GNU C-Graph

A Tool for Learning about Convolution
GNU C-Graph version 2.0.1, 19 December 2018

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This is a manual for GNU C-Graph version 2.0.1, a tool for learning about convolution.

for **ALL the VICTIMS of APARTHEID STRUGGLING to be FREE**

and

to **REGNIER**

You're sending me discrete signals from across the room,

I respond on impulse, reflecting on the sampling of events

That were a dichotomy from the day you left your mother's womb;

Multiplied in frequency, integrated in time, a weighted confluence

Of sliding, shifting trains of thought, alternative messages under transformation;

Counterpoint, duality, involution, contradistinction without confusion;

Independence in summation. Silence - this is convoluted conversation.

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1 Foreword

From the shadow cast by light to the echo in a cave convolution, like the ubiquitous Fibonacci series, is a mathematical description of naturally ocurring physical phenomena in any linear, time-invariant system capable of responding to an input signal. Today, convolution - the combination of two signals to produce a third - has wide ranging applications. Edge detection in computer vision, algorithms for robot motion, signal and image processing, crystallography, statistics and probability theory, differential equations, linear algebra, numerical analysis, and even recent innovations in music production - all utilise techniques involving convolution.

GNU C-Graph (for Convolution Graph) is a tool for visualizing the convolution of two signals. The package is a reproduction of the Fortran 77 program in my BSc. Electrical Engineering (Honours) dissertation "Interactive Computer Program Demonstrating: Sampling Convolution and the FFT", University of Aberdeen, Scotland, 1983. In this version I have included pulses, scaling of the signals, and error-handling - features that were not part of my original Thesis.

Whether student engineer or scientist, aspiring special-effects animator or roboticist, GNU C-Graph will help you find the adventure in the mathematics of convolution.

– Adrienne Gaye Thompson

2 Overview

2.1 About

GNU C-Graph computes the linear convolution of two signals in the time domain then compares their circular convolution by demonstrating the *convolution theorem* - convolution of two signals in the time domain corresponds to multiplication in the frequency domain. Each signal is modelled by a register of discrete values simulating samples of a signal, and the discrete Fourier transform (DFT) computed by means of the Fast Fourier Transform (*FFT*). See Appendix A [Appendix A], page 27, for an explanation of the convolution theorem.

GNU C-Graph is interactive, prompting the user to enter single character or numerical values from the keyboard, thereby dispensing with the learning curve for coding formulae. The user chooses 2 from a menu of 8 signal types, and up to 5 parameters to define the waveforms. The signals chosen may be periodic, aperiodic, or pulses. C-Graph then plots 3 graphs:

1. The time domain representation of both signals;

2. Their Fourier transforms;

3. A comparison of their linear and circular convolution.

See Chapter 5 [A C-Graph Session], page 13, for a typical C-Graph session.

GNU C-Graph will be useful to students of signal theory in the study of convolution and spectral analysis. This version (2.0) uses a simple FFT written by Arthur Wouk and converted to Fortran 90 by Alan Miller (see Appendix B [Appendix B], page 35).

2.2 Required Software

GNU C-Graph is written in contemporary Fortran. The package runs on GNU/Linux, was developed with GFortan and G95, and uses Gnuplot 4.2 as well as Image Magick 6.6.

Experienced users wishing to use other compilers may supply the necessary command-line options to `configure` during installation. See the file `INSTALL` for basic installation instructions.

3 Invoking C-Graph

To run GNU C-Graph, open up a terminal in X and type `c-graph`. C-Graph supports the
following options:

`--dedicate`
`-d`   Print the dedication and exit.

`--help`
`-h`   Print a summary of the command line options and exit.

`--no-splash`
`-n`   Invoke GNU C-Graph with no splash screen.

`--version`
`-v`   Print the version number and licensing information of GNU C-Graph, then exit.

4 The Signals

4.1 The Menu

C-Graph presents the following menu of signals from which the user chooses 2:

```
SIGNAL          CODE
Sine            A
Cosine          B
Triangle        C
Square          D
Sawtooth        E
Exponential     F
Ramp            G
Step            H
```

Signals 'A' to 'E' are periodic, while 'F', 'G', and 'H' are aperiodic. Pulses may also be chosen; these are a half period in duration $1/2f$, where f is the frequency of the corresponding cyclical waveform.

4.2 Parameters

The user enters up to 5 parameters to generate the signals, their FFTs and their convolution:

1. The number of samples 'N'
2. The code for the signal 'A' to 'H'
3. The wave/pulse parameter 'w' or 'p'
4. The frequency 'f'
5. The scaling coefficient 'sc'

Both signals are constructed from the same number of samples 'N'. If the user chooses a periodic signal, then he/she is prompted to select either the cyclical waveform or a derived pulse, i.e., 'w' or 'p'. For each periodic signal chosen the user is prompted to enter its frequency 'f' and a scale factor 'sc'.

Pulses are monophasic and are defined on half the period of the modulus of the corresponding periodic waveform.

4.3 Defaults & Error Handling

If the required parameter is a number and the user has erroneously entered character data, C-Graph generates an error message and gives the user another try to enter a number. Otherwise, for input outside the expected ranges C-Graph assumes default values.

Number of samples 'N'

'N' must lie in the range [64, 1024]. Values entered outside of this range will default to 512. 'N' is defined to be a power of 2. If the user enters a value that is not a power of 2 C-Graph will choose the nearest power of 2.

Signal code 'A' to 'H'
>   For input outside the range 'A' to 'H', the default codes are 'C' for the first
>   signal, and 'D' for the second.

