avoidable coloring of (not an acceptable coloring to) the contracted graph G_c and by the assumption, there exists an alternate coloring other than the above to G_c and thereby to G_1 . Repeat this process until we get alternative colorings in which $v_1, v_2, ..., v_{d(v)}$ take less than four colors while G-v is four colorable. And in these colorings graph G is four colorable by applying the fourth color, other than the colors of $v_1, v_2, ..., v_{d(v)}$ to v.

Now, let us consider different possible colorings of G-v including colorings in which $v_1, v_2, \ldots, v_{d(v)}$ take colors 1, 2, 3 and 4. See Figures 9.1 to 10.88. These are simplified figures which correspond to different colorings of G-v by taking into account of the following conditions which help to simplify and reduce the number of different possible colorings of G-v. Each figure corresponds to a coloring of G-v and hereafter numbers 1,2,3,4 and 5 represent five different colors.

Description of Figures 9.1 to 11.4

- 1. Let R_i be a set of numbers corresponding to different colors assigned to all the vertices which are adjacent to v_i in G_1 , i = 1, 2, ..., d(v).
- 2. For a given coloring of G_1 , the sets $R_1, R_2, \ldots, R_{d(v)}$ get a set of values (colors). A coloring of G-v is got by considering at first a possible coloring of G_1 and thereby a set of values $R_1, R_2, \ldots, R_{d(v)}$. For this coloring of G_1 , we find out the colorings of $v_1, v_2, \ldots, v_{d(v)}$ in G-v. This is the method of colorings of G-v in this proof.

Thus corresponding to each coloring of G - v, we get a figure that contains one set of values of $R_1, R_2, \ldots, R_{d(v)}$ and colors of $v_1, v_2, \ldots, v_{d(v)}$.

- 3. Two adjacent R_i s, say R_i and R_{i+1} (Here, sets R_i and R_j are called adjacent sets if and only if v_i and v_j are adjacent.) have at least one common number corresponding to the color of the vertex which is adjacent to both v_i and v_{i+1} in G v (v_i and v_{i+1} are also adjacent by triangulation.), subscript arithmetic modulo d(v), $i = 1, 2, \ldots, d(v)$. See Figure 2.
- 4. In G v vertex v_i is adjacent to v_{i-1} and v_{i+1} and thereby R_i is adjacent to R_{i-1} and R_{i+1} , subscript arithmetic modulo d(v), $i = 1, 2, \ldots, d(v)$.
- 5. The graph we consider comes under case-2, d(v) = 4 or 5. And G is a maximal planar graph. Therefore in G v each vertex v_i is adjacent to at least one vertex other than v_{i-1} and v_{i+1} , subscript arithmetic modulo d(v), $1 \le i \le d(v)$. Otherwise $d(v_i) = 3$ for at least one i, $1 \le i \le d(v)$. This implies R_i is non-empty for i = 1, 2, ..., d(v).
- 6. In G-v there is only one vertex which is adjacent to both v_i and v_{i+1} , subscript arithmetic modulo d(v), $i=1,2,\ldots,d(v)$. Otherwise it comes under case-1. See Figures 3.1 and 3.2.
- 7. While considering different colorings of G-v more importance is given to the colors of $v_1, v_2, \ldots, v_{d(v)}$ to prove the theorem on G.
- 8. Now color of a vertex, say v_i , of G-v depends on the colors of vertices which are adjacent to v_i in G-v. That is color of v_i in G-v should be different from the numbers of R_i and colors of v_{i-1} and v_{i+1} , subscript arithmetic modulo d(v). See Figures 4.1 and 4.2.

