

# Automated Theorem Proving in Higher-Order Logics

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Special thanks to: Chad E. Brown



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  - ▶ the set of all even naturals can be represented by the unnamed  $\lambda$ -term  $\lambda x_{\text{nat}}. \text{even } x$

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- types to avoid logical paradoxes and inconsistencies
  - ▶ “the set of all sets that do not contain themselves”
  - ▶ our choice is: Alonzo Church’s Simple Type Theory / Classical Higher-Order Logic
  - ▶ we will **not** study rich type systems supporting polymorphism or dependent types

# Focus of the Lecture

$$\exists p. p \in A \cap B$$

$$p = \{x \mid x \text{ in } A \text{ and } x \text{ in } B\}$$

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- The theorem prover LEO

# Relevance and Applications

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## Applications of Higher-Order Theorem Proving

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## Some Historical Remarks

# History

- Ancient Greek's (mostly Aristotle):  
laws of human thought; theory of well-chosen axioms and  
rules (syllogism)

Examples:

- ▶ Modus Ponens:

$$\frac{A \Rightarrow B \quad A}{B} \text{ mp}$$

- ▶ Modus Tollens:

$$\frac{A \Rightarrow B \quad \neg B}{\neg A} \text{ mt}$$

# History (Cont'd)

- Computer algebra systems
  - ▶ Roots: Abacus (approx. 500 a.D.)
  - ▶ mechanical calculators built from 15. century on
  - ▶ modern computer algebra systems are today widely used



W. Schickard's mechanical calculator  
(1592 - 1635)

A screenshot of the MathPert system interface. It shows a step-by-step solution for the equation  $\frac{5x}{2} - 5 = 3x - 7$ . The steps are:

- the problem
- $\frac{5x}{2} - 5 = 3x - 7$
- add 5
- $\frac{5x}{2} = 3x - 2$
- multiply by 2
- $5x = 6x - 4$
- subtract 6x
- $-x = -4$
- change signs
- x = 4**

A message box says "That's the answer!" with "OK" button.

MathPert system (Michael Beeson)



# History (Cont'd)

- Gottfried Wilhelm Leibniz (1646-1716)
  - ▶ Dream of formalizing and mechanizing mathematical reasoning (lingua characteristica and calculus ratiocinator)
  - ▶ Mechanization of simple arithmetical operations, e.g., mechanical calculator capable of multiplication

# History (Cont'd)

- It took until the end of the 19. century that *modern mathematical logic* was (re-)born:
  - ▶ George Boole (1815-1864),
  - ▶ Gottlob Frege (1848-1925),
  - ▶ Bertrand Russel (1872-1970),
  - ▶ and many others

stimulated the new and deep interest of many researchers in the field of mathematical logic.

# History (Cont'd)

- Frege's *Begriffsschrift* is described by Davis [Davis83] *not only as the direct ancestor of contemporary systems of mathematical logic but also as the ancestor of all formal languages, including computer programming languages.*
- First-order logic:

$$\forall x, y, z. (x + (y + z)) = ((x + y) + z)$$

# History (Cont'd)

- *Hilbert's program* at the very beginning of this century [Hilbert04,Hilbert27] aimed at the complete development of modern mathematics in a formal system.

# History (Cont'd)

- In the early 30's results came fast:
  - ▶ Kurt Gödel, Jacques Herbrand and Thoralf Skolem proved the *completeness of the (first-order) predicate calculus* in 1930 [Goedel30,Herbrand30,Skolem28]: every valid formula in the language of the predicate calculus is derivable from its axioms.
  - ▶ However, Gödel showed in his famous *incompleteness theorems* [Goedel31] that it is impossible to develop a generally complete calculus that mechanizes mathematical reasoning. More precisely, Gödel showed that *as soon as a system is rich enough to encode Peano arithmetic, one can construct sentences that are valid in Peano arithmetic but which are not derivable in the system itself.*

# History (Cont'd)

- Gerhard Gentzen [Gentzen35]: Natural Deduction Calculus (ND)

ND-Rules

(examples)

$$\frac{A \Rightarrow B \quad A}{B} \text{ mp}$$

$$\frac{A \quad B}{A \wedge B} \wedge I$$

$$\frac{}{[A]_1} \vdots$$

$$\frac{A \wedge B}{A} \wedge E_I$$

$$\frac{B}{A \Rightarrow B} \Rightarrow I^1$$

$$\frac{A \wedge B}{B} \wedge E_r$$

... etc. ....

ND-Proof for  $(A \wedge B) \Rightarrow (B \wedge (C \vee A))$

$$\frac{\frac{\frac{[A \wedge B]_1}{B} \wedge E_r \quad \frac{[A \wedge B]_1}{\frac{A}{C \vee A}} \wedge E_I}{B \wedge (C \vee A)} \wedge I}{(A \wedge B) \Rightarrow (B \wedge (C \vee A))} \Rightarrow I^1$$

- Introduced Sequent Calculus initially as tool for investigating cut elimination; however, many interactive theorem provers today employ sequent calculus and not natural deduction calculus.

# History (Cont'd)

- Development of electronic computers in the 40's and 50's: disappointment gradually gave away to attempts of developing and implementing proof procedures in practice.
- J.A. Robinson: achieved important break-through (in first-order theorem proving) with his resolution approach in 1965 [Robinson65]. The most important improvement of this approach compared to former ones is that in order to prove a theorem it tries to *refute the negated theorem in a goal directed way*, thereby *employing first-order unification as a powerful filter* instead of simply enumerating the Herbrand universe like most earlier methods.
- Remark: *This lecture (amongst others) investigates the problems of applying the resolution idea in higher-order logic.*

# History (Cont'd)

- Robinson's ideas are still employed in many state of the art theorem provers such as OTTER, EQP (which solved the Robbins Problem), or the superposition based prover SPASS.
- Even tableaux based provers like PROTEIN are rather closely related to the resolution approach and unification became an essential (filtering) tool for the whole field.

# History HOL

- Higher-order logic: any simply typed logical system that allows quantification over function and predicate variables.
- It was Bertrand Russel [Russell02, Russell03] who first pointed out in 1902 that in connection with the comprehension principles (these principles assure the existence of certain functions) this may allow for *paradoxes*: most prominent example is the *set of all non-self-containing sets*.
- As a possible solution Russel suggested a few years later in [Russell08] a theory of types as a basis for the formalization of mathematics that differentiates between objects and sets (or functions) consisting of these kinds of objects.

# History HOL (Cont'd)

- Idea was also taken up by Alonzo Church in 1940, who invented the *simply typed  $\lambda$ -calculus* [Church40] in order to prevent such paradoxes in the untyped  $\lambda$ -calculus, which he developed with Schönfinkel and Curry ten years earlier.
- Typed and untyped  $\lambda$ -calculi play an important or even central role in many research fields of modern computer science.
- The avoidance of paradoxes like Russel's paradox is also a main reason why we employ a logic based on Church's simply typed  $\lambda$ -calculus [Church40] — i.e., *classical type theory* / *classical higher-order logic* — in this lecture.

# History HOL (Cont'd)

- Relatively few researchers concentrated on the mechanization of higher-order logic.
- Robinson presents in [Robinson68, Robinson69] a higher-order proof procedure based on the tableaux idea that itself employs many ideas from the calculi given in [Schuette60] and [Takeuti53].
- The most important works to be mentioned are Peter Andrews' investigation of higher-order resolution [Andrews71], Jensen and Pietrowski's approach [JePi72] and especially Gerard Huet's constrained resolution approach [Huet72, Huet73].

# History (Cont'd)

- Big challenge for the mechanization of higher-order logic is the undecidability of higher-order unification [[Lucchesi72](#),[Huet73](#),[Goldfarb81](#)].
- Andrews' resolution approach still avoids unification (and instead employs an enumeration of the universe)
- Huet's constrained resolution approach [[Huet72](#)] solves the problem by encoding the particular unification problems as unification constraints and by delaying the application of higher-order unification until the end of a refutation.
- Huet's approach additionally gains from the higher-order pre-unification algorithm [[Huet75](#)] which avoids the guessing aspects of full higher-order unification



## Introduction

# $\lambda$ -Calculus: Motivation

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Consider the following arithmetical computations

$$(-1)^2 - 1 = 0$$

$$(1)^2 - 1 = 0$$

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A more general arithmetic expression for the LHS:

$$x^2 - 1$$

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Consider the 0's (Nullstellen) of this function; we can express the existence of two 0's in first-order logic as follows

$$\exists n, m. n^2 - 1 = 0 \wedge m^2 - 1 = 0 \wedge n \neq m$$

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Now we may want to talk about the existence of a function  $f$  with two 0's:

$$(1) \quad \exists f. \exists n, m. f(n) = 0 \wedge f(m) = 0 \wedge n \neq m$$

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This expression is not a first-order statement; however we want to be able to express such statements. We also want to prove such statements and in a constructive proof we would like to provide witnesses for  $f$  and  $n, m$ . In first-order logic we can describe  $f$  by the following equation

$$f(x) = x^2 - 1$$

# $\lambda$ -Calculus: $\lambda$ -terms

In  $\lambda$ -calculus the specified function  $f$  can be described (without giving it a name) by the witnessing  $\lambda$ -term

$$[\lambda x.x^2 - 1]$$

and the witnesses for  $n$  and  $m$  are  $-1$  and  $1$ .

# $\lambda$ -Calculus: Set of $\lambda$ -expressions

Given a countably infinite set of identifiers, say  
a, b, c, ..., x, y, z, x1, x2, .... The set of all  $\lambda$ -expressions can then be described by the following context-free grammar in BNF:

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1.  $\langle \text{expr} \rangle ::= \langle \text{identifier} \rangle$
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3.  $\langle \text{expr} \rangle ::= [\langle \text{expr} \rangle \langle \text{expr} \rangle]$  application

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We often omit brackets with the following conventions:

- $[F A B]$  means  $[[F A] B]$ . (Application associates to the left.)

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- $[FAB]$  means  $[[FA]B]$ . (Application associates to the left.)
- $[\lambda x.\lambda y. B]$  means  $[\lambda x.[\lambda y. B]]$ .
- A dot (except possibly after  $\lambda$  <identifier>) stands for a left bracket whose mate is as far to the right as possible without changing the existing bracketing.

# $\lambda$ -Calculus: $\beta$ -reduction



Consider now the instantiation of (1) with these witness terms

$$\exists f. \exists n, m. f(n) = 0 \wedge f(m) = 0 \wedge n \neq m$$

# $\lambda$ -Calculus: $\beta$ -reduction



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$$\exists f. \exists n, m. f(n) = 0 \wedge f(m) = 0 \wedge n \neq m$$

$$f \longrightarrow \exists n, m. [[\lambda x. x^2 - 1] n] = 0 \wedge [[\lambda x. x^2 - 1] m] = 0 \wedge n \neq m$$

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$$n, m \longrightarrow [[\lambda x. x^2 - 1] - 1] = 0 \wedge [[\lambda x. x^2 - 1] 1] = 0 \wedge -1 \neq 1$$

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$$n, m \longrightarrow [[\lambda x. x^2 - 1] - 1] = 0 \wedge [[\lambda x. x^2 - 1] 1] = 0 \wedge -1 \neq 1$$

Finally we can ‘evaluate’ function applications by so called  $\beta$ -reduction

$$((-1)^2 - 1) = 0 \wedge (1^2 - 1) = 0 \wedge -1 \neq 1$$

# $\lambda$ -Calculus: $\beta$ -reduction

The  $\beta$ -reduction rule expresses the idea of function application as motivated on the previous slide. Formally it states that

$$[[\lambda x. A] B] \longrightarrow_{\beta} A[x/B]$$

if all free occurrences in  $B$  remain free in  $A[x/B]$ . Here,  $A[x/B]$  means the expression  $E$  with every free occurrence of  $x$  in  $A$  replaced with  $B$ .

# $\lambda$ -Calculus: Currying

A function of two variables is expressed in lambda calculus as a function of one argument which returns a function of one argument. For instance, the function

$$f(x, y) = x^2 - y$$

is encoded as

$$[\lambda x. \lambda y. x^2 - y]$$

# $\lambda$ -Calculus: $\alpha$ -conversion

The names of the bound variables are unimportant:

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Formally, the  $\alpha$ -conversion rule states that if  $x$  and  $y$  are variables and  $A$  is a  $\lambda$ -expression then

$$[\lambda x.A] \longleftrightarrow_{\alpha} [\lambda y.A[x/y]]$$

if  $y$  does not appear freely in  $A$  and  $y$  is not bound by a  $\lambda$  in  $A$  whenever it replaces a  $x$ .

# $\lambda$ -Calculus: $\eta$ -reduction

$\eta$ -reduction expresses the idea of (functional) extensionality, which in this context is that two functions are the same iff they give the same result for all arguments:

$$[\lambda x. Fx] \longrightarrow_{\eta} F$$

whenever  $x$  does not appear free in  $F$ .

# $\lambda$ -Calculus: $\beta\eta$ -equivalence

- We define  $\longleftrightarrow_{\alpha\beta\eta}^*$  as the smallest equivalence relation closed under the reduction rules  $\longrightarrow_\beta$  and  $\longrightarrow_\eta$  and  $\alpha$ -conversion.  
(Similarly we may define  $\longleftrightarrow_M^*$  for  $M \subset \{\alpha, \beta, \eta\}$ )

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(Similarly we may define M-equivalence for  $M \subset \{\alpha, \beta, \eta\}$ )

# $\lambda$ -Calculus: Normalforms

- A  $\lambda$ -expression is called a  $\beta$ -normal form if it does not allow any  $\beta$ -reduction, i.e., has no subexpression of the form

$$[[\lambda x . A] B]$$

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- A  $\lambda$ -expression is called a  $\beta\eta$ -normal form if it satisfies both conditions.

# $\lambda$ -Calculus: Normalforms



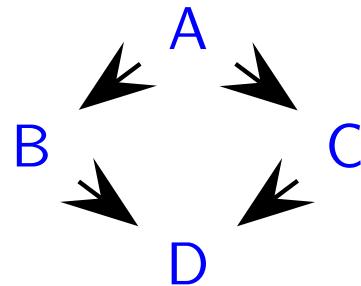
- Not every  $\lambda$ -expression is equivalent to a ?-normal form (where  $? \in \{\beta, \beta\eta\}$ )

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- Not every  $\lambda$ -expression is equivalent to a  $\beta$ -normal form (where  $\beta \in \{\beta, \beta\eta\}$ )
- The Church-Rosser theorem(s) state that if  $A \rightarrow^* B$  and  $A \rightarrow^* C$ , then there is some  $D$  such that  $B \rightarrow^* D$  and  $C \rightarrow^* D$ .

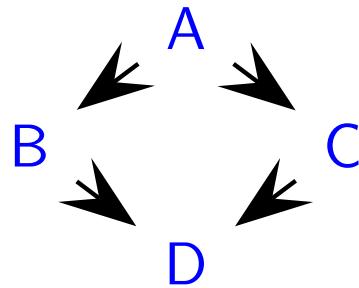
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- From Church-Rosser it follows that every term has at most one  $\beta$ -normal form (up to  $\alpha$ -conversion).

# $\lambda$ -Calculus: Iteration

Consider twofold iteration of function  $f := [\lambda x.x^2 - 1]$

$$f(f(x)) = (x^2 - 1)^2 - 1 = x^4 - 2x^2$$

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# $\lambda$ -Calculus: Church Numerals

We employ iteration to define natural numbers as Church numerals:

$$\bar{0} = [\lambda f. \lambda x. x], \quad \bar{1} = [\lambda f. \lambda x. fx], \quad \bar{2} = [\lambda f. \lambda x. f(fx)], \quad \dots$$

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Generally a natural number  $n$  is encoded as the Church numeral

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Intuitively, the number  $n$  in lambda calculus is a function that takes a function  $f$  as argument and returns the  $n$ -th iterate of  $f$ .

# $\lambda$ -Calculus: Church Numerals

We can now define a successor function  $\overline{\text{SUCC}}$ , which takes a number  $\overline{n}$  and returns  $\overline{n + 1}$ :

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Multiplication can then be defined as

$$\overline{\text{MULT}} = \lambda m. \lambda n. m[\overline{\text{PLUS}}\ n]\overline{0},$$

the idea being that multiplying  $m$  and  $n$  is the same as adding  $n$  to 0  $m$  times.

# $\lambda$ -Calculus: Church Numerals



The predecessor function is more difficult:

$$\overline{\text{PRED}} = \lambda n. \lambda f. \lambda x. n[\lambda g. \lambda h. h [g f]] [\lambda u. x] [\lambda u. u]$$

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or alternatively

$$\overline{\text{PRED}} = \lambda n. n[\lambda g. \lambda k. [g \overline{1}] [\lambda u. \overline{\text{PLUS}} [g k] \overline{1}] k] [\lambda l. \overline{0}] \overline{0}$$

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Note the trick  $[g \overline{1}] [\lambda u. \overline{\text{PLUS}} [g k] \overline{1}] k$  which evaluates to  $k$  if  $[g \overline{1}]$  is  $\overline{0}$  and to  $[g k] + \overline{1}$  otherwise.

# $\lambda$ -Calculus: Sets

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$$\exists A. \exists m, n. m \in A \wedge n \in A \wedge m \neq n$$

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Alternatively, we can express the characteristic function of A by the  $\lambda$ -term

$$[\lambda x. [x^2 - 1 = 0]]$$

# $\lambda$ -Calculus: Sets

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The characteristic function  $[\lambda x. x^2 - 1 = 0]$  provides a witness for

$$\exists P. \exists m, n. [P m] \wedge [P n] \wedge m \neq n$$

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3.  $\forall P. [P \bar{0}] \wedge [\forall x. [P x] \supset [P [\text{SUCC } x]]] \supset [\bar{\mathbb{N}} \subseteq P]$   
 “ $\bar{\mathbb{N}}$  is the least such set”

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Define  $\bar{\mathbb{N}}$  to be:

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- $[\bar{N} \bar{0}]$  since  $[P \bar{0}]$  implies  $[P \bar{0}]$
- $\forall x. [\bar{N} x] \supset [\bar{N} [\overline{\text{SUCC}} x]]$  since if  $P x$  and  $P$  is closed under successor, then  $P [\overline{\text{SUCC}} p]$

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- $\forall P. [P \bar{0}] \wedge [\forall x. [P x] \supset [P \overline{\text{SUCC}} x]]] \supset [\bar{N} \subseteq P]$   
 $\bar{N}$  is the least such set as the intersection of all such sets  $P$

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We have used quantification over sets (characteristic functions – the variable  $P$ ) to define  $\bar{N}$ .

# $\lambda$ -Calculus: Russell's Paradox



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Russell's paradox:

Consider the term R:

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As a characteristic function, R represents the set of all sets which do not contain themselves:

$$\{x | x \notin x\}$$

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which is equivalent to  $[R R]$

Thus if  $E$  holds we can infer  $\neg E$  and vice versa. This is Russell's paradox.

# $\lambda$ -Calculus: Nontermination

Note that the term  $[\lambda x.\neg.xx]$  (just as the standard example  $[\lambda x.xx]$ ) does not terminate with respect to  $\beta$ -reduction:

$$[RR] \longrightarrow_{\beta} \neg[RR] \longrightarrow_{\beta} \neg\neg[RR] \longrightarrow_{\beta} \dots$$

# Typed $\lambda$ -Calculus

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One may include arbitrarily many base types  $\iota^1, \dots, \iota^n, \dots$

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Note that the type  $(\alpha\beta\gamma)$  (or  $(\gamma \rightarrow \beta \rightarrow \alpha)$ ) is the type of a (Curried) function of two arguments which returns a value of type  $\alpha$ .

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Examples:

- $[\lambda x_\alpha. x_\alpha]$  term of type  $(\alpha\alpha)$  – identity on type  $\alpha$

# Typed $\lambda$ -Calculus: Typed Terms

- Typed Variables  $x_\alpha$
- Typed Constants and Parameters  $P_\alpha$
- Application  $[F_{\alpha\beta} B_\beta]_\alpha$  – or  $[F_{\beta \rightarrow \alpha} B_\beta]_\alpha$
- $\lambda$ -abstraction  $[\lambda y_\beta. A_\alpha]_{\alpha\beta}$  – or  $[\lambda y_\beta. A_\alpha]_{\beta \rightarrow \alpha}$

Examples:

- $[\lambda x_\alpha. x_\alpha]$  term of type  $(\alpha\alpha)$  – identity on type  $\alpha$
- $[\lambda y_\beta. x_\alpha]$  term of type  $(\alpha\beta)$  – constant  $x$ -valued function

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$$[\lambda x.x^2 - 1]$$

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where MINUS, SQUARE and 1 are constants.

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- **MINUS** should take two real numbers to a real number (type  $(\iota\iota\iota)$ )

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Typed Term:

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This term has type  $(\iota\iota)$ .

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Consider the untyped term

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- 0 should be a real number (type  $\nu$ )

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$$[\lambda x. [x^2 - 1 = 0]]$$

This is shorthand for

$$[\lambda x. [= [\text{MINUS} [\text{SQUARE} x] 1] 0]]$$

where  $=$ , MINUS, SQUARE, 0 and 1 are constants.

- Already know types of MINUS, SQUARE and 1.
- 0 should be a real number (type  $\nu$ )
- $=$  takes two real numbers and returns a truth value (type  $(\nu\nu)$ )

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This is shorthand for

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where  $=$ , MINUS, SQUARE, 0 and 1 are constants.

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$$[\lambda x_\iota. [=_{\circ\iota\iota} [\text{MINUS}_{\iota\iota} [\text{SQUARE}_\iota x_\iota] 1_\iota] 0_\iota]]$$

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Typed Term:

$$[\lambda x_\iota. [=_{\circ\iota\iota} [\text{MINUS}_{\iota\iota} [\text{SQUARE}_\iota x_\iota] 1_\iota] 0_\iota]]$$

This term has type  $(\circ\iota)$ .

# Typed $\lambda$ -Calculus: Assigning Types

General algorithm for assigning types to terms (when this is possible) – see Hindley97.

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$$\frac{\Gamma, y : \beta \vdash_{\text{TA}} A : \alpha}{\Gamma \vdash_{\text{TA}} [\lambda y. A] : \alpha\beta} \text{Lam}$$

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$$\frac{\Gamma \vdash_{\text{TA}} F : \alpha\beta \quad \Gamma \vdash_{\text{TA}} B : \beta}{\Gamma \vdash_{\text{TA}} [F B] : \alpha} \text{App}$$

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The basis for such an algorithm is the following deduction system:

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$$\frac{\Gamma \vdash_{TA} F : \alpha\beta \quad \Gamma \vdash_{TA} B : \beta}{\Gamma \vdash_{TA} [F B] : \alpha} \text{App}$$

We can assign the type  $\alpha$  to a term  $A$  in context  $\Gamma$  whenever we can derive

$$\Gamma \vdash_{TA} A : \alpha$$

# Typed $\lambda$ -Calculus: Assigning Types

Untyped Term:  $[\lambda x. \text{SQUARE } x]$

Goal: Find a type  $\alpha$  such that

$\text{SQUARE} : (\iota\iota) \vdash_{\text{TA}} [\lambda x. \text{SQUARE } x] : \alpha$

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$\alpha$  is  $(\gamma\beta)$

$$\frac{\vdots}{\text{SQUARE} : (\underline{\alpha}), x : \beta \vdash_{\text{TA}} [\text{SQUARE} x] : \gamma} \text{SQUARE} : (\underline{\alpha}) \vdash_{\text{TA}} [\lambda x. \text{SQUARE} x] : \gamma\beta \quad \text{Lam}$$

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$$\frac{\text{SQUARE} : (\underline{\alpha}), x : \beta \vdash_{\text{TA}} [\text{SQUARE} x] : \gamma}{\text{SQUARE} : (\underline{\alpha}) \vdash_{\text{TA}} [\lambda x. \text{SQUARE} x] : \gamma\beta} \text{Lam}$$

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$\gamma$  and  $\delta$  are both  $\iota$

$$\frac{\text{SQUARE} : (\iota\iota), x : \beta \vdash_{\text{TA}} \text{SQUARE} : (\iota\iota) \quad \text{Hyp} \quad \text{SQUARE} : (\iota\iota), x : \beta \vdash_{\text{TA}} x : \iota \quad \vdots}{\text{SQUARE} : (\iota\iota), x : \beta \vdash_{\text{TA}} [\text{SQUARE} x] : \iota} \text{App}$$

$$\frac{\text{SQUARE} : (\iota\iota), x : \beta \vdash_{\text{TA}} [\text{SQUARE} x] : \iota}{\text{SQUARE} : (\iota\iota) \vdash_{\text{TA}} [\lambda x. [\text{SQUARE} x]] : \iota\beta} \text{Lam}$$

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$\beta$  is  $\iota$

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$$\frac{\text{Hyp} \quad \text{Hyp}}{\text{SQUARE} : (\underline{\alpha}), x : \underline{\alpha} \vdash_{\text{TA}} \text{SQUARE} : (\underline{\alpha}) \quad \text{SQUARE} : (\underline{\alpha}), x : \underline{\alpha} \vdash_{\text{TA}} x : \underline{\alpha}}
 \frac{\text{App}}{\text{SQUARE} : (\underline{\alpha}), x : \underline{\alpha} \vdash_{\text{TA}} [\text{SQUARE} x] : \underline{\alpha}}
 \frac{\text{Lam}}{\text{SQUARE} : (\underline{\alpha}) \vdash_{\text{TA}} [\lambda x. [\text{SQUARE} x]] : \underline{\alpha}}$$

So  $[\lambda x. [\text{SQUARE} x]]$  can be assigned the type  $(\underline{\alpha})$  in context

$\text{SQUARE} : (\underline{\alpha})$

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$$\frac{\text{SQUARE} : (\underline{\alpha}), x : \underline{\alpha} \vdash_{\text{TA}} [\text{SQUARE} x] : \underline{\alpha}}{\text{SQUARE} : (\underline{\alpha}) \vdash_{\text{TA}} [\lambda x. \text{SQUARE} x] : \underline{\alpha}} \quad \text{Lam}$$

So  $[\lambda x. \text{SQUARE} x]$  can be assigned the type  $(\underline{\alpha})$  in context

$\text{SQUARE} : (\underline{\alpha})$

Corresponding Typed Term:  $[\lambda x_\alpha. \text{SQUARE}_\alpha x_\alpha]$

# Typed $\lambda$ -Calculus: Assigning Types

Untyped Term:  $[\lambda x. \neg [xx]]$

Goal: Find a type  $\alpha$  such that  $\neg : (\alpha\alpha) \vdash_{\text{TA}} [\lambda x. \neg [xx]] : \alpha$

# Typed $\lambda$ -Calculus: Assigning Types

Untyped Term:  $[\lambda x. \neg [xx]]$

Goal: Find a type  $\alpha$  such that  $\neg : (\text{oo}) \vdash_{\text{TA}} [\lambda x. \neg [xx]] : \alpha$

$$\vdots$$
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Goal: Find a type  $\alpha$  such that  $\neg : (\text{oo}) \vdash_{\text{TA}} [\lambda x. \neg [xx]] : \alpha$   
 $\alpha$  is  $(\gamma\beta)$

$$\frac{\vdots}{\neg : (\text{oo}), x : \beta \vdash_{\text{TA}} [\neg [xx]] : \gamma} \text{Lam}$$

$$\neg : (\text{oo}) \vdash_{\text{TA}} [\lambda x. \neg [xx]] : \gamma\beta$$

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 $\gamma$  and  $\delta$  are both  $\text{o}$

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Untyped Term:  $[\lambda x. \neg [xx]]$

Goal: Find a type  $\alpha$  such that  $\neg : (\text{oo}) \vdash_{\text{TA}} [\lambda x. \neg [xx]] : \alpha$

$$\frac{\vdots \quad \vdots}{\neg : (\text{oo}), x : \beta \vdash_{\text{TA}} x : (\text{o}\epsilon) \quad \neg : (\text{oo}), x : \beta \vdash_{\text{TA}} x : \epsilon} \text{App}$$

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Goal: Find a type  $\alpha$  such that  $\neg : (\text{oo}) \vdash_{\text{TA}} [\lambda x. \neg [xx]] : \alpha$   
 $\beta$  is  $(o\epsilon)$

$$\frac{\neg : (\text{oo}), x : (o\epsilon) \vdash_{\text{TA}} x : (o\epsilon) \quad \neg : (\text{oo}), x : (o\epsilon) \vdash_{\text{TA}} x : \epsilon}{\neg : (\text{oo}), x : (o\epsilon) \vdash_{\text{TA}} [xx] : o} \text{App}$$

Hyp

# Typed $\lambda$ -Calculus: Assigning Types

Untyped Term:  $[\lambda x. \neg [xx]]$

Goal: Find a type  $\alpha$  such that  $\neg : (oo) \vdash_{\text{TA}} [\lambda x. \neg [xx]] : \alpha$

Only remaining subgoal:

$$\neg : (oo), x : (o\epsilon) \vdash_{\text{TA}} x : \epsilon$$

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This goal cannot be solved since  $(o\epsilon)$  cannot equal  $\epsilon$ .

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This goal cannot be solved since  $(o\epsilon)$  cannot equal  $\epsilon$ .

Hence  $[\lambda x. \neg [xx]]$  cannot be typed – avoiding Russell's Paradox.

# Typed $\lambda$ -Calculus: $\beta\eta$

$\beta$ -reduction:

$$[[\lambda y_\beta . A_\alpha] B_\beta] \longrightarrow_\beta A_\alpha[y_\beta/B_\beta]$$

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Facts:

- $\beta\eta$ -normalization terminates for typed terms.

# Typed $\lambda$ -Calculus: $\beta\eta$

$\beta$ -reduction:

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$\eta$ -reduction:

$$[\lambda y_\beta . F_{\alpha\beta} y_\beta] \longrightarrow_\eta F_{\alpha\beta}$$

Facts:

- $\beta\eta$ -normalization terminates for typed terms.
- Every typed term has a unique  $\beta\eta$ -normal form.



## Introduction (Contd.)

# Typed $\lambda$ -Calculus: Logical Constants



We gain expressive power by combining typed  $\lambda$ -calculus with logical constants.

# Typed $\lambda$ -Calculus: Logical Constants

We gain expressive power by combining typed  $\lambda$ -calculus with logical constants.

$\top_o$  – true

$\perp_o$  – false

$\neg_{oo}$  – negation

$\vee_{ooo}$  – disjunction

$\wedge_{ooo}$  – conjunction

$\supset_{ooo}$  – implication

$\equiv_{ooo}$  – equivalence

# Typed $\lambda$ -Calculus: Logical Constants

We gain expressive power by combining typed  $\lambda$ -calculus with logical constants.

$=_{\text{o}\alpha\alpha}^{\alpha}$  – equality at type  $\alpha$

$\Pi_{\text{o}(\text{o}\alpha)}^{\alpha}$  – universal quantification over type  $\alpha$

$\Sigma_{\text{o}(\text{o}\alpha)}^{\alpha}$  – existential quantification over type  $\alpha$

Intuition:  $[\Sigma^{\alpha} . \lambda x_{\alpha} . C_o]$  is true iff  $\{x_{\alpha} | C\}$  is nonempty.

Church's Classical Type Theory: HOL

# HOL: Abbreviations

$[A_o \vee B_o]$  means  $[\vee_{ooo} A_o B_o]$

$[A_o \wedge B_o]$  means  $[\wedge_{ooo} A_o B_o]$

$[A_o \supset B_o]$  means  $[\supset_{ooo} A_o B_o]$

$[A_o \equiv B_o]$  means  $[\equiv_{ooo} A_o B_o]$

$[A_\alpha =^\alpha B_\alpha]$  means  $[=_{o\alpha\alpha}^\alpha A_\alpha B_\alpha]$

$[\forall x_\alpha. A_o]$  means  $[\Pi_{o(o\alpha)}^\alpha . \lambda x_\alpha. A_o].$

$[\exists x_\alpha. A_o]$  means  $[\Sigma_{o(o\alpha)}^\alpha . \lambda x_\alpha. A_o].$

# HOL: Expressing Properties



$$[\lambda x_\iota. x^2 - 1]$$

# HOL: Expressing Properties



$$[\lambda x_\iota. x^2 - 1]$$

$$[\lambda x_\iota. [\text{MINUS}_{\iota\iota} [\text{SQUARE}_\iota x] 1_\iota]]_\iota$$

# HOL: Expressing Properties

$$[\lambda x.x^2 - 1]$$

Term of type  $\circ$  expressing existence of an  $f$  with two roots:

$$[\exists f_\nu. \exists n_\nu. \exists m_\nu. [[f\;n] =^\nu 0_\nu] \wedge [[f\;m] =^\nu 0_\nu] \wedge \neg[n =^\nu m]]_\circ$$

# HOL: Expressing Properties

$$[\lambda x_\nu.x^2 - 1]$$

Term of type  $\circ$  expressing existence of an  $f$  with two roots:

$$[\underbrace{\exists f_{\nu\nu} \ . \ \exists n_\nu \ . \ \exists m_\nu \ . \ [[f \ n] =^\nu 0_\nu] \wedge [[f \ m] =^\nu 0_\nu] \wedge \neg[n =^\nu m]}_{\Sigma^{\nu\nu} \lambda f_{\nu\nu}}] \circ$$

# HOL: Expressing Properties

$$[\lambda x.x^2 - 1]$$

Term of type  $\circ$  expressing existence of an  $f$  with two roots:

$$[\exists f_\nu. \exists n_\nu. \exists m_\nu. \underbrace{[[f\ n] =^\nu 0_\nu] \wedge [[f\ m] =^\nu 0_\nu]}_{[=^\nu [f\ n]\ 0]} \wedge \neg[n =^\nu m]]_\circ$$

# HOL: Expressing Properties

$$[\lambda x_\nu. x^2 - 1]$$

$$[\lambda x_\nu. [x^2 - 1] = 0]$$

# HOL: Expressing Properties



$$[\lambda x_\iota. x^2 - 1]$$

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$$[\lambda x_\iota. [=^\iota [\text{MINUS}_{\iota\iota} [\text{SQUARE}_\iota x] 1_\iota] 0_\iota]]_{\circ\iota}$$

# HOL: Expressing Properties

$$[\lambda x_\nu. x^2 - 1]$$

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Term of type  $\circ$  expressing existence of a set (characteristic function) P with two elements

$$[\exists P_{\circ\nu}. \exists m_\nu. \exists n_\nu. [P m] \wedge [P n] \wedge \neg[m = n]]_\circ$$

# HOL: Expressing Properties

Suppose  $\_l$  corresponds to real numbers.

Given constants:  $\text{<}_{\text{o}\_l}$ ,  $\text{ABS}_{\text{o}\_l}$ ,  $\text{MINUS}_{\text{o}\_l}$

We can give the usual  $\epsilon - \delta$  definition of limits.

# HOL: Expressing Properties

Suppose  $\iota$  corresponds to real numbers.

Given constants:  $<_{\text{ou}}$ ,  $\text{ABS}_{\iota\iota}$ ,  $\text{MINUS}_{\iota\iota}$

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$\text{LIM}_{\text{o}u\iota(\iota)}$ :

$$\begin{aligned} [\lambda f_{\iota\iota\iota}. \lambda a_\iota. \lambda L_\iota. \forall \epsilon_\iota. [\epsilon > 0] \supset . \exists \delta_\iota. [\delta > 0] \\ \wedge . \forall x_\iota. [|x - a| < \delta] \supset [|f x - L| < \epsilon]] \end{aligned}$$

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Similarly can define continuity, differentiation, etc.

# HOL: Prefix Polymorphism

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Consider the notion of subset:

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Example: (using infix notation)

$$[\lambda U_{o\iota}. [U \subseteq_{o(o\iota)(o\iota)} X_{o\iota}]] \subseteq_{o(o(o\iota))(o(o\iota))} [\lambda U_{o\iota}. [U \subseteq_{o(o\iota)(o\iota)} Y_{o\iota}]]$$

# HOL: Cantor's Theorem

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# HOL: Cantor's Theorem

There is no surjection from a set A onto the power set  $\mathcal{P}(A)$  of A.

- Suppose A corresponds to type  $\omega$ .
- Then  $\mathcal{P}(A)$  corresponds to type  $(\omega\omega)$ .

$$\neg \exists g_{\omega\omega}. \forall f_{\omega\omega}. \exists x_\omega. g x =^{\omega\omega} f$$

# HOL: Standard Higher-Order Model



$\mathcal{D}_\iota$  (individuals)

# HOL: Standard Higher-Order Model



$\mathcal{P}(\mathcal{D}_\iota)$  (all sets)

$\mathcal{D}_\iota$  (individuals)

# HOL: Standard Higher-Order Model



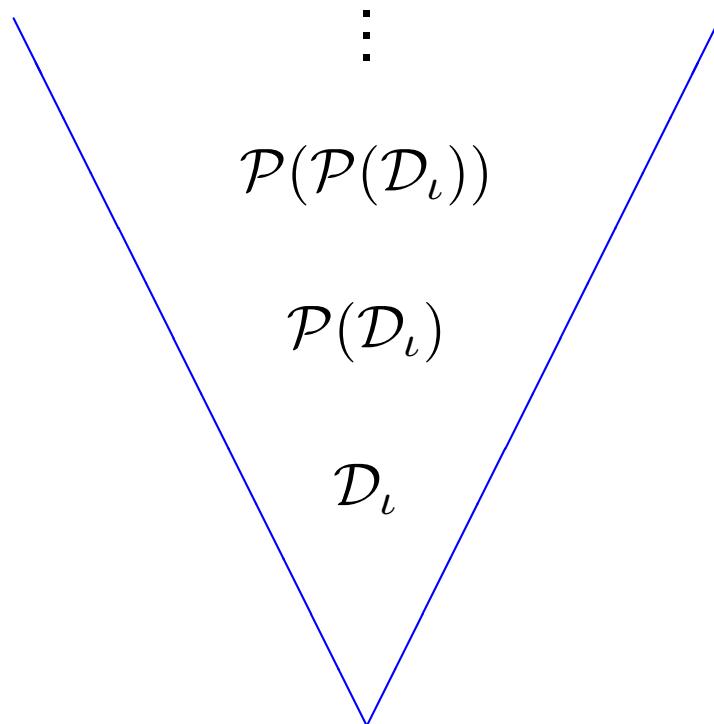
(all sets of sets)

$$\mathcal{P}(\mathcal{P}(\mathcal{D}_\iota))$$

$$\mathcal{P}(\mathcal{D}_\iota) \quad \text{(all sets)}$$

$$\mathcal{D}_\iota \quad \text{(individuals)}$$

# HOL: Standard Higher-Order Model



# HOL: Henkin-Style Model



$\mathcal{D}_{\text{o}\iota} \subseteq \mathcal{P}(\mathcal{D}_\iota)$  (some sets)

$\mathcal{D}_\iota$  (individuals)

# HOL: Henkin-Style Model

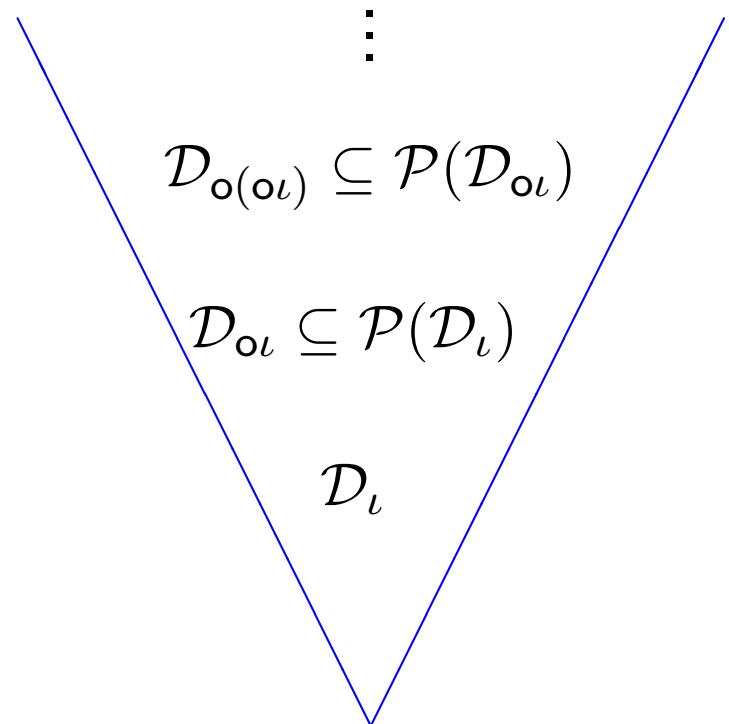
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$$\mathcal{D}_{o(o\iota)} \subseteq \mathcal{P}(\mathcal{D}_{o\iota})$$

$$\mathcal{D}_{o\iota} \subseteq \mathcal{P}(\mathcal{D}_\iota) \quad (\text{some sets})$$

$$\mathcal{D}_\iota \quad (\text{individuals})$$

# HOL: Henkin-Style Model





# Types, Frames, and Applicative Structures

# Def.: Types

Let  $\mathcal{T}$  be the least set s.t:

$$\circ \in \mathcal{T}$$

$$\iota \in \mathcal{T}$$

$$\forall \alpha, \beta \in \mathcal{T} : (\alpha\beta) \in \mathcal{T}$$

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We say that  $\alpha \in \mathcal{T}$  is a **simple type** (or type).  
 $(\alpha\beta)$  is called a **function type**.

- The set  $\mathcal{T}$  is defined inductively.
- The set  $\mathcal{T}$  is "freely generated".

# Ex.: Freely Generated

Consider the set  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

- $0 \in \mathbb{N}$

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Consider the set  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

- $0 \in \mathbb{N}$
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- $\forall n : 0 \neq s(n)$ .

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- $\forall m, n : s(m) = s(n) \Rightarrow m = n$ .

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The set  $\mathbb{N}$  is "freely generated".

Contrast  $\mathbb{N}$  to  $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ .

Note that  $\mathbb{Z}$  contains  $0$  and is closed under successor, but is not the least such set.

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The set  $\mathcal{T}$  is "freely generated":

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- $\circ \neq (\alpha\beta)$
- $\iota \neq (\alpha\beta)$
- $(\alpha\beta) = (\gamma\delta) \Rightarrow \alpha = \gamma \wedge \beta = \delta$

# Ex.: Types

- $(\text{o}\iota) \in \mathcal{T}$

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- $(o\iota) \in \mathcal{T}$
- $(o(o\iota)) \in \mathcal{T}$

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Is  $(o\iota\iota)$  also a type?

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Is  $(o\iota\iota)$  also a type? – no

But we can and will consider it shorthand by replacing missing parenthesis, associating to the left:  $(o\iota\iota) = ((o\iota)\iota) \neq (o(\iota\iota))$ .

# Def.: Functions

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- $A = \{0, 1\}, B = \{0, 1, 2\}$
- $|A^B| = 2 \cdot 2 \cdot 2 = 2^3 = 8$

# Ex.: Sets of Functions

Let  $F = \{f : B \rightarrow A \mid \forall x, y \in B : x \leq y \Rightarrow f(x) \leq f(y)\} \subseteq A^B$ .

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$A^B$	$f(0)$	$f(1)$	$f(2)$
$K_0 \in F$	0	0	0
$\in F$	0	0	1
$\notin F$	0	1	0
$\in F$	0	1	1
$g \notin F$	1	0	0
$\notin F$	1	0	1
$\notin F$	1	1	0
$K_1 \in F$	1	1	1

Consider:

$g : x = 0, y = 1, x \leq y$ , but  
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Consider:

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$|F| = 4$

# Ex.: Sets of Labelled Functions

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$$F_C = \{\langle c, f \rangle | c \in C, f \in F\}$$

$$|F_C| = 3 \cdot 4 = 12$$

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A Frame is called **standard** if

$$D_{\alpha\beta} = D_\alpha^{D_\beta} \quad \forall \alpha, \beta \in \mathcal{T}$$

# Ex.: Frames

$$D_o = \{\perp, \top\}$$

# Ex.: Frames

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$$D_o = \{\perp, \top\}$$

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D: the standard frame with  $D_o = \{\perp, \top\}$ ,  $D_i = \{1\}$

# Ex.: Frames (Contd.)

Consider the set  $D_{o(\iota\iota)}((o(\iota o)))$ . Is the set empty?

# Ex.: Frames (Contd.)

Consider the set  $D_{o(\mu)}((o(\iota o)))$ . Is the set empty? — no!

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Consider the set  $D_{o(\iota\iota)}((o(\iota o)))$ . Is the set empty? — no!

**Claim:**  $\forall \alpha \in \mathcal{T} : D_\alpha \neq \emptyset$ .

# Ex.: Frames (Contd.)

Consider the set  $D_{o(\iota\iota\iota)}((o(\iota o)))$ . Is the set empty? — no!

**Claim:**  $\forall \alpha \in \mathcal{T} : D_\alpha \neq \emptyset$ .

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(Here  $K_a$  is the constant function which always returns  $a$ . We will often use this notation for constant functions.)

# Def.: Typed Applicative Structure

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Usually we write  $f@b$  for  $@^{\alpha\beta}(f, b)$  when  $f \in D_{\alpha\beta} \wedge b \in D_\beta$

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The application operator  $@$  in an applicative structure is an abstract version of function application. It is no restriction to exclusively use a binary application operator, which corresponds to unary function application, since we can define higher-arity application operators from the binary one by setting  $f@(a^1, \dots, a^n) := (\dots (f@a^1) \dots @a^n)$  (“Currying”).

# Interesting Properties

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# Def.: Standard Applicative Structures



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Note that the definitions of functional, full, and standard impose restrictions on the domains for function types only.

# Rem.: Frames and Applicative Structures



It is easy to show that every frame is functional.

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Furthermore, an applicative structure is standard iff it is a full frame.

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$1 \in D_{oo}$  but  $1 \notin D_o^{D_o} \Rightarrow D_{oo} \not\subseteq D_o^{D_o}$

# Def.: Homomorphic Appl. Structures

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- $\kappa_\alpha : D_\alpha^1 \rightarrow D_\alpha^2 \quad \forall \alpha \in \mathcal{T}$
- $\forall \alpha, \beta \in \mathcal{T}, \quad \forall f \in D_{\alpha\beta}^1, \quad \forall b \in D_\beta^1:$

$$\kappa(f) @^2 \kappa(b) = \kappa(f @^1 b)$$

# Def.: Isomorphic Appl. Structures

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- $i$  and  $j$  are inverses (i.e  $i(j(a^2)) = a^2$  and  $j(i(a^1)) = a^1$ ).



## Simply Typed $\lambda$ -Calculus

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- $x_\alpha \in \mathcal{V}_\alpha, A_\beta \in \Lambda_\beta$  then  $(\lambda x_\alpha. A_\beta)_{\beta\alpha} \in \Lambda_{\beta\alpha}$

# Notational Conventions

- brackets may be avoided:  $A B C \rightsquigarrow ((A B) C)$

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# Def.: Positions in $\lambda$ -Terms

Consider the following term:

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... Graphics on Blackboard ...

# Def.: Position (Contd.)

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$$T_{[212]} = y$$

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Replacement of  $A_p$  in  $A$  by a term  $B$  is denoted as

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Example:

$$T[(fx)]_{[212]} = ((\lambda x.x)((\lambda y.(fx))(\lambda z.z)))$$

# Def.: Scope of $\lambda$ -Term

$(\lambda x.A)$  : We say that  $A$  is in the **scope** of  $\lambda$ -binder that binds  $x$ .

# Def.: Free and Bound Variables

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Otherwise it is called **free**.

We denote the **set of all free variables** in a  $\lambda$ -term as  $\text{FV}(A)$ .



## Syntax: Simply Typed $\lambda$ -Calculus (Contd.)

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1.  $[N_\alpha/x_\alpha]x_\alpha = N_\alpha$

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2.  $[N_\alpha/x_\alpha]a_\beta = a_\beta$  if  $a_\beta \neq x_\alpha \wedge a_\beta \in V_\beta \cup C_\beta$
3.  $[N_\alpha/x_\alpha](A_{\alpha\alpha}B_\beta) = ([N_\alpha/x_\alpha]A)([N_\alpha/x_\alpha]B)$

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$$[A_\alpha/x_\alpha] : \Lambda_\alpha \rightarrow \Lambda_\alpha \quad (\text{typed})$$

and is defined as follows:

1.  $[N_\alpha/x_\alpha]x_\alpha = N_\alpha$
2.  $[N_\alpha/x_\alpha]a_\beta = a_\beta$  if  $a_\beta \neq x_\alpha \wedge a_\beta \in V_\beta \cup C_\beta$
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5.  $[N_\alpha/x_\alpha](\lambda y_\beta.A_\gamma) = (\lambda y_\beta.[N_\alpha/x_\alpha]A_\gamma)$  if  
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6.  $[N_\alpha/x_\alpha](\lambda y_\alpha.A_\gamma) = (\lambda z_\beta.[N_\alpha/x_\alpha][z_\beta/y_\beta]A_\gamma)$  if  $x_\alpha \neq y_\beta \wedge$   
 $(y_\beta \in FV(N_\alpha) \wedge x_\alpha \in FV(A_\gamma))$  and  $z$  is a 'fresh' variable.

# Ex.: Substitution

- $[y/x](\lambda y.x) = (\lambda y.y)$  — the occurrence of  $x$  is free  
 $\neq (\lambda y.y)$  — if we replace  $x$  with  $y$ , the variable  $y$  becomes bound.

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- Further Examples on Blackboard
- Claim:  $[N/x]A = A$  if  $x \notin FV(A)$   
Proof: Induction on  $A$

# Def.: $\alpha$ -Conversion

$$[\lambda x. M] \rightarrow_{\alpha} [\lambda y. [y/x]M]$$

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From now on  $(\lambda y. y) = (\lambda z. z)$ , that is, we will say that two terms are simply equal, if they are  $\alpha$ -equal. Two terms are equal means that two terms are  $\alpha$ -convertable.

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$$\begin{aligned} M &= P[(\lambda x. A)B]_p \\ N &= P[[B/x]A]_p \end{aligned}$$

We say  $M \rightarrow_{\beta} N$ , ie.  $\beta$ -reduces in several steps, if  $\exists M^1, \dots, M^n$  for  $n \geq 1$  such that  $M = M^1$  and  $N = M^n$  and  $M^i \rightarrow_{\beta} M^{i+1}$ .

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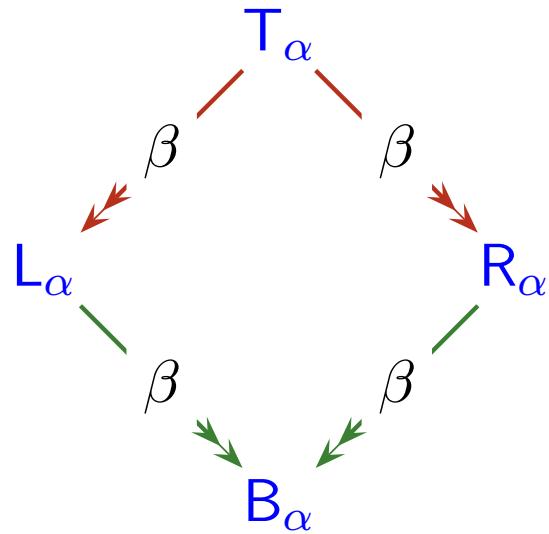
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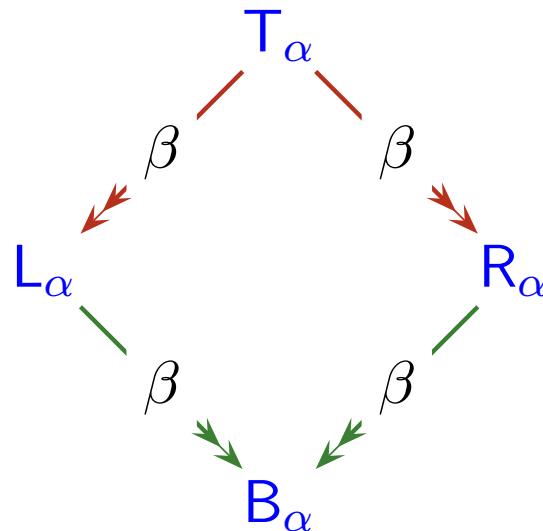
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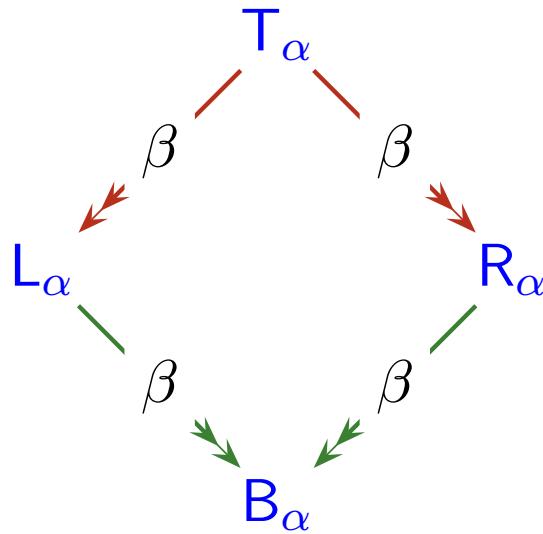


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If  $T_\alpha$   $\beta$ -reduces in multiple steps with one strategy to  $L_\alpha$  and with another strategy to  $R_\alpha$  then there exists a term  $B_\alpha$  such that  $L_\alpha$  and  $R_\alpha$   $\beta$ -reduce in multiple steps to  $B_\alpha$ .

