Universal Logical Reasoning
(with Applications in Normative Reasoning)

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Introduction

Introduction

This set of slides . . .

. . . is about (narrow picture)

- higher-order modal logic (HOML)
- classical higher-order logic (HOL)
- embedding of HOML in HOL
- mechanisation and automation with HOL ATPs
- various applications

. . . is about (wider picture)

- a quite universal approach to automate reasoning for a wide range of classical and non-classical logics
- a very broad range of possible applications
- including meta-reasoning about logical systems

Introduction

Our primary interest . . .

- is in expressive logics:
  quantification (first- and higher-order) and lambda-expressions;
- propositional fragments are trivially covered

Introduction: Outline

- Introduction
- HOL
- HOML
- Automation of HOML with LEO-II and Isabelle

The remaining slides are optional! We will use others in the upcoming lectures.

- Ontological Argument
- SUMO Ontology and HOML
- Meta-Reasoning
- Flexibility and Rigidity
- Cut-Elimination

Introduction: Expressivity Matters — Cantor's Theorem

Cantor's theorem: The set of all subsets of A, that is, the power set of A, has a strictly greater cardinality than A itself.

In HOL Cantor's theorem (surjective version) can be encoded as

$$\neg\exists F \forall G \exists X . F X = G$$

HO ATPs can solve this problem very efficiently.

Their solution includes the detection and application of the diagonalisation argument

Today: basic test example for new higher-order theorem provers.

Further reading: [AndrewsEtAl., Automating higher-order logic, 1984]

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$$\neg\exists F_{\iota\to(\iota\to o)}\forall G_{\iota\to o}\exists X_\iota.FX = G$$

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Introduction: Expressivity Matters — Boolos' Example

[George Boolos, A curious inference, J. Philosophical Logic, 16:1-12, 1987]

1. $\forall n\ f(n,1) = s(1)$
2. $\forall x\ f(1,s(x)) = s(s(f(1,x)))$
3. $\forall n \forall x\ f(s(n),s(x)) = f(n,f(s(n),x))$
4. $D(1)$
5. $\forall x\ D(x) \rightarrow D(s(x)))$

   $\therefore$

6. $D(f(s(s(s(s(1)))),s(s(s(s(1))))))$

Induction proof: from (4) and (5), we get $\forall x\, D(x)$, hence $D(f(s(s(s(s(1)))),s(s(s(s(1))))))$ by $\forall$-elimination.

But induction is not given, hence the first order proof consists of brute force modus ponens applications: infeasible number of single steps ($2^{(2^{\cdot^{\cdot^{\cdot 2}}})}$ with 64K '2s')

Introduction: Expressivity Matters — Boolos' Example

[Boolos, A curious inference, J. Philosophical Logic, 16:1-12, 1987]

Comprehension axioms $\exists N_{\overline{\alpha^n} \to \beta} \; \forall \overline{z^n} \; N(z^1, \ldots, z^n) = B_\beta$

Can be avoided: use λ-binding construct to denote N

Instances of comprehension axioms in Boolos' proof:

$$\exists N \forall z \, N(z) \leftrightarrow (\forall X \, X(1) \land \forall y \, (X(y) \to X(s(y))) \to X(z))$$

$$\exists E \forall z \, E(z) \leftrightarrow (N(z) \land D(z))$$

Central idea: "assume the induction principle holds for number z – corresponding to $N(z)$ – then we can show for any predicate X a property $X(z)$ by induction."

The proof employs the following lemmata:

Lemma 1: $N(1)$, $\forall y \, (N(y) \to N(s(y)))$, $N(s(s(s(s(1)))))$, $E(1)$, $\forall y \, (E(y) \to E(s(y)))$, $E(s(1))$

Lemma 2: $\forall n \, N(n) \to \forall x \, (N(x) \to E(f(n, x)))$

The theorem itself is then an easy application of the two lemmata.

Introduction: Expressivity Matters — Boolos' Example

[1]By the comprehension principle of second order logic, $\exists N \forall z(Nz \leftrightarrow \forall X[X1 \& \forall y(Xy \to Xsy) \to Xz])$, [2]and then for some N, $\exists E \forall z(Ez \leftrightarrow Nz \& Dz)$.

[3]**Lemma 1:** [3.1]$N1$; [3.2]$\forall y(Ny \to Nsy)$; [3.3]$Nssss1$; [3.4]$E1$; [3.5]$\forall y(Ey \to Esy)$; [3.6]$Es1$;

[4]**Lemma 2:** $\forall n(Nn \to (\forall x(Nx \to Efnx)))$

Proof: [4.1]By comprehension, $\exists M \forall n(Mn \leftrightarrow \forall x(Nx \Rightarrow Efnx))$. [4.2]We want $\forall n(Nn \to Mn)$. [4.3]Enough to show [4.3.1]$M1$ and [4.3.2]$\forall n(Mn \to Msn)$, for then if [4.4]Nn, [4.5]Mn.

[4.3.1]$M1$: [4.3.1.1]Want $\forall x(Nx \to Ef1x)$. [4.3.1.2]By comprehension, $\exists Q \forall x(Qx \leftrightarrow Ef1x)$. [4.3.1.3]Want $\forall x(Nx \to Qx)$. [4.3.1.4]Enough to show [4.3.1.4.1]$Q1$ and [4.3.1.4.2]$\forall x(Qx \to Qsx)$.

[4.3.1.4.1]$Q1$: [4.3.1.4.1.1]Want $Ef11$. [4.3.1.4.1.2]But $f11 = s1$ by (1) and [4.3.1.4.1.3]$Es1$ by Lemma 1.

[4.3.1.4.2]$\forall x(Qx \to Qsx)$: [4.3.1.4.2.1]Suppose Qx, [4.3.1.4.2.2]i.e. Ef1x. [4.3.1.4.2.3]By (2) $f1sx = ssf1x$; [4.3.1.4.2.4]by Lemma 1 twice, $Ef1sx$. [4.3.1.4.2.5]Thus Qsx and [4.3.1.4.2.6]$M1$.

[4.3.2]$\forall n(Mn \to Msn)$: [4.3.2.1]Suppose Mn, [4.3.2.2]i.e. $\forall x(Nx \to Efnx)$. [4.3.2.3]Want Msn, [4.3.2.4]i.e. $\forall x(Nx \to Efsnx)$. [4.3.2.5]By comprehension, $\exists P \forall x(Px \leftrightarrow Efsnx)$. [4.3.2.6]Want $\forall x(Nx \to Px)$. [4.3.2.7]Enough to show [4.3.2.7.1]$P1$ and [4.3.2.7.2]$\forall x(Px \to Psx)$.

[4.3.2.7.1]$P1$: [4.3.2.7.1.1]Want $Efsn1$. [4.3.2.7.1.2]But $fsn1 = s1$ by (1) and [4.3.2.7.1.3]$Es1$ by Lemma 1.

[4.3.2.7.2]$\forall x(Px \to Psx)$: [4.3.2.7.2.1]Suppose Px, [4.3.2.7.2.2]i.e. $Efsnx$; [4.3.2.7.2.3]thus $Nfsnx$. [4.3.2.7.2.4]Want $Efsnsx$. [4.3.2.7.2.5]Since $Nfsnx$ and Mn, $Efnfsnx$. [4.3.2.7.2.6]But by (3) $fnfsnx = fsnsx$; [4.3.2.7.2.7]thus $Efsnsx$.

[5]By Lemma 1, $Nssss1$. [6]By Lemma 2, $Efssss1ssss1$. [7]Thus, $Dfssss1ssss1$, as desired.

Formalization in OMEGA and Mizar: see [BenzmüllerBrown, The curious inference of Boolos in MIZAR and OMEGA, Studies in Logic, Grammar, and Rhetoric, volume 10(23), pp. 299-388, 2007.]
Earlier paper: [BenzmüllerKerber, A Lost Proof, 2001].
Earlier poster: `http://christoph-benzmueller.de/papers/poster-tphols01.pdf`

Inroduction: Higher-Order Modal Logics

$\Box P$: P is necessary, P is obligatory, P is known, P is believed, always P ...