Wave/Pulse parameter 'w' or 'p'
>   The default waveform is a pulse.

Frequency 'f'
>   C-Graph assumes a default frequency of 1Hz for values of 'f' entered outside
>   the range [0.5, $N/4$].

Scaling coefficient 'sc'
>   The scaling coefficient 'sc' may be positive or negative. The maximum absolute
>   value of 'sc' for signals a, b, f, and h is 'N',while that for signals c,d,e, and g is 1.
>   All signals will be scaled to unity for input values of 'sc' outside the permitted
>   range.
>
>   With the default scaling coefficient of 1, signals a, b, f, and h are unit functions;
>   signals d (square) an e (sawtooth) have a maximum amplitude of half the period
>   (1/(2f)) while that of c (triangle) is one-quarter the period (1/(4f)).

4.4 Frequency Selection

We can express the period P of a periodic signal as

$$P \quad = \quad N/(number\ of\ cycles)$$

$$= \quad T/(N/n)$$

where T is the duration ofthe signal register in seconds, N is the number of samples in the
register (window length), and n is the number of samples in 1 period.

The frequency of the signal f is the reciprocal of the period, so

$$f \quad = \quad N/(nT)\ samples/seconds\ or\ Hz$$

C=Graph assumes that the duration of the signal register is 1 second, so

$$f \quad = \quad N/n)\ Hz$$

The sampling rate f_s is given by

$$f_s \quad = \quad N/T\ Hz$$

The interval h between successive samples being the reciprocal

$$h \quad = \quad 1/f_s\ seconds$$

If the window length and frequency chosen are 512 and 20 Hz (approximately the lower limit of the human audible range) then the number of samples n in each period would therefore be

$$n \quad = \quad N/f$$

$$= \quad 512/20$$

$$= \quad 25.6$$

C-Graph requires that n be a multiple of 4. For each periodic signal, the frequency entered by the user is accordingly adjusted so that n approximates to the nearest multiple of 4. So the frequency of the signal used by C-Graph would become

$$f \quad = \quad N/n$$

$$= \quad 512/26$$

$$= \quad 19.7 \ Hz$$

5 A C-Graph Session

In this session, we run C-Graph twice to compare the convolution of 2 signals of equal length:

1. When the signals are finite sequences and the remaining interval is zero padded to the 'N' with at least the same number of zeros as samples in each signal;

2. When both signals extend across the full register of 'N' samples.

We use a sawtooth pulse for the first signal, and a rectangular pulse half the amplitude of the sawtooth for the second signal. We also demonstrate the use of default values for unexpected input. The keychord *ALT-TAB* is used to toggle the terminal and the Gnuplot window.

In X, type `c-graph`. The splash screen will appear for a few seconds. Pressing *ESC* will kill the display, but you may invoke C-Graph without the splash screen with the `--no-splash` option. When the splash screen disappears, the following text will appear:

```
GNU C-Graph version 2.0

Dedicated to Eliezer Regnier and all victims of apartheid.'

Copyright (c) 1982, 1983, 1996, 2008, 2009, 2010, 2011 Adrienne
Gaye Thompson, Sole Author. GNU C-Graph version 2.0. Derived from
BSc. dissertation "Interactive Computer Package Demonstrating: Sampling
Convolution and the FFT", University of Aberdeen, Scotland (1983). For
the code from the dissertation, visit
<http://codeartnow.com/law-project>.

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Public License (the GPL) version 3 or later. You are welcome to
distribute it under certain conditions.

GNU C-Graph is distributed WITHOUT ANY WARRANTY
without even the implied warranty of MERCHANTABILITY
or FITNESS FOR A PARTICULAR PURPOSE.

See the GPL section 15 regarding disclaimer of warranty.

Press <g> then <ENTER> to display the GPL, or just press <ENTER> to
continue.
C-Graph:>> RET

THIS IS GNU C-Graph - a tool for visualizing convolution.

Compare the linear convolution of two signals with their circular
convolution.
```

```
Signal            Code
======            ====
Sine              A
Cosine            B
Triangle          C
Square            D
Sawtooth          E
Unit-Exponential  F
Unit-Ramp         G
Unit-Step         H
```

Generate 2 signals from the above menu with up to 5 parameters:

1. Number of samples 'N'
2. Signal code 'A' to 'H'
3. Whether the signal, ifperiodic, is a wave 'w' or a pulse 'p'
4. The frequency 'f'
5. The scaling coefficient 'sc'

Choose a value for "N" between 64 and 1024.
C-Graph:>> *51*

The number of samples "N" is: 512

```
Signal            Code
======            ====
Sine              A
Cosine            B
Triangle          C
Square            D
Sawtooth          E
Unit-Exponential  F
Unit-Ramp         G
Unit-Step         H
```

Enter code for first signal
C-Graph:>> *e*

Is the signal periodic or is it a pulse?
Type "w" for periodic wave, or "p" for pulse
C-Graph:>> *p*