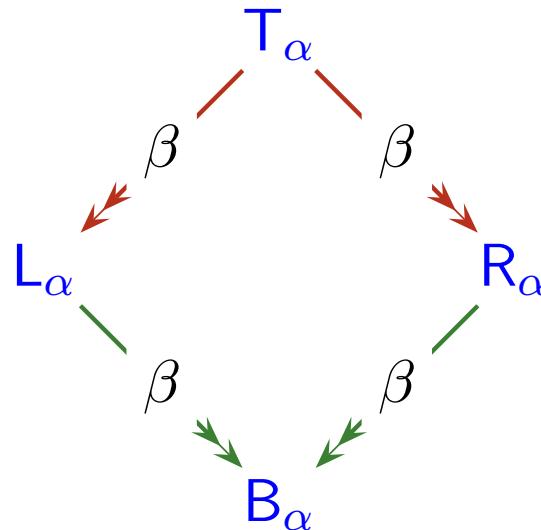
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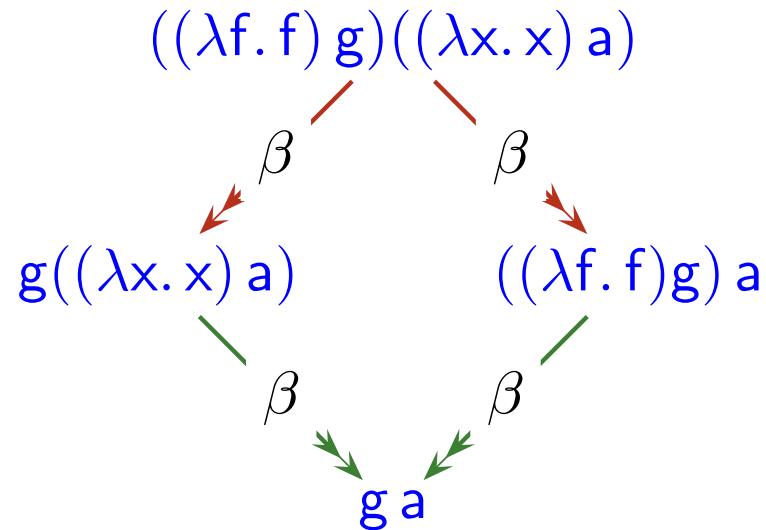


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The Church-Rosser Property for  $\rightarrow_\beta$  holds for  $\Lambda$  and  $\Lambda^\alpha$ .

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**Untyped Case:** Consider the term  $\omega = (\lambda x. xx)$

$$(\lambda x. xx)(\lambda x. xx) \xrightarrow{1} \beta \omega\omega$$

# Def.: $\eta$ -Conversion

A  **$\eta$ -redex** is a term of the form  $(\lambda x_\beta. F_{\alpha\beta} x)$  where  $x \notin FV(F)$ . The  **$\eta$ -reduct** of this term is  $F$ .

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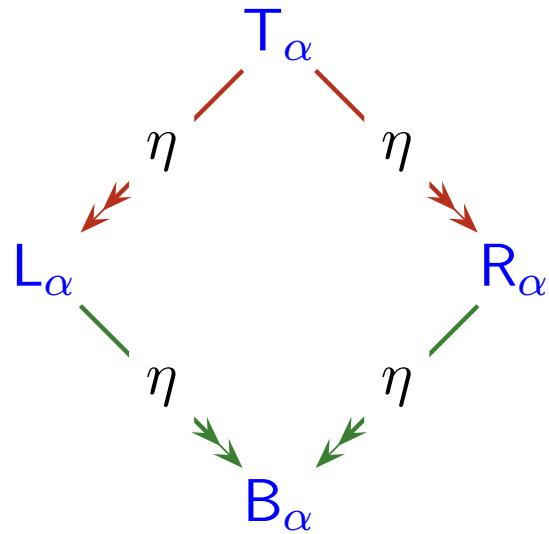
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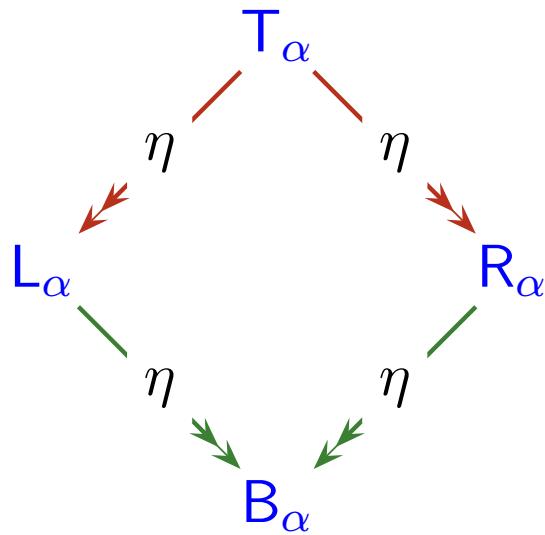
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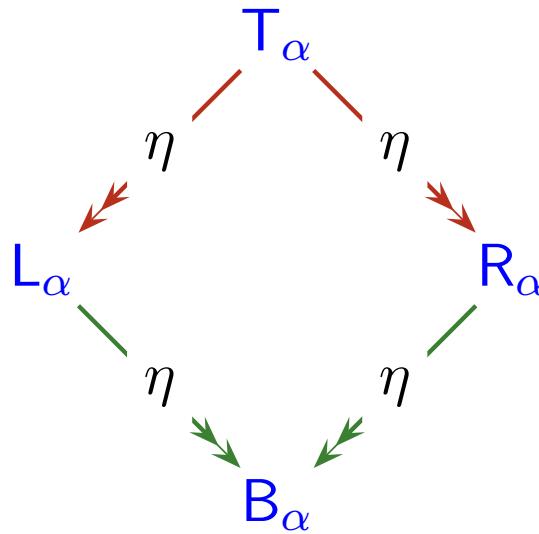


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The Church-Rosser Property for  $\rightarrow_\eta$  holds for  $\Lambda$  and  $\Lambda^\alpha$ .

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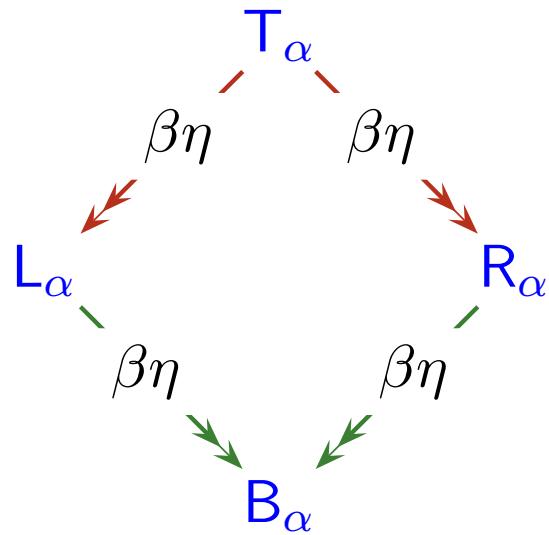
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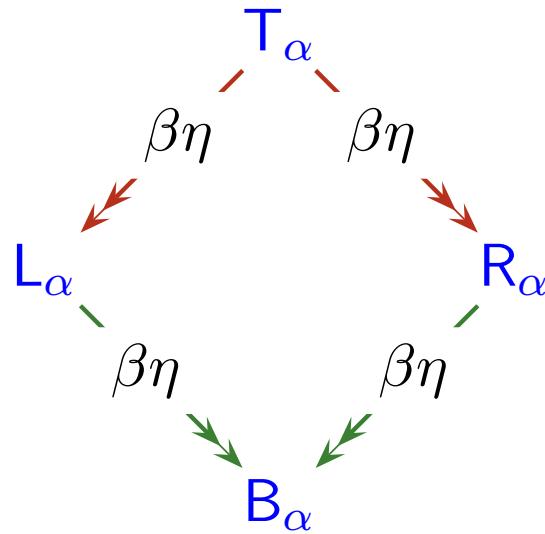
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A term is  $\beta\eta$ -normal if it contains no  $\beta$ -redexes and no  $\eta$ -redexes.

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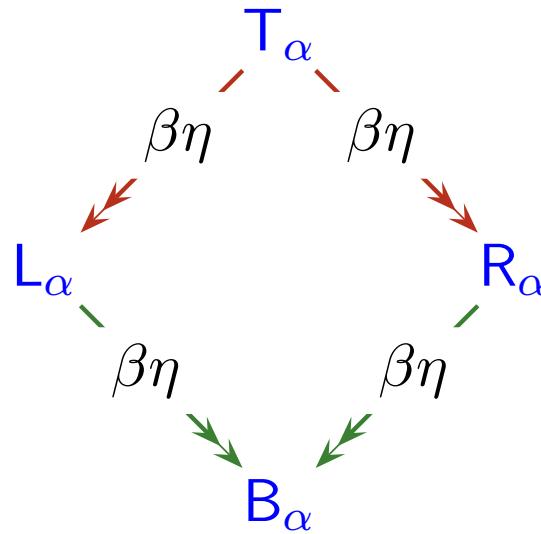


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If  $T_\alpha$   $\beta\eta$ -reduces in multiple steps with one strategy to  $L_\alpha$  and with another strategy to  $R_\alpha$  then there exists a term  $B_\alpha$  such that  $L_\alpha$  and  $R_\alpha$   $\beta\eta$ -reduce in multiple steps to  $B_\alpha$ .

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The Church-Rosser Property for  $\rightarrow_{\beta\eta}$  holds for  $\Lambda$  and  $\Lambda^\alpha$ .

# Thm.: Strong Church-Rosser Property

In  $\Lambda^\alpha$  (simply typed  $\lambda$ -calculus) the relations  $\rightarrow_\beta$  and  $\rightarrow_{\beta\eta}$  have the strong Church Rosser property:

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In  $\Lambda^\alpha$  (simply typed  $\lambda$ -calculus) the relations  $\rightarrow_\beta$  and  $\rightarrow_{\beta\eta}$  have the **strong Church Rosser property**: for every term  $A_\tau$  there exists a unique (up to  $\alpha$ -renaming)  $\beta$ -normal resp.  $\beta\eta$ -normal term  $B_\tau$  such that  $A_\tau \rightarrow_\beta B_\tau$  resp.  $A_\tau \rightarrow_{\beta\eta} B_\tau$ .

# Def.: Long $\beta\eta$ -Normal Form

Let  $n \geq 0$ ,  $\alpha^1, \dots, \alpha^n \in \mathcal{T}$ , and  $\beta \in \{o, i\}$ . A term  $A$  of type  $(\beta, \alpha^n, \dots, \alpha^1)$  is in **long  $\beta\eta$ -normal form** if it is of form

$$\lambda x_{\alpha^1}^1 \dots x_{\alpha^n}^n. (h_{\beta\gamma^m \dots \gamma^1} A_{\gamma^1}^1 \dots A_{\gamma^m}^m)$$

for a variable or constant  $h_{\beta\gamma^m \dots \gamma^1}$ ,  $m \geq 0$  and long  $\beta\eta$ -normal forms  $A_{\gamma^1}^1, \dots, A_{\gamma^m}^m$ .

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# Def.: Long $\beta\eta$ -Normal Form

Let  $n \geq 0$ ,  $\alpha^1, \dots, \alpha^n \in \mathcal{T}$ , and  $\beta \in \{o, \iota\}$ . A term  $A$  of type  $(\beta, \alpha^n, \dots, \alpha^1)$  is in **long  $\beta\eta$ -normal form** if it is of form

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for a variable or constant  $h_{\beta\gamma^m \dots \gamma^1}$ ,  $m \geq 0$  and long  $\beta\eta$ -normal forms  $A_{\gamma^1}^1, \dots, A_{\gamma^m}^m$ . Note that this is an inductive definition; the base case is when  $m = 0$ . Note that if  $\lambda \bar{x^n}. (h \bar{A^m})$  is in long  $\beta\eta$ -normal form then  $(h \bar{A^m})$  is of base type.

# Ex.: Long $\beta\eta$ -Normal Form

Consider the  $\beta\eta$ -normal term  $f_{\iota(\iota\iota)}$ .

$$\begin{array}{c} f_{\iota(\iota\iota)} \\ \uparrow^\eta \\ \lambda w_{\iota\iota}. (f_{\iota(\iota\iota)} w_{\iota\iota}) \\ \uparrow^\eta \\ \lambda w_{\iota\iota}. (f(\lambda x_\iota. w_{\iota\iota} x)) \end{array}$$

# Thm.: Long $\beta\eta$ -Normal Form

For every term A there is unique long  $\beta\eta$ -normal form B such that  $A =^{\beta\eta} B$ .

# Rem.: $\beta\eta$ -Head Normal Form

Instead of terms in long  $\beta\eta$ -normal form we often use in practice terms in  $\beta\eta$ -head normal form.

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Instead of terms in long  $\beta\eta$ -normal form we often use in practice terms in  **$\beta\eta$ -head normal form**. Definition is similar to long  $\beta\eta$ -normal, but we do not require the embedded terms  $A_{\gamma_i}^i$  to be in normal form.

# Notation

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- $A \Downarrow$  is the long  $\beta\eta$ -normal form of  $A$ .



## Semantics: $\Sigma$ -Evaluations

# Ex.: An Interesting Applicative Structure

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- Is  $(\lambda x_i x) \in D_{ii}$ ? — Yes!
- $D = (D_\alpha)_{\alpha \in \mathcal{T}}$  is not a frame!
- It requires a specific application operator  $@ : D_{\alpha\beta} \times D_\beta \rightarrow D_\alpha$

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- If  $C_i \neq \emptyset$  and  $C_o \neq \emptyset$ , then  $\forall \alpha \in \mathcal{T}. \Lambda_\alpha \neq \emptyset$ .
- Is  $D_{\alpha\beta}$  a set of functions? (ie.  $D_{\alpha\beta} \subseteq (D_\alpha)^{D_\beta}$ ) — No!
- Is  $(\lambda x_i x) \in D_{ii}$ ? — Yes!
- $D = (D_\alpha)_{\alpha \in \mathcal{T}}$  is not a frame!
- It requires a specific application operator  $@ : D_{\alpha\beta} \times D_\beta \rightarrow D_\alpha$
- If  $\Lambda_\alpha$  is non-empty for all  $\alpha \in \mathcal{T}$ , then  $\langle D, @ \rangle$  is an applicative structure.

# Ex.: Interpretation of Terms

Syntax      Semantics     $\langle D, @ \rangle$   
 $(\lambda x_i. x)$

# Ex.: Interpretation of Terms

Syntax	Semantics	$\langle D, @ \rangle$
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Remark: The variable  $y_\iota$  is a non-closed well-formed formula of type  $\iota$ . We need an assignment  $\varphi_\alpha : V_\alpha \rightarrow D_\alpha$  to give it a meaning.

# Ex.: Interesting Applicative Structures

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Claim: If  $\mathcal{C}_i \neq \emptyset$  and  $\mathcal{C}_o \neq \emptyset$  (i.e., at least one constant for each base type is given), then  $(D, @^\beta)$  is an applicative structure.

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$$(\varphi, [a/X])(X) = a$$

and

$$(\varphi, [a/X])(Y) = \varphi(Y)$$

for variables  $Y$  other than  $X$ .

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# $\Sigma$ -Evaluations

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In such models, a function is not uniquely determined by its behavior on all possible arguments.

Such models can be constructed, for example, by labeling for functions (e.g., a green and a red version of a function  $f$ ) in order to differentiate between them, even though they are functionally equivalent.

# $\Sigma$ -Evaluations

Let  $\mathcal{E}: \mathcal{F}_{\mathcal{T}}(\mathcal{V}; \mathcal{D}) \longrightarrow \mathcal{F}_{\mathcal{T}}(\text{wff}(\Sigma), \mathcal{D})$  be a total function, where  $\mathcal{F}_{\mathcal{T}}(\mathcal{V}; \mathcal{D})$  is the set of variable assignments and  $\mathcal{F}_{\mathcal{T}}(\text{wff}(\Sigma), \mathcal{D})$  is the set of typed functions mapping terms into objects in  $\mathcal{D}$ .

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What properties shall  $\mathcal{E}$  fulfill?

# Def.: Evaluation Function

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2.  $\mathcal{E}_\varphi(\mathbf{F}\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{F}) @ \mathcal{E}_\varphi(\mathbf{A})$  for any  $\mathbf{F} \in \text{wff}_{\alpha \rightarrow \beta}(\Sigma)$  and  $\mathbf{A} \in \text{wff}_\alpha(\Sigma)$  and types  $\alpha$  and  $\beta$ .

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4.  $\mathcal{E}_\varphi(\mathbf{A}) = \mathcal{E}_\varphi(\mathbf{A}\downarrow_\beta)$  for all  $\mathbf{A} \in \mathit{wff}_\alpha(\Sigma)$ .

# Def.: $\Sigma$ -Evaluation

We call  $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$  a  **$\Sigma$ -evaluation** if  $(\mathcal{D}, @)$  is an applicative structure and  $\mathcal{E}$  is an evaluation function for  $(\mathcal{D}, @)$ . We call  $\mathcal{E}_\varphi(\mathbf{A}_\alpha) \in \mathcal{D}_\alpha$  the **denotation** of  $\mathbf{A}_\alpha$  in  $\mathcal{J}$  for  $\varphi$ .

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Remark: since  $\mathcal{E}$  is a function, the denotation in  $\mathcal{J}$  is unique. However, for a given applicative structure  $\mathcal{A}$ , there may be many possible evaluation functions.

If  $\mathbf{A}$  is a closed formula, then  $\mathcal{E}_\varphi(\mathbf{A})$  is independent of  $\varphi$ , since  $\text{Free}(\mathbf{A}) = \emptyset$ . In these cases we sometimes drop the reference to  $\varphi$  from  $\mathcal{E}_\varphi(\mathbf{A})$  and simply write  $\mathcal{E}(\mathbf{A})$ .

# Def.: Functional/Full/Standard $\Sigma$ -Eval.



We call a  $\Sigma$ -evaluation  $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$  **functional** [**full, standard**] if the applicative structure  $(\mathcal{D}, @)$  is **functional** [**full, standard**].

# Def.: Functional/Full/Standard $\Sigma$ -Eval.



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We say  $\mathcal{J}$  is a  $\Sigma$ -evaluation over a frame if  $(\mathcal{D}, @)$  is a frame.

# What is the Idea?

$\Sigma$ -evaluations **generalize**  $\Sigma$ -evaluations over frames, which are the basis for Henkin models, **to the non-functional case**.

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The existence of an evaluation function that meets the conditions as presented seems to be the weakest situation where one would like to speak of a model.

We cannot in general assume the evaluation function is uniquely determined by its values on constants as this requires functionality.  
Example: two evaluation functions  $\mathcal{E}$  and  $\mathcal{E}'$  on the same applicative structure may agree on all constants, but give a different value to the term  $(\lambda x.x)$ .

# Lemma: $\Sigma$ -Evaluations respect $\beta$ -Equality

Let  $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$  be a  $\Sigma$ -evaluation and  $\mathbf{A} =_{\beta} \mathbf{B}$ . For all assignments  $\varphi$  into  $(\mathcal{D}, @)$ , we have

# Lemma: $\Sigma$ -Evaluations respect $\beta$ -Equality

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- $\eta$ -functionality simply means the evaluation respects  $\eta$ -conversion.
- $\xi$ -functionality means we have functionality (only) with respect to  $\lambda$ -abstractions.

# Def.: $\eta$ -Functional

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i.e.,  $\neg \exists g : A \rightarrow \mathcal{P}(A)$  with  $g$  surjective.

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Note: for this term to be in the set  $cwff_\alpha(\Sigma)$ , the constants  $\neg_{oo}$ ,  $\Sigma_{o(o(o\iota\iota))}^{o\iota\iota}$ ,  $\Pi_{o(o(o\iota))}^{o\iota}$ ,  $\Sigma^\iota$  and  $=^{o\iota}$  have to be in the set  $\mathcal{C}$ .

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Note that the proof uses  $\neg$ .



## Semantics: $\Sigma$ -Models

# Def.: Properties of Logical Constants

Let  $(D, @)$  be an applicative structure and let  $v : D_o \rightarrow \{T, F\}$  be a function (for given  $T \neq F$ ).

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# Def.: Properties of Logical Constants

Let  $(D, @)$  be an applicative structure and let  $v : D_o \rightarrow \{T, F\}$  be a function (for given  $T \neq F$ ). For each logical constant  $c_\beta$  and for  $a \in D_\beta$ , we define the proposition  $\mathcal{L}_c((a))$  with respect to  $v$ :

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$=^\alpha$	$o\alpha\alpha$	$v(a@b@c) = T \quad \text{iff } b = c \quad \forall b, c \in D_o$

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$=^\alpha$	$o\alpha\alpha$	$v(a@b@c) = T \quad \text{iff } b = c \quad \forall b, c \in D_o$
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$\Leftrightarrow$	$ooo$	$v(a@b@c) = T \quad \text{iff } v(b) = v(c) \quad \forall b, c \in D_o$
$=^\alpha$	$o\alpha\alpha$	$v(a@b@c) = T \quad \text{iff } b = c \quad \forall b, c \in D_o$
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$=^\alpha$	$o\alpha\alpha$	$v(a@b@c) = T \quad \text{iff } b = c \quad \forall b, c \in D_o$
$\Pi^\alpha$	$o(o\alpha)$	$v(a@f) = T \quad \text{iff } \forall b \in D_\alpha : v(f@b) = T \quad \forall f \in D_{o\alpha}$

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$\vee$	$ooo$	$v(a@b@c) = T$	iff $v(b) = T$ or $v(c) = T \ \forall b, c \in D_o$
$\wedge$	$ooo$	$v(a@b@c) = T$	iff $v(b) = T$ and $v(c) = T \ \forall b, c \in D_o$
$\supset$	$ooo$	$v(a@b@c) = T$	iff $v(b) = F$ or $v(c) = T \ \forall b, c \in D_o$
$\Leftrightarrow$	$ooo$	$v(a@b@c) = T$	iff $v(b) = v(c) \ \forall b, c \in D_o$
$=^\alpha$	$o\alpha\alpha$	$v(a@b@c) = T$	iff $b = c \ \forall b, c \in D_o$
$\Pi^\alpha$	$o(o\alpha)$	$v(a@f) = T$	iff $\forall b \in D_\alpha : v(f@b) = T \ \forall f \in D_{o\alpha}$
$\Sigma^\alpha$			

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$\Leftrightarrow$	$ooo$	$v(a@b@c) = T \quad \text{iff } v(b) = v(c) \quad \forall b, c \in D_o$
$=^\alpha$	$o\alpha\alpha$	$v(a@b@c) = T \quad \text{iff } b = c \quad \forall b, c \in D_o$
$\Pi^\alpha$	$o(o\alpha)$	$v(a@f) = T \quad \text{iff } \forall b \in D_\alpha : v(f@b) = T \quad \forall f \in D_{o\alpha}$
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# Def.: Properties of Logical Constants

Let  $(D, @)$  be an applicative structure and let  $v : D_o \rightarrow \{T, F\}$  be a function (for given  $T \neq F$ ). For each logical constant  $c_\beta$  and for  $a \in D_\beta$ , we define the proposition  $\mathcal{L}_c((a))$  with respect to  $v$ :

$c$	$\beta$	$\mathcal{L}_c((a))$ holds when	
$T$	$o$	$v(a) = T$	
$\perp$	$o$	$v(a) = F$	
$\neg$	$oo$	$v(a@b) = T$	iff $v(b) = F \ \forall b \in D_o$
$\vee$	$ooo$	$v(a@b@c) = T$	iff $v(b) = T$ or $v(c) = T \ \forall b, c \in D_o$
$\wedge$	$ooo$	$v(a@b@c) = T$	iff $v(b) = T$ and $v(c) = T \ \forall b, c \in D_o$
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$\Sigma^\alpha$	$o(o\alpha)$	$v(a@f) = T$	iff $\exists b \in D_\alpha : v(f@b) = T \ \forall f \in D_{o\alpha}$

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We use  $\equiv^*$  in the following to refer to any of the above

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Note: In the [JSC04]-paper,  $b$  is defined as  $D_o = \{T, F\}$ , but here we are using the injectivity criterion, because we are varying the signature. If the signature is too sparse, we could have a  $D_o$  with two elements which both valuate via  $v$  to  $T$ . Another ill case would be  $D_o$  with just one element.

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Note: This basically says that for each type  $\alpha$  the identity relation over  $\alpha$  is already present in the model. If we require  $=_{o\alpha\alpha} \in \mathcal{C}$  with  $\mathcal{L}_{=\alpha}(\mathcal{E}_\varphi(=_{o\alpha\alpha}))$ , then this property is automatically ensured, but not for weaker signatures. See [Andrew71] for a detailed discussion of property  $q$ . Andrews constructs a Henkin model where Leibniz equality  $\doteq$  does not evaluate to the intended identity relation. This is resolved by property  $q$ .

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## Semantics: HOL-CUBE

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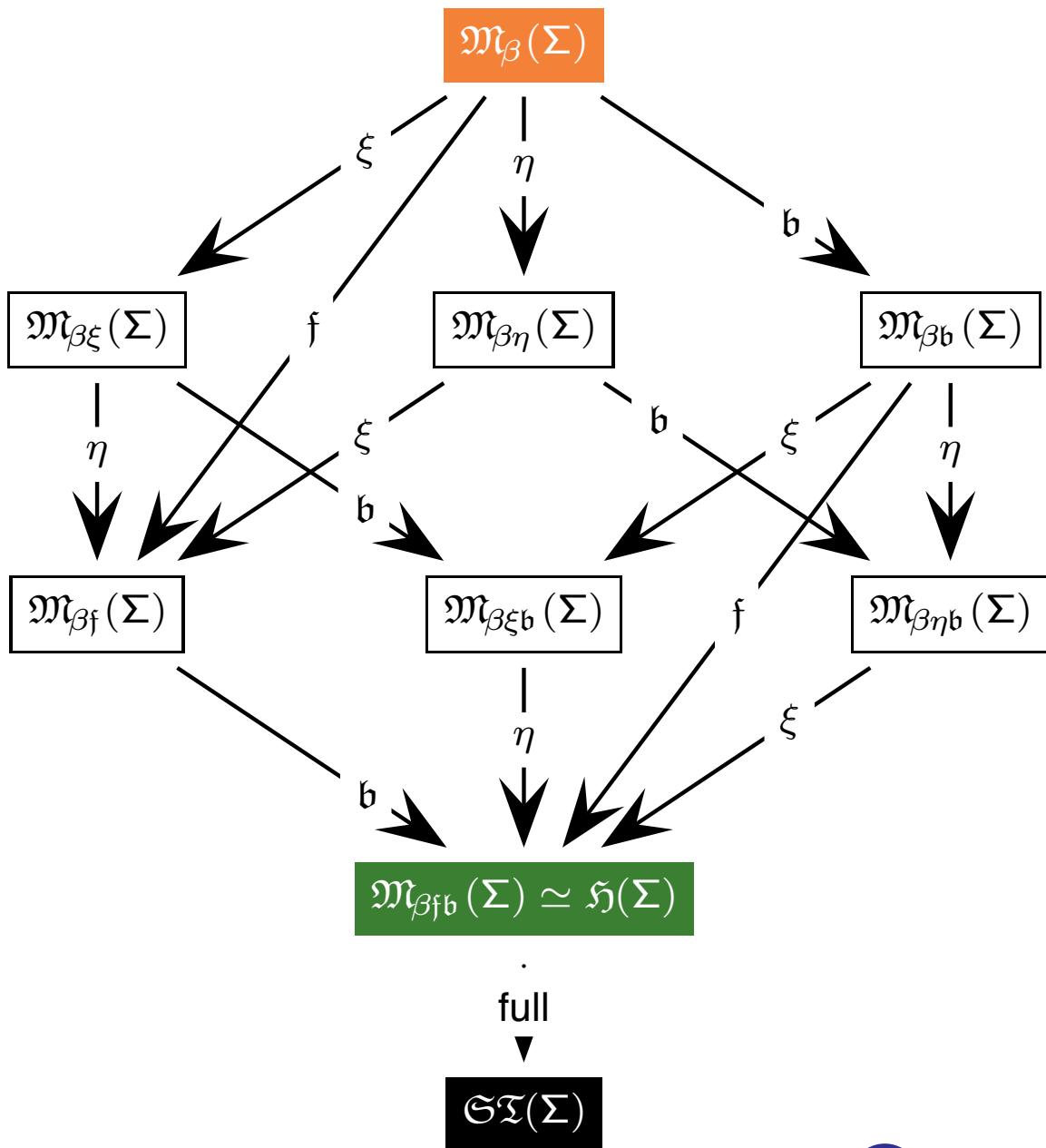
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We say that  $A$  is **valid** in  $\mathcal{M}$  (and write  $\mathcal{M} \models A$ ) if  $\mathcal{M} \models_{\varphi} A$  for all assignments  $\varphi$ . When  $A \in cwff_o(\Sigma)$ , we drop the reference to the assignment and use the notation  $\mathcal{M} \models A$ .

Finally, we say that  $\mathcal{M}$  is a  $\Sigma$ -**model for a set**  $\Phi \subseteq cwff_o(\Sigma)$  (we write  $\mathcal{M} \models \Phi$ ) if  $\mathcal{M} \models A$  for all  $A \in \Phi$ .

# Semantics: HOL-CUBE



## Landscape of HOL model classes

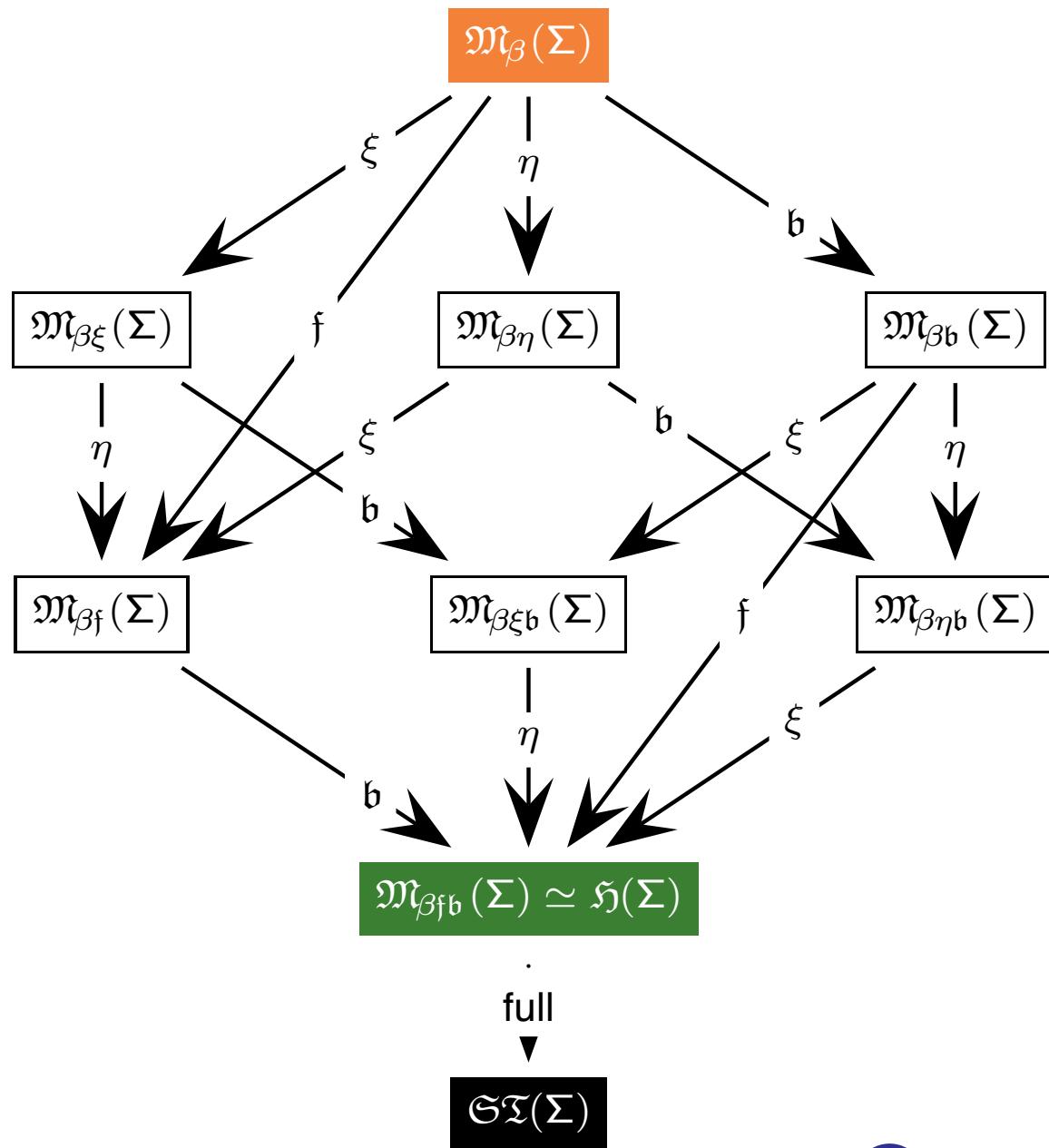
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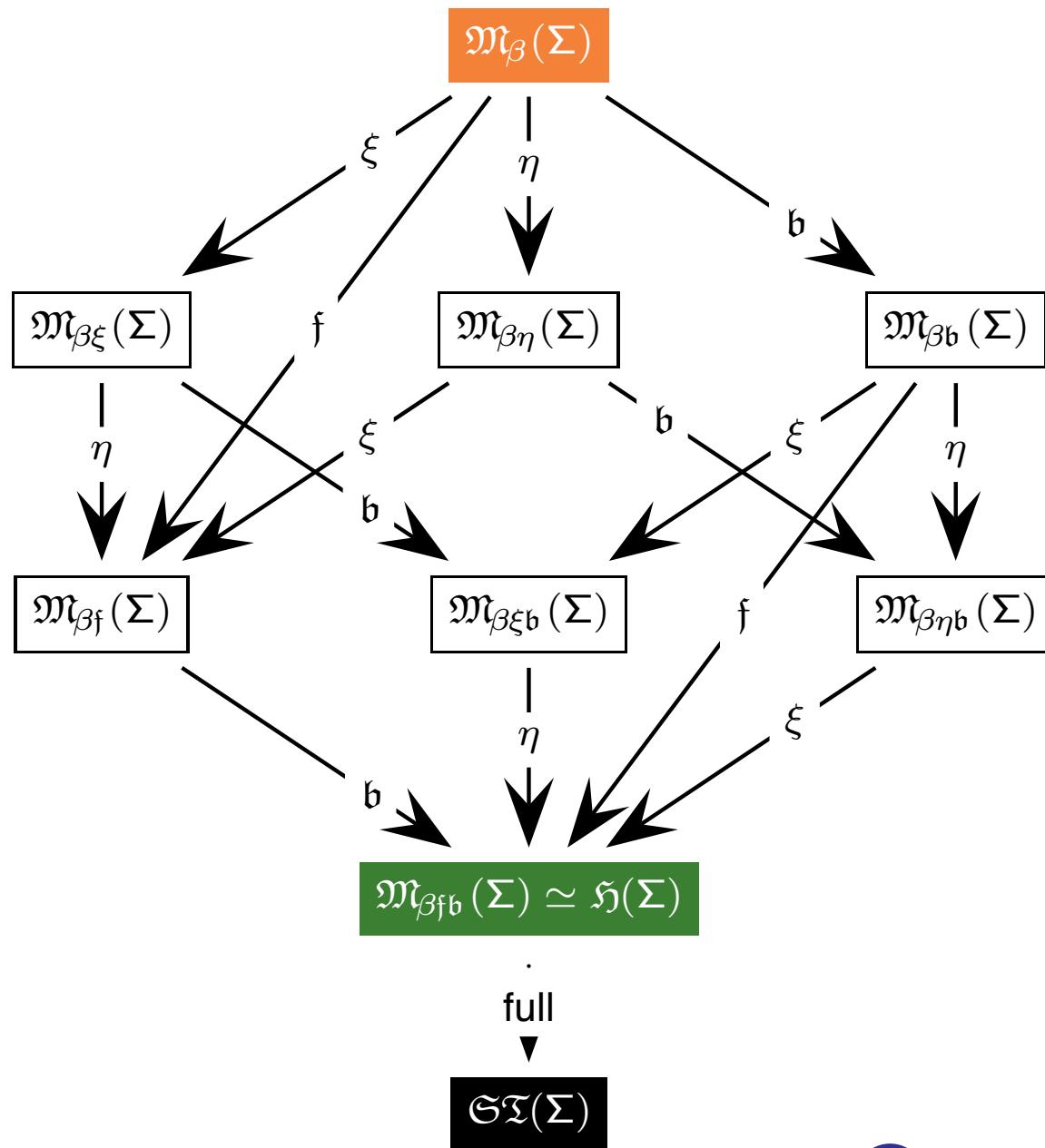
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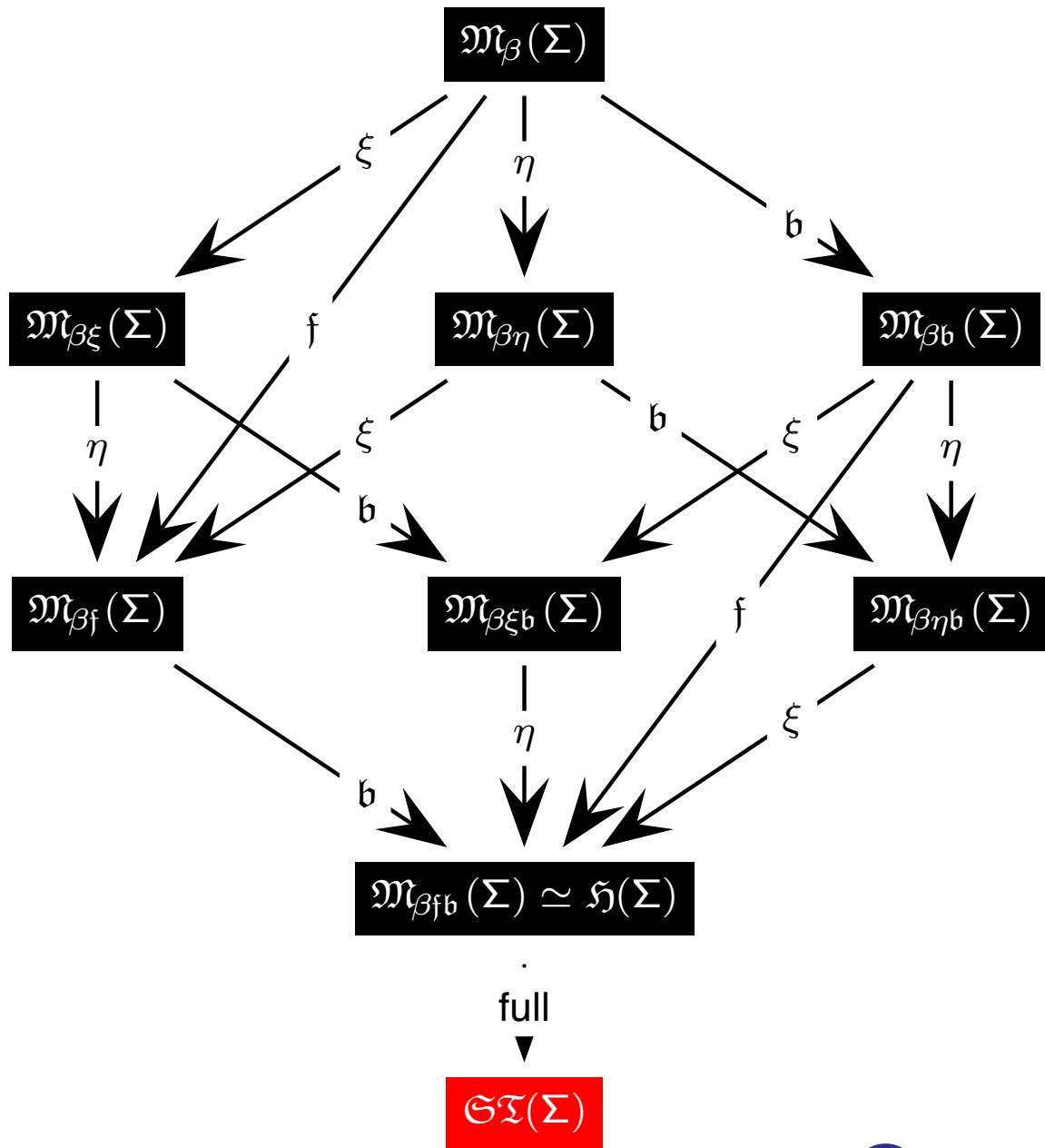
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$\mathfrak{M}_\beta(\Sigma)$  model class for  $\Sigma$ -fragment of elementary type theory

$\mathfrak{M}_{\beta f b}(\Sigma)$  model class for  $\Sigma$ -fragment of extensional type theory (Henkin models)

# Semantics: HOL-CUBE

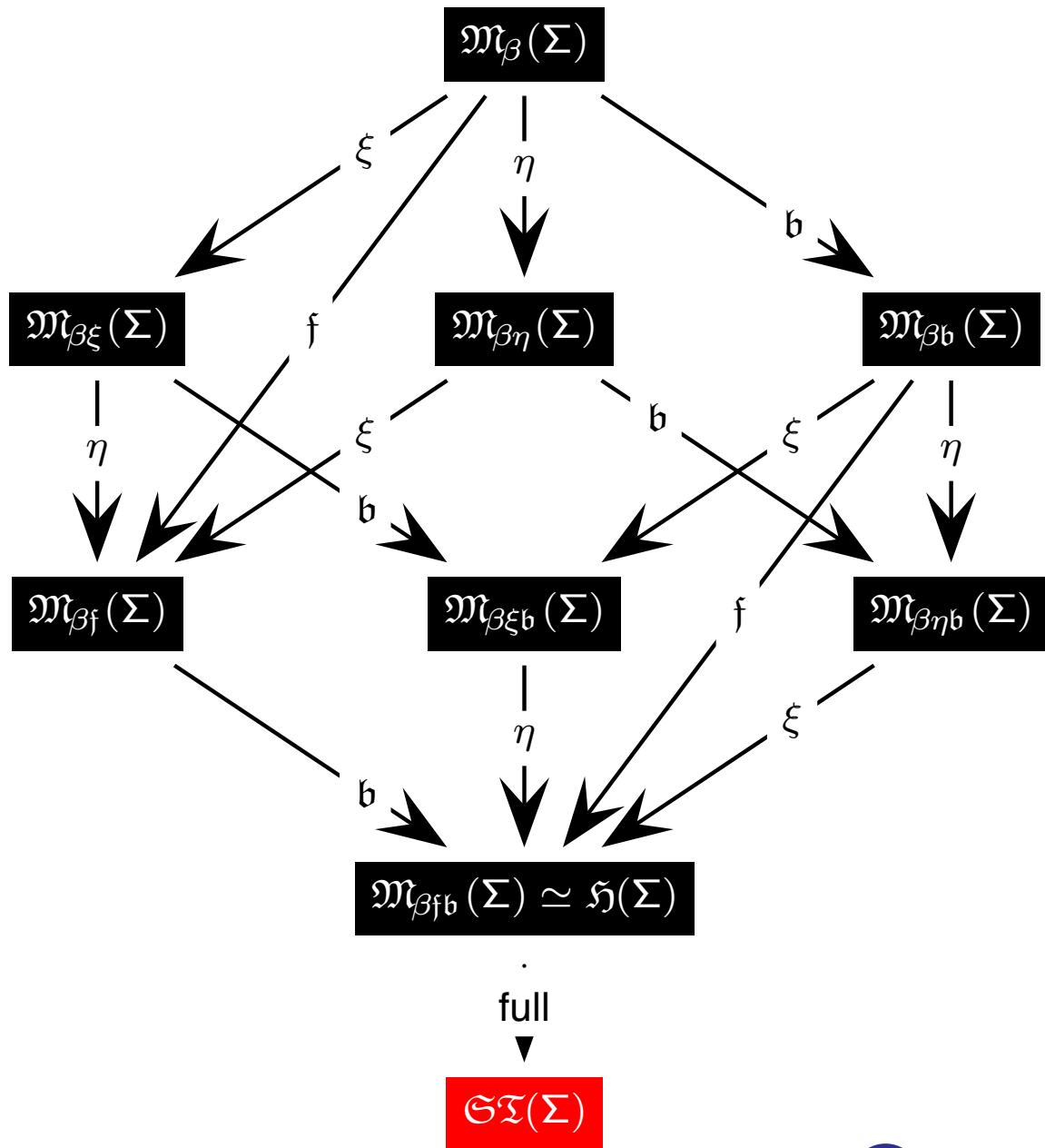


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- [Andrews72]: without property  $q$  Leibniz equality  $\doteq$  not necessarily evaluates to identity relation even in Henkin semantics ( $\mathfrak{H}(\Sigma)$ )

# Standard Models and Henkin Models

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Instead of requiring  $\mathcal{D}_{\alpha\beta}$  (and thus in particular,  $\mathcal{D}_{\alpha\alpha}$ ) to be the full set of functions (predicates), it is sufficient to require that  $\mathcal{D}_{\alpha\beta}$  has enough members that any well-formed formula can be evaluated (in other words, the domains of function types are rich enough to satisfy comprehension).

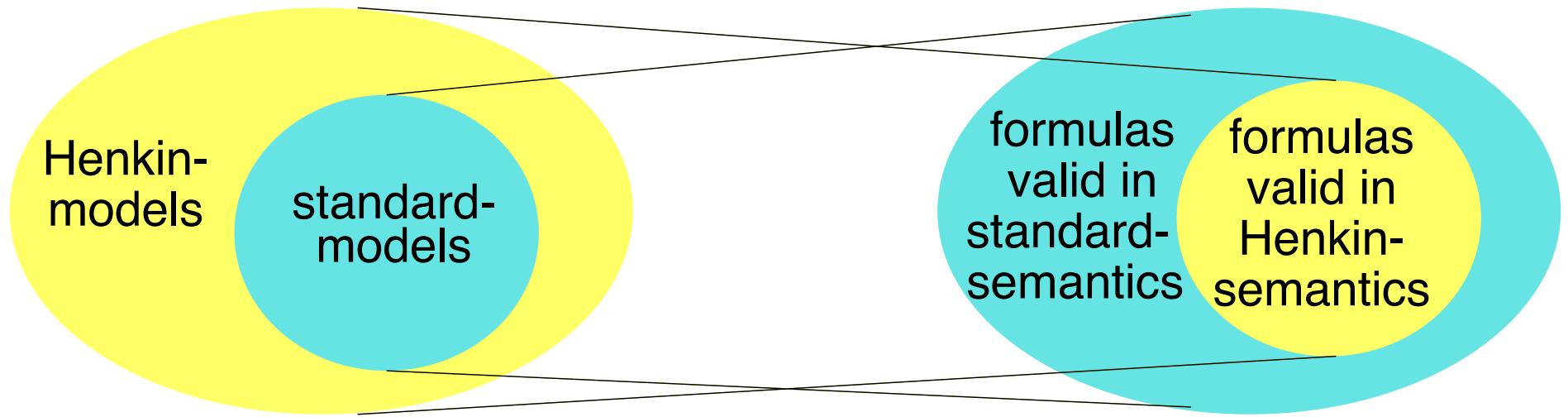
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Note that with this generalized notion of a model, there are fewer formulae that are valid in all models (intuitively, for any given formula there are more possibilities for counter-models).

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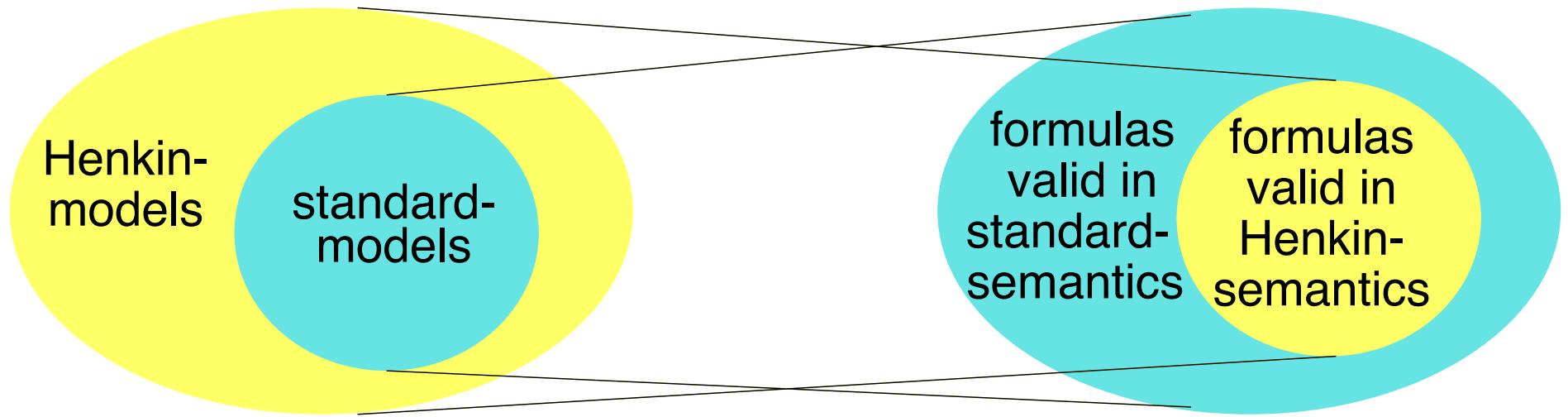
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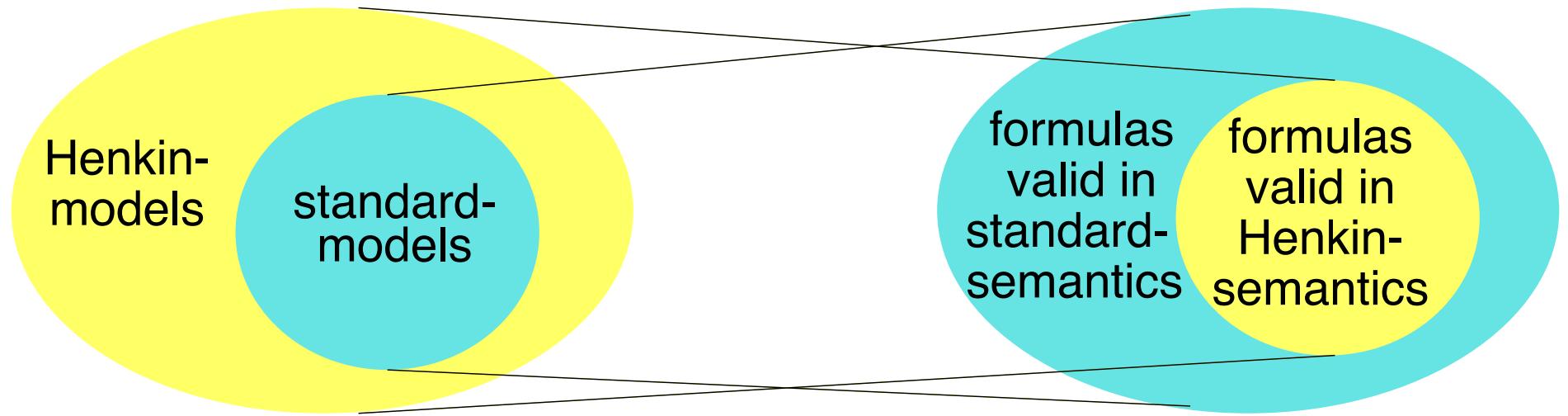
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Note that even though we can consider model classes with richer and richer function spaces, **we can never reach standard models where function spaces are full while maintaining complete (recursively axiomatizable) calculi.**

# Standard Models and Henkin Models



# Standard Models and Henkin Models



What has been our motivation for further generalization of Henkin semantics with respect to Boolean and functional extensionality?

