

# **Universal Logical Reasoning (with Applications in Normative Reasoning)**

**Christoph Benzmüller**  
Freie Universität Berlin

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## Introduction

## Introduction

This set of slides ...

... is about (narrow picture)

- ▶ higher-order modal logic (HOML)
- ▶ classical higher-order logic (HOL)
- ▶ embedding of HOML in HOL
- ▶ mechanisation and automation with HOL ATPs
- ▶ various applications

... is about (wider picture)

- ▶ a quite universal approach to automate reasoning for a wide range of classical and non-classical logics
- ▶ a very broad range of possible applications
- ▶ including meta-reasoning about logical systems

## Introduction

Our primary interest ...

- ▶ is in expressive logics:  
quantification (first- and higher-order) and lambda-expressions;
- ▶ propositional fragments are trivially covered

## Introduction: Outline

- ▶ Introduction
- ▶ HOL
- ▶ HOML
- ▶ Automation of HOML with LEO-II and Isabelle

**The remaining slides are optional! We will use others in the upcoming lectures.**

- ▶ Ontological Argument
- ▶ SUMO Ontology and HOML
- ▶ Meta-Reasoning
- ▶ Flexibility and Rigidity
- ▶ Cut-Elimination

## Introduction: Expressivity Matters — Cantor's Theorem

**Cantor's theorem:** The set of all subsets of A, that is, the power set of A, has a strictly greater cardinality than A itself.

In HOL Cantor's theorem (surjective version) can be encoded as

$$\neg \exists F \forall G \exists X. FX = G$$

HO ATPs can solve this problem very efficiently.

Their solution includes the detection and application of the diagonalisation argument

Today: basic test example for new higher-order theorem provers.

Further reading: [AndrewsEtAl., Automating higher-order logic, 1984]

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$$\neg \exists F_{\iota \rightarrow (\iota \rightarrow o)} \forall G_{\iota \rightarrow o} \exists X_\iota. FX = G$$

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## Introduction: Expressivity Matters — Boolos' Example

[George Boolos, A curious inference, J. Philosophical Logic, 16:1-12, 1987]

1.  $\forall n f(n, 1) = s(1)$
2.  $\forall x f(1, s(x)) = s(s(f(1, x)))$
3.  $\forall n \forall x f(s(n), s(x)) = f(n, f(s(n), x))$
4.  $D(1)$
5.  $\forall x D(x) \rightarrow D(s(x))$
- ⋮
6.  $D(f(s(s(s(s(1))))), s(s(s(s(1))))))$

Induction proof: from (4) and (5), we get  $\forall x D(x)$ , hence  $D(f(s(s(s(s(1))))), s(s(s(s(1))))))$  by  $\forall$ -elimination.

But induction is not given, hence the first order proof consists of brute force modus ponens applications: infeasible number of single steps ( $2^{(2^{\dots^2})}$  with 64K '2s')

## Introduction: Expressivity Matters — Boolos' Example

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Comprehension axioms

$$\exists N_{\overline{\alpha^n} \rightarrow \beta} \forall \overline{z^n} N(z^1, \dots, z^n) = B_\beta$$

Can be avoided: use  $\lambda$ -binding construct to denote  $N$

Instances of comprehension axioms in Boolos' proof:

$$\exists N \forall z N(z) \leftrightarrow (\forall X X(1) \wedge \forall y (X(y) \rightarrow X(s(y))) \rightarrow X(z))$$

$$\exists E \forall z E(z) \leftrightarrow (N(z) \wedge D(z))$$

Central idea: “assume the induction principle holds for number  $z$  – corresponding to  $N(z)$  – then we can show for any predicate  $X$  a property  $X(z)$  by induction.”

The proof employs the following lemmata:

**Lemma 1:**  $N(1), \forall y (N(y) \rightarrow N(s(y))), N(s(s(s(s(1))))), E(1), \forall y (E(y) \rightarrow E(s(y))), E(s(1))$

**Lemma 2:**  $\forall n N(n) \rightarrow \forall x (N(x) \rightarrow E(f(n, x)))$

The theorem itself is then an easy application of the two lemmata.

## Introduction: Expressivity Matters — Boolos' Example

<sup>1</sup>By the comprehension principle of second order logic,  $\exists N \forall z(Nz \leftrightarrow \forall X[X1\&\forall y(Yx \rightarrow Xsy) \rightarrow Xz])$ , <sup>2</sup>and then for some  $N$ ,  $\exists E \forall z(Ez \leftrightarrow Nz \& Dz)$ .

<sup>3</sup>**Lemma 1:**  $\frac{3.1}{3}N1$ ;  $\frac{3.2}{3}\forall y(Ny \rightarrow Nsy)$ ;  $\frac{3.3}{3}Nssss1$ ;  $\frac{3.4}{3}E1$ ;  $\frac{3.5}{3}\forall y(Ey \rightarrow Esy)$ ;  $\frac{3.6}{3}Es1$ ;

<sup>4</sup>**Lemma 2:**  $\forall n(Nn \rightarrow (\forall x(Nx \rightarrow Efnx)))$

*Proof:* <sup>4.1</sup>By comprehension,  $\exists M \forall n(Mn \leftrightarrow \forall x(Nx \rightarrow Efnx))$ . <sup>4.2</sup>We want  $\forall n(Nn \rightarrow Mn)$ . <sup>4.3</sup>Enough to show  $\frac{4.3.1}{4.3}M1$  and  $\frac{4.3.2}{4.3}\forall n(Mn \rightarrow Msn)$ , for then if  $\frac{4.4}{4.4}Nn$ ,  $\frac{4.5}{4.5}Mn$ .

<sup>4.3.1</sup>**M1:**  $\frac{4.3.1.1}{4.3.1.1}$ Want  $\forall x(Nx \rightarrow Ef1x)$ .  $\frac{4.3.1.2}{4.3.1.2}$ By comprehension,  $\exists Q \forall x(Qx \leftrightarrow Ef1x)$ .

<sup>4.3.1.3</sup>Want  $\forall x(Nx \rightarrow Qx)$ .  $\frac{4.3.1.4}{4.3.1.4}$ Enough to show  $\frac{4.3.1.4.1}{4.3.1.4.1}Q1$  and  $\frac{4.3.1.4.2}{4.3.1.4.2}\forall x(Qx \rightarrow Qsx)$ .

<sup>4.3.1.4.1</sup>**Q1:**  $\frac{4.3.1.4.1.1}{4.3.1.4.1.1}$ Want  $Ef11$ .  $\frac{4.3.1.4.1.2}{4.3.1.4.1.2}$ But  $f11 = s1$  by (1) and  $\frac{4.3.1.4.1.3}{4.3.1.4.1.3}Es1$  by Lemma 1.

<sup>4.3.1.4.2</sup> $\forall x(Qx \rightarrow Qsx)$ :  $\frac{4.3.1.4.2.1}{4.3.1.4.2.1}$ Suppose  $Qx$ ,  $\frac{4.3.1.4.2.2}{4.3.1.4.2.2}$ i.e.  $Ef1x$ .  $\frac{4.3.1.4.2.3}{4.3.1.4.2.3}$ By (2)  $f1sx = ssf1x$ ;  $\frac{4.3.1.4.2.4}{4.3.1.4.2.4}$ by Lemma 1 twice,  $Ef1sx$ .  $\frac{4.3.1.4.2.5}{4.3.1.4.2.5}$ Thus  $Qsx$  and  $\frac{4.3.1.4.2.6}{4.3.1.4.2.6}M1$ .

<sup>4.3.2</sup> $\forall n(Mn \rightarrow Msn)$ :  $\frac{4.3.2.1}{4.3.2.1}$ Suppose  $Mn$ ,  $\frac{4.3.2.2}{4.3.2.2}$ i.e.  $\forall x(Nx \rightarrow Efnx)$ .

<sup>4.3.2.3</sup>Want  $Msn$ ,  $\frac{4.3.2.4}{4.3.2.4}$ i.e.  $\forall x(Nx \rightarrow Efnsx)$ .  $\frac{4.3.2.5}{4.3.2.5}$ By comprehension,  $\exists P \forall x(Px \leftrightarrow Efnsx)$ .  $\frac{4.3.2.6}{4.3.2.6}$ Want  $\forall x(Nx \rightarrow Px)$ .  $\frac{4.3.2.7}{4.3.2.7}$ Enough to show  $\frac{4.3.2.7.1}{4.3.2.7.1}P1$  and  $\frac{4.3.2.7.2}{4.3.2.7.2}\forall x(Px \rightarrow Psx)$ .

<sup>4.3.2.7.1</sup>**P1:**  $\frac{4.3.2.7.1.1}{4.3.2.7.1.1}$ Want  $Efsn1$ .  $\frac{4.3.2.7.1.2}{4.3.2.7.1.2}$ But  $fsn1 = s1$  by (1) and  $\frac{4.3.2.7.1.3}{4.3.2.7.1.3}Es1$  by Lemma 1.

<sup>4.3.2.7.2</sup> $\forall x(Px \rightarrow Psx)$ :  $\frac{4.3.2.7.2.1}{4.3.2.7.2.1}$ Suppose  $Px$ ,  $\frac{4.3.2.7.2.2}{4.3.2.7.2.2}$ i.e.  $Efsnx$ ;  $\frac{4.3.2.7.2.3}{4.3.2.7.2.3}$ thus  $Nfsnx$ .  $\frac{4.3.2.7.2.4}{4.3.2.7.2.4}$ Want  $Efsnsx$ .  $\frac{4.3.2.7.2.5}{4.3.2.7.2.5}$ Since  $Nfsnx$  and  $Mn$ ,  $Efnfsnx$ .  $\frac{4.3.2.7.2.6}{4.3.2.7.2.6}$ But by (3)  $fnsnx = fsnx$ ;  $\frac{4.3.2.7.2.7}{4.3.2.7.2.7}$ thus  $Efsnsx$ .

<sup>5</sup>By Lemma 1,  $Nssss1$ . <sup>6</sup>By Lemma 2,  $Efssss1ssss1$ . <sup>7</sup>Thus,  $Dfssss1ssss1$ , as desired.

**Formalization in OMEGA and Mizar:** see [BenzmüllerBrown, The curious inference of Boolos in MIZAR and OMEGA, Studies in Logic, Grammar, and Rhetoric, volume 10(23), pp. 299-388, 2007.]

**Earlier paper:** [BenzmüllerKerber, A Lost Proof, 2001].

**Earlier poster:** <http://christoph-benzmueller.de/papers/poster-tphols01.pdf>

## Introduction: Higher-Order Modal Logics

$\Box P$ : P is necessary, P is obligatory, P is known, P is believed, always P ...

$\Diamond P$ : P is possible, P is permissible, P is epistemically possible, P is doxastically possible, eventually P ...

$\Box$  and  $\Diamond$  are not truth-functional

HOL can be extended by  $\Box$  and  $\Diamond$  to obtain HOML

There are many interesting applications of such logics, e.g. the Ontological Argument

## Introduction: Embedding Approach — Idea

Your-logic (object-logic)

$\psi ::=$  

HOL (meta-logic)

$\varphi ::=$  

Embedding of  in 

 = 

 = 

 = 

 = 

Embedding of meta-logical notions on  in 

*valid* = 

*satisfiable* = 

*...* = 

Pass this set of equations to a higher-order automated theorem prover

## Introduction: Embedding Approach — HOML in HOL

HOML       $\varphi, \psi ::= \dots | \neg\varphi | \varphi \wedge \psi | \varphi \rightarrow \psi | \Box\varphi | \Diamond\varphi | \forall x_\gamma \varphi | \exists x_\gamma \varphi$

HOL       $s, t ::= C_\alpha | x_\alpha | (\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta} | (s_{\alpha \rightarrow \beta} t_\alpha)_\beta | \neg s_o | s_o \vee t_o | \forall x_\alpha t_o$

HOML in HOL: HOML formulas  $\varphi$  are mapped to HOL predicates  $\varphi_{\iota \rightarrow o}$   
(explicit representation of labelled formulas)

$\neg$	$= \lambda \varphi_{\iota \rightarrow o} \lambda w_\iota \neg \varphi w$
$\wedge$	$= \lambda \varphi_{\iota \rightarrow o} \lambda \psi_{\iota \rightarrow o} \lambda w_\iota (\varphi w \wedge \psi w)$
$\rightarrow$	$= \lambda \varphi_{\iota \rightarrow o} \lambda \psi_{\iota \rightarrow o} \lambda w_\iota (\neg \varphi w \vee \psi w)$
$\forall$	$= \lambda h_{\gamma \rightarrow (\iota \rightarrow o)} \lambda w_\iota \forall d_\gamma hdw$
$\exists$	$= \lambda h_{\gamma \rightarrow (\iota \rightarrow o)} \lambda w_\iota \exists d_\gamma hdw$
$\Box$	$= \lambda \varphi_{\iota \rightarrow o} \lambda w_\iota \forall u_\iota (\neg rwu \vee \varphi u)$
$\Diamond$	$= \lambda \varphi_{\iota \rightarrow o} \lambda w_\iota \exists u_\iota (rwu \wedge \varphi u)$
valid	$= \lambda \varphi_{\iota \rightarrow o} \forall w_\iota \varphi w$

Ax (polymorphic over  $\gamma$ )

The equations in Ax are given as axioms to the HOL provers!

## Introduction: Embedding Approach — A “Lean” Prover for HOML KB

```
1 %----The base type $i (already built-in) stands here for worlds and
2 %----mu for individuals; $o (also built-in) is the type of Booleans
3 thf(mu_type,type,(mu:$tType)).
4 %----Reserved constant r for accessibility relation
5 thf(r,type,(r:$i>$i>$o)).
6 %----Modal logic operators not, or, and, implies, box, diamond
7 thf(mnot_type,type,(mnot:($i>$o)>$i>$o)).
8 thf(mnot_definition,(mnot = (^[A:$i>$o,W:$i]:~(A@W)))). 
9 thf(mor_type,type,(mor:($i>$o)>($i>$o)>$i>$o)).
10 thf(mor_definition,(mor = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W) | (Psi@W))))).
11 thf(mand_type,type,(mand:($i>$o)>($i>$o)>$i>$o)).
12 thf(mand_definition,(mand = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W) & (Psi@W))))).
13 thf(mimplies_type,type,(mimplies:($i>$o)>($i>$o)>$i>$o)).
14 thf(mimplies_definition,(mimplies = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W) & (Psi@W))))).
15 thf(mbox_type,type,(mbox:($i>$o)>$i>$o)).
16 thf(mbox_definition,(mbox = (^[A:$i>$o,W:$i]:![V:$i]:(~(r@W@V) | (A@V))))).
17 thf(mdia_type,type,(mdia:($i>$o)>$i>$o)).
18 thf(mdia_definition,(mdia = (^[A:$i>$o,W:$i]:?[V:$i]:((r@W@V) & (A@V))))).
19 %----Quantifiers (constant domains) for individuals and propositions
20 thf(mforall_ind_type,type,(mforall_ind:(mu:$i>$o)>$i>$o)).
21 thf(mforall_ind_definition,(mforall_ind = (^[A:mu:$i>$o,W:$i]:![X:mu]: (A@X@W)))). 
22 thf(mforall_indset_type,type,(mforall_indset:(mu:$i>$o)>$i>$o)>$i>$o)).
23 thf(mforall_indset_definition,(mforall_indset = (^[A:(mu:$i>$o)>$i>$o,W:$i]:![X:mu:$i>$o]: (A@X@W)))). 
24 thf(mexists_ind_type,type,(mexists_ind:(mu:$i>$o)>$i>$o)).
25 thf(mexists_ind_definition,(mexists_ind = (^[A:mu:$i>$o,W:$i]:?[X:mu]: (A@X@W)))). 
26 thf(mexists_indset_type,type,(mexists_indset:(mu:$i>$o)>$i>$o)>$i>$o)).
27 thf(mexists_indset_definition,(mexists_indset = (^[A:(mu:$i>$o)>$i>$o,W:$i]:?[X:mu:$i>$o]: (A@X@W)))). 
28 %----Definition of validity (grounding of lifted modal formulas)
29 thf(v_type,type,(v:($i>$o)>$o)).
30 thf(mvalid_definition,(v = (^[A:$i>$o]:![W:$i]: (A@W)))). 
31 %----Properties of accessibility relations: symmetry
32 thf(msymmetric_type,type,(msymmetric:($i>$i>$o)>$o)).
33 thf(msymmetric_definition,(msymmetric = (^[R:$i>$i>$o]:![S:$i,T:$i]:((R@S@T) => (R@T@S))))).
34 %----Here we work with logic KB, i.e., we postulate symmetry for r
35 thf(sym_axiom,(msymmetric@r)).
```

Reading on THF0 syntax: [SutcliffeBenzmüller, J. Formalized Reasoning, 2010]

## Advantages of the Approach

### Pragmatics and convenience.

- \* 'Implementing' a new theorem prover made very simple
- \* even for very challenging quantified non-classical logics

### Flexibility

- \* Rapid experimentations with logic variations: quantifiers for constant, varying and cumulative domains; rigid or non-rigid terms; etc.
- \* Sahlqvist axioms may be postulated or (preferably) the corresponding conditions of the accessibility relation
- \* prominent connections between logics can be verified and exploited

### Availability

- \* Option 1: Reuse and adapt our TPTP THF0 encodings
- \* Option 2: Reuse and adapt our Isabelle encodings
- \* Option 3: Reuse and adapt our Coq encodings
- \* Option X: Adapt our encodings for your preferred HO prover
- \* Options can be employed simultaneously.

## Advantages of the Approach

### Relation to labelled deductive systems

- \* Labelled deductive systems: meta-level (world-)labelling techniques
- \* Embedding approach: labels are encoded directly in HOL logic
- \* No extra-logical annotations in embedding approach

### Relation to the standard translation

- \* Intra-logical formalisation and implementation of the standard translation
- \* Extension: various other logics
- \* Extension: quantifiers (different domain conditions)
- \* Future work: functional translation as an alternative

### Soundness and completeness

- \* Sound and complete (Henkin semantics)

### Meta-reasoning

- \* Example: Verification of the modal logic cube in Isabelle
- \* Example: Soundness of the usual ALC tableaux rules
- \* Example: Correspondence between ALC and base modal logic K
- \* Example: Some meta-level results for conditional logics
- \* Future work: Completeness results via abstract consistency

## Advantages of the Approach

### Direct calculi and user intuition

- \* Implementation of 'direct' proof calculi on top of logic embeddings
- \* Example: ND calculus for HOML as tactics in Coq
- \* Human intuitive proofs enabled at the interaction layer
- \* Automation via embedding and ATPs for HOL
- \* Interesting perspective for mixed proof developments
- \* Future work: proof planning to automate the abstract-level direct proof calculi; proof assistants in the style of  $\Omega$ mega could eventually be adapted for this (support for 3-dimensional proof objects!)

### Cut-elimination

- \* Very generic result: combine soundness and completeness of the embedding with the fact that HOL already enjoys cut-elimination for Henkin semantics



**Higher-order Logic (HOL)**

# Classical Higher-Order Logic (HOL)

Expressivity	FOL	HOL	Example
Quantification over			
- Individuals	✓	✓	$\forall X p(f(X))$
- Functions	✗	✓	$\forall F p(F(a))$
- Predicates/Sets/Relns	✗	✓	$\forall P P(f(a))$
Unnamed			
- Functions	✗	✓	$(\lambda X X)$
- Predicates/Sets/Relns	✗	✓	$(\lambda X X \neq a)$
Statements about			
- Functions	✗	✓	<i>continuous</i> $(\lambda X X)$
- Predicates/Sets/Relns	✗	✓	<i>reflexive</i> $(=)$
Powerful abbreviations	✗	✓	<i>reflexive</i> = $\lambda R \forall X R(X, X)$

# Classical Higher-Order Logic (HOL)

Expressivity	FOL	HOL	Example
Quantification over			
- Individuals	✓	✓	$\forall X_\iota p_{\iota \rightarrow o}(f_{\iota \rightarrow \iota}(X_\iota))$
- Functions	✗	✓	$\forall F_{\iota \rightarrow \iota} p_{\iota \rightarrow o}(F_{\iota \rightarrow o}(a_\iota))$
- Predicates/Sets/Relns	✗	✓	$\forall P_{\iota \rightarrow o} P_{\iota \rightarrow o}(f_{\iota \rightarrow \iota}(a_\iota))$
Unnamed			
- Functions	✗	✓	$(\lambda X_\iota X_\iota)$
- Predicates/Sets/Relns	✗	✓	$(\lambda X_{\iota \rightarrow \iota} X_{\iota \rightarrow \iota} \neq_{\iota \rightarrow \iota \rightarrow o} a)_\iota$
Statements about			
- Functions	✗	✓	$continuous_{(\iota \rightarrow \iota) \rightarrow o}(\lambda X_\iota X_\iota)$
- Predicates/Sets/Relns	✗	✓	$reflexive_{(\iota \rightarrow \iota \rightarrow o) \rightarrow o}(=_{\iota \rightarrow \iota \rightarrow o})$
Powerful abbreviations	✗	✓	$reflexive_{(\iota \rightarrow \iota \rightarrow o) \rightarrow o} =$ $\lambda R_{(\iota \rightarrow \iota \rightarrow o)} \forall X_\iota R(X, X)$

**Simple Types:** Prevent Some Paradoxes and Inconsistencies

## HOL: Syntax

Simple Types:

$$\alpha, \beta ::= \iota \mid o \mid (\alpha \rightarrow \beta)$$

(we may add further base types, e.g.  $\mu$ )

HOL Language:

$$s, t ::= p_\alpha \mid X_\alpha \mid (\lambda X_\alpha s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta} t_\alpha)_\beta \mid \\ (\neg_{o \rightarrow o} s_o) \mid ((\vee_{o \rightarrow o \rightarrow o} s_o) t_o) \mid \forall_{(\alpha \rightarrow o) \rightarrow o} (\lambda X_\alpha s_o)$$

constant symbols

variable symbols

lambda abstraction

application

negation

disjunction

universal quantification

Terms of type  $o$ : formulas

Other logical connectives can be defined, e.g.  $\exists X s$  stands for  $\neg \forall X \neg s$

Equality may also be defined:  $s \doteq t$  stands for  $\forall P (Ps \Rightarrow Pt)$   
(but it is strongly recommended to add primitive equality!)

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Equality may also be defined:  $s \doteq t$  stands for  $\forall P (Ps \Rightarrow Pt)$   
(but it is strongly recommended to add primitive equality!)

## HOL: Syntax

Simple Types:

$$\alpha, \beta ::= \iota \mid o \mid (\alpha \rightarrow \beta)$$

(we may add further base types, e.g.  $\mu$ )

HOL Language:

$$s, t ::= p_\alpha \mid X_\alpha \mid (\lambda X_\alpha s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta} t_\alpha)_\beta \mid$$

$(\neg_{o \rightarrow o} s_o)$

$$| ((\vee_{o \rightarrow o \rightarrow o} s_o) t_o) \mid \forall_{(\alpha \rightarrow o) \rightarrow o} (\lambda X_\alpha s_o)$$

constant symbols

variable symbols

lambda abstraction

application

**negation**

disjunction

universal quantification

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disjunction (we may use infix notation, omit types and brackets:  $s \vee t$ )

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(we may use syntactical sugar:  $\forall X_\alpha s$ )

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## HOL: Syntax

### $\alpha$ -conversion

is considered implicitly, e.g.  $(\lambda X pX)$  is identified with  $(\lambda Y pY)$ .

### Substitution

of a term  $s_\alpha$  for  $X_\alpha$  in  $t_\beta$  is denoted by  $[s/X]t$ .

We assume the bound variables of  $t$  avoid variable capture.

### $\beta$ -reduction and $\eta$ -reduction

$\beta$ -redex has the form  $(\lambda X s)t$  and  $\beta$ -reduces to  $[t/X]s$ .

$\eta$ -redex has the form  $(\lambda X sX)$  where  $X$  is not free in  $s$ ; it  $\eta$ -reduces to  $s$ .

$s \equiv_\beta t$  means  $s$  can be converted to  $t$  by  $\beta$ -reductions and expansions.

$s \equiv_{\beta\eta} t$  means  $s$  can be converted to  $t$  using both  $\beta$  and  $\eta$ .