Select the frequency "f" of this signal.
C-Graph:>> *1.0*

```
The frequency of this signal is 1.00 Hz.

Do you wish to scale this signal?
Enter a value for the scaling coefficient "sc".
A coefficient of 1 will give the unit function.
C-Graph:>> 1

Signal              Code
======              ====
Sine                A
Cosine              B
Triangle            C
Square              D
Sawtooth            E
Unit-Exponential    F
Unit-Ramp           G
Unit-Step           H

Enter code for second signal
C-Graph:>> s

Is the signal periodic or is it a pulse?
Type "w" for periodic wave, or "p" for pulse
C-Graph:>> p

Select the frequency "f" of this signal.
C-Graph:>> 1

The frequency of this signal is 1.00 Hz.


Do you wish to scale this signal?
Enter a value for the scaling coefficient "sc".
A coefficient of 1 will give the unit function.
C-Graph:>> .5

You selected a square signal by default.
```
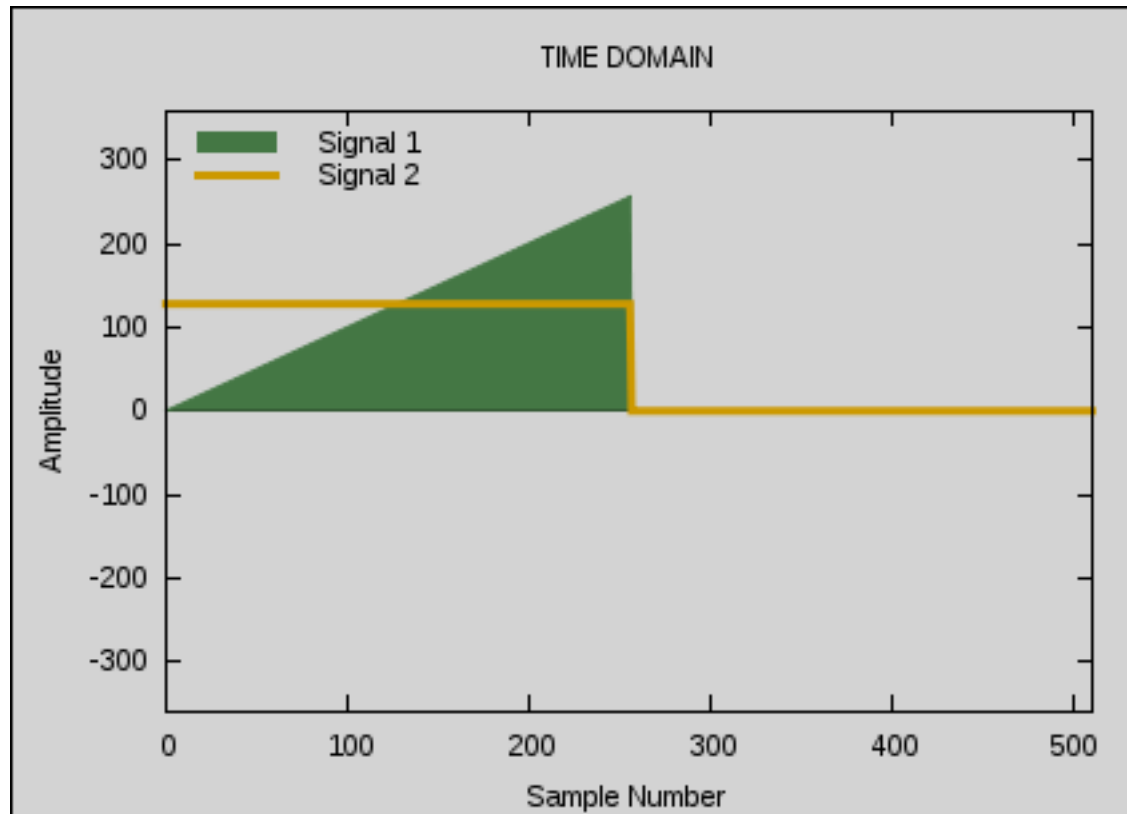
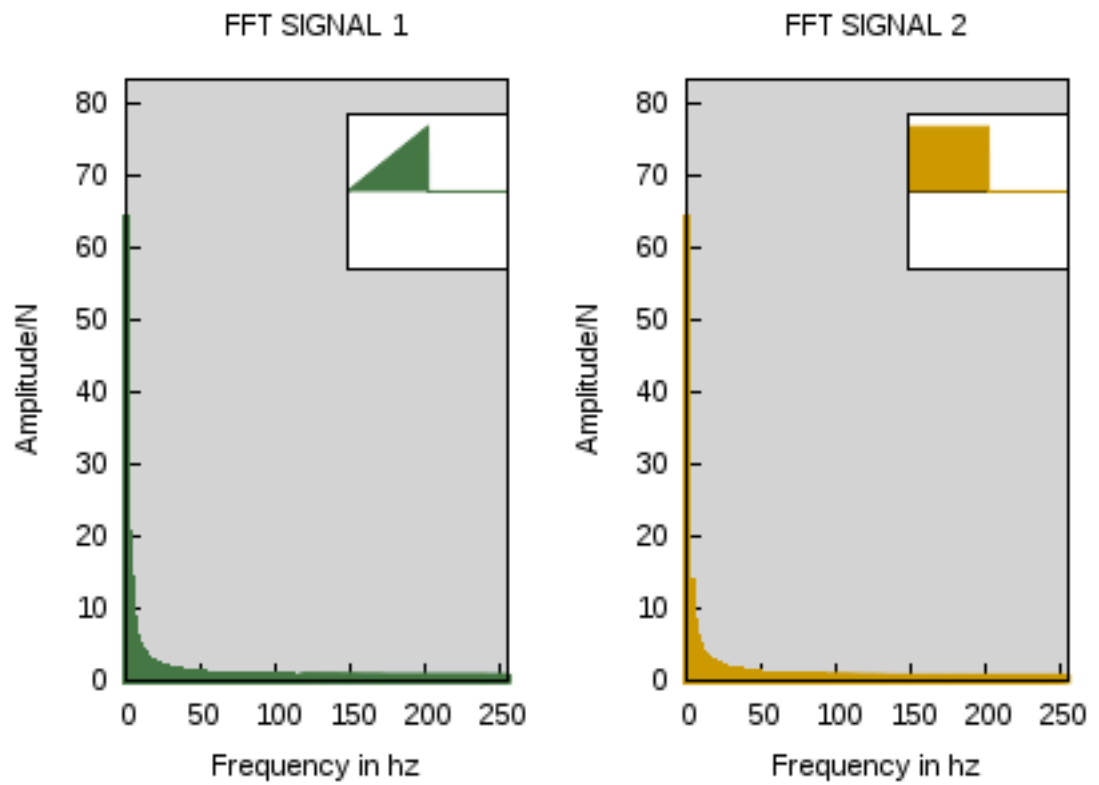Press <Enter> to see the signals in the time domain:>> RET