# Models without Functional Extensionality

Motivation: modeling programs as (higher-order) functions

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- The identity function has constant complexity, the function  $\text{rev}$  is linear in the length of its argument.

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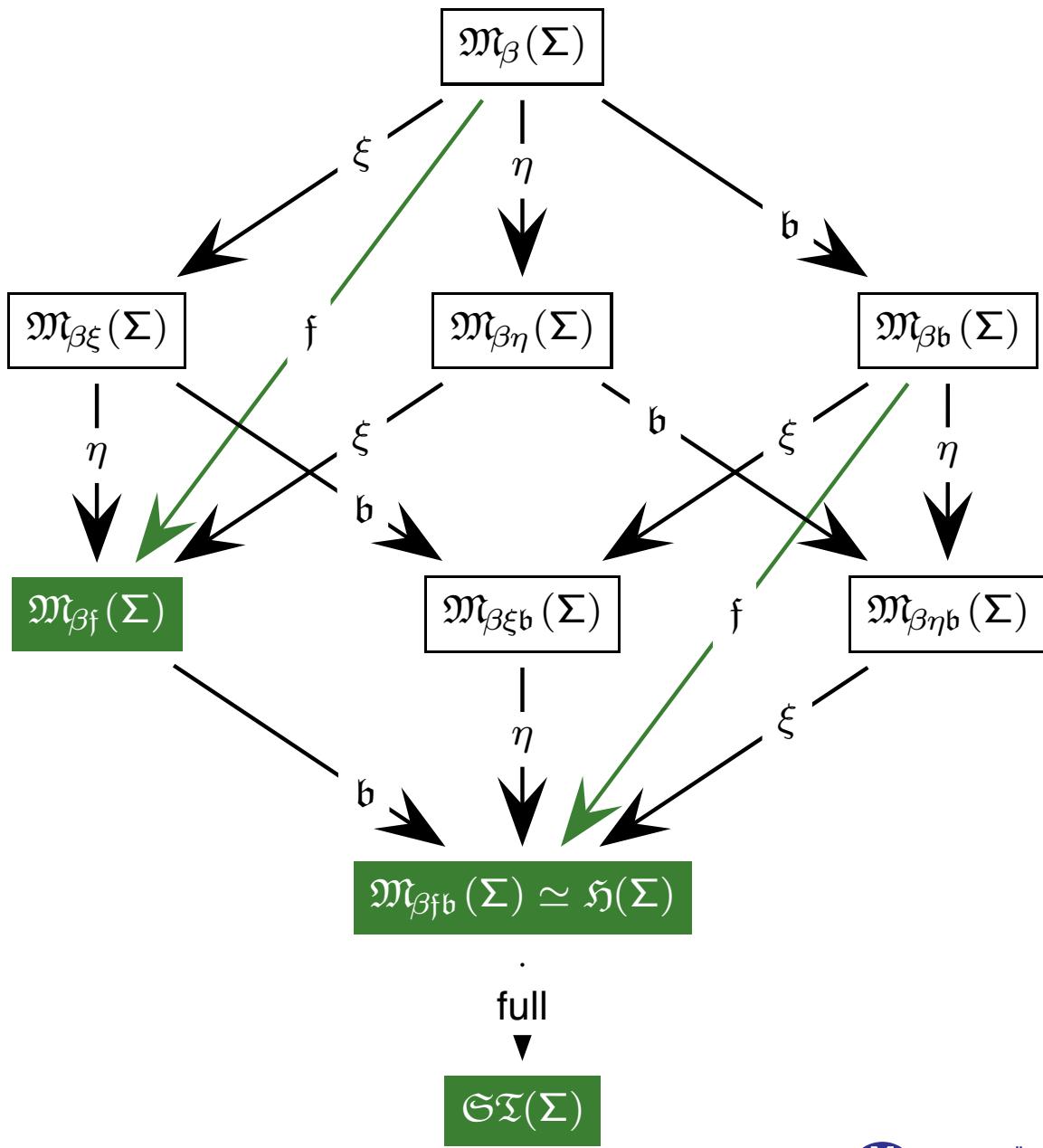
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- In such models, a function is not uniquely determined by its behavior on all possible arguments.

# Semantics: HOL-CUBE



$f$ : models are functional

$\forall f, g \in \mathcal{D}_{\beta\alpha} :$   
 $f = g \text{ iff } f@a = g@a \ (\forall a \in \mathcal{D}_\alpha)$

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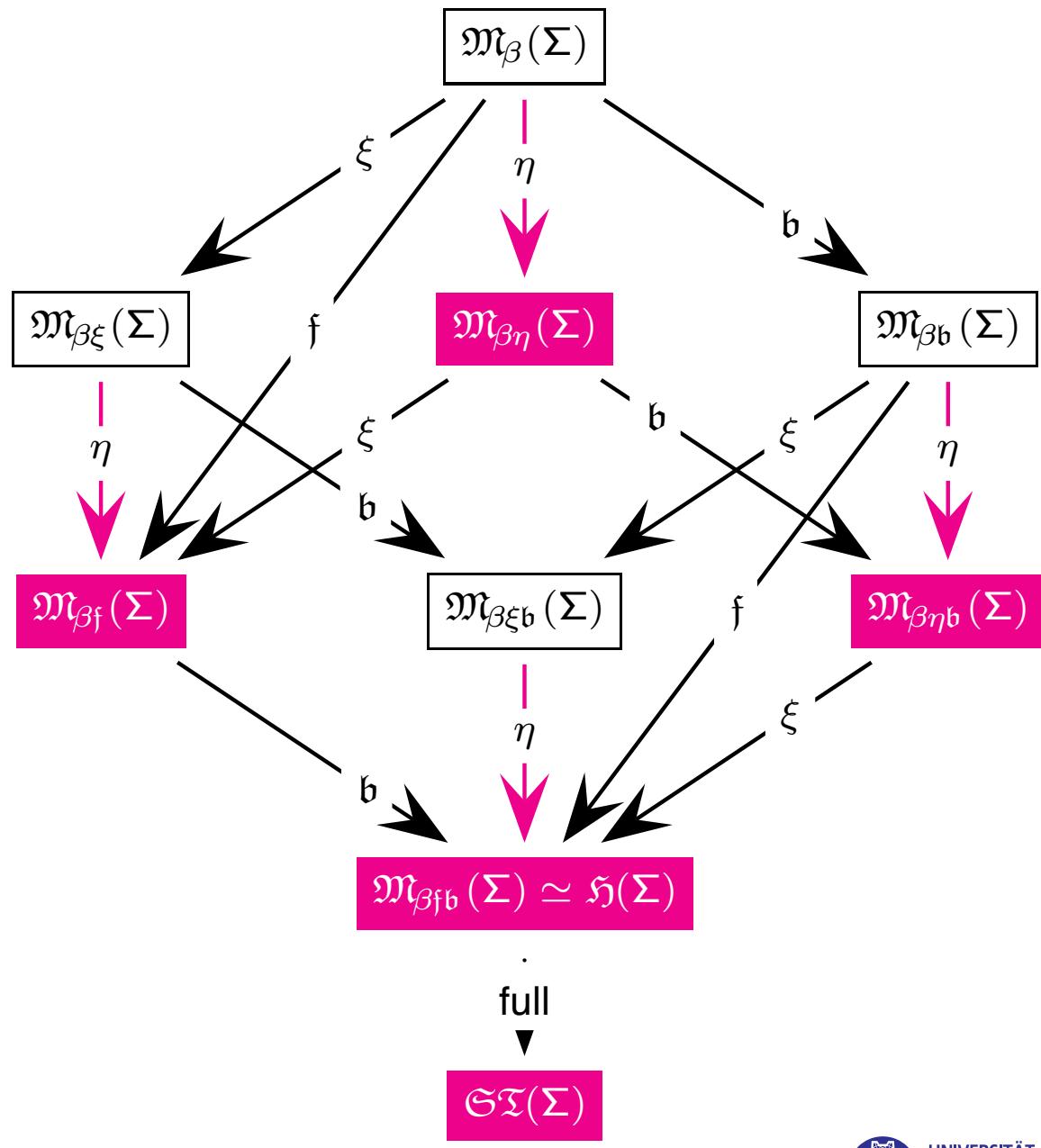
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- Therefore, we integrated these two cases in our landscape.

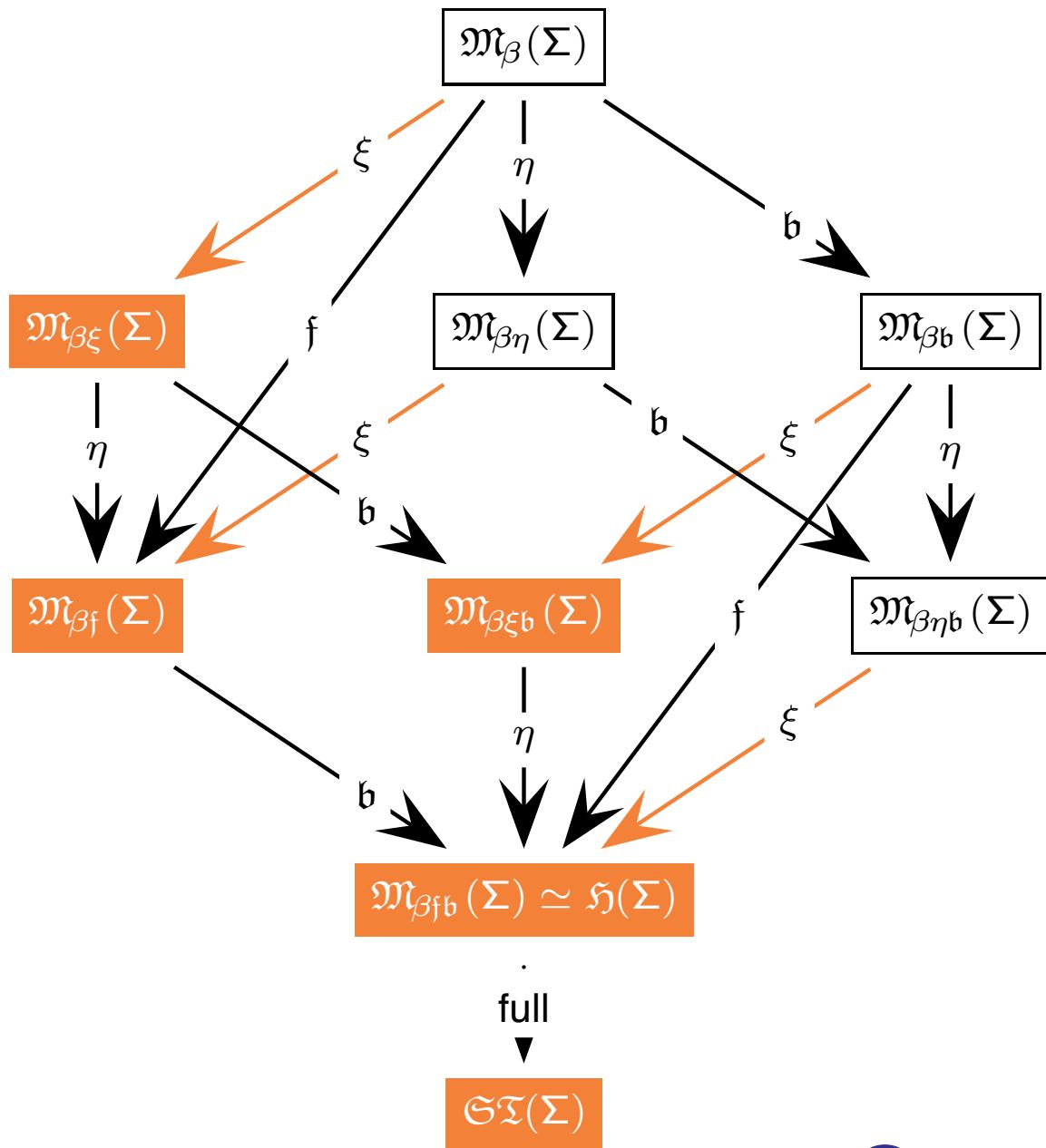
# Semantics: HOL-CUBE



$\eta$ : models are  $\eta$ -functional

$$\mathcal{E}_\varphi(A) = \mathcal{E}_\varphi(A \downarrow_{\beta\eta})$$

# Semantics: HOL-CUBE



$\xi$ : models are  $\xi$ -functional

$$\begin{aligned}\mathcal{E}_\varphi(\lambda X_\alpha.M_\beta) &= \mathcal{E}_\varphi(\lambda X_\alpha.N_\beta) \text{ iff} \\ \mathcal{E}_{\varphi,[a/X]}(M) &= \mathcal{E}_{\varphi,[a/X]}(N) \ (\forall a \in D_\alpha)\end{aligned}$$

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- Another example:  $\text{obvious(O)}$  and  $\text{obvious(F)}$  where  $O := 2 + 2 = 4$  and  $F := \forall n > 2 \cdot x^n + y^n = z^n \Rightarrow x = y = z = 0$  should not be equivalent, even if their arguments are.

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- Another example: **obvious(O)** and **obvious(F)** where **O := 2 + 2 = 4** and **F :=  $\forall n > 2 \cdot x^n + y^n = z^n \Rightarrow x = y = z = 0$**  should not be equivalent, even if their arguments are.
- Such phenomena have been studied under the heading of **“hyper-intensional semantics”** in theoretical semantics.

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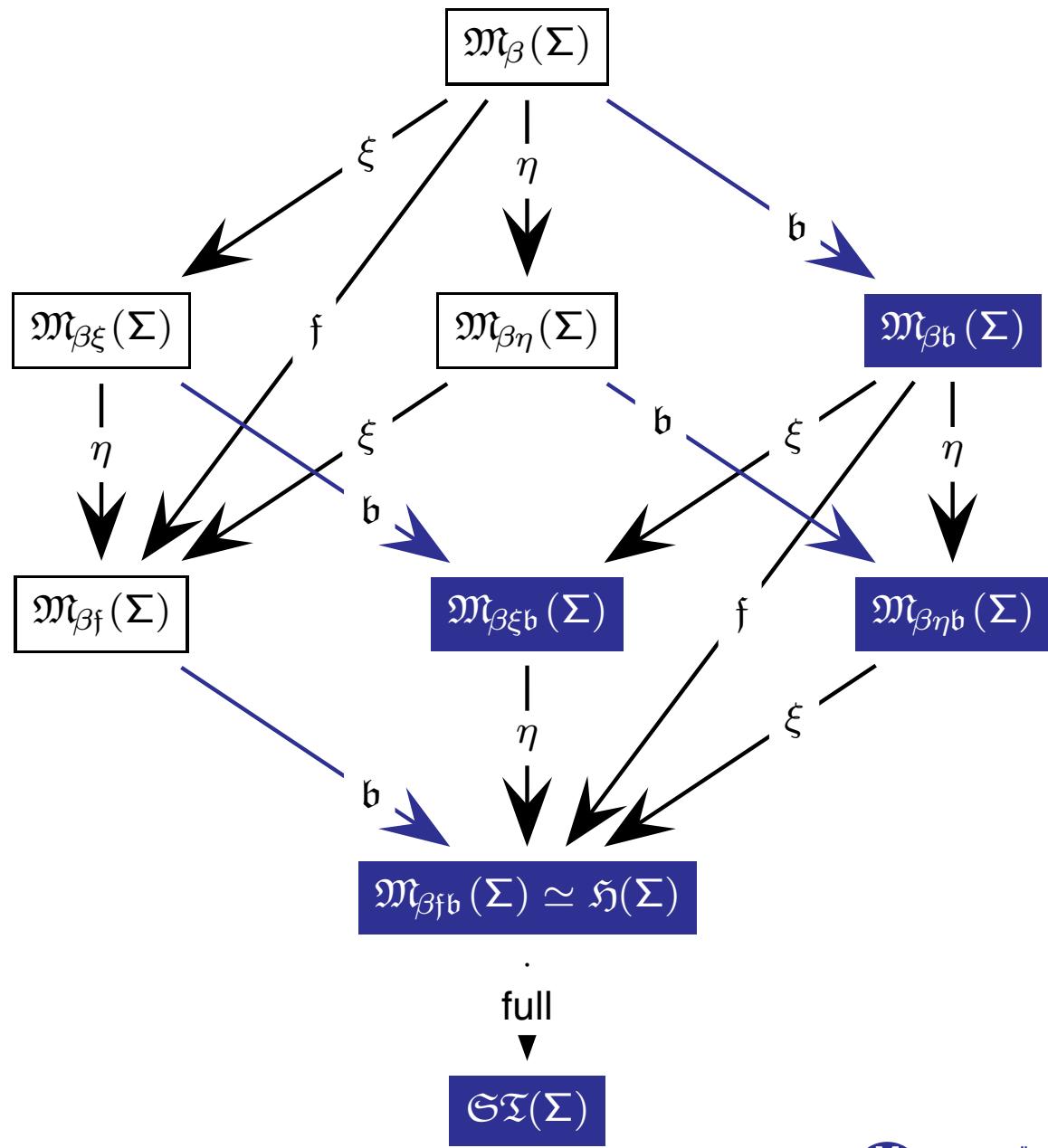
- We have weakened the assumption that  $\mathcal{D}_o = \{\text{T}, \text{F}\}$ , since this entails that the values of **O** and **F** are identical.
- In our  **$\Sigma$ -models** without property **b** we only insist that there is a division of the truth values into “good” and “bad” ones, which we express by insisting on the existence of a valuation  $v$  of  $\mathcal{D}_o$ , i.e., a function  $v: \mathcal{D}_o \rightarrow \{\text{T}, \text{F}\}$  that is coordinated with the interpretations of the logical constants  $\neg$ ,  $\vee$ , and  $\Pi^\alpha$  (for each type  $\alpha$ ).

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- Notion of validity: we call a sentence **A** valid in such a model if  $v(a) = \text{T}$ , where  $a \in \mathcal{D}_o$  is the denotation of the sentence **A**.

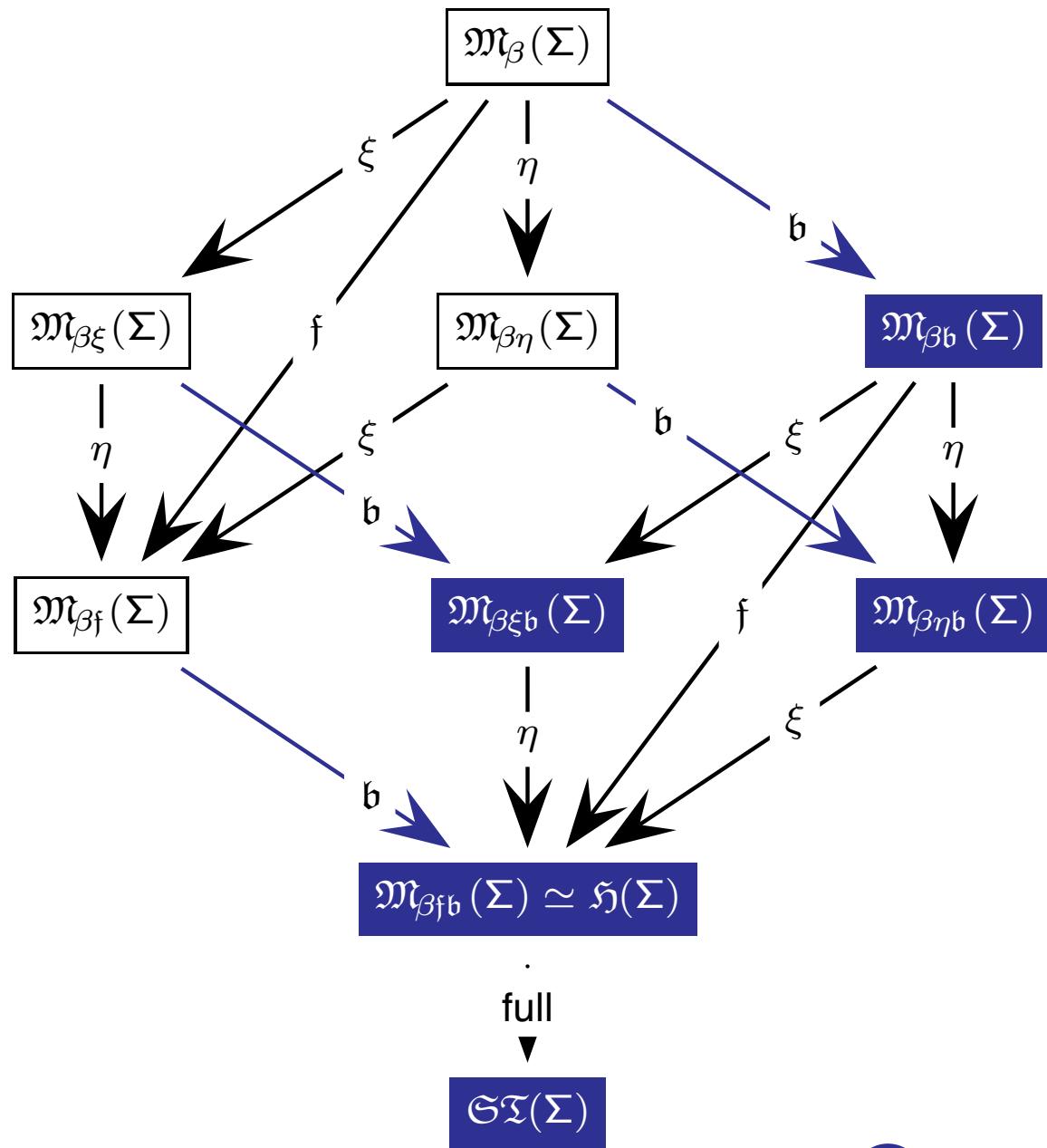
# Semantics: HOL-CUBE



$b$ : models are Boolean extensional

$v$  is injective

# Semantics: HOL-CUBE



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If  $\Sigma$  contains sufficiently many logical constants:

$$\mathcal{D}_o = \{\perp, \top\}$$



# Semantics and Theorem Proving: Test Problems for Theorem Provers

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- (Some more challenging examples are also added in [TPHOLS-05])

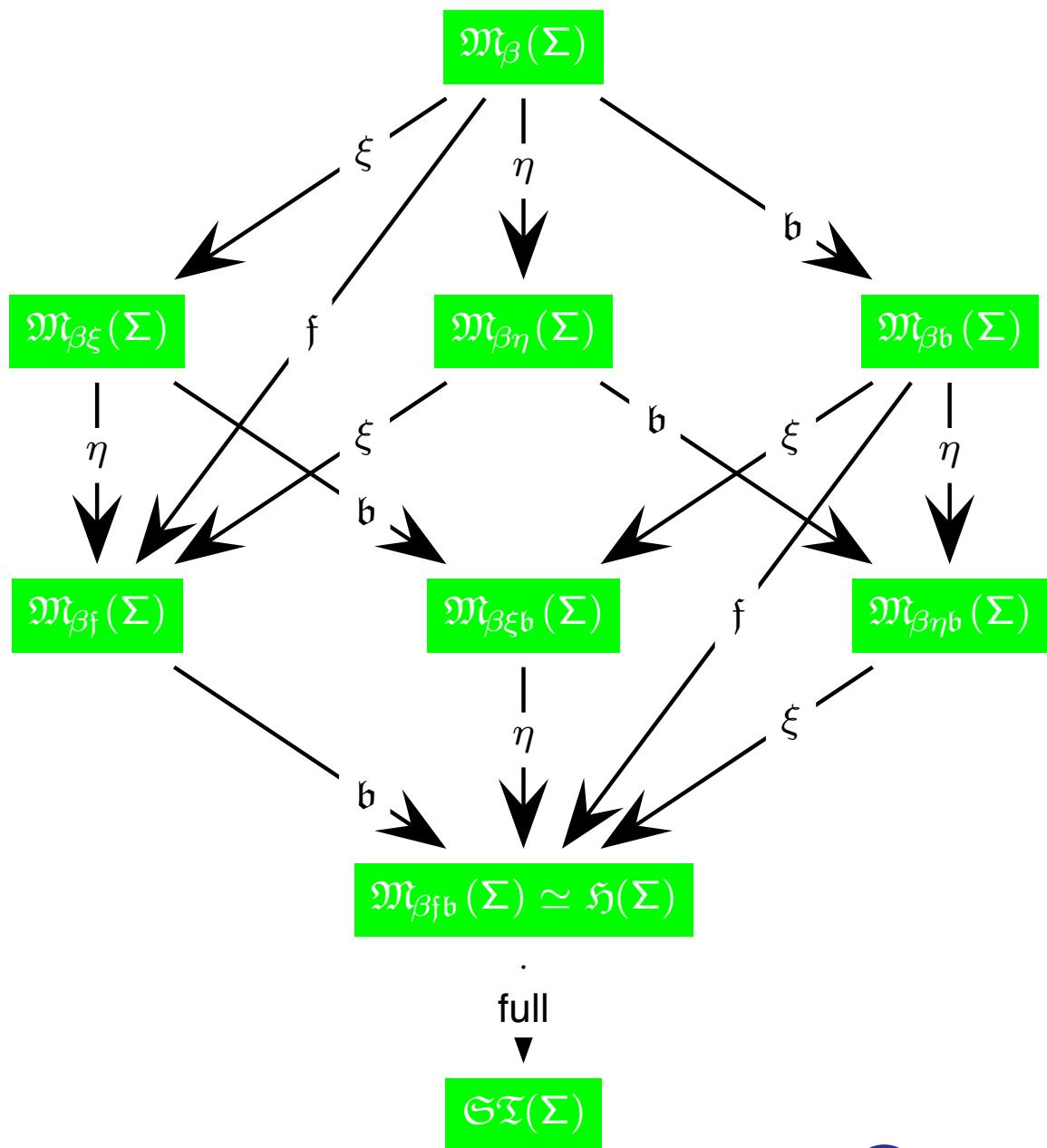
# Remark: Signature

Unless stated otherwise we assume on the following slides that our signature  $\Sigma$  contains the following logical connectives:

$$\{\top, \perp, \neg, \wedge, \vee, \supset, \Leftrightarrow\} \cup \{\Pi^\alpha, \Sigma^\alpha, =^\alpha\}$$

(less logical connectives are possible)

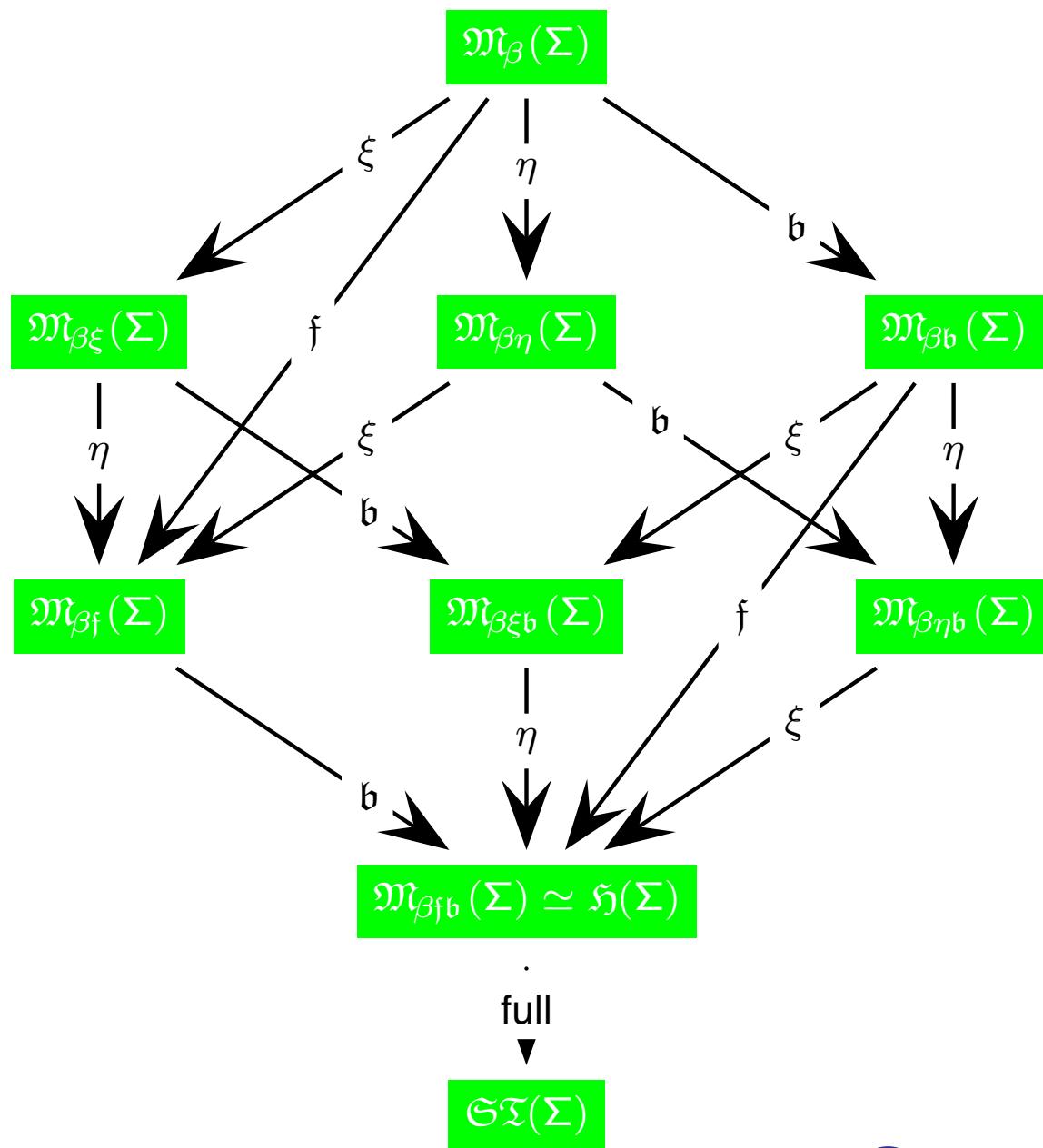
# HOL-Problems: $\beta$



$\stackrel{*}{=}$  is equivalence relation

- $\forall X_\alpha. X \stackrel{*}{=} X$
- $\forall X_\alpha, Y_\alpha. X \stackrel{*}{=} Y \supset Y \stackrel{*}{=} X$
- $\forall X_\alpha, Y_\alpha, Z_\alpha. (X \stackrel{*}{=} Y \wedge Y \stackrel{*}{=} Z) \supset X \stackrel{*}{=} Z$

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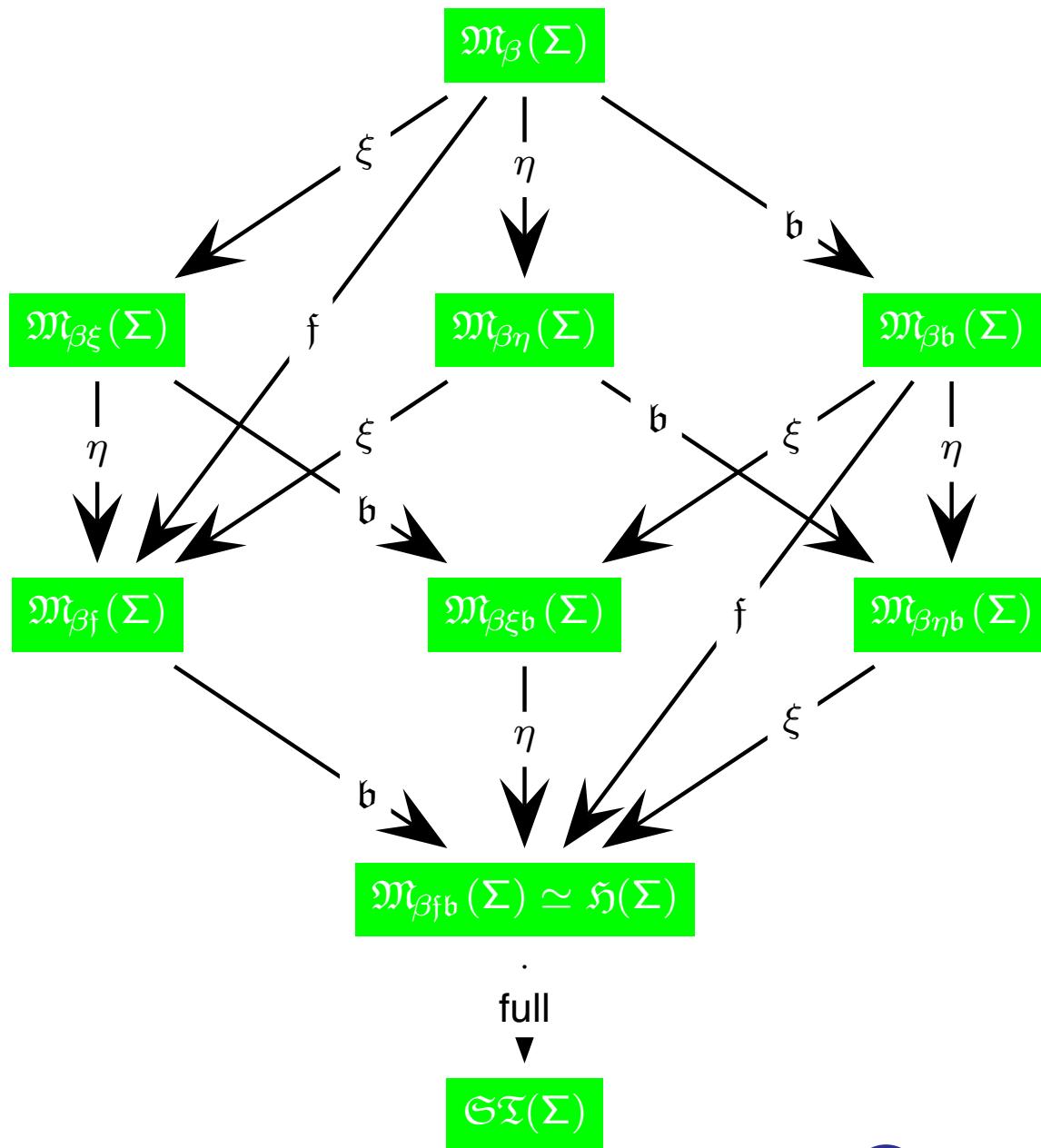
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$\stackrel{*}{=}$  is congruence relation

- $\forall X_\alpha, Y_\alpha, F_{\alpha\alpha}. X \stackrel{*}{=} Y \supset (FX) \stackrel{*}{=} (FY)$
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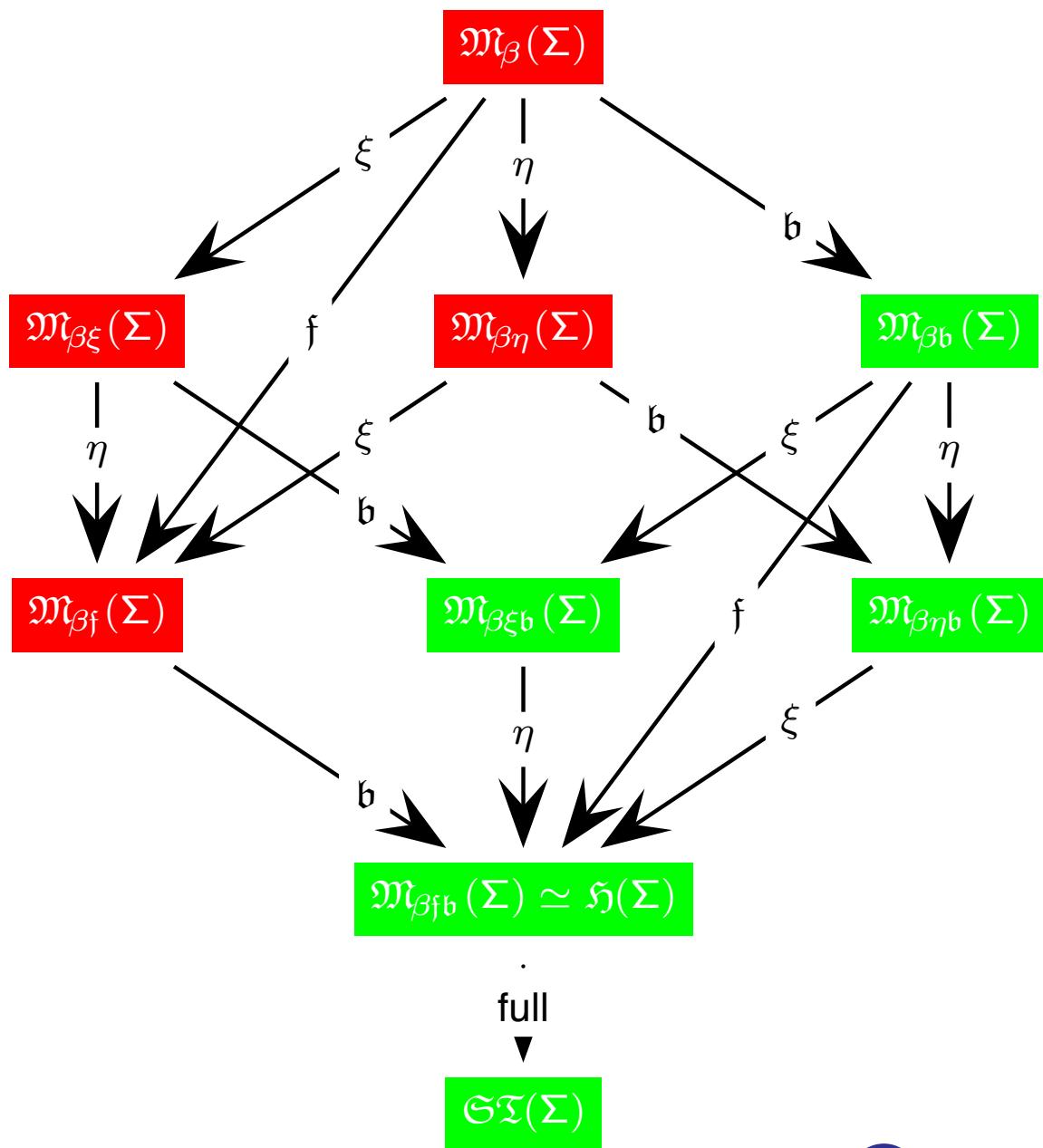
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Trivial directions of Boolean and functional extensibility

- $\forall A_o, B_o. A \stackrel{*}{=} B \supset (A \Leftrightarrow B)$
- $\forall F_{\beta\alpha}, G_{\beta\alpha}. F \stackrel{*}{=} G \supset (\forall X_\alpha. FX \stackrel{*}{=} GX)$

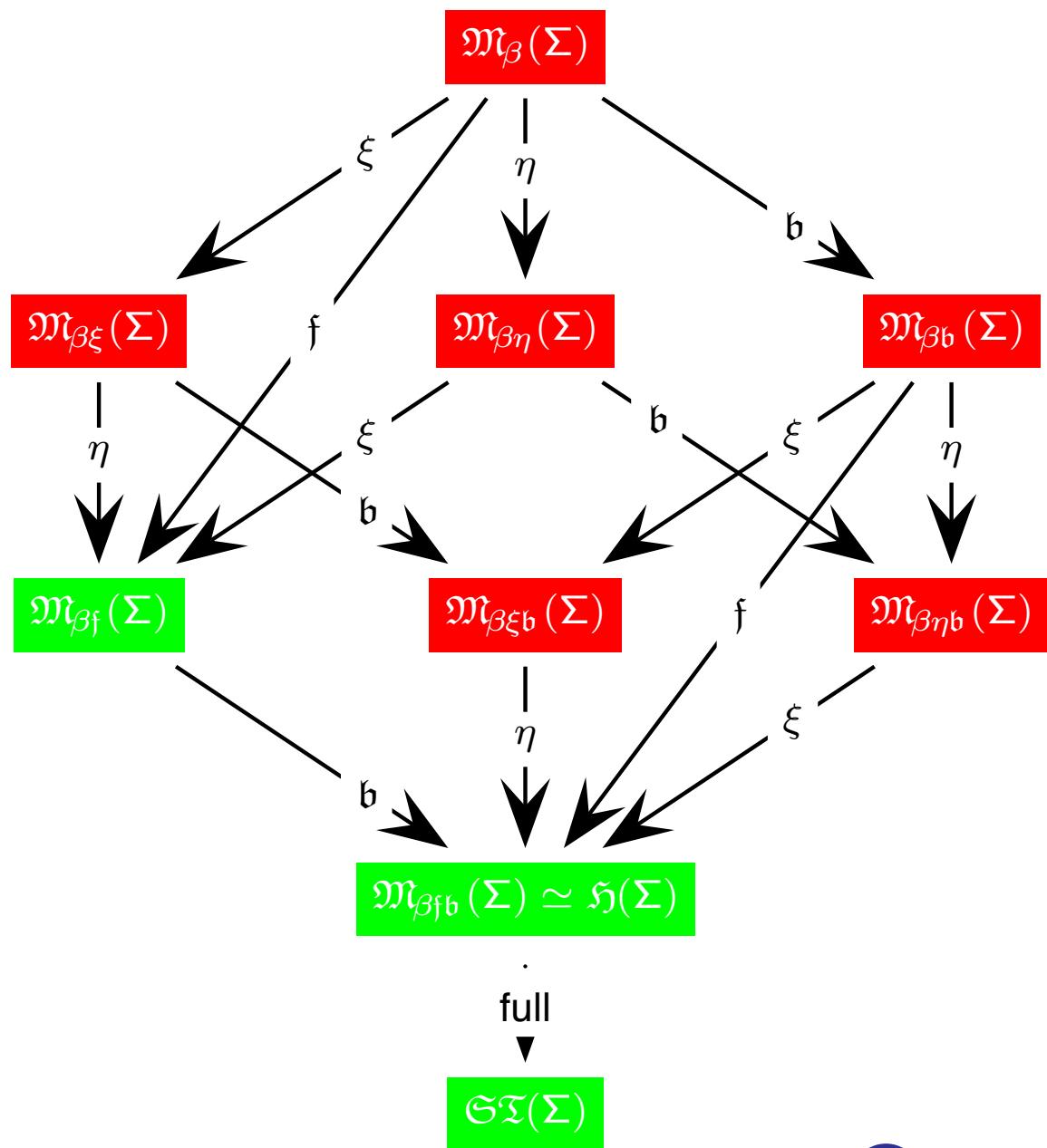
# HOL-Problems: $\mathfrak{b}$



Non-trivial direction of Boolean extensionality

- $\forall A_o, B_o. (A \Leftrightarrow B) \supset A \stackrel{*}{=} B$

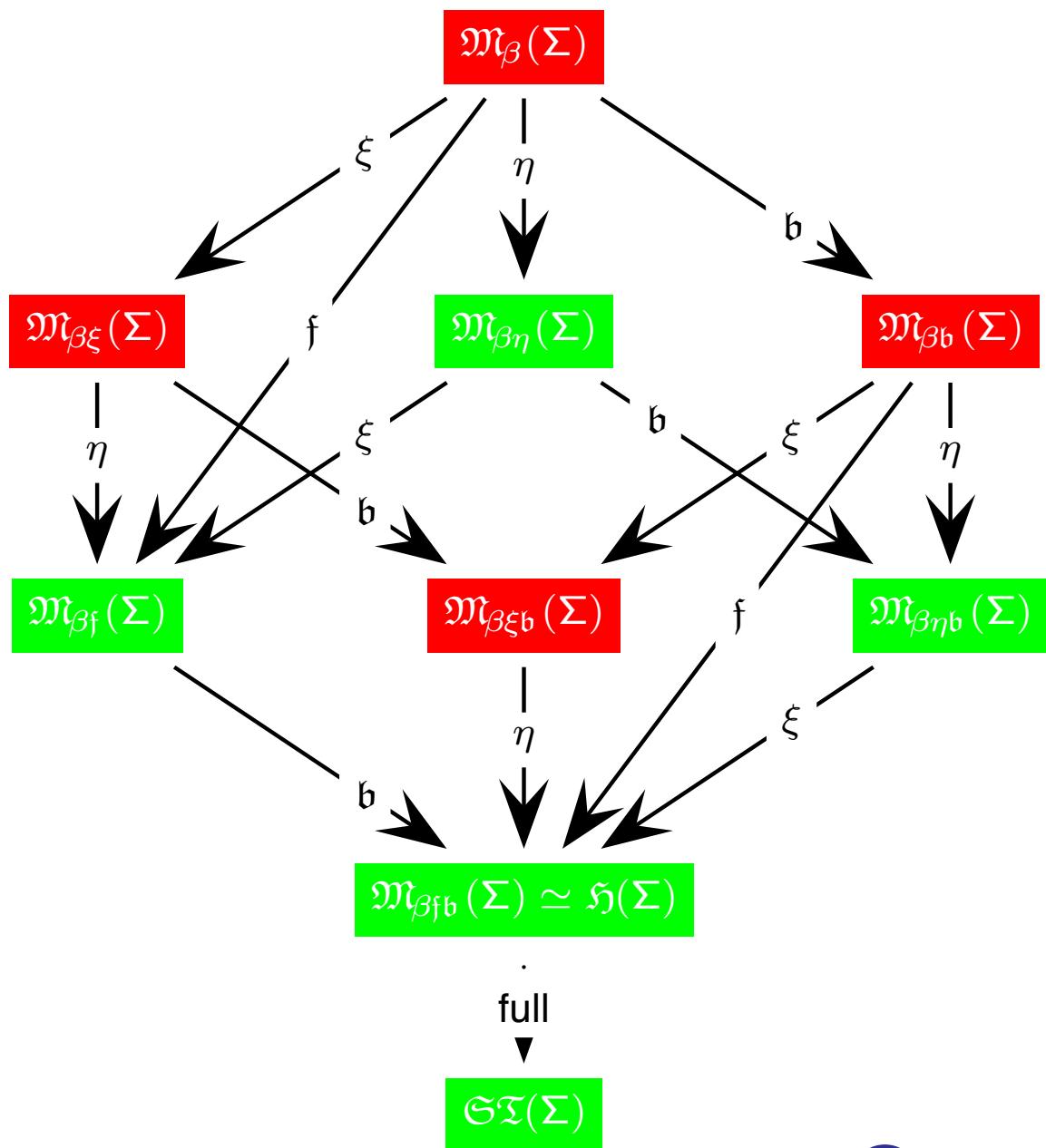
# HOL-Problems: $\mathfrak{f}$



**Non-trivial direct. of functional extensionality**

- $\forall F_{\beta\alpha}, G_{\beta\alpha}. (\forall X_\alpha. FX \stackrel{*}{=} GX) \supset F \stackrel{*}{=} G$

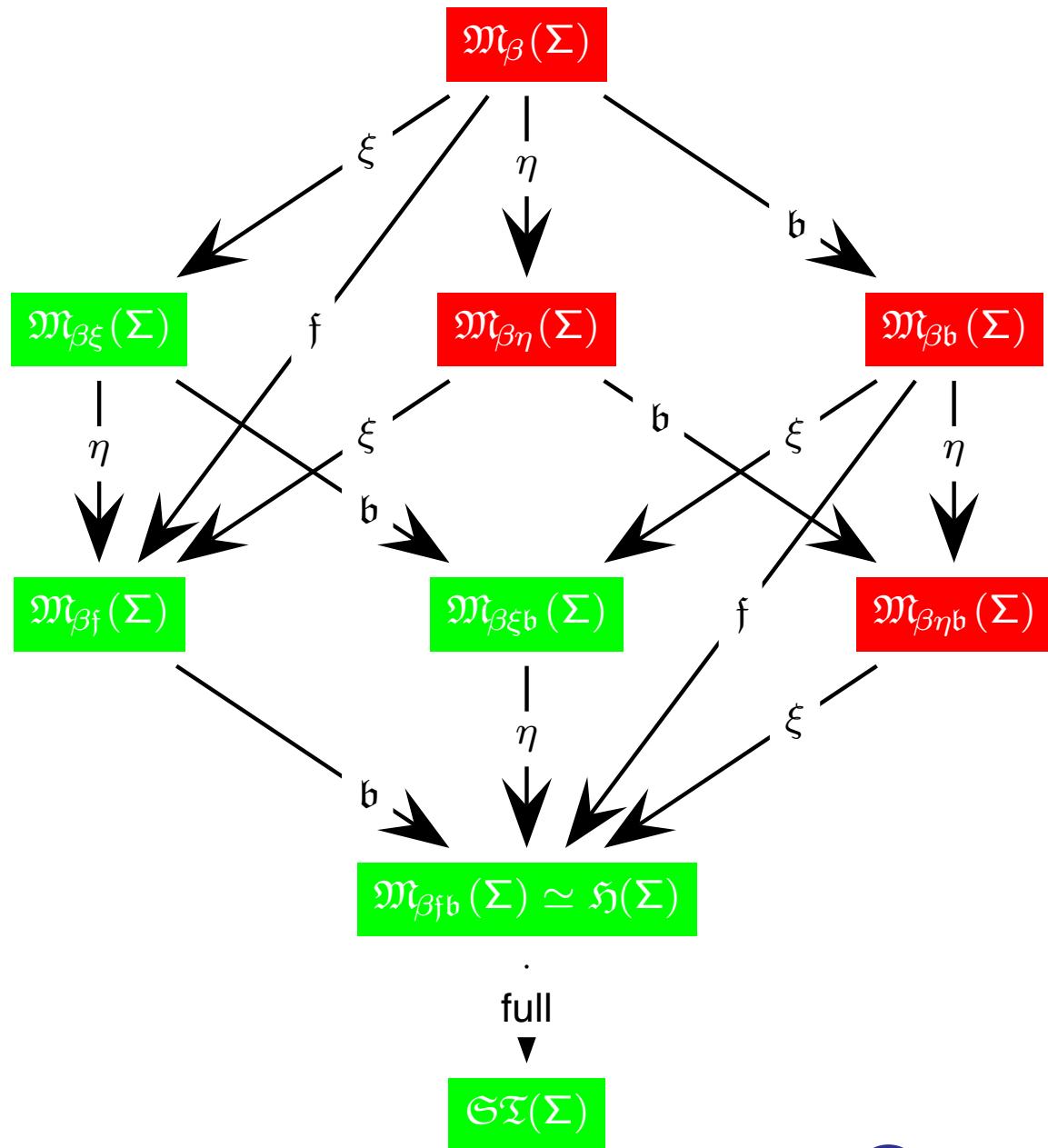
# HOL-Problems: $\eta$



Example requiring property  $\eta$

- $(p_{o(\nu\nu)}(\lambda X_\nu.f_{\nu\nu}X)) \supset (p f)$

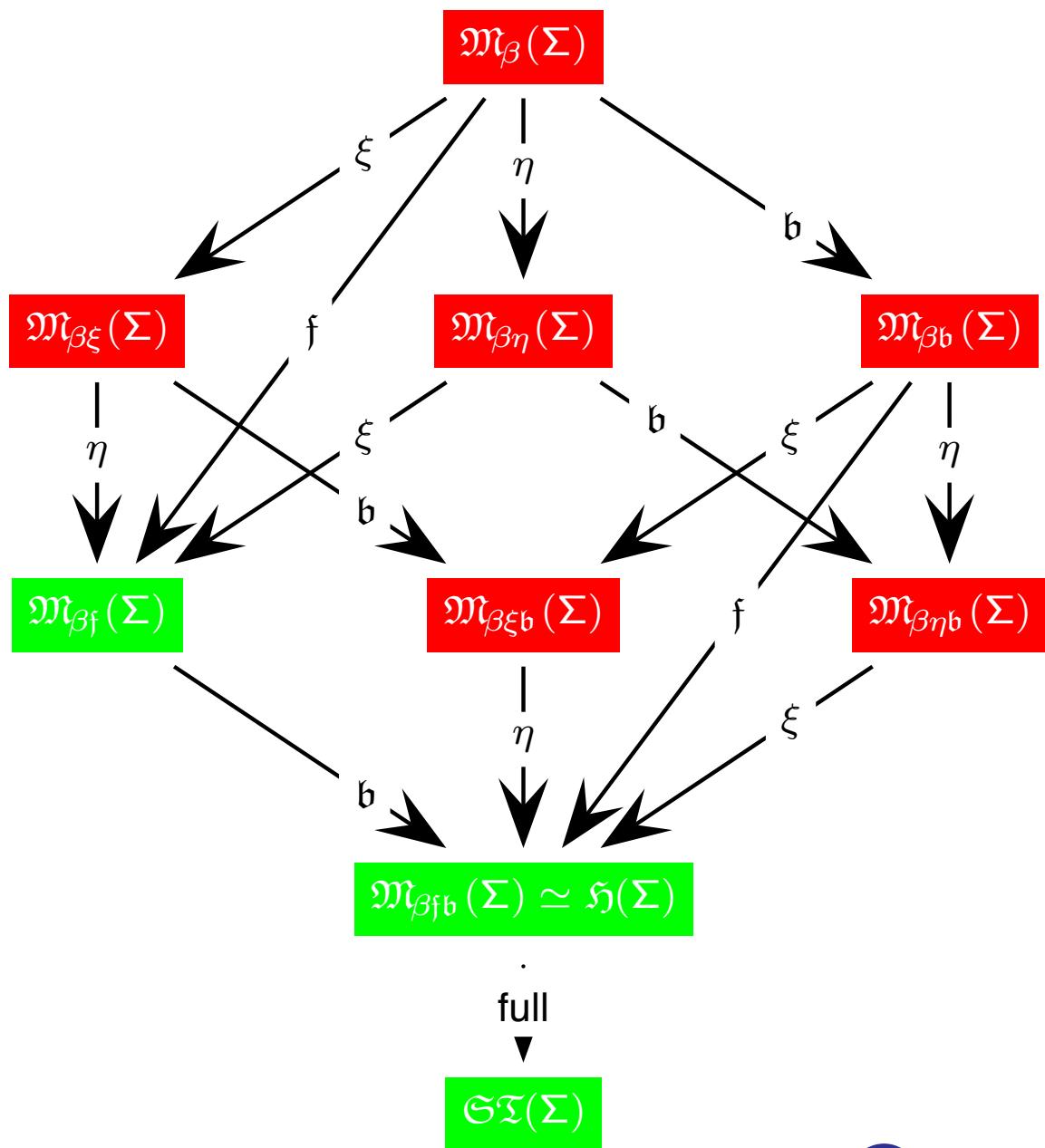
# HOL-Problems: $\xi$



Example requiring property  $\xi$  (and  $q!$ )