- 9. Corresponding to each coloring of G-v one set of values of R_1 , R_2 , ..., $R_{d(v)}$ and a set of possible colors of $v_1, v_2, \ldots, v_{d(v)}$ are considered.
- 10. G_1 has n d(v) number of vertices and using the assumption G_1 is four colorable. Hence the elements of R_i s are 1, 2, 3, 4 at the most, i = 1, 2, ..., d(v).
- 11. In each figure number just out side of R_i corresponds to the color of vertex v_i in G v, i = 1, 2, ..., d(v).
- 12. Coloring in which $v_1, v_2, \ldots, v_{d(v)}$ take the 5th color can be avoided since it is a bad coloring of G v, by the assumption.
- 13. Each R_i contains less than 4 different numbers since otherwise v_i requires the 5^{th} color in G v, i = 1, 2, ..., d(v).
- 14. Colorings in which two adjacent $R_i s$, say, R_i and R_{i+1} each having same set of 3 different numbers can be avoided, subscript arithmetic modulo d(v). Otherwise v_i or v_{i+1} requires the 5^{th} color since v_i and v_{i+1} are adjacent in G-v. See Figure 2.
- 15. A coloring is considered as a repetition when mutual interchange of numbers (colors) 1, 2, 3, 4 give rise to one of the colorings considered already. We avoid repetition of same colorings while considering different possible colorings of graph G-v. See Figure 8.
- 16. If the vertices $v_1, v_2, \ldots, v_{d(v)}$ of G v are contracted to a vertex, say v_c , and let G_2 be the resultant contracted graph of G v. Then graphs G_1 and $G_2 v_c$ are the same.
- 17. In each coloring of G-v, $R_1 \cup R_2 \cup \ldots \cup R_{d(v)}$ may take 2 or 3 or 4 colors out of the colors 1, 2, 3 and 4. But $R_1 \cup R_2 \cup \ldots \cup R_{d(v)}$ taking 4 different colors can be avoided by considering the same coloring of G_1 (in which 1, 2, 3 and 4 are the elements of $R_1 \cup R_2 \cup \ldots \cup R_{d(v)}$.) of G-v to G_2-v_c which leads to the 5th color to v_c and hence it is a bad coloring to G_2 . And by the assumption there exists another coloring to G_2 (and to G_1) and thereby to G-v in which $G_1 \cup G_2 \cup \ldots \cup G_d(v)$ gets 2 or 3 colors out of the colors 1, 2, 3 and 4. Thus colorings of G-v in which 1, 2 and 3 are the elements of G_1 are considered, $G_2 \cap G_2 \cap G_1$.

By all the above conditions, when d(v) = 4, it is always possible to find colorings of G - v in which G - v is 4 colorable and v_1 , v_2 , v_3 and v_4 take at the most three colors out of the 4 colors. Figures 9.1 to 9.5 are the different colorings under d(v) = 4 (in the sense, if the number of elements of R_i s are reduced further, then it is easy to see that v_1 , v_2 , v_3 and v_4 require at the most 3 colors out of the colors 1, 2, 3, 4).

18. When d(v) = 5 and $1 \le i \le 5$, each R_i will contain at least two different numbers. Otherwise $d(v_i) \le 4$ in G, $1 \le i \le 5$. See Figure 5. Already colorings under $d(v) \le 4$ have been considered and $d(v) \le d(v_i)$, i = 1, 2, 3, 4, 5. Therefore when $d(v_i) \ge d(v) = 5$, each R_i