$\Diamond P$: P is possible, P is permissible, P is epistemically possible, P is doxastically possible, eventually P ...

$\Box$ and $\Diamond$ are not truth-functional

HOL can be extended by $\Box$ and $\Diamond$ to obtain HOML

There are many interesting applications of such logics, e.g. the Ontological Argument

Introduction: Embedding Approach — Idea

Your-logic (object-logic) ψ ::=

HOL (meta-logic) φ ::=

Embedding of in

$=$

$=$

$=$

$=$

Embedding of meta-logical notions on in

valid $=$

satisfiable $=$

... $=$

Pass this set of equations to a higher-order automated theorem prover

Introduction: Embedding Approach — HOML in HOL

HOML $\quad \varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

HOL $\quad s, t \quad ::= \quad C_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \mid \neg s_o \mid s_o \vee t_o \mid \forall x_\alpha\, t_o$

HOML in HOL: HOML formulas φ are mapped to HOL predicates $\varphi_{\iota\rightarrow o}$
(explicit representation of labelled formulas)

$$
\boxed{
\begin{array}{lcl}
\neg & = & \lambda\varphi_{\iota\rightarrow o}\lambda w_\iota \neg\varphi w \\
\wedge & = & \lambda\varphi_{\iota\rightarrow o}\lambda\psi_{\iota\rightarrow o}\lambda w_\iota(\varphi w \wedge \psi w) \\
\rightarrow & = & \lambda\varphi_{\iota\rightarrow o}\lambda\psi_{\iota\rightarrow o}\lambda w_\iota(\neg\varphi w \vee \psi w) \\
\forall & = & \lambda h_{\gamma\rightarrow(\iota\rightarrow o)}\lambda w_\iota \forall d_\gamma\, hdw \\
\exists & = & \lambda h_{\gamma\rightarrow(\iota\rightarrow o)}\lambda w_\iota \exists d_\gamma\, hdw \\
\Box & = & \lambda\varphi_{\iota\rightarrow o}\lambda w_\iota \forall u_\iota\, (\neg rwu \vee \varphi u) \\
\Diamond & = & \lambda\varphi_{\iota\rightarrow o}\lambda w_\iota \exists u_\iota\, (rwu \wedge \varphi u) \\
\text{valid} & = & \lambda\varphi_{\iota\rightarrow o}\forall w_\iota\, \varphi w
\end{array}
}
$$

Ax (polymorphic over γ)

The equations in Ax are given as axioms to the HOL provers!

Introduction: Embedding Approach — A “Lean” Prover for HOML KB

```
%----The base type $i (already built-in) stands here for worlds and
%----mu for individuals; $o (also built-in) is the type of Booleans
thf(mu_type,type,(mu:$tType)).
%----Reserved constant r for accessibility relation
thf(r,type,(r:$i>$i>$o)).
%----Modal logic operators not, or, and, implies, box, diamond
thf(mnot_type,type,(mnot:($i>$o)>$i>$o)).
thf(mnot,definition,(mnot = (^[A:$i>$o,W:$i]:~(A@W)))).
thf(mor_type,type,(mor:($i>$o)>($i>$o)>$i>$o)).
thf(mor,definition,(mor = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)|(Psi@W))))).
thf(mand_type,type,(mand:($i>$o)>($i>$o)>$i>$o)).
thf(mand,definition,(mand = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)&(Psi@W))))).
thf(mimplies_type,type,(mimplies:($i>$o)>($i>$o)>$i>$o)).
thf(mimplies,definition,(mimplies = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)&(Psi@W))))).
thf(mbox_type,type,(mbox:($i>$o)>$i>$o)).
thf(mbox,definition,(mbox = (^[A:$i>$o,W:$i]:![V:$i]:(~(r@W@V)|(A@V))))).
thf(mdia_type,type,(mdia:($i>$o)>$i>$o)).
thf(mdia,definition,(mdia = (^[A:$i>$o,W:$i]:?[V:$i]:((r@W@V)&(A@V))))).
%----Quantifiers (constant domains) for individuals and propositions
thf(mforall_ind_type,type,(mforall_ind:(mu>$i>$o)>$i>$o)).
thf(mforall_ind,definition,(mforall_ind = (^[A:mu>$i>$o,W:$i]:![X:mu]:(A@X@W)))).
thf(mforall_indset_type,type,(mforall_indset:((mu>$i>$o)>$i>$o)>$i>$o)).
thf(mforall_indset,definition,(mforall_indset = (^[A:(mu>$i>$o)>$i>$o,W:$i]:![X:mu>$i>$o]:(A@X@W)))).
thf(mexists_ind_type,type,(mexists_ind:(mu>$i>$o)>$i>$o)).
thf(mexists_ind,definition,(mexists_ind = (^[A:mu>$i>$o,W:$i]:?[X:mu]:(A@X@W)))).
thf(mexists_indset_type,type,(mexists_indset:((mu>$i>$o)>$i>$o)>$i>$o)).
thf(mexists_indset,definition,(mexists_indset = (^[A:(mu>$i>$o)>$i>$o,W:$i]:?[X:mu>$i>$o]:(A@X@W)))).
%----Definition of validity (grounding of lifted modal formulas)
thf(v_type,type,(v:($i>$o)>$o)).
thf(mvalid,definition,(v = (^[A:$i>$o]:![W:$i]:(A@W)))).
%----Properties of accessibility relations: symmetry
thf(msymmetric_type,type,(msymmetric:($i>$i>$o)>$o)).
thf(msymmetric,definition,(msymmetric = (^[R:$i>$i>$o]:![S:$i,T:$i]:((R@S@T)=>(R@T@S))))).
%----Here we work with logic KB, i.e., we postulate symmetry for r
thf(sym,axiom,(msymmetric@r)).
```

Reading on THF0 syntax: [SutcliffeBenzmüller, J.Formalized Reasoning, 2010]

Advantages of the Approach

Pragmatics and convenience.
* 'Implementing' a new theorem prover imade very simple
* even for very challenging quantified non-classical logics

Flexibility
* Rapid experimentations with logic variations: quantifiers for constant, varying and cumulative domains; rigid or non-rigid terms; etc.
* Sahlqvist axioms may be postulated or (preferably) the corresponding conditions of the accessibility relation
* prominent connections between logics can be verified and exploited

Availability
* Option 1: Reuse and adapt our TPTP THF0 encodings
* Option 2: Reuse and adapt our Isabelle encodings
* Option 3: Reuse and adapt our Coq encodings
* Option X: Adapt our encodings for your preferred HO prover
* Options can be employed simultaneously.

Advantages of the Approach

Relation to labelled deductive systems
* Labelled deductive systems: meta-level (world-)labelling techniques
* Embedding approach: labels are encoded directly in HOL logic
* No extra-logical annotations in embedding approach

Relation to the standard translation
* Intra-logical formalisation and implementation of the standard translation
* Extension: various other logics
* Extension: quantifiers (different domain conditions)
* Future work: functional translation as an alternative

Soundness and completeness
* Sound and complete (Henkin semantics)

Meta-reasoning
* Example: Verification of the modal logic cube in Isabelle
* Example: Soundness of the usual ALC tableaux rules
* Example: Correspondence between ALC and base modal logic K
* Example: Some meta-level results for conditional logics
* Future work: Completeness results via abstract consistency

Advantages of the Approach

Direct calculi and user intuition

* Implementation of ‘direct’ proof calculi on top of logic embeddings
* Example: ND calculus for HOML as tactics in Coq
* Human intuitive proofs enabled at the interaction layer
* Automation via embedding and ATPs for HOL
* Interesting perspective for mixed proof developments
* Future work: proof planning to automate the abstract-level direct proof calculi; proof assistants in the style of Ωmega could eventually be adapted for this (support for 3-dimensional proof objects!)