- $(\forall X_\nu.(f_{\nu\nu}X) \stackrel{*}{=} X) \wedge p_{o(\nu\nu)}(\lambda X_\nu.X)$   
 $\supset p(\lambda X_\nu.fX)$

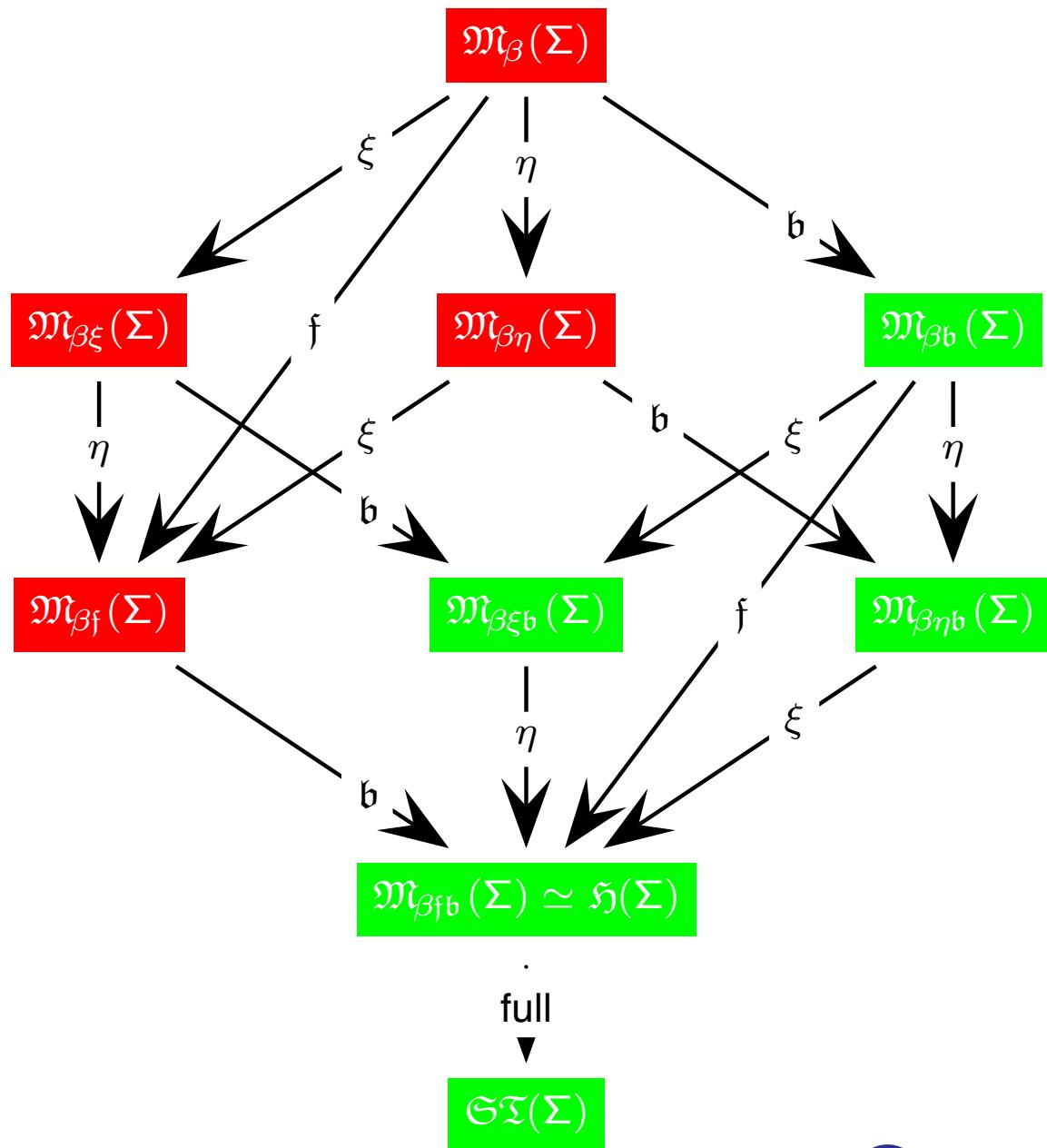
# HOL-Problems: $\mathfrak{f}$



Example requiring property  $\mathfrak{f}$  (and  $\mathfrak{q}!$ )

- ( $\forall X_\nu . (f_{\nu\nu} X) \stackrel{*}{=} X \wedge p_{o(\nu\nu)}(\lambda X_\nu . X)$ )  
 $\supset (p f)$

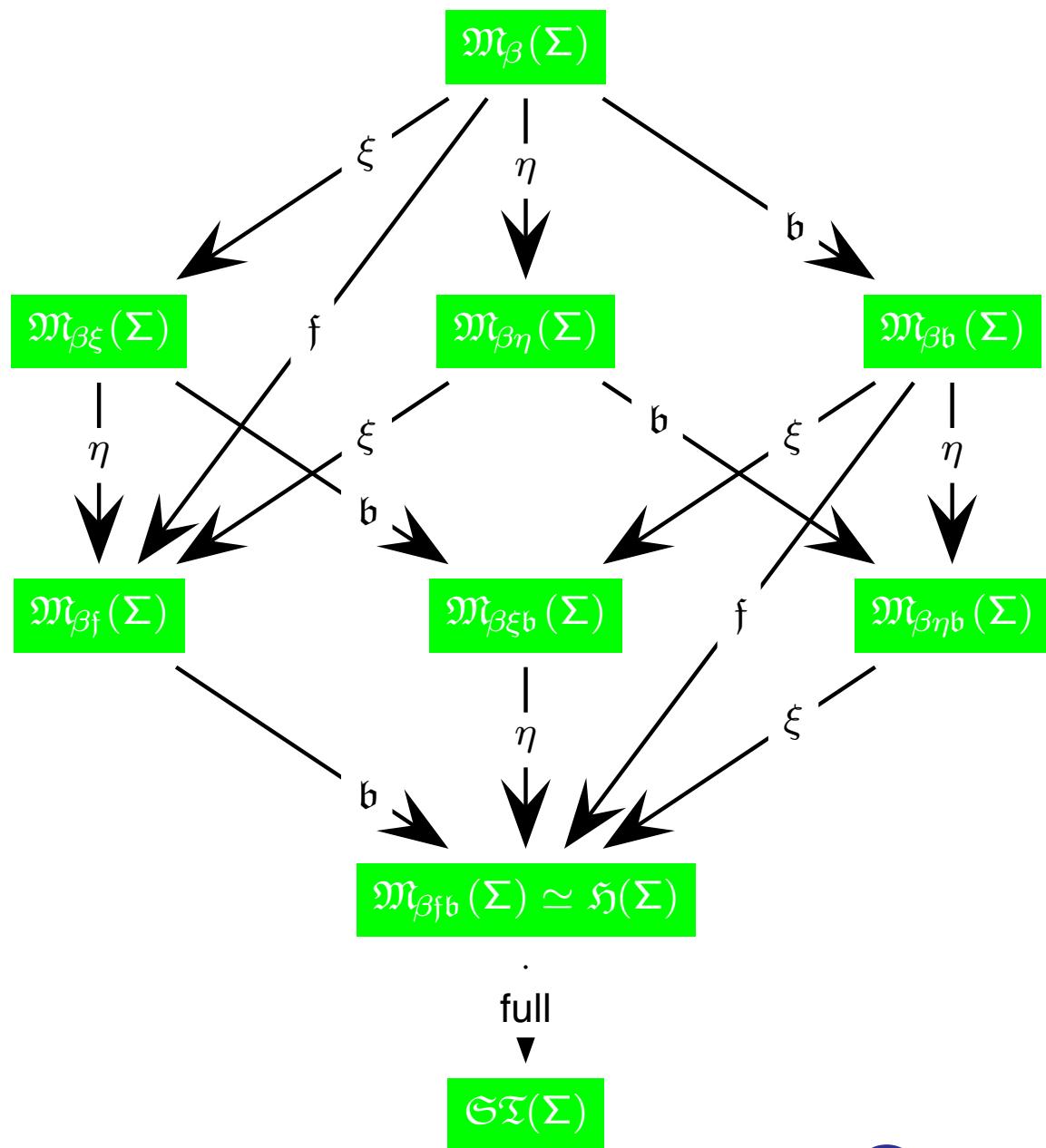
# HOL-Problems: $\mathfrak{b}$



## Examples requiring property $\mathfrak{b}$

- $(p_{oo} a_o) \wedge (p b_o) \Rightarrow (p (a \wedge b))$
- $\neg(a \stackrel{*}{=} \neg a)$  (in particular  $\neg(a = \neg a)$ )
- $(h_{\iota o}((h\top) \stackrel{*}{=} (h\perp))) \stackrel{*}{=} (h\perp)$

# HOL-Problems: Other Examples

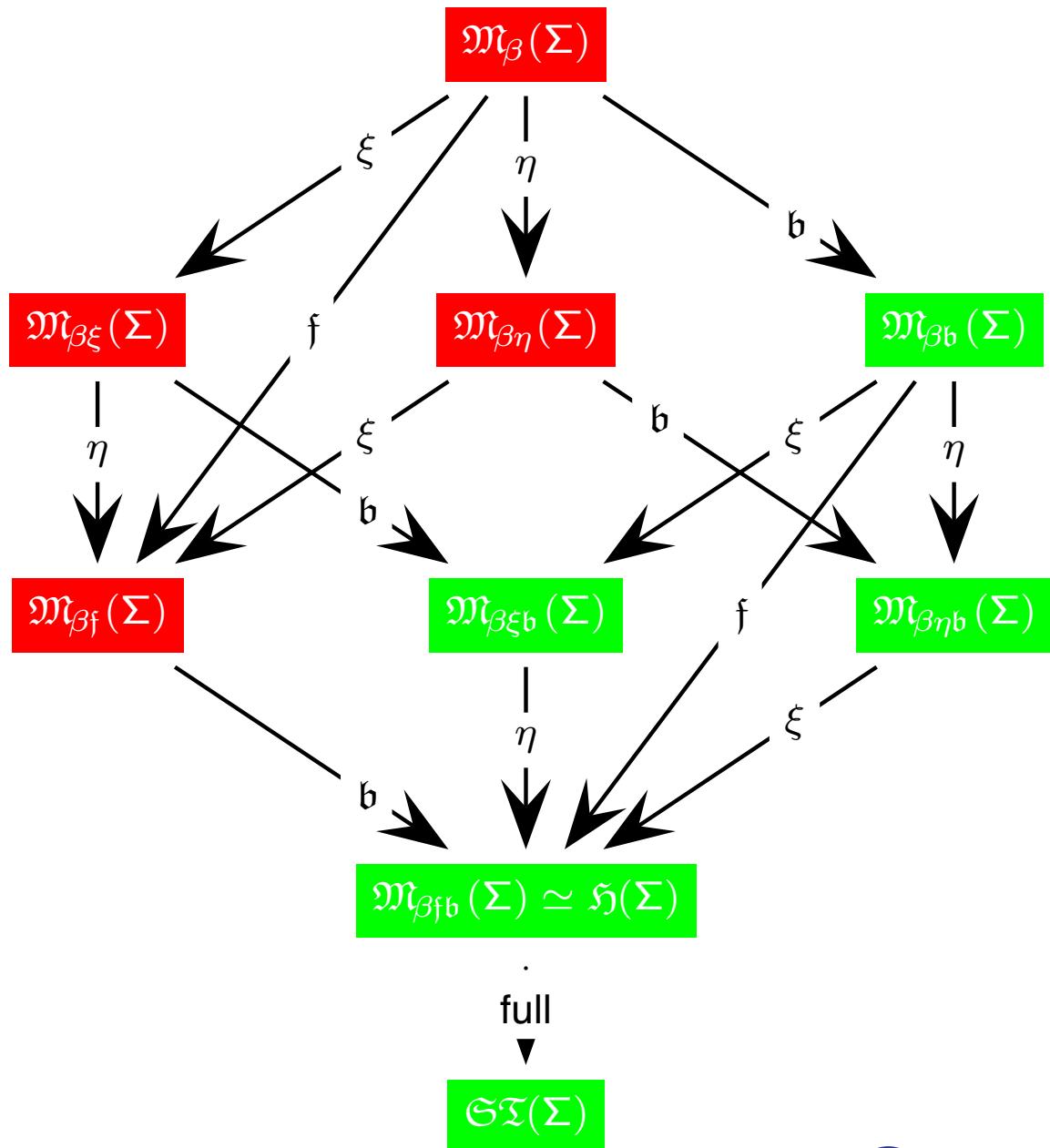


Playing with DeMorgan's Law:

- $\forall X, Y. X \wedge Y \Leftrightarrow \neg(\neg X \vee \neg Y)$

'Ok' for all model classes

# HOL-Problems: DeMorgan's Law

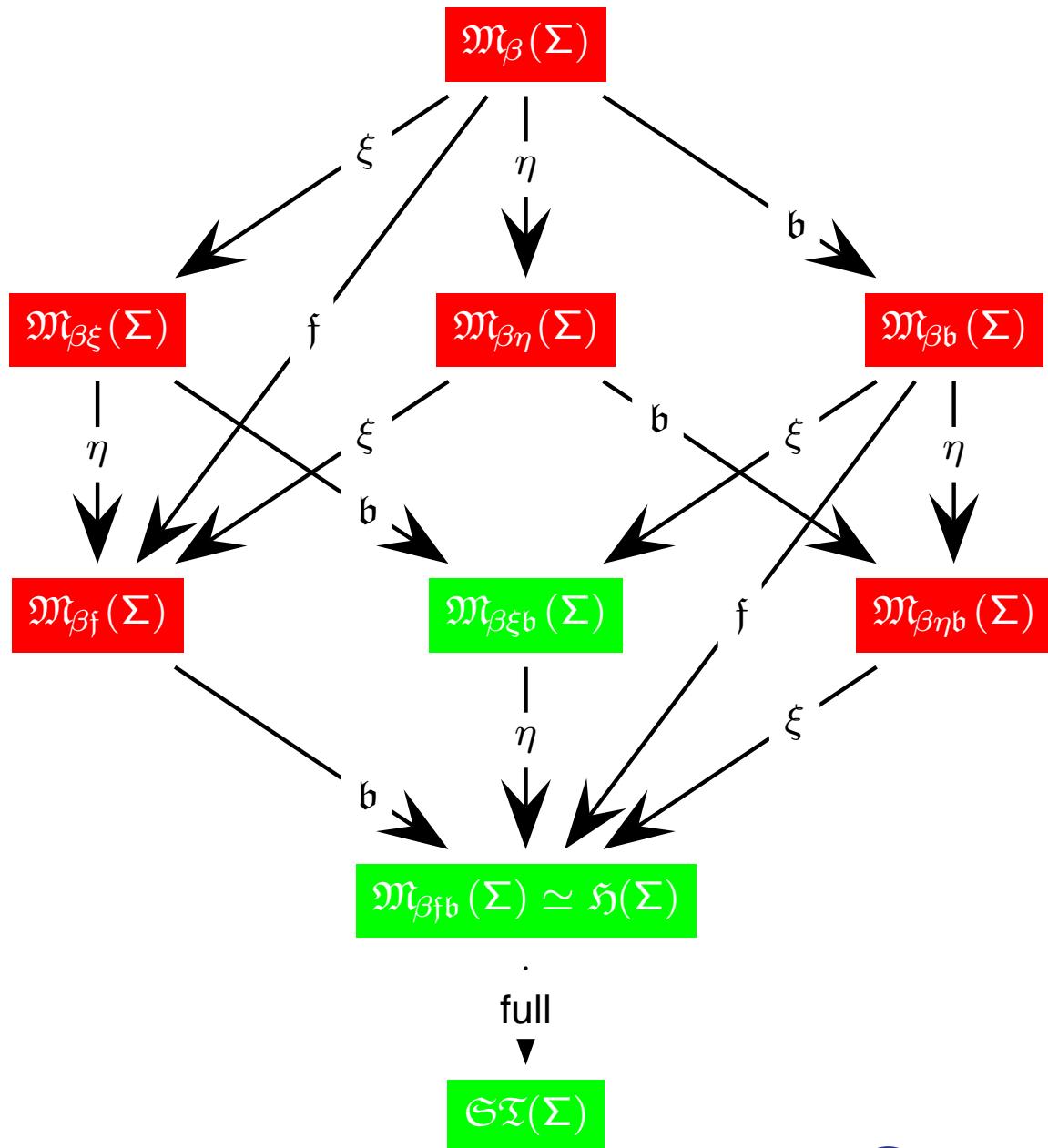


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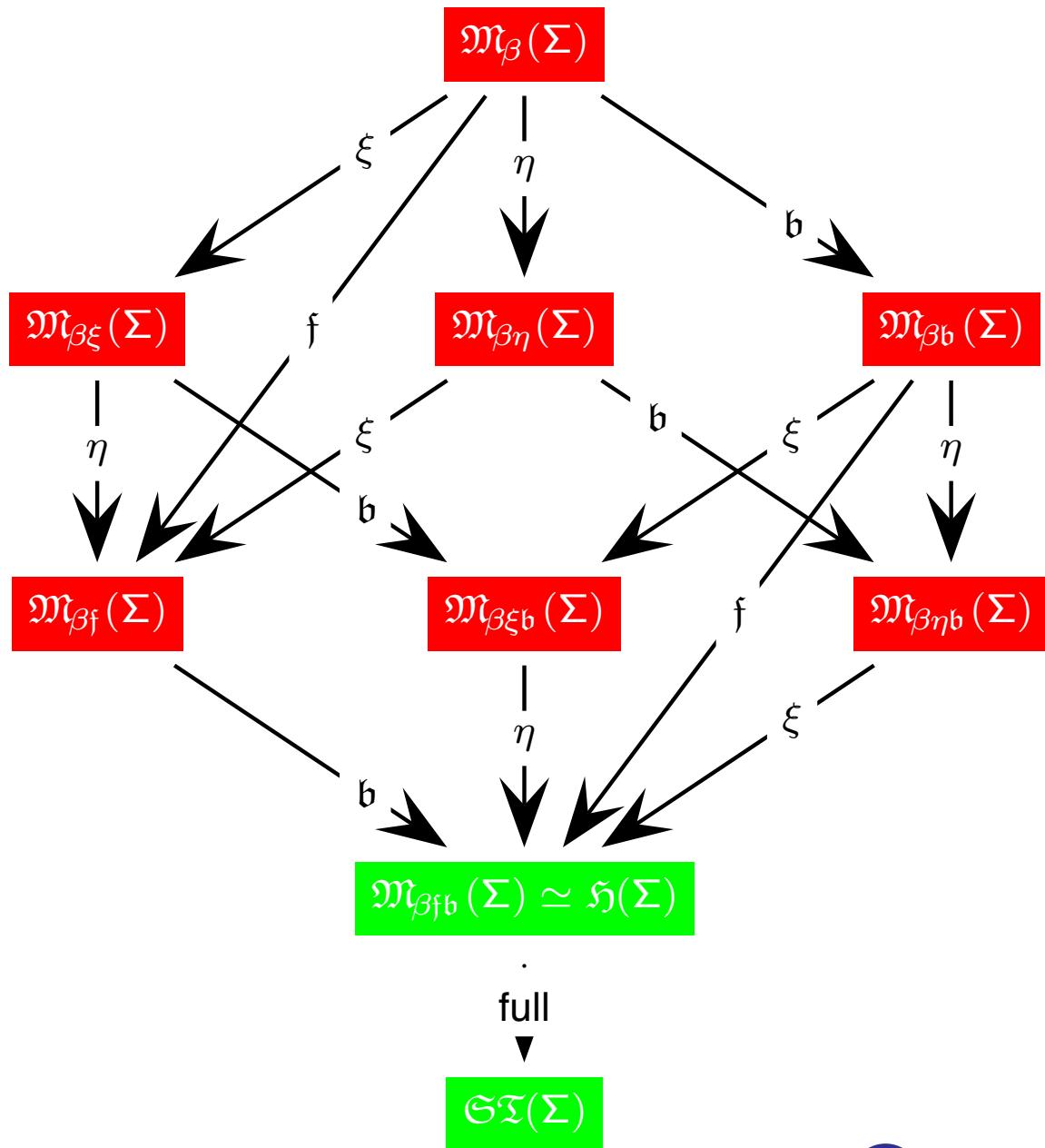


Playing with DeMorgan's Law:

- $\forall X, Y. X \wedge Y \Leftrightarrow \neg(\neg X \vee \neg Y)$
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- $(\lambda U \lambda V. U \wedge V) \stackrel{*}{=} (\lambda X \lambda Y. \neg(\neg X \vee \neg Y))$

requires  $b$  and  $\xi$

# HOL-Problems: DeMorgan's Law

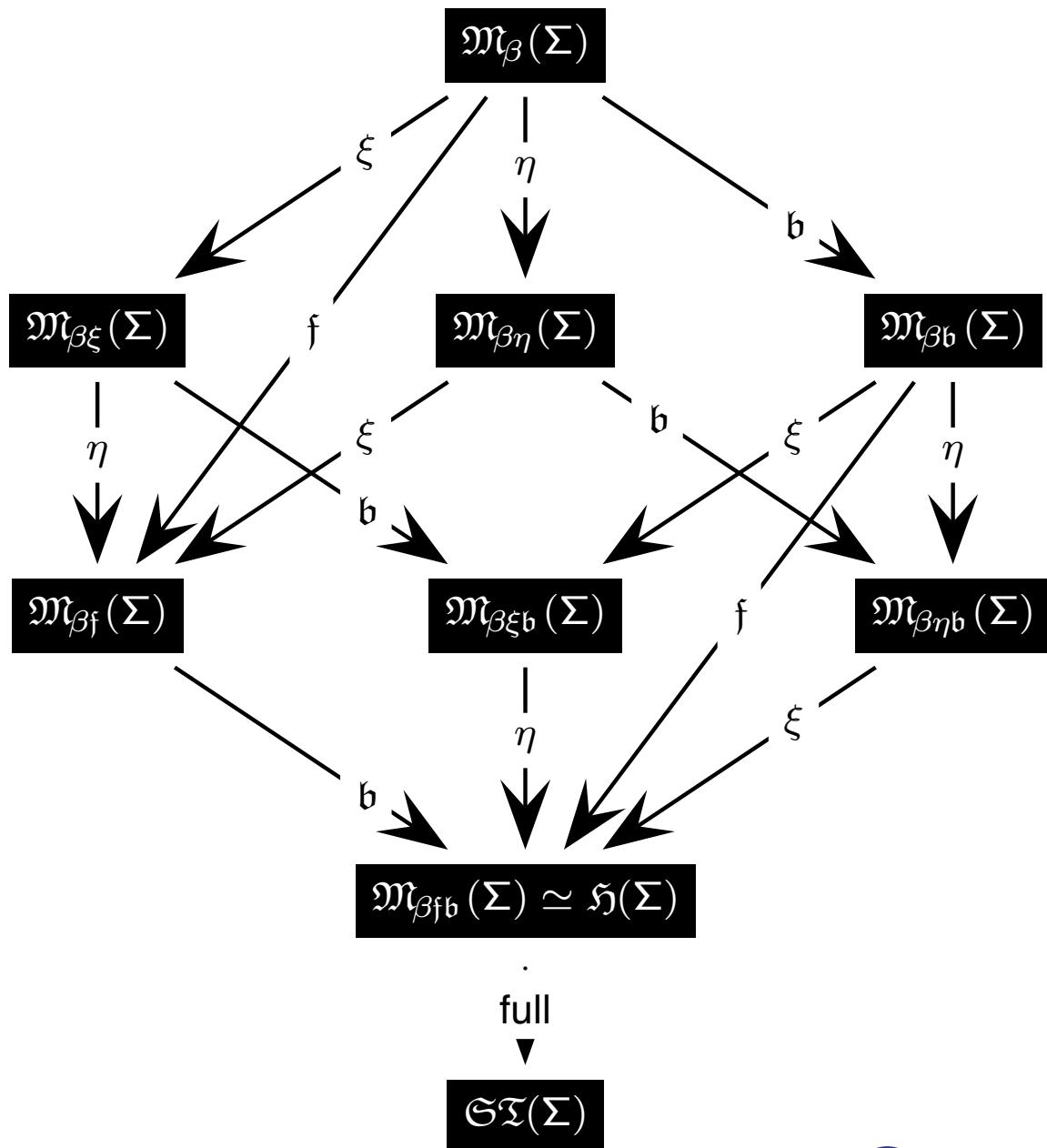


**Playing with DeMorgan's Law:**

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- $\wedge \stackrel{*}{=} (\lambda X \lambda Y. \neg(\neg X \vee \neg Y))$

requires  $b$  and  $f$

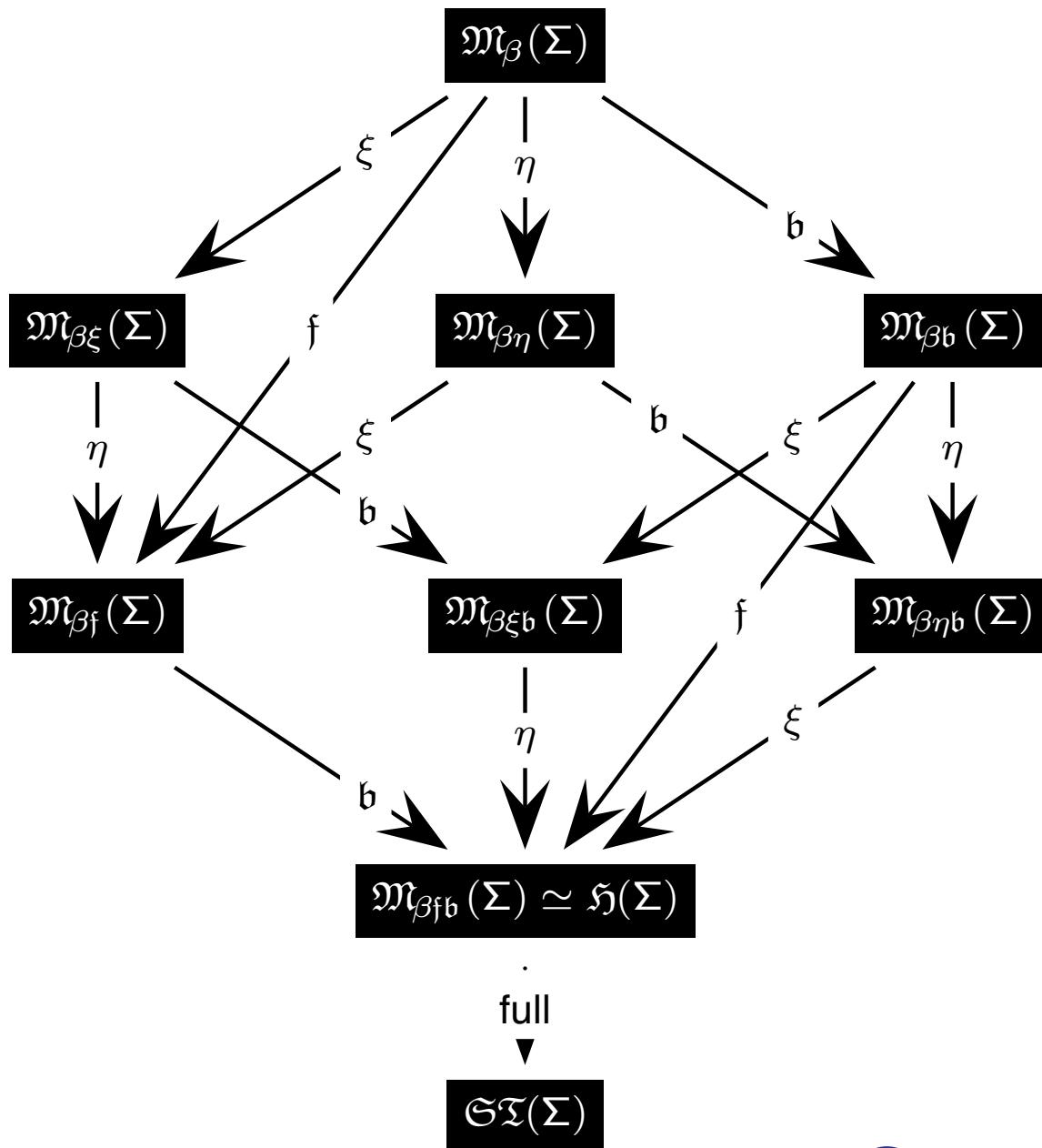
# HOL-Problems: Set Comprehension



## Set comprehension

- big challenge for automation
- [Benzm.BrownKohlhase-Draft-05] set instantiations can be used to simulate cut-rule if one of the following axioms is given: comprehension, induction, extensionality, choice, description
- dependend on logical constants in  $\Sigma$

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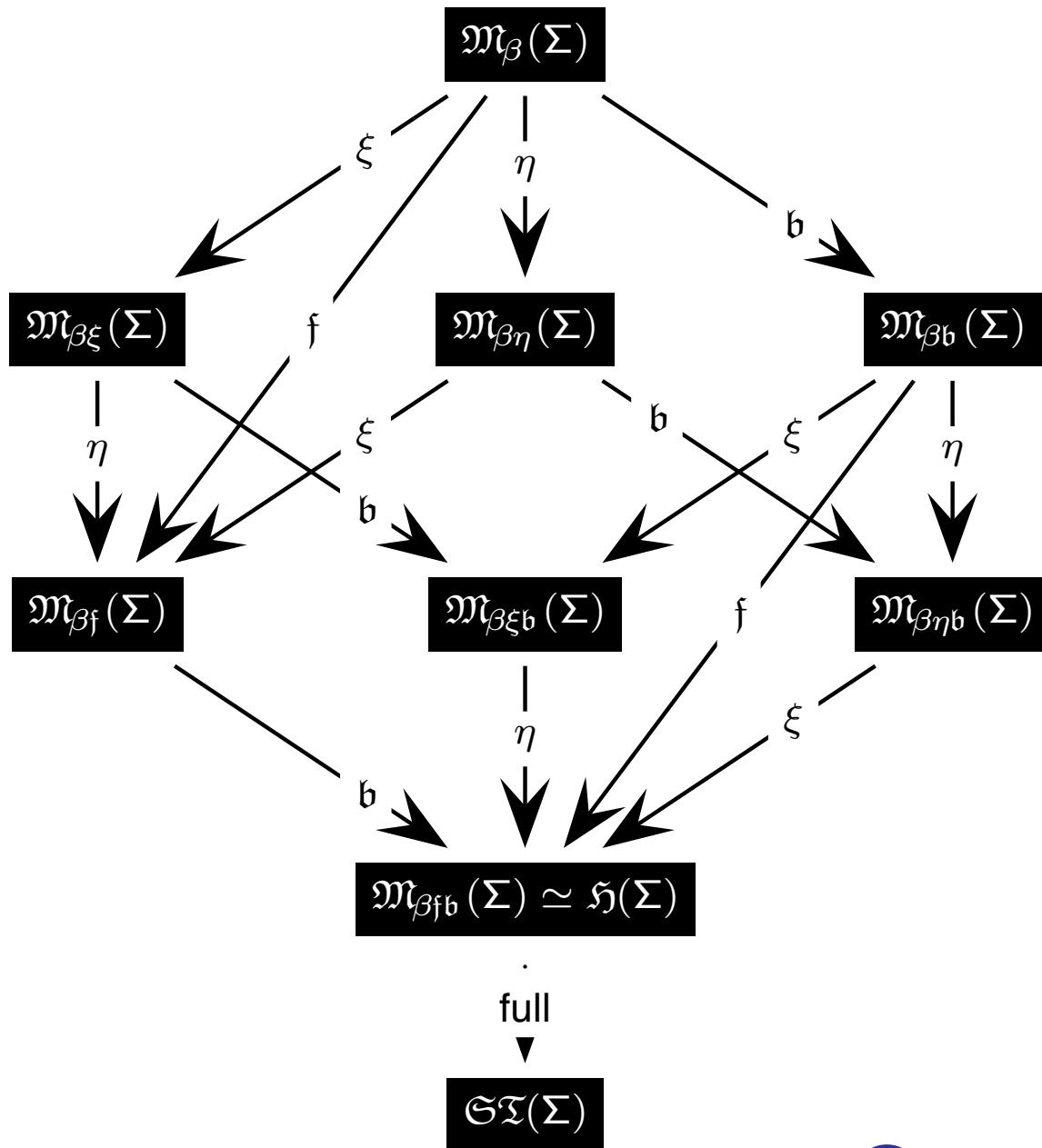
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- signature  $\Sigma$  varying
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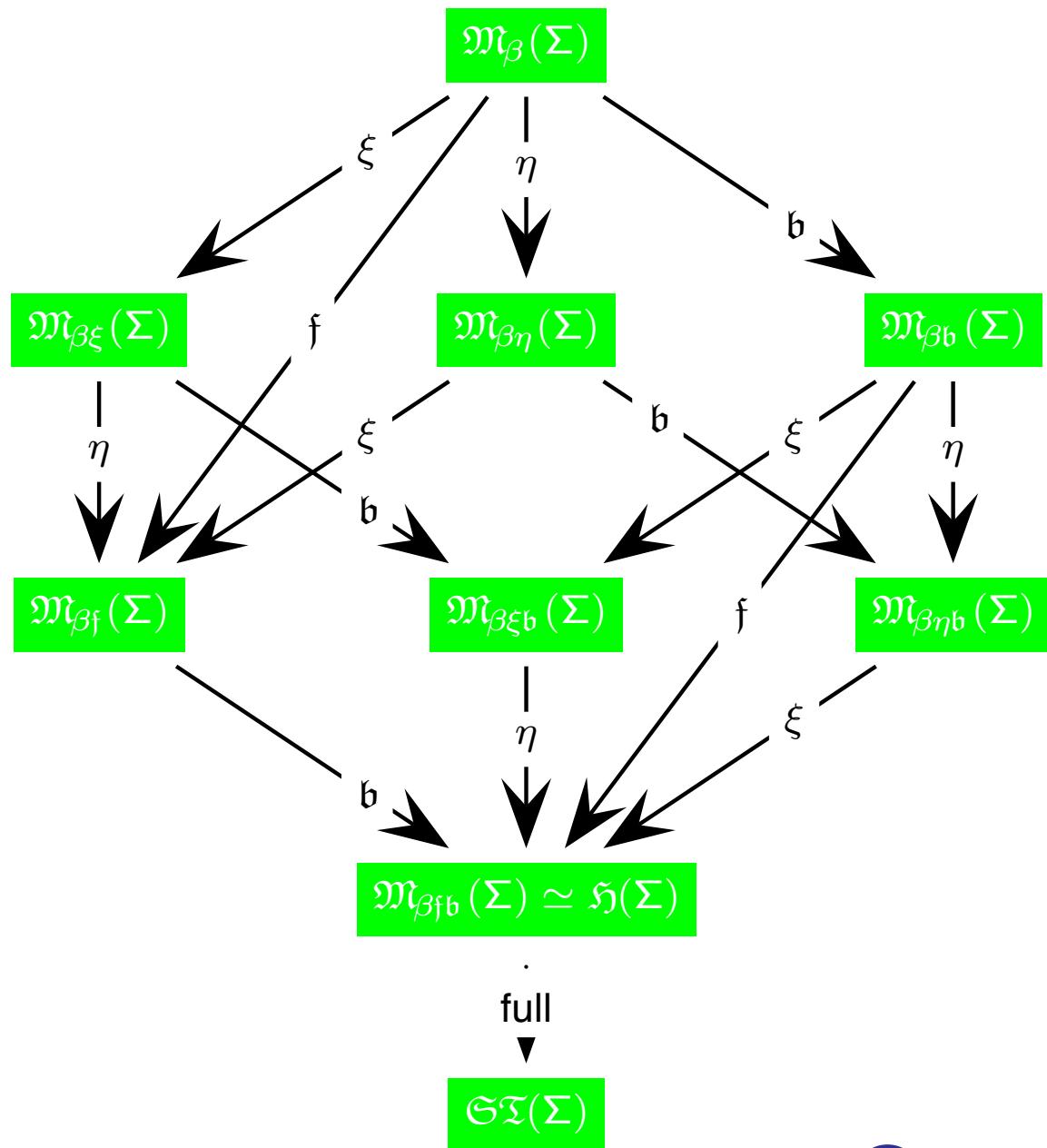
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- signature  $\Sigma$  varying
- no property  $q$  assumed

## External vs. internal logical constants

- if  $\neg \notin \Sigma$ :  
 $\neg$  refers to 'external' symbol  
 $\mathcal{M} \models \neg A$  means  $\mathcal{M} \not\models A$

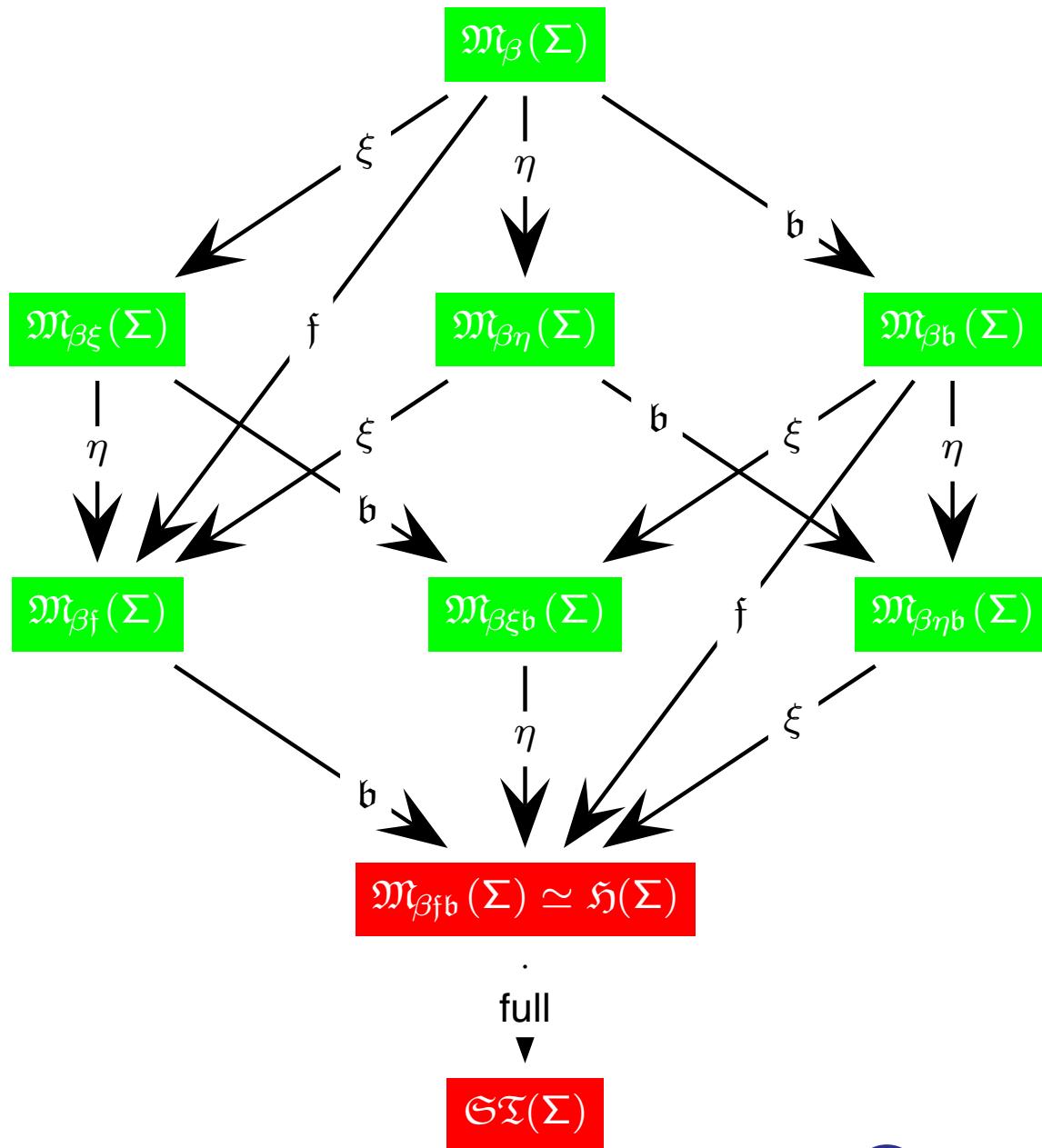
# HOL-Problems: Set Comprehension



## Set comprehension

- $\exists N_{oo} \forall P_o. NP \Leftrightarrow \neg P$ 
  - ▶ if  $\neg \in \Sigma$  or  $\{\perp, \supset\} \subseteq \Sigma$  or  $\{\perp, \Leftrightarrow\} \subseteq \Sigma$
  - ▶ e.g.:  $N_{oo} \leftarrow \lambda X_o. \neg X$
  - ▶ e.g.:  $N_{oo} \leftarrow \lambda X_o. X \supset \perp$

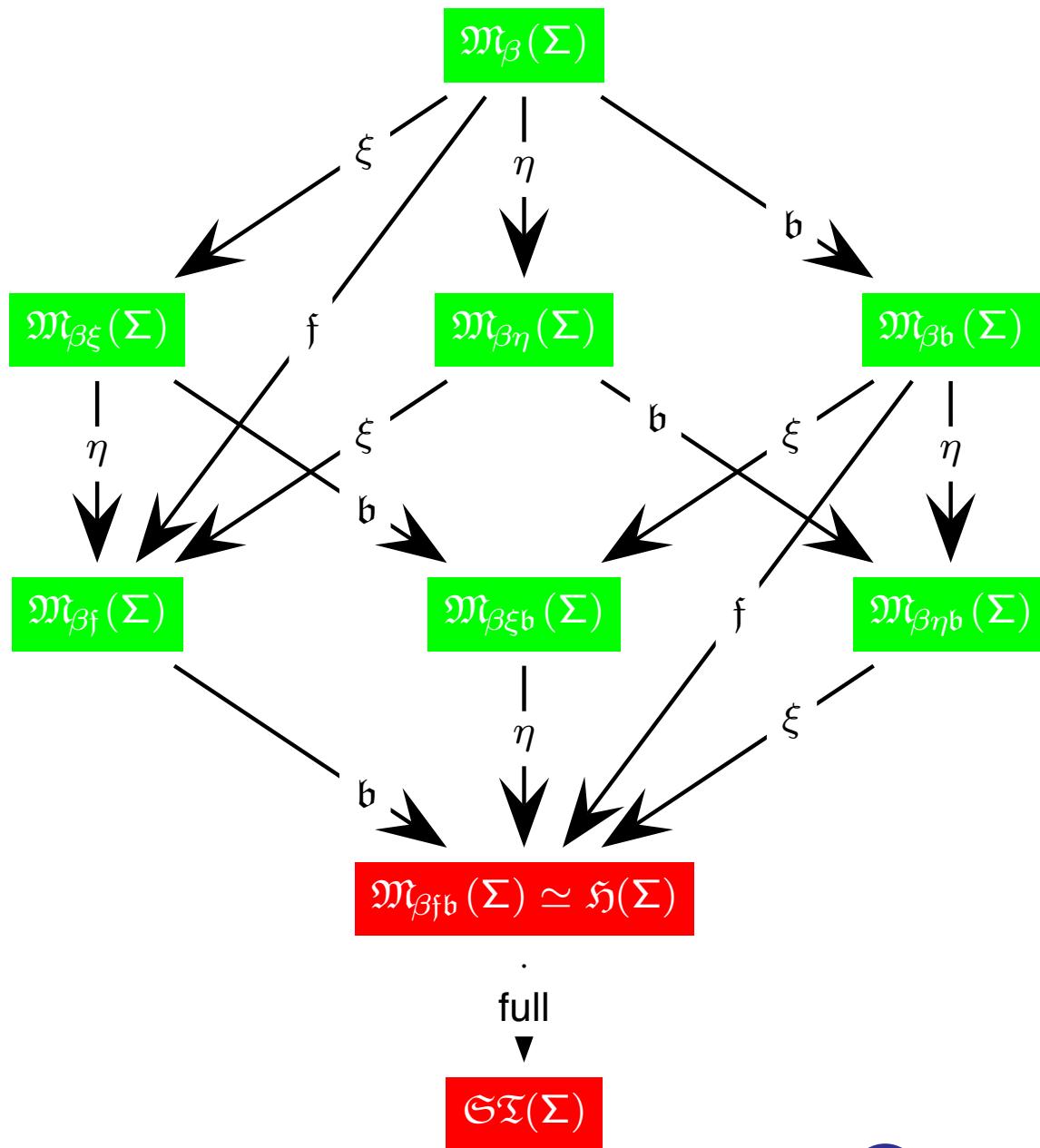
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## Other examples from [Brown-PhD-04]

- Surjective Cantor Theorem
- Injective Cantor Theorem



## Semantics: Examples of $\Sigma$ -Models

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We now sketch the construction of models in the model classes  $\mathfrak{M}_*(\Sigma)$  to demonstrate concretely how properties for Boolean, strong and weak functional extensionality can fail.

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We now sketch the construction of models in the model classes  $\mathfrak{M}_*(\Sigma)$  to demonstrate concretely how properties for Boolean, strong and weak functional extensionality can fail.

We need this to show that the inclusions of the model classes in our landscape are proper, and we indeed need all of them.

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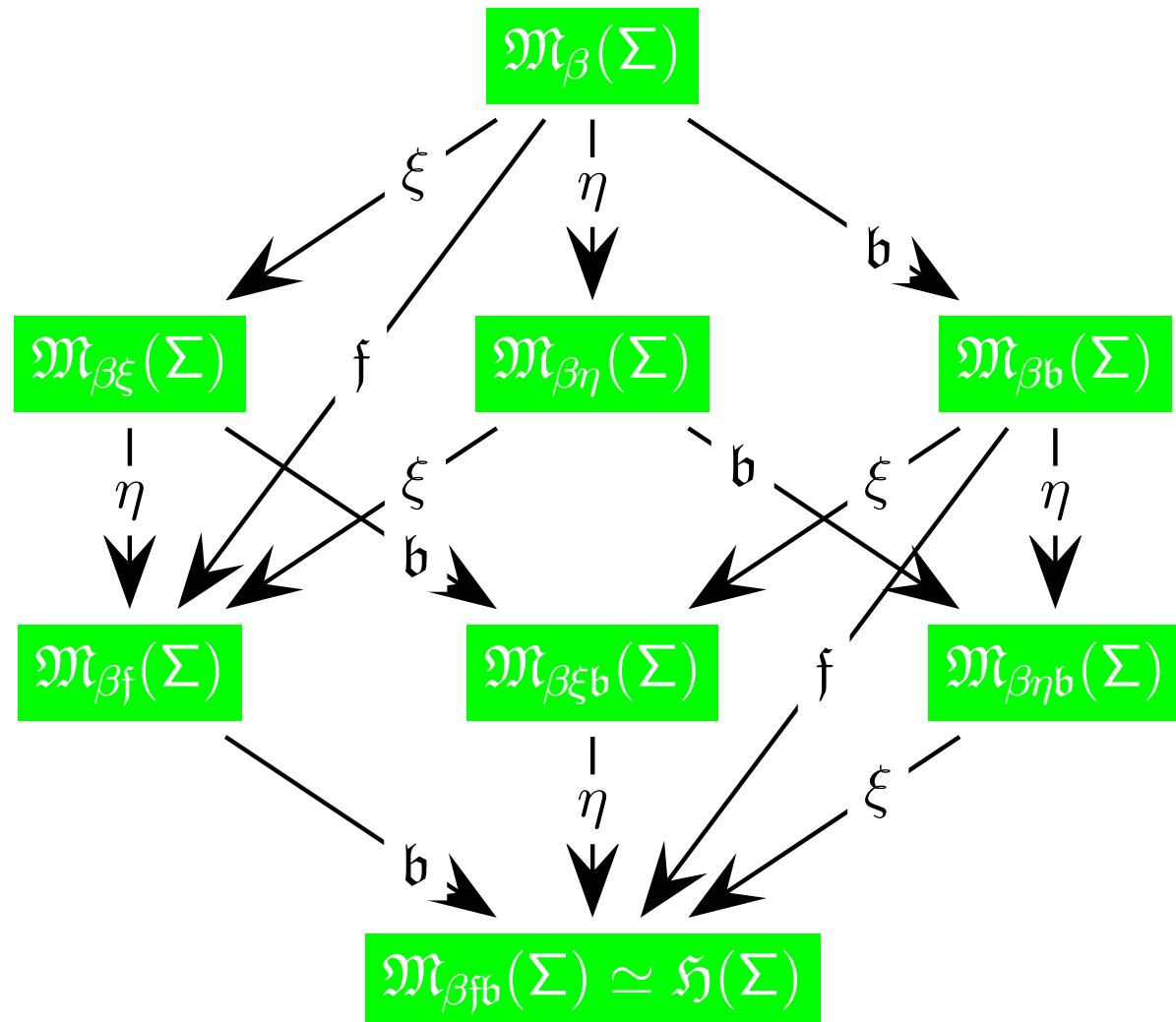
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- So,  $\mathcal{M}^{\beta\text{fb}} \in \mathfrak{ST}(\Sigma) \subseteq \mathfrak{H}(\Sigma) \subseteq \mathfrak{M}_{\beta\text{fb}}(\Sigma) \subseteq \dots$

# Ex.: Singleton Model



full

$\mathfrak{ST}(\Sigma)$



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# Ex.: Model without Boolean Extensionality

- Assume  $\Sigma$  contains only the connectives  $\neg, \vee, \Pi^\alpha$ ; other connectives defined as usual, e.g.,  $\forall X, Y. X \wedge Y \Leftrightarrow \neg(\neg X \vee \neg Y)$ .