ALT-TAB

Hit <Enter> to continue:>> RET

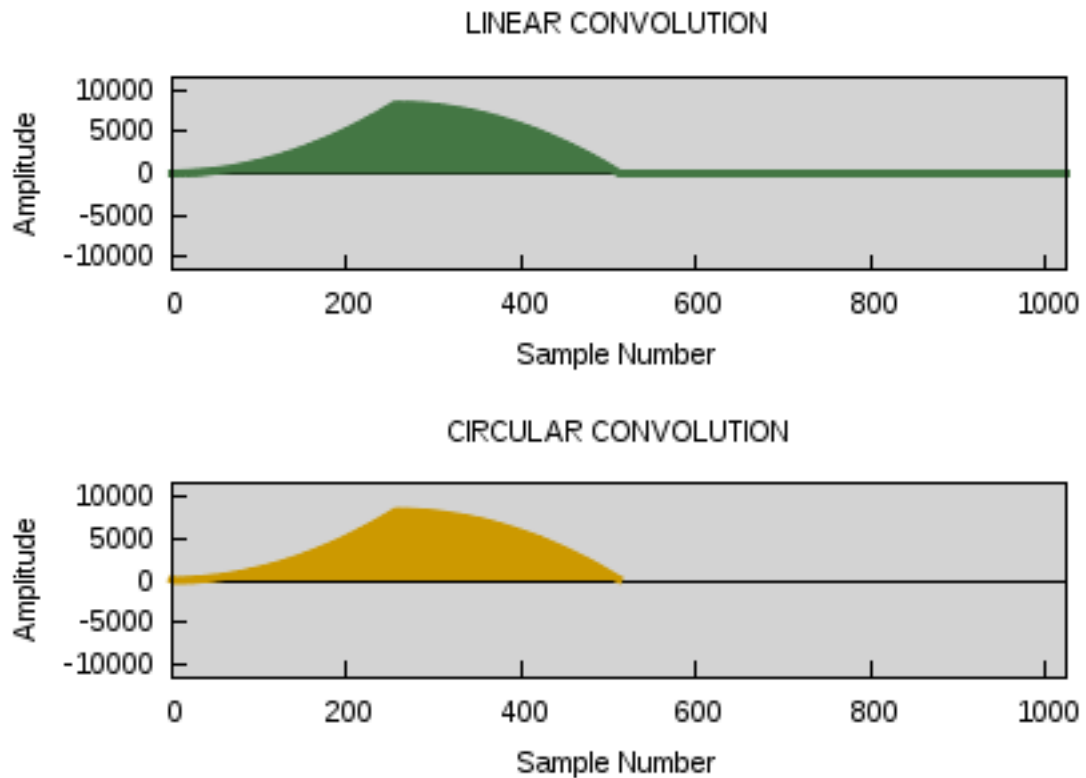View the frequency-domain representation of the signals.

Press <Enter> to see their FFTs:>> RET

FFT SIGNAL 1

FFT SIGNAL 2

ALT-TAB

Hit <Enter> to continue:>> RET

Press <Enter> to compare linear and circular convolution:>> RET

LINEAR CONVOLUTION

CIRCULAR CONVOLUTION

ALT-TAB

Hit <Enter> to continue:>> RET

Exiting GNU C-Graph ...
Bye.

galactica@regnier:~$ c-graph --no-splash

GNU C-Graph version 2.0

Dedicated to Eliezer Regnier and all victims of apartheid.'