Cut-elimination

* Very generic result: combine soundness and completeness of the embedding with the fact that HOL already enjoys cut-elimination for Henkin semantics

λ

Higher-order Logic (HOL)

Classical Higher-Order Logic (HOL)

Expressivity	FOL	HOL	Example
Quantification over			
- Individuals	✓	✓	$\forall X\, p(f(X))$
- Functions	-	✓	$\forall F\, p(F(a))$
- Predicates/Sets/Rels	-	✓	$\forall P\, P(f(a))$
Unnamed			
- Functions	-	✓	$(\lambda X\, X)$
- Predicates/Sets/Rels	-	✓	$(\lambda X\, X \neq a)$
Statements about			
- Functions	-	✓	$continuous(\lambda X\, X)$
- Predicates/Sets/Rels	-	✓	$reflexive(=)$
Powerful abbreviations	-	✓	$reflexive = \lambda R\, \forall X R(X, X)$

Classical Higher-Order Logic (HOL)

Expressivity	FOL	HOL	Example
Quantification over			
- Individuals	✓	✓	$\forall X_\iota\, p_{\iota\to o}(f_{\iota\to\iota}(X_\iota))$
- Functions	-	✓	$\forall F_{\iota\to\iota}\, p_{\iota\to o}(F_{\iota\to o}(a_\iota))$
- Predicates/Sets/Rels	-	✓	$\forall P_{\iota\to o}\, P_{\iota\to o}(f_{\iota\to\iota}(a_\iota))$
Unnamed			
- Functions	-	✓	$(\lambda X_\iota\, X_\iota)$
- Predicates/Sets/Rels	-	✓	$(\lambda X_{\iota\to\iota}\, X_{\iota\to\iota} \neq_{\iota\to\iota\to p} a)_\iota)$
Statements about			
- Functions	-	✓	$\mathit{continuous}_{(\iota\to\iota)\to o}(\lambda X_\iota\, X_\iota)$
- Predicates/Sets/Rels	-	✓	$\mathit{reflexive}_{(\iota\to\iota\to o)\to o}(=_{\iota\to\iota\to o})$
Powerful abbreviations	-	✓	$\mathit{reflexive}_{(\iota\to\iota\to o)\to o} = \lambda R_{(\iota\to\iota\to o)}\, \forall X_\iota R(X, X)$

Simple Types: Prevent Some Paradoxes and Inconsistencies

HOL: Syntax

Simple Types: $\alpha, \beta \quad ::= \quad \iota \mid o \mid (\alpha \to \beta)$

(we may add further base types, e.g. μ)

HOL Language:

$$s,t \quad ::= \quad p_\alpha \mid X_\alpha \mid (\lambda X_\alpha\, s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \mid$$
$$(\neg_{o\to o}\, s_o) \mid ((\vee_{o\to o\to o}\, s_o)\, t_o) \mid \forall_{(\alpha\to o)\to o}(\lambda X_\alpha\, s_o)$$

constant symbols
variable symbols
lambda abstraction
application
negation
disjunction
universal quantification

Terms of type o: formulas

Other logical connectives can be defined, e.g. $\exists X s$ stands for $\neg\forall X \neg s$

Equality may also be defined: $s \doteq t$ stands for $\forall P(Ps \Rightarrow Pt)$
(but it is stongly recommended to add primitive equality!)

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universal quantification

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$$(\neg_{o\to o}\, s_o) \mid \boxed{((\vee_{o\to o\to o}\, s_o)\, t_o)} \mid \forall_{(\alpha\to o)\to o}(\lambda X_\alpha\, s_o)$$

constant symbols
variable symbols
lambda abstraction
application
negation
disjunction (we may use infix notation, omit types and brackets: $s \vee t$)
universal quantification

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application
negation
disjunction
universal quantification (we may use syntactical sugar: $\forall X_\alpha s$)

Terms of type o: formulas

Other logical connectives can be defined, e.g. $\exists X s$ stands for $\neg \forall X \neg s$

Equality may also be defined: $s \doteq t$ stands for $\forall P (P s \Rightarrow P t)$
(but it is stongly recommended to add primitive equality!)

HOL: Syntax

Simple Types: $\alpha, \beta \quad ::= \quad \iota \mid o \mid (\alpha \to \beta)$
(we may add further base types, e.g. μ)

HOL Language:

$$s, t \quad ::= \quad p_\alpha \mid X_\alpha \mid (\lambda X_\alpha\, s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \mid$$
$$(\neg_{o\to o}\, s_o) \mid ((\lor_{o\to o\to o}\, s_o)\, t_o) \mid \forall_{(\alpha\to o)\to o}(\lambda X_\alpha\, s_o)$$

constant symbols
variable symbols
lambda abstraction
application
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universal quantification

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constant symbols
variable symbols
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Equality may also be defined: $s \doteq t$ stands for $\forall P (Ps \Rightarrow Pt)$
(but it is stongly recommended to add primitive equality!)

α-conversion
is considered implicitly, e.g. $(\lambda X\, pX)$ is identified with $(\lambda Y\, pY)$.

Substitution
of a term s_α for X_α in t_β is denoted by $[s/X]t$.
We assume the bound variables of t avoid variable capture.

β-reduction and η-reduction
β-redex has the form $(\lambda X\, s)t$ and β-reduces to $[t/X]s$.
η-redex has the form $(\lambda X\, sX)$ where X is not free in s; it η-reduces to s.
$s \equiv_\beta t$ means s can be converted to t by β-reductions and expansions.
$s \equiv_{\beta\eta} t$ means s can be converted to t using both β and η.
For each $s_\alpha \in \mathrm{HOML}$ there is a unique β-normal form and a unique $\beta\eta$-normal form.

α-conversion

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HOL: Semantics

Semantics of HOL is (meanwhile) well understood

- Origin [Church, J.Symb.Log., 1940]
- Henkin-Semantics [Henkin, J.Symb.Log., 1950]
  [Andrews, J.Symb.Log., 1971, 1972]
- Extensionality/Intensionality [BenzmüllerEtAl., J.Symb.Log., 2004]
  [Muskens, J.Symb.Log., 2007]

HOL with Henkin-Semantics: semi-decidable & compact (like FOL)

Recommended Reading: [BenzmüllerMiller, HandbookHistoryOfLogic, Vol.9, 2014]

HOL: Semantics

A Frame D
is a collection $\{D_\alpha\}_{\alpha \in \mathrm{T}}$ of nonempty sets D_α, such that

- D_ι can be chosen freely
- $D_o = \{T, F\}$ (for truth and falsehood), and
- $D_{\alpha \to \beta}$ are collections of functions mapping D_α into D_β

A Model M
for HOL is a tuple $\boldsymbol{M} = \langle \boldsymbol{D}, \boldsymbol{I} \rangle$, where

- $\boldsymbol{D}$ is a frame;
- $\boldsymbol{I}$ is a family of typed interpretation functions mapping constant symbols $\boldsymbol{p_\alpha}$ to appropriate elements of $\boldsymbol{D_\alpha}$, called the denotation of $\boldsymbol{p_\alpha}$;
- the logical connectives $\neg$, $\vee$, and $\forall$ are always given the standard denotations;
- moreover, we assume that the domains $\boldsymbol{D_{\alpha \to \alpha \to o}}$ contain the respective identity relations.

HOL: Semantics

Variable Assignment

A <u>variable assignment</u> g maps variables X_α to elements in D_α.
$g[d/W]$ denotes the assignment that is identical to g, except for variable W, which is now mapped to d.

Interpretation/Value of a HOL term

The <u>value</u> $\|s_\alpha\|^{M,g}$ of a HOL term s_α on a model $M = \langle D, I\rangle$ under assignment g is an element $d \in D_\alpha$ defined in the following way:

1. $\|p_\alpha\|^{M,g} = I(p_\alpha)$
2. $\|X_\alpha\|^{M,g} = g(X_\alpha)$
3. $\|(s_{\alpha\to\beta}\, t_\alpha)_\beta\|^{M,g} = \|s_{\alpha\to\beta}\|^{M,g}(\|t_\alpha\|^{M,g})$
4. $\|(\lambda X_\alpha\, s_\beta)_{\alpha\to\beta}\|^{M,g} =$ the function f from D_α to D_β such that $f(d) = \|s_\beta\|^{M,g[d/X_\alpha]}$ for all $d \in D_\alpha$ Text
5. $\|(\neg_{o\to o}\, s_o)_o\|^{M,g} = T$ iff $\|s_o\|^{M,g} = F$
6. $\|((\vee_{o\to o\to o}\, s_o)\, t_o)_o\|^{M,g} = T$ iff $\|s_o\|^{M,g} = T$ or $\|t_o\|^{M,g} = T$
7. $\|(\forall_{(\alpha\to o)\to o}(\lambda X_\alpha\, s_o))_o\|^{M,g} = T$ iff for all $d \in D_\alpha$ we have $\|s_o\|^{M,g[d/X_\alpha]} = T$

HOL: Semantics

Variable Assignment

A variable assignment g maps variables x_α to elements in D_α.
$g[d/W]$ denotes the assignment that is identical to g, except for variable W, which is now mapped to d.