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$\mathcal{E}(\neg)$	a	b	c
a	c	c	a
b	a	c	c
c	a	a	c

$\mathcal{E}(\vee)$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	c

$$\mathcal{E}(\Pi^\alpha)@f = \begin{cases} a, & \text{if } f@g \in \{a, b\} \text{ for all } g \in \mathcal{D}_\alpha \\ c, & \text{if } f@g = c \text{ for some } g \in \mathcal{D}_\alpha \end{cases}$$

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- We can choose  $\mathcal{E}(w)$  to be arbitrary for parameters  $w \in \Sigma$ .

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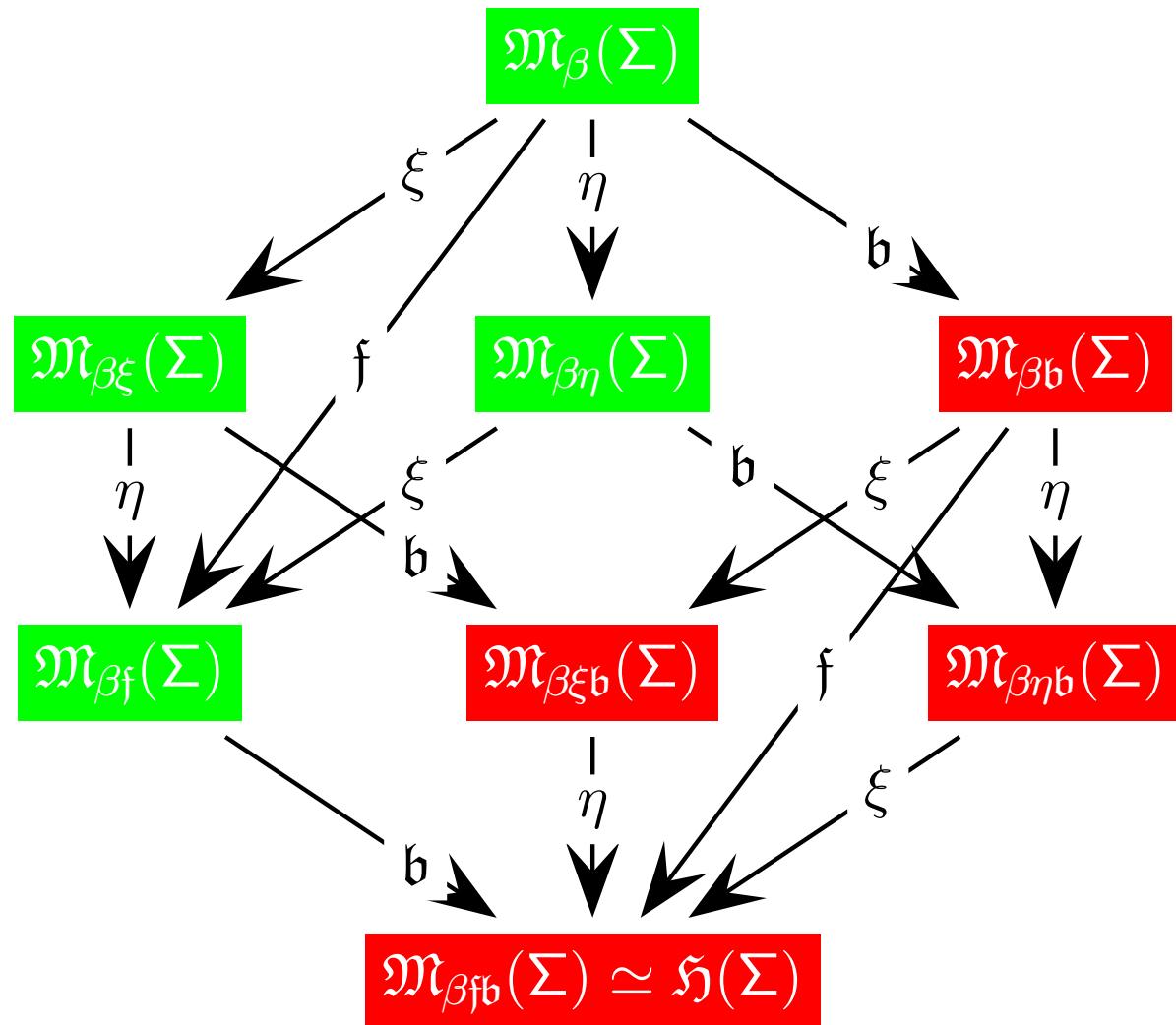
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- Since this frame is full, we know property  $q$  holds.
- Clearly property  $b$  fails.
- So,  $\mathcal{M}^{\beta f} \in \mathfrak{M}_{\beta f}(\Sigma) \setminus \mathfrak{M}_{\beta fb}(\Sigma)$ .

# Ex.: Model without Boolean Extensionality



full

$\mathfrak{ST}(\Sigma)$



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# Ex.: Model without Boolean Extensionality

In the previous model one can easily verify, if  $d := \mathcal{E}_\varphi(\mathbf{D}_o)$  and  $e := \mathcal{E}_\varphi(\mathbf{E}_o)$ , then the values  $\mathcal{E}_\varphi(\mathbf{D} \wedge \mathbf{E})$ ,  $\mathcal{E}_\varphi(\mathbf{D} \Rightarrow \mathbf{E})$ , and  $\mathcal{E}_\varphi(\mathbf{D} \Leftrightarrow \mathbf{E})$  are given by the following tables:

$\mathcal{E}(\mathbf{D} \wedge \mathbf{E})$			$\mathcal{E}(\mathbf{D} \Rightarrow \mathbf{E})$			$\mathcal{E}(\mathbf{D} \Leftrightarrow \mathbf{E})$					
e:			e:			e:					
	a	b	c		a	b	c		a	b	c
d: a	a	a	c	d: a	a	a	c	d: a	a	a	c
b	a	a	c	b	a	a	c	b	a	a	c
c	c	c	c	c	a	a	a	c	c	c	a

Now we show that one can properly model the woodchuck/groundhog example.

# Ex.: Groundhogs and Woodchucks

- Let  $\mathcal{M}^{\beta_f}$  be given as above and suppose  $\text{woodchuck}_{\iota \rightarrow o}$ ,  $\text{groundhog}_{\iota \rightarrow o}$ ,  $\text{john}_\iota$ , and  $\text{phil}_\iota$  are in the signature  $\Sigma$ .

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- Let  $\mathcal{E}(\text{groundhog})$  be the function  $g \in \mathcal{D}_{\iota \rightarrow o}$  with  $g(0) = a$  and  $g(1) = c$ .

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- Let  $\mathcal{E}(\text{phil}) := 0$  and  $\mathcal{E}(\text{john}) := 1$ .
- Let  $\mathcal{E}(\text{woodchuck})$  be the function  $w \in \mathcal{D}_{\iota \rightarrow o}$  with  $w(0) = b$  and  $w(1) = c$ .
- Let  $\mathcal{E}(\text{groundhog})$  be the function  $g \in \mathcal{D}_{\iota \rightarrow o}$  with  $g(0) = a$  and  $g(1) = c$ .
- One can show that the sentence  $\forall X_\iota. (\text{woodchuck } X) \Leftrightarrow (\text{groundhog } X)$  is valid.

# Ex.: Groundhogs and Woodchucks

- Let  $\mathcal{M}^{\beta_f}$  be given as above and suppose  $\text{woodchuck}_{\iota \rightarrow o}$ ,  $\text{groundhog}_{\iota \rightarrow o}$ ,  $\text{john}_\iota$ , and  $\text{phil}_\iota$  are in the signature  $\Sigma$ .
- Let  $\mathcal{E}(\text{phil}) := 0$  and  $\mathcal{E}(\text{john}) := 1$ .
- Let  $\mathcal{E}(\text{woodchuck})$  be the function  $w \in \mathcal{D}_{\iota \rightarrow o}$  with  $w(0) = b$  and  $w(1) = c$ .
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- One can show that the sentence  
 $\forall X_\iota. (\text{woodchuck } X) \Leftrightarrow (\text{groundhog } X)$  is valid.
- Also,  $\mathcal{E}(\text{woodchuck } \text{phil}) = b$  and  $\mathcal{E}(\text{groundhog } \text{phil}) = a$ , so the propositions  $(\text{woodchuck } \text{phil})$  and  $(\text{groundhog } \text{phil})$  are valid.

# Ex.: Groundhogs and Woodchucks

- Suppose  $\text{believe}_{\iota \rightarrow o \rightarrow o} \in \Sigma$  and  $\mathcal{E}(\text{believe})$  is the (Curried) function  $\text{bel} \in \mathcal{D}_{\iota \rightarrow o \rightarrow o}$  such that  $\text{bel}(1)(b) = b$  and  $\text{bel}(1)(a) = \text{bel}(1)(c) = \text{bel}(0)(a) = \text{bel}(0)(b) = \text{bel}(0)(c) = c$ .

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- Intuitively, John believes propositions with value **b**, but not those with value **a** or **c**.

# Ex.: Groundhogs and Woodchucks

- Suppose  $\text{believe}_{\iota \rightarrow o \rightarrow o} \in \Sigma$  and  $\mathcal{E}(\text{believe})$  is the (Curried) function  $\text{bel} \in \mathcal{D}_{\iota \rightarrow o \rightarrow o}$  such that  $\text{bel}(1)(b) = b$  and  $\text{bel}(1)(a) = \text{bel}(1)(c) = \text{bel}(0)(a) = \text{bel}(0)(b) = \text{bel}(0)(c) = c$ .
- Intuitively, John believes propositions with value **b**, but not those with value **a** or **c**.
- So,  $\text{believes john}(\text{woodchuck phil})$  is valid, while  $\text{believes john}(\text{groundhog phil})$  is not.

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These semantic constructions are similar to those in multi-valued logics. In contrast to these logics where the logical connectives are adapted to talk about multiple truth values, in our setting we are mainly interested in multiple truth values as diverse  $v$ -pre-images of  $\text{T}$  and  $\text{F}$ .



## Semantics: Examples of $\Sigma$ -Models (Contd.)

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- We define application by

$$(i, f)@a := f(a)$$

whenever  $(i, f) \in \mathcal{D}_{\beta\alpha}$  and  $a \in \mathcal{D}_\alpha$

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( $k^T$  is the constant  $T$  function)  
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  - ▶  $\mathcal{E}(\Pi^\alpha) := (0, \pi^\alpha)$  where for each  $(i, f) \in \mathcal{D}_{o\alpha}$ ,  $\pi^\alpha((i, f)) := T$  if  
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  - ▶  $\mathcal{E}(w) \in \mathcal{D}_\alpha$  arbitrary for parameters  $w \in \Sigma_\alpha$ .

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- Hence,  $\mathcal{M}^{\mathcal{B}} := (\mathcal{D}, @, \mathcal{E}, v)$  is a  $\Sigma$ -model

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- Hence,  $\mathcal{M}^B := (\mathcal{D}, @, \mathcal{E}, v)$  is a  $\Sigma$ -model with property  $q$

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$$(i, u)@* = *$$

although  $(0, u) \neq (1, u)$

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for every  $a \in \mathcal{D}_\alpha$ , then

$$\mathcal{E}_\varphi(\lambda X_\alpha.M) = (0, f) = \mathcal{E}_\varphi(\lambda X.N)$$

where  $f(a) = \mathcal{E}_{\varphi,[a/X]}(\mathbf{M}) = \mathcal{E}_{\varphi,[a/X]}(\mathbf{N})$  for every  $a \in \mathcal{D}_\alpha$ .

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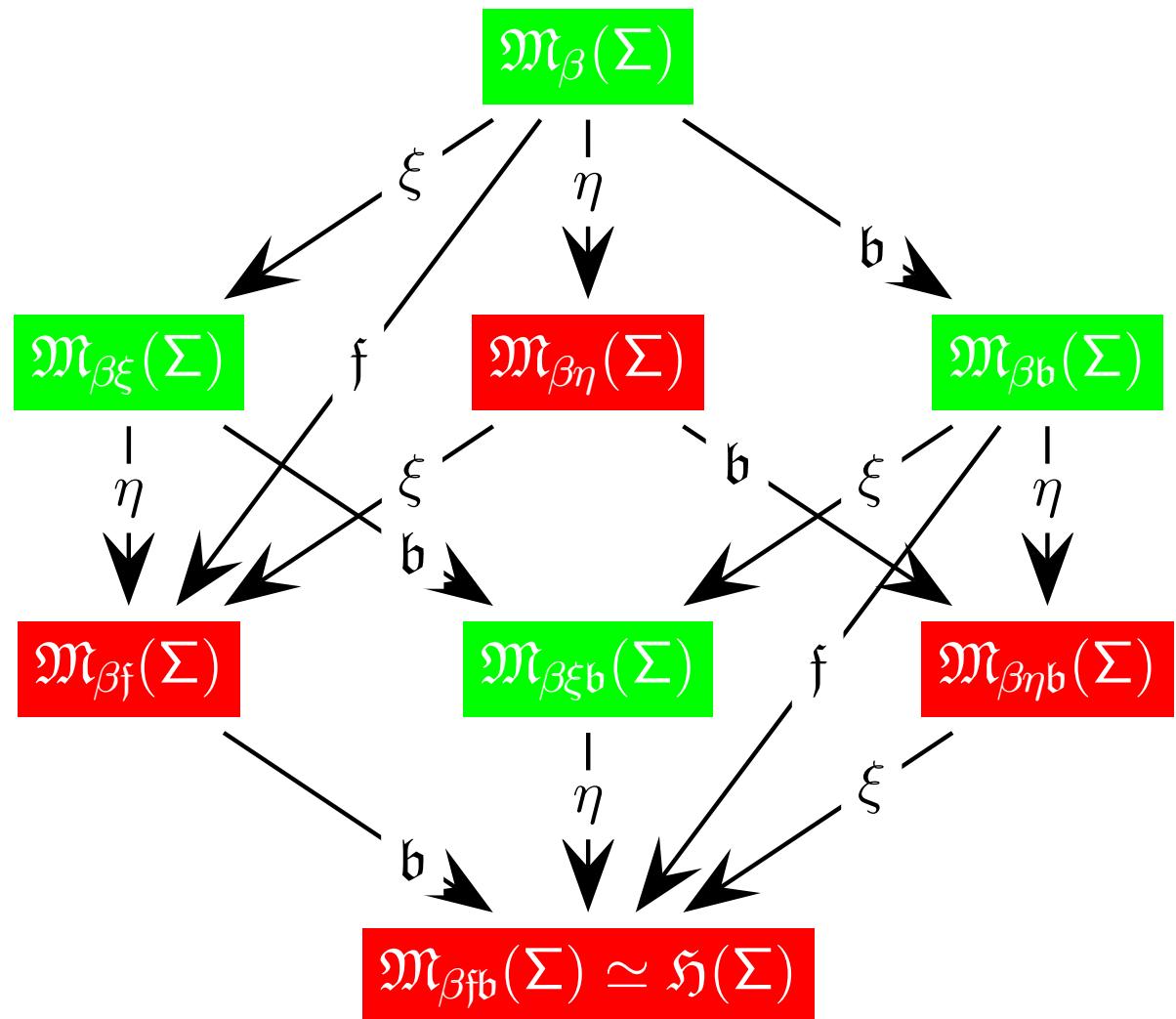
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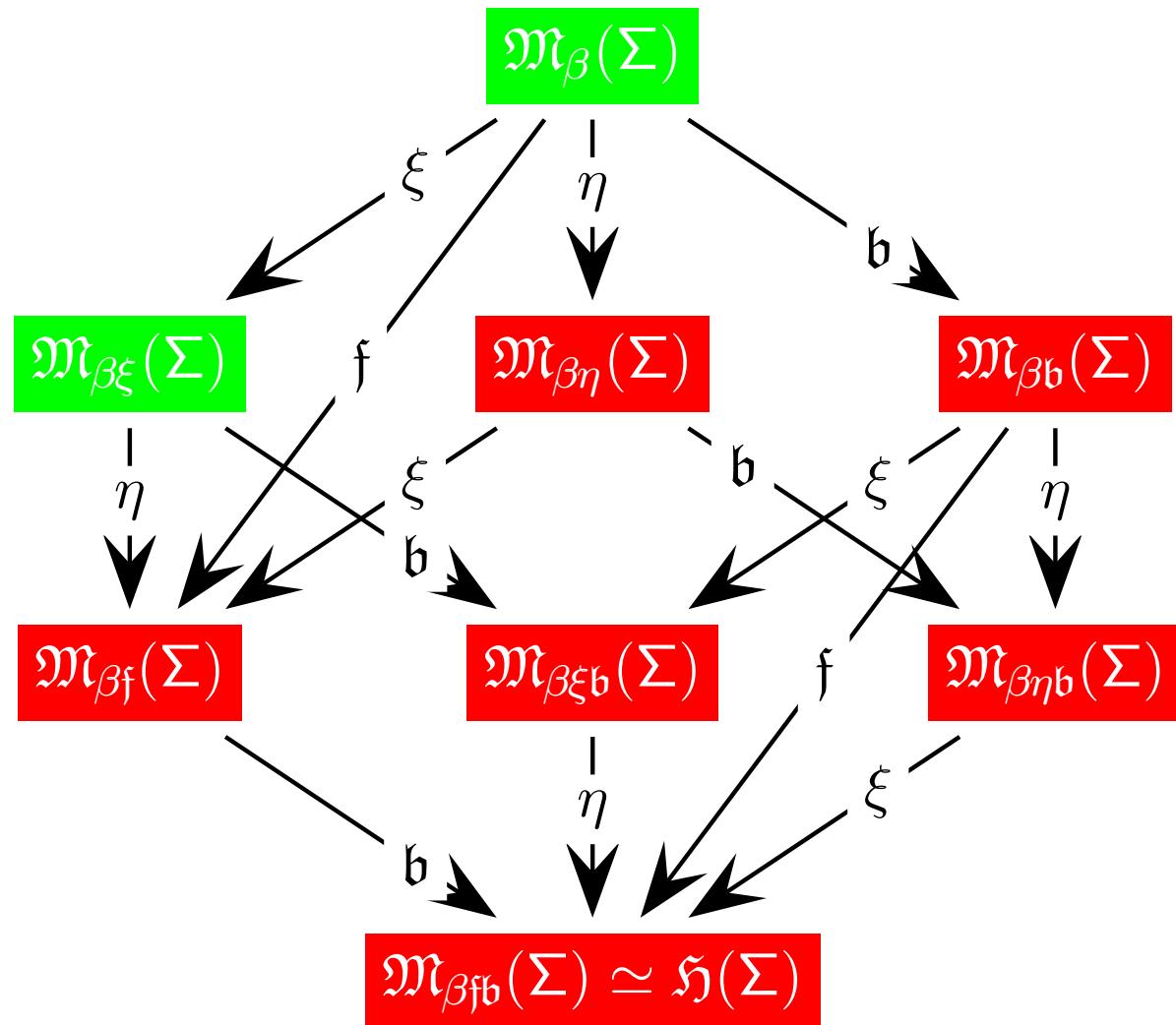
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- $\mathcal{E}$  and  $\mathcal{E}'$  agree on all constants, they are different though:

$$\mathcal{E}(\lambda X_{\nu}.X) = (0, \text{id}) \neq (1, \text{id}) = \mathcal{E}'(\lambda X_{\nu}.X)$$

where  $\text{id} : \mathcal{D}_{\nu} \longrightarrow \mathcal{D}_{\nu}$  is the identity function

# Ex.: Models without $\eta$ and $f$

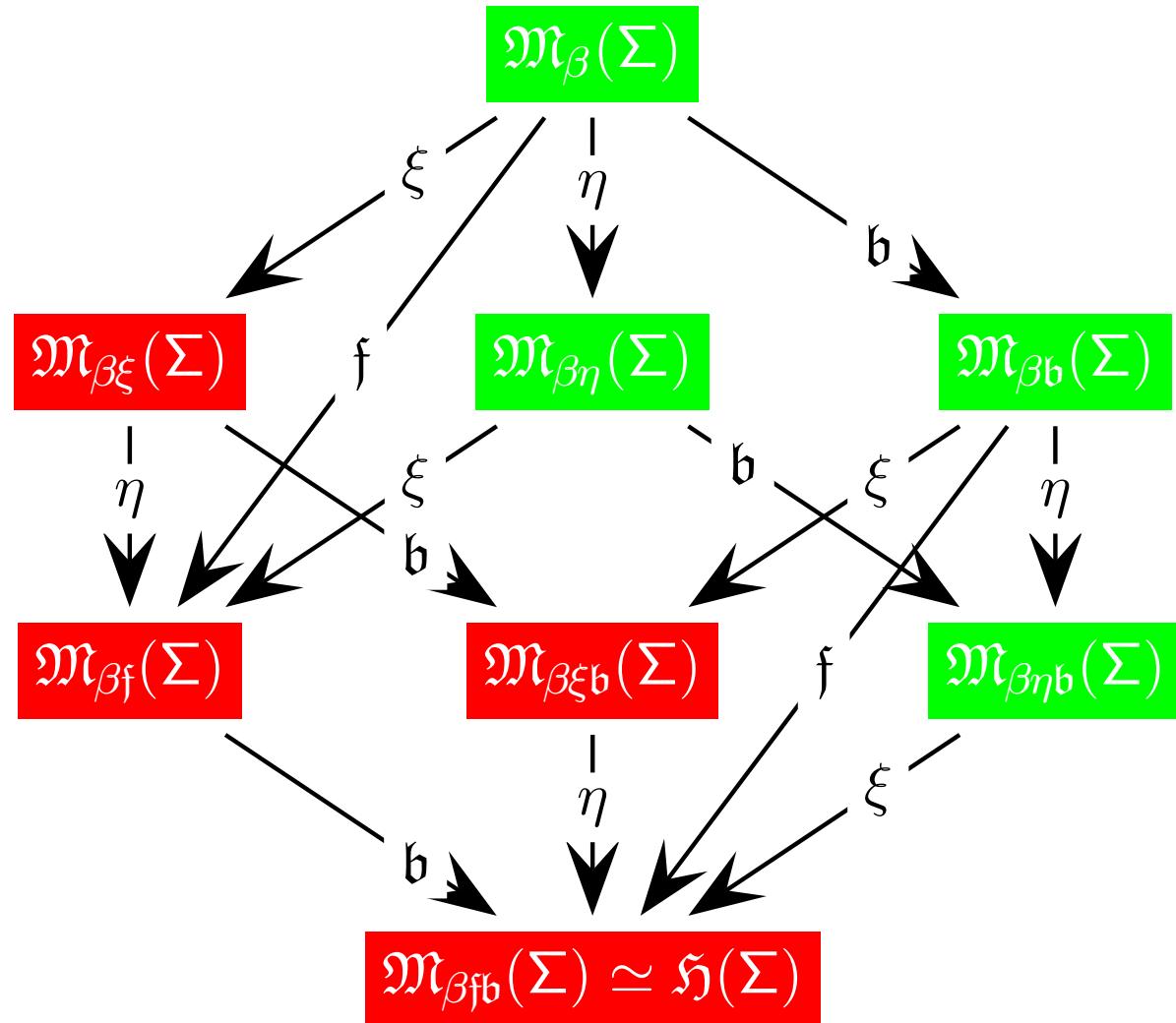
- Let  $\mathcal{M}^{\mathcal{B}}$  be the  $\Sigma$ -model  $(\mathcal{D}, @, \mathcal{E}, v)$  as constructed before
- Define an alternative evaluation function  $\mathcal{E}'$  by induction:
  - ▶ For all  $w \in \Sigma$ , let  $\mathcal{E}'(w) := \mathcal{E}(w)$
  - ▶ For variables we define  $\mathcal{E}'_{\varphi}(X) := \varphi(X)$
  - ▶ We must define  $\mathcal{E}'_{\varphi}(FA) := \mathcal{E}'_{\varphi}(F) @ \mathcal{E}'_{\varphi}(A)$
  - ▶ We choose  $\mathcal{E}'_{\varphi}(\lambda X_{\alpha}.B_{\beta}) := (1, f)$  where  $f: \mathcal{D}_{\alpha} \longrightarrow \mathcal{D}_{\beta}$  is the function such that  $f(a) = \mathcal{E}_{\varphi, [a/X]}(B)$  for all  $a \in \mathcal{D}_{\alpha}$
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- Thus, in non-functional models evaluation functions are not uniquely determined by their values on constants

# Ex.: Models without $\xi$



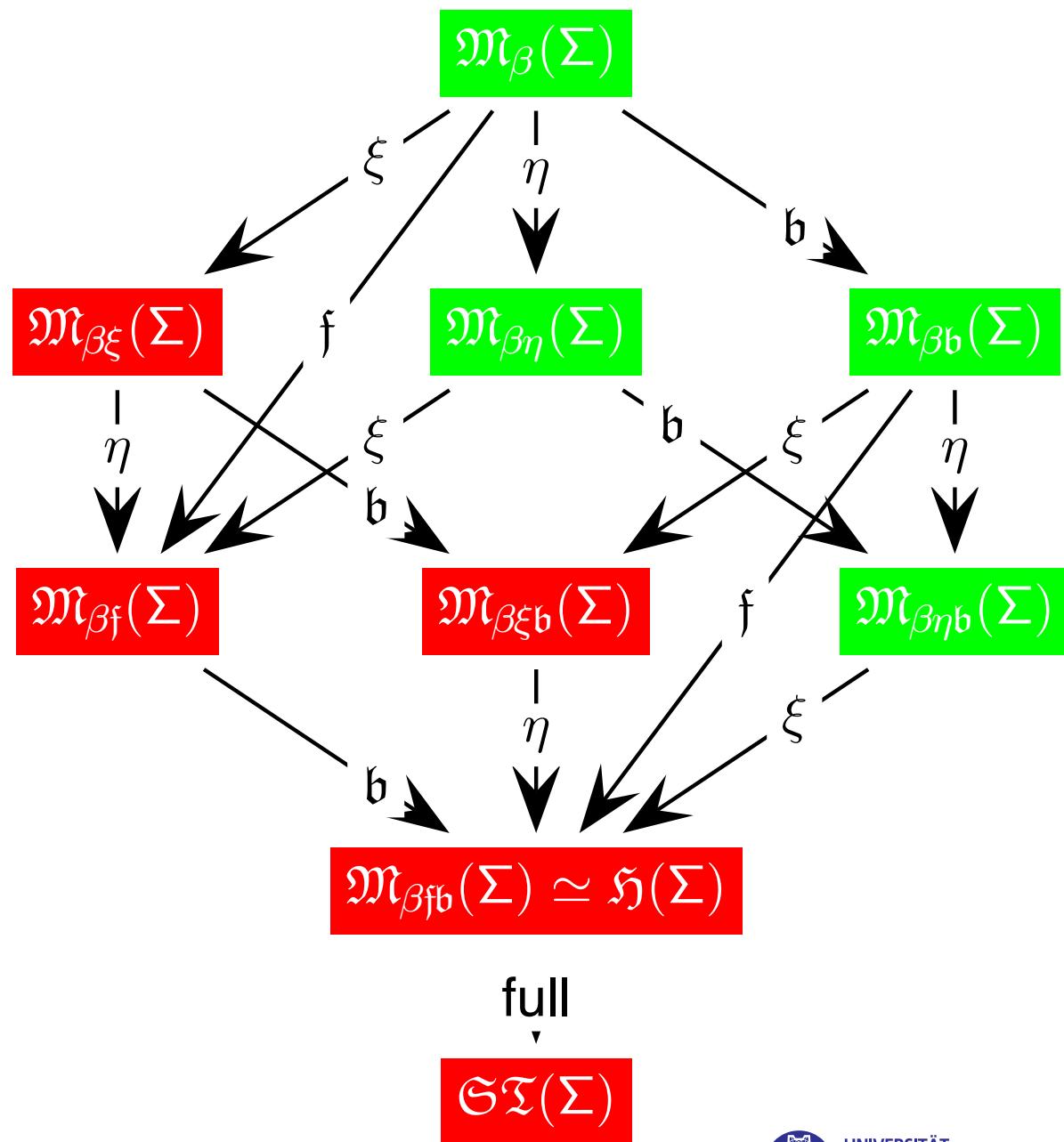
full

$\mathfrak{ST}(\Sigma)$



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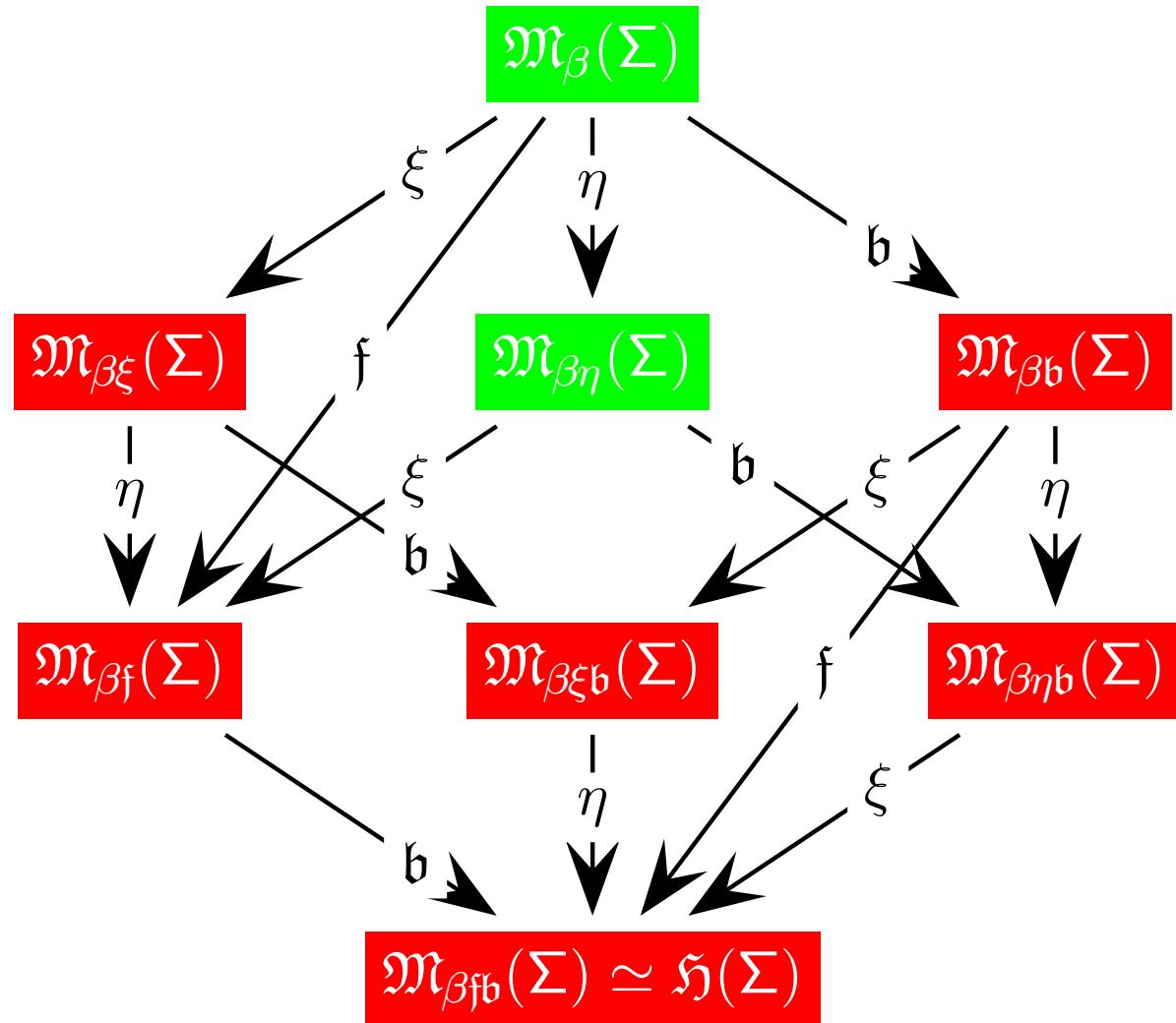
# Ex.: Models without $\xi$



Not here!

See [JSL-04]

# Ex.: Models without $\xi$



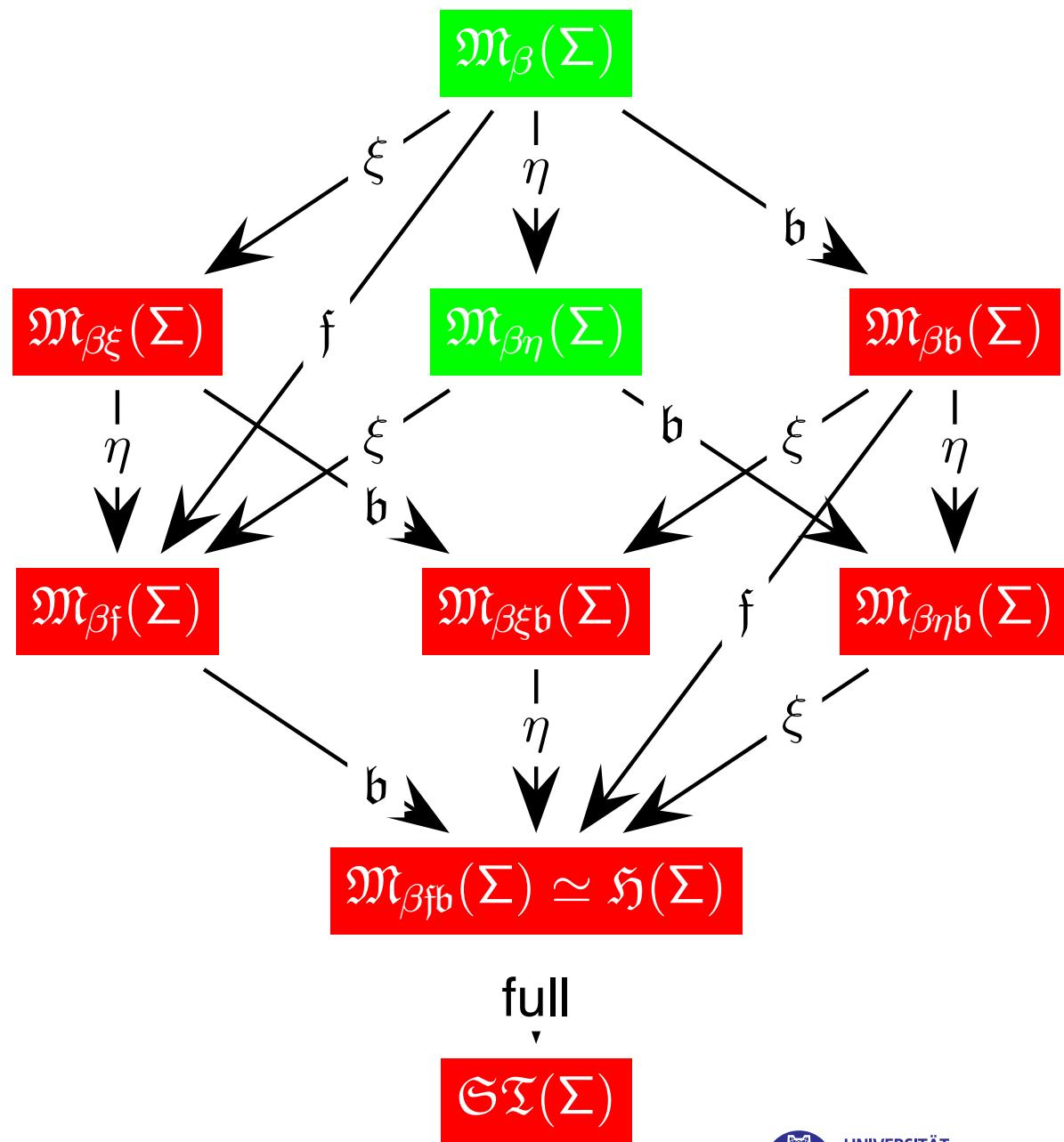
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# Ex.: Models without $\xi$



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# Calculi: First-Order Natural Deduction and Sequent Calculus

# From Natural Deduction to Sequent Calculus and Back

## From Natural Deduction to Sequent Calculus and Back

Remark: We first illustrate the correspondence between natural deduction and sequent calculus in first-order logic. Later we will present natural deduction calculi for HOL. More precisely we will present one sound and complete calculus for each class in our landscape of semantics as presented before.

# Reading

---

- F. Pfenning: Automated Theorem Proving, Course at Carnegie Mellon University. Draft. 1999.
- A.S. Troelstra and H. Schwichtenberg: Basic Proof Theory. Cambridge. 2nd Edition 2000.
- John Byrnes: Proof Search and Normal Forms in Natural Deduction. PhD Thesis. Carnegie Mellon University. 1999.
- ... many more books on Proof Theory ...

# Natural Deduction: Motivation

- Frege, Russel, Hilbert: Predicate calculus and type theory as formal basis for mathematics

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- Frege, Russel, Hilbert: Predicate calculus and type theory as formal basis for mathematics
- Gentzen: Natural deduction (ND) as intuitive formulation of predicate calculus; introduction and elimination rules for each logical connective

*The formalization of logical deduction, especially as it has been developed by Frege, Russel, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. . . . In contrast I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a calculus of natural deduction (NJ for intuitionist, NK for classical predicate logic).*

[Gentzen: Investigations into logical deduction]

# Sequent Calculus: Motivation

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# Sequent Calculus: Motivation

- Gentzen had a pure technical motivation for sequent calculus:
  - ▶ same theorems as natural deduction
  - ▶ prove of the Hauptsatz (all sequent proofs can be found with a simple strategy)
  - ▶ corollary: consistency of formal system(s)

*The Hauptsatz says that every purely logical proof can be reduced to a definite, though not unique, normal form. Perhaps we may express the essential properties of such a normal proof by saying: it is not round-about. . . .*

*In order to be able to prove the Hauptsatz in a convenient form, I had to provide a logical calculus especially for the purpose. For this the natural calculus proved unsuitable.*

[Gentzen: Investigations into logical deduction]

# Sequent Calculus: Introduction

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  - ▶ Forward: classical resolution, inverse method

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- Don't be afraid of the many variants of sequent calculi.

# Sequent Calculus: Introduction

- Sequent calculus exposes many details of fine structure of proofs in a very clear manner. Therefore it is well suited to serve as a basic representation formalism for many automation oriented search procedures
  - ▶ Backward: tableaux, connection methods, matrix methods, some forms of resolution
  - ▶ Forward: classical resolution, inverse method
- Don't be afraid of the many variants of sequent calculi.
- Choose the one that is most suited for you.

# Natural Deduction

---

Natural deduction rules operate on proof trees.

Example:

# Natural Deduction

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► Conjunction:

$$\frac{D_1 \quad D_2}{A \wedge B} \wedge I \quad \frac{D_1}{A \wedge B} \wedge E_l \quad \frac{D_1}{B} \wedge E_r$$

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The presentation on the next slides treats the proof tree aspects implicit.

Example:

# Natural Deduction

Natural deduction rules operate on proof trees.

Example:

- ▶ Conjunction:

$$\frac{\begin{array}{c} D_1 \\ \hline A \end{array} \quad \begin{array}{c} D_2 \\ \hline B \end{array}}{A \wedge B} \wedge I \quad \frac{D_1}{\begin{array}{c} A \wedge B \\ \hline A \end{array}} \wedge E_l \quad \frac{D_1}{\begin{array}{c} A \wedge B \\ \hline B \end{array}} \wedge E_r$$

The presentation on the next slides treats the proof tree aspects implicit.

Example:

- ▶ Conjunction:

$$\frac{\begin{array}{c} A \\ \hline B \end{array}}{A \wedge B} \wedge I \quad \frac{A \wedge B}{\begin{array}{c} A \\ \hline \end{array}} \wedge E_l \quad \frac{A \wedge B}{\begin{array}{c} B \\ \hline \end{array}} \wedge E_r$$

# Natural Deduction Rules Ia

- Conjunction:

$$\frac{A \quad B}{A \wedge B} \wedge I \quad \frac{A \wedge B}{A} \wedge E_l \quad \frac{A \wedge B}{B} \wedge E_r$$

# Natural Deduction Rules Ia

- Conjunction:

$$\frac{A \quad B}{A \wedge B} \wedge I \quad \frac{A \wedge B}{A} \wedge E_l \quad \frac{A \wedge B}{B} \wedge E_r$$

$[A]_1 \quad [B]_2$

- Disjunction:  $\frac{A}{A \vee B} \vee I_l \quad \frac{B}{A \vee B} \vee I_r \quad \frac{A \vee B \quad C}{C} \vee E^{1,2}$

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- Implication:

$$\frac{[A]_1 \quad \dots \quad B}{A \Rightarrow B} \Rightarrow I^1 \quad \frac{A \Rightarrow B \quad A}{B} \Rightarrow E$$

# Natural Deduction Rules Ia

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- Implication:

$$\frac{[A]_1 \quad \vdots \quad B}{A \Rightarrow B} \Rightarrow I^1 \quad \frac{A \Rightarrow B \quad A}{B} \Rightarrow E$$

- Truth and Falsehood:

$$\top \top I \quad \frac{\perp}{C} \perp E$$

# Natural Deduction Rules IIa

- Negation:

$$\frac{[\mathbf{A}]_1 \quad \vdots \quad \perp}{\neg \mathbf{A}} \neg I^1 \quad \frac{\neg \mathbf{A} \quad \mathbf{A}}{\perp} \neg E$$

# Natural Deduction Rules IIa

- Negation:
- Universal Quantif.:

$$\begin{array}{c}
 [A]_1 \\
 \vdots \\
 \vdots \\
 \frac{\perp}{\neg A} \neg I^1 \quad \frac{\neg A \quad A}{\perp} \neg E \\
 \\ 
 \frac{A[x/P^*]}{\forall x.A} \forall I \quad \frac{\forall x.A}{A[x/T]} \forall E \\
 \\ 
 (*: \text{parameter } P \text{ must be new in context})
 \end{array}$$

# Natural Deduction Rules Ila

- Negation:
- Universal Quantif.:
- Existential Quantif.:

$$\frac{[\mathbf{A}]_1 \quad \vdots \quad \perp}{\neg \mathbf{A}} \neg I^1 \quad \frac{\neg \mathbf{A} \quad \mathbf{A}}{\perp} \neg E$$

$$\frac{\mathbf{A}[x/P^*]}{\forall x. \mathbf{A}} \forall I \quad \frac{\forall x. \mathbf{A}}{\mathbf{A}[x/T]} \forall E$$

(\*: parameter P must be new in context)

$$\frac{[\mathbf{A}[x/P^*]] \quad \vdots \quad \mathbf{A}[x/T]}{\exists x. \mathbf{A}} \exists I \quad \frac{\exists x. \mathbf{A} \quad C}{C} \exists E$$

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# Natural Deduction Rules IIIa

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- For classical logic choose one of the following

- ▶ Excluded Middle

$$\frac{}{A \vee \neg A} XM$$

- ▶ Double Negation

$$\frac{\neg\neg A}{A} \neg\neg C$$

- ▶ Proof by Contradiction

$$\frac{\begin{array}{c} [\neg A] \\ \vdots \\ \bot \end{array}}{A} \perp_C$$

# Natural Deduction

---

- Structural properties

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  - ▶ Exchange hypotheses order is irrelevant

# Natural Deduction

---

## ■ Structural properties

- ▶ Exchange
- ▶ Weakening

hypotheses order is irrelevant  
hypothesis need not be used

# Natural Deduction

---

- Structural properties
  - ▶ Exchange hypotheses order is irrelevant
  - ▶ Weakening hypothesis need not be used
  - ▶ Contraction hypotheses can be used more than once

# Natural Deduction Proofs

$$\frac{\frac{[\mathbf{A}]_1 \quad [\mathbf{A}]_2}{\mathbf{A} \wedge \mathbf{A}} \wedge I}{\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A})} \Rightarrow I^2 \\ \frac{}{\mathbf{A} \Rightarrow (\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A}))} \Rightarrow I^1$$

or

$$\frac{\frac{[\mathbf{A}]_1 \quad [\mathbf{A}]_1}{\mathbf{A} \wedge \mathbf{A}} \wedge I}{\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A})} \Rightarrow I^2 \\ \frac{}{\mathbf{A} \Rightarrow (\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A}))} \Rightarrow I^1$$

$$\frac{\frac{[\mathbf{A} \wedge \mathbf{B}]_1}{\mathbf{B}} \wedge E_r \quad \frac{[\mathbf{A} \wedge \mathbf{B}]_1}{\mathbf{A}} \wedge E_l}{\frac{\mathbf{C} \vee \mathbf{A}}{\mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A})} \vee I_r} \wedge I \\ \frac{}{(\mathbf{A} \wedge \mathbf{B}) \Rightarrow (\mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A}))} \Rightarrow I^1$$

# Natural Deduction with Contexts

- FO-Soundness of ND: Let  $F$  be a first-order formula such that there is a ND proof of  $F$ . Then  $F$  is valid.  $(\vdash F \Rightarrow \models F)$   
(Proof: Standard textbooks)

# Natural Deduction with Contexts

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(Proof: Standard textbooks)
- FO-Completeness of ND: Let  $F$  be a valid first-order formula then there is a ND proof of  $F$   $(\models F \Rightarrow \vdash F)$ .  
(Proof: Standard textbooks)

# Natural Deduction with Contexts

Idea: Localizing hypotheses; explicit representation of the available assumptions for each formula occurrence in a ND proof:

$$\Gamma \vdash A$$

$\Gamma$  is a multiset of the (uncanceled) assumptions on which formula  $A$  depends.  $\Gamma$  is called context.

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Example proof in context notation:

$$\begin{array}{c}
 \dfrac{\overline{A_1 \vdash A} \quad \overline{A_2 \vdash A}}{A_1, A_2 \vdash A \wedge A} \wedge I \\
 \dfrac{}{A_1 \vdash A \Rightarrow (A \wedge A)} \Rightarrow I_2 \\
 \hline
 \vdash A \Rightarrow (A \Rightarrow (A \wedge A)) \Rightarrow I_1
 \end{array}$$

# Natural Deduction with Contexts

Another Idea: Consider sets of assumptions instead of multisets.

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Example proof:

$$\frac{\frac{\frac{A \vdash A \quad A \vdash A}{A \vdash A \wedge A} \wedge I}{A \vdash A \Rightarrow (A \wedge A)} \Rightarrow I}{\vdash A \Rightarrow (A \Rightarrow (A \wedge A))} \Rightarrow I$$

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# Natural Deduction Rules Ib

- Hypotheses:

$$\frac{}{\Gamma, A, \Delta \vdash A}$$

# Natural Deduction Rules Ib

- Hypotheses:
- Conjunction:

$$\frac{}{\Gamma, A, \Delta \vdash A}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge I \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge E_l \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge E_r$$

# Natural Deduction Rules Ib

- Hypotheses:
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- Disjunction:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee I_l \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee I_r$$

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} \vee E_r$$

# Natural Deduction Rules Ib

- Hypotheses:
- Conjunction:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge I \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge E_l \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge E_r$$

- Disjunction:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee I_l \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee I_r$$

- Implication:

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} \vee E_r$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow I \quad \frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow E$$

# Natural Deduction Rules IIb

- Truth and Falsehood:

$$\frac{}{\Gamma \vdash T} T\text{I} \quad \frac{\Gamma \vdash \perp}{\Gamma \vdash C} \perp\text{E}$$

# Natural Deduction Rules IIb

- Truth and Falsehood:

$$\frac{}{\Gamma \vdash \top} \top I \quad \frac{\Gamma \vdash \perp}{\Gamma \vdash C} \perp E$$

- Negation:

$$\frac{\Gamma, A \vdash \perp}{\Gamma \vdash \neg A} \neg I \quad \frac{\Gamma \vdash \neg A \quad \Gamma \vdash A}{\Gamma \vdash \perp} \neg E$$

# Natural Deduction Rules IIb

- Truth and Falsehood:
- Negation:
- Universal Quantif.:

$$\frac{}{\Gamma \vdash \top} \top I \quad \frac{\Gamma \vdash \perp}{\Gamma \vdash C} \perp E$$

$$\frac{\Gamma, A \vdash \perp}{\Gamma \vdash \neg A} \neg I \quad \frac{\Gamma \vdash \neg A \quad \Gamma \vdash A}{\Gamma \vdash \perp} \neg E$$

$$\frac{\Gamma \vdash A[x/P^*]}{\Gamma \vdash \forall x.A} \forall I \quad \frac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[x/T]} \forall E$$

(\*: parameter P must be new in context)

# Natural Deduction Rules IIb

- Truth and Falsehood:

$$\frac{}{\Gamma \vdash \top} \top I \quad \frac{\Gamma \vdash \perp}{\Gamma \vdash C} \perp E$$

- Negation:

$$\frac{\Gamma, A \vdash \perp}{\Gamma \vdash \neg A} \neg I \quad \frac{\Gamma \vdash \neg A \quad \Gamma \vdash A}{\Gamma \vdash \perp} \neg E$$

- Universal Quantif.:

$$\frac{\Gamma \vdash A[x/P^*]}{\Gamma \vdash \forall x.A} \forall I \quad \frac{\Gamma \vdash \forall x.A}{\Gamma \vdash A[x/T]} \forall E$$

(\*: parameter P must be new in context)

- Existential Quantif.:

$$\frac{\Gamma \vdash A[x/T]}{\Gamma \vdash \exists x.A} \exists I \quad \frac{\Gamma \vdash \exists x.A \quad \Gamma, A[x/P^*] \vdash C}{\Gamma \vdash C} \exists E$$

(\*: parameter P must be new in context)

# Natural Deduction Rules IIIb

For classical logic add:

- Proof by Contradiction:

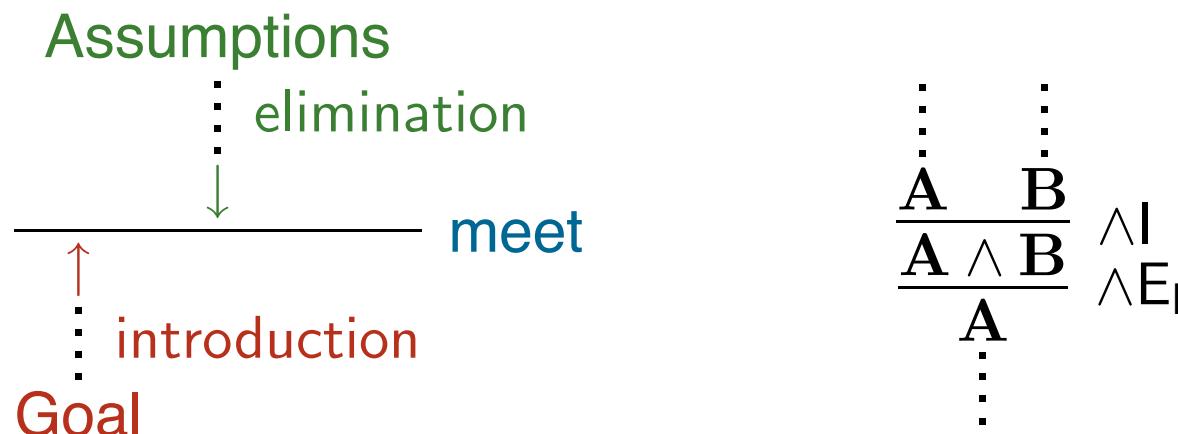
$$\frac{\Gamma, \neg A \vdash \perp}{\Gamma \vdash A} \perp_c$$

# Intercalation

- Idea (Prawitz, Sieg & Scheines, Byrnes & Sieg): Detour free proofs: strictly use introduction rules bottom up (from proposed theorem to hypothesis) and elimination rules top down (from assumptions to proposed theorem). When they meet in the middle we have found a **proof in normal form**.