For each  $s_\alpha \in \text{HOML}$  there is a unique  $\beta$ -normal form and a unique  $\beta\eta$ -normal form.

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### Semantics of HOL is (meanwhile) well understood

- ▶ Origin [Church, J.Symb.Log., 1940]
- ▶ Henkin-Semantics [Henkin, J.Symb.Log., 1950]  
[Andrews, J.Symb.Log., 1971, 1972]
- ▶ Extensionality/Intensionality [BenzmüllerEtAl., J.Symb.Log., 2004]  
[Muskens, J.Symb.Log., 2007]

HOL with Henkin-Semantics: semi-decidable & compact (like FOL)

Recommended Reading: [BenzmüllerMiller, HandbookHistoryOfLogic, Vol.9, 2014]

### A Frame $D$

is a collection  $\{D_\alpha\}_{\alpha \in T}$  of nonempty sets  $D_\alpha$ , such that

- ▶  $D_t$  can be chosen freely
- ▶  $D_o = \{T, F\}$  (for truth and falsehood), and
- ▶  $D_{\alpha \rightarrow \beta}$  are collections of functions mapping  $D_\alpha$  into  $D_\beta$

### A Model $M$

for HOL is a tuple  $M = \langle D, I \rangle$ , where

- ▶  $D$  is a frame;
- ▶  $I$  is a family of typed interpretation functions mapping constant symbols  $p_\alpha$  to appropriate elements of  $D_\alpha$ , called the denotation of  $p_\alpha$ ;
- ▶ the logical connectives  $\neg$ ,  $\vee$ , and  $\forall$  are always given the standard denotations;
- ▶ moreover, we assume that the domains  $D_{\alpha \rightarrow \alpha \rightarrow o}$  contain the respective identity relations.

### Variable Assignment

A variable assignment  $g$  maps variables  $X_\alpha$  to elements in  $D_\alpha$ .

$g[d/W]$  denotes the assignment that is identical to  $g$ , except for variable  $W$ , which is now mapped to  $d$ .

### Interpretation/Value of a HOL term

The value  $\|s_\alpha\|^{M,g}$  of a HOL term  $s_\alpha$  on a model  $M = \langle D, I \rangle$  under assignment  $g$  is an element  $d \in D_\alpha$  defined in the following way:

1.  $\|p_\alpha\|^{M,g} = I(p_\alpha)$
2.  $\|X_\alpha\|^{M,g} = g(X_\alpha)$
3.  $\|(s_{\alpha \rightarrow \beta} t_\alpha)_\beta\|^{M,g} = \|s_{\alpha \rightarrow \beta}\|^{M,g} (\|t_\alpha\|^{M,g})$
4.  $\|(\lambda X_\alpha s_\beta)_{\alpha \rightarrow \beta}\|^{M,g} =$  the function  $f$  from  $D_\alpha$  to  $D_\beta$  such that  
 $f(d) = \|s_\beta\|^{M,g[d/X_\alpha]}$  for all  $d \in D_\alpha$
5.  $\|(\neg_{o \rightarrow o} s_o)_o\|^{M,g} = T$  iff  $\|s_o\|^{M,g} = F$
6.  $\|((\vee_{o \rightarrow o \rightarrow o} s_o) t_o)_o\|^{M,g} = T$  iff  $\|s_o\|^{M,g} = T$  or  $\|t_o\|^{M,g} = T$
7.  $\|(\forall_{(\alpha \rightarrow o) \rightarrow o} (\lambda X_\alpha s_o))_o\|^{M,g} = T$  iff for all  $d \in D_\alpha$  we have  
 $\|s_o\|^{M,g[d/X_\alpha]} = T$

### Standard and Henkin Models

In a standard model  $M = \langle D, I \rangle$  we have

- ▶  $D_{\alpha \rightarrow \beta} = \{f \mid f : D_\alpha \longrightarrow D_\beta\}$  (for all types  $\alpha, \beta$ )

In a Henkin model  $M = \langle D, I \rangle$  we only require

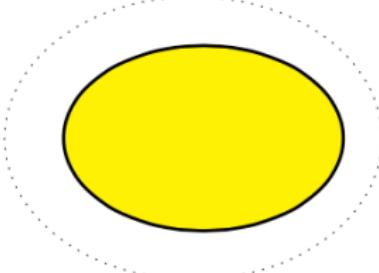
- ▶  $D_{\alpha \rightarrow \beta} \subseteq \{f \mid f : D_\alpha \longrightarrow D_\beta\}$  (for all types  $\alpha, \beta$ )
- ▶ the valuation function  $\| \cdot \|^{M,g}$  from above is total (every term denotes)

Any standard model is obviously also a Henkin model.

We consider Henkin models in the remainder.

## HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



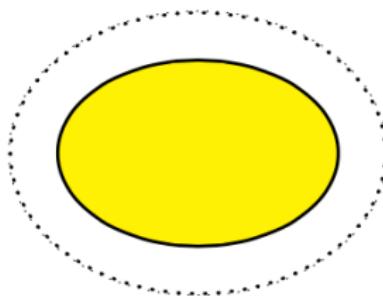
### ■ Idea of Standard Semantics:

- $\iota \longrightarrow \mathcal{D}_\iota$  (choose)
- $\circ \longrightarrow \mathcal{D}_\circ = \{\text{T}, \text{F}\}$  (fixed)
- $(\alpha \rightarrow \beta) \longrightarrow \mathcal{D}_{\alpha \rightarrow \beta} = \mathcal{F}(\mathcal{D}_\alpha, \mathcal{D}_\beta)$  (fixed)

Standard Models  $\mathfrak{ST}(\Sigma)$

## HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



Standard Models  $\mathfrak{S}\mathfrak{T}(\Sigma)$

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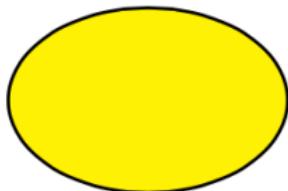
### ■ Henkin's Generalization:

- $\mathcal{D}_{\alpha \rightarrow \beta} \subseteq \mathcal{F}(\mathcal{D}_\alpha, \mathcal{D}_\beta)$  (choose)  
but elements are still functions!

[Henkin-50]

## HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])

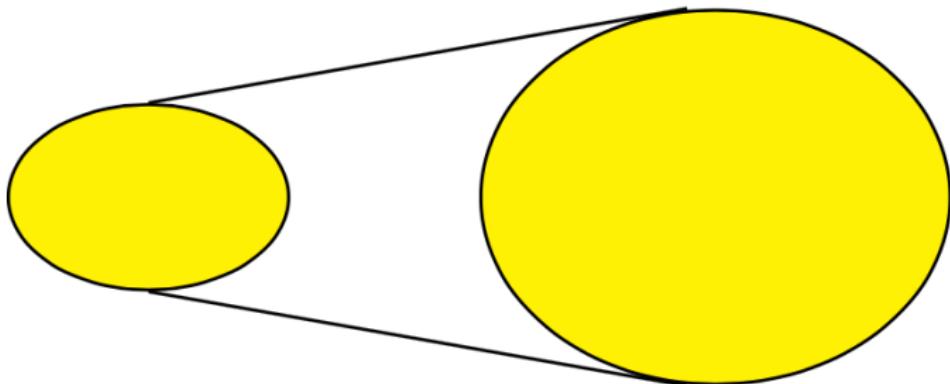


Standard Models  $\mathfrak{S}\mathfrak{T}(\Sigma)$

choose:  $\mathcal{D}_t$   
fixed:  $\mathcal{D}_o, \mathcal{D}_{\alpha \rightarrow \beta}$ , functions

## HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



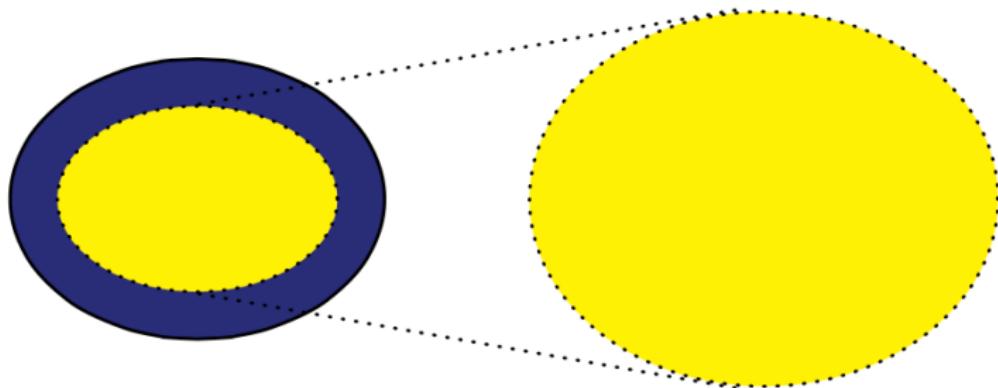
Standard Models  $\mathfrak{S}\mathfrak{T}(\Sigma)$

Formulas valid in  $\mathfrak{S}\mathfrak{T}(\Sigma)$

choose:  $\mathcal{D}_t$   
fixed:  $\mathcal{D}_o, \mathcal{D}_{\alpha \rightarrow \beta}$ , functions

## HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



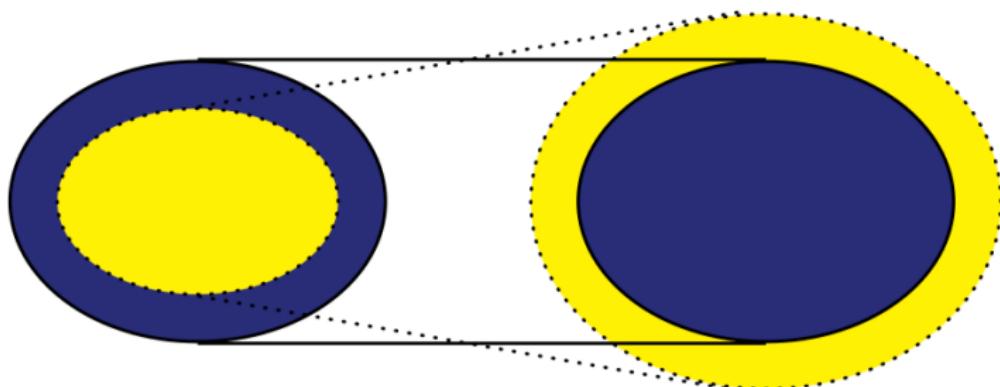
Henkin Models  $\mathfrak{H}(\Sigma) = \mathfrak{M}_{\text{bf}}(\Sigma)$

Formulas valid in  $\mathfrak{M}_{\text{bf}}(\Sigma)$  ?

choose:  $\mathcal{D}_\ell, \mathcal{D}_{\alpha \rightarrow \beta}$   
fixed:  $\mathcal{D}_o$ , functions

## HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



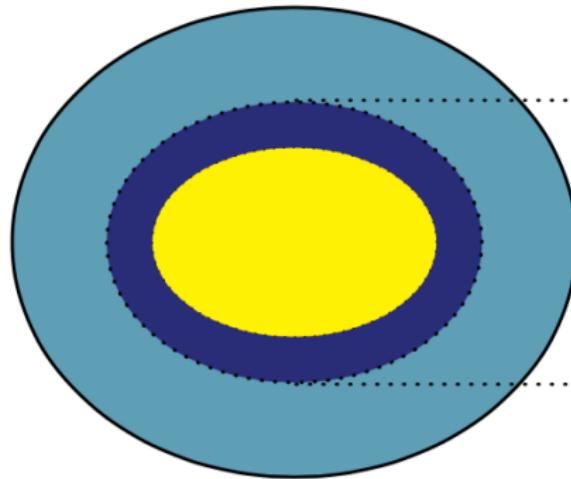
Henkin Models  $\mathfrak{H}(\Sigma) = \mathfrak{M}_{\beta\text{fb}}(\Sigma)$

Formulas valid in  $\mathfrak{M}_{\beta\text{fb}}(\Sigma)$

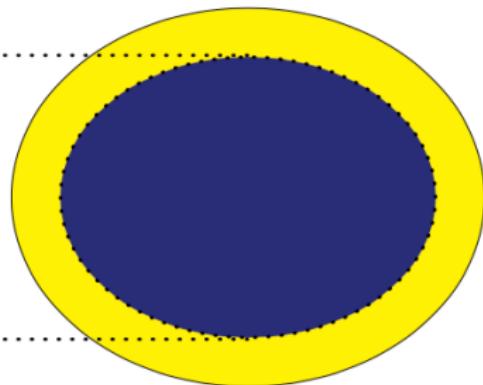
choose:  $\mathcal{D}_t, \mathcal{D}_{\alpha \rightarrow \beta}$   
fixed:  $\mathcal{D}_o$ , functions

## HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



Non-Extensional Models  $\mathfrak{M}_\beta(\Sigma)$

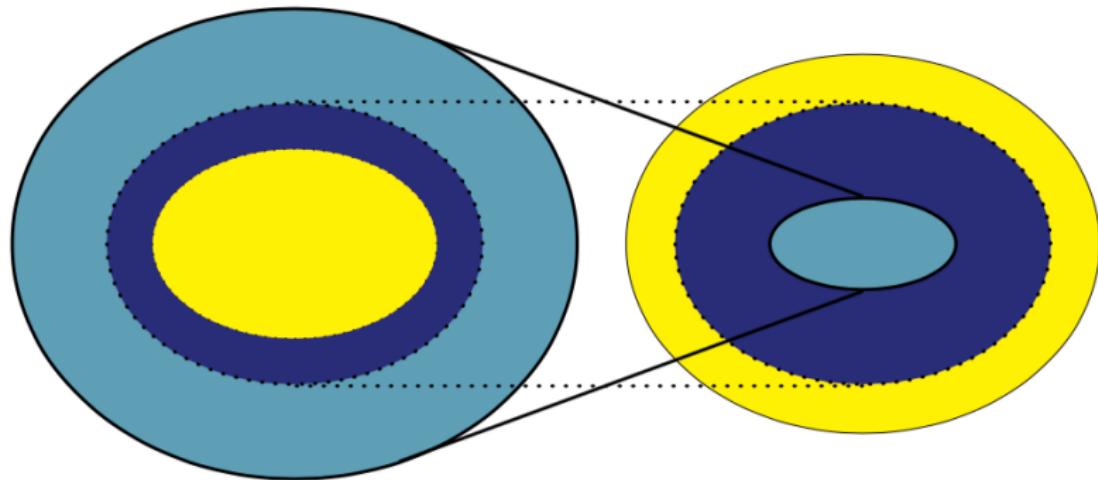


Formulas valid in  $\mathfrak{M}_\beta(\Sigma)$  ?

choose:  $\mathcal{D}_\iota, \mathcal{D}_{\alpha \rightarrow \beta}$ , also non-functions,  $\mathcal{D}_o$   
fixed:

## HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



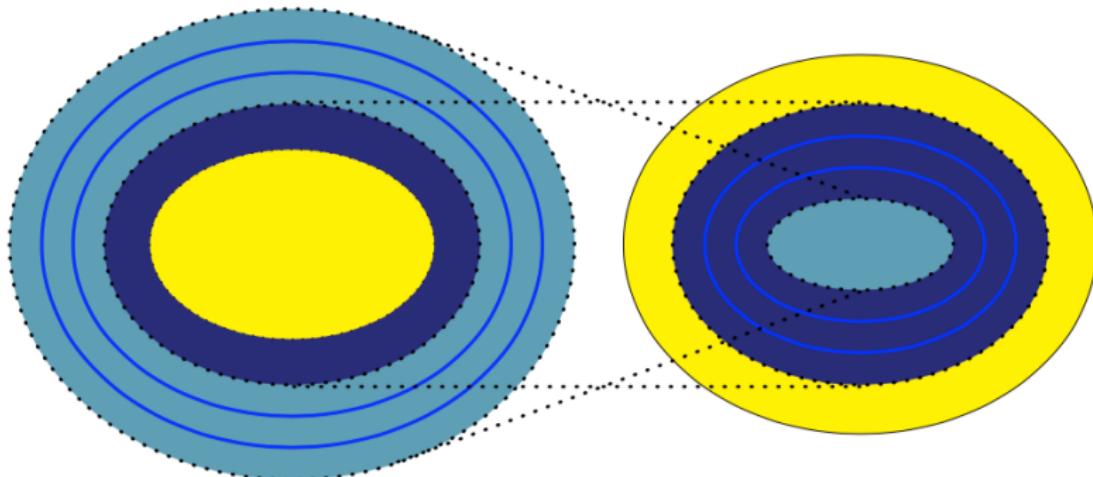
Non-Extensional Models  $\mathfrak{M}_\beta(\Sigma)$

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## HOL: Semantics

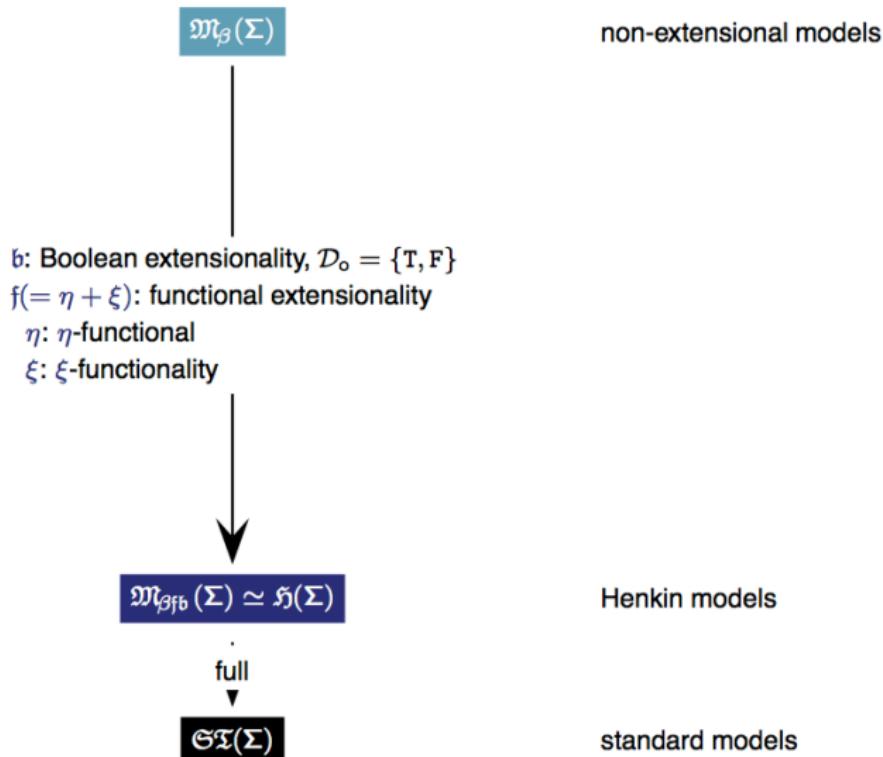
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We additionally studied different model classes with 'varying degrees of extensionality'