Copyright © 1982, 1983, 1996, 2008, 2009, 2010, 2011 Adrienne
Gaye Thompson, Sole Author. GNU C-Graph version 2.0. Derived from
BSc. dissertation "Interactive Computer Package Demonstrating: Sampling
Convolution and the FFT", University of Aberdeen, Scotland (1983). For

```
the code from the dissertation, visit
<http://codeartnow.com/law-project>.

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without even the implied warranty of MERCHANTABILITY
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See the GPL section 15 regarding disclaimer of warranty.

Press <g> then <ENTER> to display the GPL, or just press <ENTER> to
continue.
C-Graph:>> RET

THIS IS GNU C-Graph - a tool for visualizing convolution.


Compare the linear convolution of two signals with their circular
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Signal              Code
======              ====
Sine                A
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Sawtooth            E
Unit-Exponential    F
Unit-Ramp           G
Unit-Step           H


Generate 2 signals from the above menu with up to 5 parameters:

1. Number of samples 'N'
2. Signal code 'A' to 'H'
3. Whether the signal, ifperiodic, is a wave 'w' or a pulse 'p'
4. The frequency 'f'
5. The scaling coefficient 'sc'

Choose a value for "N" between 64 and 1024.
C-Graph:>> 521


The number of samples "N" is: 512.
```

```
Signal              Code
======              ====
Sine                A
Cosine              B
Triangle            C
Square              D
Sawtooth            E
Unit-Exponential    F
Unit-Ramp           G
Unit-Step           H


Enter code for first signal
C-Graph:>> g


Do you wish to scale this signal?
Enter a value for the scaling coefficient "sc".
A coefficient of 1 will give the unit function.
C-Graph:>> 1


Signal              Code
======              ====
Sine                A
Cosine              B
Triangle            C
Square              D
Sawtooth            E
Unit-Exponential    F
Unit-Ramp           G
Unit-Step           H


Enter code for second signal
C-Graph:>> h


Do you wish to scale this signal?
Enter a value for the scaling coefficient "sc".
A coefficient of 1 will give the unit function.
C-Graph:>> q56


That was not a number. Try again!
C-Graph:>> 256
```
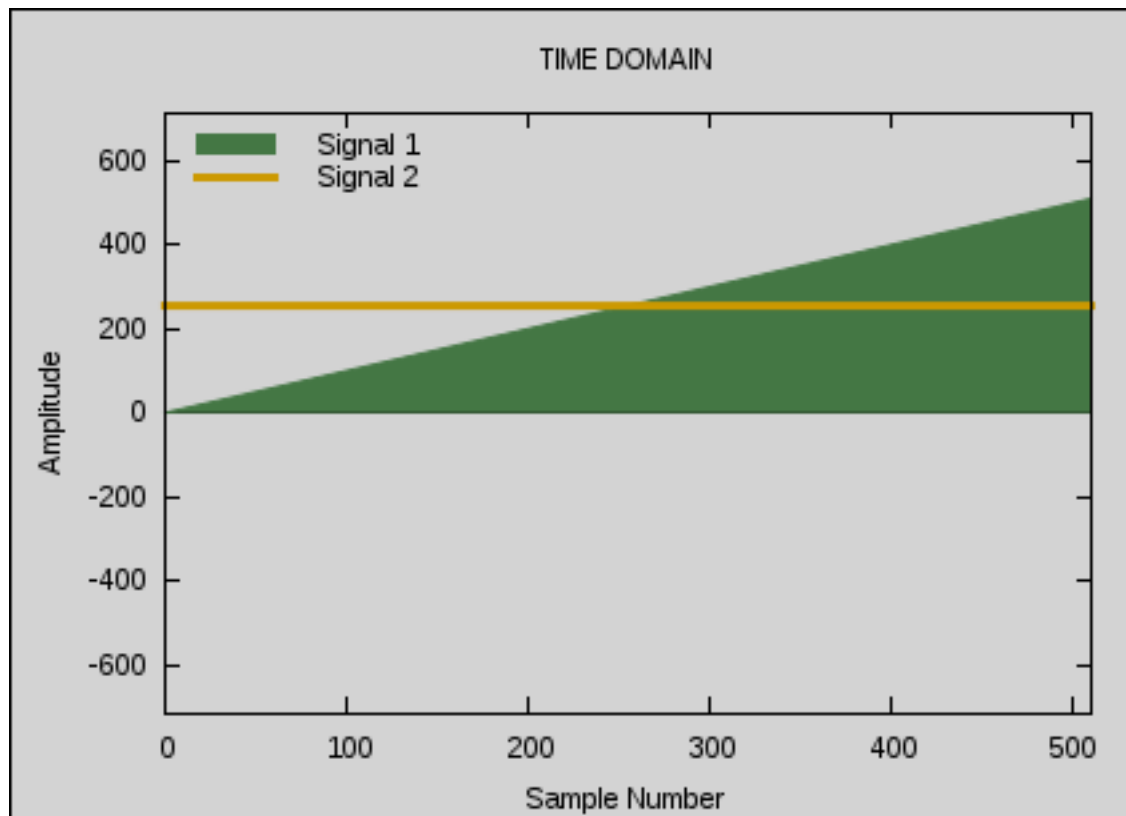
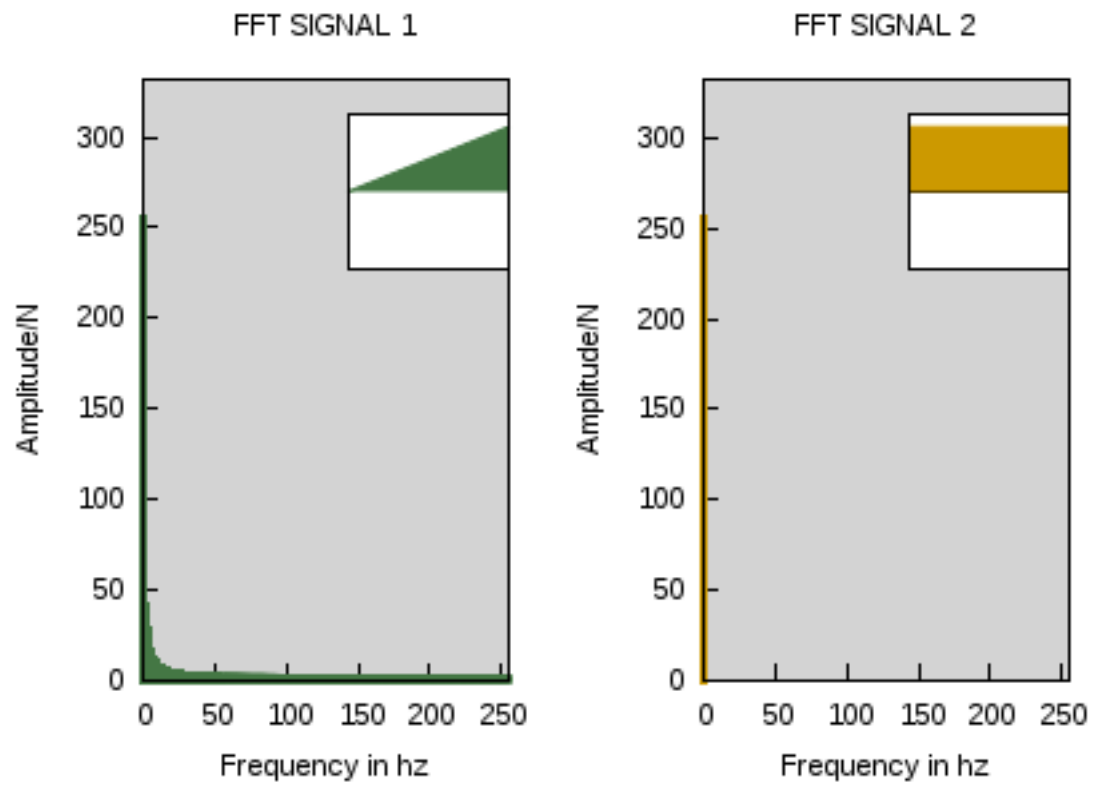Press <Enter> to see the signals in the time domain:>> RET