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# Intercalating Natural Deductions

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## ■ New annotations:

- ▶  $A \uparrow$ :  $A$  is obtained by an introduction derivation
- ▶  $A \downarrow$ :  $A$  is extracted from a hypothesis by an elimination derivation

## ■ Example:

$$\frac{\Gamma, A \vdash_{\text{IC}} B \uparrow}{\Gamma \vdash_{\text{IC}} A \Rightarrow B \uparrow} \Rightarrow I \quad \frac{\Gamma \vdash_{\text{IC}} A \Rightarrow B \downarrow \quad \Gamma \vdash_{\text{IC}} A \uparrow}{\Gamma \vdash_{\text{IC}} B \uparrow} \Rightarrow E$$

# ND Intercalation Rules I

- Hypotheses:

$$\overline{\Gamma, A, \Delta \vdash_{\text{IC}} A \downarrow}$$

# ND Intercalation Rules I

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- Conjunction:

$$\frac{}{\Gamma, A, \Delta \vdash_{\text{IC}} A \downarrow}$$

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$$\frac{\Gamma \vdash_{\text{IC}} A \wedge B \downarrow}{\Gamma \vdash_{\text{IC}} A \downarrow} \wedge E_l$$

$$\frac{\Gamma \vdash_{\text{IC}} A \wedge B \downarrow}{\Gamma \vdash_{\text{IC}} B \downarrow} \wedge E_r$$

# ND Intercalation Rules I

- Hypotheses:
- Conjunction:

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$$\frac{\Gamma \vdash_{\text{IC}} A \wedge B \downarrow}{\Gamma \vdash_{\text{IC}} B \downarrow} \wedge E_r$$

- Disjunction:

$$\frac{\Gamma \vdash_{\text{IC}} A \uparrow}{\Gamma \vdash_{\text{IC}} A \vee B \uparrow} \vee I_l$$

$$\frac{\Gamma \vdash_{\text{IC}} B \uparrow}{\Gamma \vdash_{\text{IC}} A \vee B \uparrow} \vee I_r$$

$$\frac{\Gamma \vdash_{\text{IC}} A \vee B \downarrow \quad \Gamma, A \vdash_{\text{IC}} C \uparrow \quad \Gamma, B \vdash_{\text{IC}} C \uparrow}{\Gamma \vdash_{\text{IC}} C \uparrow} \vee E$$

# ND Intercalation Rules I

- Hypotheses:
- Conjunction:

$$\frac{}{\Gamma, A, \Delta \vdash_{\text{IC}} A \downarrow}$$

$$\frac{\Gamma \vdash_{\text{IC}} A \uparrow \quad \Gamma \vdash_{\text{IC}} B \uparrow}{\Gamma \vdash_{\text{IC}} A \wedge B \uparrow} \wedge I$$

$$\frac{\Gamma \vdash_{\text{IC}} A \wedge B \downarrow}{\Gamma \vdash_{\text{IC}} A \downarrow} \wedge E_l$$

$$\frac{\Gamma \vdash_{\text{IC}} A \wedge B \downarrow}{\Gamma \vdash_{\text{IC}} B \downarrow} \wedge E_r$$

- Disjunction:

$$\frac{\Gamma \vdash_{\text{IC}} A \uparrow}{\Gamma \vdash_{\text{IC}} A \vee B \uparrow} \vee I_l$$

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$$\frac{\Gamma \vdash_{\text{IC}} A \vee B \downarrow \quad \Gamma, A \vdash_{\text{IC}} C \uparrow \quad \Gamma, B \vdash_{\text{IC}} C \uparrow}{\Gamma \vdash_{\text{IC}} C \uparrow} \vee E$$

- Implication:

$$\frac{\Gamma, A \vdash_{\text{IC}} B \uparrow}{\Gamma \vdash_{\text{IC}} A \Rightarrow B \uparrow} \Rightarrow I$$

$$\frac{\Gamma \vdash_{\text{IC}} A \Rightarrow B \downarrow \quad \Gamma \vdash_{\text{IC}} A \uparrow}{\Gamma \vdash_{\text{IC}} B \uparrow} \Rightarrow E$$

# ND Intercalation Rules II

- Truth and Falsehood:

$$\frac{}{\Gamma \vdash_{\text{IC}} \top \uparrow} \top I \quad \frac{\Gamma \vdash_{\text{IC}} \perp \downarrow}{\Gamma \vdash_{\text{IC}} C \uparrow} \perp E$$

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- Negation:

$$\frac{\Gamma, A \vdash_{\text{IC}} \perp \uparrow}{\Gamma \vdash_{\text{IC}} \neg A \uparrow} \neg I \quad \frac{\Gamma \vdash_{\text{IC}} \neg A \downarrow \quad \Gamma \vdash_{\text{IC}} A \uparrow}{\Gamma \vdash_{\text{IC}} \perp \uparrow} \neg E$$

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- Universal Quantif.:

$$\frac{\Gamma \vdash_{\text{IC}} A[x/P^*] \uparrow}{\Gamma \vdash_{\text{IC}} \forall x.A \uparrow} \forall I \quad \frac{\Gamma \vdash_{\text{IC}} \forall x.A \downarrow}{\Gamma \vdash_{\text{IC}} A[x/T] \downarrow} \forall E$$

(\*: parameter P must be new in context)

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(\*: parameter P must be new in context)

- Existential Quantif.:

$$\frac{\Gamma \vdash_{\text{IC}} A[x/T] \uparrow}{\Gamma \vdash_{\text{IC}} \exists x.A \uparrow} \exists I \quad \frac{\Gamma \vdash_{\text{IC}} \exists x.A \downarrow \quad \Gamma, A[x/P^*] \vdash_{\text{IC}} C \uparrow}{\Gamma \vdash C \uparrow} \exists E$$

(\*: parameter P must be new in context)

# ND Intercalation Rules III

For classical logic add:

- Proof by Contradiction:

$$\frac{\Gamma, \neg A \vdash_{\text{IC}} \perp \uparrow}{\Gamma \vdash_{\text{IC}} A \uparrow} \perp_c$$

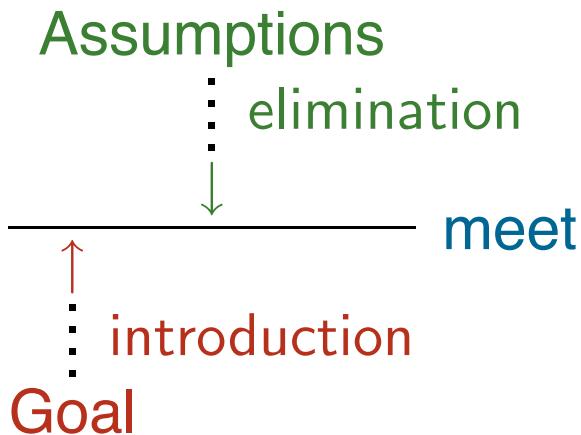
# Intercalation and ND

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- Normal form proofs

# Intercalation and ND

## ■ Normal form proofs



guaranteed by

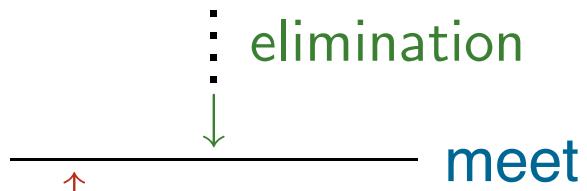
$$\frac{\Gamma \vdash_{\text{IC}} A \downarrow}{\Gamma \vdash_{\text{IC}} A \uparrow} \text{ meet}$$

# Intercalation and ND

## ■ Normal form proofs

Assumptions

elimination



introduction

Goal

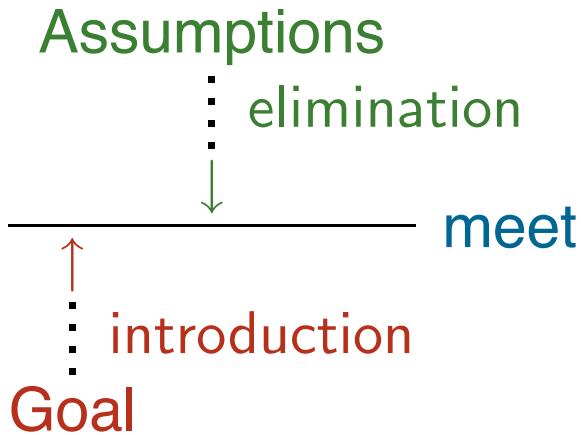
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$$\frac{\Gamma \vdash_{\text{IC}} A \downarrow}{\Gamma \vdash_{\text{IC}} A \uparrow} \text{ meet}$$

... proofs without detour ...

# Intercalation and ND

- Normal form proofs



guaranteed by

$$\frac{\Gamma \vdash_{\text{IC}} A \downarrow}{\Gamma \vdash_{\text{IC}} A \uparrow} \text{ meet}$$

... proofs without detour ...

- To model all ND proofs add

$$\frac{\Gamma \vdash_{\text{IC}} A \uparrow}{\Gamma \vdash_{\text{IC}} A \downarrow} \text{ roundabout}$$

# Example Proofs

- In normal form

$$\frac{\frac{\frac{\frac{M \wedge Q \vdash_{IC} M \wedge Q \downarrow}{M \wedge Q \vdash_{IC} Q \downarrow} \wedge E_r}{M \wedge Q \vdash_{IC} Q \uparrow} \text{meet}}{M \wedge Q \vdash_{IC} Q \vee S \uparrow} \vee I_I}{\vdash_{IC} (M \wedge Q) \Rightarrow (Q \vee S) \uparrow} \Rightarrow I$$

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- With detour

$$\frac{\vdots}{\frac{\frac{M \wedge Q \vdash_{\text{IC}} Q \uparrow \quad M \wedge Q \vdash_{\text{IC}} M \uparrow}{M \wedge Q \vdash_{\text{IC}} Q \wedge M \uparrow} \wedge I}{\frac{M \wedge Q \vdash_{\text{IC}} Q \wedge M \downarrow}{M \wedge Q \vdash_{\text{IC}} Q \downarrow} \wedge E_l} \text{roundabout}}$$

# Soundness and Completeness

Let  $\vdash_{\text{IC}}$  denote the intercalation calculus with rule **roundabout** and  $\vdash_{\text{IC}}$  the calculus without this rule.

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- ▶ Theorem 2 (Completeness of  $\Gamma \vdash_{\text{IC}}^{\pm} A \uparrow$  relative to  $\vdash$ ): If  $\Gamma \vdash A$  then  $\Gamma \vdash_{\text{IC}}^{\pm} A \uparrow$ .

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- ▶ Is normal form proof search also complete?:

If  $\Gamma \vdash_{\text{IC}}^\pm A \uparrow$  then  $\Gamma \vdash_{\text{IC}} A \uparrow$ ?

We will investigate this question within the sequent calculus.

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# From ND to Sequent Calculus

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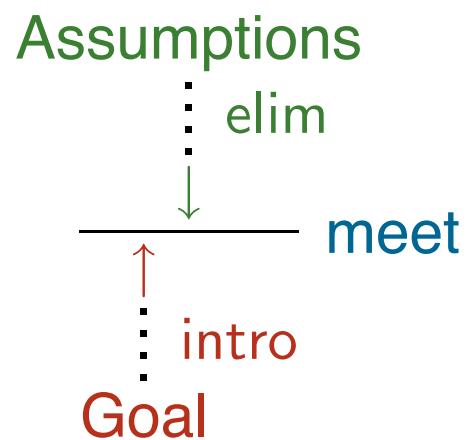
Normal form ND proofs

Sequent proofs

# From ND to Sequent Calculus

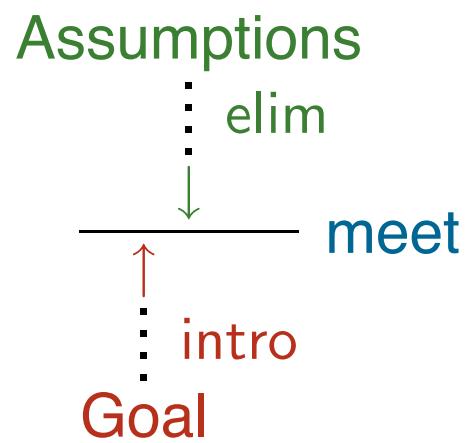
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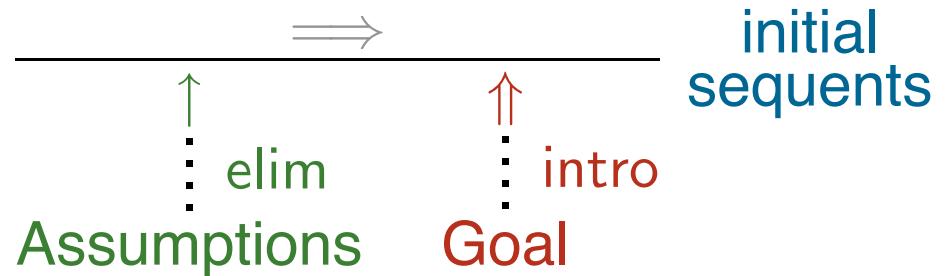


# From ND to Sequent Calculus

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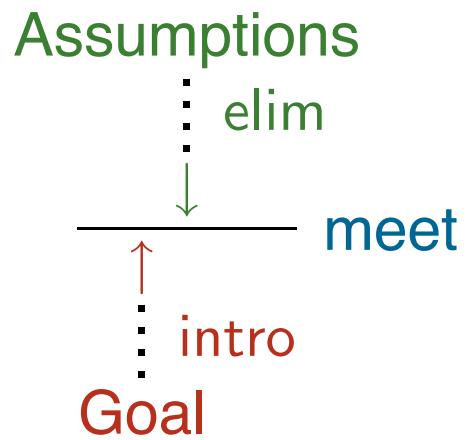


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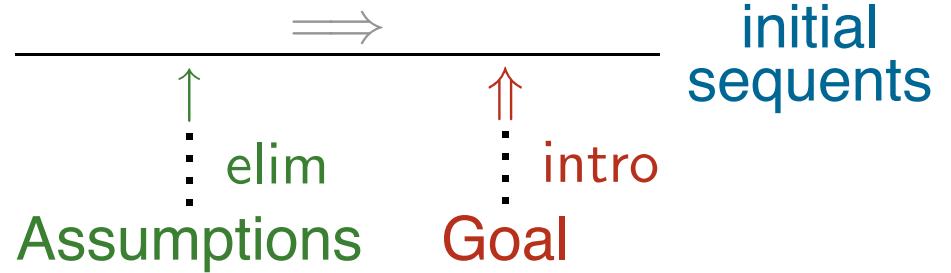


# From ND to Sequent Calculus

Normal form ND proofs



Sequent proofs



Sequents pair  $\langle \Gamma, \Delta \rangle$  of finite lists, multisets, or sets of formulas

Notation:  $\Gamma \Rightarrow \Delta$        $\Gamma$  conjunctiv and  $\Delta$  disjunctive

Intuitive: a kind of implication,  $\Delta$  “follows from”  $\Gamma$

# Sequent Calculus Rules I

- Initial Sequents:

$$\frac{}{\Gamma, A \implies \Delta, A} \text{ init} \quad (A \text{ atomic})$$

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$$\frac{\Gamma \implies \Delta, A \quad \Gamma, B \implies \Delta}{\Gamma, A \Rightarrow B \implies \Delta} \Rightarrow L \quad \frac{\Gamma, A \implies \Delta, B}{\Gamma \implies \Delta, A \Rightarrow B} \Rightarrow R$$

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- Truth and Falsehood

$$\frac{}{\Gamma, \perp \implies \Delta} \perp L \quad \frac{\Gamma \implies \Delta, \top}{\Gamma \implies \Delta, \top} \top R$$

# Sequent Calculus Rules II

- Negation:

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma, \neg A \Rightarrow \Delta} \neg L$$

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$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \vee R \quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \vee L$$

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- Universal Quantification:

$$\frac{\Gamma, \forall x.A, A[x/T] \Rightarrow \Delta}{\Gamma, \forall x.A \Rightarrow \Delta} \forall L \quad \frac{\Gamma \Rightarrow \Delta, A[x/P^*]}{\Gamma \Rightarrow \Delta, \forall x.A} \forall R$$

(\*: parameter P must be new in context)

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# Example Proof

---

$$\frac{\frac{\frac{\frac{\frac{\frac{A, B \Rightarrow B}{A \wedge B \Rightarrow B} \text{ init}}{\frac{A \wedge B \Rightarrow B}{A \wedge B \Rightarrow B \wedge (C \vee A)} \text{ } \wedge L}}{\frac{A, B \Rightarrow C, A}{A \wedge B \Rightarrow C, A} \text{ init}} \text{ } \wedge L}{\frac{A \wedge B \Rightarrow C, A}{A \wedge B \Rightarrow C \vee A} \text{ } \vee R} \text{ } \wedge R}{A \wedge B \Rightarrow B \wedge (C \vee A)} \Rightarrow R$$

# Sequent Calculus: Cut-rule

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- The question whether normal form proof search ( $\vdash_{\text{IC}}$ ) is complete corresponds to the question whether the cut-rule can be eliminated (is *admissible*) in sequent calculus.

# Sequent Calculus

Let  $\Rightarrow^+$  denote the sequent calculus with cut-rule and  $\Rightarrow$  the sequent calculus without the cut-rule.

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Theorem 3 (Soundness of  $\Rightarrow$  relative to  $\Gamma \vdash_{\text{IC}}$  and  $\Gamma \models_{\text{IC}}^+$ )

- (a) If  $\Gamma \Rightarrow C$  then  $\Gamma \vdash_{\text{IC}} C \uparrow$ .
- (b) If  $\Gamma \Rightarrow^+ C$  then  $\Gamma \models_{\text{IC}}^+ C \uparrow$ .

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Theorem 4 (Completeness of  $\Rightarrow$  relative to  $\Gamma \vdash_{\text{IC}}$  and  $\Gamma \models_{\text{IC}}^+$ )

- (a) If  $\Gamma \vdash_{\text{IC}} C \uparrow$  then  $\Gamma \Rightarrow C$ .
- (b) If  $\Gamma \models_{\text{IC}}^+ C \uparrow$  then  $\Gamma \Rightarrow^+ C$ .

# Gentzen's Hauptsatz

Theorem 5 (Cut-Elimination): Cut-elimination holds for the sequent calculus. In other words: The cut rule is *admissible* in the sequent calculus.

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Proof non-trivial; main means: nested inductions and case distinctions over rule applications

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Proof non-trivial; main means: nested inductions and case distinctions over rule applications

This result qualifies the sequent calculus as suitable for automating proof search.

# Applications of Cut-Elimination

Theorem (Normalization for ND):

If  $\Gamma \vdash C$  then  $\Gamma \vdash_{\text{IC}} C \uparrow$ .

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- ▶ Assume  $\Gamma \vdash C$ .

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Theorem (Normalization for ND):

$$\text{If } \Gamma \vdash C \text{ then } \Gamma \vdash_{\text{IC}} C^{\uparrow}.$$

Proof sketch:

- ▶ Assume  $\Gamma \vdash C$ .
- ▶ Then  $\Gamma \vdash_{\text{IC}} C^{\uparrow}$  by completeness of  $\vdash_{\text{IC}}$ .

# Applications of Cut-Elimination

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- ▶ Then  $\Gamma \Rightarrow^+ C$  by completeness of  $\Rightarrow^+$ .

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- ▶ Then  $\Gamma \vdash_{\text{ic}} C^{\uparrow}$  by completeness of  $\vdash_{\text{ic}}$ .
- ▶ Then  $\Gamma \Rightarrow^{+} C$  by completeness of  $\Rightarrow^{+}$ .
- ▶ Then  $\Gamma \Rightarrow C$  by cut-elimination.

# Applications of Cut-Elimination

Theorem (Normalization for ND):

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- ▶ Then  $\Gamma \Rightarrow^+ C$  by completeness of  $\Rightarrow^+$ .
- ▶ Then  $\Gamma \Rightarrow C$  by cut-elimination.
- ▶ Then  $\Gamma \vdash_{\text{IC}} C \uparrow$  by soundness of  $\Rightarrow$ .

# What have we done?

Natural Deduction		
$\vdash$ (with detours)  →		

# What have we done?

Natural Deduction	Intercalation	
$\vdash$ (with detours)	$\vdash_{ic}$ (with roundabout)	

# What have we done?

Natural Deduction	Intercalation	Sequent Calculus
$\vdash$ (with detours)	$\vdash_{\text{ic}}$ (with roundabout)	$\Rightarrow^+$ (with cut)
		↓

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Natural Deduction	Intercalation	Sequent Calculus
$\vdash$ (with detours)	$\vdash_{\text{ic}}$ (with roundabout)	$\Rightarrow^+$ (with cut)
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Natural Deduction	Intercalation	Sequent Calculus
$\vdash$ (with detours)	$\vdash_{\text{ic}}$ (with roundabout)	$\Rightarrow^+$ (with cut)
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- ▶ Then  $\Rightarrow^+ \perp$  by completeness of  $\Rightarrow^+$  and  $\vdash_{\text{ic}}$ .
- ▶ Then  $\Rightarrow \perp$  by cut-elimination.

# Applications of Cut-Elimination

Theorem (Consistency of ND): There is no natural deduction derivation  $\vdash \perp$ .

Proof sketch:

- ▶ Assume there is a proof of  $\vdash \perp$ .
- ▶ Then  $\Rightarrow^+ \perp$  by completeness of  $\Rightarrow^+$  and  $\vdash_{\text{ic}}$ .
- ▶ Then  $\Rightarrow \perp$  by cut-elimination.
- ▶ But  $\Rightarrow \perp$  cannot be the conclusion of any sequent rule.

# Summary

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- We have illustrated the connection of
  - ▶ natural deduction and sequent calculus
  - ▶ normal form natural deductions and cut-free sequent calculus.
- Fact: Sequent calculus often employed as meta-theory for specialized proof search calculi and strategies.

# Summary

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- We have illustrated the connection of
  - ▶ natural deduction and sequent calculus
  - ▶ normal form natural deductions and cut-free sequent calculus.
- Fact: Sequent calculus often employed as meta-theory for specialized proof search calculi and strategies.
- Question: Can these calculi and strategies be transformed to natural deduction proof search?



# Calculi: Higher-Order Natural Deduction

# ND Calculi for HOL

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Some conventions for this part:

- signature  $\Sigma$  contains only the logical constants  $\neg, \vee, \Pi^\alpha$  unless stated otherwise.

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Some conventions for this part:

- signature  $\Sigma$  contains only the logical constants  $\neg, \vee, \Pi^\alpha$  unless stated otherwise.
- $\Phi * A := \Phi \cup \{A\}$
- context representation of ND calculi

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}\mathfrak{K}_\beta$

$$\frac{A \in \Phi}{\Phi \Vdash A} \mathfrak{N}\mathfrak{K}(Hyp)$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}_\beta$

$$\frac{A \in \Phi}{\Phi \Vdash A} \mathfrak{N}(Hyp)$$

$$\frac{A =_\beta B \quad \Phi \Vdash A}{\Phi \Vdash B} \mathfrak{N}(\beta)$$

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Inference rules for  $\mathfrak{N}_\beta$

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Inference rules for  $\mathfrak{N}_\beta$

$$\frac{A \in \Phi}{\Phi \Vdash A} \mathfrak{N}(Hyp)$$

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$$\frac{\Phi \Vdash \neg A \quad \Phi \Vdash A}{\Phi \Vdash C} \mathfrak{N}(\neg E)$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}_\beta$

$$\frac{A \in \Phi}{\Phi \Vdash A} \mathfrak{N}(Hyp)$$

$$\frac{\Phi * A \Vdash F_o}{\Phi \Vdash \neg A} \mathfrak{N}(\neg I)$$

$$\frac{\Phi \Vdash A}{\Phi \Vdash A \vee B} \mathfrak{N}(\vee I_L)$$

$$\frac{A =_\beta B \quad \Phi \Vdash A}{\Phi \Vdash B} \mathfrak{N}(\beta)$$

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$$\frac{A =_\beta B \quad \Phi \Vdash A}{\Phi \Vdash B} \mathfrak{N}(\beta)$$

$$\frac{\Phi \Vdash \neg A \quad \Phi \Vdash A}{\Phi \Vdash C} \mathfrak{N}(\neg E)$$

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Inference rules for  $\mathfrak{N}_\beta$

$$\frac{A \in \Phi}{\Phi \Vdash A} \mathfrak{N}(Hyp)$$

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$$\frac{\Phi \Vdash A}{\Phi \Vdash A \vee B} \mathfrak{N}(\vee I_L)$$

$$\frac{\Phi \Vdash A \vee B \quad \Phi * A \Vdash C \quad \Phi * B \Vdash C}{\Phi \Vdash C} \mathfrak{N}(\vee E)$$

$$\frac{A =_\beta B \quad \Phi \Vdash A}{\Phi \Vdash B} \mathfrak{N}(\beta)$$

$$\frac{\Phi \Vdash \neg A \quad \Phi \Vdash A}{\Phi \Vdash C} \mathfrak{N}(\neg E)$$

$$\frac{\Phi \Vdash B}{\Phi \Vdash A \vee B} \mathfrak{N}(\vee I_R)$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}\kappa_\beta$

$$\frac{A \in \Phi}{\Phi \Vdash A} \mathfrak{N}\kappa(Hyp)$$

$$\frac{\Phi * A \Vdash F_o}{\Phi \Vdash \neg A} \mathfrak{N}\kappa(\neg I)$$

$$\frac{\Phi \Vdash A}{\Phi \Vdash A \vee B} \mathfrak{N}\kappa(\vee I_L)$$

$$\frac{\Phi \Vdash A \vee B \quad \Phi * A \Vdash C \quad \Phi * B \Vdash C}{\Phi \Vdash C} \mathfrak{N}\kappa(\vee E)$$

$$\frac{\Phi \Vdash G_{w_\alpha} \quad w \text{ new parameter}}{\Phi \Vdash \Pi^\alpha G} \mathfrak{N}\kappa(\Pi)^w$$

$$\frac{A =_\beta B \quad \Phi \Vdash A}{\Phi \Vdash B} \mathfrak{N}\kappa(\beta)$$

$$\frac{\Phi \Vdash \neg A \quad \Phi \Vdash A}{\Phi \Vdash C} \mathfrak{N}\kappa(\neg E)$$

$$\frac{\Phi \Vdash B}{\Phi \Vdash A \vee B} \mathfrak{N}\kappa(\vee I_R)$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}\kappa_\beta$

$$\frac{A \in \Phi}{\Phi \Vdash A} \mathfrak{N}\kappa(Hyp)$$

$$\frac{\Phi * A \Vdash F_o}{\Phi \Vdash \neg A} \mathfrak{N}\kappa(\neg I)$$

$$\frac{\Phi \Vdash A}{\Phi \Vdash A \vee B} \mathfrak{N}\kappa(\vee I_L)$$

$$\frac{\Phi \Vdash A \vee B \quad \Phi * A \Vdash C \quad \Phi * B \Vdash C}{\Phi \Vdash C} \mathfrak{N}\kappa(\vee E)$$

$$\frac{\Phi \Vdash G_{w_\alpha} \quad w \text{ new parameter}}{\Phi \Vdash \Pi^\alpha G} \mathfrak{N}\kappa(\Pi I)^w$$

$$\frac{\Phi \Vdash \Pi^\alpha G}{\Phi \Vdash GA} \mathfrak{N}\kappa(\Pi E)$$

$$\frac{A =_\beta B \quad \Phi \Vdash A}{\Phi \Vdash B} \mathfrak{N}\kappa(\beta)$$

$$\frac{\Phi \Vdash \neg A \quad \Phi \Vdash A}{\Phi \Vdash C} \mathfrak{N}\kappa(\neg E)$$

$$\frac{\Phi \Vdash B}{\Phi \Vdash A \vee B} \mathfrak{N}\kappa(\vee I_R)$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}_\beta$

$$\frac{A \in \Phi}{\Phi \Vdash A} \mathfrak{N}(Hyp)$$

$$\frac{\Phi * A \Vdash F_o}{\Phi \Vdash \neg A} \mathfrak{N}(\neg I)$$

$$\frac{\Phi \Vdash A}{\Phi \Vdash A \vee B} \mathfrak{N}(\vee I_L)$$

$$\frac{\Phi \Vdash A \vee B \quad \Phi * A \Vdash C \quad \Phi * B \Vdash C}{\Phi \Vdash C} \mathfrak{N}(\vee E)$$

$$\frac{\Phi \Vdash G w_\alpha \quad w \text{ new parameter}}{\Phi \Vdash \Pi^\alpha G} \mathfrak{N}(\Pi)^w$$

$$\frac{\Phi \Vdash \Pi^\alpha G}{\Phi \Vdash G A} \mathfrak{N}(\Pi E)$$

$$\frac{A =_\beta B \quad \Phi \Vdash A}{\Phi \Vdash B} \mathfrak{N}(\beta)$$

$$\frac{\Phi \Vdash \neg A \quad \Phi \Vdash A}{\Phi \Vdash C} \mathfrak{N}(\neg E)$$

$$\frac{\Phi \Vdash B}{\Phi \Vdash A \vee B} \mathfrak{N}(\vee I_R)$$

$$\frac{\Phi \Vdash A \vee B \quad \Phi * \neg A \Vdash F_o}{\Phi \Vdash A} \mathfrak{N}(Contr)$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}\mathfrak{K}_\beta$  (for richer signatures)

$$\frac{\Phi \Vdash A \wedge B}{\Phi \Vdash A} \mathfrak{N}\mathfrak{K}(\wedge E_L)$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}\kappa_\beta$  (for richer signatures)

$$\frac{\Phi \models A \wedge B}{\Phi \models A} \mathfrak{N}\kappa(\wedge E_L)$$

$$\frac{\Phi \models A \wedge B}{\Phi \models B} \mathfrak{N}\kappa(\wedge E_R)$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}_\beta$  (for richer signatures)

$$\frac{\Phi \Vdash A \wedge B}{\Phi \Vdash A} \mathfrak{N}_\beta(\wedge E_L)$$

$$\frac{\Phi \Vdash A \wedge B}{\Phi \Vdash B} \mathfrak{N}_\beta(\wedge E_R)$$

$$\frac{\Phi \Vdash A \quad \Phi \Vdash B}{\Phi \Vdash A \wedge B} \mathfrak{N}_\beta(\wedge I)$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}\kappa_\beta$  (for richer signatures)

$$\begin{array}{c}
 \frac{\Phi \Vdash A \wedge B}{\Phi \Vdash A} \text{ } \mathfrak{N}\kappa(\wedge E_L) \quad \frac{\Phi \Vdash A \wedge B}{\Phi \Vdash B} \text{ } \mathfrak{N}\kappa(\wedge E_R) \quad \frac{\Phi \Vdash A \quad \Phi \Vdash B}{\Phi \Vdash A \wedge B} \text{ } \mathfrak{N}\kappa(\wedge I) \\
 \\ 
 \frac{\Phi \Vdash A \Rightarrow B \quad \Phi \Vdash A}{\Phi \Vdash B} \text{ } \mathfrak{N}\kappa(\Rightarrow E)
 \end{array}$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}\kappa_\beta$  (for richer signatures)

$$\begin{array}{c}
 \frac{\Phi \models A \wedge B}{\Phi \models A} \mathfrak{N}\kappa(\wedge E_L) \quad \frac{\Phi \models A \wedge B}{\Phi \models B} \mathfrak{N}\kappa(\wedge E_R) \quad \frac{\Phi \models A \quad \Phi \models B}{\Phi \models A \wedge B} \mathfrak{N}\kappa(\wedge I) \\
 \\ 
 \frac{\Phi \models A \Rightarrow B \quad \Phi \models A}{\Phi \models B} \mathfrak{N}\kappa(\Rightarrow E) \quad \frac{\Phi, A \models B}{\Phi \models A \Rightarrow B} \mathfrak{N}\kappa(\Rightarrow I)
 \end{array}$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}_\beta$  (for richer signatures)

$$\frac{\Phi \models A \wedge B}{\Phi \models A} \mathfrak{N}(\wedge E_L) \quad \frac{\Phi \models A \wedge B}{\Phi \models B} \mathfrak{N}(\wedge E_R) \quad \frac{\Phi \models A \quad \Phi \models B}{\Phi \models A \wedge B} \mathfrak{N}(\wedge I)$$

$$\frac{\Phi \models A \Rightarrow B \quad \Phi \models A}{\Phi \models B} \mathfrak{N}(\Rightarrow E) \quad \frac{\Phi, A \models B}{\Phi \models A \Rightarrow B} \mathfrak{N}(\Rightarrow I)$$

$$\frac{\Phi \models GT_\alpha}{\Phi \models \Sigma^\alpha G} \mathfrak{N}(\Sigma I)$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}\kappa_\beta$  (for richer signatures)

$$\frac{\Phi \Vdash A \wedge B}{\Phi \Vdash A} \text{ } \mathfrak{N}\kappa(\wedge E_L) \quad \frac{\Phi \Vdash A \wedge B}{\Phi \Vdash B} \text{ } \mathfrak{N}\kappa(\wedge E_R) \quad \frac{\Phi \Vdash A \quad \Phi \Vdash B}{\Phi \Vdash A \wedge B} \text{ } \mathfrak{N}\kappa(\wedge I)$$

$$\frac{\Phi \Vdash A \Rightarrow B \quad \Phi \Vdash A}{\Phi \Vdash B} \text{ } \mathfrak{N}\kappa(\Rightarrow E) \quad \frac{\Phi, A \Vdash B}{\Phi \Vdash A \Rightarrow B} \text{ } \mathfrak{N}\kappa(\Rightarrow I)$$

$$\frac{\Phi \Vdash GT_\alpha}{\Phi \Vdash \Sigma^\alpha G} \text{ } \mathfrak{N}\kappa(\Sigma I) \quad \frac{\Phi \Vdash \Sigma^\alpha G \quad \Phi * Gw_\alpha \Vdash C \quad w \text{ new parameter}}{\Phi \Vdash C} \text{ } \mathfrak{N}\kappa(\Sigma E)$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}\kappa_\beta$  (for richer signatures)

$$\frac{\Phi \Vdash A \wedge B}{\Phi \Vdash A} \text{ } \mathfrak{N}\kappa(\wedge E_L) \quad \frac{\Phi \Vdash A \wedge B}{\Phi \Vdash B} \text{ } \mathfrak{N}\kappa(\wedge E_R) \quad \frac{\Phi \Vdash A \quad \Phi \Vdash B}{\Phi \Vdash A \wedge B} \text{ } \mathfrak{N}\kappa(\wedge I)$$

$$\frac{\Phi \Vdash A \Rightarrow B \quad \Phi \Vdash A}{\Phi \Vdash B} \text{ } \mathfrak{N}\kappa(\Rightarrow E) \quad \frac{\Phi, A \Vdash B}{\Phi \Vdash A \Rightarrow B} \text{ } \mathfrak{N}\kappa(\Rightarrow I)$$

$$\frac{\Phi \Vdash GT_\alpha}{\Phi \Vdash \Sigma^\alpha G} \text{ } \mathfrak{N}\kappa(\Sigma I) \quad \frac{\Phi \Vdash \Sigma^\alpha G \quad \Phi * Gw_\alpha \Vdash C \quad w \text{ new parameter}}{\Phi \Vdash C} \text{ } \mathfrak{N}\kappa(\Sigma E)$$

$$\frac{\Phi \Vdash T =^\alpha W \quad \Phi \Vdash A[T]}{\Phi \Vdash A[W]} \text{ } \mathfrak{N}\kappa(= Subst)$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}\kappa_\beta$  (for richer signatures)

$$\frac{\Phi \Vdash A \wedge B}{\Phi \Vdash A} \text{ } \mathfrak{N}\kappa(\wedge E_L) \quad \frac{\Phi \Vdash A \wedge B}{\Phi \Vdash B} \text{ } \mathfrak{N}\kappa(\wedge E_R) \quad \frac{\Phi \Vdash A \quad \Phi \Vdash B}{\Phi \Vdash A \wedge B} \text{ } \mathfrak{N}\kappa(\wedge I)$$

$$\frac{\Phi \Vdash A \Rightarrow B \quad \Phi \Vdash A}{\Phi \Vdash B} \text{ } \mathfrak{N}\kappa(\Rightarrow E) \quad \frac{\Phi, A \Vdash B}{\Phi \Vdash A \Rightarrow B} \text{ } \mathfrak{N}\kappa(\Rightarrow I)$$

$$\frac{\Phi \Vdash GT_\alpha}{\Phi \Vdash \Sigma^\alpha G} \text{ } \mathfrak{N}\kappa(\Sigma I) \quad \frac{\Phi \Vdash \Sigma^\alpha G \quad \Phi * Gw_\alpha \Vdash C \quad w \text{ new parameter}}{\Phi \Vdash C} \text{ } \mathfrak{N}\kappa(\Sigma E)$$

$$\frac{\Phi \Vdash T =^\alpha W \quad \Phi \Vdash A[T]}{\Phi \Vdash A[W]} \text{ } \mathfrak{N}\kappa(= Subst) \quad \frac{}{\Phi \Vdash A = A} \text{ } \mathfrak{N}\kappa(= Refl)$$

# ND Calculi for HOL

Inference rules for  $\mathfrak{N}_\beta$  (for richer signatures)

$\frac{\Phi \models A \wedge B}{\Phi \models A} \mathfrak{N}(\wedge E_L)$	$\frac{\Phi \models A \wedge B}{\Phi \models B} \mathfrak{N}(\wedge E_R)$	$\frac{\Phi \models A \quad \Phi \models B}{\Phi \models A \wedge B} \mathfrak{N}(\wedge I)$
$\frac{\Phi \models A \Rightarrow B \quad \Phi \models A}{\Phi \models B} \mathfrak{N}(\Rightarrow E)$	$\frac{\Phi, A \models B}{\Phi \models A \Rightarrow B} \mathfrak{N}(\Rightarrow I)$	
$\frac{\Phi \models GT_\alpha}{\Phi \models \Sigma^\alpha G} \mathfrak{N}(\Sigma I)$	$\frac{\Phi \models \Sigma^\alpha G \quad \Phi * Gw_\alpha \models C \quad w \text{ new parameter}}{\Phi \models C} \mathfrak{N}(\Sigma E)$	
$\frac{\Phi \models T =^\alpha W \quad \Phi \models A[T]}{\Phi \models A[W]} \mathfrak{N}(= Subst)$		$\frac{\Phi \models A = A}{\Phi \models A = A} \mathfrak{N}(= Refl)$

Alternative: Define logical constants  $\wedge, \Rightarrow, \Sigma$ , etc. in terms of  $\neg, \vee, \Pi$  as usual and strictly use Leibniz equality instead of primitive equality; then the above rules are not needed.

# ND Calculi for HOL

Inference rules for extensionality (rules for  $\xi, \eta, \mathfrak{f}, \mathfrak{b}$ )

$$\frac{A \stackrel{\beta\eta}{=} B \quad \Phi \vdash A}{\Phi \vdash B} \mathfrak{M}\mathfrak{R}(\eta)$$

# ND Calculi for HOL

Inference rules for extensionality (rules for  $\xi, \eta, \mathfrak{f}, \mathfrak{b}$ )

$$\frac{A \stackrel{\beta\eta}{=} B \quad \Phi \Vdash A}{\Phi \Vdash B} \mathfrak{M}(\eta)$$

$$\frac{\Phi \Vdash \forall x_\alpha.M \stackrel{\cdot\beta}{=} N}{\Phi \Vdash (\lambda x_\alpha.M) \stackrel{\cdot\beta\alpha}{=} (\lambda x_\alpha.N)} \mathfrak{M}(\xi)$$

# ND Calculi for HOL

Inference rules for extensionality (rules for  $\xi, \eta, \mathfrak{f}, \mathfrak{b}$ )

$$\frac{A \stackrel{\beta\eta}{=} B \quad \Phi \Vdash A}{\Phi \Vdash B} \mathfrak{N}\mathfrak{R}(\eta)$$

$$\frac{\Phi \Vdash \forall x_\alpha.M \stackrel{\cdot\beta}{=} N}{\Phi \Vdash (\lambda x_\alpha.M) \stackrel{\cdot\beta\alpha}{=} (\lambda x_\alpha.N)} \mathfrak{N}\mathfrak{R}(\xi)$$

$$\frac{\Phi \Vdash \forall x_\alpha.Gx \stackrel{\cdot\beta}{=} Hx}{\Phi \Vdash G \stackrel{\cdot\beta\alpha}{=} H} \mathfrak{N}\mathfrak{R}(\mathfrak{f})$$

# ND Calculi for HOL

Inference rules for extensionality (rules for  $\xi, \eta, \mathfrak{f}, \mathfrak{b}$ )

$$\frac{A \stackrel{\beta\eta}{=} B \quad \Phi \Vdash A}{\Phi \Vdash B} \mathfrak{N}\mathfrak{R}(\eta)$$

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$$\frac{\Phi \Vdash \forall x_\alpha.Gx \stackrel{\cdot\beta}{=} Hx}{\Phi \Vdash G \stackrel{\cdot\beta\alpha}{=} H} \mathfrak{N}\mathfrak{R}(\mathfrak{f})$$

$$\frac{\Phi * A \Vdash B \quad \Phi * B \Vdash A}{\Phi \Vdash A \stackrel{\cdot\circ}{=} B} \mathfrak{N}\mathfrak{R}(\mathfrak{b})$$

# ND Calculi for HOL

Inference rules for extensionality (rules for  $\xi, \eta, \mathfrak{f}, \mathfrak{b}$ )

$$\frac{A \stackrel{\beta\eta}{=} B \quad \Phi \Vdash A}{\Phi \Vdash B} \mathfrak{N}\mathfrak{R}(\eta)$$

$$\frac{\Phi \Vdash \forall x_\alpha.M \stackrel{\beta}{=} N}{\Phi \Vdash (\lambda x_\alpha.M) \stackrel{\beta\alpha}{=} (\lambda x_\alpha.N)} \mathfrak{N}\mathfrak{R}(\xi)$$

$$\frac{\Phi \Vdash \forall x_\alpha.Gx \stackrel{\beta}{=} Hx}{\Phi \Vdash G \stackrel{\beta\alpha}{=} H} \mathfrak{N}\mathfrak{R}(\mathfrak{f})$$

$$\frac{\Phi * A \Vdash B \quad \Phi * B \Vdash A}{\Phi \Vdash A \stackrel{\circ}{=} B} \mathfrak{N}\mathfrak{R}(\mathfrak{b})$$

In case of a primitive notion of equality we define respective extensionality rules also for  $=$ .

## ■ The Calculi $\mathcal{M}_*$

# ND Calculi for HOL

## ■ The Calculi $\mathfrak{N}_*$

- ▶ The calculus  $\mathfrak{N}_\beta$  consists of the inference rules for  $\mathfrak{N}_\beta$  for the provability judgment  $\Vdash$  between sets of sentences  $\Phi$  and sentences  $\mathbf{A}$ . (We write  $\Vdash \mathbf{A}$  for  $\emptyset \Vdash \mathbf{A}$ .) The rule  $\mathfrak{N}(\beta)$  incorporates  $\beta$ -equality into  $\Vdash$ . The others characterize ‘the semantics of the connectives and quantifiers.

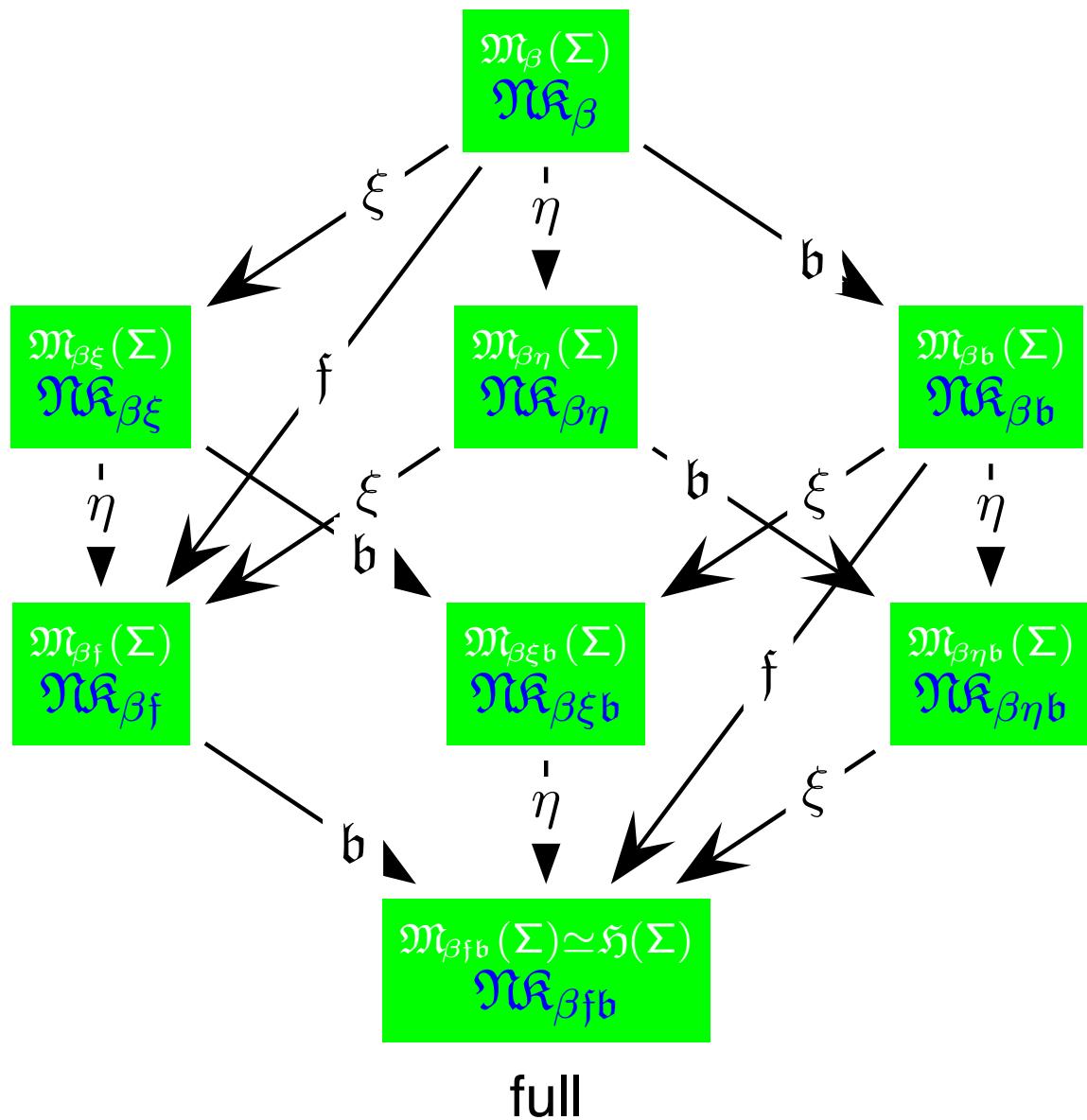
# ND Calculi for HOL

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## ■ The Calculi $\mathfrak{N}_*$

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- ▶ For  $* \in \{\beta\eta, \beta\xi, \beta f, \beta b, \beta\eta b, \beta\xi b, \beta f b\}$  we obtain the calculus  $\mathfrak{N}_*$  by adding the respective extensionality rules when specified in  $*$ .

# ND Calculi for HOL



# ND Calculi for HOL

- Note that  $\mathcal{M}_\beta$  and  $\mathcal{M}_{\beta fb}$  correspond to the extremes of the model classes in our landscape of model classes. For example,  $\mathcal{M}_{\beta fb}$  will be proven sound and complete for Henkin models, and  $\mathcal{M}_\beta$  will be proven sound and complete for  $\mathfrak{M}_\beta(\Sigma)$ .

# ND Calculi for HOL

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- Standard models do not admit (recursively axiomatizable) calculi that are sound and complete.

# ND Calculi for HOL

- Note that  $\mathfrak{N}_\beta$  and  $\mathfrak{N}_{\beta\text{fb}}$  correspond to the extremes of the model classes in our landscape of model classes. For example,  $\mathfrak{N}_{\beta\text{fb}}$  will be proven sound and complete for Henkin models, and  $\mathfrak{N}_\beta$  will be proven sound and complete for  $\mathfrak{M}_\beta(\Sigma)$ .
- Standard models do not admit (recursively axiomatizable) calculi that are sound and complete.
- In the following we will develop the abstract consistency proof method for HOL (wrt all the different semantic classes  $\mathfrak{M}_*(\Sigma)$  in our landscape) and we will analyse soundness and completeness of each  $\mathfrak{N}_*$  with respect to each corresponding model class  $\mathfrak{M}_*(\Sigma)$  with the help of the abstract consistency method.

# ND Calculi for HOL

## ■ (Soundness for $\mathfrak{N}_*$ )

$\mathfrak{N}_*$  is sound for  $\mathfrak{M}_*(\Sigma)$  for  $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ .  
That is, if  $\Phi \Vdash_{\mathfrak{N}_*} C$  is derivable, then  $\mathcal{M} \models C$  for all models  
 $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, v)$  in  $\mathfrak{M}_*(\Sigma)$  such that  $\mathcal{M} \models \Phi$ .

Proof: ... exercise ...

# ND Calculi for HOL

---

- (Soundness for  $\mathfrak{N}_*$ )

$\mathfrak{N}_*$  is sound for  $\mathfrak{M}_*(\Sigma)$  for  $* \in \{\beta, \beta\eta, \beta\xi, \beta f, \beta b, \beta\eta b, \beta\xi b, \beta f b\}$ . That is, if  $\Phi \Vdash_{\mathfrak{N}_*} C$  is derivable, then  $\mathcal{M} \models C$  for all models  $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, v)$  in  $\mathfrak{M}_*(\Sigma)$  such that  $\mathcal{M} \models \Phi$ .

Proof: ... exercise ...

- (Completeness for  $\mathfrak{N}_*$ )

Let  $\Phi$  be a sufficiently  $\Sigma$ -pure set of sentences,  $A$  be a sentence, and  $* \in \{\beta, \beta\eta, \beta\xi, \beta f, \beta b, \beta\eta b, \beta\xi b, \beta f b\}$ . If  $A$  is valid in all models  $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$  that satisfy  $\Phi$ , then  $\Phi \Vdash_{\mathfrak{N}_*} A$ .

Proof: ... will follow ...

# ND Calculi for HOL

---

- (Soundness for  $\mathfrak{N}_*$ )

$\mathfrak{N}_*$  is sound for  $\mathfrak{M}_*(\Sigma)$  for  $* \in \{\beta, \beta\eta, \beta\xi, \beta f, \beta b, \beta\eta b, \beta\xi b, \beta f b\}$ . That is, if  $\Phi \Vdash_{\mathfrak{N}_*} C$  is derivable, then  $\mathcal{M} \models C$  for all models  $\mathcal{M} = (\mathcal{D}, @, \mathcal{E}, v)$  in  $\mathfrak{M}_*(\Sigma)$  such that  $\mathcal{M} \models \Phi$ .

Proof: ... exercise ...

- (Completeness for  $\mathfrak{N}_*$ )

Let  $\Phi$  be a sufficiently  $\Sigma$ -pure set of sentences,  $A$  be a sentence, and  $* \in \{\beta, \beta\eta, \beta\xi, \beta f, \beta b, \beta\eta b, \beta\xi b, \beta f b\}$ . If  $A$  is valid in all models  $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$  that satisfy  $\Phi$ , then  $\Phi \Vdash_{\mathfrak{N}_*} A$ .

Proof: ... will follow ...

# ND Calculi for HOL

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Derivation of  $\neg(p \vee \neg p) \Vdash (p \vee \neg p)$

$$\frac{\frac{\frac{\neg(p \vee \neg p), p \Vdash \neg(p \vee \neg p)}{\neg(p \vee \neg p), p \Vdash \neg(p \vee \neg p)} \mathfrak{N}(Hyp)}{\neg(p \vee \neg p), p \Vdash (p \vee \neg p)} \mathfrak{N}(\neg E)}{\frac{\frac{\frac{\neg(p \vee \neg p), p \Vdash F_o}{\neg(p \vee \neg p) \Vdash \neg p} \mathfrak{N}(\neg I)}{\neg(p \vee \neg p) \Vdash (p \vee \neg p)} \mathfrak{N}(\vee I_R)}{\frac{\neg(p \vee \neg p), p \Vdash p}{\neg(p \vee \neg p), p \Vdash (p \vee \neg p)} \mathfrak{N}(\vee I_L)} \mathfrak{N}(Hyp)}$$



Completeness (of  $\mathfrak{M}_*$ )

# Completeness (of $\mathcal{M}_*$ )

- How to prove completeness?

# Completeness (of $\mathcal{MR}_*$ )

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- Completeness can be proven rather easily for propositional logic calculi.

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# Completeness (of $\mathcal{MR}_*$ )

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- For first-order and especially higher-order logic completeness proofs become increasingly difficult and technical.
- Here we will introduce a **strong proof tool** that uniformly supports completeness proofs (and many other things): **abstract consistency**.
- This proof tool is based on a strong theorem which connects syntax and semantics: **model existence theorem**.