- contains colors of at least two adjacent vertices, i = 1, 2, 3, 4, 5. See Figure 6.
- 19. Using the above conditions we obtain Figures 10.1 to 10.88 when d(v) = 5. The number of figures under Figure 10 can be reduced to 23 using condition-15 and only for clarity 88 figures (type of colorings) are considered. The different and equivalent figures (colorings) are as follows (Here two figures F_1 and F_2 are said to be equivalent if mutual interchange of numbers (colors) 1, 2, 3, 4 in one figure gives rise to the other figure and in this case it is denoted by $F_1 \cong F_2$.).
 - (a) Fig. 10.1;
 - (b) Fig. 10.2;
 - (c) Fig. 10.3;
 - (d) Fig. 10.4;
 - (e) Fig. $10.5 \cong Fig. \ 10.36 \cong Fig. \ 10.49;$
 - (f) Fig. $10.6 \cong Fig. \ 10.7 \cong Fig. \ 10.17 \cong Fig. \ 10.28 \cong Fig. \ 10.37 \cong Fig. \ 10.43;$
 - (g) Fig. $10.8 \cong Fig$. $10.11 \cong Fig$. $10.31 \cong Fig$. $10.38 \cong Fig$. $10.46 \cong Fig$. 10.48;
 - (h) Fig. $10.9 \cong Fig. \ 10.13 \cong Fig. \ 10.39$;
 - (i) Fig. $10.10 \cong Fig. \ 10.12 \cong Fig. 10.21 \cong Fig. \ 10.24 \cong Fig. \ 10.34 \cong Fig. \ 10.35;$
 - (j) Fig. $10.14 \cong Fig. \ 10.23 \cong Fig. \ 10.41$;
 - (k) Fig. $10.15 \cong Fig. \ 10.20 \cong Fig. 10.22 \cong Fig. \ 10.25 \cong Fig. \ 10.32 \cong Fig. \ 10.40;$
 - (l) Fig. $10.16 \cong Fig. \ 10.26 \cong Fig. \ 10.29 \cong Fig. \ 10.42 \cong Fig. \ 10.44 \cong Fig. \ 10.47;$
 - (m) Fig. $10.18 \cong Fig. \ 10.27 \cong Fig. \ 10.30;$
 - (n) Fig. $10.19 \cong Fig. 10.33 \cong Fig. 10.45$;
 - (o) Fig. 10.50;
 - (p) Fig. $10.51 \cong Fig. \ 10.52 \cong Fig. \ 10.53 \cong Fig. \ 10.88$;
 - (q) Fig. $10.54 \cong Fig. \ 10.58 \cong Fig. \ 10.61 \cong Fig. \ 10.62 \cong Fig. \ 10.71 \cong Fig. \ 10.73;$
 - (r) Fig. $10.55 \cong Fig$. $10.56 \cong Fig$. $10.57 \cong Fig$. $10.64 \cong Fig$. $10.77 \cong Fig$. 10.82;
 - (s) Fig. $10.59 \cong Fig$. $10.68 \cong Fig$. $10.74 \cong Fig$. $10.76 \cong Fig$. $10.83 \cong Fig$. 10.85;
 - (t) Fig. $10.60 \cong Fig. \ 10.67 \cong Fig. \ 10.75 \cong Fig. \ 10.78 \cong Fig. \ 10.87$;
 - (u) Fig. $10.63 \cong Fig. \ 10.66 \cong Fig. \ 10.69 \cong Fig. \ 10.72 \cong Fig. \ 10.79;$
 - (v) Fig. $10.65 \cong Fig. 10.70 \cong Fig. 10.81 \cong Fig. 10.86$ and
 - (w) Fig. $10.80 \cong Fig. 10.84$.

