# An Introduction to Bayesian Statistics

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- 6) Repeat 3—5 as necessary

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  - Step 2: Prior information
  - Step 5: Prior information $\rightarrow$ posterior information

# Economic Applications of Bayesian Statistics

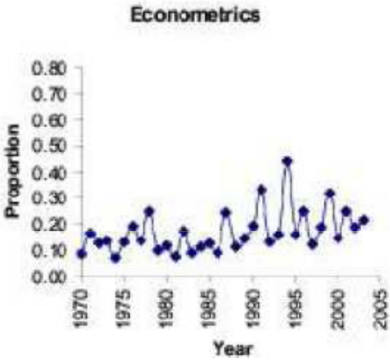


Figure 5: Econometrica Containing "Bayes" or "Bayesian"

# Common Applications of Bayesian Statistics

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- Using as much prior information as possible as well as personal judgment (placing a bet)
- Once an outcome is revealed, prior information is updated

# Table of Frequentist vs. Bayesian Interpretations

| | Frequentist statistics | Bayesian statistics |
|---|---|---|
| Definition of the $p$ value | The probability of observing the same or more extreme data assuming that the null hypothesis is true in the population | The probability of the (null) hypothesis |
| Large samples needed? | Usually, when normal theory-based methods are used | Not necessarily |
| Inclusion of prior knowledge possible? | No | Yes |
| Nature of the parameters in the model | Unknown but fixed | Unknown and therefore random |
| Population parameter | One true value | A distribution of values reflecting uncertainty |
| Uncertainty is defined by | The sampling distribution based on the idea of infinite repeated sampling | Probability distribution for the population parameter |
| Estimated intervals | Confidence interval: Over an infinity of samples taken from the population, 95% of these contain the true population value | Credibility interval: A 95% probability that the population value is within the limits of the interval |

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- Objective prior: Pure ignorance
- Prior reflects knowledge about parameters before observing current data
- Science can be accumulative!

# Bayes' Theorem

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(B|A) \times \mathrm{P}(A)}{\mathrm{P}(B)}$$

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- Let $\mathrm{A} = \{\text{the patient is positive}\}$. Is it proper to use $\Pr(\mathrm{A})$?
- How should we update the uncertainty after a test?

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# Some Common Conjugate Priors

| Likelihood | Prior | Posterior |
|---|---|---|
| $X\|\theta \sim \mathcal{N}(\theta, \sigma^2)$ | $\theta \sim \mathcal{N}(\mu, \tau^2)$ | $\theta\|X \sim \mathcal{N}(\frac{\tau^2}{\sigma^2+\tau^2}X + \frac{\sigma^2}{\sigma^2+\tau^2}\mu, \frac{\sigma^2\tau^2}{\sigma^2+\tau^2})$ |
| $X\|\theta \sim \mathcal{B}(n, \theta)$ | $\theta \sim \mathcal{B}e(\alpha, \beta)$ | $\theta\|X \sim \mathcal{B}e(\alpha + x, n - x + \beta)$ |
| $X_1, \ldots, X_n\|\theta \sim \mathcal{P}(\theta)$ | $\theta \sim \mathcal{G}a(\alpha, \beta)$ | $\theta\|X_1, \ldots, X_n \sim \mathcal{G}a(\sum_i X_i + \alpha, n + \beta)$. |
| $X_1, \ldots, X_n\|\theta \sim \mathcal{NB}(m, \theta)$ | $\theta \sim \mathcal{B}e(\alpha, \beta)$ | $\theta\|X_1, \ldots, X_n \sim \mathcal{B}e(\alpha + mn, \beta + \sum_{i=1}^n x_i)$ |
| $X \sim \mathcal{G}(n/2, 2\theta)$ | $\theta \sim \mathcal{IG}(\alpha, \beta)$ | $\theta\|X \sim \mathcal{IG}(n/2 + \alpha, (x/2 + \beta^{-1})^{-1})$ |
| $X_1, \ldots, X_n\|\theta \sim \mathcal{U}(0, \theta)$ | $\theta \sim \mathcal{P}a(\theta_0, \alpha)$ | $\theta\|X_1, \ldots, X_n \sim \mathcal{P}a(\max\{\theta_0, x_1, \ldots, x_n\}\alpha + n)$ |
| $X\|\theta \sim \mathcal{N}(\mu, \theta)$ | $\theta \sim \mathcal{IG}(\alpha, \beta)$ | $\theta\|X \sim \mathcal{IG}(\alpha + 1/2, \beta + (\mu - X)^2/2)$ |
| $X\|\theta \sim \mathcal{G}a(\nu, \theta)$ | $\theta \sim \mathcal{G}a(\alpha, \beta)$ | $\theta\|X \sim \mathcal{G}a(\alpha + \nu, \beta + x)$ |

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- Data: An experiment is conducted and 9 heads and 3 tails are observed

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- Bayesian method has no difficulty $\rightarrow$ the same conclusion under both scenarios

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  - This is the same as if we observed both batches together

# Simulate Normal Random Variables: Box–Muller Transformation

- We require two random variables, $U$ and $V$, uniformly distributed on $[0, 1]$. Set

$$\begin{aligned} R &= \sqrt{-2 \log V}, \\ \theta &= 2\pi U, \end{aligned}$$

and

$$\begin{aligned} Z_1 &= R \cos\theta, \\ Z_2 &= R \sin\theta. \end{aligned}$$

Then they are independent standard normal variables. To obtain two standard normal variables with correlation $\rho$, take

$$\begin{aligned} X &= Z_1 \\ Y &= \rho Z_1 + \sqrt{1-\rho^2} Z_2 \end{aligned}$$