## Abstract Consistency

# Abstract Consistency: History

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- The technique has been extended to our landscape of HOL model classes in [[Benzmueller-PhD-99](#),[JSL04](#)].

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*If a set of sentences  $\Phi$  of  $L$  is a member of an (saturated) abstract consistency class  $\Gamma$ , then there exists a model for  $\Phi$ .*

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  - ▶ This shows refutation completeness of  $C$ .
  - ▶ For many calculi  $C$ , this also shows  $A$  is provable, thus establishing completeness of  $C$ .

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Thus,  $S \in C$  by compactness.

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This can be obtained in practice by enriching the signature with spurious parameters.

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- as a matter of convenience we will write  $\varphi * A$  for  $\varphi \cup \{A\}$ .

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Let  $\Gamma_\Sigma$  be a class of sets of  $\Sigma$ -sentences. We define (where  $\Phi \in \Gamma_\Sigma$ ,  $\alpha, \beta \in \mathcal{T}$ ,  $\mathbf{A}, \mathbf{B} \in \text{cwff}_o(\Sigma)$ ,  $\mathbf{F} \in \text{cwff}_{\alpha \rightarrow o}(\Sigma)$  are arbitrary):

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- $\nabla_\forall$  If  $\Pi^\alpha \mathbf{F} \in \Phi$ , then  $\Phi * \mathbf{F} \mathbf{W} \in \Gamma_\Sigma$  for each  $\mathbf{W} \in \text{cwff}_\alpha(\Sigma)$ .

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(These properties are going back to Hintikka, Smullyan, and Andrews)

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$\nabla_\xi$  If  $\neg(\lambda X_\alpha.M \doteq^{\alpha \rightarrow \beta} \lambda X_\alpha.N) \in \Phi$ , then

$\Phi * \neg([w/X]M \doteq^\beta [w/X]N) \in \Gamma_\Sigma$  for any parameter  $w_\alpha \in \Sigma_\alpha$  which does not occur in any sentence of  $\Phi$ .

# Def.: Abstract Consistency Properties

Let  $\Gamma_\Sigma$  be a class of sets of  $\Sigma$ -sentences. We define (where  $\Phi \in \Gamma_\Sigma$ ,  $\alpha, \beta \in \mathcal{T}$ ,  $\mathbf{A}, \mathbf{B} \in \text{cwff}_o(\Sigma)$ ,  $\mathbf{G}, \mathbf{H}, (\lambda X_\alpha.M), (\lambda X_\alpha.N) \in \text{cwff}_{\alpha \rightarrow \beta}(\Sigma)$  are arbitrary):

- $\nabla_b$  If  $\neg(\mathbf{A} \doteq^\circ \mathbf{B}) \in \Phi$ , then  $\Phi * \mathbf{A} * \neg\mathbf{B} \in \Gamma_\Sigma$  or  $\Phi * \neg\mathbf{A} * \mathbf{B} \in \Gamma_\Sigma$ .
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- $\nabla_f$  If  $\neg(\mathbf{G} \doteq^{\alpha \rightarrow \beta} \mathbf{H}) \in \Phi$ , then  $\Phi * \neg(\mathbf{G}_w \doteq^\beta \mathbf{H}_w) \in \Gamma_\Sigma$  for any  
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(These properties are new in [Benzmueller-PhD-99, JSL04])

# Def.: Abstract Consistency Classes

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We will denote the collection of abstract consistency classes by  $\mathfrak{Acc}_\beta$ .

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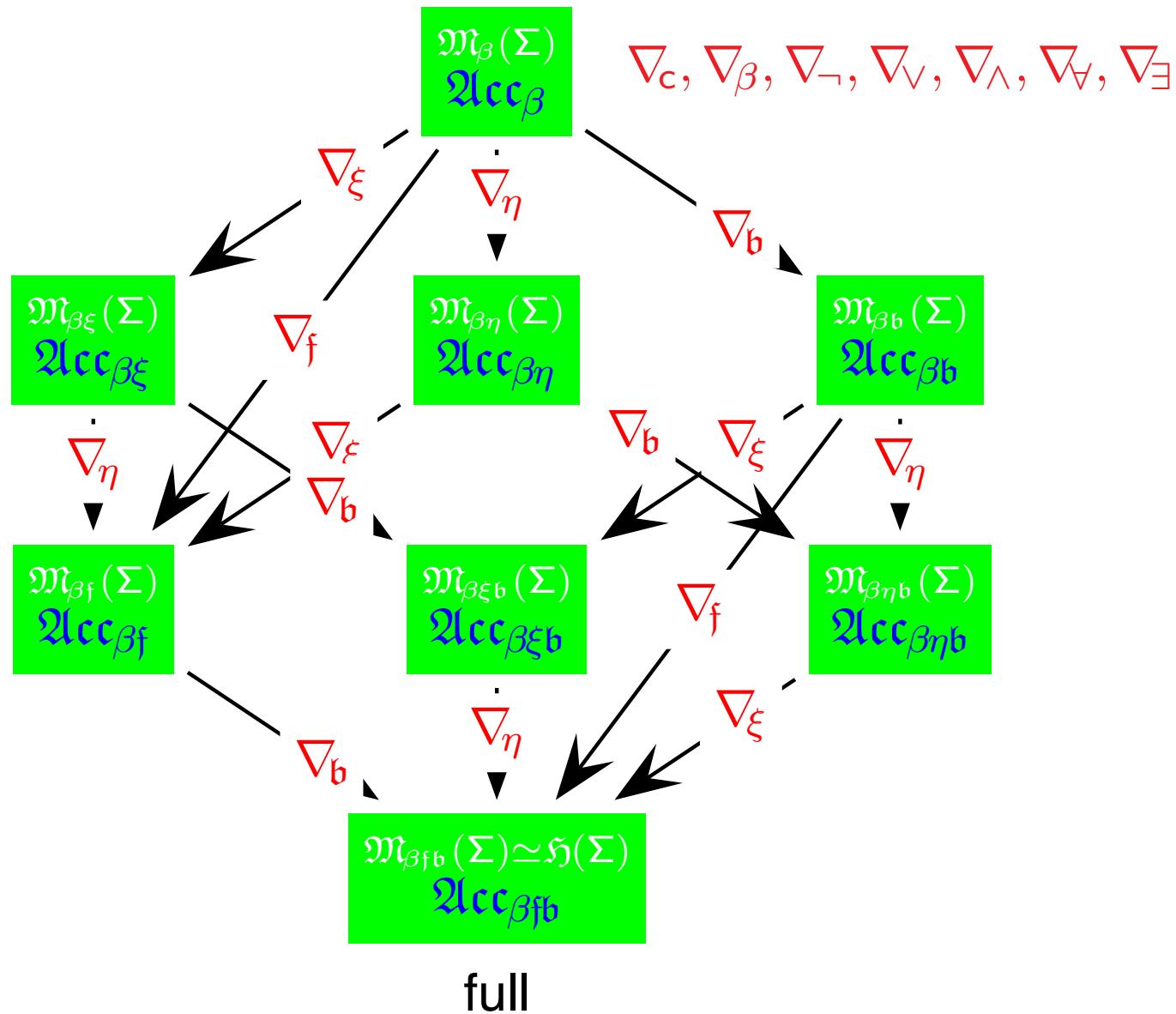
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We will denote the collection of abstract consistency classes by  $\mathcal{Acc}_\beta$ .

Similarly, we introduce the following collections of specialized abstract consistency classes (with primitive equality):  $\mathcal{Acc}_{\beta\eta}, \mathcal{Acc}_{\beta\xi}, \mathcal{Acc}_{\beta f}, \mathcal{Acc}_{\beta b}, \mathcal{Acc}_{\beta\eta b}, \mathcal{Acc}_{\beta\xi b}, \mathcal{Acc}_{\beta f b}$ , where we indicate by indices which additional properties from  $\{\nabla_\eta, \nabla_\xi, \nabla_f, \nabla_b\}$  are required.

# Abstract Consistency Classes



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- not an abstract consistency class:

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- how about this one:

$\Gamma := \{\{\neg(A \vee B), \neg A, \neg B\}, \{\neg(A \vee B), \neg A\}, \{\neg(A \vee B), \neg B\}, \{\neg A, \neg B\}, \{\neg(A \vee B)\}, \{\neg A\}, \{\neg B\}, \{\}\}$

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- and how about this:

$$\Gamma_0 := \Gamma$$

$$\Phi \in \Gamma_i \wedge A \in \Phi \wedge B =_{\beta\eta} A \wedge B \neq A \wedge (\Phi * B) \notin \Gamma_i \longrightarrow$$

$$\Gamma_{i+1} := \text{close-under-subsets}(\Gamma_i * (\Phi * B))$$

$$\Gamma^* := \Gamma_\infty$$

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**$\alpha$ -case** If  $\alpha \in \Phi$ , then  $\Phi * \alpha_1 * \alpha_2 \in \Gamma_\Sigma$ .

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**$\alpha$ -case** If  $\alpha \in \Phi$ , then  $\Phi * \alpha_1 * \alpha_2 \in \Gamma_\Sigma$ .

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**$\gamma$ -case** If  $\gamma \in \Phi$ , then  $\Phi * \gamma \mathbf{W} \in \Gamma_\Sigma$  for each  $\mathbf{W} \in \text{cwff}_\alpha(\Sigma)$ .

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Consider the following property (where  $\Phi \in \Gamma_\Sigma$ ,  $\Lambda \in \text{cwff}_o(\Sigma)$ ):

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We call an abstract consistency class  $\Gamma_\Sigma$  **saturated** if  $\nabla_{\text{sat}}$  holds for all  $\mathbf{A}$ .

# Ex.: Saturated

- consider  $\Gamma$  (and  $\Gamma^*$ ) from before:

$$\{\{\neg(A \vee B), \neg A, \neg B\}, \{\neg(A \vee B), \neg A\}, \{\neg(A \vee B), \neg B\}, \{\neg A, \neg B\}, \{\neg(A \vee B)\}, \{\neg A\}, \{\neg B\}, \{\}\}$$

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- if there is another symbol  $C_o$  in the signature, then  $\Gamma$  (and  $\Gamma^*$ ) is not atomically saturated anymore
- $\Gamma$  (and  $\Gamma^*$ ) is not saturated: for instance, it does not provide information on the formulas  $(\neg A \vee B) \vee A$  and  $\Pi^o(\lambda X_o.X)$

# Thm.: Model Existence Theorem

Let  $\Gamma_\Sigma$  be a saturated abstract consistency class and let  $\Phi \in \Gamma_\Sigma$  be a sufficiently  $\Sigma$ -pure set of sentences.

For all  $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$  we have:

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Proof: ... we are not yet ready for this ...

# Thm.: Model Existence for Henkin Models

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Proof: ... we are not yet ready for this ...



## Completeness of $\mathcal{M}_*$ via Abstract Consistency

# Def.: $\mathfrak{M}_*$ -Consistent/Inconsistent

A set of sentences  $\Phi$  is  $\mathfrak{M}_*$ -inconsistent if  $\Phi \models_{\mathfrak{M}_*} F_o$ , and  $\mathfrak{M}_*$ -consistent otherwise.

# Lemma: Saturated $\mathfrak{Acc}_*$

The class  $\Gamma_\Sigma^* := \{\Phi \subseteq \text{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{N}\mathfrak{K}_* \text{-consistent}\}$  is a saturated  $\mathfrak{Acc}_*$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

The class  $\Gamma_\Sigma^* := \{\Phi \subseteq \text{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{N}_* \text{-consistent}\}$  is a saturated  $\mathfrak{Acc}_*$ .

Proof: Obviously  $\Gamma_\Sigma^*$  is closed under subsets, since any subset of an  $\mathfrak{N}_*$ -consistent set is  $\mathfrak{N}_*$ -consistent.

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# Lemma: Saturated $\mathfrak{Acc}_*$

The class  $\Gamma_\Sigma^* := \{\Phi \subseteq \text{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{N}\mathfrak{R}_*\text{-consistent}\}$  is a saturated  $\mathfrak{Acc}_*$ .

Proof: Obviously  $\Gamma_\Sigma^*$  is closed under subsets, since any subset of an  $\mathfrak{N}\mathfrak{R}_*$ -consistent set is  $\mathfrak{N}\mathfrak{R}_*$ -consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

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$\nabla_c$  Suppose  $A, \neg A \in \Phi$ . We have  $\Phi \Vdash F_o$  by  $\mathfrak{N}\mathfrak{R}(Hyp)$  and  $\mathfrak{N}\mathfrak{R}(\neg E)$ .

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Hence  $\Phi$  is  $\mathfrak{N}\mathfrak{R}_*$ -inconsistent.

$\nabla_\beta$  Let  $\mathbf{A} \in \Phi$ ,  $\mathbf{A} =_\beta \mathbf{B}$  and  $\Phi * \mathbf{B}$  be  $\mathfrak{N}\mathfrak{R}_*$ -inconsistent.

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$\nabla_c$  Suppose  $\mathbf{A}, \neg\mathbf{A} \in \Phi$ . We have  $\Phi \Vdash F_o$  by  $\mathfrak{N}\mathfrak{R}(Hyp)$  and  $\mathfrak{N}\mathfrak{R}(\neg E)$ .

Hence  $\Phi$  is  $\mathfrak{N}\mathfrak{R}_*$ -inconsistent.

$\nabla_\beta$  Let  $\mathbf{A} \in \Phi$ ,  $\mathbf{A} =_\beta \mathbf{B}$  and  $\Phi * \mathbf{B}$  be  $\mathfrak{N}\mathfrak{R}_*$ -inconsistent. That is,  
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$\Phi * \mathbf{B} \Vdash F_o$ . By  $\mathfrak{N}\mathfrak{R}(\neg I)$ , we know  $\Phi \Vdash \neg \mathbf{B}$ . Since  $\mathbf{A} \in \Phi$ , we know  $\Phi \Vdash \mathbf{B}$  by  $\mathfrak{N}\mathfrak{R}(Hyp)$  and  $\mathfrak{N}\mathfrak{R}(\beta)$ . Using  $\mathfrak{N}\mathfrak{R}(\neg E)$ , we know  $\Phi \Vdash F_o$ .

$\nabla_\perp$  Suppose  $\neg\neg\mathbf{A} \in \Phi$  and  $\Phi * \mathbf{A}$  is  $\mathfrak{N}\mathfrak{R}_*$ -inconsistent.

# Lemma: Saturated $\mathfrak{Acc}_*$

The class  $\Gamma_\Sigma^* := \{\Phi \subseteq \text{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{N}\mathfrak{R}_*\text{-consistent}\}$  is a saturated  $\mathfrak{Acc}_*$ .

Proof: Obviously  $\Gamma_\Sigma^*$  is closed under subsets, since any subset of an  $\mathfrak{N}\mathfrak{R}_*$ -consistent set is  $\mathfrak{N}\mathfrak{R}_*$ -consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

$\nabla_c$  Suppose  $\mathbf{A}, \neg\mathbf{A} \in \Phi$ . We have  $\Phi \Vdash F_o$  by  $\mathfrak{N}\mathfrak{R}(Hyp)$  and  $\mathfrak{N}\mathfrak{R}(\neg E)$ . Hence  $\Phi$  is  $\mathfrak{N}\mathfrak{R}_*$ -inconsistent.

$\nabla_\beta$  Let  $\mathbf{A} \in \Phi$ ,  $\mathbf{A} =_\beta \mathbf{B}$  and  $\Phi * \mathbf{B}$  be  $\mathfrak{N}\mathfrak{R}_*$ -inconsistent. That is,  $\Phi * \mathbf{B} \Vdash F_o$ . By  $\mathfrak{N}\mathfrak{R}(\neg I)$ , we know  $\Phi \Vdash \neg \mathbf{B}$ . Since  $\mathbf{A} \in \Phi$ , we know  $\Phi \Vdash \mathbf{B}$  by  $\mathfrak{N}\mathfrak{R}(Hyp)$  and  $\mathfrak{N}\mathfrak{R}(\beta)$ . Using  $\mathfrak{N}\mathfrak{R}(\neg E)$ , we know  $\Phi \Vdash F_o$ .

$\nabla_\perp$  Suppose  $\neg\neg\mathbf{A} \in \Phi$  and  $\Phi * \mathbf{A}$  is  $\mathfrak{N}\mathfrak{R}_*$ -inconsistent. From  $\Phi * \mathbf{A} \Vdash F_o$  and  $\mathfrak{N}\mathfrak{R}(\neg I)$ , we have  $\Phi \Vdash \neg\mathbf{A}$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

The class  $\Gamma_\Sigma^* := \{\Phi \subseteq \text{cwff}_o(\Sigma) \mid \Phi \text{ is } \mathfrak{N}\mathfrak{R}_*\text{-consistent}\}$  is a saturated  $\mathfrak{Acc}_*$ .

Proof: Obviously  $\Gamma_\Sigma^*$  is closed under subsets, since any subset of an  $\mathfrak{N}\mathfrak{R}_*$ -consistent set is  $\mathfrak{N}\mathfrak{R}_*$ -consistent. We now check the remaining conditions. We prove all the properties by proving their contrapositive.

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$\nabla_\beta$  Let  $\mathbf{A} \in \Phi$ ,  $\mathbf{A} =_\beta \mathbf{B}$  and  $\Phi * \mathbf{B}$  be  $\mathfrak{N}\mathfrak{R}_*$ -inconsistent. That is,  $\Phi * \mathbf{B} \Vdash F_o$ . By  $\mathfrak{N}\mathfrak{R}(\neg I)$ , we know  $\Phi \Vdash \neg \mathbf{B}$ . Since  $\mathbf{A} \in \Phi$ , we know  $\Phi \Vdash \mathbf{B}$  by  $\mathfrak{N}\mathfrak{R}(Hyp)$  and  $\mathfrak{N}\mathfrak{R}(\beta)$ . Using  $\mathfrak{N}\mathfrak{R}(\neg E)$ , we know  $\Phi \Vdash F_o$ .

$\nabla_\perp$  Suppose  $\neg\neg\mathbf{A} \in \Phi$  and  $\Phi * \mathbf{A}$  is  $\mathfrak{N}\mathfrak{R}_*$ -inconsistent. From  $\Phi * \mathbf{A} \Vdash F_o$  and  $\mathfrak{N}\mathfrak{R}(\neg I)$ , we have  $\Phi \Vdash \neg\mathbf{A}$ . Since  $\neg\neg\mathbf{A} \in \Phi$ , we can apply  $\mathfrak{N}\mathfrak{R}(Hyp)$  and  $\mathfrak{N}\mathfrak{R}(\neg E)$  to obtain  $\Phi \Vdash F_o$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla$  Suppose  $(A \vee B) \in \Phi$  and both  $\Phi * A$  and  $\Phi * B$  are  $\mathfrak{NR}_*$ -inconsistent.

# Lemma: Saturated $\mathfrak{Acc}_*$

▽ Suppose  $(A \vee B) \in \Phi$  and both  $\Phi * A$  and  $\Phi * B$  are  $\mathfrak{N}\mathfrak{K}_*$ -inconsistent.  
By  $\mathfrak{N}\mathfrak{K}(Hyp)$  and  $\mathfrak{N}\mathfrak{K}(\vee E)$ , we have  $\Phi \Vdash F_\circ$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

▽ Suppose  $(A \vee B) \in \Phi$  and both  $\Phi * A$  and  $\Phi * B$  are  $\mathfrak{N}\mathfrak{K}_*$ -inconsistent.  
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# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla \vee$  Suppose  $(A \vee B) \in \Phi$  and both  $\Phi * A$  and  $\Phi * B$  are  $\mathfrak{NR}_*$ -inconsistent.  
By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(\vee E)$ , we have  $\Phi \Vdash F_o$ .
- $\nabla \wedge$  Suppose  $\neg(A \vee B) \in \Phi$  and  $\Phi * \neg A * \neg B$  is  $\mathfrak{NR}_*$ -inconsistent.

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla \vee$  Suppose  $(A \vee B) \in \Phi$  and both  $\Phi * A$  and  $\Phi * B$  are  $\mathfrak{NR}_*$ -inconsistent.  
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with  $\neg(A \vee B) \in \Phi$ , we have  $\Phi, \neg A \Vdash F_o$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_V$  Suppose  $(A \vee B) \in \Phi$  and both  $\Phi * A$  and  $\Phi * B$  are  $\mathfrak{NR}_*$ -inconsistent.  
By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(\vee E)$ , we have  $\Phi \Vdash F_o$ .
- $\nabla_A$  Suppose  $\neg(A \vee B) \in \Phi$  and  $\Phi * \neg A * \neg B$  is  $\mathfrak{NR}_*$ -inconsistent. By  
 $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_R)$ , we have  $\Phi, \neg A \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$   
with  $\neg(A \vee B) \in \Phi$ , we have  $\Phi, \neg A \Vdash F_o$ . By  $\mathfrak{NR}(Contr)$  and  
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- $\nabla_\wedge$  Suppose  $\neg(A \vee B) \in \Phi$  and  $\Phi * \neg A * \neg B$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_R)$ , we have  $\Phi, \neg A \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ , we have  $\Phi, \neg A \Vdash F_o$ . By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_L)$ , we have  $\Phi \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ ,  $\Phi$  is  $\mathfrak{NR}_*$ -inconsistent.

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- $\nabla_A$  Suppose  $\neg(A \vee B) \in \Phi$  and  $\Phi * \neg A * \neg B$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_R)$ , we have  $\Phi, \neg A \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ , we have  $\Phi, \neg A \Vdash F_o$ . By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_L)$ , we have  $\Phi \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ ,  $\Phi$  is  $\mathfrak{NR}_*$ -inconsistent.
- $\nabla_V$  Suppose  $(\Pi^\alpha G) \in \Phi$  and  $\Phi * (GA)$  is  $\mathfrak{NR}_*$ -inconsistent.

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_V$  Suppose  $(A \vee B) \in \Phi$  and both  $\Phi * A$  and  $\Phi * B$  are  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(\vee E)$ , we have  $\Phi \Vdash F_o$ .
- $\nabla_A$  Suppose  $\neg(A \vee B) \in \Phi$  and  $\Phi * \neg A * \neg B$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_R)$ , we have  $\Phi, \neg A \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ , we have  $\Phi, \neg A \Vdash F_o$ . By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_L)$ , we have  $\Phi \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ ,  $\Phi$  is  $\mathfrak{NR}_*$ -inconsistent.
- $\nabla_V$  Suppose  $(\Pi^\alpha G) \in \Phi$  and  $\Phi * (GA)$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(\neg I)$ ,  $\Phi \Vdash \neg(GA)$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_V$  Suppose  $(A \vee B) \in \Phi$  and both  $\Phi * A$  and  $\Phi * B$  are  $\mathfrak{NR}_*$ -inconsistent.  
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- $\nabla_A$  Suppose  $\neg(A \vee B) \in \Phi$  and  $\Phi * \neg A * \neg B$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_R)$ , we have  $\Phi, \neg A \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ , we have  $\Phi, \neg A \Vdash F_o$ . By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_L)$ , we have  $\Phi \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ ,  $\Phi$  is  $\mathfrak{NR}_*$ -inconsistent.
- $\nabla_V$  Suppose  $(\Pi^\alpha G) \in \Phi$  and  $\Phi * (GA)$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(\neg I)$ ,  $\Phi \Vdash \neg(GA)$ . By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(\Pi E)$ ,  $\Phi \Vdash GA$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_V$  Suppose  $(A \vee B) \in \Phi$  and both  $\Phi * A$  and  $\Phi * B$  are  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(\vee E)$ , we have  $\Phi \Vdash F_o$ .
- $\nabla_A$  Suppose  $\neg(A \vee B) \in \Phi$  and  $\Phi * \neg A * \neg B$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_R)$ , we have  $\Phi, \neg A \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ , we have  $\Phi, \neg A \Vdash F_o$ . By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_L)$ , we have  $\Phi \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ ,  $\Phi$  is  $\mathfrak{NR}_*$ -inconsistent.
- $\nabla_V$  Suppose  $(\Pi^\alpha G) \in \Phi$  and  $\Phi * (GA)$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(\neg I)$ ,  $\Phi \Vdash \neg(GA)$ . By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(\Pi E)$ ,  $\Phi \Vdash GA$ . Finally,  $\mathfrak{NR}(\neg E)$  implies  $\Phi \Vdash F_o$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_V$  Suppose  $(A \vee B) \in \Phi$  and both  $\Phi * A$  and  $\Phi * B$  are  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(\vee E)$ , we have  $\Phi \Vdash F_o$ .
- $\nabla_A$  Suppose  $\neg(A \vee B) \in \Phi$  and  $\Phi * \neg A * \neg B$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_R)$ , we have  $\Phi, \neg A \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ , we have  $\Phi, \neg A \Vdash F_o$ . By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_L)$ , we have  $\Phi \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ ,  $\Phi$  is  $\mathfrak{NR}_*$ -inconsistent.
- $\nabla_V$  Suppose  $(\Pi^\alpha G) \in \Phi$  and  $\Phi * (GA)$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(\neg I)$ ,  $\Phi \Vdash \neg(GA)$ . By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(\Pi E)$ ,  $\Phi \Vdash GA$ . Finally,  $\mathfrak{NR}(\neg E)$  implies  $\Phi \Vdash F_o$ .
- $\nabla_\exists$  Suppose  $\neg(\Pi^\alpha G) \in \Phi$ ,  $w_\alpha$  is a parameter which does not occur in  $\Phi$ , and  $\Phi * \neg(Gw)$  is  $\mathfrak{NR}_*$ -inconsistent.

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_V$  Suppose  $(A \vee B) \in \Phi$  and both  $\Phi * A$  and  $\Phi * B$  are  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(\vee E)$ , we have  $\Phi \Vdash F_o$ .
- $\nabla_A$  Suppose  $\neg(A \vee B) \in \Phi$  and  $\Phi * \neg A * \neg B$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_R)$ , we have  $\Phi, \neg A \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ , we have  $\Phi, \neg A \Vdash F_o$ . By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_L)$ , we have  $\Phi \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ ,  $\Phi$  is  $\mathfrak{NR}_*$ -inconsistent.
- $\nabla_V$  Suppose  $(\Pi^\alpha G) \in \Phi$  and  $\Phi * (GA)$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(\neg I)$ ,  $\Phi \Vdash \neg(GA)$ . By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(\Pi E)$ ,  $\Phi \Vdash GA$ . Finally,  $\mathfrak{NR}(\neg E)$  implies  $\Phi \Vdash F_o$ .
- $\nabla_\exists$  Suppose  $\neg(\Pi^\alpha G) \in \Phi$ ,  $w_\alpha$  is a parameter which does not occur in  $\Phi$ , and  $\Phi * \neg(Gw)$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Contr)$ ,  $\Phi \Vdash Gw$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_V$  Suppose  $(A \vee B) \in \Phi$  and both  $\Phi * A$  and  $\Phi * B$  are  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(\vee E)$ , we have  $\Phi \Vdash F_o$ .
- $\nabla_A$  Suppose  $\neg(A \vee B) \in \Phi$  and  $\Phi * \neg A * \neg B$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_R)$ , we have  $\Phi, \neg A \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ , we have  $\Phi, \neg A \Vdash F_o$ . By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_L)$ , we have  $\Phi \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ ,  $\Phi$  is  $\mathfrak{NR}_*$ -inconsistent.
- $\nabla_V$  Suppose  $(\Pi^\alpha G) \in \Phi$  and  $\Phi * (GA)$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(\neg I)$ ,  $\Phi \Vdash \neg(GA)$ . By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(IE)$ ,  $\Phi \Vdash GA$ . Finally,  $\mathfrak{NR}(\neg E)$  implies  $\Phi \Vdash F_o$ .
- $\nabla_\exists$  Suppose  $\neg(\Pi^\alpha G) \in \Phi$ ,  $w_\alpha$  is a parameter which does not occur in  $\Phi$ , and  $\Phi * \neg(Gw)$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Contr)$ ,  $\Phi \Vdash Gw$ . By  $\mathfrak{NR}(II)^w$ ,  $\Phi \Vdash (\Pi^\alpha G)$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_V$  Suppose  $(A \vee B) \in \Phi$  and both  $\Phi * A$  and  $\Phi * B$  are  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(\vee E)$ , we have  $\Phi \Vdash F_o$ .
- $\nabla_A$  Suppose  $\neg(A \vee B) \in \Phi$  and  $\Phi * \neg A * \neg B$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_R)$ , we have  $\Phi, \neg A \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ , we have  $\Phi, \neg A \Vdash F_o$ . By  $\mathfrak{NR}(Contr)$  and  $\mathfrak{NR}(\vee I_L)$ , we have  $\Phi \Vdash A \vee B$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(A \vee B) \in \Phi$ ,  $\Phi$  is  $\mathfrak{NR}_*$ -inconsistent.
- $\nabla_V$  Suppose  $(\Pi^\alpha G) \in \Phi$  and  $\Phi * (GA)$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(\neg I)$ ,  $\Phi \Vdash \neg(GA)$ . By  $\mathfrak{NR}(Hyp)$  and  $\mathfrak{NR}(IE)$ ,  $\Phi \Vdash GA$ . Finally,  $\mathfrak{NR}(\neg E)$  implies  $\Phi \Vdash F_o$ .
- $\nabla_\exists$  Suppose  $\neg(\Pi^\alpha G) \in \Phi$ ,  $w_\alpha$  is a parameter which does not occur in  $\Phi$ , and  $\Phi * \neg(Gw)$  is  $\mathfrak{NR}_*$ -inconsistent. By  $\mathfrak{NR}(Contr)$ ,  $\Phi \Vdash Gw$ . By  $\mathfrak{NR}(II)^w$ ,  $\Phi \Vdash (\Pi^\alpha G)$ . Using  $\mathfrak{NR}(\neg E)$  with  $\neg(\Pi^\alpha G) \in \Phi$ ,  $\Phi$  is  $\mathfrak{NR}_*$ -inconsistent.

# Lemma: Saturated $\mathfrak{Acc}_*$

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{NR}_*$ -inconsistent.

# Lemma: Saturated $\mathfrak{A}\text{cc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{M}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{M}_*$ -inconsistent.

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{M}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{M}_*$ -inconsistent. Using  $\mathfrak{M}(\neg I)$ , we know  $\Phi \Vdash \neg A$  and  $\Phi \Vdash \neg\neg A$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{N}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{N}_*$ -inconsistent. Using  $\mathfrak{N}(\neg I)$ , we know  $\Phi \Vdash \neg A$  and  $\Phi \Vdash \neg\neg A$ . By  $\mathfrak{N}(\neg E)$ , we have  $\Phi \Vdash F_\circ$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{N}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{N}_*$ -inconsistent. Using  $\mathfrak{N}(\neg I)$ , we know  $\Phi \Vdash \neg A$  and  $\Phi \Vdash \neg\neg A$ . By  $\mathfrak{N}(\neg E)$ , we have  $\Phi \Vdash F_\circ$ .

Thus we have shown that  $\Gamma_\Sigma^\beta$  is saturated and in  $\mathfrak{Acc}_\beta$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{N}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{N}_*$ -inconsistent. Using  $\mathfrak{N}(\neg I)$ , we know  $\Phi \Vdash \neg A$  and  $\Phi \Vdash \neg\neg A$ . By  $\mathfrak{N}(\neg E)$ , we have  $\Phi \Vdash F_\circ$ .

Thus we have shown that  $\Gamma_\Sigma^\beta$  is saturated and in  $\mathfrak{Acc}_\beta$ .

Now let us check the conditions for the additional properties  $\eta$ ,  $\xi$ ,  $f$ , and  $b$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{N}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{N}_*$ -inconsistent. Using  $\mathfrak{N}(\neg I)$ , we know  $\Phi \Vdash \neg A$  and  $\Phi \Vdash \neg\neg A$ . By  $\mathfrak{N}(\neg E)$ , we have  $\Phi \Vdash F_\circ$ .

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# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{N}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{N}_*$ -inconsistent. Using  $\mathfrak{N}(\neg I)$ , we know  $\Phi \Vdash \neg A$  and  $\Phi \Vdash \neg\neg A$ . By  $\mathfrak{N}(\neg E)$ , we have  $\Phi \Vdash F_\circ$ .

Thus we have shown that  $\Gamma_\Sigma^\beta$  is saturated and in  $\mathfrak{Acc}_\beta$ .

Now let us check the conditions for the additional properties  $\eta$ ,  $\xi$ ,  $f$ , and  $b$ .

$\nabla_\eta$  If  $*$  includes  $\eta$ , then the proof proceeds as in  $\nabla_\beta$  above, but with the rule  $\mathfrak{N}(\eta)$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{N}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{N}_*$ -inconsistent. Using  $\mathfrak{N}(\neg I)$ , we know  $\Phi \Vdash \neg A$  and  $\Phi \Vdash \neg\neg A$ . By  $\mathfrak{N}(\neg E)$ , we have  $\Phi \Vdash F_\circ$ .

Thus we have shown that  $\Gamma_\Sigma^\beta$  is saturated and in  $\mathfrak{Acc}_\beta$ .

Now let us check the conditions for the additional properties  $\eta$ ,  $\xi$ ,  $f$ , and  $b$ .

$\nabla_\eta$  If  $*$  includes  $\eta$ , then the proof proceeds as in  $\nabla_\beta$  above, but with the rule  $\mathfrak{N}(\eta)$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{N}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{N}_*$ -inconsistent. Using  $\mathfrak{N}(\neg I)$ , we know  $\Phi \Vdash \neg A$  and  $\Phi \Vdash \neg\neg A$ . By  $\mathfrak{N}(\neg E)$ , we have  $\Phi \Vdash F_\circ$ .

Thus we have shown that  $\Gamma_\Sigma^\beta$  is saturated and in  $\mathfrak{Acc}_\beta$ .

Now let us check the conditions for the additional properties  $\eta$ ,  $\xi$ ,  $f$ , and  $b$ .

$\nabla_\eta$  If  $*$  includes  $\eta$ , then the proof proceeds as in  $\nabla_\beta$  above, but with the rule  $\mathfrak{N}(\eta)$ .

$\nabla_\xi$  Suppose  $*$  includes  $\xi$ ,  $\neg(\lambda X.M \doteq^{\alpha \rightarrow \beta} \lambda X.N) \in \Phi$ , and  $\Phi * \neg([w/X]M \doteq^\beta [w/X]N)$  is  $\mathfrak{N}_*$ -inconsistent for some parameter  $w_\alpha$  which does not occur in any sentence of  $\Phi$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * \mathbf{A}$  and  $\Phi * \neg \mathbf{A}$  be  $\mathfrak{N}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{N}_*$ -inconsistent. Using  $\mathfrak{N}(\neg I)$ , we know  $\Phi \Vdash \neg \mathbf{A}$  and  $\Phi \Vdash \neg \neg \mathbf{A}$ . By  $\mathfrak{N}(\neg E)$ , we have  $\Phi \Vdash \mathbf{F}_o$ .

Thus we have shown that  $\Gamma_\Sigma^\beta$  is saturated and in  $\mathfrak{Acc}_\beta$ .

Now let us check the conditions for the additional properties  $\eta$ ,  $\xi$ ,  $f$ , and  $b$ .

$\nabla_\eta$  If  $*$  includes  $\eta$ , then the proof proceeds as in  $\nabla_\beta$  above, but with the rule  $\mathfrak{N}(\eta)$ .

$\nabla_\xi$  Suppose  $*$  includes  $\xi$ ,  $\neg(\lambda X.M \doteq^{\alpha \rightarrow \beta} \lambda X.N) \in \Phi$ , and  $\Phi * \neg([w/X]M \doteq^\beta [w/X]N)$  is  $\mathfrak{N}_*$ -inconsistent for some parameter  $w_\alpha$  which does not occur in any sentence of  $\Phi$ . By  $\mathfrak{N}(Contr)$ , we have  $\Phi \Vdash ([w/X]M \doteq^\beta [w/X]N)$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{N}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{N}_*$ -inconsistent. Using  $\mathfrak{N}(\neg I)$ , we know  $\Phi \Vdash \neg A$  and  $\Phi \Vdash \neg\neg A$ . By  $\mathfrak{N}(\neg E)$ , we have  $\Phi \Vdash F_\circ$ .

Thus we have shown that  $\Gamma_\Sigma^\beta$  is saturated and in  $\mathfrak{Acc}_\beta$ .

Now let us check the conditions for the additional properties  $\eta$ ,  $\xi$ ,  $f$ , and  $b$ .

$\nabla_\eta$  If  $*$  includes  $\eta$ , then the proof proceeds as in  $\nabla_\beta$  above, but with the rule  $\mathfrak{N}(\eta)$ .

$\nabla_\xi$  Suppose  $*$  includes  $\xi$ ,  $\neg(\lambda X.M \doteq^{\alpha \rightarrow \beta} \lambda X.N) \in \Phi$ , and  $\Phi * \neg([w/X]M \doteq^\beta [w/X]N)$  is  $\mathfrak{N}_*$ -inconsistent for some parameter  $w_\alpha$  which does not occur in any sentence of  $\Phi$ . By  $\mathfrak{N}(Contr)$ , we have  $\Phi \Vdash ([w/X]M \doteq^\beta [w/X]N)$ . By  $\mathfrak{N}(\beta)$ , we have  $\Phi \Vdash ((\lambda X.M \doteq^\beta N)w)$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{N}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{N}_*$ -inconsistent. Using  $\mathfrak{N}(\neg I)$ , we know  $\Phi \Vdash \neg A$  and  $\Phi \Vdash \neg\neg A$ . By  $\mathfrak{N}(\neg E)$ , we have  $\Phi \Vdash F_\circ$ .

Thus we have shown that  $\Gamma_\Sigma^\beta$  is saturated and in  $\mathfrak{Acc}_\beta$ .

Now let us check the conditions for the additional properties  $\eta$ ,  $\xi$ ,  $f$ , and  $b$ .

$\nabla_\eta$  If  $*$  includes  $\eta$ , then the proof proceeds as in  $\nabla_\beta$  above, but with the rule  $\mathfrak{N}(\eta)$ .

$\nabla_\xi$  Suppose  $*$  includes  $\xi$ ,  $\neg(\lambda X.M \doteq^{\alpha \rightarrow \beta} \lambda X.N) \in \Phi$ , and  $\Phi * \neg([w/X]M \doteq^\beta [w/X]N)$  is  $\mathfrak{N}_*$ -inconsistent for some parameter  $w_\alpha$  which does not occur in any sentence of  $\Phi$ . By  $\mathfrak{N}(Contr)$ , we have  $\Phi \Vdash ([w/X]M \doteq^\beta [w/X]N)$ . By  $\mathfrak{N}(\beta)$ , we have  $\Phi \Vdash ((\lambda X.M \doteq^\beta N)w)$ . By  $\mathfrak{N}(III)$ ,  $\Phi \Vdash (\forall X.M \doteq^\beta N)$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{N}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{N}_*$ -inconsistent. Using  $\mathfrak{N}(\neg I)$ , we know  $\Phi \Vdash \neg A$  and  $\Phi \Vdash \neg\neg A$ . By  $\mathfrak{N}(\neg E)$ , we have  $\Phi \Vdash F_\circ$ .

Thus we have shown that  $\Gamma_\Sigma^\beta$  is saturated and in  $\mathfrak{Acc}_\beta$ .

Now let us check the conditions for the additional properties  $\eta$ ,  $\xi$ ,  $f$ , and  $b$ .

$\nabla_\eta$  If  $*$  includes  $\eta$ , then the proof proceeds as in  $\nabla_\beta$  above, but with the rule  $\mathfrak{N}(\eta)$ .

$\nabla_\xi$  Suppose  $*$  includes  $\xi$ ,  $\neg(\lambda X.M \doteq^{\alpha \rightarrow \beta} \lambda X.N) \in \Phi$ , and  $\Phi * \neg([w/X]M \doteq^\beta [w/X]N)$  is  $\mathfrak{N}_*$ -inconsistent for some parameter  $w_\alpha$  which does not occur in any sentence of  $\Phi$ . By  $\mathfrak{N}(Contr)$ , we have  $\Phi \Vdash ([w/X]M \doteq^\beta [w/X]N)$ . By  $\mathfrak{N}(\beta)$ , we have  $\Phi \Vdash ((\lambda X.M \doteq^\beta N)w)$ . By  $\mathfrak{N}(II)$ ,  $\Phi \Vdash (\forall X.M \doteq^\beta N)$ . By  $\mathfrak{N}(\xi)$ ,  $\Phi \Vdash (\lambda X.M \doteq^{\alpha \rightarrow \beta} \lambda X.N)$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_{\text{sat}}$  Let  $\Phi * A$  and  $\Phi * \neg A$  be  $\mathfrak{N}_*$ -inconsistent. We show that  $\Phi$  is  $\mathfrak{N}_*$ -inconsistent. Using  $\mathfrak{N}(\neg I)$ , we know  $\Phi \Vdash \neg A$  and  $\Phi \Vdash \neg\neg A$ . By  $\mathfrak{N}(\neg E)$ , we have  $\Phi \Vdash F_\circ$ .

Thus we have shown that  $\Gamma_\Sigma^\beta$  is saturated and in  $\mathfrak{Acc}_\beta$ .

Now let us check the conditions for the additional properties  $\eta$ ,  $\xi$ ,  $f$ , and  $b$ .

$\nabla_\eta$  If  $*$  includes  $\eta$ , then the proof proceeds as in  $\nabla_\beta$  above, but with the rule  $\mathfrak{N}(\eta)$ .

$\nabla_\xi$  Suppose  $*$  includes  $\xi$ ,  $\neg(\lambda X.M \doteq^{\alpha \rightarrow \beta} \lambda X.N) \in \Phi$ , and  $\Phi * \neg([w/X]M \doteq^\beta [w/X]N)$  is  $\mathfrak{N}_*$ -inconsistent for some parameter  $w_\alpha$  which does not occur in any sentence of  $\Phi$ . By  $\mathfrak{N}(Contr)$ , we have  $\Phi \Vdash ([w/X]M \doteq^\beta [w/X]N)$ . By  $\mathfrak{N}(\beta)$ , we have  $\Phi \Vdash ((\lambda X.M \doteq^\beta N)w)$ . By  $\mathfrak{N}(II)$ ,  $\Phi \Vdash (\forall X.M \doteq^\beta N)$ . By  $\mathfrak{N}(\xi)$ ,  $\Phi \Vdash (\lambda X.M \doteq^{\alpha \rightarrow \beta} \lambda X.N)$ . By  $\mathfrak{N}(\neg E)$ ,  $\Phi$  is  $\mathfrak{N}_*$ -inconsistent.

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_f$  This case is analogous to the previous one, generalizing  $\lambda X.M \doteq \lambda X.N$  to arbitrary  $G \doteq H$  and using the extensionality rule  $\mathfrak{M}(f)$  instead of  $\mathfrak{M}(\xi)$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

$\nabla_f$  This case is analogous to the previous one, generalizing  $\lambda X.M \doteq \lambda X.N$  to arbitrary  $G \doteq H$  and using the extensionality rule  $\mathfrak{M}(f)$  instead of  $\mathfrak{M}(\xi)$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_f$  This case is analogous to the previous one, generalizing  $\lambda X.M \doteq \lambda X.N$  to arbitrary  $G \doteq H$  and using the extensionality rule  $\mathfrak{ER}(f)$  instead of  $\mathfrak{ER}(\xi)$ .
- $\nabla_b$  Suppose  $*$  includes  $b$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_f$  This case is analogous to the previous one, generalizing  $\lambda X.M \doteq \lambda X.N$  to arbitrary  $G \doteq H$  and using the extensionality rule  $\mathfrak{M}(\mathfrak{f})$  instead of  $\mathfrak{M}(\xi)$ .
- $\nabla_b$  Suppose  $*$  includes  $b$ . Assume that  $\neg(A \doteq^{\circ} B) \in \Phi$  but both  $\Phi * \neg A * B \notin \Gamma_{\Sigma}^*$  and  $\Phi * A * \neg B \notin \Gamma_{\Sigma}^*$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_f$  This case is analogous to the previous one, generalizing  $\lambda X.M \doteq \lambda X.N$  to arbitrary  $G \doteq H$  and using the extensionality rule  $\mathfrak{M}(f)$  instead of  $\mathfrak{M}(\xi)$ .
- $\nabla_b$  Suppose  $*$  includes  $b$ . Assume that  $\neg(A \doteq^{\circ} B) \in \Phi$  but both  $\Phi * \neg A * B \notin \Gamma_{\Sigma}^*$  and  $\Phi * A * \neg B \notin \Gamma_{\Sigma}^*$ . So both are  $\mathfrak{M}_*$ -inconsistent and we have  $\Phi * A \Vdash B$  and  $\Phi * B \Vdash A$  by  $\mathfrak{M}(Contr)$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_f$  This case is analogous to the previous one, generalizing  $\lambda X.M \doteq \lambda X.N$  to arbitrary  $G \doteq H$  and using the extensionality rule  $\mathfrak{M}(f)$  instead of  $\mathfrak{M}(\xi)$ .
- $\nabla_b$  Suppose  $*$  includes  $b$ . Assume that  $\neg(A \doteq^{\circ} B) \in \Phi$  but both  $\Phi * \neg A * B \notin \Gamma_{\Sigma}^*$  and  $\Phi * A * \neg B \notin \Gamma_{\Sigma}^*$ . So both are  $\mathfrak{M}_*$ -inconsistent and we have  $\Phi * A \Vdash B$  and  $\Phi * B \Vdash A$  by  $\mathfrak{M}(Contr)$ . By  $\mathfrak{M}(b)$ , we have  $\Phi \Vdash (A \doteq^{\circ} B)$ .

# Lemma: Saturated $\mathfrak{Acc}_*$

- $\nabla_f$  This case is analogous to the previous one, generalizing  $\lambda X.M \doteq \lambda X.N$  to arbitrary  $G \doteq H$  and using the extensionality rule  $\mathfrak{M}(f)$  instead of  $\mathfrak{M}(\xi)$ .
- $\nabla_b$  Suppose  $*$  includes  $b$ . Assume that  $\neg(A \doteq^{\circ} B) \in \Phi$  but both  $\Phi * \neg A * B \notin \Gamma_{\Sigma}^*$  and  $\Phi * A * \neg B \notin \Gamma_{\Sigma}^*$ . So both are  $\mathfrak{M}_*$ -inconsistent and we have  $\Phi * A \Vdash B$  and  $\Phi * B \Vdash A$  by  $\mathfrak{M}(Contr)$ . By  $\mathfrak{M}(b)$ , we have  $\Phi \Vdash (A \doteq^{\circ} B)$ . Since  $\neg(A \doteq^{\circ} B) \in \Phi$ ,  $\Phi$  is  $\mathfrak{M}_*$ -inconsistent.

# Thm.: Henkin's Theorem for $\mathfrak{M}_*$

Let  $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ . Every sufficiently  $\Sigma$ -pure  $\mathfrak{M}_*$ -consistent set of sentences has an  $\mathfrak{M}_*(\Sigma)$ -model.

Proof:

# Thm.: Henkin's Theorem for $\mathfrak{M}_*$

Let  $* \in \{\beta, \beta\eta, \beta\xi, \beta f, \beta b, \beta\eta b, \beta\xi b, \beta f b\}$ . Every sufficiently  $\Sigma$ -pure  $\mathfrak{M}_*$ -consistent set of sentences has an  $\mathfrak{M}_*(\Sigma)$ -model.

Proof: Let  $\Phi$  be a sufficiently  $\Sigma$ -pure  $\mathfrak{M}_*$ -consistent set of sentences.

# Thm.: Henkin's Theorem for $\mathfrak{N}_*$

Let  $* \in \{\beta, \beta\eta, \beta\xi, \beta f, \beta b, \beta\eta b, \beta\xi b, \beta f b\}$ . Every sufficiently  $\Sigma$ -pure  $\mathfrak{N}_*$ -consistent set of sentences has an  $\mathfrak{M}_*(\Sigma)$ -model.

Proof: Let  $\Phi$  be a sufficiently  $\Sigma$ -pure  $\mathfrak{N}_*$ -consistent set of sentences. By the previous lemma we know that the class of sets of  $\mathfrak{N}_*$ -consistent sentences constitute a saturated  $\mathfrak{Acc}_*$ ,

# Thm.: Henkin's Theorem for $\mathfrak{N}_*$

Let  $* \in \{\beta, \beta\eta, \beta\xi, \beta f, \beta b, \beta\eta b, \beta\xi b, \beta f b\}$ . Every sufficiently  $\Sigma$ -pure  $\mathfrak{N}_*$ -consistent set of sentences has an  $\mathfrak{M}_*(\Sigma)$ -model.

Proof: Let  $\Phi$  be a sufficiently  $\Sigma$ -pure  $\mathfrak{N}_*$ -consistent set of sentences. By the previous lemma we know that the class of sets of  $\mathfrak{N}_*$ -consistent sentences constitute a saturated  $\mathfrak{Acc}_*$ , thus the Model Existence Theorem guarantees an  $\mathfrak{M}_*(\Sigma)$  model for  $\Phi$ .

# Thm.: Completeness Theorem for $\mathfrak{M}_*$

Let  $\Phi$  be a sufficiently  $\Sigma$ -pure set of sentences,  $A$  be a sentence, and  $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ . If  $A$  is valid in all models  $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$  that satisfy  $\Phi$ , then  $\Phi \Vdash_{\mathfrak{M}_*} A$ .

Proof:

# Thm.: Completeness Theorem for $\mathfrak{M}_*$

Let  $\Phi$  be a sufficiently  $\Sigma$ -pure set of sentences,  $\mathbf{A}$  be a sentence, and  $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ . If  $\mathbf{A}$  is valid in all models  $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$  that satisfy  $\Phi$ , then  $\Phi \Vdash_{\mathfrak{M}_*} \mathbf{A}$ .

Proof: Let  $\mathbf{A}$  be given such that  $\mathbf{A}$  is valid in all  $\mathfrak{M}_*(\Sigma)$  models that satisfy  $\Phi$ .

# Thm.: Completeness Theorem for $\mathfrak{M}_*$

Let  $\Phi$  be a sufficiently  $\Sigma$ -pure set of sentences,  $\mathbf{A}$  be a sentence, and  $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ . If  $\mathbf{A}$  is valid in all models  $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$  that satisfy  $\Phi$ , then  $\Phi \Vdash_{\mathfrak{M}_*} \mathbf{A}$ .

Proof: Let  $\mathbf{A}$  be given such that  $\mathbf{A}$  is valid in all  $\mathfrak{M}_*(\Sigma)$  models that satisfy  $\Phi$ . So,  $\Phi * \neg\mathbf{A}$  is unsatisfiable in  $\mathfrak{M}_*(\Sigma)$ . Since only finitely many constants occur in  $\neg\mathbf{A}$ ,  $\Phi * \neg\mathbf{A}$  is sufficiently  $\Sigma$ -pure.

# Thm.: Completeness Theorem for $\mathfrak{M}_*$

Let  $\Phi$  be a sufficiently  $\Sigma$ -pure set of sentences,  $\mathbf{A}$  be a sentence, and  $* \in \{\beta, \beta\eta, \beta\xi, \beta\mathfrak{f}, \beta\mathfrak{b}, \beta\eta\mathfrak{b}, \beta\xi\mathfrak{b}, \beta\mathfrak{f}\mathfrak{b}\}$ . If  $\mathbf{A}$  is valid in all models  $\mathcal{M} \in \mathfrak{M}_*(\Sigma)$  that satisfy  $\Phi$ , then  $\Phi \Vdash_{\mathfrak{M}_*} \mathbf{A}$ .

Proof: Let  $\mathbf{A}$  be given such that  $\mathbf{A}$  is valid in all  $\mathfrak{M}_*(\Sigma)$  models that satisfy  $\Phi$ . So,  $\Phi * \neg\mathbf{A}$  is unsatisfiable in  $\mathfrak{M}_*(\Sigma)$ . Since only finitely many constants occur in  $\neg\mathbf{A}$ ,  $\Phi * \neg\mathbf{A}$  is sufficiently  $\Sigma$ -pure. So,  $\Phi * \neg\mathbf{A}$  must be  $\mathfrak{M}_*$ -inconsistent by Henkin's theorem above.

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We can use the completeness theorems obtained so far to prove a compactness theorem for our semantics:

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# Approaches to Higher-Order Resolution

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- **multiple step derivations** in calculus  $R$  are abbreviated by  $\Phi_1 \vdash_R \Phi_k$  (or  $C_1 \vdash_R C_k$ )

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Since we need both imitation and projection bindings for higher-order unification, we collect them in the set of **general bindings for  $h$  and  $\alpha$**  ( $\mathcal{AB}_\alpha^h := \{G_\alpha^h\} \cup \{G_\alpha^j \mid j \leq l\}$ ).