Interpretation/Value of a HOL term

The value $||s_\alpha||^{M,g}$ of a HOL term s_α on a model $M = \langle D, I\rangle$ under assignment g is an element $d \in D_\alpha$ defined in the following way:

1. $||P_\alpha||^{M,g} = I(P_\alpha)$
2. $||x_\alpha||^{M,g} = g(x_\alpha)$
3. $||(s_{\alpha\to\beta}\, t_\alpha)_\beta||^{M,g} = ||s_{\alpha\to\beta}||^{M,g}(||t_\alpha||^{M,g})$
4. $||(\lambda x_\alpha\, s_\beta)_{\alpha\to\beta}||^{M,g}$ = the function f from D_α to D_β such that $f(d) = ||s_\beta||^{M,g[d/x_\alpha]}$ for all $d \in D_\alpha$
5. $I(\neg_{o\to o})$ = not $\in D_{o\to o}$ such that: not$(F) = T$ and not$(T) = F$
6. $I(\vee_{o\to o\to o})$ = or $\in D_{o\to o\to o}$ such that: or$(a, b) = T$ iff $a = T$ or $b = T$
7. $I(\forall_{(\alpha\to o)\to o})$ = All $\in D_{(\alpha\to o)\to o}$ such that: for $p \in D_{\alpha\to o}$ we have All$(p) = T$ iff $p(d) = T$ for all $d \in D_\alpha$

Standard and Henkin Models

In a standard model $\boldsymbol{M} = \langle \boldsymbol{D}, \boldsymbol{I} \rangle$ we have

- $\boldsymbol{D_{\alpha \to \beta}} = \{f \mid f : D_\alpha \longrightarrow D_\beta\}$ (for all types α, β)

In a Henkin model $\boldsymbol{M} = \langle \boldsymbol{D}, \boldsymbol{I} \rangle$ we only require

- $\boldsymbol{D_{\alpha \to \beta}} \subseteq \{f \mid f : \boldsymbol{D_\alpha} \longrightarrow \boldsymbol{D_\beta}\}$ (for all types α, β)
- the valuation function $\| \cdot \|^{M,g}$ from above is total (every term denotes)

Any standard model is obviously also a Henkin model.
We consider Henkin models in the remainder.

HOL: Semantics

Truth in model / Validity

A formula s_o is true in model M under assignment g if and only if $\|s_o\|^{M,g} = T$; this is also denoted as $M, g \models s_o$.

A formula s_o is called valid in M if and only if $M, g \models s_o$ for all assignments g; this is also denoted as $M \models s_o$.

A formula s_o is called valid, which we denote by $\models s_o$, if and only if $M \models s_o$ for all M.

We define $\models \varphi$, where φ is a set of HOL formulas, if and only if $\models s$ for all $s \in \varphi$.

Logical Consequence

Let φ and ψ be set of HOL formulas. We define

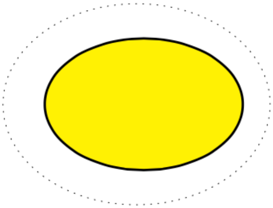
$$\varphi \models \psi \qquad \text{(logical consequence)}$$

if and only if for each model M we have:

$$M \models \varphi \quad \text{implies} \quad M \models \psi$$

HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009]



Standard Models $\mathfrak{ST}(\Sigma)$

- Idea of Standard Semantics:

$\iota \longrightarrow \mathcal{D}_\iota$ (choose)

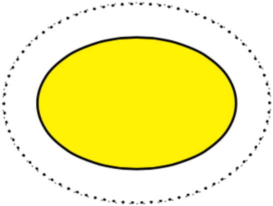
$o \longrightarrow \mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$ (fixed)

$(\alpha \rightarrow \beta) \longrightarrow$

$\mathcal{D}_{\alpha\rightarrow\beta} = \mathcal{F}(\mathcal{D}_\alpha, \mathcal{D}_\beta)$ (fixed)

HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009]



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  $(\alpha \to \beta) \longrightarrow$

  $\mathcal{D}_{\alpha\to\beta} = \mathcal{F}(\mathcal{D}_\alpha, \mathcal{D}_\beta)$ (fixed)

- Henkin's Generalization:

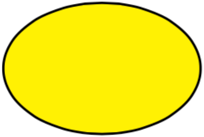
  $\mathcal{D}_{\alpha\to\beta} \subseteq \mathcal{F}(\mathcal{D}_\alpha, \mathcal{D}_\beta)$ (choose)

  but elements are still functions!

[Henkin-50]

HOL: Semantics

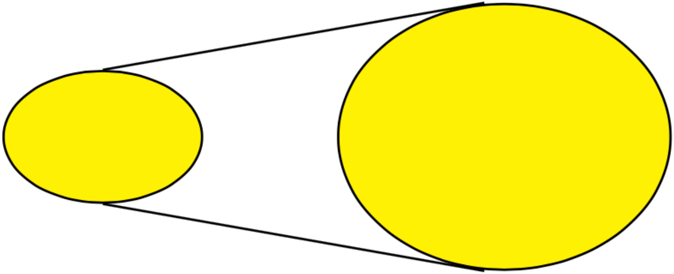
(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009]



Standard Models $\mathfrak{ST}(\Sigma)$

choose: $\mathcal{D}_\iota$
fixed: $\mathcal{D}_o, \mathcal{D}_{\alpha\to\beta}$, functions

HOL: Semantics
(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009]



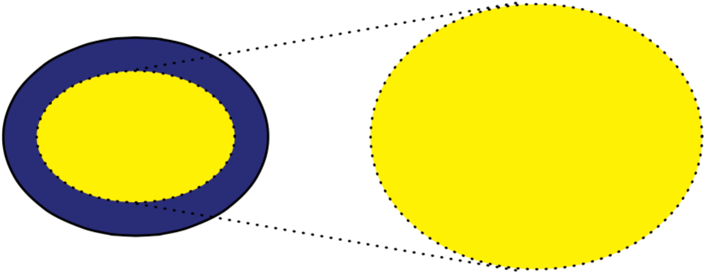
Standard Models $\mathfrak{ST}(\Sigma)$

Formulas valid in $\mathfrak{ST}(\Sigma)$

choose: $\mathcal{D}_\iota$
fixed: $\mathcal{D}_o, \mathcal{D}_{\alpha\to\beta}$, functions

HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009]



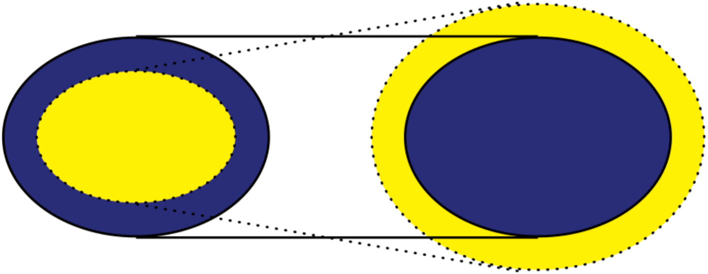
Henkin Models $\mathfrak{H}(\Sigma) = \mathfrak{M}_{\beta\mathfrak{fb}}(\Sigma)$

Formulas valid in $\mathfrak{M}_{\beta\mathfrak{fb}}(\Sigma)$?

choose: $\mathcal{D}_\iota, \mathcal{D}_{\alpha\to\beta}$
fixed: $\mathcal{D}_o$, functions

HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009]