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# Frequentist Inference

- Ordinary least squares

$$\hat{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$$
$$\hat{\sigma}^2 = s^2 = \frac{1}{n-k}(\boldsymbol{y} - \boldsymbol{X}\hat{\beta})'(\boldsymbol{y} - \boldsymbol{X}\hat{\beta})$$

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- Marginal posterior of $\boldsymbol{\beta} | \boldsymbol{y}$ is the multivariate $t$-distribution with $n-k$ degrees of freedom

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  - Hierarchical models can often fit data with a small number of parameters but can also do well in prediction

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  - Third-stage prior: $\alpha \sim \mathit{Exp}(1)$

# Hierarchical Linear Model

$$
\begin{aligned}
Y|X, \beta, \Sigma &\sim \mathcal{N}(X\beta, \Sigma) \\
\beta|X_\beta, \alpha, \Sigma_\beta &\sim \mathcal{N}(X_\beta \alpha, \Sigma_\beta) \\
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  - $\ldots$
  - Draw $\theta_k$ from $p(\theta_k|\theta_1, \theta_2, \ldots, \theta_{k-1}, y)$

# Gibbs Sampler (cont.)

- Full conditional distribution $p(\theta_j|\theta_{-j}, y)$, where $\theta_{-j} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_k)$

# Gibbs Sampler (cont.)

- Full conditional distribution $p(\theta_j|\theta_{-j}, y)$, where $\theta_{-j} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_k)$
- In iteration $t$, draw $\theta_j^t = p(\theta_j|\theta_{-j}^t, y)$, where $(\theta_1^t, \ldots, \theta_{j-1}^t, \theta_{j+1}^{t-1}, \ldots, \theta_k^{t-1})$

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- Each $\theta_j$ is updated conditional on the latest values of $\theta$

# Example: Simulate from a Bivariate Normal Distribution

S

- Joint distribution

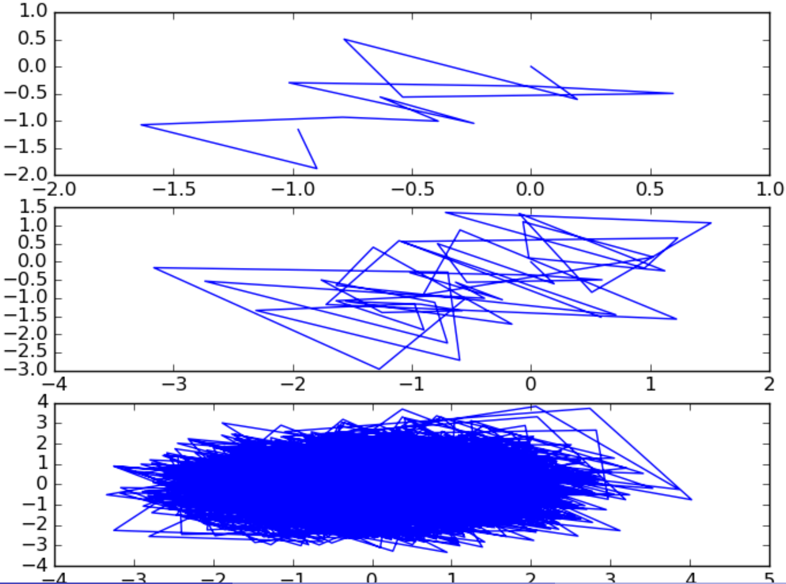
$$\boldsymbol{Z} = (X, Y)' \sim \mathcal{N}(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix})$$

# Python Code

```
import numpy as np
x = [0]
y = [0]
rho = .9
c = np.sqrt(1 - rho**2)
for i in range (6000):
    x.append(rho*y[i - 1] + c*np.random.normal(0, 1, 1))
    y.append(rho*x[i] + c*np.random.normal(0, 1, 1))

%matplotlib qt5
import matplotlib as mpl
import matplotlib.pyplot as plt
plt.style.use('classic')
plt.figure()
plt.subplot(3, 1, 1)
plt.plot(x[0:14], y[0:14], '-')
plt.subplot(3, 1, 2)
```

# The Resulting Posterior



## General Property of Gibbs Sampler

- Output: A dependent sequence

$$\begin{aligned}
\theta^{(1)} &= \{\theta_1^{(1)}, \ldots, \theta_p^{(1)}\} \\
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- This is called a Markov property, and the sequence a Markov chain
- For the models in this class, the sampling distribution of $\theta^{(S)}$ approaches the target distribution as $S \to \infty$, regardless of starting value

$$Pr(\theta^{(S)} \in A \to \int_A p(\theta) d\theta \text{ as } S \to \infty$$

# General Property of Gibbs Sampler (cont.)

- More importantly, for most functions $g$ of interest,

$$\frac{1}{S}\sum_{s=1}^{S} g(\theta^{(s)}) \rightarrow E[g(\theta)] = \int g(\theta)p(\theta)d\theta \text{ as } S \rightarrow \infty$$

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- Hence, we call this method Markov chain Monte Carlo (MCMC)

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- When the posterior distribution is complicated, we can "look at" the posterior by studying Monte Carlo samples from the posterior

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- Approximation: The use of Monte Carlo procedures to approximate integrals

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- We will deal with these later