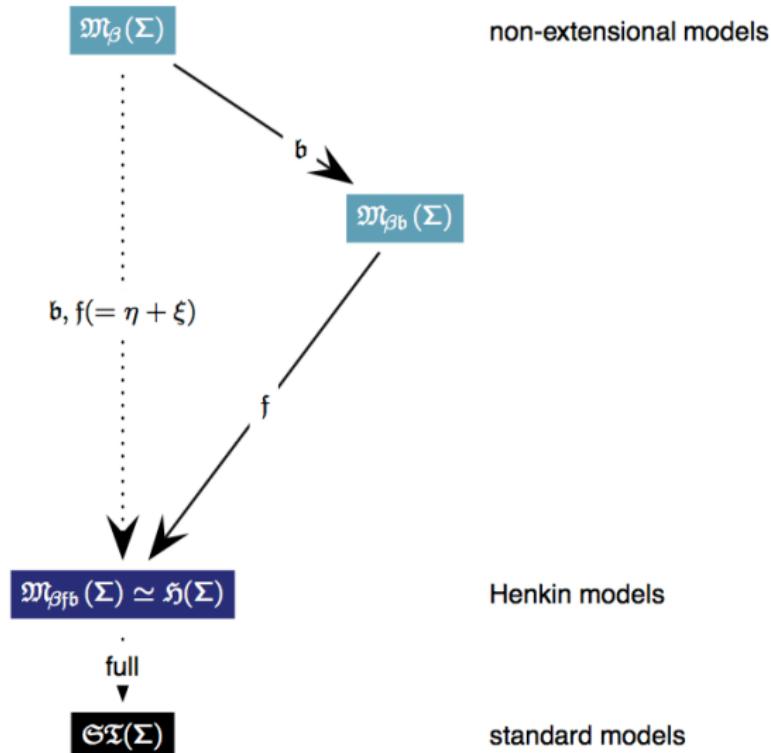
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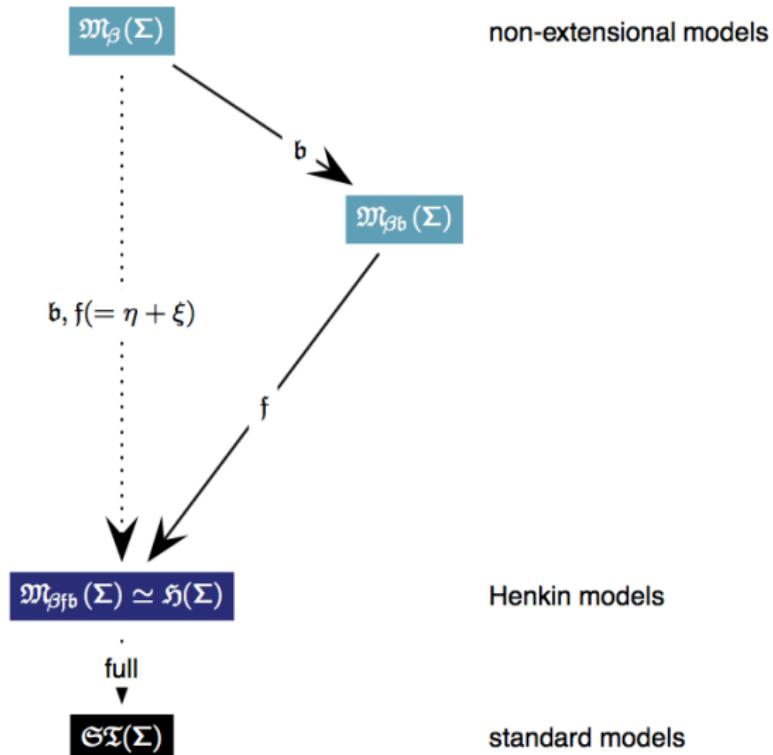
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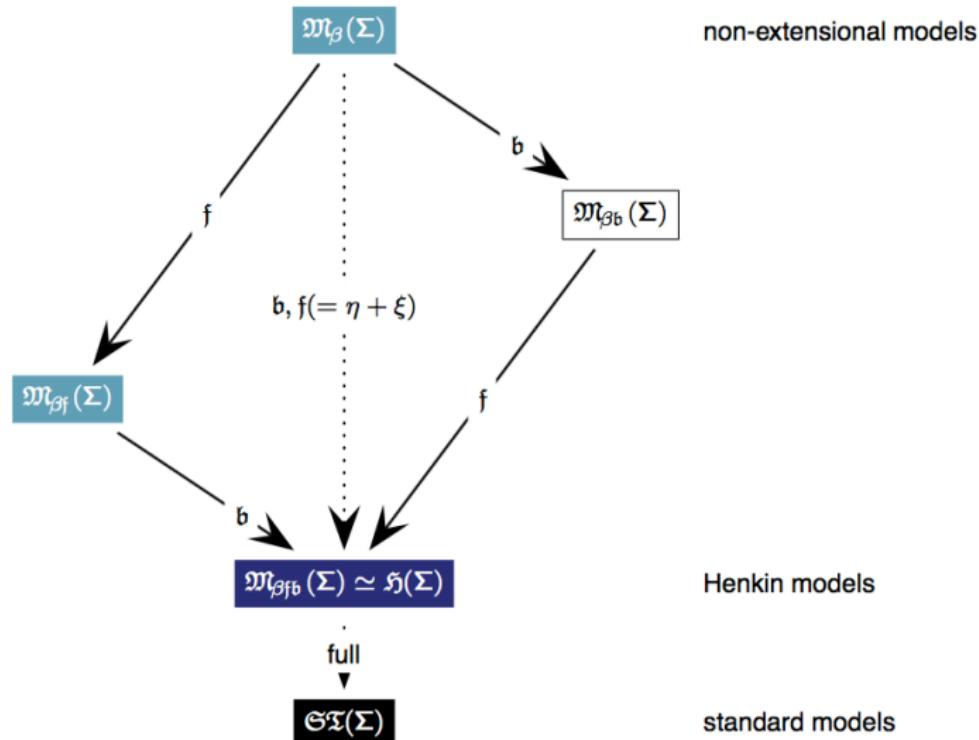
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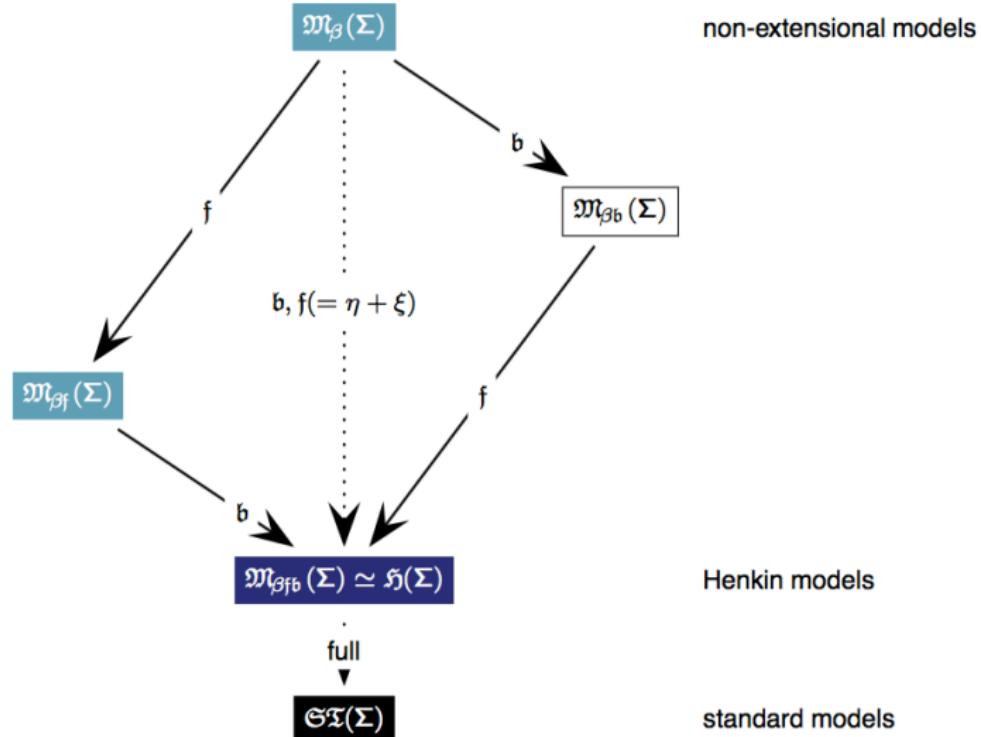
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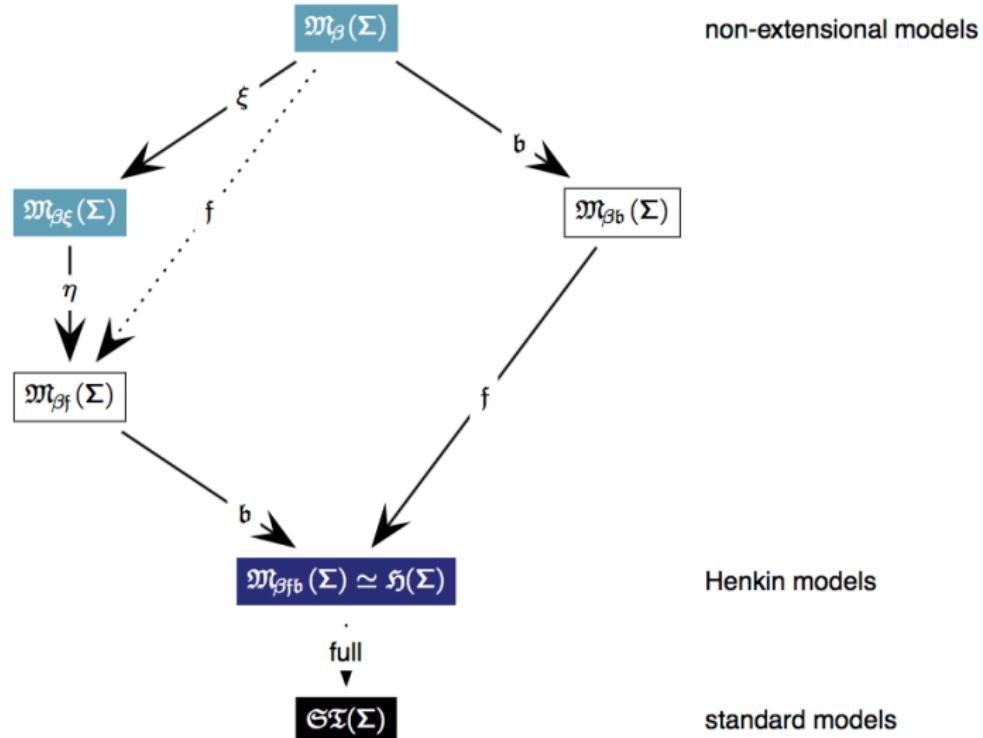
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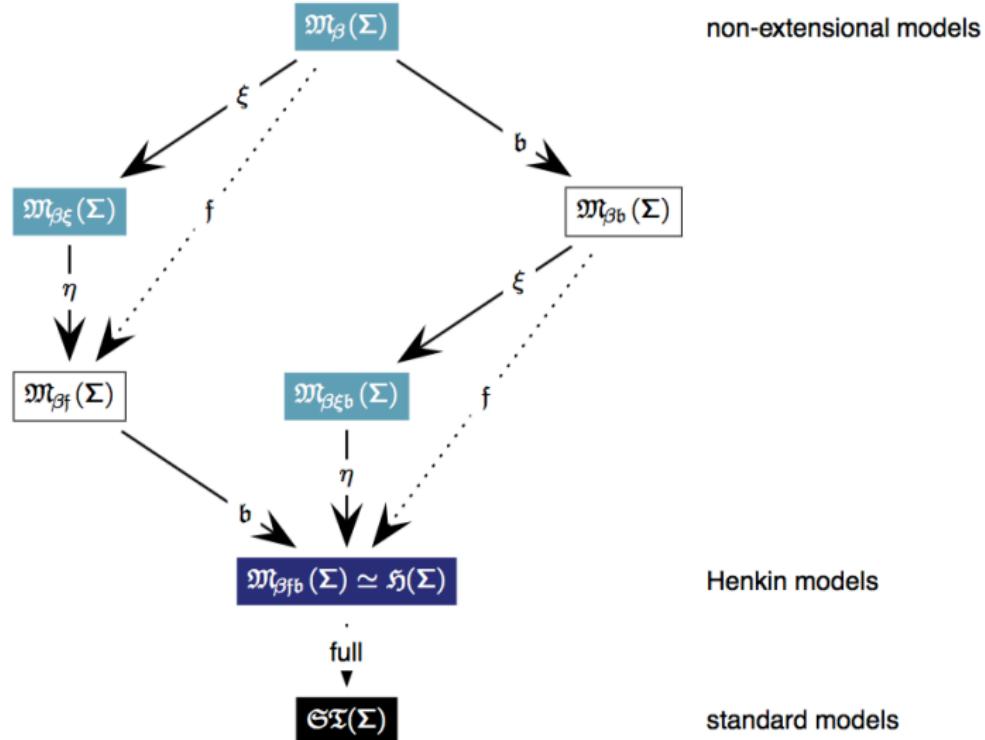
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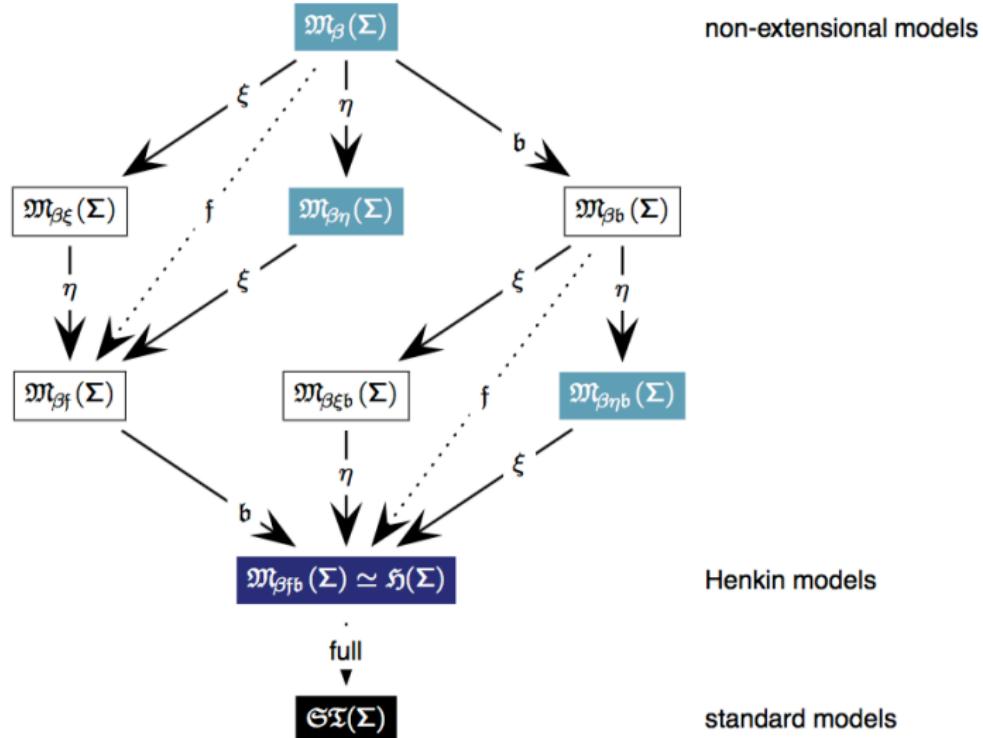
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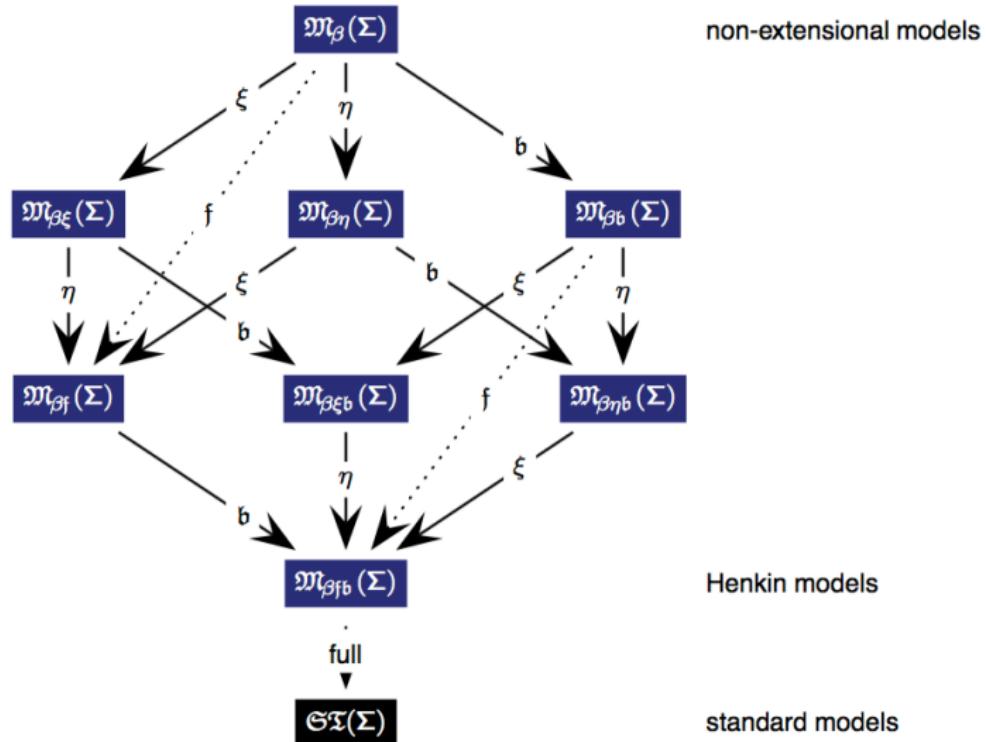
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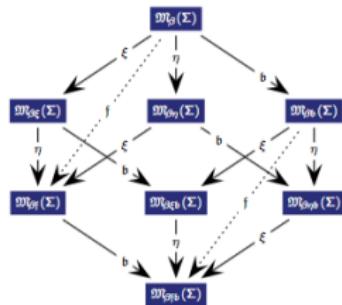
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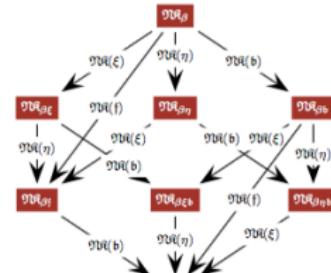
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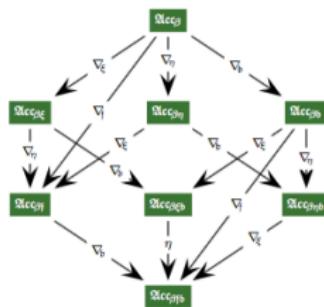
## Semantics: Model Classes (Extensionality)



## Reference Calculi: ND (and others)



Abstract Consistency / Unifying Principle:  
Extensions of Smullyan-63 and Andrews-71





## Higher-order Modal Logic (HOML)

## HOML: Motivation

$\Box P$

P is necessary, P is obligatory, P is known,  
P is believed, always P . . .

$\Diamond P$

P is possible, P is permissible, P is epistemically possible,  
P is doxastically possible, eventually P . . .

$\Box$  and  $\Diamond$  are not truth-functional

HOL can be extended by  $\Box P$  and  $\Diamond P$  to obtain HOML

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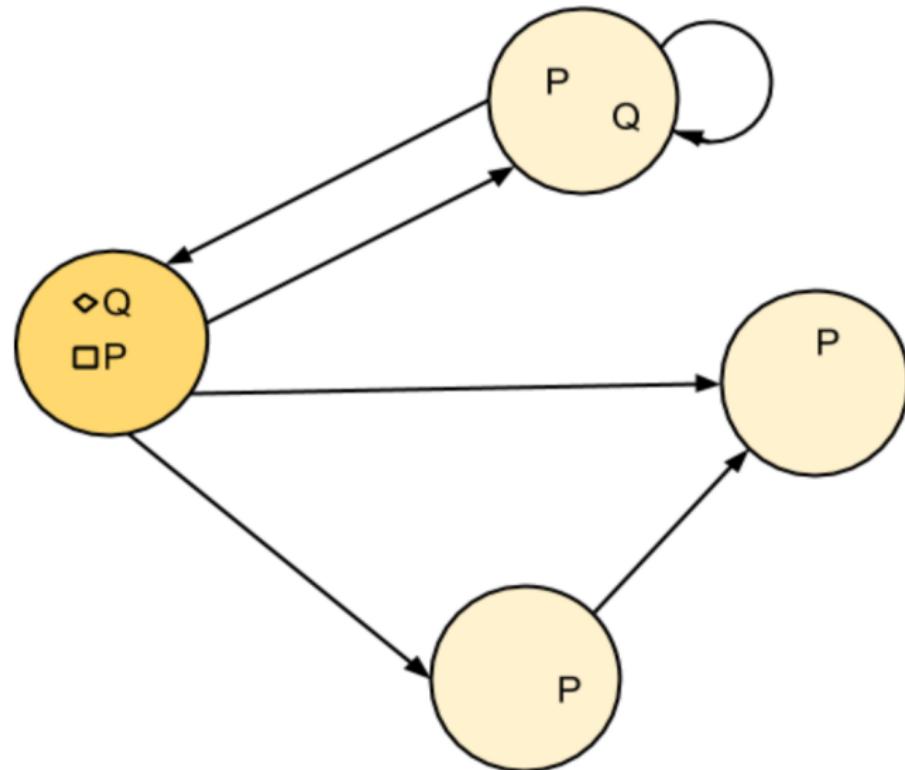
P is possible, P is permissible, P is epistemically possible,  
P is doxastically possible, eventually P ...

$\Box$  and  $\Diamond$  are not truth-functional

HOL can be extended by  $\Box P$  and  $\Diamond P$  to obtain HOML

## HOML: Motivation

### Kripke Semantics - Possible Worlds



## HOML: Syntax

Simple Types:

$$\alpha, \beta ::= \iota \mid o \mid (\alpha \rightarrow \beta)$$

HOML Language:

$$\begin{aligned} s, t ::= & p_\alpha \mid X_\alpha \mid (\lambda X_\alpha s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta} t_\alpha)_\beta \mid \\ & (\neg_{o \rightarrow o} s_o) \mid ((\vee_{o \rightarrow o \rightarrow o} s_o) t_o) \mid \forall_{(\alpha \rightarrow o) \rightarrow o} (\lambda X_\alpha s_o) \\ & (\Box_{o \rightarrow o} s_o) \end{aligned}$$

constant symbols

variable symbols

lambda abstraction

application

negation

disjunction

universal quantification

modal box operator

Terms of type  $o$ : formulas

Other logical connectives can be defined, e.g.  $\Diamond s$  stands for  $\neg \Box \neg s$

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## HOML: Syntax

### $\alpha$ -conversion

... as before ...

### Substitution

... as before ...

### $\beta$ -reduction and $\eta$ -reduction

... as before ...

### Semantics of HOML investigated in

- ▶ [D. Gallin, Intensional and Higher-Order Modal Logic, North Holland, 1975]
- ▶ [R. Muskens, Higher Order Modal Logic, Handbook of Modal Logic, 2006]
- ▶ [Benzmüller and Woltzenlogel Paleo, Automating Gödel's Ontological Proof . . . , ECAI, 2014]
- ▶ our interest: combination of Kripke style models and Henkin semantics

### A Frame $D$

... as before ...

### A Model $M$

for HOML is a quadruple  $M = \langle W, R, D, \{I_w\}_{w \in W} \rangle$ , where

- ▶  $W$  is a set of worlds (or states);
- ▶  $R$  is an accessibility relation between the worlds in  $W$ ;
- ▶  $D$  is a frame;
- ▶ for each  $w \in W$ ,  $\{I_w\}_{w \in W}$  is a family of typed interpretation functions mapping constant symbols  $p_\alpha$  to appropriate elements of  $D_\alpha$ , called the denotation of  $p_\alpha$  in world  $w$ ;
- ▶ the logical connectives  $\neg$ ,  $\vee$ ,  $\forall$ , and  $\square$  are always given the standard denotations;
- ▶ moreover, it is assumed that the domains  $D_{\alpha \rightarrow \alpha \rightarrow o}$  contain the respective identity relations on objects of type  $\alpha$ .

## Variable Assignment

... as before ...

## Interpretation/Value of a HOML term

The value  $\|s_\alpha\|^{M,g,w}$  of a HOML term  $s_\alpha$  on a model  $M = \langle W, R, D, \{I_w\}_{w \in W} \rangle$  in a world  $w \in W$  under variable assignment  $g$  is an element  $d \in D_\alpha$  defined in the following way:

1.  $\|p_\alpha\|^{M,g,w} = I_w(p_\alpha)$
2.  $\|X_\alpha\|^{M,g,w} = g(X_\alpha)$
3.  $\|(s_{\alpha \rightarrow \beta} t_\alpha)_\beta\|^{M,g,w} = \|s_{\alpha \rightarrow \beta}\|^{M,g,w}(\|t_\alpha\|^{M,g,w})$
4.  $\|(\lambda X_\alpha s_\beta)_{\alpha \rightarrow \beta}\|^{M,g,w} = \text{the function } f \text{ from } D_\alpha \text{ to } D_\beta \text{ such that } f(d) = \|s_\beta\|^{M,g[d/X_\alpha],w} \text{ for all } d \in D_\alpha$
5.  $\|(\neg_{o \rightarrow o} s_o)_o\|^{M,g,w} = T \text{ iff } \|s_o\|^{M,g,w} = F$
6.  $\|((\vee_{o \rightarrow o \rightarrow o} s_o) t_o)_o\|^{M,g,w} = T \text{ iff } \|s_o\|^{M,g,w} = T \text{ or } \|t_o\|^{M,g,w} = T$
7.  $\|(\forall_{(\alpha \rightarrow o) \rightarrow o} (\lambda X_\alpha s_o))_o\|^{M,g,w} = T \text{ iff for all } d \in D_\alpha \text{ we have } \|s_o\|^{M,g[d/X_\alpha],w} = T$
8.  $\|(\Box_{o \rightarrow o} s_o)_o\|^{M,g,w} = T \text{ iff for all } v \in W \text{ with } wRv \text{ we have } \|s_o\|^{M,g,v} = T$

### Standard and Henkin Models (as before)

In a standard model  $M = \langle W, R, D, \{I_w\}_{w \in W} \rangle$  we have

- ▶  $D_{\alpha \rightarrow \beta} = \{f \mid f : D_\alpha \longrightarrow D_\beta\}$  (for all types  $\alpha, \beta$ )

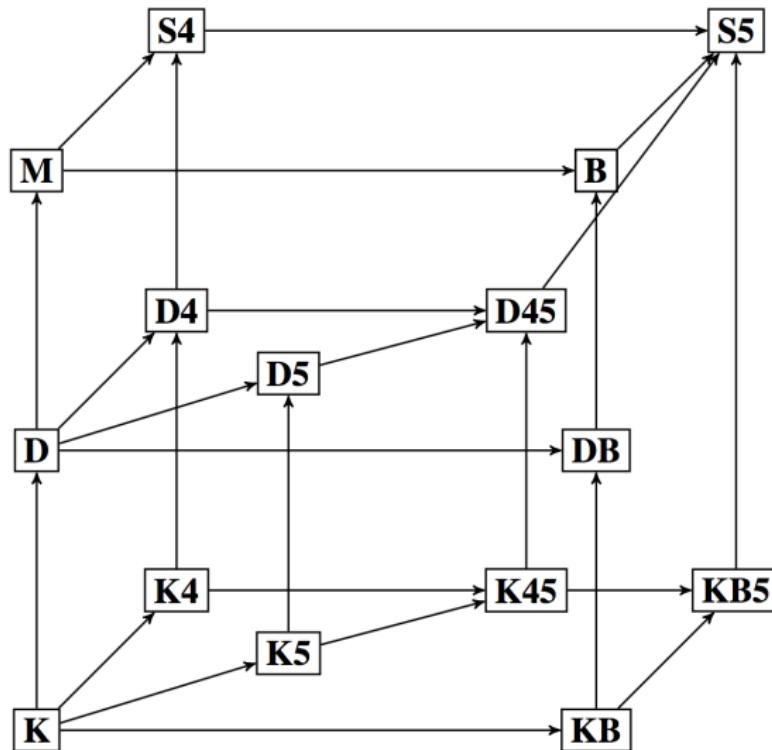
In a Henkin model  $M = \langle W, R, D, \{I_w\}_{w \in W} \rangle$  we only require

- ▶  $D_{\alpha \rightarrow \beta} \subseteq \{f \mid f : D_\alpha \longrightarrow D_\beta\}$  (for all types  $\alpha, \beta$ )
- ▶ the valuation function  $\|\cdot\|^{M,g,w}$  from above is total (every term denotes)

Any standard model is obviously also a Henkin model.

We consider Henkin models in the remainder.