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# Def.: Unification Constraints

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- thus, the Skolem terms only serve as descriptions of the existential witnesses and never appear as functions proper
- without this additional restriction the calculi do not really become unsound, but one can prove an instance of the axiom of choice ([Andrews73]), which we want to treat as an optional axiom for the resolution calculi presented here



# Approaches to Higher-Order Resolution: $\mathcal{R}$

# Andrews' Higher-Order Resolution $\mathcal{R}$

We present and discuss Andrews' higher-order resolution calculus [Andrews71] in our uniform notation; we call this calculus  $\mathcal{R}$

## $\lambda$ -Conversion

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- we omit explicit rules for  $\alpha$ - and  $\beta$ -convertibility and instead treat them implicitly, i.e. we assume that the presented rules operate on input and generate output in  $\beta$ -normal form and we automatically identify terms which differ only with respect to the names of bound variables

# Andrews' Higher-Order Resolution $\mathcal{R}$

## Clause Normalisation

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# Andrews' Higher-Order Resolution $\mathcal{R}$

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- negation elimination:

$$\frac{\mathbf{C} \vee [\neg A]^T}{\mathbf{C} \vee [A]^F} \quad \neg^T \quad \frac{\mathbf{C} \vee [\neg A]^F}{\mathbf{C} \vee [A]^T} \quad \neg^F$$

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- conjunction/disjunction elimination:

$$\frac{\mathbf{C} \vee [A \vee B]^T}{\mathbf{C} \vee [A]^T \vee [B]^T} \vee^T$$

$$\frac{\mathbf{C} \vee [A \vee B]^F}{\mathbf{C} \vee [A]^F} \vee^F_l$$

$$\frac{\mathbf{C} \vee [A \vee B]^F}{\mathbf{C} \vee [B]^F} \vee^F_r$$

# Andrews' Higher-Order Resolution $\mathcal{R}$

Clause Normalisation (contd.)

- existential/universal elimination:

$$\frac{\mathbf{C} \vee [\Pi^\alpha \mathbf{A}]^T}{\mathbf{C} \vee [\mathbf{A} \, \mathbf{x}_\alpha]^T} \quad \Pi^T \quad \frac{\mathbf{C} \vee [\Pi^\alpha \mathbf{A}]^F}{\mathbf{C} \vee [\mathbf{A} \, \mathbf{s}_\alpha]^F} \quad \Pi^F$$

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- additionally Andrews presents rules addressing commutativity and associativity of the  $\vee$ -operator connecting the clauses literals; we have already mentioned the implicit treatment of these aspects here
- we refer with  $\text{Cnf}(\mathbf{A})$  to the set of clauses obtained from formula  $\mathbf{A}$  by exhaustive clause normalisation

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## Resolution & Factorisation

- Instead of a resolution and a factorisation rule — which work in connection with unification — Andrews presents a simplification and a cut rule. The cut rule is only applicable to clauses with two complementary literals which have identical atoms. Similarly Sim is defined only for clauses with two identical literals. In order to generate identical literal atoms during the refutation process these two rules have to be combined with the substitution rule Sub presented below.

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## Unification & Primitive Substitution

- As higher-order unification was still an open problem in 1971 calculus  $\mathcal{R}$  employs the British museum method instead, i.e. it provides a substitution rule that allows to blindly instantiate free variables by arbitrary terms. As the instantiated terms may contain logical constants, instantiation of variables in proper clauses may lead to pre-clauses, which must be normalised again with the clause normalisation rules.

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- Extensionality axioms

$$\text{EXT}_{\alpha \rightarrow \beta}^{\dot{=}} : \forall F_{\alpha \rightarrow \beta} \forall G_{\alpha \rightarrow \beta} (\forall X_{\beta} F X \dot{=} G X) \Rightarrow F \dot{=} G$$

$$\text{EXT}_o^{\dot{=}} : \forall A_o \forall B_o (A \Leftrightarrow B) \Rightarrow A \dot{=}^o B$$

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## Extensionality Treatment (contd.)

- The extensionality clauses derived from the extensionality axioms have the following form (note the many free variables, especially at literal head position, that are introduced into the search space – they heavily increase the amount of blind search in any attempt to automate the calculus):

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$p_{\beta \rightarrow o}$ ,  $s_\alpha$  are Skolem terms and  $A_o$ ,  $B_o$ ,  $P_{o \rightarrow o}$ ,  $Q_{(\alpha \rightarrow \beta) \rightarrow o}$  are new free variables.

# Andrews' Higher-Order Resolution $\mathcal{R}$



## Proof Search

- initially the proof problem is negated and normalised

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- the proof search can be graphically illustrated as follows:

ext.  
axioms

proof search & blind variable instantiation

# Andrews' Higher-Order Resolution $\mathcal{R}$

## Completeness

- [Andrews71] gives a completeness proof for calculus  $\mathcal{R}$  with respect to the semantical notion of  $\vee$ -complexes (corresponds to our weakest model class  $\mathfrak{M}_\beta(\Sigma)$ )

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## Completeness

- [Andrews71] gives a completeness proof for calculus  $\mathcal{R}$  with respect to the semantical notion of V-complexes (corresponds to our weakest model class  $\mathfrak{M}_\beta(\Sigma)$ )
- as the extensionality principles are not valid in this rather weak semantical structures, the extensionality axioms are not needed in this completeness proof
- Theorem: (V-completeness of  $\mathcal{R}$ ) The calculus  $\mathcal{R}$  is (sound and) complete with respect to the notion of V-complexes.

Proof: [Andrews71].

# Andrews' Higher-Order Resolution $\mathcal{R}$



## Henkin Completeness

- We can also prove Henkin completeness of calculus  $\mathcal{R}$ .

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Proof: exercise

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# Example Proofs

Exercise: How are the following theorems proved in calculus  $\mathcal{R}$ ?

- Leibniz equality and  $\eta$ -equality:

$$f_{\iota \rightarrow \iota} \doteq \lambda X_\iota . f \; X$$

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- The set of all red balls equals the set of all balls that are red:  
 $\{X | \text{red } X \wedge \text{ball } X\} = \{X | \text{ball } X \wedge \text{red } X\}$ . This problem can be encoded as

$$(\lambda X_\iota . \text{red } X \wedge \text{ball } X) = (\lambda X_\iota . \text{ball } X \wedge \text{red } X)$$

# Example Proofs

Exercise: How are the following theorems proved in calculus  $\mathcal{R}$ ?

- All unary logical operators  $O_{o \rightarrow o}$  which map the propositions  $a$  and  $b$  to  $\top$  consequently also map  $a \wedge b$  to  $\top$ :

$$\forall O_{o \rightarrow o} \cdot (O a_o) \wedge (O b_o) \Rightarrow (O (a_o \wedge b_o))$$

# Example Proofs

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Exercise: How are the following theorems proved in calculus  $\mathcal{R}$ ?

- In Henkin semantics the domain  $\mathcal{D}_o$  of all Booleans contains exactly the truth values  $\perp$  and  $\top$ . Consequently the domain of all mappings from Booleans to Booleans contains exactly four elements. And because of the requirement, that the function domains in Henkin models must be rich enough such that every term has a denotation, it follows that  $\mathcal{D}_{o \rightarrow o}$  contains exactly the pairwise distinct denotations of the following four terms:  $\lambda X_o.X_o$ ,  $\lambda X_o.\neg X_o$ ,  $\lambda X_o.\perp$ , and  $\lambda X_o.\top$ . This theorem can be formulated as follows (where  $f_{o \rightarrow o}$  is a constant):

$$(f = \lambda X_o.X_o) \vee (f = \lambda X_o.\neg X_o) \vee (f = \lambda X_o.\perp) \vee (f = \lambda X_o.\top)$$



# Approaches to Higher-Order Resolution: $\mathcal{CR}$

# Huet's Constrained Resolution $\mathcal{CR}$

We transform Huet's constrained resolution approach [Huet72,Huet73] in our uniform notation. The calculus here is the unsorted fragment of the variant of Huet's approach as presented in [Kohlhase94]. In the remainder of this paper we refer to this calculus with  $\mathcal{CR}$ .

## $\lambda$ -Conversion

- Calculus  $\mathcal{CR}$  assumes that terms, literals, and clauses are implicitly reduced to  $\beta$ -normal form.

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## $\lambda$ -Conversion

- Calculus  $\mathcal{CR}$  assumes that terms, literals, and clauses are implicitly reduced to  $\beta$ -normal form.
- Furthermore, we assume that  $\alpha$ -equality is treated implicitly, i.e. we identify all terms that differ only with respect to the names of bound variables.

# Huet's Constrained Resolution $\mathcal{CR}$

## Clause Normalisation

- [Huet72] does not explicitly present clause normalisation rules but assumes that they are given. Here we employ the rules  $\neg^T$ ,  $\neg^F$ ,  $\vee^T$ ,  $\vee_l^F$ ,  $\vee_r^F$ ,  $\Pi^T$ , and  $\Pi^F$  as already defined for calculus  $\mathcal{R}$  before.

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- As first-order unification is decidable and unitary it can be employed as a strong filter in first-order resolution [Robinson65].

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- Unfortunately higher-order unification is not decidable (cf. [Lucchesi72, Huet73, Goldfarb81]) and thus it can not be applied in the sense of a terminating side computation in higher-order theorem proving.

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## Resolution & Factorisation

- As first-order unification is decidable and unitary it can be employed as a strong filter in first-order resolution [Robinson65].
- Unfortunately higher-order unification is not decidable (cf. [Lucchesi72, Huet73, Goldfarb81]) and thus it can not be applied in the sense of a terminating side computation in higher-order theorem proving.
- Huet therefore suggests in [Huet72, Huet73] to delay the unification process and to explicitly encode unification problems occurring during the refutation search as unification constraints.

# Huet's Constrained Resolution $\mathcal{CR}$

## Resolution & Factorisation (contd.)

- In his original approach Huet presented a hyper-resolution rule which simultaneously resolves on the resolution literals  $A^1, \dots, A^n$  ( $1 \leq n$ ) and  $B^1, \dots, B^m$  ( $1 \leq m$ ) of two given clauses and adds the unification constraint  $[ \neq? (A^1, \dots, A^n, B^1, \dots, B^m) ]$  to the resolvent:

$$\frac{[A^1]^\mu \vee \dots \vee [A^n]^\mu \vee C \quad [B^1]^\nu \vee \dots \vee [B^m]^\nu \vee D}{C \vee D \vee [ \neq? (A^1, \dots, A^n, B^1, \dots, B^m) ]} \text{ Hres}$$

(where  $\mu \neq \nu$ ).

# Huet's Constrained Resolution $\mathcal{CR}$

## Resolution & Factorisation (contd.)

- In order to ease the comparison with the two other approaches discussed in this paper we instead employ a resolution rule Res and a factorisation rule Fac.

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- Constrained factorisation:

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# Huet's Constrained Resolution $\mathcal{CR}$

## Resolution & Factorisation (contd.)

- One can easily prove by induction on  $n + m$  that each proof step applying rule Hres can be replaced by a corresponding derivation employing Res and Fac.

# Huet's Constrained Resolution $\mathcal{CR}$

## Resolution & Factorisation (contd.)

- One can easily prove by induction on  $n + m$  that each proof step applying rule Hres can be replaced by a corresponding derivation employing Res and Fac.
- For a formal proof note that the unification constraint  $[\neq^? (A^1, \dots, A^n, B^1, \dots, B^m)]$  is equivalent to  $[A^1 \neq^? A^2] \vee [A^2 \neq^? A^3] \vee \dots \vee [A^{n-1} \neq^? A^n] \vee [A^n \neq^? B^1] \vee [B^1 \neq^? B^2] \vee [B^2 \neq^? B^3] \vee \dots \vee [B^{n-1} \neq^? B^n]$ .

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## Unification & Splitting

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- Decomposition

$$\frac{\mathbf{C} \vee [h\overline{\mathbf{U}^n} \neq? h\overline{\mathbf{V}^n}]}{\mathbf{C} \vee [\mathbf{U}^1 \neq? \mathbf{V}^1] \vee \dots \vee [\mathbf{U}^n \neq? \mathbf{V}^n]} \text{Dec}$$

# Huet's Constrained Resolution $\mathcal{CR}$

## Unification & Splitting (contd.)

- Elimination of  $\lambda$ -binders:
- (weak functional extensionality)

$$\frac{\mathbf{C} \vee [\mathbf{M}_{\alpha \rightarrow \beta} \not\equiv? \mathbf{N}_{\alpha \rightarrow \beta}]}{\mathbf{C} \vee [\mathbf{M} s_\alpha \not\equiv? \mathbf{N} s_\alpha]} \text{ Func}$$

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$s_\alpha$  is a new Skolem term.

- Imitation of rigid heads:

$$\frac{\mathbf{C} \vee [F_\gamma \overline{U^n} \neq? h \overline{V^m}] \quad G \in \mathcal{AB}_\gamma^h}{\mathbf{C} \vee [F \neq? G] \vee [F \overline{U^n} \neq? h \overline{V^m}]} \text{ FlexRigid}$$

# Huet's Constrained Resolution $\mathcal{CR}$

## Unification & Splitting (contd.)

Elimination of  $\lambda$ -binders:

- (weak functional extensionality)

$$\frac{\mathbf{C} \vee [\mathbf{M}_{\alpha \rightarrow \beta} \neq? \mathbf{N}_{\alpha \rightarrow \beta}]}{\mathbf{C} \vee [\mathbf{M} s_\alpha \neq? \mathbf{N} s_\alpha]} \text{ Func}$$

$s_\alpha$  is a new Skolem term.

- Imitation of rigid heads:

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$\mathcal{AB}_\gamma^h$  is the set of general bindings of type  $\gamma$  for head  $h$ .

# Huet's Constrained Resolution $\mathcal{CR}$

## Unification & Splitting (contd.)

- Huet points to the usefulness of eager unification to filter out clauses with non-unifiable unification constraints or to back-propagate the solutions of easily solvable constraints (e.g., in case of first-order unification problems occurring during the proof search): many of the higher-order unification problems occurring in practice are decidable and have only finitely many solutions.

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- Hence, even though higher-order unification is generally not decidable it is sensible in practice to apply the unification algorithm with a particular resource, such that only those unification problems which may have further solutions beyond this bound need to be delayed.

# Huet's Constrained Resolution $\mathcal{CR}$

## Unification & Splitting (contd.)

- In our presentation of calculus  $\mathcal{CR}$  we explicitly address the aspect of eager unification and substitution by rule Subst. This rule back-propagates eagerly computed unifiers to the literal part of a clause.

# Huet's Constrained Resolution $\mathcal{CR}$

## Unification & Splitting (contd.)

- In our presentation of calculus  $\mathcal{CR}$  we explicitly address the aspect of eager unification and substitution by rule Subst. This rule back-propagates eagerly computed unifiers to the literal part of a clause.
- Eager unification & substitution:

$$\frac{\mathbf{C} \vee [X \neq? A] \quad X \notin \text{free}(A)}{\mathbf{C}_{[A/X]}} \text{ Subst}$$

# Huet's Constrained Resolution $\mathcal{CR}$

## Unification & Splitting (contd.)

- The literal heads of our clauses may consist of set variables and it may be necessary to instantiate them with terms introducing new logical constant at head position in order to find a refutation.

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- The literal heads of our clauses may consist of set variables and it may be necessary to instantiate them with terms introducing new logical constant at head position in order to find a refutation.
- Unfortunately not all appropriate instantiations can be computed with the calculus rules presented so far.

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- Unfortunately not all appropriate instantiations can be computed with the calculus rules presented so far.
- To address this problem Huet's approach provides the following splitting rules:

# Huet's Constrained Resolution $\mathcal{CR}$

## Unification & Splitting (contd.)

- Instantiate  
set variables:

# Huet's Constrained Resolution $\mathcal{CR}$

## Unification & Splitting (contd.)

- Instantiate set variables:

$$\frac{[P \ A]^T \vee C}{[Q]^T \vee [R]^T \vee C \vee [P \ A \neq? (Q_o \vee R_o)]} S^T_{\vee}$$

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## Unification & Splitting (contd.)

- Instantiate set variables:

$$\frac{[P\ A]^{\mu} \vee C}{[Q]^{\nu} \vee C \vee [P\ A \neq? \neg Q_o]} S_{\neg}^{TF}$$

(where  $\mu \neq \nu$ )

$$\frac{[P\ A]^T \vee C}{[Q]^T \vee [R]^T \vee C \vee [P\ A \neq? (Q_o \vee R_o)]} S_{\vee}^T$$

$$\frac{\begin{array}{c} [P\ A]^F \vee C \\ [Q]^F \vee C \vee [P\ A \neq? (Q_o \vee R_o)] \\ [R]^F \vee C \vee [P\ A \neq? (Q_o \vee R_o)] \end{array}}{S_{\vee}^F}$$

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$$\frac{[P A_{\alpha \rightarrow o}]^T \vee C}{[M_{\alpha \rightarrow o} Z]^T \vee C \vee [P A \neq? \Pi^\alpha M]} S_\Pi^T$$

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# Huet's Constrained Resolution $\mathcal{CR}$

## Unification & Splitting (contd.)

- Instantiate set variables:

$$\frac{[P A]^{\mu} \vee C}{[Q]^{\nu} \vee C \vee [P A \neq? \neg Q_o]} S_{\sqsupset}^T \quad (\text{where } \mu \neq \nu)$$

$$\frac{[P A]^T \vee C}{[Q]^T \vee [R]^T \vee C \vee [P A \neq? (Q_o \vee R_o)]} S_{\vee}^T$$

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$$\frac{[P A_{\alpha \rightarrow o}]^T \vee C}{[M_{\alpha \rightarrow o} Z]^T \vee C \vee [P A \neq? \Pi^{\alpha} M]} S_{\Pi}^T$$

$$\frac{[P A_{\alpha \rightarrow o}]^F \vee C}{[M_{\alpha \rightarrow o} s]^F \vee C \vee [P A \neq? \Pi^{\alpha} M]} S_{\Pi}^F$$

- $S_{\Pi}^T$  and  $S_{\Pi}^F$  are infinitely branching as they are parameterised over type  $\alpha$ .  $Q_o, R_o, M_{\alpha \rightarrow o}, Z_{\alpha}$  are new variables and  $s_{\alpha}$  is a new Skolem constant.

# Huet's Constrained Resolution $\mathcal{CR}$

## Unification & Splitting (contd.)

- A theorem which is not refutable in  $\mathcal{CR}$  if the splitting rules are not available is  $\exists A_o.A$ :

# Huet's Constrained Resolution $\mathcal{CR}$

## Unification & Splitting (contd.)

- A theorem which is not refutable in  $\mathcal{CR}$  if the splitting rules are not available is  $\exists A_o.A$ :
- After negation this statement normalises to clause  $\mathcal{C}_1 : [A]^F$ , such that none but the splitting rules are applicable. With the help of rule  $S_{\neg}^{TF}$  and eager unification, however, we can derive  $\mathcal{C}_2 : [A']^T$  which is then successfully resolvable against  $\mathcal{C}_1$ .

# Huet's Constrained Resolution $\mathcal{CR}$

## Extensionality Treatment

- On the one hand  $\eta$ -convertibility is built-in in higher-order unification, such that calculus  $\mathcal{CR}$  already supports functional extensionality reasoning to a certain extend.

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## Extensionality Treatment

- On the one hand  $\eta$ -convertibility is built-in in higher-order unification, such that calculus  $\mathcal{CR}$  already supports functional extensionality reasoning to a certain extend.
- On the other hand  $\mathcal{CR}$  nevertheless fails to address full extensionality as it does not realise the required subtle interplay between the functional and Boolean extensionality principles.
- Without employing additional (Boolean and functional!) extensionality axioms  $\mathcal{CR}$  is, e.g., not able to prove the rather simple examples presented before.

# Huet's Constrained Resolution $\mathcal{CR}$

## Proof Search

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## Proof Search

- Initially the proof problem is negated and normalised. The main proof search then operates on the generated clauses by applying the resolution, factorisation, and splitting rules.
- Despite the possibility of eager unification  $\mathcal{CR}$  generally foresees to delay the higher-order unification process in order to overcome the undecidability problem.
- When deriving a potentially empty clause (no normal literals),  $\mathcal{CR}$  then tests whether the accumulated unification constraints justifying this particular refutation are solvable.

# Huet's Constrained Resolution $\mathcal{CR}$

## Proof Search (contd.)

- Like  $\mathcal{R}$ , the extensionality treatment of  $\mathcal{CR}$  requires to add infinitely many extensionality axioms to the search space.

# Huet's Constrained Resolution $\mathcal{CR}$

## Proof Search (contd.)

- Like  $\mathcal{R}$ , the extensionality treatment of  $\mathcal{CR}$  requires to add infinitely many extensionality axioms to the search space.
- The following figure graphically illustrates the main ideas of the proof search in  $\mathcal{CR}$ .



# Huet's Constrained Resolution $\mathcal{CR}$

## Completeness Results

- [Huet72,Huet73] analyses completeness of  $\mathcal{CR}$  formally only with respect to Andrews V-complexes, i.e. Huet verifies that the set of non-refutable sentences in  $\mathcal{CR}$  is an abstract consistency class for V-complexes.

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- Theorem (V-completeness of  $\mathcal{CR}$ ): The calculus  $\mathcal{CR}$  is complete with respect to the notion of V-complexes.

Proof: [Huet72,Huet73]

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- Theorem (V-completeness of  $\mathcal{CR}$ ): The calculus  $\mathcal{CR}$  is complete with respect to the notion of V-complexes.

Proof: [Huet72,Huet73]

- Theorem (Henkin completeness of  $\mathcal{CR}$ ): The calculus  $\mathcal{CR}$  is complete wrt. Henkin semantics provided that the infinitely many extensionality axioms are given.

Proof: exercise

# Example Proofs

Exercise: How are the following theorems proved in calculus *CR*?

- Leibniz equality and  $\eta$ -equality:

$$f_{\iota \rightarrow \iota} \doteq \lambda X_\iota . f \; X$$

# Example Proofs

Exercise: How are the following theorems proved in calculus CR?

- Leibniz equality and  $\eta$ -equality:

$$f_{\iota \rightarrow \iota} \doteq \lambda X_\iota . f X$$

- The set of all red balls equals the set of all balls that are red:  
 $\{X | \text{red } X \wedge \text{ball } X\} = \{X | \text{ball } X \wedge \text{red } X\}$ . This problem can be encoded as

$$(\lambda X_\iota . \text{red } X \wedge \text{ball } X) = (\lambda X_\iota . \text{ball } X \wedge \text{red } X)$$

# Example Proofs

Exercise: How are the following theorems proved in calculus CR?

- All unary logical operators  $O_{o \rightarrow o}$  which map the propositions  $a$  and  $b$  to  $\top$  consequently also map  $a \wedge b$  to  $\top$ :

$$\forall O_{o \rightarrow o} \cdot (O a_o) \wedge (O b_o) \Rightarrow (O (a_o \wedge b_o))$$

# Example Proofs

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Exercise: How are the following theorems proved in calculus  $\text{CR}$ ?

- In Henkin semantics the domain  $\mathcal{D}_o$  of all Booleans contains exactly the truth values  $\perp$  and  $\top$ . Consequently the domain of all mappings from Booleans to Booleans contains exactly four elements. And because of the requirement, that the function domains in Henkin models must be rich enough such that every term has a denotation, it follows that  $\mathcal{D}_{o \rightarrow o}$  contains exactly the pairwise distinct denotations of the following four terms:  $\lambda X_o.X_o$ ,  $\lambda X_o.\neg X_o$ ,  $\lambda X_o.\perp$ , and  $\lambda X_o.\top$ . This theorem can be formulated as follows (where  $f_{o \rightarrow o}$  is a constant):

$$(f = \lambda X_o.X_o) \vee (f = \lambda X_o.\neg X_o) \vee (f = \lambda X_o.\perp) \vee (f = \lambda X_o.\top)$$



## Approaches to Higher-Order Resolution: $\mathcal{ER}$

# Extensional HO Resolution $\mathcal{ER}$

Clause normalization

$$\frac{C \vee [A \vee B]^T}{C \vee [A]^T \vee [B]^T} \vee^T$$

$$\frac{C \vee [A \vee B]^F}{C \vee [A]^F} \vee^F_l$$

$$\frac{C \vee [A \vee B]^F}{C \vee [B]^F} \vee^F_r$$

$$\frac{C \vee [\neg A]^T}{C \vee [A]^F} \neg^T$$

$$\frac{C \vee [\neg A]^F}{C \vee [A]^T} \neg^F$$

$$\frac{C \vee [\Pi^\alpha A]^T \quad X_\alpha \text{ new variable}}{C \vee [A X]^T} \Pi^T$$

$$\frac{C \vee [\Pi^\alpha A]^F \quad sk_\alpha \text{ Skolem term}}{C \vee [A sk_\alpha]^F} \Pi^F$$

This rules may be combined into a single rule  $\text{Cnf.}$

# Extensional HO Resolution $\mathcal{ER}$

## Resolution and Factorisation

$$\frac{[N]^\alpha \vee C \quad [M]^\beta \vee D \quad \alpha \neq \beta}{C \vee D \vee [N \neq? M]} \text{ Res}$$

$$\frac{[N]^\alpha \vee [M]^\alpha \vee C \quad \alpha \in \{\top, \perp\}}{[N]^\alpha \vee C \vee [N \neq? M]} \text{ Fac}$$

$$\frac{[Q_\gamma \overline{U^k}]^\alpha \vee C \quad P \in \mathcal{GB}_\gamma^{\{\neg, \vee\} \cup \{\Pi^\beta | \beta \in T^k\}}}{[Q_\gamma \overline{U^k}]^\alpha \vee C \vee [Q \neq? P]} \text{ Prim}^k$$

# Extensional HO Resolution $\mathcal{ER}$

(Pre-)unification rules

$$\frac{C \vee [M_{\alpha \rightarrow \beta} \neq? N_{\alpha \rightarrow \beta}]^F \quad s_\alpha \text{ Skolem-Term}}{C \vee [M s \neq? N s]} \text{ Func}$$

$$\frac{C \vee [h \overline{U^n} \neq? h \overline{V^n}]}{C \vee [U^1 \neq? V^1] \vee \dots \vee [U^n \neq? V^n]} \text{ Dec} \quad \frac{C \vee [A \neq? A]}{C} \text{ Triv}$$

$$\frac{C \vee [F_\gamma \overline{U^n} \neq? h \overline{V^n}] \quad G \in \mathcal{GB}_\gamma^h}{C \vee [F \neq? G] \vee [F \overline{U^n} \neq? h \overline{V^n}]} \text{ Flex/Rigid}$$

$$\frac{C \vee E \quad E \text{ solved for } C}{\text{Cnf}(\text{subst}_E(C))} \text{ Subst}$$

# Extensional HO Resolution $\mathcal{ER}$

## Extensionality rules

$$\frac{C \vee [M_o \neq? N_o]^F}{\text{Cnf}(C \vee [M_o \Leftrightarrow N_o]^F)} \text{ Equiv}$$

$$\frac{C \vee [M_\alpha \neq? N_\alpha]^F \quad \alpha \in \{o, \iota\}}{\text{Cnf}(C \vee [\forall P_{\alpha \rightarrow o}. PM \Rightarrow PN]^F)} \text{ Leib}$$

# Extensional HO Resolution $\mathcal{ER}$

## Extensionality Treatment

- Instead of adding infinitely many extensionality axioms to the search space  $\textcolor{blue}{CR}$  provides two new extensionality rules which closely connect refutation search and eager unification.

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# Extensional HO Resolution $\mathcal{ER}$

## Extensionality Treatment

- Instead of adding infinitely many extensionality axioms to the search space  $\mathcal{CR}$  provides two new extensionality rules which closely connect refutation search and eager unification.
- The idea is to allow for recursive calls from higher-order unification to the overall refutation process.
- This turns the rather weak syntactical higher-order unification approach considered so far into a most general approach for *dynamic* higher-order theory unification.

# Extensional HO Resolution $\mathcal{ER}$

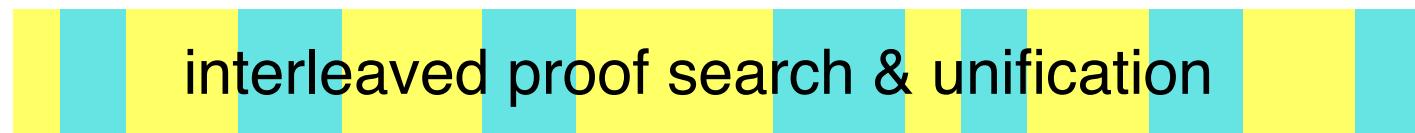
## Proof Search

- Initially the proof problem is negated and normalised. The main proof search then closely interleaves the refutation process on resolution layer and unification, i.e. the main proof search rules Res, Fac, and Prim and the unification rules are integrated at a common conceptual level. The calls from unification to the overall refutation process with rules *Leib* and *Equiv* introduce new clauses into the search space which can be resolved against already given ones.

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- The following figure graphically illustrates the main ideas of the proof search in  $\mathcal{ER}$ .



# Ex.: Extensional HO Resolution $\mathcal{ER}$

$$\forall B_{\alpha \rightarrow o}, C_{\alpha \rightarrow o}, D_{\alpha \rightarrow o} \cdot B \cup (C \cap D) = (B \cup C) \cap (B \cup D)$$

Negation and definition expansion with

$$\cup = \lambda A_{\alpha \rightarrow o}, B_{\alpha \rightarrow o}, X_{\alpha \rightarrow o} \cdot (A \ X) \vee (B \ X) \quad \cap = \lambda A_{\alpha \rightarrow o}, B_{\alpha \rightarrow o}, X_{\alpha \rightarrow o} \cdot (A \ X) \wedge (B \ X)$$

leads to:

$$C_1 : [\lambda X_{\alpha \rightarrow o} \cdot (b \ X) \vee ((c \ X) \wedge (d \ X)) \neq? \lambda X_{\alpha \rightarrow o} \cdot ((b \ X) \vee (c \ X)) \wedge ((b \ X) \vee (d \ X))]$$

Goal directed functional and Boolean extensionality treatment:

$$C_2 : [(b \ x) \vee ((c \ x) \wedge (d \ x)) \Leftrightarrow ((b \ x) \vee (c \ x)) \wedge ((b \ x) \vee (d \ x))]^F$$

Clause normalization results then in a pure propositional, i.e. decidable, set of clauses. Only these clauses are still in the search space of LEO (in total there are 33 clauses generated and LEO finds the proof on a 2,5GHz PC in 820ms).

Similar proof in case of embedded propositions:

$$\forall P_{(\alpha \rightarrow o) \rightarrow o}, B_{\alpha \rightarrow o}, C_{\alpha \rightarrow o}, D_{\alpha \rightarrow o} \cdot P(B \cup (C \cap D)) \Rightarrow P((B \cup C) \cap (B \cup D))$$

# Ex.: Extensional HO Resolution $\mathcal{ER}$

$$\forall P_{o \rightarrow o} (P a_o) \wedge (P b_o) \Rightarrow (P (a_o \wedge b_o))$$

Negation and clause normalization

$$\mathcal{C}_1 : [p\ a]^T \quad \mathcal{C}_2 : [p\ b]^T \quad \mathcal{C}_3 : [p\ (a \wedge b)]^F$$

Resolution between  $\mathcal{C}_1$  and  $\mathcal{C}_3$  and between  $\mathcal{C}_2$  and  $\mathcal{C}_3$

$$\mathcal{C}_4 : [p\ a \neq? p\ (a \wedge b)] \quad \mathcal{C}_5 : [p\ b \neq? p\ (a \wedge b)]$$

Decomposition

$$\mathcal{C}_6 : [a \neq? (a \wedge b)] \quad \mathcal{C}_7 : [b \neq? (a \wedge b)]$$

Recursive call of proof process with rules Equiv and Cnf

$$\mathcal{C}_8 : [a]^F \vee [b]^F \quad \mathcal{C}_9 : [a]^T \vee [b]^T \quad \mathcal{C}_{10} : [a]^T \quad \mathcal{C}_{11} : [b]^T$$

# Ex.: Extensional HO Resolution $\mathcal{ER}$



Further small examples which test Henkin completeness:

$$\forall F_{o \rightarrow o} \cdot (F \doteq \lambda X_o. X_o) \vee (F \doteq \lambda X_o. \neg X_o) \vee (F \doteq \lambda X_o. \perp) \vee (F \doteq \lambda X_o. \top)$$

$$\forall H_{o \rightarrow o} \cdot H \perp \doteq H \quad (H \top \doteq H \perp)$$

...



# Higher-Order Sequent Calculi, Cut, and Saturation

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vs.

$$\frac{\Gamma \implies \Delta * \mathbf{A} * \mathbf{B}}{\Gamma \implies \Delta * (\mathbf{A} \vee \mathbf{B})} \vee R$$

$$\frac{}{\Delta * \neg(\mathbf{A} \vee \mathbf{B})} \mathcal{G}(\vee_-)$$

vs.

$$\frac{\Gamma * \mathbf{A} \vee \mathbf{B} \implies \Delta}{\Gamma * \mathbf{A} \vee \mathbf{B} \implies \Delta} \vee L$$

# Def.: Sequent Calculi

- A sequent calculus  $\mathcal{G}$  provides an inductive definition for when  $\Vdash_{\mathcal{G}} \Delta$  holds.
- Here we consider a one-sided sequent calculus, thus a sequent is defined as a finite set  $\Delta$  of  $\beta$ -normal sentences from  $cwff_o(\Sigma)$ .
- Let's compare our one-sided rules to standard two-sided sequent calculus rules (we use  $\Delta * \mathbf{A}$  to denote the set  $\Delta \cup \{\mathbf{A}\}$  (which is simply  $\Delta$  if  $\mathbf{A} \in \Delta$ ):

$$\frac{\Delta * \mathbf{A} * \mathbf{B}}{\Delta * (\mathbf{A} \vee \mathbf{B})} \mathcal{G}(\vee_+)$$

vs.

$$\frac{\Gamma \implies \Delta * \mathbf{A} * \mathbf{B}}{\Gamma \implies \Delta * (\mathbf{A} \vee \mathbf{B})} \vee R$$

$$\frac{\Delta * \neg \mathbf{A} \quad \Delta * \neg \mathbf{B}}{\Delta * \neg(\mathbf{A} \vee \mathbf{B})} \mathcal{G}(\vee_-)$$

vs.

$$\frac{\Gamma * \mathbf{B} \implies \Delta \quad \Gamma * \mathbf{A} \implies \Delta}{\Gamma * \mathbf{A} \vee \mathbf{B} \implies \Delta} \vee L$$

# Def.: Validity of Sequents

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- Given a sequent  $\Delta$ , a model  $\mathcal{M}$ , and a class  $\mathfrak{M}$  of models, we say

$\Delta$  is valid for  $\mathcal{M}$  (or valid for  $\mathfrak{M}$ ),

if  $\mathcal{M} \models D$  for some  $D \in \Delta$  (or  $\Delta$  is valid for every  $\mathcal{M} \in \mathfrak{M}$ ).

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if any instance of  $r$  can be replaced by a derivation with at most  $k$  additional proof steps.

- Remark: We use admissibility to obtain more general results.  
In fact all rules that are later shown to be  $k$ -admissible are actually even  $k$ -derivable.

# Def.: Sequent Calculus Rules

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# ACC for Sequent Calculi

- For any sequent calculus  $\mathcal{G}$  we can define a class  $\Gamma_{\Sigma}^{\mathcal{G}}$  of sets of sentences.

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- For any sequent calculus  $\mathcal{G}$  we can define a class  $\Gamma_{\Sigma}^{\mathcal{G}}$  of sets of sentences.
- Under certain assumptions,  $\Gamma_{\Sigma}^{\mathcal{G}}$  is an abstract consistency class.
- First we adopt the notation  $\neg\Phi$  and  $\Phi \downarrow_{\beta}$  for the sets  $\{\neg A \mid A \in \Phi\}$  and  $\{A \downarrow_{\beta} \mid A \in \Phi\}$ , resp., where  $\Phi \subseteq \text{cwff}_o(\Sigma)$ .

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- Furthermore, we assume this use of  $\neg$  binds more strongly than  $\cup$  or  $*$ , so that  $\neg\Phi \cup \Delta$  means  $(\neg\Phi) \cup \Delta$  and  $\neg\Phi * A$  means  $(\neg\Phi) * A$ .

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Let  $\mathcal{G}$  be a sequent calculus.

# Def.: ACC for Sequent Calculi

Let  $\mathcal{G}$  be a sequent calculus. We define  $\Gamma_{\Sigma}^{\mathcal{G}}$  to be the class of all finite  $\Phi \subset \text{cwff}_o(\Sigma)$  such that  $\vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta}$  does not hold.

# Lemma: Consequence of $\mathcal{G}(Inv^\vdash)$



Let  $\mathcal{G}$  be a sequent calculus such that  $\mathcal{G}(Inv^\vdash)$  is admissible.

# Lemma: Consequence of $\mathcal{G}(Inv^\perp)$

Let  $\mathcal{G}$  be a sequent calculus such that  $\mathcal{G}(Inv^\perp)$  is admissible. For any finite sets  $\Phi$  and  $\Delta$  of sentences, if  $\Phi \cup \neg\Delta \notin \Gamma_\Sigma^{\mathcal{G}}$ , then  $\vdash_{\mathcal{G}} \neg\Phi \downarrow_\beta \cup \Delta \downarrow_\beta$  holds.

# Lemma: Consequence of $\mathcal{G}(Inv^\perp)$

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Proof:

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Proof: Suppose  $\Phi \cup \neg\Delta \notin \Gamma_\Sigma^{\mathcal{G}}$ . By definition,  $\vdash_{\mathcal{G}} \neg\Phi \downarrow_\beta \cup \neg\neg\Delta \downarrow_\beta$  holds.

# Lemma: Consequence of $\mathcal{G}(Inv^\neg)$

Let  $\mathcal{G}$  be a sequent calculus such that  $\mathcal{G}(Inv^\neg)$  is admissible. For any finite sets  $\Phi$  and  $\Delta$  of sentences, if  $\Phi \cup \neg\Delta \notin \Gamma_\Sigma^{\mathcal{G}}$ , then  $\vdash_{\mathcal{G}} \neg\Phi \downarrow_\beta \cup \Delta \downarrow_\beta$  holds.

Proof: Suppose  $\Phi \cup \neg\Delta \notin \Gamma_\Sigma^{\mathcal{G}}$ . By definition,  $\vdash_{\mathcal{G}} \neg\Phi \downarrow_\beta \cup \neg\neg\Delta \downarrow_\beta$  holds. Applying  $\mathcal{G}(Inv^\neg)$  to each member of  $\Delta \downarrow_\beta$ , we have  $\vdash_{\mathcal{G}} \neg\Phi \downarrow_\beta \cup \Delta \downarrow_\beta$ .

# Thm.: Sufficient Conditions for $\Gamma_{\Sigma}^{\mathcal{G}} \in \mathfrak{Acc}_{\beta}$



Let  $\mathcal{G}$  be a sequent calculus.

# Thm.: Sufficient Conditions for $\Gamma_{\Sigma}^{\mathcal{G}} \in \mathfrak{Acc}_{\beta}$

Let  $\mathcal{G}$  be a sequent calculus. If the rules  $\mathcal{G}(Inv^{\neg})$ ,  $\mathcal{G}(\neg)$ ,  $\mathcal{G}(weak)$ ,  $\mathcal{G}(init)$ ,  $\mathcal{G}(\vee_-)$ ,  $\mathcal{G}(\vee_+)$ ,  $\mathcal{G}(\Pi_-^C)$  and  $\mathcal{G}(\Pi_+^c)$  are admissible in  $\mathcal{G}$ , then  $\Gamma_{\Sigma}^{\mathcal{G}} \in \mathfrak{Acc}_{\beta}$ .

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Proof: We prove  $\Gamma_{\Sigma}^{\mathcal{G}}$  is closed under subsets and satisfies  $\nabla_c$ ,  $\nabla_{\neg}$ ,  $\nabla_{\vee}$ ,  $\nabla_{\wedge}$  and  $\nabla_{\beta}$ . The remaining conditions are proven analogously.

# Thm.: Sufficient Conditions for $\Gamma_\Sigma^G \in \mathfrak{Acc}_\beta$

Let  $\mathcal{G}$  be a sequent calculus. If the rules  $\mathcal{G}(Inv^\neg)$ ,  $\mathcal{G}(\neg)$ ,  $\mathcal{G}(weak)$ ,  $\mathcal{G}(init)$ ,  $\mathcal{G}(\vee_-)$ ,  $\mathcal{G}(\vee_+)$ ,  $\mathcal{G}(\Pi_-^C)$  and  $\mathcal{G}(\Pi_+^c)$  are admissible in  $\mathcal{G}$ , then  $\Gamma_\Sigma^G \in \mathfrak{Acc}_\beta$ .

Proof: We prove  $\Gamma_\Sigma^G$  is closed under subsets and satisfies  $\nabla_c$ ,  $\nabla_{\neg}$ ,  $\nabla_{\vee}$ ,  $\nabla_{\wedge}$  and  $\nabla_\beta$ . The remaining conditions are proven analogously.

- Suppose  $\Phi \in \Gamma_\Sigma^G$ , If  $\Phi_0 \subseteq \Phi$  and  $\Phi_0 \notin \Gamma_\Sigma^G$ , then  $\vdash_{\mathcal{G}} \neg \Phi_0 \downarrow_\beta$  and so  $\vdash_{\mathcal{G}} \neg \Phi \downarrow_\beta$  by admissibility of  $\mathcal{G}(weak)$ . Hence  $\Gamma_\Sigma^G$  is closed under subsets.

# Thm.: Sufficient Conditions for $\Gamma_\Sigma^G \in \mathfrak{Acc}_\beta$

Let  $\mathcal{G}$  be a sequent calculus. If the rules  $\mathcal{G}(Inv^\neg)$ ,  $\mathcal{G}(\neg)$ ,  $\mathcal{G}(weak)$ ,  $\mathcal{G}(init)$ ,  $\mathcal{G}(\vee_-)$ ,  $\mathcal{G}(\vee_+)$ ,  $\mathcal{G}(\Pi_-^C)$  and  $\mathcal{G}(\Pi_+^c)$  are admissible in  $\mathcal{G}$ , then  $\Gamma_\Sigma^G \in \mathfrak{Acc}_\beta$ .

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- Suppose  $\Phi \in \Gamma_\Sigma^G$ , If  $\Phi_0 \subseteq \Phi$  and  $\Phi_0 \notin \Gamma_\Sigma^G$ , then  $\models_G \neg \Phi_0 \downarrow_\beta$  and so  $\models_G \neg \Phi \downarrow_\beta$  by admissibility of  $\mathcal{G}(weak)$ . Hence  $\Gamma_\Sigma^G$  is closed under subsets.
- Suppose  $\Phi \in \Gamma_\Sigma^G$  and  $\mathbf{A}, \neg \mathbf{A} \in \Phi$  where  $\mathbf{A}$  is atomic. By admissibility of  $\mathcal{G}(init)$ ,  $\models_G \neg \Phi \downarrow_\beta * \mathbf{A} \downarrow_\beta$  since  $\neg \mathbf{A} \downarrow_\beta \in \neg \Phi \downarrow_\beta$ . By admissibility of  $\mathcal{G}(\neg)$ ,  $\models_G \neg \Phi \downarrow_\beta$  since  $\neg \neg \mathbf{A} \downarrow_\beta \in \neg \Phi \downarrow_\beta$ , contradicting  $\Phi \in \Gamma_\Sigma^G$ . Thus  $\nabla_c$  holds.

# Thm.: Sufficient Condition for $\Gamma_{\Sigma}^{\mathcal{G}} \in \text{Acc}_{\beta}$



Proof (contd.):

# Thm.: Sufficient Condition for $\Gamma_{\Sigma}^{\mathcal{G}} \in \text{Acc}_{\beta}$

Proof (contd.):

- Suppose  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ ,  $\neg\neg A \in \Phi$  and  $\Phi * A \notin \Gamma_{\Sigma}^{\mathcal{G}}$ . Hence  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta} * \neg A \downarrow_{\beta}$  and so  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta} * \neg\neg\neg A \downarrow_{\beta}$  by admissibility of  $\mathcal{G}(\neg)$ . Since  $\neg\neg A \in \Phi$ , we know  $\neg\Phi \downarrow_{\beta}$  is equal to  $\neg\Phi \downarrow_{\beta} * \neg\neg\neg A \downarrow_{\beta}$ . Hence  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta}$ , contradicting  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ . Thus  $\nabla_{\neg}$  holds.

# Thm.: Sufficient Condition for $\Gamma_{\Sigma}^{\mathcal{G}} \in \text{Acc}_{\beta}$

Proof (contd.):

- Suppose  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ ,  $\neg\neg A \in \Phi$  and  $\Phi * A \notin \Gamma_{\Sigma}^{\mathcal{G}}$ . Hence  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta} * \neg A \downarrow_{\beta}$  and so  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta} * \neg\neg\neg A \downarrow_{\beta}$  by admissibility of  $\mathcal{G}(\neg)$ . Since  $\neg\neg A \in \Phi$ , we know  $\neg\Phi \downarrow_{\beta}$  is equal to  $\neg\Phi \downarrow_{\beta} * \neg\neg\neg A \downarrow_{\beta}$ . Hence  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta}$ , contradicting  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ . Thus  $\nabla_{\neg}$  holds.
- Suppose  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ ,  $(A \vee B) \in \Phi$ ,  $\Phi * A \notin \Gamma_{\Sigma}^{\mathcal{G}}$  and  $\Phi * B \notin \Gamma_{\Sigma}^{\mathcal{G}}$ . Hence  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta} * \neg A \downarrow_{\beta}$  and  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta} * \neg B \downarrow_{\beta}$ . Applying  $\mathcal{G}(\vee_{-})$ , we have  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta}$  since  $\neg(A \vee B) \downarrow_{\beta} \in \neg\Phi \downarrow_{\beta}$ , contradicting  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ . Thus  $\nabla_{\vee}$  holds.

# Thm.: Sufficient Condition for $\Gamma_{\Sigma}^{\mathcal{G}} \in \text{Acc}_{\beta}$

Proof (contd.):

- Suppose  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ ,  $\neg\neg A \in \Phi$  and  $\Phi * A \notin \Gamma_{\Sigma}^{\mathcal{G}}$ . Hence  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta} * \neg A \downarrow_{\beta}$  and so  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta} * \neg\neg\neg A \downarrow_{\beta}$  by admissibility of  $\mathcal{G}(\neg)$ . Since  $\neg\neg A \in \Phi$ , we know  $\neg\Phi \downarrow_{\beta}$  is equal to  $\neg\Phi \downarrow_{\beta} * \neg\neg\neg A \downarrow_{\beta}$ . Hence  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta}$ , contradicting  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ . Thus  $\nabla_{\neg}$  holds.
- Suppose  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ ,  $(A \vee B) \in \Phi$ ,  $\Phi * A \notin \Gamma_{\Sigma}^{\mathcal{G}}$  and  $\Phi * B \notin \Gamma_{\Sigma}^{\mathcal{G}}$ . Hence  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta} * \neg A \downarrow_{\beta}$  and  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta} * \neg B \downarrow_{\beta}$ . Applying  $\mathcal{G}(\vee_{-})$ , we have  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta}$  since  $\neg(A \vee B) \downarrow_{\beta} \in \neg\Phi \downarrow_{\beta}$ , contradicting  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ . Thus  $\nabla_{\vee}$  holds.
- By a similar argument, admissibility of  $\mathcal{G}(\Pi_{-}^C)$  implies  $\nabla_{\forall}$ .

# Thm.: Sufficient Condition for $\Gamma_{\Sigma}^{\mathcal{G}} \in \text{Acc}_{\beta}$



Proof (contd.):

# Thm.: Sufficient Condition for $\Gamma_{\Sigma}^{\mathcal{G}} \in \text{Acc}_{\beta}$

Proof (contd.):

- Suppose  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ ,  $\neg(A \vee B) \in \Phi$  and  $\Phi * \neg A * \neg B \notin \Gamma_{\Sigma}^{\mathcal{G}}$ . By Lemma 'Consequence of  $\mathcal{G}(Inv^-)$ ',  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta} * A \downarrow_{\beta} * B \downarrow_{\beta}$ . Applying  $\mathcal{G}(\vee_+)$ , we have  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta} * (A \vee B) \downarrow_{\beta}$ . Applying  $\mathcal{G}(\neg)$ , we have  $\Vdash_{\mathcal{G}} \neg\Phi \downarrow_{\beta}$  since  $\neg(A \vee B) \in \Phi$ , contradicting  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ . Thus  $\nabla_{\wedge}$  holds.

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- Suppose  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ ,  $A \in \Phi$ ,  $A =_{\beta} B$  and  $\Phi * B \notin \Gamma_{\Sigma}^{\mathcal{G}}$ . Hence  $\Vdash_{\mathcal{G}} \neg \Phi \downarrow_{\beta} * \neg B \downarrow_{\beta}$ , contradicting  $A \downarrow_{\beta} \in \Phi \downarrow_{\beta}$  and  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ . Thus  $\nabla_{\beta}$  holds.

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Proof: Suppose  $\mathcal{G}(cut)$  is admissible,  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ ,  $\mathbf{A} \in \text{cwff}_o(\Sigma)$ ,  $\Phi * \mathbf{A} \notin \Gamma_{\Sigma}^{\mathcal{G}}$  and  $\Phi * \neg \mathbf{A} \notin \Gamma_{\Sigma}^{\mathcal{G}}$ . Hence  $\vdash_{\mathcal{G}} \neg \Phi \downarrow_{\beta} * \neg \mathbf{A} \downarrow_{\beta}$  and  $\vdash_{\mathcal{G}} \neg \Phi \downarrow_{\beta} * \neg \neg \mathbf{A} \downarrow_{\beta}$ . Using  $\mathcal{G}(cut)$ , we have  $\vdash_{\mathcal{G}} \neg \Phi \downarrow_{\beta}$ , contradicting  $\Phi \in \Gamma_{\Sigma}^{\mathcal{G}}$ .

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Suppose  $\Gamma_\Sigma^\mathcal{G}$  is saturated,  $\Vdash_{\mathcal{G}} \Delta * \mathbf{C}$  and  $\Vdash_{\mathcal{G}} \Delta * \neg\mathbf{C}$  hold but  $\Vdash_{\mathcal{G}} \Delta$  does not. Applying  $\mathcal{G}(\neg)$  to every member of  $\Delta$  and to  $\mathbf{C}$  we have

$\Vdash_{\mathcal{G}} \neg\neg\Delta * \neg\neg\mathbf{C}$  and  $\Vdash_{\mathcal{G}} \neg\neg\Delta * \neg\mathbf{C}$ . By Lemma 'Consequence of  $\mathcal{G}(Inv^\neg)$ ', we know  $\neg\Delta \in \Gamma_\Sigma^\mathcal{G}$ . By saturation, we must have  $\neg\Delta * \mathbf{C} \in \Gamma_\Sigma^\mathcal{G}$  or  $\neg\Delta * \neg\mathbf{C} \in \Gamma_\Sigma^\mathcal{G}$ . The first case contradicts  $\Vdash_{\mathcal{G}} \neg\neg\Delta * \neg\mathbf{C}$  while the second case contradicts  $\Vdash_{\mathcal{G}} \neg\neg\Delta * \neg\neg\mathbf{C}$ .

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Def. (Saturated Extension): Let  $\Gamma_\Sigma, \Gamma'_\Sigma \in \mathfrak{Acc}_*$  be abstract consistency classes. We say  $\Gamma'_\Sigma$  is an **extension** of  $\Gamma_\Sigma$  if  $\Phi \in \Gamma'_\Sigma$  for every sufficiently  $\Sigma$ -pure  $\Phi \in \Gamma_\Sigma$ . We say  $\Gamma'_\Sigma$  is a **saturated extension** of  $\Gamma_\Sigma$  if  $\Gamma'_\Sigma$  is saturated and an extension of  $\Gamma_\Sigma$ .

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