Henkin Models $\mathfrak{H}(\Sigma) = \mathfrak{M}_{\beta\mathfrak{fb}}(\Sigma)$

Formulas valid in $\mathfrak{M}_{\beta\mathfrak{fb}}(\Sigma)$

choose: $\mathcal{D}_\iota, \mathcal{D}_{\alpha\to\beta}$
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HOL: Semantics

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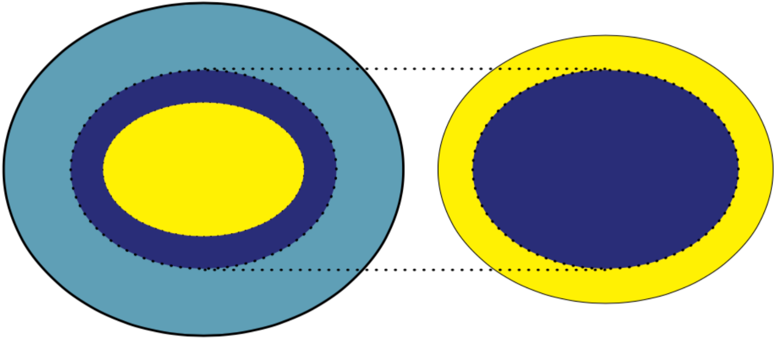


Non-Extensional Models $\mathfrak{M}_\beta(\Sigma)$

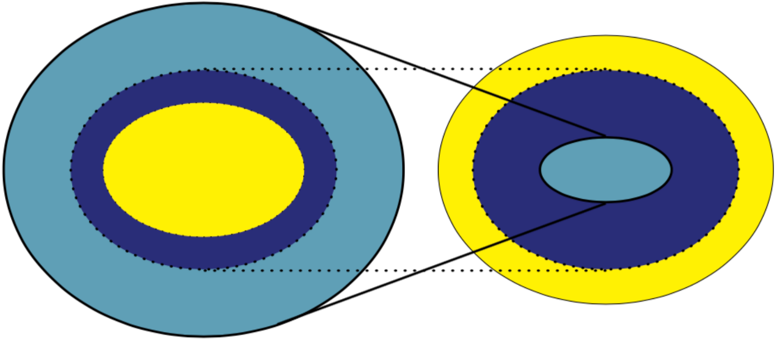
Formulas valid in $\mathfrak{M}_\beta(\Sigma)$?

choose: $\mathcal{D}_\iota, \mathcal{D}_{\alpha\to\beta}$, also non$-$functions, $\mathcal{D}_o$

fixed:

HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009]



Non-Extensional Models $\mathfrak{M}_\beta(\Sigma)$

Formulas valid in $\mathfrak{M}_\beta(\Sigma)$?

choose: $\mathcal{D}_\iota, \mathcal{D}_{\alpha\rightarrow\beta}$, also non−functions, $\mathcal{D}_o$

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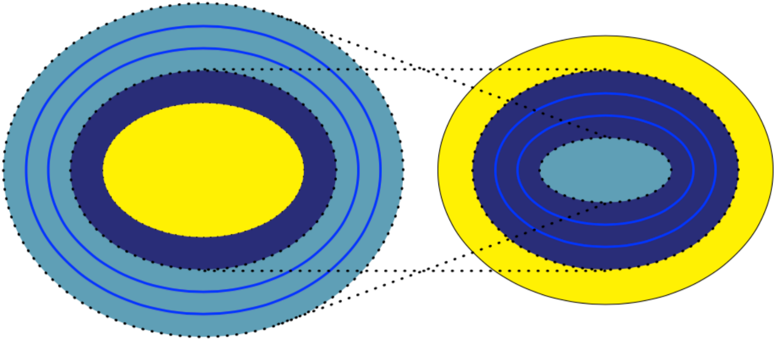


We additionally studied different model classes with 'varying degrees of extensionality'

HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009]

$\mathfrak{M}_\beta(\Sigma)$ — non-extensional models

$\mathfrak{b}$: Boolean extensionality, $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$

$\mathfrak{f}(= \eta + \xi)$: functional extensionality

η: η-functional

ξ: ξ-functionality

$\downarrow$

$\mathfrak{M}_{\beta\mathfrak{fb}}(\Sigma) \simeq \mathfrak{H}(\Sigma)$ — Henkin models

full

$\downarrow$

$\mathfrak{ST}(\Sigma)$ — standard models

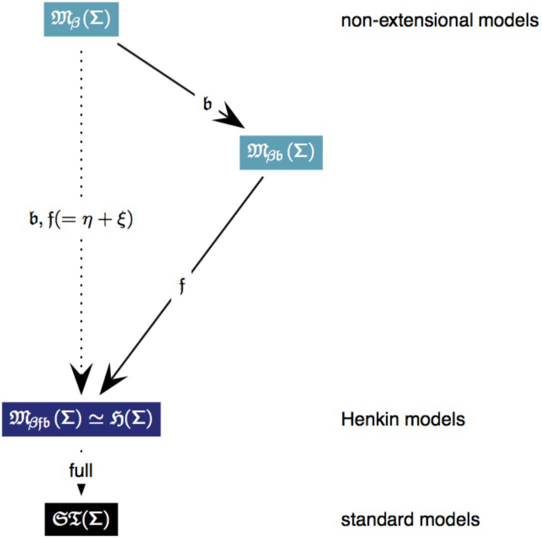
HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009]

$$
\begin{tikzcd}
\mathfrak{M}_{\beta}(\Sigma) \arrow[dr, "\mathfrak{b}"] \arrow[dd, dotted, "{\mathfrak{b}, \mathfrak{f}(= \eta + \xi)}"'] & & \text{non-extensional models} \\
& \mathfrak{M}_{\beta\mathfrak{b}}(\Sigma) \arrow[dl, "\mathfrak{f}"] & \\
\mathfrak{M}_{\beta\mathfrak{f}\mathfrak{b}}(\Sigma) \simeq \mathfrak{H}(\Sigma) \arrow[d, dotted, "\text{full}"] & & \text{Henkin models} \\
\mathfrak{ST}(\Sigma) & & \text{standard models}
\end{tikzcd}
$$

HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009])



HOL: Semantics

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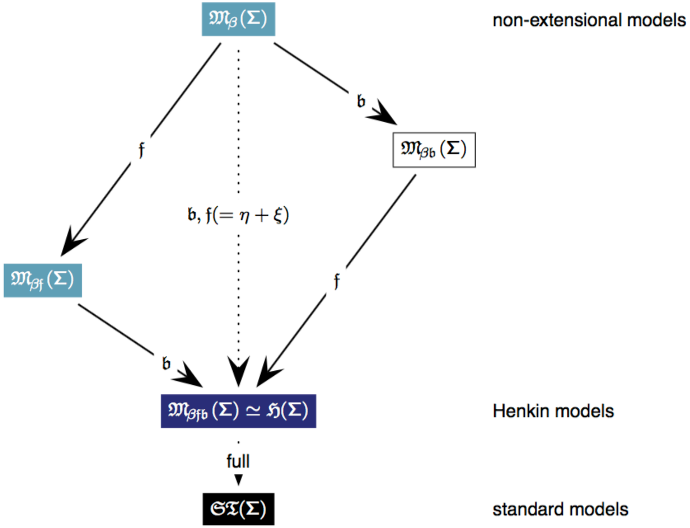

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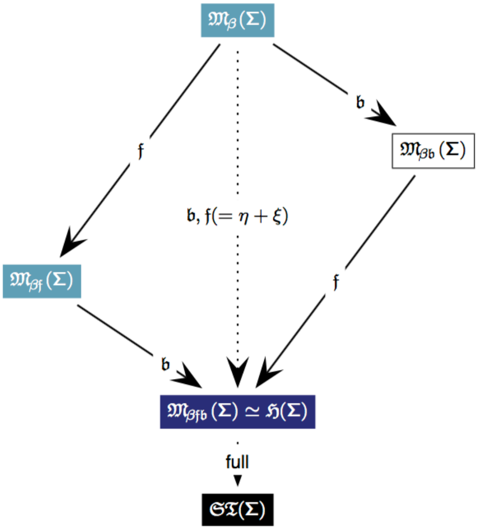