# HOML: The Modal Logic Cube



$$\equiv M5 \equiv MB5 \equiv M4B5$$

$$\equiv M45 \equiv M4B \equiv D4B$$

$$\equiv D4B5 \equiv DB5$$

$$M: \Box P \rightarrow P$$

$$B: P \rightarrow \Box \Diamond P$$

$$D: \Box P \rightarrow \Diamond P$$

$$4: \Box P \rightarrow \Box \Box P$$

$$5: \Diamond P \rightarrow \Box \Diamond P$$

M



$$\equiv K4B5 \equiv K4B$$



**How to automate HOML?**

## Embedding HOML in HOL

Challenge: No provers for Higher-order Modal Logic (HOML)

Church's Simple Type Theory

Our solution: Embedding in Higher-order Classical Logic (HOL)

Then use existing HOL theorem provers for reasoning in HOML

[BenzmüllerPaulson, Logica Universalis, 2013]

Assumption: Henkin semantics for both HOML and HOL

Previous empirical findings:

Embedding of First-order Modal Logic in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[Benzmüller, LPAR, 2013]

## Embedding HOML in HOL

HOML       $\varphi, \psi ::= \dots | \neg\varphi | \varphi \wedge \psi | \varphi \rightarrow \psi | \Box\varphi | \Diamond\varphi | \forall x_\gamma \varphi | \exists x_\gamma \varphi$

HOL       $s, t ::= C_\alpha | x_\alpha | (\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta} | (s_{\alpha \rightarrow \beta} t_\alpha)_\beta | \neg s_o | s_o \vee t_o | \forall x_\alpha t_o$

HOML in HOL: HOML formulas  $\varphi$  are mapped to HOL predicates  $\varphi_{\iota \rightarrow o}$   
(explicit representation of labelled formulas)

$\neg$	$= \lambda \varphi_{\iota \rightarrow o} \lambda w_\iota \neg \varphi w$
$\wedge$	$= \lambda \varphi_{\iota \rightarrow o} \lambda \psi_{\iota \rightarrow o} \lambda w_\iota (\varphi w \wedge \psi w)$
$\rightarrow$	$= \lambda \varphi_{\iota \rightarrow o} \lambda \psi_{\iota \rightarrow o} \lambda w_\iota (\neg \varphi w \vee \psi w)$
$\forall$	$= \lambda h_{\gamma \rightarrow (\iota \rightarrow o)} \lambda w_\iota \forall d_\gamma hdw$
$\exists$	$= \lambda h_{\gamma \rightarrow (\iota \rightarrow o)} \lambda w_\iota \exists d_\gamma hdw$
$\Box$	$= \lambda \varphi_{\iota \rightarrow o} \lambda w_\iota \forall u_\iota (\neg rwu \vee \varphi u)$
$\Diamond$	$= \lambda \varphi_{\iota \rightarrow o} \lambda w_\iota \exists u_\iota (rwu \wedge \varphi u)$
valid	$= \lambda \varphi_{\iota \rightarrow o} \forall w_\iota \varphi w$

Ax (polymorphic over  $\gamma$ )

The equations in Ax are given as axioms to the HOL provers!

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## Embedding HOML in HOL

### Example

HOML formula

HOML formula in HOL

expansion,  $\beta\eta$ -conversion

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expansion,  $\beta\eta$ -conversion

$\diamond \exists x G(x)$

valid  $(\diamond \exists x G(x))_{t \rightarrow o}$

$\forall w_L (\diamond \exists x G(x))_{t \rightarrow o} w$

$\forall w_L \exists u_t (rwu \wedge (\exists x G(x))_{t \rightarrow o} u)$

$\forall w_L \exists u_t (rwu \wedge \exists x Gxu)$

Expansion: user or prover may flexibly choose expansion depth

### What are we doing?

In order to prove that  $\varphi$  is valid in HOML,

→ we instead prove that  $\text{valid } \varphi_{t \rightarrow o}$  can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

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## Embedding HOML in HOL

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## Embedding HOML in HOL: Logics beyond K

### Modal logic axioms

- M: valid  $\forall\varphi(\Box\varphi \rightarrow \varphi)$
- B: valid  $\forall\varphi(\varphi \rightarrow \Box\Diamond\varphi)$
- D: valid  $\forall\varphi(\Box\varphi \rightarrow \Diamond\varphi)$
- 4: valid  $\forall\varphi(\Box\varphi \rightarrow \Box\Box\varphi)$
- 5: valid  $\forall\varphi(\Diamond\varphi \rightarrow \Box\Diamond\varphi)$

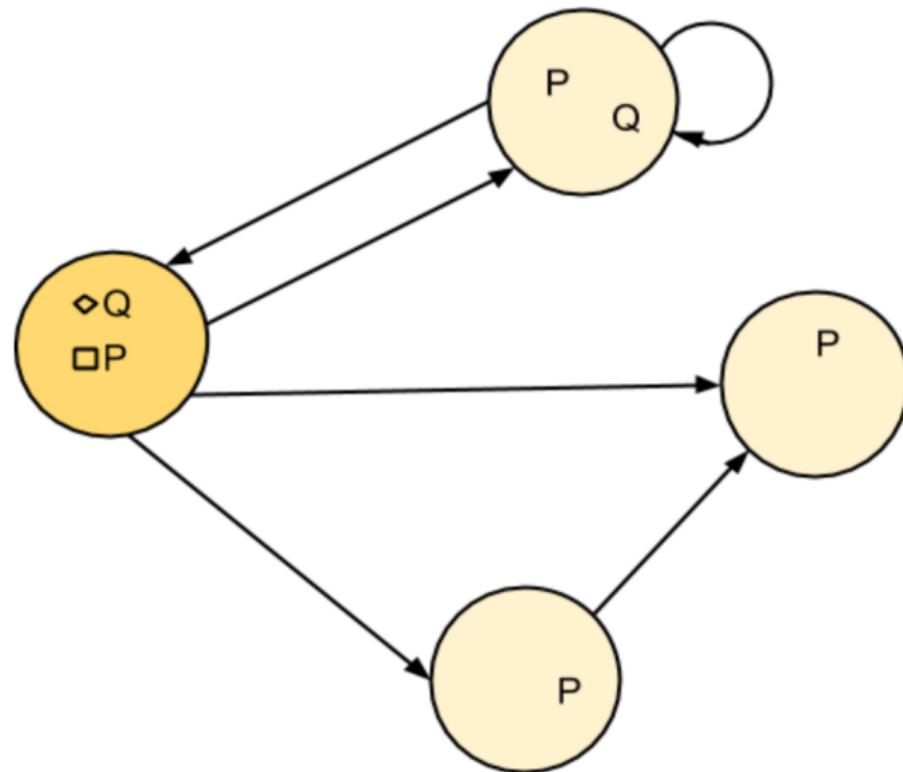
### Semantical conditions

- $\forall x(rxy)$
- $\forall x\forall y(rxy \rightarrow ryx)$
- $\forall x\exists y(rxy)$
- $\forall x\forall y\forall z(rxy \wedge ryz \rightarrow rxz)$
- $\forall x\forall y\forall z(rxy \wedge rxz \rightarrow ryz)$

## HOML: Motivation

### Kripke Semantics - Possible Worlds

Possibilist vs. Actualist Quantification



## Embedding HOML in HOL: Possibilist vs. Actualist Quantification

$$\forall = \lambda h_{\gamma \rightarrow (\iota \rightarrow o)} \lambda w_\iota \forall d_\gamma h d w \quad (\text{possibilist / constant dom.})$$

becomes

$$\forall^{va} = \lambda h_{\gamma \rightarrow (\iota \rightarrow o)} \lambda w_\iota \forall d_\gamma (\mathbf{ExInW} d w \rightarrow h d w) \quad (\text{actualist / varying dom.})$$

where **ExInW** is an existence predicate.

Additional axioms (optional):

- ▶ domains are non-empty  $\forall w_\iota \exists x_\mu \mathbf{exInW}_{xw}$
- ▶ denotation (constants & functions)  $\forall w_\iota (\mathbf{exInW} t^1 w \wedge \dots \wedge \mathbf{exInW} t^n w \supset \mathbf{exInW}(f t^1 \dots t^n) w)$

Cumulative domains:

$$\forall x \forall v \forall w (\mathbf{exInW}_{xv} \wedge r v w \supset \mathbf{exInW}_{xw})$$

## Embedding HOML in HOL: Theoretical Results

### Soundness and Completeness

$$\models_{\text{Henkin}}^{\text{HOML}} s_o \quad \text{iff} \quad \text{Ax} \models_{\text{Henkin}}^{\text{HOL}} \text{valid } s_{\nu \rightarrow o} \quad (\text{iff} \quad \text{Ax} \vdash_{\text{cut-free}}^{\text{HOL}} \text{valid } \varphi_{\nu \rightarrow o})$$

## Embedding HOML in HOL: Theoretical Results

### Soundness and Completeness

$$\models_{\text{Henkin}}^{\text{HOML}} s_o \text{ iff } \text{Ax} \models_{\text{Henkin}}^{\text{HOL}} \text{valid } s_{\iota \rightarrow o}$$

**Proof sketch (adapts [Benzmüller and Paulson, Logica Universalis, 2013]):**  
By contraposition it is sufficient to show

$$\not\models_{\text{Henkin}}^{\text{HOML}} s_o \text{ iff } \text{Ax} \not\models_{\text{Henkin}}^{\text{HOL}} \text{valid } s_{\iota \rightarrow o}$$

One easily gets the proof by choosing the obvious correspondences between  $D$  and  $D$ ,  $W$  and  $D_\iota$ ,  $I$  and  $I$ ,  $g$  and  $g$ ,  $R$  and  $r_{\iota \rightarrow \iota \rightarrow o}$ , and  $w$  and  $w$ .  $\square$

## Embedding HOML in HOL: Theoretical Results

### Soundness and Completeness (and Cut-elimination)

$$\models_{\text{Henkin}}^{\text{HOML}} s_o \text{ iff } \mathbf{Ax} \models_{\text{Henkin}}^{\text{HOL}} \text{valid } s_{\iota \rightarrow o}$$

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## Embedding of Other Logics in HOL: Theoretical Results

### Soundness and Completeness (and Cut-elimination)

$$\models^L s_o \text{ iff } \text{Ax} \models_{\text{Henkin}}^{\text{HOL}} \text{valid } s_{\iota \rightarrow o} \quad (\text{iff } \text{Ax} \vdash_{\text{cut-free}}^{\text{HOL}} \text{valid } \varphi_{\iota \rightarrow o})$$

Logic L:

- ▶ Higher-order Modal Logics
- ▶ First-order Multimodal Logics
- ▶ Propositional Multimodal Logics
- ▶ Quantified Conditional Logics
- ▶ Propositional Conditional Logics
- ▶ Intuitionistic Logics
- ▶ Access Control Logics
- ▶ Logic Combinations
- ▶ ...more is on the way ... including:
  - ▶ Description Logics
  - ▶ Nominal Logics
  - ▶ Multivalued Logics (SIXTEEN)
  - ▶ Logics based on Neighborhood Semantics
  - ▶ (Mathematical) Fuzzy Logics
  - ▶ Paraconsistent Logics

## Embedding HOL in HOL: TPTP THF

```
1 %----The base type $i (already built-in) stands here for worlds and
2 %----mu for individuals; $o (also built-in) is the type of Booleans
3 thf(mu_type,type,(mu:$tType)).
4 %----Reserved constant r for accessibility relation
5 thf(r,type,(r:$i>$i>$o)).
6 %----Modal logic operators not, or, and, implies, box, diamond
7 thf(mnot_type,type,(mnot:($i>$o)>$i>$o)).
8 thf(mnot_definition,(mnot = (^[A:$i>$o,W:$i]:~(A@W)))). 
9 thf(mor_type,type,(mor:($i>$o)>($i>$o)>$i>$o)).
10 thf(mor_definition,(mor = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W) | (Psi@W))))).
11 thf(mand_type,type,(mand:($i>$o)>($i>$o)>$i>$o)).
12 thf(mand_definition,(mand = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W) & (Psi@W))))).
13 thf(mimplies_type,type,(mimplies:($i>$o)>($i>$o)>$i>$o)).
14 thf(mimplies_definition,(mimplies = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W) & (Psi@W))))).
15 thf(mbox_type,type,(mbox:($i>$o)>$i>$o)).
16 thf(mbox_definition,(mbox = (^[A:$i>$o,W:$i]:![V:$i]:(~(r@W@V) | (A@V))))).
17 thf(mdia_type,type,(mdia:($i>$o)>$i>$o)).
18 thf(mdia_definition,(mdia = (^[A:$i>$o,W:$i]:?[V:$i]:((r@W@V) & (A@V))))).
19 %----Quantifiers (constant domains) for individuals and propositions
20 thf(mforall_ind_type,type,(mforall_ind:(mu:$i>$o)>$i>$o)).
21 thf(mforall_ind_definition,(mforall_ind = (^[A:mu>$i>$o,W:$i]:![X:mu]: (A@X@W)))). 
22 thf(mforall_indset_type,type,(mforall_indset:(mu:$i>$o)>$i>$o)>$i>$o)).
23 thf(mforall_indset_definition,(mforall_indset = (^[A:(mu>$i>$o)>$i>$o,W:$i]:![X:mu>$i>$o]: (A@X@W)))). 
24 thf(mexists_ind_type,type,(mexists_ind:(mu:$i>$o)>$i>$o)).
25 thf(mexists_ind_definition,(mexists_ind = (^[A:mu>$i>$o,W:$i]:?[X:mu]: (A@X@W)))). 
26 thf(mexists_indset_type,type,(mexists_indset:(mu:$i>$o)>$i>$o)>$i>$o)).
27 thf(mexists_indset_definition,(mexists_indset = (^[A:(mu>$i>$o)>$i>$o,W:$i]:?[X:mu>$i>$o]: (A@X@W)))). 
28 %----Definition of validity (grounding of lifted modal formulas)
29 thf(v_type,type,(v:($i>$o)>$o)).
30 thf(mvalid_definition,(v = (^[A:$i>$o]:![W:$i]: (A@W)))). 
31 %----Properties of accessibility relations: symmetry
32 thf(msymmetric_type,type,(msymmetric:($i>$i>$o)>$o)).
33 thf(msymmetric_definition,(msymmetric = (^[R:$i>$i>$o]:![S:$i,T:$i]:((R@S@T) => (R@T@S))))).
34 %----Here we work with logic KB, i.e., we postulate symmetry for r
35 thf(sym_axiom,(msymmetric@r)).
```

Reading on THF0 syntax: [SutcliffeBenzmüller, J. Formalized Reasoning, 2010]

## Proof Automation with LEO-II

```
>  
>  
> Beweise-mit-Leo2 Notwendigerweise-existiert-Gott.p
```

```
Leo-II tries to prove  
=====
```

```
Goedel's Theorem T3: "Necessarily, God exists"
```

```
thf(thmT3,conjecture,  
    ( v  
      @ ( mbox  
        @ ( mexists_ind  
          @ ^ [X: mu] :  
            ( g @ X ) ) ) )).
```

```
Assumptions: D1, C, T2, D3, A5
```

```
. searching for proof ..
```

```
*****
```

```
* Proof found *
```

```
*****
```

```
% Szs status Theorem for Notwendigerweise-existiert-Gott.p
```

```
. generating proof object □
```

Provers can be called remotely in Miami — no local installation needed!

Download our experiments from

[https:](https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/THF)

//github.com/FormalTheology/GoedelGod/tree/master/Formalizations/THF

## Embedding HOML in HOL: Evaluation — How good is the approach?

- ▶ There are no other provers for HOML
- ▶ There are some provers for first-order modal logic (FML)
- ▶ Comparative evaluation done in 2014 for the proof problems in the QMLTP (v1.1) library [Otten and Raths, <http://www.iltp.de/qmltp/>]
- ▶ This library contains 580 FML problems (in fact,  $5 \times 3 \times 580 = 8700$  problems).
- ▶ Metaprover HOL-P: sequentially schedules LEO-II—1.6.2, Satallax—2.7, Isabelle—2013, Nitrox—2013, agsyHOI—1.0
- ▶ Timeout for each HOL prover 120sec of CPU time (HOL-P: 600sec)
- ▶ Timeout for competitor systems: 600sec

## HOML in HOL: Evaluation — FML's (D — constant/varying/cumulative)

No. of solved problems in the QMLTP library

	<b>M</b> ean <b>S</b> e <b>P</b> labelled sequents	<b>M</b> ean <b>T</b> AP labelled tableaux	<b>f</b> 2 <b>p</b> - <b>M</b> SPASS instant. & transform.	<b>M</b> ean <b>C</b> o <b>P</b> labelled connections	<b>H</b> OL-P
<b>Logic D, constant domains</b>					
Theorem	135	134	76	<b>217</b>	208
Non-Theorem	1	4	107	209	<b>250</b>
Solved	136	138	183	426	<b>458</b>
<b>Logic D, cumulative domains</b>					
Theorem	130	120	79	<b>200</b>	184
Non-Theorem	4	4	108	224	<b>269</b>
Solved	134	124	187	424	<b>453</b>
<b>Logic D, varying domains</b>					
Theorem	-	100	-	<b>170</b>	163
Non-Theorem	-	4	-	243	<b>295</b>
Solved	-	104	-	413	<b>458</b>

## HOML in HOL: Evaluation — FML's (S4 — constant/varying/cumulative)

No. of solved problems in the QMLTP library

	MleanSeP labelled sequents	MleanTAP labelled tableaux	f2p-MSPASS instant. & transform.	MleanCoP labelled connections	HOL-P
<b>Logic S4, constant domains</b>					
Theorem	197	220	111	<b>352</b>	300
Non-Theorem	1	4	36	82	<b>132</b>
Solved	198	224	147	<b>434</b>	432
<b>Logic S4, cumulative domains</b>					
Theorem	197	205	121	<b>338</b>	278
Non-Theorem	4	4	41	94	<b>146</b>
Solved	201	209	162	<b>432</b>	424
<b>Logic S4, varying domains</b>					
Theorem	-	169	-	<b>274</b>	245
Non-Theorem	-	4	-	119	<b>184</b>
Solved	-	173	-	393	<b>429</b>

**The above results are meanwhile outdated; see more recent experiments in:**

Theorem Provers for Every Normal Modal Logic (Tobias Gleißner, Alexander Steen, Christoph Benzmüller), In LPAR-21. 21st International Conference on Logic for Programming, Artificial Intelligence and Reasoning (Thomas Eiter, David Sands, eds.), EasyChair, EPiC Series in Computing, volume 46, pp. 14-30, 2017. <http://dx.doi.org/10.29007/jsb9>



**The remaining slides are optional!  
We will use others in the upcoming lectures.**

...BUT WOULDN'T A GOD  
WHO COULD FIND A FLAW IN  
THE ONTOLOGICAL ARGUMENT  
BE EVEN GREATER?



## The Ontological Argument

## SCIENCE NEWS

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[HOME](#) / [SCIENCE NEWS](#) / RESEARCHERS SAY THEY USED MACBOOK TO PROVE GOEDEL'S GOD THEOREM

# Researchers say they used MacBook to prove Goedel's God theorem

Oct. 23, 2013 | 8:14 PM | [1 comments](#)

See more serious and funny news links at

**https:**

**//github.com/FormalTheology/GoedelGod/tree/master/Press**

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Nachrichten > Wissenschaft > Mensch > Mathematik > Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

## Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hüller



Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis Jahrzehntlang geheim  
picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

**Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.**

Montag, 09.09.2013 – 12:03 Uhr

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- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
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[English Site](#) > [Germany](#) > [Science](#) > Scientists Use Computer to Mathematically Prove Gödel God Theorem

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## Holy Logic: Computer Scientists 'Prove' God Exists

By David Knight



Austrian mathematician Kurt Gödel kept his proof of God's existence a secret for decades. Now two scientists say they have proven it mathematically using a computer.

picture-alliance/Imago/Wiener Stadt- und Landesbibliothek

**Two scientists have formalized a theorem regarding the existence of God penned by mathematician Kurt Gödel. But the God angle is somewhat of a red herring -- the real step forward is the example it sets of how computers can make scientific progress simpler.**

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**DIE WELT**

FREITAG, 18. OKTOBER 2013

**Zippert rapt**

Die Welt erhält gute Noten für die Qualität ihres politischen Berichterstattung. Die Presse und die Wiener Zeitung sind ebenfalls gut eingestuft. Die ORF und die Süddeutsche Zeitung sind ebenfalls gut eingestuft. Die Presse und die Wiener Zeitung sind ebenfalls gut eingestuft. Die Presse und die Wiener Zeitung sind ebenfalls gut eingestuft.

**THEMEN**

**Sport**

Bundestrainer Löw ist der neue Chef der Nationalmannschaft. Er wird den Posten bis zum Ende des Jahres übernehmen. Löw ist 50 Jahre alt.

**Wirtschaft**

Der Junge wird den Boing zum Ladehüter machen.

**Kultur**

Bravo! Bravo! So legt sie sich einen dicken Pausen zu.

**Aus aller Welt**

Mit dem neuen Präsidenten Donald Trump ist es nun offiziell. Er hat die USA verlassen.

**Technik**

Haben Forscher mit einem Computerprogramm die Existenz Gottes bewiesen?

**Götliche Mathematik**

Einige Forscher haben eine Software geschrieben, die die Existenz Gottes beweisen kann. Sie haben die Software auf einer CD-ROM veröffentlicht.

**UNION GLASHÜTTE/SA**

Ein eleganter Uhrenhersteller aus Sachsen-Anhalt, der seit über 100 Jahren Uhren herstellt. Die Uhren sind handgefertigt und haben eine hohe Qualität.

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# A Long History of Ontological Arguments

pros and cons



Anselm's notion of God (Proslogion, 1078):

**“God is that, than which nothing greater can be conceived.”**

Gödel's notion of God:

“A God-like being possesses all ‘positive’ properties.”

To show by logical, deductive reasoning:

**“God exists.”**

$$\exists x G(x)$$

# A Long History of Ontological Arguments

pros and cons



Anselm's notion of God (Proslogion, 1078):

**“God is that, than which nothing greater can be conceived.”**

Gödel's notion of God:

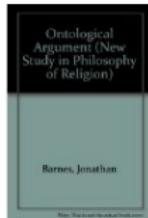
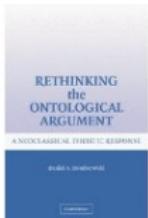
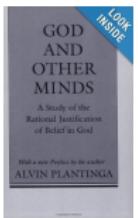
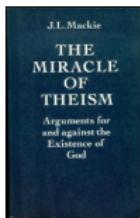
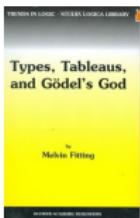
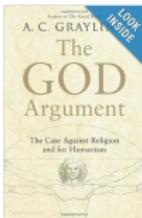
**“A God-like being possesses all ‘positive’ properties.”**

To show by logical, deductive reasoning:

**“Necessarily, God exists.”**

$$\Box \exists x G(x)$$

# The Ontological Proof Today



See more recent papers at my website.

## Various Different Interests in Ontological Arguments

- ▶ Philosophical: Boundaries of Metaphysics & Epistemology
- ▶ Theistic: Successful argument could convince atheists?
- ▶ **Ours:** Can computers (theorem provers) be used ...
  - ... to formalize the definitions, axioms and theorems?
  - ... to verify/falsify the arguments step-by-step?
  - ... to automate (sub-)arguments?
  - ... to discover new philosophical knowledge?

## Vision of Leibniz (1646–1716): *Calculemus!*



If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.

(Translation by Russell)

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo . . . dicere: calculemus.  
(Leibniz, 1684)



Required:  
**characteristica universalis** and **calculus ratiocinator**

# Gödel's Manuscript: 1930's, 1941, 1946-1955, 1970

Ontologischer Begriff

Feb 10, 1970

$P(\varphi)$   $\varphi$  is positive ( $\Leftrightarrow \varphi \in P$ )

At 1  $P(\varphi), P(\psi) \supset P(\varphi \wedge \psi)$  At 2  $P(\varphi) \supset P(\neg \varphi)$

$P_1 G(x) = (\varphi)[P(\varphi) \supset \varphi(x)]$  (God)

$P_2 \varphi_{\text{Ex}x} = (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$  (Existence)

$P \supset_N q = N(p \supset q)$  Necessity

At 2  $\begin{cases} P(\varphi) \supset N P(\varphi) \\ \neg P(\varphi) \supset N \neg P(\varphi) \end{cases}$  because it follows from the nature of the property

Th.  $G(x) \supset G_{\text{Ex}x}$

Df.  $E(x) = (\varphi)[\varphi_{\text{Ex}x} \supset N \exists x \varphi(x)]$  necessary Existence

At 3  $P(E)$

Th.  $G(x) \supset N(\exists y) G(y)$

hence  $(\exists x) G(x) \supset N(\exists y) G(y)$

"  $M(\exists x) G(x) \supset M N(\exists y) G(y)$  "  $\supset N(\exists y) G(y)$  Mi = pertaining

any two elements of  $X$  are nec. equivalent

exclusive or \* and for any number of nonmembers

$M(x) G(x)$  means all pos. prop. w.r.t. com-  
patible This is true because of:

At 4:  $P(\varphi), q, \neg \psi \supset P(\psi)$  which impl.