- 20. Coloring corresponding to Figures 10.1, 10.3, 10.5, 10.9, 10.13, 10.36, 10.39, 10.49 and 10.50 are avoidable colorings of G-v since each one contains the 5th color. And all other figures (type of colorings) under Figure 10, other than figures each inside a square, correspond to possible colorings with which G gets only 4 or less colors.
- 21. Figure inside a square, in Figure 10, corresponds to coloring (of G-v) in which v₁, v₂, v₃, v₄, v₅ require colors 1, 2, 3 and 4. These figures are 10.4, 10.10, 10.12, 10.18, 10.21, 10.24, 10.27, 10.30, 10.34, 10.35, 10.63, 10.66, 10.69, 10.72 and 10.79. These are separately considered under Figure 11 and we will show by contraction that these colorings are avoidable colorings of G v and by assumption graph G-v has alternate colorings other than these. For doing this, at first, we attack colorings corresponding to Figure 11.1 and then we attack Figures 11.2 to 11.4 simultaneously that will also cover Figure 11.1.
- 22. Consider graph G v and coloring in which R_i = {1,3} and R_{i+1} = {2,3}. If edge v_iv_{i+1} is contracted to a vertex, say v_{i,i+1}, then by conditions (3), (6) and (18), the contracted vertex v_{i,i+1} will be adjacent to vertices of R_i ∪ R_{i+1} = {1,2,3} = R_{i,i+1}, say, of G₁, subscript arithmetic modulo d(v). This is also true when R_i = {1,2,3}. See Figure 7.1.
- 23. Consider the coloring corresponding to Figure 11.1. Figure 11.1 contains two R_is each = {1,2,3} and they are non-adjacent. Now, in G − v, make contraction on any two non-adjacent edges of the cycle (v₁v₂v₃v₄v₅). Then the contracted cycle of cycle (v₁v₂v₃v₄v₅) (in the contracted graph, say G₃, of G − v) requires 5th color and thereby Figure 11.1 (type of coloring) is an avoidable coloring to the contracted graph G₃. And by the assumption there exists another coloring (to G₃ (and to G₁, different from coloring corresponding to Figure 11.1) and thereby) to G − v in which G − v is 4 colorable and R₁ ∪ R₂ ∪ ... ∪ R_{d(v)} takes colors different from Figure 11.1. These alternate coloring(s) may include Figures 11.2, 11.3 and 11.4.
- 24. Consider colorings of the graph G − v corresponding to Figures 11.2, 11.3 and 11.4. Each figure contains a pair of adjacent R_is each of order 2, say, R_{i+1} and R_{i+2} such that R_{i+1} = {1, 2} = R_{i+2}, R_i ≠ R_{i+1} and R_{i+3} ≠ R_{i+2}, 1 ≤ i ≤ 5, under subscript addition modulo 5. Make contraction on the non-adjacent edges v_iv_{i+1} and v_{i+2}v_{i+3} of the cycle (v₁v₂v₃v₄v₅) and let v_{i,i+1} and v_{i+2,i+3} be the contracted vertices so that they are adjacent in the contracted graph and R_{i,i+1} = R_i ∪ R_{i+1} = {1, 2, 3} and R_{i+2,i+3} = R_{i+2} ∪ R_{i+3} = {1, 2, 3}, using conditions (3), (6) and (18), subscript arithmetic modulo d(v). This case also covers the case corresponding to Figure 11.1 (since under Figure 11.1 we make contraction on two non adjacent edges of the cycle (v₁v₂v₃v₄v₅) of the graph G − v, the two contracted vertices

are themselves adjacent and also each adjacent to vertices with colors 1, 2 and 3 in the contracted graph). Now the contracted cycle of cycle $(v_1v_2v_3v_4v_5)$ (in the contracted graph, say G_4 , of G-v) requires 5^{th} color and thereby these type of colorings corresponding to Figures 11.2, 11.3, 11.4 and 11.1 are avoidable colorings of G_4 . And by the assumption that any simple planar graph with number of vertices \leq n is 4 colorable, there exists coloring(s) (to G_4 (and to G_1 , different from Figures 11.1, 11.2, 11.3 and 11.4) and thereby) to G-v in which G-v is 4 colorable, $R_1 \cup R_2 \cup \ldots \cup R_{d(v)}$ takes colors from Figures 10 (or simpler figures in the sense G-v is four colorable, $R_1 \cup R_2 \cup \ldots \cup R_{d(v)}$ takes 3 colors and $v_1 \cup v_2 \cup \ldots \cup v_{d(v)}$ takes 3 or less colors out of the 4 colors of G-v), other than Figures 11.1, 11.2, 11.3 and 11.4 and v_1, v_2, v_3, v_4, v_5 take only 3 colors out of the colors 1, 2, 3 and 4. See Figure 7.2.

25. While searching for acceptable colorings for a given graph H under given conditions, if a coloring is found as a bad coloring to H, then we avoid such colorings to the graph H thereafter.

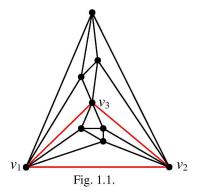
This implies, G - v is 4 colorable with at most 3 colors (out of the 4 colors) to $v_1, v_2, \ldots, v_{d(v)}$. Now by giving the 4^{th} color other than that of $v_1, v_2, \ldots, v_{d(v)}$ to v the graph G is 4 colorable. Therefore by the Principle of Mathematical Induction the theorem is true for any simple planar graph and hence for any planar graph.