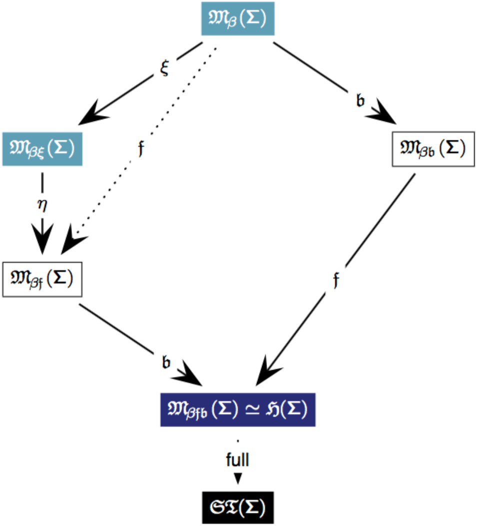
non-extensional models

Henkin models

standard models

HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009]



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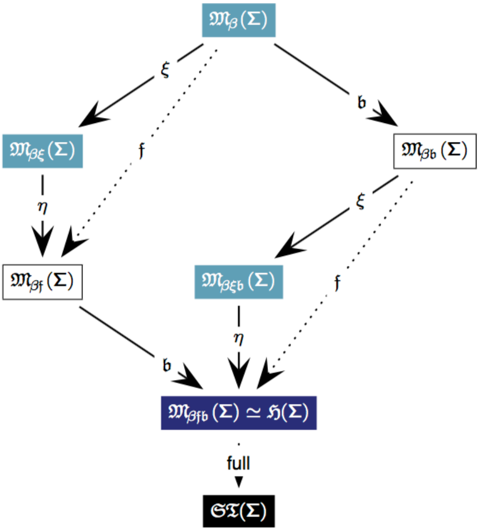

non-extensional models

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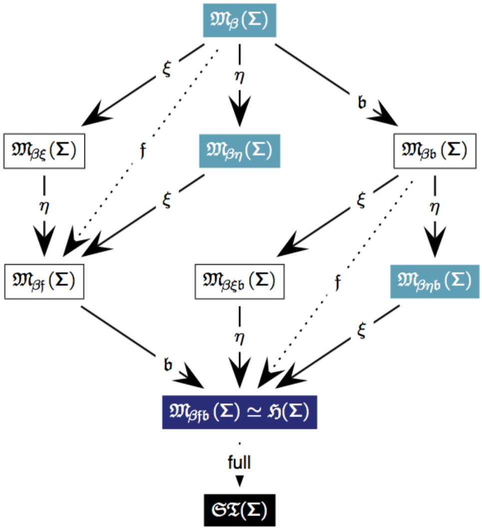

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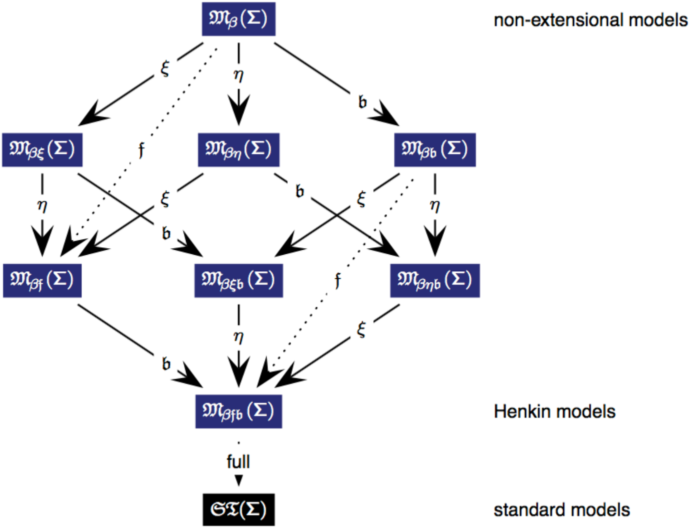
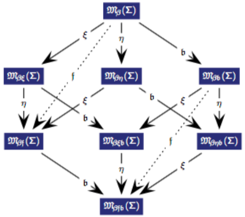


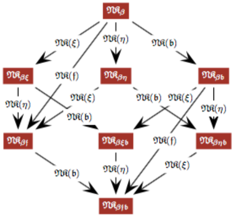
HOL: Semantics

(see [BenzmüllerBrownKohlhase, J.Symb.Log., 2004] and [BenzmüllerBrownKohlhase, LMCS, 2009]

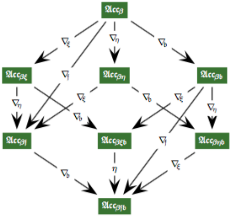
Semantics:
Model Classes (Extensionality)



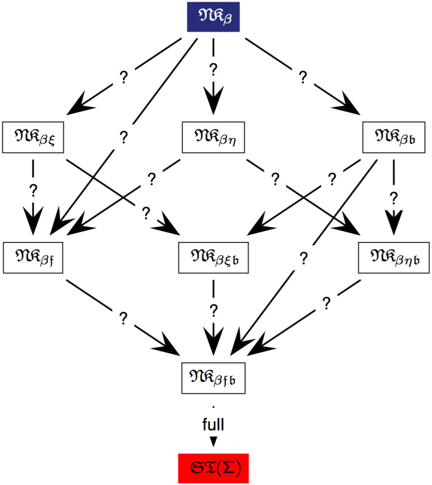
Reference Calculi:
ND (and others)



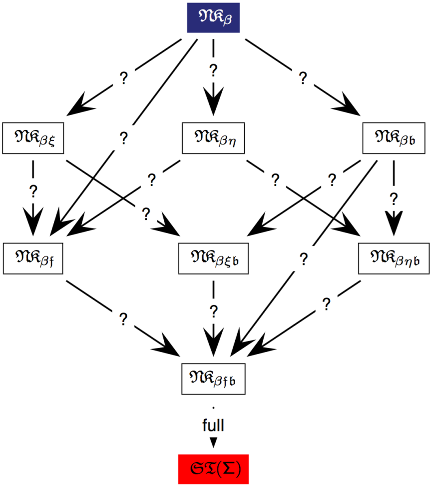
Abstract Consistency / Unifying Principle:
Extensions of Smullyan-63 and Andrews-71



HOL: ND Calculi



HOL: ND Calculi



Base Calculus $\mathfrak{NK}_\beta$

— $\mathfrak{NK}(Hyp)$ — $\mathfrak{NK}(\beta)$

— $\mathfrak{NK}(\neg I)$ — $\mathfrak{NK}(\neg E)$

— $\mathfrak{NK}(\vee I_L)$ — $\mathfrak{NK}(\vee I_R)$

— $\mathfrak{NK}(\vee E)$

— $\mathfrak{NK}(\Pi I)^{\mathsf{w}}$

— $\mathfrak{NK}(\Pi E)$ — $\mathfrak{NK}(Contr)$

HOL: ND Calculi

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \mathbf{\Phi}}{\mathbf{\Phi} \Vdash \mathbf{A}} \mathfrak{NK}(Hyp)$$

$\mathbf{\Phi} * \mathbf{A} := \mathbf{\Phi} \cup \{\mathbf{A}\}$

HOL: ND Calculi

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\beta)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

HOL: ND Calculi

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(\neg I)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

HOL: ND Calculi

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

HOL: ND Calculi

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_L)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

HOL: ND Calculi

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}} \mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg\mathbf{A}} \mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}} \mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}} \mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}} \mathfrak{NK}(\vee I_R)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

HOL: ND Calculi

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_R)$$

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\vee E)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

HOL: ND Calculi

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\ \mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_\circ}{\Phi \Vdash \neg\mathbf{A}}\ \mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\ \mathfrak{NK}(\vee I_R)$$

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\ \mathfrak{NK}(\vee E)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w}\ \text{new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\ \mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

HOL: ND Calculi

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}} \mathfrak{NK}(Hyp) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg \mathbf{A}} \mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg \mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}} \mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}} \mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}} \mathfrak{NK}(\vee I_R)$$

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}} \mathfrak{NK}(\vee E)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w}\text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}} \mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

$$\frac{\Phi \Vdash \Pi^\alpha \mathbf{G}}{\Phi \Vdash \mathbf{G}\mathbf{A}} \mathfrak{NK}(\Pi E)$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