$\begin{cases} x=x & \text{is positive} \\ x \neq x & \text{is negative} \end{cases}$

But if a system  $S$  of pos. prop. were incon-  
sistent it would mean, that the same prop.  $s$  (which  
is positive) would be  $x \neq x$

Positive means positive in the moral aesth.  
sense (independently of the accidental structure of  
the world). Only ~~in the~~ the at. time. It may  
also mean "affirmation" as opposed to "privatation"  
(or containing privation). This supports the pl. part

$\neg \varphi$  is positive  $\neg (\varphi \supset \psi)$  otherwise  $\varphi(x) \supset_N x \neq x$   
hence  $x \neq x$  (positive)  $\neg x = x$  (negative) At  
or the opposite of  $\varphi$  (At)

i.e. the formal form in terms of elem. prop. contains a  
Member without negation.

## Proof Overview

**T3:**  $\Box \exists x.G(x)$

## Proof Overview

**C1:**  $\Diamond \exists z. G(z)$

---

**T3:**  $\Box \exists x. G(x)$

## Proof Overview

$$\frac{\mathbf{C1: } \diamond \exists z. G(z) \quad \mathbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\mathbf{T3: } \square \exists x. G(x)}$$

## Proof Overview

**L2:**  $\Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$

$$\frac{\begin{array}{c} \textbf{C1: } \Diamond \exists z. G(z) \\ \textbf{L2: } \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x) \end{array}}{\textbf{T3: } \Box \exists x. G(x)}$$

## Proof Overview

$$\frac{\begin{array}{c} \mathbf{C1: } \diamond \exists z. G(z) \\ \mathbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x) \end{array}}{\mathbf{T3: } \square \exists x. G(x)}$$
$$\frac{\overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}}{\mathbf{S5}}$$

## Proof Overview

$$\frac{\diamond \exists z. G(z) \rightarrow \diamond \Box \exists x. G(x)}{\text{L2: } \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}$$

**S5**  
 $\neg \forall \xi. [\diamond \Box \xi \rightarrow \Box \xi]$

$$\frac{\text{C1: } \diamond \exists z. G(z) \quad \text{L2: } \diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\text{T3: } \Box \exists x. G(x)}$$

## Proof Overview

$$\frac{\begin{array}{c} \mathbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x) \\ \hline \diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x) \end{array}}{\mathbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}$$
$$\frac{\mathbf{S5} \quad \neg \forall \xi. [\neg \diamond \square \xi \rightarrow \neg \square \xi]}{\neg}$$

$$\frac{\mathbf{C1: } \diamond \exists z. G(z) \quad \mathbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\mathbf{T3: } \square \exists x. G(x)}$$

## Proof Overview

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$

$$\frac{\mathbf{L1:} \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)} \quad \frac{\mathbf{S5}}{\neg \forall \xi. [\Diamond \Box \xi \rightarrow \Box \xi]}$$

$$\mathbf{L2:} \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)$$

$$\frac{\mathbf{C1:} \Diamond \exists z. G(z) \quad \mathbf{L2:} \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)}{\mathbf{T3:} \Box \exists x. G(x)}$$

## Proof Overview

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$

**D3<sup>\*</sup>:**  $E(x) \equiv \square \exists y. G(y)$

$$\frac{\begin{array}{c} P(E) \\ \hline \textbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x) \\ \hline \diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x) \end{array}}{\textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)} \quad \frac{\textbf{S5}}{\forall \xi. [\neg \diamond \square \xi \rightarrow \square \xi]}$$

---

$$\frac{\textbf{C1: } \diamond \exists z. G(z) \quad \textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\textbf{T3: } \square \exists x. G(x)}$$

## Proof Overview

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$

**D3<sup>\*</sup>:**  $E(x) \equiv \square \exists y. G(y)$  (cheating!)

$$\frac{\begin{array}{c} P(E) \\ \hline \textbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x) \\ \hline \diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x) \end{array}}{\textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)} \quad \frac{\textbf{S5}}{\forall \xi. [\neg \diamond \square \xi \rightarrow \square \xi]}$$
$$\frac{\textbf{C1: } \diamond \exists z. G(z) \quad \textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\textbf{T3: } \square \exists x. G(x)}$$

## Proof Overview

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$

**D3<sup>\*</sup>:**  $E(x) \equiv \square \exists y. G(y)$

**D3:**  $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\begin{array}{c} \mathbf{T2:} \forall y. [G(y) \rightarrow G \text{ ess } y] & P(E) \\ \hline \mathbf{L1:} \exists z. G(z) \rightarrow \square \exists x. G(x) \\ \hline \frac{\square \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\mathbf{L2:} \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)} \end{array}}{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]} \quad \mathbf{S5}$$

$$\frac{\mathbf{C1:} \diamond \exists z. G(z) \quad \mathbf{L2:} \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\mathbf{T3:} \square \exists x. G(x)}$$

## Proof Overview

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$

**D3<sup>\*</sup>:**  $E(x) \equiv \square \exists y. G(y)$

**D3:**  $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\begin{array}{c} \mathbf{T2:} \forall y. [G(y) \rightarrow G \text{ ess } y] \\ \hline \mathbf{L1:} \exists z. G(z) \rightarrow \square \exists x. G(x) \\ \hline \frac{\square \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\mathbf{L2:} \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)} \end{array}}{\frac{\mathbf{A5} \quad \overline{P(E)}}{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}} \quad \mathbf{S5}$$

$$\frac{\mathbf{C1:} \diamond \exists z. G(z) \quad \mathbf{L2:} \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\mathbf{T3:} \square \exists x. G(x)}$$

## Proof Overview

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$

**D3<sup>\*</sup>:**  $E(x) \equiv \square \exists y. G(y)$

**D3:**  $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\begin{array}{c} \mathbf{T2:} \forall y. [G(y) \rightarrow G \text{ ess } y] \\ \hline \mathbf{L1:} \exists z. G(z) \rightarrow \square \exists x. G(x) \\ \hline \frac{\square \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}{\mathbf{L2:} \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)} \end{array}}{\frac{\begin{array}{c} \mathbf{A5} \\ \overline{P(E)} \end{array}}{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}} \quad \mathbf{S5}$$

$$\frac{\mathbf{C1:} \diamond \exists z. G(z) \quad \mathbf{L2:} \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\mathbf{T3:} \square \exists x. G(x)}$$

## Proof Overview

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$

**D2:**  $\varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \square \forall x. (\varphi(x) \rightarrow \psi(x)))$

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**D3:**  $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\begin{array}{c} \mathbf{A1b} \\ \hline \neg \forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)] \end{array} \quad \begin{array}{c} \mathbf{A4} \\ \hline \neg \forall \varphi. [P(\varphi) \longrightarrow \square P(\varphi)] \end{array} \quad \begin{array}{c} \mathbf{A5} \\ \hline P(E) \end{array}}{\mathbf{T2}: \forall y. [G(y) \rightarrow G \text{ ess } y]}$$

$$\frac{\mathbf{L1}: \exists z. G(z) \rightarrow \square \exists x. G(x)}{\frac{\mathbf{L2}: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\frac{\mathbf{S5}}{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}}}$$

$$\frac{\mathbf{C1}: \diamond \exists z. G(z) \quad \mathbf{L2}: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}{\mathbf{T3}: \square \exists x. G(x)}$$

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**C1:**  $\diamond \exists z. G(z)$

$$\frac{\begin{array}{c} \textbf{A1b} \\ \overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]} \end{array} \quad \begin{array}{c} \textbf{A4} \\ \overline{\forall \varphi. [P(\varphi) \longrightarrow \square P(\varphi)]} \end{array} \quad \begin{array}{c} \textbf{A5} \\ \overline{P(E)} \end{array}}{\textbf{T2: } \forall y. [G(y) \rightarrow G \text{ ess } y]}$$

$$\frac{\begin{array}{c} \textbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x) \\ \textbf{L2: } \diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x) \end{array}}{\textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}$$

$$\frac{\begin{array}{c} \textbf{S5} \\ \overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]} \end{array}}{\textbf{L2: } \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}$$

**C1:**  $\diamond \exists z. G(z)$

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

**T3:**  $\square \exists x. G(x)$

## Proof Overview

**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$

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**D3:**  $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$P(G)$$

---

**C1:**  $\diamond \exists z. G(z)$

$$\frac{}{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]} \quad \textbf{A1b}$$

$$\frac{}{\forall \varphi. [P(\varphi) \longrightarrow \square P(\varphi)]} \quad \textbf{A4}$$

$$\frac{}{P(E)} \quad \textbf{A5}$$

---

**T2:**  $\forall y. [G(y) \rightarrow G \text{ ess } y]$

$$\frac{\textbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x)}{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}$$

$$\frac{}{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]} \quad \textbf{S5}$$

---

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

---

**C1:**  $\diamond \exists z. G(z)$

---

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

---

**T3:**  $\square \exists x. G(x)$

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**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$

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**D3\*:**  $E(x) \equiv \square \exists y. G(y)$

**D3:**  $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

**A3**  
 $\overline{P(G)}$

---

**C1:**  $\diamond \exists z. G(z)$

**A1b**

$\overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}$

**A4**

$\overline{\forall \varphi. [P(\varphi) \longrightarrow \square P(\varphi)]}$

**A5**

$\overline{P(E)}$

**T2:**  $\forall y. [G(y) \rightarrow G \text{ ess } y]$

**L1:**  $\exists z. G(z) \rightarrow \square \exists x. G(x)$

$\overline{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}$

**S5**

$\overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}$

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

**C1:**  $\diamond \exists z. G(z)$

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

---

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**A3**  
 $\overline{P(G)}$

**T1:**  $\forall \varphi. [P(\varphi) \rightarrow \diamond \exists x. \varphi(x)]$

**C1:**  $\diamond \exists z. G(z)$

**A1b**

$\overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}$

**A4**

$\overline{\forall \varphi. [P(\varphi) \rightarrow \square P(\varphi)]}$

**A5**

$\overline{P(E)}$

**T2:**  $\forall y. [G(y) \rightarrow G \text{ ess } y]$

**L1:**  $\exists z. G(z) \rightarrow \square \exists x. G(x)$

$\overline{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}$

**S5**

$\overline{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}$

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

**C1:**  $\diamond \exists z. G(z)$

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**D3:**  $E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\overline{P(G)} \quad \frac{\overline{\forall \varphi. \forall \psi. [(P(\varphi) \wedge \square \forall x. [\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]} \quad \overline{\forall \varphi. [P(\neg \varphi) \rightarrow \neg P(\varphi)]}}{\textbf{T1: } \forall \varphi. [P(\varphi) \rightarrow \diamond \exists x. \varphi(x)]}$$

**C1:**  $\diamond \exists z. G(z)$

$$\frac{\overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]} \quad \overline{\forall \varphi. [P(\varphi) \rightarrow \square P(\varphi)]} \quad \overline{P(E)}}{\textbf{T2: } \forall y. [G(y) \rightarrow G \text{ ess } y]}$$

$$\frac{\textbf{L1: } \exists z. G(z) \rightarrow \square \exists x. G(x)}{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}$$

$$\frac{}{\forall \xi. [\diamond \square \xi \rightarrow \square \xi]}$$

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

**C1:**  $\diamond \exists z. G(z)$

**L2:**  $\diamond \exists z. G(z) \rightarrow \square \exists x. G(x)$

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**D1:**  $G(x) \equiv \forall \varphi. [P(\varphi) \longrightarrow \varphi(x)]$

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**D3:**  $NE(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \square \exists y. \varphi(y)]$

$$\frac{\frac{\frac{\frac{\frac{\overline{A3}}{P(G)}}{\overline{\forall \varphi. \forall \psi. [\overline{[P(\varphi) \wedge \square \forall x. [\varphi(x) \rightarrow \psi(x)]} \rightarrow P(\psi)]}}}{\overline{\forall \varphi. [P(\neg \varphi) \rightarrow \neg P(\varphi)]}}}{\overline{\forall \varphi. [P(\varphi) \rightarrow \diamond \exists x. \varphi(x)]}}}{\overline{\textbf{C1}: \diamond \exists z. G(z)}}$$

$$\frac{\frac{\frac{\frac{\overline{A1b}}{\overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}}}{\overline{\forall \varphi. [P(\varphi) \longrightarrow \square P(\varphi)]}}}{\overline{\textbf{T2}: \forall y. [G(y) \rightarrow G \text{ ess } y]}}}{\frac{\frac{\overline{A4}}{\overline{\textbf{L1}: \exists z. G(z) \rightarrow \square \exists x. G(x)}}}{\frac{\overline{\diamond \exists z. G(z) \rightarrow \diamond \square \exists x. G(x)}}{\overline{\textbf{L2}: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}}}}}{\frac{\overline{A5}}{\overline{P(NE)}}}$$

$$\frac{\frac{\textbf{C1}: \diamond \exists z. G(z)}{\textbf{L2}: \diamond \exists z. G(z) \rightarrow \square \exists x. G(x)}}{\textbf{T3}: \square \exists x. G(x)}$$

## Scott's Version of Gödel's Axioms, Definitions and Theorems

**Axiom A1** Either a property or its negation is positive, but not both:  $\forall\varphi[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$

**Axiom A2** A property necessarily implied by a positive property is positive:

$$\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

**Thm. T1** Positive properties are possibly exemplified:

$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

**Def. D1** A God-like being possesses all positive properties:

$$G(x) \leftrightarrow \forall\varphi[P(\varphi) \rightarrow \varphi(x)]$$

**Axiom A3** The property of being God-like is positive:

$$P(G)$$

**Cor. C** Possibly, God exists:

$$\Diamond\exists xG(x)$$

**Axiom A4** Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \rightarrow \Box P(\varphi)]$$

**Def. D2** An essence of an individual is a property possessed by it and necessarily implying any of its properties:  $\varphi \text{ ess } x \leftrightarrow \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$

**Thm. T2** Being God-like is an essence of any God-like being:

$$\forall x[G(x) \rightarrow G \text{ ess } x]$$

**Def. D3** Necessary existence of an individual is the necessary exemplification of all its essences:

$$NE(x) \leftrightarrow \forall\varphi[\varphi \text{ ess } x \rightarrow \Box\exists y\varphi(y)]$$

**Axiom A5** Necessary existence is a positive property:

$$P(NE)$$

**Thm. T3** Necessarily, God exists:

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Difference to Gödel (who omits this conjunct)

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$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

$$G(x) \leftrightarrow \forall\varphi[P(\varphi) \rightarrow \varphi(x)]$$

$$P(G)$$

$$\Diamond\exists xG(x)$$

$$\forall\varphi[P(\varphi) \rightarrow \Box P(\varphi)]$$

$$\forall x[G(x) \rightarrow G \text{ ess } x]$$

$$NE(x) \leftrightarrow \forall\varphi[\varphi \text{ ess } x \rightarrow \Box\exists y\varphi(y)]$$

$$P(NE)$$

$$\Box\exists xG(x)$$

Modal operators are used

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$$NE(x) \leftrightarrow \forall\varphi[\varphi \text{ ess } x \rightarrow \Box\exists y\varphi(y)]$$

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$$P(NE)$$

**Thm. T3** Necessarily, God exists:

$$\Box\exists xG(x)$$

second-order quantifiers

# The Ontological Argument in TPTP THF0

```
1 %-----  
2 %----Axioms for Quantified Modal Logic KB.  
3 include('Quantified_KB.ax').  
4 %-----  
5 %----constant symbol for positive (p), God-like (g), essence (ess), necessary existence (ne)  
6 thf(p_tp,type,(p:(mu:$i>$o)>$i>$o)).  
7 thf(g_tp,type,(g:mu>$i>$o)).  
8 thf(ess_tp,type,(ess:(mu>$i>$o)>mu>$i>$o)).  
9 thf(ne_tp,type,(ne:mu>$i>$o)).  
10 %----D1:A God-like being possesses all positive properties.  
11 thf(defD1,definition,(g = (^[X:mu]:(mforall_indset@^[_Phi:mu>$i>$o]:(mimplies@(p@Phi)@(Phi@X))))).  
12 %----C: Possibly, God exists. (Proved in C.p)  
13 thf(corC,axiom,(v@(media@(mexists_ind@^[_X:mu]:(g@X))))).  
14 %----T2: Being God-like is an essence of any God-like being. (Proved in T2.p)  
15 thf(thmT2,axiom,(v@(mforall_ind@^[_X:mu]:(mimplies@(g@X)@(ess@g@X))))).  
16 %----D3: Necessary existence of an individual is the necessary exemplification of all its essences  
17 thf(defD3,definition,(ne = (^[_X:mu]:(mforall_indset@^[_Phi:mu>$i>$o]:  
18 (mimplies@(ess@Phi@X)@(mbox@(mexists_ind@^[_Y:mu]:(Phi@Y))))))).  
19 %----A5:Necessary existence is positive.  
20 thf(axA5,axiom,(v@(p@ne))).  
21 %----T3: Necessarily God exists.  
22 thf(thmT3,conjecture,(v@(mbox@(mexists_ind@^[_X:mu]:(g@X))))).
```

# The Ontological Argument in Isabelle/HOL

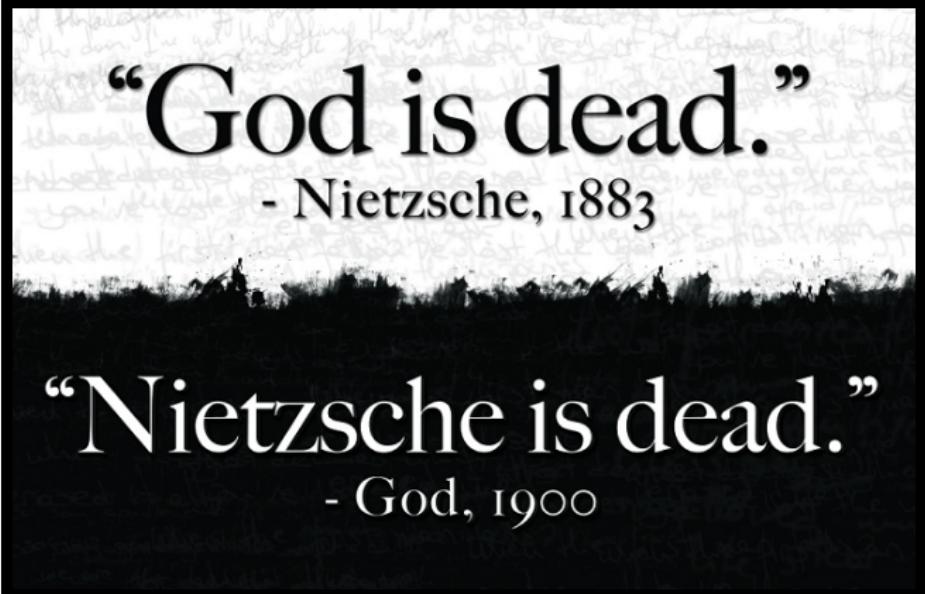
```
GoedelGod.thy (~/Varia/Talks/2015-ReasoningWeb/DemoMaterial/DEMO-Isabelle/)

text {* QML formulas are translated as HOL terms of type @{typ "i ⇒ bool"}.  
This type is abbreviated as @{text "σ"}. *}  
  
type_synonym σ = "i ⇒ bool"  
  
text {* The classical connectives $\neg$, $\wedge$, $\rightarrow$, and $\forall$ (over individuals and over sets of individuals) and $\exists$ (over individuals) are lifted to type $\sigma$. The lifted connectives are @{text "m¬"}, @{text "m\wedge"}, @{text "m→"}, @{text "m\exists"}, and @{text "m\equiv"} (the latter two are modeled as constant symbols). Other connectives can be introduced analogously. We exemplarily do this for @{text "m\vee"}, @{text "m\equiv"}, and @{text "mL="} (Leibniz equality on individuals). Moreover, the modal operators @{text "m□"} and @{text "m○"} are introduced. Definitions could be used instead of abbreviations. *}  
  
abbreviation mnot :: "σ ⇒ σ" ("m¬") where "m¬ φ ≡ (λw. ¬ φ w)"  
abbreviation mand :: "σ ⇒ σ ⇒ σ" (infixr "m\wedge" 51) where "φ m\wedge ψ ≡ (λw. φ w ∧ ψ w)"  
abbreviation mor :: "σ ⇒ σ ⇒ σ" (infixr "m→" 50) where "φ m→ ψ ≡ (λw. φ w ∨ ψ w)"  
abbreviation mimplies :: "σ ⇒ σ ⇒ σ" (infixr "m⇒" 49) where "φ m⇒ ψ ≡ (λw. φ w → ψ w)"  
abbreviation mequiv :: "σ ⇒ σ ⇒ σ" (infixr "m\equiv" 48) where "φ m\equiv ψ ≡ (λw. φ w ← ψ w)"  
abbreviation mforall :: "('a ⇒ σ) ⇒ σ" ("m\forall") where "m\forall Φ ≡ (λw. ∀x. Φ x w)"  
abbreviation mexists :: "('a ⇒ σ) ⇒ σ" ("m\exists") where "m\exists Φ ≡ (λw. ∃x. Φ x w)"  
abbreviation mLeibeq :: "μ ⇒ μ ⇒ σ" (infixr "mL=" 52) where "x mL= y ≡ ∀(λp. (p x m⇒ p y))"  
abbreviation mbox :: "σ ⇒ σ" ("m□") where "m□ φ ≡ (λw. ∀v. w r v → φ v)"  
abbreviation mdia :: "σ ⇒ σ" ("m○") where "m○ φ ≡ (λw. ∃v. w r v ∧ φ v)"  
  
text {* For grounding lifted formulas, the meta-predicate @{text "valid"} is introduced. *}  
  
(*<*) no_syntax "_list" :: "args ⇒ 'a list" ("[_]") (*>*)  
abbreviation valid :: "σ ⇒ bool" ("[p]") where "[p] ≡ ∀w. p w"
```

See verifiable Isabelle/HOL document (Archive of Formal Proofs) at:  
<http://afp.sourceforge.net/entries/GoedelGod.shtml>



## Findings from Our Experiments



“God is dead.”

- Nietzsche, 1883

“Nietzsche is dead.”

