On Equivalent Color Transform and Four Coloring Theorem

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Abstract

In this paper, we apply an equivalent color transform (ECT) for a minimal k-coloring of any graph G. It contracts each color class of the graph to a single vertex and produces a complete graph K_k for G by removing redundant edges between any two vertices. Based on ECT, a simple proof for four color theorem for planar graph is then proposed.

Keywords: Four color theorem, Equivalent color transform (ECT)

1. Introduction

Four Coloring Theorem (FCT) states that every planar map is four colorable so that no two adjacent regions have the same color, given any separation of a plane into contiguous regions in the map. Proving four colors theorem is a very hard and challenging task since the first statement of the four color theorem in 1852.

The four color theorem was proved in 1976 by Kenneth Appel and Wolfgang Haken [3] [4] with computer assistance. It was the first major theorem to be proved using a computer.

To dispel remaining doubt about the Appel-Haken proof, a simpler proof using the same ideas still with computer-assistance was published in 1996 by Robertson, Sanders, Seymour, and Thomas [7] and the part which required hand verification was not as tedious as in Appel-Haken proof. A quadratic algorithm to 4-color planar graph was proposed in [5] by Appel and Haken. An efficient algorithm for four-coloring planar graph was also proposed in [6]. In this paper, we apply an equivalent color transform (ECT) for planar graph and propose a simpler proof for four color theorem.

2. Methods

Definition 1. Equivalent Color Class (ECC). For coloring a simple graph, node i and j can have the same color if they are non-adjacent, node i and j are called in equivalent color class.

Definition 2. Equivalent Color Graph (ECG) is a composite graph, where composite nodes are formed by moving those non-adjacent nodes in original graph together and

edges are kept the same as in the original graph.

Definition 3. Equivalent Color Transform (ECT) is contracting each equivalent color class in ECG to a single vertex and keep just one edge between any adjacent vertex.

There may be more than one way to form ECG for any given graph. Figure 1 shows an example of 6-node simple graph (a) and its corresponding ECG (b) and ECT (c).
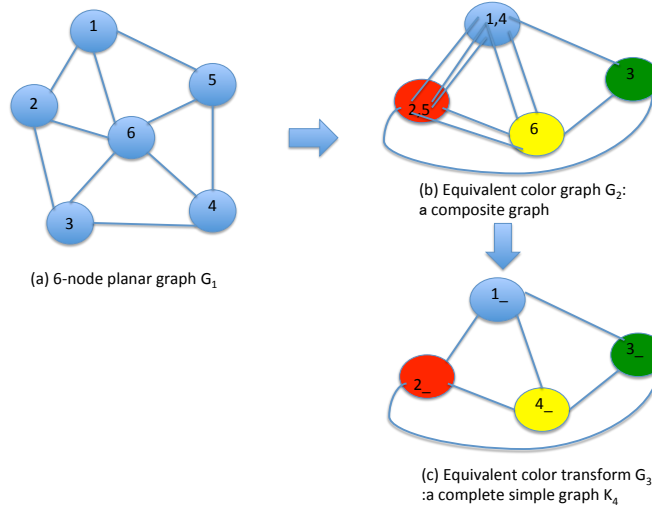

(a) 6-node planar graph G_1

(b) Equivalent color graph G_2: a composite graph

(c) Equivalent color transform G_3 : a complete simple graph K_4

Figure 1: An instance of ECG and ECT

Definition 4. Planar Graph is a graph that can be embedded in the plane, i.e., it can be drawn in such a way that no edges cross each other [1].

Lemma 1. There exists an efficient ECT for planar graph.

There are exact algorithms to determine if a graph is 4-colorable such as one suggested in [2]; there are also efficient algorithms such as one suggested in [6] for efficiently four-coloring a planar graph. Notice that the RSST(abbreviated by the first letter of four authors in [6]) algorithm has computational complexity of $O(n^2)$, where n is the number of nodes in the graph. We therefore can conduct ECT efficiently by using RSST algorithm or similar algorithm.

Theorem 1. For a planar graph, there exist an ECT which does not change the planarity of the graph.

Proof. Building ECG just moves the place of original nodes and does not alter any node or edge or their connection relationship, so we can apply RSST algorithm to find an ECT,

which do not change the planarity of the original graph.

□

Notice that there may be more than one ECG and ECT for any given planar graph; we can apply RSST algorithm and similar algorithm to find an ECG and ECT, which do not change the planarity of the graph. To do this, we may need running RSST algorithm a few times until the ECG and ECT are found to keep the planarity of the original graph.

Theorem 2. A k-coloring graph which needs the minimum number of k colors, can be converted to a complete simple graph K_k by ECT.

Proof. We prove by contradiction. Let us assume the original k-coloring graph G_1 is converted to a simple graph G_2 with k nodes by ECT and G_2 is not complete graph. Since G_2 is not complete graph, there must exist at least two nodes n_i and n_j which are not adjacent. According to the definition of ECC and ECG, n_i and n_j can be contracted to a single vertex (by equivalent color class and ECT) and a new graph with (k-1) nodes is formed. Therefore there only need (k-1) colors for G_1 and G_2, this contradicts the fact that G_1 is k-coloring and needs the minimum k colors. □

Lemma 1. Wagner's Theorem. A finite graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor [1].

Theorem 3. Planar graph is four colorable.

Proof. We prove by contradiction. Set k as the minimum number of colors needed for coloring planar graph G. Let assume the planar graph G needs more than four colors, i.e., $k > 4$. From Theorem 2, we know that G can be converted to a complete graph K_k ($k > 4$) in this case. This means G is transformed by ECT to a complete graph K_k ($k > 4$) which is not planar because any complete graph with more than 4 nodes will have K_5 as a minor and is not planar (by Lemma 1). But this contradicts the fact that G is planar and there exits an ECT which does not change the planarity of the original graph (by Theorem 1). So $k \leq 4$ and planar graph is four colorable.

□

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