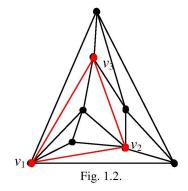
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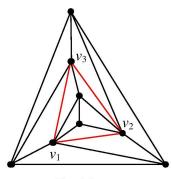
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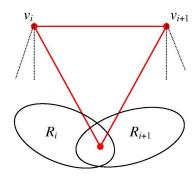
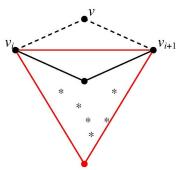




Fig. 2.



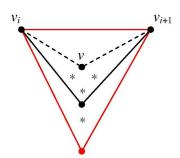


Fig. 3.1.

Fig. 3.2.

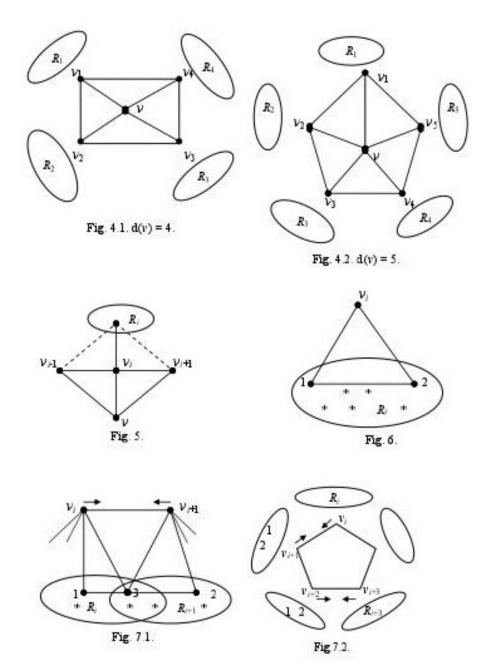


Figure 8:

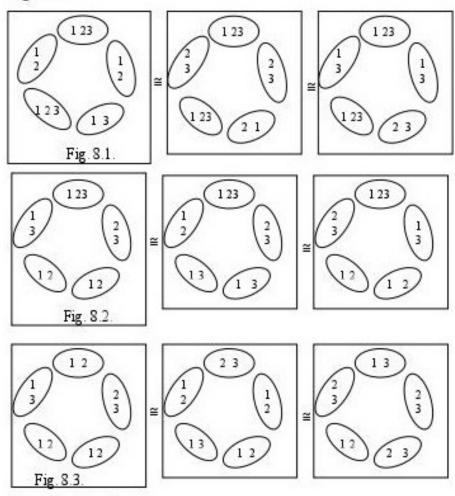
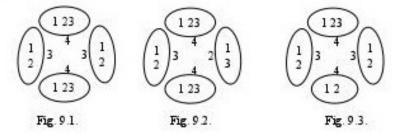