- God, 1900

## Experimental Results and Philosophical Discoveries

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p\phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu. (\phi X \dot{\wedge} \psi X)) \dot{>} p\psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\exists} X_\mu. \phi X$	A1( $\supset$ ), A2 A1, A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$		K	THM	0.1/0.1	0.0/5.2	—/—
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\phi} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p\phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$						
T2	$\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
K			K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess } \phi X \dot{\wedge} \dot{\Box} \dot{\exists} Y_\mu. \phi Y)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$	D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5 D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	K K KB KB	CSA CSA THM THM	—/— —/— 0.0/0.1 —/—	—/— —/— 0.1/5.3 —/—	3.8/6.2 8.2/7.5 —/— —/—
MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$	D2, T2, T3 A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	17.9/—	3.3/3.2	—/—
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\wedge} \dot{\neg}(\phi X)))$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	16.5/— 12.8/15.1	0.0/0.0 0.0/5.4	—/— —/—
MT	$\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X \dot{=} Y))$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	—/— —/—	0.0/3.3 —/—	—/— —/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$	A1( $\supset$ ), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
CO'	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II sat/unsat	Satallax sat/unsat	Nitpick sat/unsat
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p\phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu. (\phi X \dot{\wedge} \psi X)) \dot{>} p\psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{>} \dot{\exists} X_\mu. \phi X$	A1( $\supset$ ), A2 A1, A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{>} \phi X$		K	THM	0.1/0.1	0.0/5.2	—/—
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\phi} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{>} \dot{\Box} p\phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$						
T2	$\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{>} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess } \phi X \dot{\wedge} \dot{\Box} \dot{\exists} Y_\mu. \phi Y)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$	D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5 D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	K K KB KB	CSA CSA THM THM	—/— —/— 0.0/0.1 —/—	—/— —/— 0.1/5.3 —/—	3.8/6.2 8.2/7.5 —/— —/—
MC	$[s_\sigma \dot{>} \dot{\Box} s_\sigma]$	D2, T2, T3 A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	17.9/—	3.3/3.2	—/—
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu. (g_{\mu \rightarrow \sigma} X \dot{>} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{>} \dot{\neg}(\phi X)))$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	16.5/— 12.8/15.1	0.0/0.0 0.0/5.4	—/— —/—
MT	$\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{>} (g_{\mu \rightarrow \sigma} Y \dot{>} X \dot{= Y}))$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	—/— —/—	0.0/3.3 —/—	—/— —/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$	A1( $\supset$ ), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
CO'	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p\phi)$						
A2	$\dot{\forall} \psi_{\mu \rightarrow \sigma^*} \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\forall} X_\mu. (\phi X \dot{\rightarrow} \phi X)) \dot{=} p\psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\forall} \exists X_\mu. \phi X$	A1( $\supset$ ), A2 A1, A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \phi X$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\forall} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\supset} \dot{\neg} p\phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\supset} \dot{\forall} Y_\mu. (\phi Y \dot{\supset} \psi Y))$						
T2	$\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{\supset} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1 A2 D1 A3 A4 D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \phi)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\forall} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$						

- MC  $[s_\sigma \dot{\supset} \square_\sigma]$
- FG  $[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu. (g_{\mu \rightarrow \sigma} X \dot{\supset} (\phi X \dot{\supset} \phi X))]$
- MT  $[\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{\supset} (g_{\mu \rightarrow \sigma} Y \dot{\supset} Y))]$
- CO  $\emptyset$  (no goal, check for cons)
- D2'  $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda$
- CO'  $\emptyset$  (no goal, check for cons)

## Automating Scott's proof script

**T1: "Positive properties are possibly exemplified"** proved by LEO-II and Satallax

- ▶ in logic: K
- ▶ from assumptions:
  - ▶ A1 and A2
  - ▶ A1( $\supset$ ) and A2
- ▶ notion of quantification
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p \phi)]$						
A2	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu. (\phi X \dot{\wedge} \psi X)) \dot{=} p \psi]$						
T1	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\exists} X_\mu. \phi X]$	A1( $\supset$ ), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$						
A3	$[\dot{\forall} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \Box p \phi]$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$						
T2	$[\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \phi \dot{\wedge} \phi X)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
D3	$[\dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$						
MC	$[s_\sigma \dot{\supset} \dot{\neg} s_\sigma]$						
FG	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu. (g_\mu. X \dot{\wedge} (g_\mu. X \dot{\supset} \phi X))]$						
MT	$[\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\supset} X))]$						
CO	$\emptyset$ (no goal, check for cons)						
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda$						
CO'	$\emptyset$ (no goal, check for cons)						

## Automating Scott's proof script

**C: "Possibly, God exists"**  
proved by LEO-II and Satallax

- ▶ in logic: K
- ▶ from assumptions:
  - ▶ T1, D1, A3
- ▶ for domain conditions:
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu. (\phi X \dot{\wedge} \psi X)) \dot{=} p \psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\exists} \dot{\exists} X_\mu. \phi X$	A1( $\dot{\exists}$ ), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\exists} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p \phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \phi X \wedge \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\supset} \dot{\Box} \dot{\forall} Y_\mu. (\psi Y \dot{\supset} \psi X))$						
T2	$\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{\supset} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \phi X)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\exists} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$						

MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu. (g_\mu. X \dot{\supset} ($
MT	$\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_\mu. X \dot{\supset} (g_\mu.$
CO	$\emptyset$ (no goal, check for cons)
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda$
CO'	$\emptyset$ (no goal, check for cons)

## Automating Scott's proof script

**T2: "Being God-like is an ess. of any God-like being"**  
 proved by LEO-II and Satallax

- ▶ in logic: K
- ▶ from assumptions:
  - ▶ A1, D1, A4, D2
- ▶ for domain conditions:
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p\phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu. (\phi X \dot{\wedge} \psi X)) \dot{=} p\psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\exists} \dot{\exists} X_\mu. \phi X$	A1( $\dot{\exists}$ ), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\exists} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p\phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$						
T2	$[\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g_X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.0/14.0	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \phi_X)$						
A5	$[\dot{\exists} \dot{\exists} X_\mu. \text{NE}_{\mu \rightarrow \sigma} X]$						
T3	$[\dot{\Box} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$						

## Automating Scott's proof script

**T3: "Necessarily, God exists"**  
 proved by LEO-II and Satallax

- ▶ in logic: KB
- ▶ from assumptions:
  - ▶ D1, C, T2, D3, A5
- ▶ for domain conditions:
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

For logic K we got a **countermodel** by Nitpick

MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$
FG	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \phi_X))]$
MT	$[\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} \text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} (X \dot{\wedge} Y)))]$
CO	∅ (no goal, check for cons)
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} Y)$
CO'	∅ (no goal, check for cons)

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II sat/very	Satallax sat/very	Nitpick sat/very
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p\phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu. (\phi X \dot{\wedge} \psi X)) \dot{>} p\psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{>} \dot{\exists} \dot{\exists} X_\mu. \phi X$	A1( $\dot{\exists}$ ), A2 A1, A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{>} \phi X$		K	THM	0.1/0.1	0.0/5.2	—/—
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\exists} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{>} \dot{\Box} p\phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{>} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$						
T2	$[\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{>} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)]$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \phi X)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\exists} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$						
MC	$[s_\sigma \dot{>} \dot{\Box} s_\sigma]$						
FG	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu. (s_\mu \dot{>} X) \dot{>} (s_\mu \dot{>} X)]$						
MT	$[\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{>} (g_{\mu \rightarrow \sigma} Y) \dot{>} (X \dot{>} Y))]$						
CO	$\emptyset$ (no goal, check for cons)						
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda$						
CO'	$\emptyset$ (no goal, check for cons)						

## Automating Scott's proof script

### Summary

- ▶ proof verified and automated
- ▶ KB is sufficient (criticized logic S5 not needed!)
- ▶ possibilist and actualist quantifiers (individuals)
- ▶ exact dependencies determined experimentally
- ▶ ATPs have found alternative proofs  
e.g. self-identity  $\lambda x(x = x)$  is not needed

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p\phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu. (\phi X \dot{\wedge} \psi X)) \dot{=} p\psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\exists} X_\mu. \dot{\Box} \dot{\neg} \phi$	A1, A2	KB	THM	0.1/0.1	0.0/0.0	/
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\forall} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$						
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p\phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda$						
T2	$\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} X)$						
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \phi)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$						
<b>Consistency check: Gödel vs. Scott</b>							
<ul style="list-style-type: none"> <li>▶ Scott's assumptions are consistent; shown by Nitpick</li> <li>▶ Gödel's assumptions are inconsistent; shown by LEO-II (new philosophical result?)</li> </ul>							
MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$	A1, A2, D1, A3, A4, D2, D3, A5 D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	KB KB KB	CSA THM THM	—/— 0.0/0.1 —/—	—/— 0.1/5.3 —/—	8.2/7.5 —/— —/—
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\wedge} \dot{\neg}(\phi X)))$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	KB KB KB	THM THM THM	17.9/— 16.5/— 12.8/15.1	3.3/3.2 0.0/0.0 0.0/5.4	—/— —/— —/—
MT	$\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X \dot{=} Y))$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	—/— —/—	0.0/3.3 —/—	—/— —/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$	A1, D1	KB	UNSAT	7.5/7.8	—/—	—/—
CO'	∅ (no goal, check for consistency)	A1(∅), A2, D2', D3, A5 A1, A2, D1, A3, A4, D2', D3, A5	KB	UNSAT	—/—	—/—	—/—

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p\phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma} \dot{\forall} \psi_{\mu \rightarrow \sigma} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu. (\phi X \dot{\wedge} \psi X)) \dot{=} p\psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\exists} X_\mu. \phi X$	A1( $\supset$ ), A2 A1, A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\exists} X_\mu. \phi X$		K	THM	0.1/0.1	0.0/5.2	—/—
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$						
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p\phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma}. \lambda$						
T2	$\dot{\forall} X_\mu. p_{(\mu \rightarrow \sigma) \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} X)$						
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \phi X)$	D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5 D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	K K KB KB	CSA CSA THM THM	—/— —/— 0.0/0.1 —/—	—/— —/— 0.1/5.3 —/—	3.8/6.2 8.2/7.5 —/— —/—
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$						
MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$	D2, T2, T3 A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	17.9/—	3.3/3.2	—/—
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma} \dot{\forall} X_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\wedge} \dot{\neg}(\phi X)))$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	16.5/— 12.8/15.1	0.0/0.0 0.0/5.4	—/— —/—
MT	$\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X \dot{=} Y))$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	—/— —/—	0.0/3.3 —/—	—/— —/—
CO	Ø (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma}. \lambda X_\mu. \dot{\forall} \psi_{\mu \rightarrow \sigma} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$	A1( $\supset$ ), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
CO'	Ø (no goal, check for consistency)	A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

## Further Results

- ▶ Monotheism holds
- ▶ God is flawless

# Main Findings

	HOL encoding
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\exists} ($
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma} \dot{\forall} \psi_{\mu \rightarrow \sigma} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma}$
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\exists} \dot{\forall} \exists$
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}$
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$
C	$[\dot{\forall} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\exists} \dot{\forall} p q$
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma}. \lambda$
T2	$\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{\exists} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma})$
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} [p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}])$
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$
T3	$[\dot{\forall} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$

## Modal Collapse (Sobel)

$$\forall \varphi (\varphi \supset \Box \varphi)$$

- ▶ proved by LEO-II and Satallax
- ▶ for possibilist and actualist quantification (ind.)

## Main critique on Gödel's ontological proof:

- ▶ there are no contingent truths
- ▶ everything is determined / no free will

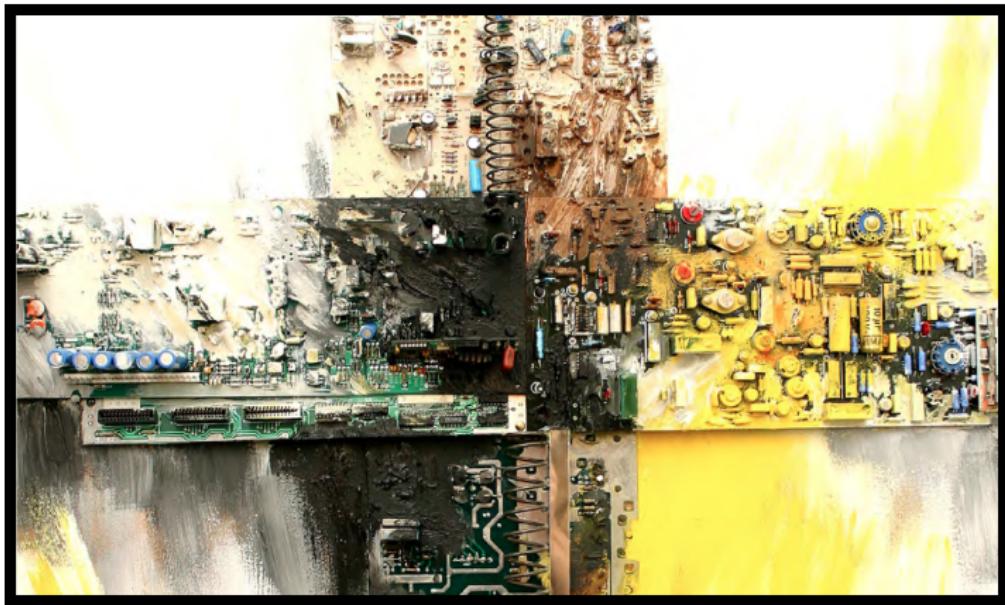
MC	$[s_\sigma \dot{\exists} \dot{\forall} s_\sigma]$		D2, T2, T3 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	17.9/— —/—	3.3/3.2 —/—				
FG	$[\dot{\forall} \phi_{\mu \rightarrow \sigma} \dot{\forall} X_\mu. (g_{\mu \rightarrow \sigma} X \dot{\exists} (\dot{\neg} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\exists} \dot{\neg} (\phi X)))]$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	16.5/— 12.8/15.1	0.0/0.0 0.0/5.4					
MT	$[\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{\exists} (g_{\mu \rightarrow \sigma} Y \dot{\exists} X \dot{=} Y))]$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	—/— —/—	0.0/3.3 —/—					
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4				
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma}. \lambda X_\mu. \dot{\forall} \psi_{\mu \rightarrow \sigma} ((\psi X \dot{\exists} \dot{\forall} Y_\mu. (\psi Y \dot{\exists} \psi Y)))$	A1(∅), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—					
CO'	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—					

# Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p\phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu. (\phi X \dot{\wedge} \psi X)) \dot{>} p\psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{>} \dot{\exists} X_\mu. \phi X$	A1( $\supset$ ), A2 A1, A2	K	THM	0.1/0.1 0.1/0.1	0.0/0.0 0.0/5.2	—/— —/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{>} \phi X$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0 0.0/0.0	0.0/0.0 5.2/31.3	—/— —/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{>} \dot{\Box} p\phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{>} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$						
T2	$[\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{>} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)]$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3 12.9/14.0	0.0/0.0 0.0/0.0	—/— —/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess } \phi X \dot{>} \dot{\Box} \dot{\exists} Y_\mu. \phi Y)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \dot{\exists} X_\mu. g_{\mu \rightarrow \sigma} X]$	D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5 D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	K K KB KB	CSA CSA THM THM	—/— —/— 0.0/0.1 —/—	—/— —/— 0.1/5.3 —/—	3.8/6.2 8.2/7.5 —/— —/—
			KB	THM	17.9/—	3.3/3.2	—/—
		3, A5	KB	THM	—/—	—/—	—/—
			KB	THM	16.5/—	0.0/0.0	—/—
		3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
			KB	THM	—/—	0.0/3.3	—/—
		3, A5	KB	SAT	—/—	—/—	7.3/7.4
			KB	UNS	7.5/7.8	—/—	—/—
		3, A5	KB	UNS	—/—	—/—	—/—

## Observation

- ▶ good performance of ATPs
- ▶ excellent match between argumentation granularity in papers and the reasoning strength of the ATPs



## Reconstruction of the Inconsistency of Gödel's Axioms

# Inconsistency (Gödel): Proof by LEO-II in KB

DemoMaterial — bash — 166x52

```
@SV8)@SV3)=$false) | (((p@(^{[SX0:mu,SX1:$i]: $false})@SV3)=$true))), inference(prim_subst,[status(thm)], [66:[bind(SV11,$thf(^{[SV23:mu,SV24:$i]: $false}))]])).  
thf(44,plain,!([SV22:{mu:(\$i>$o)},SV3:$i,SV8:{mu:(\$i>$o)}): ((({SV8@(^{[sk2_SK33@SV3]@(^{[SX0:mu,SX1:$i]: (~ ({SV22@SX0]@($X1))))}@$V8)}@((isK1_SK31@(^{[SX0:mu,SX1:$i]: (~ ({SV22@SX0]@($X1))))}@$V3))=true))), inference(prim_subst,[status(thm)], [66:[bind(SV11,$thf(^{[SV20:mu,SV21:$i]: (~ ({SV22@SX0]@($V21))))}))]).  
thf(45,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4)=$false) | (((p@SV9)@SV4) = ((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4) = ((p@(^{[SY29:mu,SY30:$i]: (~ ({SV9@SV29]@($V30))))}@$V4)=true) | (((p@SV9)@SV4) = ((p@(^{[SY29:mu,SY30:$i]: (~ ({SV9@SV29]@($V30))))}@$V4)=false)), inference(fac_restr,[status(thm)], [57]))).  
thf(46,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((p@(^{[SY29:mu,SY30:$i]: (~ ({SV9@SV29]@($V30))))}@$V4)=$true) | (((p@SV9)@SV4) = ((p@(^{[SY29:mu,SY30:$i]: (~ ({SV9@SV29]@($V30))))}@$V4)=false)), inference(fac_restr,[status(thm)], [57])).  
thf(47,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4)) | (~ ((~ ((p@SV9)@SV4)) | (~ ((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4)))=false) | (((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4)=false), inference(extnf_equal_neg,[status(thm)], [85])).  
thf(48,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((p@(^{[SY29:mu,SY30:$i]: (~ ({SV9@SV29]@($V30))))}@$V4)=$false) | (~ ((~ ((p@SV9)@SV4)) | (~ ((p@(^{[SY29:mu,SY30:$i]: (~ ({SV9@SV29]@($V30))))}@$V4))=true)), inference(extnf_equal_neg,[status(thm)], [86])).  
thf(49,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4))=false) | (((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4)=false)), inference(extnf_or_neg,[status(thm)], [87])).  
thf(50,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((~ ((p@SV9)@SV4) | ((p@(^{[SY29:mu,SY30:$i]: (~ ({SV9@SV29]@($V30))))}@$V4))=false) | (((p@(^{[SY29:mu,SY30:$i]: (~ ({SV9@SV29]@($V30))))}@$V4)=true)), inference(extnf_or_neg,[status(thm)], [89])).  
thf(51,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((~ ((p@SV9)@SV4)) | (~ ((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4))=true) | (((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4)=true)), inference(extnf_or_neg,[status(thm)], [91])).  
thf(52,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((~ ((p@SV9)@SV4)) | ((p@(^{[SY29:mu,SY30:$i]: (~ ({SV9@SV29]@($V30))))}@$V4))=true) | (((p@(^{[SY29:mu,SY30:$i]: (~ ({SV9@SV29]@($V30))))}@$V4))=true)), inference(extnf_or_neg,[status(thm)], [93])).  
thf(53,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((~ ((p@SV9)@SV4))=true) | (~ ((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4))=true) | (((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4)=false)), inference(extnf_or_pos,[status(thm)], [96])).  
thf(54,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((p@SV9)@SV4)=true) | ((p@(^{[SY29:mu,SY30:$i]: (~ ({SV9@SV29]@($V30))))}@$V4)=true) | (((p@(^{[SY29:mu,SY30:$i]: (~ ({SV9@SV29]@($V30))))}@$V4)=true)), inference(extnf_or_pos,[status(thm)], [97])).  
thf(55,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((p@SV9)@SV4)=false) | (~ ((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4))=true) | (((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4)=true)), inference(extnf_not_pos,[status(thm)], [100])).  
thf(56,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4))=false) | ((p@SV9)@SV4)=false) | (((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4)=false)), inference(extnf_not_pos,[status(thm)], [103])).  
thf(57,plain,!([SV8:{mu:(\$i>$o)},SV3:$i,SV22:{mu:(\$i>$o)}): (((SV22@(({sk2_SK33@SV3}@(^{[SX0:mu,SX1:$i]: (~ ({SV22@SX0]@($X1))))}@$V8))@((isK1_SK31@(^{[SX0:mu,SX1:$i]: (~ ({SV22@SX0]@($X1))))}@$V3))=true))), inference(extnf_not_neg,[status(thm)], [78])).  
thf(58,plain,!([SV11:{mu:(\$i>$o)},SV3:$i,SV15:{mu:(\$i>$o)}): (((SV15@(({sk2_SK33@SV3}@(^{[SX0:mu,SX1:$i]: (~ ({SV15@SX0]@($X1))))}@$V11))@((isK1_SK31@(^{[SX0:mu,SX1:$i]: (~ ({SV15@SX0]@($X1))))}@$V11))=true) | ((p@(^{[SX0:mu,SX1:$i]: (~ ({SV15@SX0]@($X1))))}@$V3))=false) | (((p@(^{[SX0:mu,SX1:$i]: (~ ({SV15@SX0]@($X1))))}@$V3))=true)), inference(extnf_not_pos,[status(thm)], [81])).  
thf(59,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((p@(^{[SY27:mu,SY28:$i]: (~ ({SV9@SV27]@($V28))))}@$V4))=false) | ((p@SV9)@SV4)=false)), inference(sim,[status(thm)], [105])).  
thf(60,plain,!([SV4:$i,SV9:{mu:(\$i>$o)}): (((p@SV9)@SV4)=true) | ((p@(^{[SY29:mu,SY30:$i]: (~ ({SV9@SV29]@($V30))))}@$V4))=true)), inference(sim,[status(thm)], [101])).  
thf(61,plain,!([SV3:$i,SV8:{mu:(\$i>$o)}): (((p@SV8)@SV3)=false) | (((p@(^{[SX0:mu,SX1:$i]: $true}))@SV3)=true)), inference(sim,[status(thm)], [76])).  
thf(62,plain,!([SV11:{mu:(\$i>$o)},SV3:$i]: (((p@(^{[SX0:mu,SX1:$i]: $false}))@SV3)=false) | (((p@SV11)@SV3)=true)), inference(sim,[status(thm)], [80])).  
thf(63,plain,(!$false)=true), inference(fo_atp_e,[status(thm)], [25,112,111,110,109,108,107,84,83,82,75,74,73,72,71,70,69,68,67,66,65,62,57,56,51,42,29]).  
thf(64,plain,(!$false), inference(solved_all_splits,[solved_all_splits(join,[])]), [113]).  
% S25 output end CNFRefutation  
  
***** End of derivation protocol *****  
***** no. of clauses in derivation: 97 *****  
***** clause counter: 113 *****  
  
% S25 status Unsatisfiable for ConsistencyWithoutFirstConjunctionD02.p : (rf:0,axioms:6,ps:3,u:6,ude:false,rLeibE0:true,rAndE0:true,use_choice:true,use_extuni:true,use_extcnf_combined:true,expand_extuni:false,foatp:e,atp_timeout:25,atp_calls_frequency:10,ordering:none,proof_output:1,clause_count:113,loop_count:0,foatp_calls:2,translati  
on:fof_full)  
ontoleo:DemoMaterial cbenzmueller$ □
```

## Inconsistency (Gödel): Reconstruction of Informal Argument (KB)

(special thanks to Chad Brown for a fruitful discussion)

**Axiom A1**( $\Box$ )

$$\forall \varphi [P(\neg\varphi) \rightarrow \neg P(\varphi)]$$

**Axiom A2**

$$\forall \varphi \forall \psi [(P(\varphi) \wedge \Box \forall x [\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

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**Theorem 1** Positive Properties are possibly exemplified.  $\forall \varphi [P(\varphi) \rightarrow \Diamond \exists x \varphi(x)]$

by A1( $\Box$ ), A2

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$$\varphi \text{ ess } x \leftrightarrow \cancel{\varphi(x)} \wedge \forall \psi (\psi(x) \rightarrow \Box \forall y (\varphi(y) \rightarrow \psi(y)))$$

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$$\forall x (\emptyset \text{ ess } x)$$

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by B, D3, Lemma1

**Axiom A5**

$$P(NE)$$

Inconsistency

$\perp$

by A5, T1, Lemma2

# Inconsistency (Gödel): Verification in Isabelle/HOL (KB)

The screenshot shows the Isabelle/HOL IDE interface with the following details:

- Title Bar:** GoedelGodWithoutConjunctInEss\_KB.thy
- Toolbar:** Standard file operations (New, Open, Save, Print, etc.) and navigation icons.
- Text Area:** The code for `GoedelGodWithoutConjunctInEss_KB.thy`. The code defines a theory `GoedelGodWithoutConjunctInEss_KB` that imports `QML`. It includes constants `P`, axioms `A1a` and `A2`, a theorem `T1`, a definition `ess`, a lemma `Lemmal`, a definition `NE`, a lemma `Lemma2`, an axiomatization `A5`, and a lemma `False`. Proofs are carried out using the `metis` tactic.
- Right Panel:** Navigation tabs: Documentation, Sidekick, Theories.
- Bottom Bar:** Output, Query, Sledgehammer, Symbols buttons. Status bar: 11,1 (477/1095) (isabelle,sidekick,UTF-8-Isabelle)Nr m o UG 263/347 MB 17:18

```
theory GoedelGodWithoutConjunctInEss_KB imports QML
begin
consts P :: "(μ ⇒ σ) ⇒ σ"
axiomatization where A1a: "[∀(λΦ. P (λx. m¬ (Φ x)) m→ m¬ (P Φ))]"
  and A2: "[∀(λΦ. ∀(λΨ. (P Φ m ∧ □ (∀(λx. Φ x m→ Ψ x))) m→ P Ψ))]"
-- {* Positive properties are possibly exemplified. *}
theorem T1: "[∀(λΦ. P Φ m→ ◇ (exists Φ))]" by (metis A1a A2)
definition ess (infixr "ess" 85) where "Φ ess x = ∀(λΨ. Ψ x m→ □ (forall(λy. Φ y m→ Ψ y)))"
-- {* The empty property is an essence of every individual. *}
lemma Lemmal: "[((forall(λx.( λy. λw. False) ess x)))]" by (metis ess_def)
definition NE where "NE x = ∀(λΦ. Φ ess x m→ □ (exists Φ))"
axiomatization where sym: "x r y → y r x"
-- {* Exemplification of necessary existence is not possible. *}
lemma Lemma2: "[m¬ (◇ (exists NE))]" by (metis sym Lemmal NE_def)
axiomatization where A5: "[P NE]"
-- {* Now the inconsistency follows from A5, T1 and Lemma2 *}
lemma False by (metis A5 T1 Lemma2)
end
```

# Inconsistency (Gödel): Verification in Isabelle/HOL (K)

The screenshot shows the Isabelle/HOL interface with the following details:

- Title Bar:** GoedelGodWithoutConjunctInEss\_K.thy
- Toolbar:** Standard icons for file operations, search, and navigation.
- Text Area:** Displays the source code of the theory GoedelGodWithoutConjunctInEss\_K.thy. The code defines constants, functions, and axioms related to Gödel's incompleteness theorem. It includes proofs for theorems T1 and Lemma1, and a lemma False.
- Right Panel:** A vertical panel with tabs: Documentation, Sidekick, and Theories.
- Status Bar:** Shows the current state: (isabelle,sidekick,UTF-8-Isabelle)Nr m o UG 302/343MB 11:37

```
theory GoedelGodWithoutConjunctInEss_K imports QML
begin
  consts P :: " $(\mu \Rightarrow \sigma) \Rightarrow \sigma$ "
  definition ess (infixr "ess" 85) where " $\Phi \text{ ess } x = \forall(\lambda\Psi. \Psi x \rightarrow \square (\forall(\lambda y. \Phi y \rightarrow \Psi y)))$ "
  definition NE where " $\text{NE } x = \forall(\lambda\Phi. \Phi \text{ ess } x \rightarrow \square (\exists \Phi))$ "
  axiomatization where A1a: "[ $\forall(\lambda\Phi. P (\lambda x. \text{m}\neg (\Phi x)) \rightarrow \text{m}\neg (P \Phi))$ ]"
    and A2: "[ $\forall(\lambda\Phi. \forall(\lambda\Psi. (P \Phi \wedge \square (\forall(\lambda x. \Phi x \rightarrow \Psi x)) \rightarrow P \Psi))$ ]"

  -- {* Positive properties are possibly exemplified. *}
  theorem T1: "[ $\forall(\lambda\Phi. P \Phi \rightarrow \diamond (\exists \Phi))$ ]" by (metis A1a A2)

  -- {* The empty property is an essence of every individual. *}
  lemma Lemmal: "[ $(\forall(\lambda x. (\lambda y. \lambda w. \text{False}) \text{ ess } x))$ ]" by (metis ess_def)

  axiomatization where A5: "[P NE]"

  -- {* Now the inconsistency follows from A5, Lemmal, NE_def and T1 *}
  lemma False
    -- {* sledgehammer [remote_leo2] *}
    by (metis A5 Lemmal NE_def T1)
end
```

## Summary: Results of Experiments

GÃºdel's version	K		KB		S5	
	constant	varying	constant	varying	constant	varying
Consistency	×	×	×	×	×	×
T1	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
C	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
T2	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
T3	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
Flawless God	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
Monotheism	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
Modal Collapse	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete

## Further logic details

- ▶ Henkin semantics
- ▶ full comprehension
- ▶ rigid constant symbols

Question:

Has this inconsistency been reported before?

If not, then LEO-II deserves (part of) the credit!

## Summary: Results of Experiments

Scott's version	K constant	K varying	KB constant	KB varying	S5 constant	S5 varying
Consistency	✓	✓	✓	✓	✓	✓
T1	✓	✓	✓	✓	✓	✓
C	✓	✓	✓	✓	✓	✓
T2	✓	✓	✓	✓	✓	✓
T3	✗	✗	✓	✓	✓	✓
Flawless God	✓	✓	✓	✓	✓	✓
Monotheism	✓	✓	✓	✓	✓	✓
Modal Collapse	✓	✓	✓	✓	✓	✓

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# C. Anthony Anderson Avoiding the Modal Collapse: Recent Variants

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

Gödel's version of the modal ontological argument for the existence of God has been criticized by J. Howard Sobel [5] and modified by C. Anthony Anderson [1]. In the present paper we consider the extent to which Anderson's emendation is defeated by the type of objection first offered by the Monk Gaunilo to St. Anselm's original Ontological Argument. And we try to push the analysis of this Gödelian argument a bit further to bring it into closer agreement with the details of Gödel's own formulation. Finally, we indicate what seems to be the main weakness of this emendation of Gödel's attempted proof.

## Magari and others on Gödel's ontological proof

Petr Hájek

Institute of Computer Science, Academy of Sciences  
182 07 Prague, Czech Republic  
e-mail: hajek@uivt.cas.cz

## 1 Introduction

This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variant by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe

## Der Mathematiker und die Frage der Existenz Gottes (betreffend Gödels ontologischen Beweis)

Es ist gut, daß wir nicht wissen,  
andersen glauben, daß er Gott sei.  
(Karl Naegel)

### 1. Einführung

Gödels zu Lebzeiten unveröffentlichter Beweis für die notwendige Existenz eines Gott-ähnlichen Wesens hat sowohl philosophisches als auch mathematisches Interesse geweckt. Zweck der vorliegenden Arbeit ist es, zu einer Deutung des Gödelischen Textes beizutragen, 1. durch Kommentierung der einschlägigen Literatur und 2. durch Bereitstellung von etwas Modelltheorie. Die Arbeit enthält keinen philosophischen Beitrag. Während der letzten Jahre habe ich etliche Male über Gödels Götteshinweis vortragen, insbesondere auf dem Symposium zur Feier von Professor Gert Müller (Heidelberg, Januar 1991), doch habe ich niemals behauptet, eine Verallgemeinerung über das Thema zu machen. Da ich wiederholt um eine schriftliche Verarbeitung gebeten wurde, entschied ich mich, schnell eine „erweiterte Kurzfassung“<sup>1</sup> zu schreiben, ohne aus ihr einen

PETR HÁJEK

## A New Small Emendation of Gödel's Ontological Proof

Keywords: Ontological proof, Gödel, modal logic, comprehension, positive properties.

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Gödel's ontological proof of necessary existence of a godlike being was finally published in the third volume of Gödel's collected works [7]; but it became known in 1970 when Gödel showed the proof to Dana Scott and Scott presented it (in fact a variant of it) at a seminar at Princeton. Detailed history is found in Adams' introductory remarks to the ontological proof in [7]. The proof uses modal logic and its analysis is an exciting exercise in systems of formal modal logic. Needless to say, formal modal logic has found several

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## Results Obtained with Fully Automated Reasoners

A controversy between Magari, Hájek and Anderson regarding the redundancy of some axioms

Proof	D1'	A:A1'	A2'	A3'	A4	A4'	H:A4	A5	A5'	H:A5	T3	T3'	MC
Scott (const)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Scott (var)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Anderson (const)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (var)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (mix)	-	-	-	-	R (K4B)	-	-	-	I	-	-	CS	CS
Hájek AOE' (var)	-	-	CS	-	S/I	-	-	-	S/I	-	-	P (KB)	CS
Hájek AOE'_0 (var)	-	-	-	CS	R	-	-	-	S/U	-	-	P (KB)	CS
Hájek AOE'' (var)	-	-	-	-	-	S/I	-	-	S/I	-	-	P (KB)	CS
Anderson (simp) (var)	-	R	R	-	R (K4B)	-	-	-	-	-	-	-	-
Bjørdal (const)	R (K4)	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS
Bjørdal (var)	CS	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS

# Results Obtained with Fully Automated Reasoners

A controversy between Magari, Hájek and Anderson regarding the redundancy of some axioms

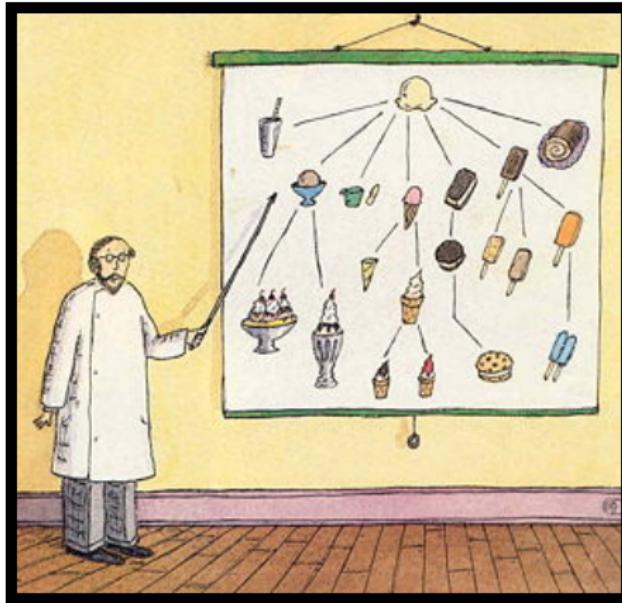
Proof	D1'	A:A1'	A2'	A3'	A4	A4'	H:A4	A5	A5'	H:A5	T3	T3'	MC
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Hájek AOE' (var)	-	-	CS	-	S/I	-	-	-	S/I	-	-	P (KB)	CS
Hájek AOE'_0 (var)	-	-	-	CS	R	-	-	-	S/U	-	-	P (KB)	CS
Hájek AOE'' (var)	-	-	-	-	-	S/I	-	-	S/I	-	-	P (KB)	CS
Anderson (simp) (var)	-	R	R	-	R (K4B)	-	-	-	-	-	-	-	-
Bjørdal (const)	R (K4)	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS
Bjørdal (var)	CS	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS



Leibniz (1646–1716)

## characteristica universalis and calculus ratiocinator

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.



## SUMO Ontology and HOML

### Context

- ▶ prominent challenge in AI (CS, Philosophy)
- ▶ McCarthy: modeling of contexts as first-class objects

```
ist(context_of("Ben's Knowledge), likes(Sue, Bill))
```

```
ist(context_of("Ben's Knowledge),  
     ist(context_of(...), ...))
```

- ▶ McCarthy's [McCarthy, Comm.ACM 1987] [McCarthy, IJCAI 1993] approach has been followed by many others
- ▶ Giunchiglia's contextual reasoning [Giunchiglia, Epistemologia 1993] emphasizes the locality aspect; structured knowledge
- ▶ McCarthy and Giunchiglia avoid modal logics
- ▶ they also avoid a HOL perspective

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- ▶ they also avoid a HOL perspective

Our approach is complementary:  
takes a HOL perspective and integrates modal logics

### Expressive ontologies

- ▶ SUMO and Cyc
- ▶ modeling of contexts:

```
(holdsDuring (yearFn 2009) (loves Bill Mary))
```

```
(believes Bill  
  (knows Ben  
    (forall (?X)  
      ((woman ?X) => (loves Bill ?X))))
```

- ▶ relation to McCarthy's approach is obvious
- ▶ often a questionable semantics assumed for embedded formulas and modal predicates (also in Common Logic)

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      ((woman ?X) => (loves Bill ?X))))
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- ▶ relation to McCarthy's approach is obvious
- ▶ often a questionable semantics assumed for embedded formulas and modal predicates (also in Common Logic)

### Our approach:

HOL-based semantics, but holdsDuring, believes, knows and alike are associated with modal logic connectives

### Combining logics

- ▶ prominent challenge in AI (CS, Philosophy)
- ▶ epistemic, deontic, temporal, intuitionistic, relevant, linear, conditional, security ...
- ▶ wide literature—few implementations
- ▶ some propositional systems exists: Logic Workbench, LoTREC, Tableaux Workbench, FaCT, ileanCoP, MSPASS
- ▶ no implemented systems for combinations of first-order logics
- ▶ combination is typically approached **bottom-up**

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- ▶ no implemented systems for combinations of first-order logics
- ▶ combination is typically approached bottom-up

Our approach is complementary:

works top-down starting from classical higher-order logic (HOL)

## SUMO Ontology and HOML

### SUMO — Suggested Upper Merged Ontology

[NilesPease FOIS 2001, Pease 2011]

- ▶ open source, formal ontology: [www.ontologyportal.org](http://www.ontologyportal.org)
- ▶ has been extended for a number of domain specific ontologies
- ▶ altogether approx. 20,000 terms and 70,000 axioms
- ▶ employs the SUO-KIF representation language, a simplification of Genesereth's original Knowledge Interchange Format (KIF)

### Sigma

[PeaseBenzmüller AI Comm. 2013]

- ▶ browsing and inference system for ontology development
- ▶ integrates KIF-Vampire and SystemOnTPTP

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### Sigma

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- ▶ browsing and inference system for ontology development
- ▶ integrates KIF-Vampire and SystemOnTPTP

SUMO (and similarly Cyc) contains **higher-order representations**, but there is only very limited automation support so far

## Higher-Order Aspects in SUO-KIF and SUMO: Examples

- ▶ Embedded formulas

*term* ::= *variable*|*word*|*string*|*funterm*|*number*|*sentence*

(holdsDuring (YearFn 2009) (likes Mary Bill))

## Higher-Order Aspects in SUO-KIF and SUMO: Examples

- ▶ Embedded formulas
- ▶ ... often in combination with modal operators such as `holdsDuring`, `knows`, `believes`, etc.

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- ▶ Predicate variables, function variables, propositional variables

`funterm ::= (funword arg+)`   `relsent ::= (relword arg+)`

`funword, relword ::= initialchar wordchar*` | `variable`

( $\Leftarrow$ )  
    (instance ?REL TransitiveRelation)  
    (forall (?INST1 ?INST2 ?INST3)  
          ( $\Rightarrow$   
            (and  
              (?REL ?INST1 ?INST2)  
              (?REL ?INST2 ?INST3))  
              (?REL ?INST1 ?INST3))))

## Higher-Order Aspects in SUO-KIF and SUMO: Examples

- ▶ Embedded formulas
- ▶ ... often in combination with modal operators such as `holdsDuring`, `knows`, `believes`, etc.
- ▶ Predicate variables, function variables, propositional variables
- ▶ Lambda-Abstraction with KappaFn

```
(=>
  (attribute ?X Celebrity)
  (greaterThan
    (CardinalityFn
      (KappaFn ?A
        (knows ?A (exists (?P) (equal ?P ?X))))))
    1000))
```

## Higher-Order Aspects in SUO-KIF and SUMO: Examples

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Our focus in the remainder:

embedded formulas and modal operators

## Some FO translation 'tricks'

### First-order reasoning on a large ontology

[PeaseSutcliffe, CEUR 257, 2007]

- ▶ Quoting of embedded formulas

**A:** (holdsDuring (YearFn 2009) (likes Mary Bill))

**Q:** (holdsDuring (YearFn ?Y) (likes ?X Bill))

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Answer with FO-ATPs (?Y ← 2009, ?X ← Mary)

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**A:** (holdsDuring (YearFn 2009)

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Failure with FO-ATP

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First-order reasoning on a large ontology

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- ▶ Expansion of predicate variables

Why not trying higher-order automated theorem proving directly?

Our focus:

embedded formulas and modal operators



The SUO-KIF to TPTP THF0 Translation

## The SUO-KIF to TPTP THF0 Translation

- ▶ THF0: TPTP format for simple type theory  
[SutcliffeBenzmüller, J.Formalized Reasoning, 2010]
- ▶ THF0 ATPs: LEO-II, TPS, Isabelle, Satallax, ...  
THF0 (counter-)model finders: Nitpick, Satallax

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- ▶ THF0 ATPs: LEO-II, TPS, Isabelle, Satallax, ...  
THF0 (counter-)model finders: Nitpick, Satallax
- ▶ achieved:  
**SUO-KIF → TPTP THF0**  
translation mechanism for SUMO as part of Sigma
- ▶ so far only exploits base type  $\iota$  and  $o$  in THF0 ( $\rightarrow$  improvable)
- ▶ generally applicable to SUO-KIF representations
- ▶ translation example (for SUMO) available at:

<http://christoph-benzmueller.de/papers/SUMO.thf>

## The SUO-KIF to TPTP THF Translation

Main challenge: find consistent typing for untyped SUO-KIF

(instance instance BinaryPredicate)

## The SUO-KIF to TPTP THF Translation

Main challenge: find consistent typing for untyped SUO-KIF

```
(p_instance t_instance BinaryPredicate)
```



Higher-Order Automated Theorem Proving in Ontology Reasoning

## Embedded Formulas — An Easy Task for HO-ATP

### Example (A: Embedded Formulas)

During 2009 Mary liked Bill and Sue liked Bill. Who liked Bill in 2009?

**A:** (holdsDuring (YearFn 2009)  
      (and (likes Mary Bill) (likes Sue Bill)))

**Q:** (holdsDuring (YearFn 2009) (likes ?X Bill))

Proof by LEO-II in milliseconds

## Embedded Formulas — An Easy Task for HO-ATP

### Example (B: Embedded Formulas)

During 2009 Mary liked Bill and Sue liked Bill. Who liked Bill in 2009?

**A:** (holdsDuring (YearFn 2009)  
    (not (or (not (likes Mary Bill))  
              (not (likes Sue Bill))))))

**Q:** (holdsDuring (YearFn 2009) (likes ?X Bill))

Proof by LEO-II in milliseconds

## Embedded Formulas — An Easy Task for HO-ATP

### Example (C: Embedded Formulas)

At all times Mary likes Bill. During 2009 Sue liked whomever Mary liked. Is there a year in which Sue has liked somebody?

**A:** (holdsDuring ?Y (likes Mary Bill))

**B:** (holdsDuring (YearFn 2009)  
    (forall (?X) (=> (likes Mary ?X) (likes Sue ?X))))

**Q:** (holdsDuring (YearFn ?Y) (likes Sue ?X))

Proof by LEO-II in milliseconds

## Embedded Formulas — An Easy Task for HO-ATP

In the above examples we have (silently) assumed that the semantics of the logic underlying SUMO is a classical, bivalent logic, meaning that Boolean extensionality is valid:

```
(<=> (<=> ?P ?Q) (equal ?P ?Q))
```

### Example (D: Embedded Formulas)

During 2009 Mary liked Bill and Sue liked Bill. Who liked Bill in 2009?

**A:** (holdsDuring (YearFn 2009)  
    (and (likes Mary Bill) (likes Sue Bill)))

**Q:** (holdsDuring (YearFn 2009)  
    (and (likes Sue Bill) (likes Mary Bill)))

Proof by LEO-II in milliseconds

Boolean extensionality seems fine for the particular temporal contexts of our previous examples.

However, as we will show next, it quickly leads to counterintuitive inferences in other modal contexts.

## Problem: Boolean Extensionality versus Modal Operators

### Example (E: Embedded Formulas – Epistemic Contexts)

**A:** (`knows` Chris (equal Chris Chris))

**B:** (likes Mary Bill)

**C:** (`knows` Chris  
    (forall (?X) (=> (likes Mary ?X) (likes Sue ?X))))

**Q:** (`knows` Chris (likes Sue Bill))

Proof by LEO-II in milliseconds

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Proof by LEO-II in milliseconds

Boolean extensionality is in conflict with (epistemic) modalities!

(Has Boolean extensionality ever been questioned for KIF?)

## Proposed Solution: Possible World Semantics for SUMO

SUMO → HOML → TPTP THF

- ▶ T-Box like information in SUMO:

(instance holdsDuring AsymmetricRelation) →  
 $\forall W_t \text{ (instance holdsDuring AsymmetricRelation)}_{t \rightarrow o} W$

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SUMO → HOML → TPTP THF

- ▶ T-Box like information in SUMO:

(instance holdsDuring AsymmetricRelation) →  
 $\forall W_t \text{ (instance holdsDuring AsymmetricRelation)}_{t \rightarrow o} W$

- ▶ A-Box like information as in query problem: current world  $cw_t$

(likes Mary Bill) →  $(\text{likes Mary Bill})_{t \rightarrow o} cw$

(knows Chris (likes Sue Bill)) →  $(\square_{\text{Chris}} (\text{likes Sue Bill}))_{t \rightarrow o} cw$

## Challenge: Embedded Formulas — Epistemic Context

### Example (F: Embedded Formulas – Epistemic Contexts)

A:  $(\square_{Chris} (equal Chris Chris)) \text{ cw}$

B:  $(likes Mary Bill) \text{ cw}$

C:  $(\square_{Chris} (\forall^i X_\mu. ((likes Mary X) \supset (likes Sue X)))) \text{ cw}$

Q:  $(\square_{Chris} (likes Sue Bill)) \text{ cw}$

## Challenge: Embedded Formulas — Epistemic Context

### Example (F: Embedded Formulas – Epistemic Contexts)

A:  $(\Box_{Chris} (equal Chris Chris)) \text{ cw}$

B:  $(likes Mary Bill) \text{ cw}$

C:  $(\Box_{Chris} (\forall^i X_\mu ((likes Mary X) \supset (likes Sue X)))) \text{ cw}$

Q:  $(\Box_{Chris} (likes Sue Bill)) \text{ cw}$

Axioms for  $\Box_{Chris}$  can be added:

M:  $\forall W_L (\forall^p \varphi_{L \rightarrow o} \Box_{Chris} \varphi \supset \varphi) W$

4:  $\forall W_L (\forall^p \varphi_{L \rightarrow o} \Box_{Chris} \varphi \supset \Box_{Chris} \Box_{Chris} \varphi) W$

5:  $\forall W_L (\forall^p \varphi_{L \rightarrow o} \Box_{Chris} \neg \varphi \supset \Box_{Chris} \neg \Box_{Chris} \varphi) W$

## Challenge: Embedded Formulas — Epistemic Context

### Example (F: Embedded Formulas – Epistemic Contexts)

A:  $(\Box_{Chris} (equal Chris Chris)) \text{ cw}$

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Q:  $(\Box_{Chris} (likes Sue Bill)) \text{ cw}$

Axioms for  $\Box_{Chris}$  can be added:

M:  $\forall W_\iota (\forall^p \varphi_{\iota \rightarrow o} \Box_{Chris} \varphi \supset \varphi) W$

4:  $\forall W_\iota (\forall^p \varphi_{\iota \rightarrow o} \Box_{Chris} \varphi \supset \Box_{Chris} \Box_{Chris} \varphi) W$

5:  $\forall W_\iota (\forall^p \varphi_{\iota \rightarrow o} \Box_{Chris} \neg \varphi \supset \Box_{Chris} \neg \Box_{Chris} \varphi) W$

LEO-II cannot solve this problem anymore! Countermodel exists.

## Challenge: Embedded Formulas — Epistemic Context

### Example (F: Embedded Formulas – Epistemic Contexts)

A:  $(\Box_{Chris} (equal Chris Chris)) \text{ cw}$

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But LEO-II(+E) can solve this problem in milliseconds!

## Challenge: Embedded Formulas — Epistemic Context

### Example (F: Embedded Formulas – Epistemic Contexts)

A:  $(\Box_{Chris} (equal Chris Chris)) \text{ cw}$

B:  $(\Box_{fool} (likes Mary Bill)) \text{ cw}$

C:  $(\Box_{Chris} (\forall^i X_\mu ((\text{likes Mary } X) \supset (\text{likes Sue } X)))) \text{ cw}$

Q:  $(\Box_{Chris} (\text{likes Sue Bill})) \text{ cw}$

Axioms for  $\Box_{Chris}$  can be added:

M:  $\forall W_L (\forall^p \varphi_{L \rightarrow o} \Box_{Chris} \varphi \supset \varphi) W$

4:  $\forall W_L (\forall^p \varphi_{L \rightarrow o} \Box_{Chris} \varphi \supset \Box_{Chris} \Box_{Chris} \varphi) W$

5:  $\forall W_L (\forall^p \varphi_{L \rightarrow o} \Box_{Chris} \neg \varphi \supset \Box_{Chris} \neg \Box_{Chris} \varphi) W$

Axioms for  $\Box_{fool}$  can be added ...