ALT-TAB

Hit <Enter> to continue:>> RET

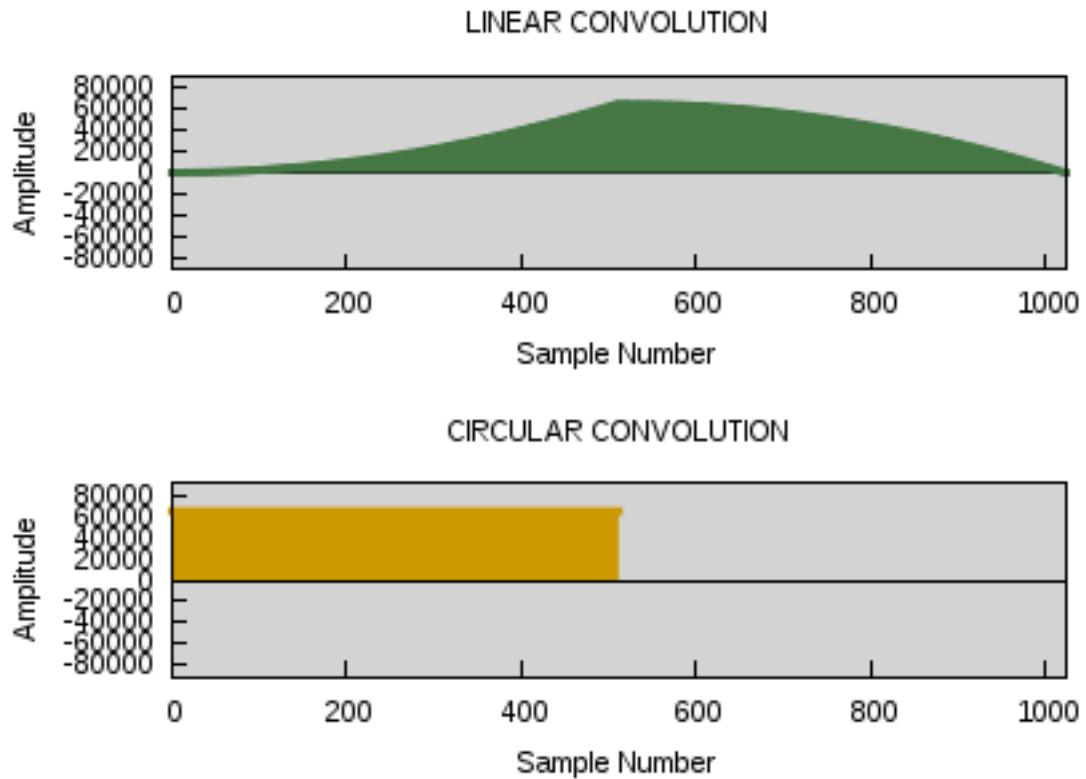View the frequency-domain representation of the signals.

Press <Enter> to see their FFTs:>> RET

FFT SIGNAL 1

FFT SIGNAL 2

ALT-TAB

Hit <Enter> to continue:>> RET

```
Press <Enter> to compare linear and circular convolution:>> RET
```

LINEAR CONVOLUTION

CIRCULAR CONVOLUTION


```
ALT-TAB

Hit <Enter> to continue:>> RET

Exiting GNU C-Graph ...
Bye.
```

On exit, the directory `c-graphs` will be created. Directory `c-graphs` will have 2 subdirectories containing the graphs, Gnuplot command files, and the data used for plotting generated by the last run. The subdirectories and files are:

- c-graphs/graphs: signals.png, transforms.png, convolutions.png
- c-graphs/coms: signals.cg, transforms.cg, convolutions.cg time.dat trans.dat

One can then print the graphs and display them either by using a graphics editor like Image Magick, or by executing the command files with Gnuplot.

6 Reporting Bugs

To report bugs or suggest enhancements for GNU C-Graph, please send electronic mail to `agt@codeartnow.com`.

For bug reports, please include enough information for the maintainers to reproduce the problem. Generally speaking, that means:

- The version numbers of GNU C-Graph (which you can find by running '`c-graph --version`') and any other program(s) or manual(s) involved.
- Hardware and operating system names and versions.
- The contents of any input files necessary to reproduce the bug.
- The expected behavior and/or output.
- A description of the problem and samples of any erroneous output.
- Options you gave to `configure` other than specifying installation directories.
- Anything else that you think would be helpful.