HOL: ND Calculi

$\mathfrak{NK}_\beta$:

$$\frac{\mathbf{A} \in \Phi}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\mathit{Hyp}) \qquad \frac{\mathbf{A} =_\beta \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\beta)$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \neg\mathbf{A}}\,\mathfrak{NK}(\neg I) \qquad \frac{\Phi \Vdash \neg\mathbf{A} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\neg E)$$

$$\frac{\Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_L) \qquad \frac{\Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \vee \mathbf{B}}\,\mathfrak{NK}(\vee I_R)$$

$$\frac{\Phi \Vdash \mathbf{A} \vee \mathbf{B} \quad \Phi * \mathbf{A} \Vdash \mathbf{C} \quad \Phi * \mathbf{B} \Vdash \mathbf{C}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\vee E)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathsf{w}_\alpha \quad \mathsf{w}\text{ new parameter}}{\Phi \Vdash \Pi^\alpha \mathbf{G}}\,\mathfrak{NK}(\Pi I)^{\mathsf{w}}$$

$$\frac{\Phi \Vdash \Pi^\alpha \mathbf{G}}{\Phi \Vdash \mathbf{G}\mathbf{A}}\,\mathfrak{NK}(\Pi E) \qquad \frac{\Phi * \neg\mathbf{A} \Vdash \mathbf{F}_o}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\mathit{Contr})$$

$\Phi * \mathbf{A} := \Phi \cup \{\mathbf{A}\}$

HOL: ND Calculi

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\wedge E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}\,\mathfrak{NK}(\wedge I)$$

$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\Rightarrow E) \qquad \frac{\Phi, \mathbf{A} \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B}}\,\mathfrak{NK}(\Rightarrow I)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \Vdash \Sigma^\alpha \mathbf{G}}\,\mathfrak{NK}(\Sigma I) \qquad \frac{\Phi \Vdash \Sigma^\alpha \mathbf{G} \quad \Phi * \mathbf{G}w_\alpha \Vdash \mathbf{C} \quad \text{w new parameter}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\Sigma E)$$

$$\frac{\Phi \Vdash \mathbf{T} =^\alpha \mathbf{W} \quad \Phi \Vdash \mathbf{A}[\mathbf{T}]}{\Phi \Vdash \mathbf{A}[\mathbf{W}]}\,\mathfrak{NK}(=Subst) \qquad \frac{}{\Phi \Vdash \mathbf{A} = \mathbf{A}}\,\mathfrak{NK}(=Refl)$$

HOL: ND Calculi

Inference rules for $\mathfrak{NK}_\beta$ (for richer signatures)

$$\frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{A}}\,\mathfrak{NK}(\wedge E_L) \qquad \frac{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\wedge E_R) \qquad \frac{\Phi \Vdash \mathbf{A} \quad \Phi \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \wedge \mathbf{B}}\,\mathfrak{NK}(\wedge I)$$

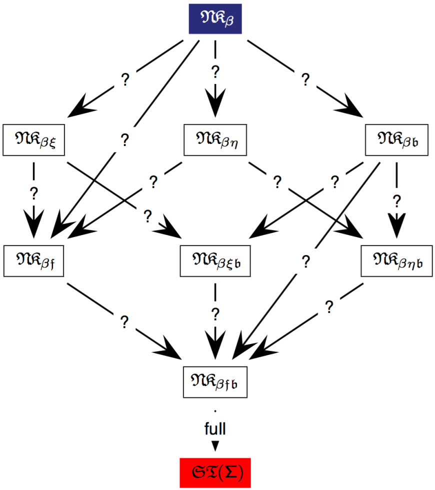
$$\frac{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\Rightarrow E) \qquad \frac{\Phi, \mathbf{A} \Vdash \mathbf{B}}{\Phi \Vdash \mathbf{A} \Rightarrow \mathbf{B}}\,\mathfrak{NK}(\Rightarrow I)$$

$$\frac{\Phi \Vdash \mathbf{G}\mathbf{T}_\alpha}{\Phi \Vdash \Sigma^\alpha \mathbf{G}}\,\mathfrak{NK}(\Sigma I) \qquad \frac{\Phi \Vdash \Sigma^\alpha \mathbf{G} \quad \Phi * \mathbf{G}\mathsf{w}_\alpha \Vdash \mathbf{C} \quad \text{w new parameter}}{\Phi \Vdash \mathbf{C}}\,\mathfrak{NK}(\Sigma E)$$

$$\frac{\Phi \Vdash \mathbf{T} =^\alpha \mathbf{W} \quad \Phi \Vdash \mathbf{A}[\mathbf{T}]}{\Phi \Vdash \mathbf{A}[\mathbf{W}]}\,\mathfrak{NK}(=Subst) \qquad \frac{}{\Phi \Vdash \mathbf{A} = \mathbf{A}}\,\mathfrak{NK}(=Refl)$$

Here: we define logical constants $\wedge, \Rightarrow, \Sigma$, etc. in terms of $\neg, \vee, \Pi$ as usual and strictly use Leibniz equality instead of primitive equality; then the above rules are not needed

HOL: ND Calculi



Base Calculus $\mathfrak{NK}_{\beta}$

— $\mathfrak{NK}(Hyp)$ — $\mathfrak{NK}(\beta)$

— $\mathfrak{NK}(\neg I)$ — $\mathfrak{NK}(\neg E)$

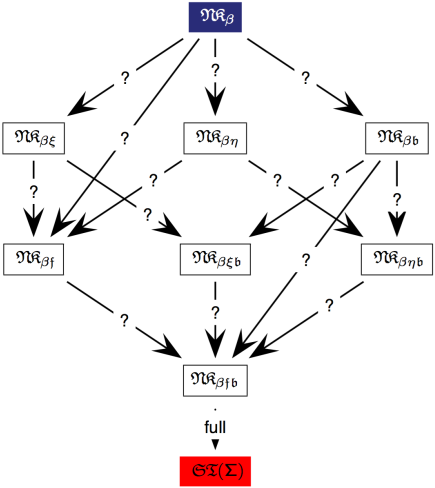
— $\mathfrak{NK}(\vee I_L)$ — $\mathfrak{NK}(\vee I_R)$

— $\mathfrak{NK}(\vee E)$

— $\mathfrak{NK}(\Pi I)^{\mathsf{w}}$

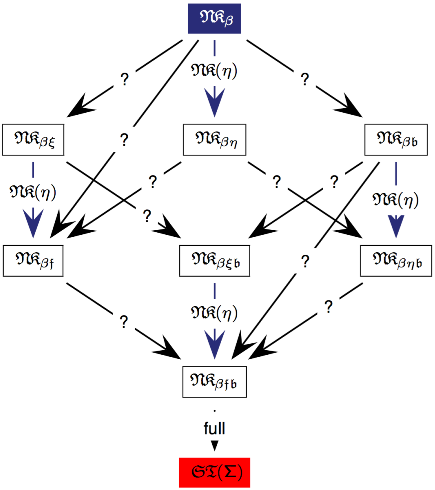
— $\mathfrak{NK}(\Pi E)$ — $\mathfrak{NK}(Contr)$

HOL: ND Calculi



Optional Extensionality Rules

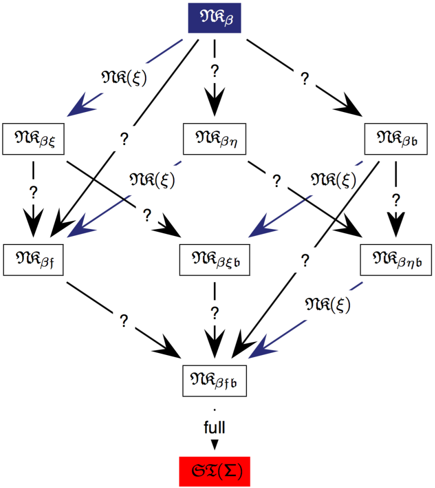
HOL: ND Calculi



Optional Extensionality Rules

$$\frac{\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\ \mathfrak{NK}(\eta)$$