Figure 9: d(v) = 4



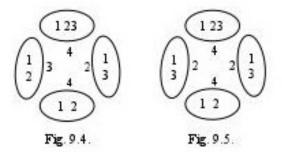
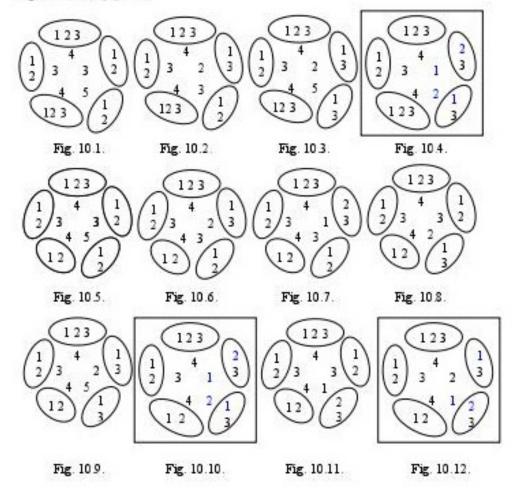
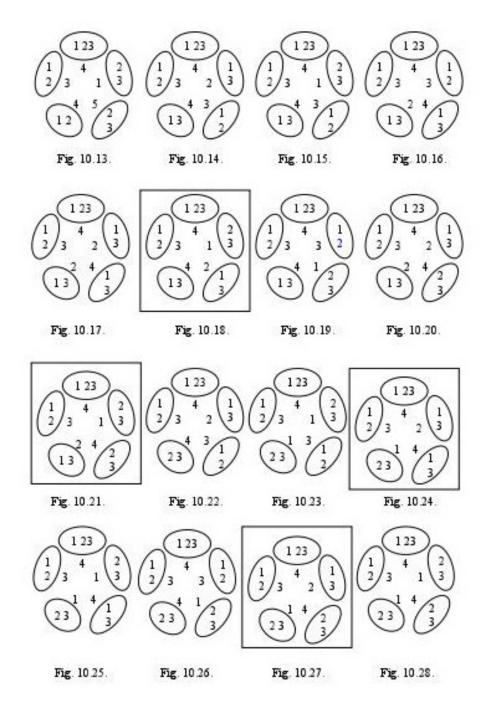
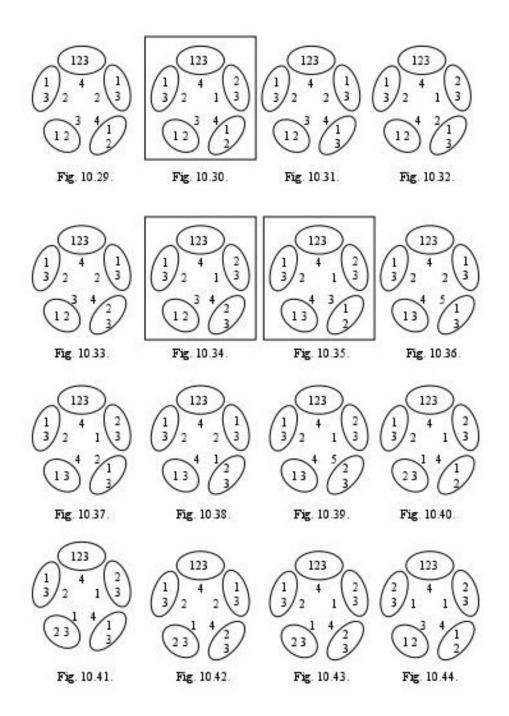
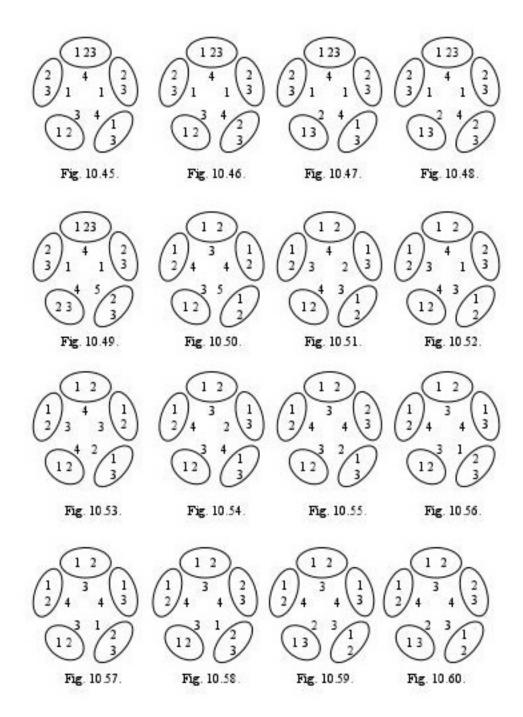