$\forall W_L (\forall^p \varphi_{L \rightarrow o} \Box_{fool} \varphi \supset \Box_{Chris} \varphi) W$

...

## SUMO Ontology — Proposal

Redevelop entire SUMO Ontology in HOML (multimodal, eventually other logics)

Should give a proper semantics for SUMO

Employ HOML embedding in HOL to automated reasoning



Experiments with Large Knowledge Bases (SUMO)

## Significant Improvements for Large Theories (PAAR-2010)

### LEO-II(+E) version v1.1

Ex.	A	B	C	D	E			F	
local	.19	.19	.13	.16	.08	.34	.18	.04	2642.55
SInE	—	—	—	—	—	—	—	—	—
global	—	—	—	—	—	—	—	—	—

global: all SUMO axioms given to LEO-II

SInE: filters SUMO axioms for problem — ~400 axioms given to LEO-II

local: only handselected axioms given to LEO-II

Further reading and more experiments [BenzmüllerPease J.WebSemantics 2012]

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local: only handselected axioms given to LEO-II

### LEO-II(+E) version v1.2.1 (with relevance filtering)

Ex.	A	B	C	D	E			F	
local	.19	.18	.11	.08	.10	.38	.32	.14	.18
SInE	.43	.40	.21	.54	.37	.12	.70	.06	.26
global	2.8	2.7	1.6	4.9	1.4	0.9	4.7	1.3	0.9

Further reading and more experiments [BenzmüllerPease J.WebSemantics 2012]



## Meta-Reasoning

## Description logic $\mathcal{ALC}$

Syntax	Semantics	Description	Example
$A$	$A^I \subseteq \Delta^I$	atomic concept	$Human, Female, \dots$
$r$	$r^I \subseteq \Delta^I \times \Delta^I$	binary relation	$married, \dots$
$\perp$	$\emptyset$	empty concept	
$\top$	$\Delta^I$	universal concept	
$\sim A$	$\Delta^I \setminus A^I$	complement	$\sim Female$
$A \sqcup B$	$A^I \cup B^I$	disjunction	$Female \sqcup Male$
$A \sqcap B$	$A^I \cap B^I$	conjunction	$Female \sqcap Human$
$\exists r C$	$\{x   \exists y. r^I(x, y) \wedge C^I(y)\}$	existential restriction	$\exists married Female$
$\forall r C$	$\{x   \forall y. r^I(x, y) \rightarrow C^I(y)\}$	universal restriction	$\forall married Female$

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$\perp$	$\emptyset$	empty concept	
$\top$	$\Delta^I$	universal concept	
$\sim A$	$\Delta^I \setminus A^I$	complement	$\sim Female$
$A \sqcup B$	$A^I \cup B^I$	disjunction	$Female \sqcup Male$
$A \sqcap B$	$A^I \cap B^I$	conjunction	$Female \sqcap Human$
$\exists r C$	$\{x   \exists y. r^I(x, y) \wedge C^I(y)\}$	existential restriction	$\exists married Female$
$\forall r C$	$\{x   \forall y. r^I(x, y) \rightarrow C^I(y)\}$	universal restriction	$\forall married Female$

### Simple exercises (useful lemmata)

$$\top = A \sqcup \sim A \tag{L1}$$

$$\perp = \sim \top \tag{L2}$$

$$A \sqcap B = \sim (\sim A \sqcup \sim B) \tag{L3}$$

$$\forall r C = \sim (\exists r \sim C) \tag{L4}$$

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$A$ $r$	$A^I \subseteq \Delta^I$ $r^I \subseteq \Delta^I \times \Delta^I$	atomic concept binary relation	$Human, Female, \dots$ $married, \dots$
$\sim A$ $A \sqcup B$	$\Delta^I \setminus A^I$ $A^I \cup B^I$	complement disjunction	$\sim Female$ $Female \sqcup Male$
$\exists r C$	$\{x   \exists y. r^I(x, y) \wedge C^I(y)\}$	existential restriction	$\exists married Female$

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## Description logic $\mathcal{ALC}$

Syntax	Semantics	Description	Example
$A \sqsubseteq B$	$A^I \subseteq B^I$	$B$ subsumes $A$	$Doctor \sqsubseteq Human$
$A \doteq B$	$A^I \sqsubseteq B^I$ und $B^I \sqsubseteq A^I$	$A$ defined by $B$	$Parent \doteq Human \sqcap \exists hasChild Human$

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### Simple exercises:

$$A \sqsubseteq B \quad gdw. \quad \exists C. A \doteq C \sqcap B \tag{L5}$$

$$A \sqsubseteq B \quad gdw. \quad (A \sqcap \neg B) \sqsubseteq \perp \tag{L6}$$

*gdw.*  $\exists x. (A \sqcap \neg B)(x)$  unerfüllbar

## Knowledge representation in $\mathcal{ALC}$

### TBox (terminological knowledge, taxonomy)

Example:

$HappyMan \doteq Human \sqcap \sim Female \sqcap$

$(\exists married Doctor) \sqcap (\forall hasChild(Doctor \sqcup Professor))$

$Doctor \sqsubseteq Human$

## Knowledge representation in $\mathcal{ALC}$

### TBox (terminological knowledge, taxonomy)

Example:

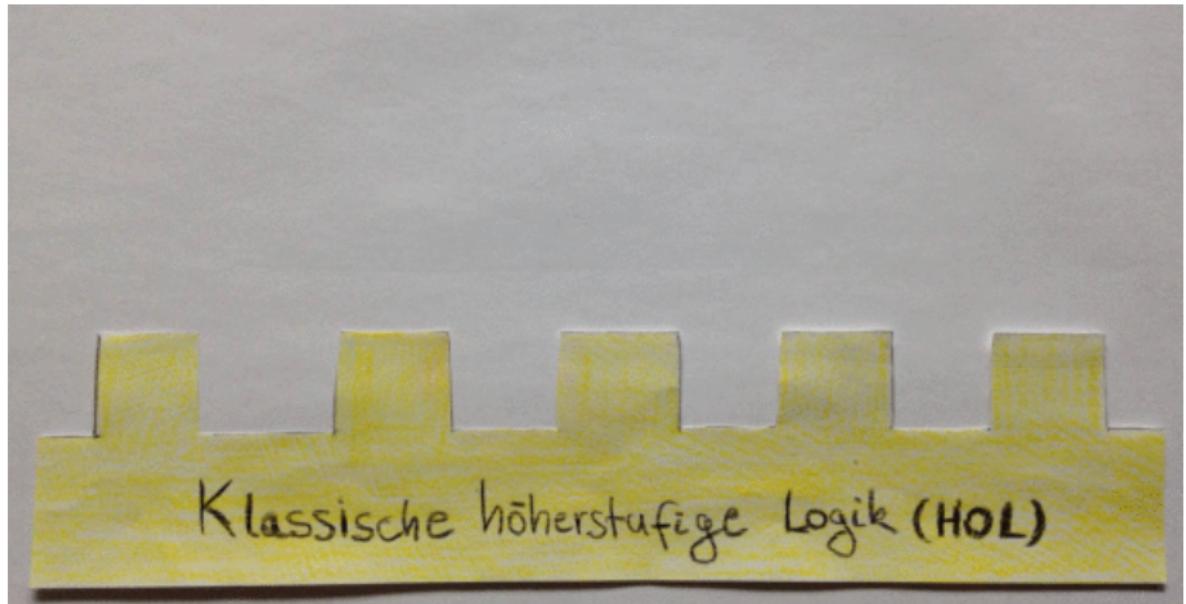
$$\begin{aligned} HappyMan &\doteq Human \sqcap \neg Female \sqcap \\ &(\exists married Doctor) \sqcap (\forall hasChild(Doctor \sqcup Professor)) \\ Doctor &\sqsubseteq Human \end{aligned}$$

### ABox (assertional knowledge, e.g. assumptions on individuals)

Example:

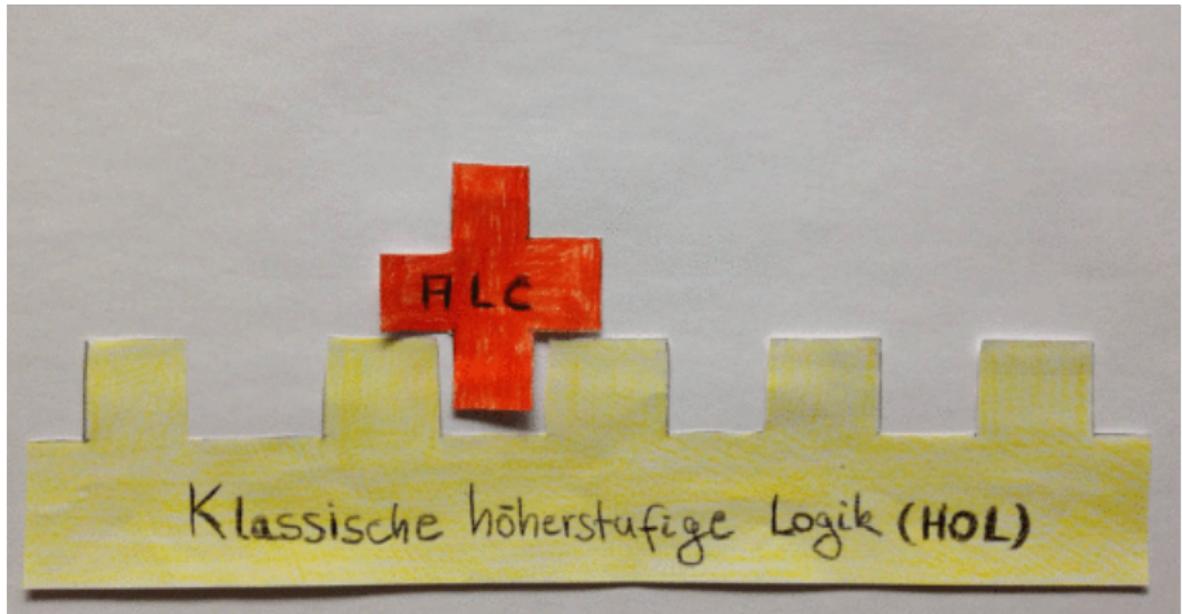
$$HappyMan(BOB), \quad hasChild(BOB, MARY), \quad \neg Doctor(MARY)$$

## EINSCHUB: Knowledge representation & Inference about ALC in HOL



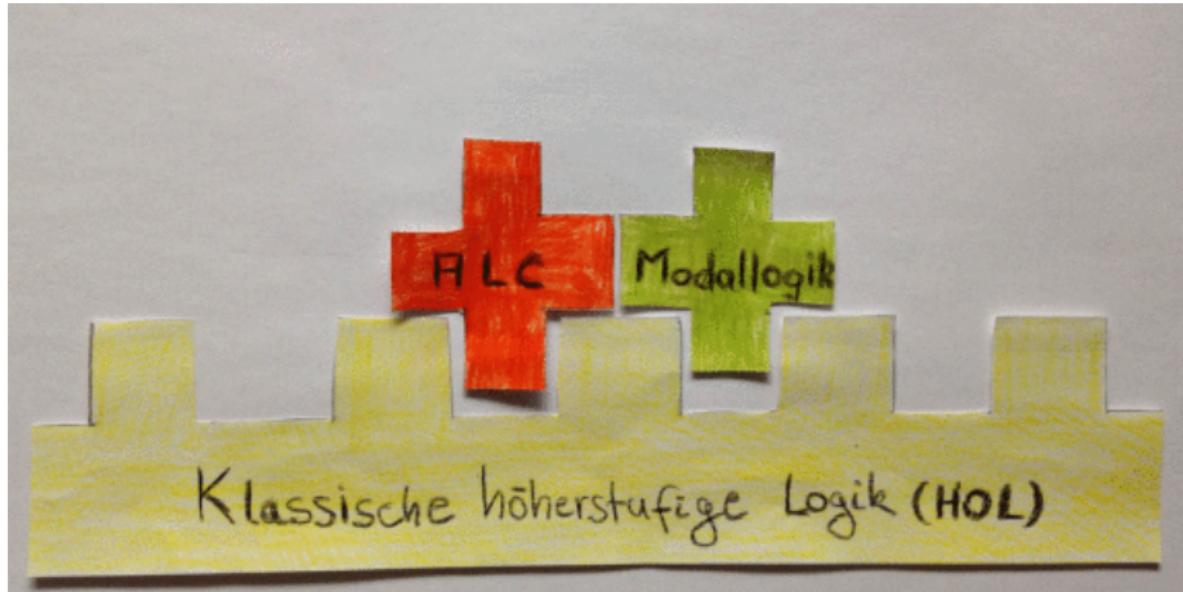
Animation: Max Benzmüller (5 years)

## EINSCHUB: Knowledge representation & Inference about ALC in HOL



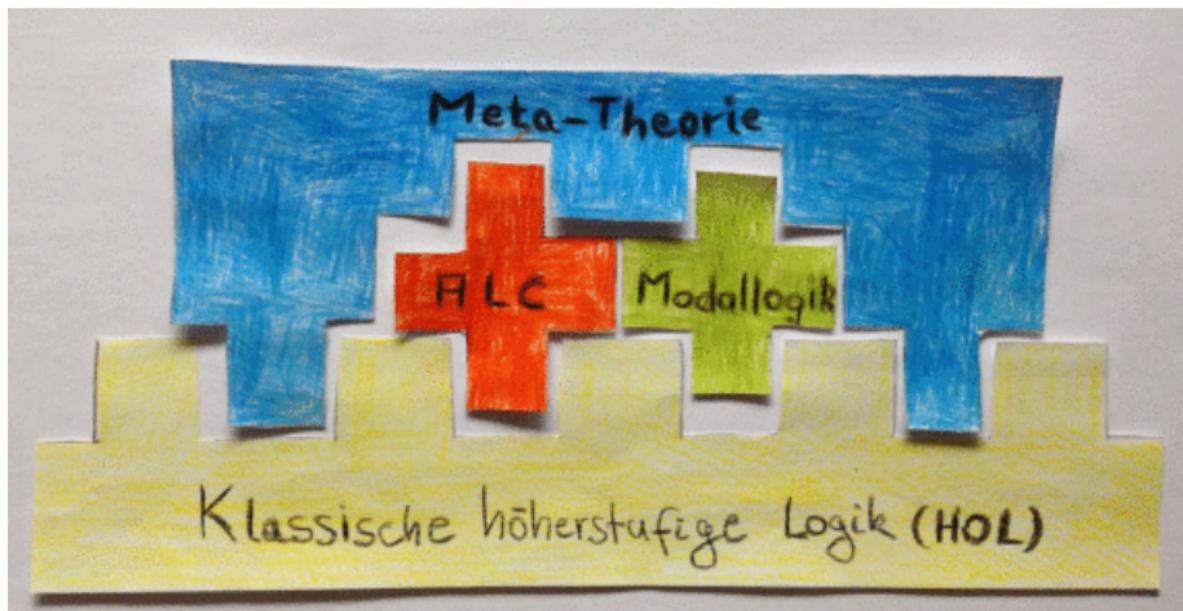
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## EINSCHUB: Knowledge representation & Inference about ALC in HOL



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## EINSCHUB: Knowledge representation & Inference about ALC in HOL



Animation: Max Benzmüller (5 years)

# EINSCHUB: Meta theory of ALC in Isabelle/HOL

The screenshot shows the Isabelle/HOL Sidekick interface with the ALC.thy theory file open. The code defines the ALC logic with various abbreviations for propositional connectives and quantifiers, along with their type signatures and where-clauses. It also includes three lemmas related to the equivalence of different logical connectives.

```
theory ALC imports Main begin

  typedecl i type_synonym τ = "(i ⇒ bool)" type_synonym σ = "(i ⇒ i ⇒ bool)"

  abbreviation bot :: "τ" ("⊥")
  abbreviation top :: "τ" ("⊤")
  abbreviation neg ("¬")
  abbreviation disj (infixr "⊔" 40)
  abbreviation conj (infixr "⊓" 41)
  abbreviation exi_r ("∃")
  abbreviation all_r ("∀")

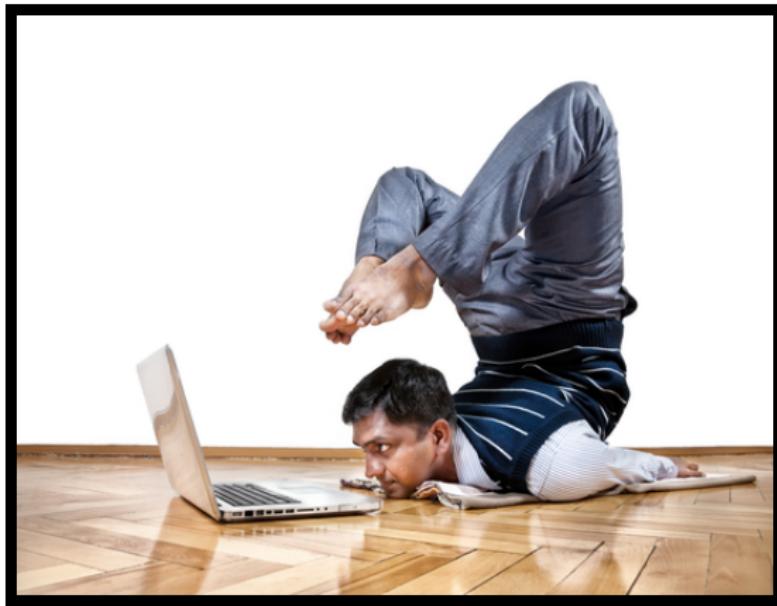
  abbreviation sub (infixr "⊑" 39)
  abbreviation eq (infixr "⊓" 38)

  (* Einfaches Beispiele für etwas Meta-Theorie *)
  lemma "A ⊓ B ≡ ∼(A ⊎ ∼B)" by metis
  lemma "∃ r C ≡ ∼(∀ r (∼C))" by metis (* sledgehammer [remote_leo2] *)
  lemma "A ⊓ B ≡ A ⊎ B" nitpick oops

Nitpicking formula...
Nitpick found a counterexample for card 'a = 2:

  Free variables:
    A = (λx. _)(a1 := False, a2 := False)
    B = (λx. _)(a1 := False, a2 := True)
```

Bottom status bar: 19,22 (939/2329) (isabelle,sidekick,UTF-8-Isabelle)N m r o UG 85/124 MB 9:12 PM



## Flexibility in HOML

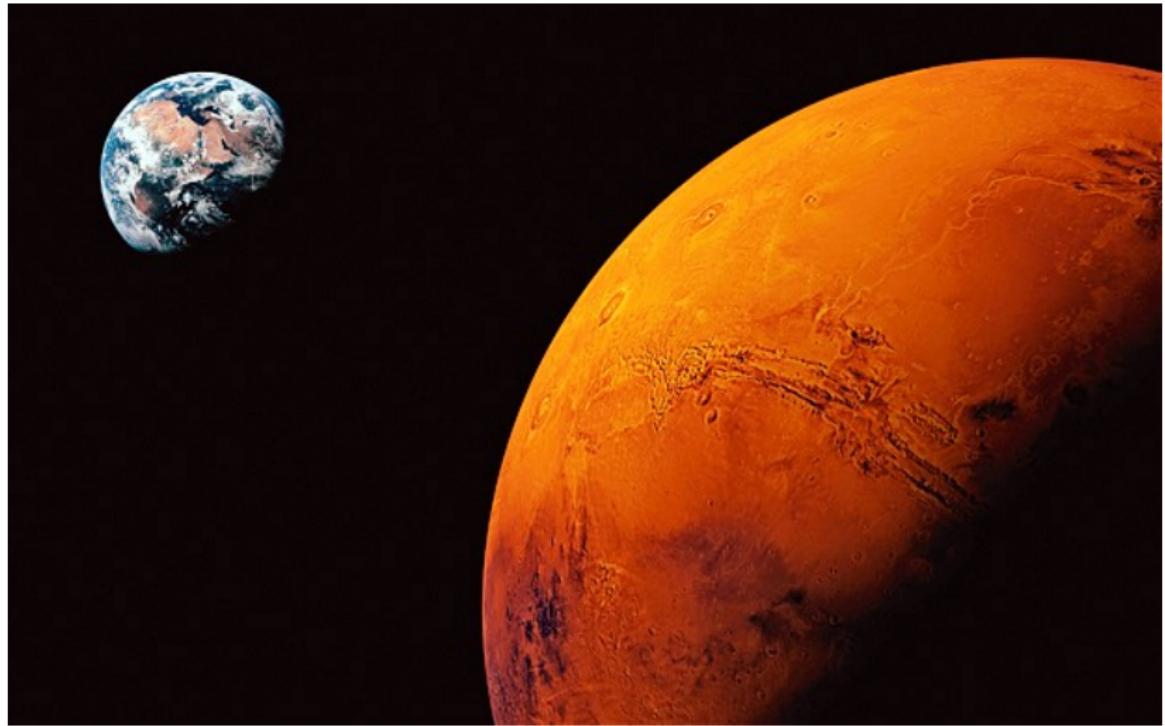
## Many Variations of Higher-Order Modal Logics

- ▶ (Meta-)Axioms for the accessibility relation  
(e.g. reflexivity, transitivity, symmetry)
- ▶ Axioms (e.g.  $\Box A \rightarrow A$ ,  $\Box A \rightarrow \Box\Box A$ ,  $A \rightarrow \Box\Diamond A$ )
- ▶ Possibilistic vs. Actualistic Quantifiers
- ▶ Constant Domains vs. Varying Domains vs. Cumulative Domains
- ▶ Rigidity vs. **Flexibility**
- ▶ Simple Types vs. Dependent Types (e.g.  $\mu$  vs.  $\mu(w)$ )
- ▶ ...

## Flexibility

### A Funny Example

*Blue(sky)*



## Flexibility

### A Funny Example

$\square Blue(sky)$



## Flexibility

### A Funny Example

$\square Blue(sky)$



Human: "Earth's sky is blue"



Martian: "Mars' sky is blue"

## Flexibility

$\square Blue(sky)$

- Rigid embedding:

$$[\mu] = \mu \quad [\textcolor{red}{o}] = \textcolor{blue}{\iota} \rightarrow o \quad [\alpha \rightarrow \beta] = [\alpha] \rightarrow [\beta]$$

$$\begin{aligned} [\textcolor{red}{sky}_\mu] &= \textcolor{blue}{sky}_\mu \\ [\textcolor{red}{Blue}_{\mu \rightarrow o}] &= \textcolor{blue}{Blue}_{\mu \rightarrow (\iota \rightarrow o)} \end{aligned}$$

- Flexible embedding:

$$[\mu] = \textcolor{blue}{\iota} \rightarrow \mu \quad [\textcolor{red}{o}] = \textcolor{blue}{\iota} \rightarrow o \quad [\alpha \rightarrow \beta] = \textcolor{blue}{\iota} \rightarrow [\alpha] \rightarrow [\beta]$$

$$\begin{aligned} [\textcolor{red}{sky}_\mu] &= \textcolor{blue}{sky}_{\iota \rightarrow \mu} \\ [\textcolor{red}{Blue}_{\mu \rightarrow o}] &= \textcolor{blue}{Blue}_{\iota \rightarrow (\iota \rightarrow \mu) \rightarrow (\iota \rightarrow o)} \end{aligned}$$

## Flexibility

### Flexible Embedding is Ambiguous

$[\Box \text{Blue}(\text{sky})]$

$\forall w. \forall w'. \text{Blue } w (\text{sky } w) w'$

$\forall w. \forall w'. \text{Blue } w' (\text{sky } w') w'$

$\forall w. \forall w'. \text{Blue } w' (\text{sky } w) w'$

$\forall w. \forall w'. \text{Blue } w (\text{sky } w') w'$

## Flexibility

### Applications of Flexible HOML

Reasoning with Inconsistencies!



**The End**