When in doubt whether something is needed or not, include it. It's better to include too much than to leave out something important.

Patches are welcome. Please follow the existing coding style.

Appendix A Sketch of Convolution Theory

A.1 Introductory Ideas

GNU C-Graph compares the linear and circular convolution of two signals. Subroutine `convo` computes the linear convolution directly in the time domain, while the FFT is exploited to compute circular convolution through the convolution theorem, which is defined below (see Section A.2.2 [The Convolution Theorem], page 28).

Convolution is an operation by which two functions combine to produce a third that represents a kind of moving average. This is a naturally ocurring phenomenon that presents itself whenever there is a linear system obeying the principles of superposition and shift/time invariance. Accordingly, the mathematics of convolution has found application to much of science and engineering in areas ranging from statistics to computer vision.

The output of any linear shift invariant system may be described as the convolution of the input with the impulse response of the system. In computer vision, for example, where the system being considered is a 2-dimensional image, the output of the system may be blurred as a result of the relative motion of the camera and the object. This blurred image can be modelled by convolution of the static image with the 2-dimensional impulse response.

The 2-dimensional impulse response is called a *pointspread function* (PSF). Each pixel in the image produces a copy of the PSF, scaled according to the strength of the pixel and spatially shifted. Superposition of these copies form the resultant output signal, the system being linear and shift invariant. The output blurred image is then a convolution that is, in fact, a linear combination of the PSFs. The design of a filter for image restoration must then rely on inverse convolution.

A thorough treatment of the mathematics of convolution is beyond the scope of this manual. See Appendix B [Appendix B], page 35, for some references on the subject, and related engineering theory.

A.2 The Mathematics

Consideration of the 1-dimensional case simplifies the arithmetic. To prove the convolution theorem, we first derive an expression for the convolution of 2 signals, then apply the Fourier transform to this expansion.

A.2.1 Deriving the Convolution Sum

A discrete-time signal may be modelled as a series of piecewise rectangular pulses. The summation of all such rectangular pulses approximates the signal f:

$$f(n) \quad = \quad \sum_m \ f(m) \ \ \text{rect}(n \ - \ m)$$

where $n - m$ denotes the rectangle whose base on the n axis is centred at sample $n = m$.

In the limit, the series of rectangular pulses approaches a continuous signal as the pulse width tends to zero and each pulse becomes an impulse signal. Each impulse signal can then be represented as a scaled and shifted unit impulse simulating one sample of the discrete signal.

$$f(n) \quad = \quad \sum_m \ f(m) \ \delta(n \ - \ m)$$

Applying a system transform M that maps the input signal f to the output signal g,

$$g(n) \quad = \quad M[f(n)]$$

$$g(n) \quad = \quad M \ [f(n) \ = \ \textstyle\sum_m \ f(m) \ \delta(n \ - \ m)]$$

$$= \quad \textstyle\sum_m \ f(m) \ M[\delta(n \ - \ m)]$$

Since the system transforms a delta function to the system *impulse response* h

$$g(n) \quad = \quad \textstyle\sum_m \ f(m) \ h(n \ - \ m) \qquad (1)$$

The above expression called the *convolution sum*, denoted by $f(*)h$, defines the output $g(n)$ of the system.

A.2.2 The Convolution Theorem

GNU C-Graph demonstrates the *convolution theorem*. The convolution of 2 signals in the time domain is equal to the inverse Fourier transform of the product of their transforms in the frequency domain.

Just as a signal can be represented by a linear combination of scaled and shifted impulses, we can also describe the signal as a linear combination of sinusoidal basis functions. The Fourier transform exploits this representation to deconstruct the signal into frequency components, each corresponding to a basis sinusoid.

Using Euler's identity

$$e^{j\theta} \quad = \quad \cos(\theta) \ + \ j \ \sin(\theta)$$

The sinusoidal sum representing the discrete time signal may be written in the form

$$f(n) \quad = \quad 1/N \ \sum_{k=0}^{N-1} \ F(k) \ e^{j\omega_k n}$$

where $\omega_k = 2k\pi/N$, and the $F(k)$ are Fourier transform coefficients indicating the strength of the kth spectral sample of frequency ω_k (how much of the each basis sinusoid is present in the signal).[1]

Accordingly, the $F(k)$ may be computed from the signal

$$F(k) \quad = \quad \sum_{n=0}^{N-1} \ f(n) \ e^{-j\omega_k n}$$

This is the Fourier transform description of the signal as a function of frequency.

From eqn (1), the convolution of an input signal f with the system impulse response h to give an output g is defined as:

$$g(n) \quad = \quad f(*)h \ = \ \textstyle\sum_m \ f(m) \ h(n \ - \ m)$$

---

[1] For a periodic signal, these coefficients are the δ functions of the Fourier transform.

Let the Fourier transform of $g(n)$ be denoted by

$$\Gamma[g(n)] \;=\; g(n)\ e^{-j\omega n}, \text{ then}$$

$$\Gamma[f(*)g] = \; \sum_n\ [f(*)g]\ e^{-j\omega n}$$

$$\Gamma[f(*)h] = \; \sum_n\ \sum_m\ f(m)\ h(n\ -\ m)]\ e^{-j\omega n}$$

$$= \; \sum_m\ f(m)\ \sum_n\ h(n\ -\ m)\ e^{-j\omega n}$$

Changing the variable to $p = n - m$

$$\Gamma[g(n)] \;=\; \sum_m\ f(m)\ \sum_p\ h(p)\ e^{-j\omega(m+p)}$$

$$= \; \sum_m\ f(m)e^{-jm\omega}\ \sum_p\ h(p)e^{-j\omega p}$$

Taking the inverse Fourier transform,
$$f(*)g \quad = \; \Gamma^{-1}[\ \Gamma[f]\ \Gamma[h]]$$

This is the convolution theorem.