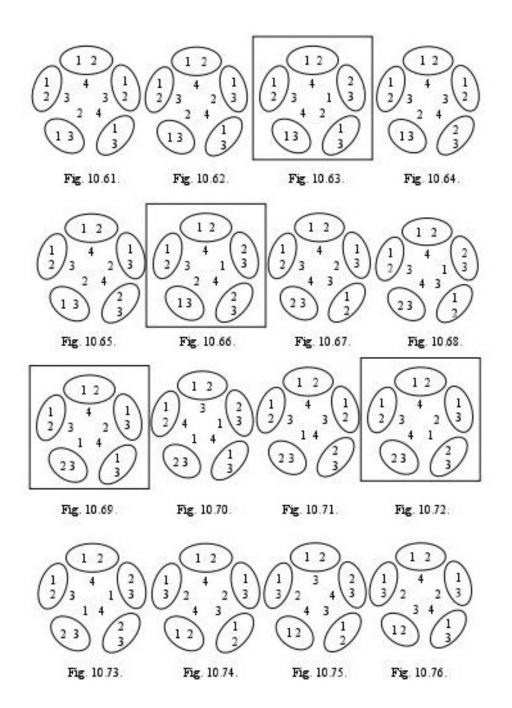
Figure 10: d(v) = 5











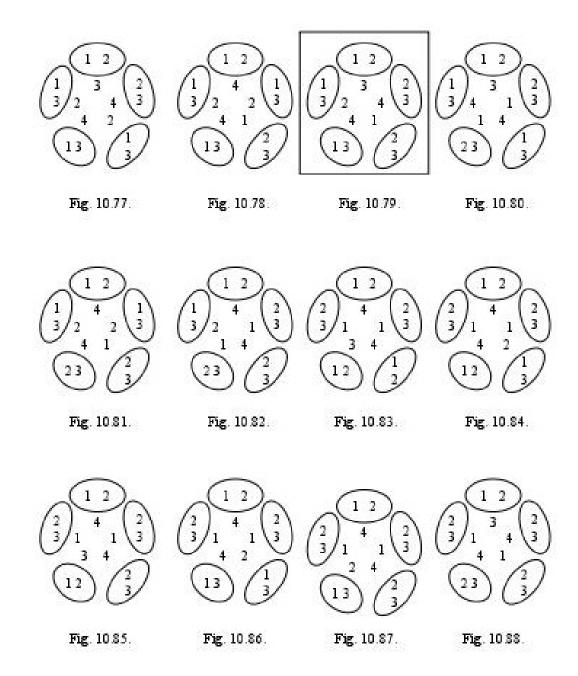


Figure 11:

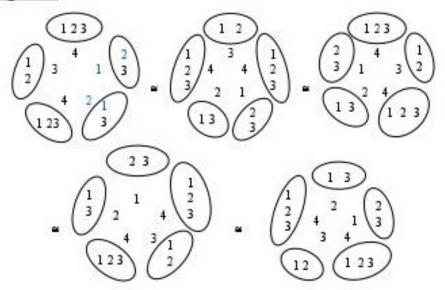
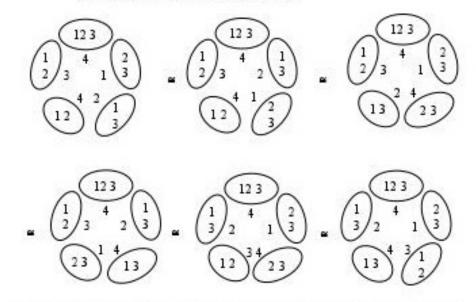


Fig. 11.1. Equivalent figures of Fig. 10.4.



 $\label{eq:Fig.11.2} \text{Fig.10.10} \cong \text{Fig.10.12} \cong \text{Fig.10.21} \cong \text{Fig.10.24} \cong \text{Fig.10.34} \cong \text{Fig.10.35}.$

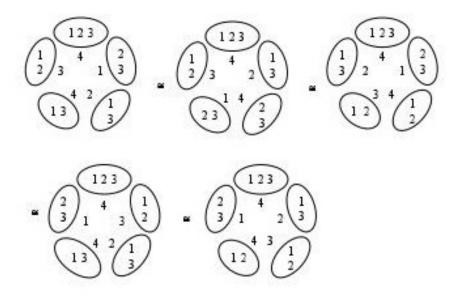


Fig. 11.3 a Fig. 10.18 a Fig. 10.27 a Fig. 10.30 and their equivalent figures.

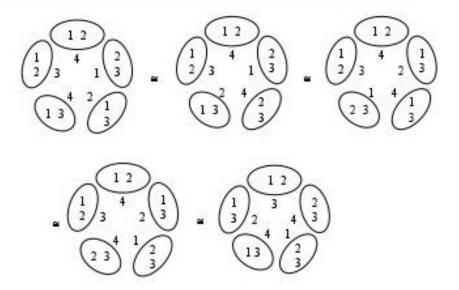


Fig.11.4 = Fig.10.63 = Fig.10.66 = Fig.10.69 = Fig.10.72 = Fig.10.79