A.3 Linear and Circular Convolution

The simulation of actual or linear convolution requires a sequence of multiplications and additions that are computationally too slow for high speed operations such as deblurring filters for precision robotic vision control systems. The *FFT*, an algorithm for efficiently computing the DFT, dramatically overcomes the computational load by successively decomposing the multiplication of two sequences into subsequences of half the length thereby reducing the number of artithmetic operations by roughly $N/\log N$.

The cost of this additional computational power is the treatment of the convolving signals as periodic with N samples per period. The resulting convolution is termed *circular convolution*. It can be shown that circular convolution and linear convolution are equivalent if $N \geq L + P - 1$ where L, M are the unpadded lengths of the sequences being convolved.

We illustrate the difference between linear and circular convolution using abbreviated sequences for the pulses demonstrated in Chapter 5 [A C-Graph Session], page 13,

A.3.1 Linear Convolution

As noted above (see Section A.1 [Introductory Ideas], page 27), in linear convolution, each sample of f contributes a scaled and shifted copy of the h. This is accomplished by the multiplication of the particular sample m of f by each sample of h.

This sequential multiplication can be visualized as a physical reflection of $h(m)$) about the vertical axis to obtain $h(-m)$ followed by discrete shifts of 1 sample interval ($\delta n = 1$) along the time axis with no overlap of the signals at the beginning and end of the translation. As the impulse response moves along the time axis, the point by point multiplication of coincident samples is summed. The sum at each point in the translation is the value of the convolution sum $g(n)$ at that point, and the length of the convolution is $L + M$.

For the sequences

$$f(m) \quad = \quad [1, 1, 1], \text{ and}$$

$$h(m) \quad = \quad [0, 1, 2]$$

The series of operations for linear convolution

$$f(*)h \quad = \quad [0 \ 1 \ 3 \ 3 \ 2], \text{ are:}$$

```
0.                             1  1  1

                            2  1  0

                            ---------

g(0) = f(m)h(0-m)          0+0+0+0+0 = 0


1.                             1  1  1

                            2  1  0

                            -------

g(1) = f(m)h(1-m)          0+1+0+0 = 1


2.                             1  1  1

                            2  1  0

                            -----

g(2) = f(m)h(2-m) =          2+1+0 = 3


3.                             1  1  1

                              2  1  0

                            -------

g(3) = f(m)h(3-m) =          0+2+1+0 = 3
```

```
4.                              1 1 1

                                  2 1 0

                                ---------

g(4) = f(m)h(4-m) =             0+0+2+0+0 = 2
```

A.3.2 Circular Convolution

We can imagine circular convolution in terms of the relative rotation of two concentric cylinders whose circumferences are of length N. A copy of the N samples comprising f is wrapped anticlockwise round one cylinder, while a copy of h is wrapped clockwise round the other cylinder, reflecting h . Rotating the second cylinder anticlockwise by 1 sample interval each time, multiplying the coincident samples and summing will give corresponding values of the convolved signal g.

$$f(m) \quad = \quad [1,1,1], \texttt{ and}$$

$$h(m) \quad = \quad [0,1,2]$$

The series of operations for circular convolution

$$f(*)h \quad = \quad [0\ 1\ 3\ 3\ 2], \texttt{ are:}$$

```
0.                              1 1 1

                                0 2 1

                                -----

g(0) = f(m)h(0-m)               0+2+1 = 3


1.                              1 1 1

                                1 0 2

                                -----

g(1) = f(m)h(1-m)               1+0+2 = 3
```

```
2.                              1 1 1

                                2 1 0

                                -----

g(2) = f(m)h(2-m)         2+1+0 = 3
```

By zero-padding each sequence of length $L = 3$ to length N so that $N \geq L + L - 1$ (see Section A.3 [Linear and Circular Convolution], page 29), we obtain the sequences

$$f(m) \quad = \quad [1, 1, 1, 0, 0]$$

$$h(m) \quad = \quad [0, 1, 2, 0, 0]$$

Circular convolution then achieves the same as result as linear convolution:

$$f(*)h \quad = \quad [0 \ 1 \ 3 \ 3 \ 2]$$

The operations are:

```
0.                              1 1 1 0 0

                                0 0 0 2 1

                                ---------

g(0) = f(m)h(0-m)         0+0+0+2+1 = 0


1.                              1 1 1 0 0

                                1 0 0 0 2

                                -----

g(1) = f(m)h(1-m)         1+0+0+0+2 = 1


2.                              1 1 1 0 0

                                2 1 0 0 0

                                ---------

g(2) = f(m)h(2-m)         2+1+0+0+0 = 3
```

```
3.                      1 1 1 0 0

                        0 2 1 0 0

                        ---------

g(3) = f(m)h(3-m)       0+2+1+0+0 = 3

4.                      1 1 1 0 0

                        0 0 2 1 0

                        ---------

g(4) = f(m)h(4-m)       0+0+2+1+0 = 2
```

Appendix B References

The sources below were consulted in the preparation of GNU C-Graph and/or the 1983 dissertation [1] from which GNU C-Graph is derived.

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