HOL: ND Calculi

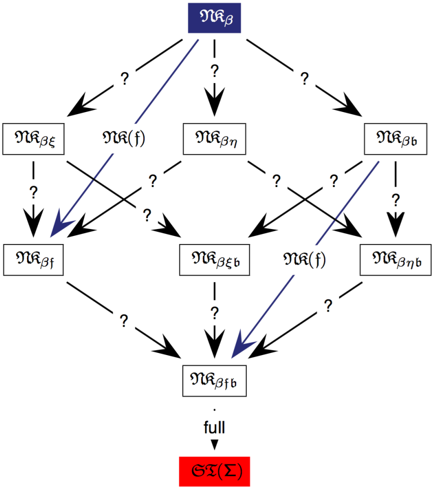


Optional Extensionality Rules

$$\frac{\mathbf{A} \stackrel{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \mathfrak{NK}(\eta)$$

$$\frac{\Phi \vdash \forall x_\alpha.\mathbf{M} \doteq^{\beta} \mathbf{N}}{\Phi \vdash (\lambda x_\alpha.\mathbf{M}) \doteq^{\beta\alpha} (\lambda x_\alpha.\mathbf{N})} \mathfrak{NK}(\xi)$$

HOL: ND Calculi



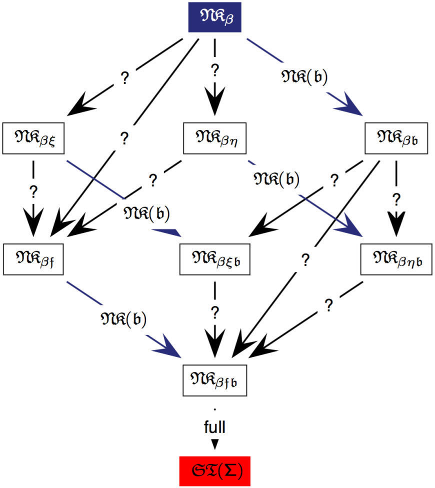
Optional Extensionality Rules

$$\frac{\mathbf{A} \overset{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}} \,\mathfrak{NK}(\eta)$$

$$\frac{\Phi \Vdash \forall X_\alpha . \mathbf{M} \doteq^{\beta} \mathbf{N}}{\Phi \Vdash (\lambda X_\alpha . \mathbf{M}) \doteq^{\beta\alpha} (\lambda X_\alpha . \mathbf{N})} \,\mathfrak{NK}(\xi)$$

$$\frac{\Phi \Vdash \forall X_\alpha . \mathbf{G}X \doteq^{\beta} \mathbf{H}X}{\Phi \Vdash \mathbf{G} \doteq^{\beta\alpha} \mathbf{H}} \,\mathfrak{NK}(\mathfrak{f})$$

HOL: ND Calculi



Optional Extensionality Rules

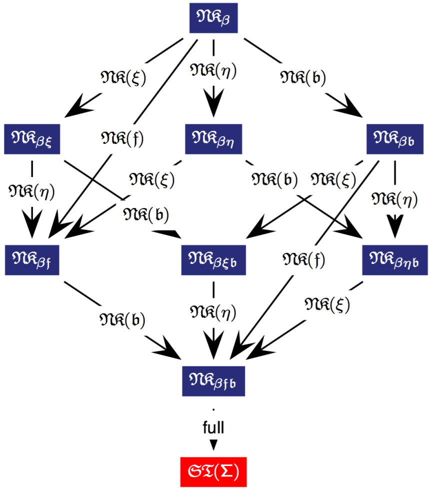
$$\frac{\mathbf{A} \overset{\beta\eta}{=} \mathbf{B} \quad \Phi \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{B}}\,\mathfrak{NK}(\eta)$$

$$\frac{\Phi \Vdash \forall X_\alpha.\mathbf{M} \doteq^{\beta} \mathbf{N}}{\Phi \Vdash (\lambda X_\alpha.\mathbf{M}) \doteq^{\beta\alpha} (\lambda X_\alpha.\mathbf{N})}\,\mathfrak{NK}(\xi)$$

$$\frac{\Phi \Vdash \forall X_\alpha.\mathbf{G}X \doteq^{\beta} \mathbf{H}X}{\Phi \Vdash \mathbf{G} \doteq^{\beta\alpha} \mathbf{H}}\,\mathfrak{NK}(\mathfrak{f})$$

$$\frac{\Phi * \mathbf{A} \Vdash \mathbf{B} \quad \Phi * \mathbf{B} \Vdash \mathbf{A}}{\Phi \Vdash \mathbf{A} \doteq^{o} \mathbf{B}}\,\mathfrak{NK}(\mathfrak{b})$$

HOL: ND Calculi



Base Calculus $\mathfrak{NK}_\beta$

$\mathfrak{NK}(Hyp)$ $\mathfrak{NK}(\beta)$

$\mathfrak{NK}(\neg I)$ $\mathfrak{NK}(\neg E)$

$\mathfrak{NK}(\vee I_L)$ $\mathfrak{NK}(\vee I_R)$

$\mathfrak{NK}(\vee E)$

$\mathfrak{NK}(\Pi I)^w$

$\mathfrak{NK}(\Pi E)$ $\mathfrak{NK}(Contr)$

Optional Extensionality Rules

$\mathfrak{NK}(\eta)$ $\mathfrak{NK}(\xi)$

$\mathfrak{NK}(\mathfrak{f})$ $\mathfrak{NK}(\mathfrak{b})$

HOL: ND Calculi

Derivation of:

$$(\mathsf{A} \doteq^{\alpha} \mathsf{A})$$

HOL: ND Calculi

$$\frac{}{\Vdash_{\mathfrak{NK}_\beta} (\mathsf{A} \doteq^\alpha \mathsf{A}) := \Pi^\alpha(\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))}\ \mathfrak{NK}(\Pi I)$$

Derivation of:

$$(\mathsf{A} \doteq^\alpha \mathsf{A})$$

HOL: ND Calculi

$$\dfrac{\dfrac{}{\Vdash_{\mathfrak{NK}_\beta} ((\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))\mathsf{q})}\,\mathfrak{NK}(\beta)}{\Vdash_{\mathfrak{NK}_\beta} (\mathsf{A} \doteq^{\alpha} \mathsf{A}) := \Pi^{\alpha}(\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))}\,\mathfrak{NK}(\Pi I)$$

Derivation of:

$$(\mathsf{A} \doteq^{\alpha} \mathsf{A})$$

HOL: ND Calculi

$$\dfrac{\dfrac{\dfrac{}{\Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{q}\mathsf{A}) \vee (\mathsf{q}\mathsf{A})}\,\mathfrak{NK}(Contr)}{\Vdash_{\mathfrak{NK}_\beta} ((\lambda \mathsf{P}(\neg(\mathsf{P}\mathsf{A}) \vee (\mathsf{P}\mathsf{A})))\mathsf{q})}\,\mathfrak{NK}(\beta)}{\Vdash_{\mathfrak{NK}_\beta} (\mathsf{A} \doteq^{\alpha} \mathsf{A}) := \Pi^{\alpha}(\lambda \mathsf{P}(\neg(\mathsf{P}\mathsf{A}) \vee (\mathsf{P}\mathsf{A})))}\,\mathfrak{NK}(\Pi I)$$

Derivation of:

$$(\mathsf{A} \doteq^{\alpha} \mathsf{A})$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\Phi^1 := \{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}\,\mathfrak{NK}(\neg E)}{\Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})}\,\mathfrak{NK}(Contr)}{\Vdash_{\mathfrak{NK}_\beta} ((\lambda\mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))\mathsf{q})}\,\mathfrak{NK}(\beta)}{\Vdash_{\mathfrak{NK}_\beta} (\mathsf{A} \doteq^\alpha \mathsf{A}) := \Pi^\alpha(\lambda\mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))}\,\mathfrak{NK}(\Pi I)$$

Derivation of:

$$(\mathsf{A} \doteq^\alpha \mathsf{A})$$

HOL: ND Calculi

$$\dfrac{\dfrac{\dfrac{\dfrac{\Phi^1 \Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})}{\Phi^1 := \{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_\circ}\ \mathfrak{NK}(\neg E)}{\Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})}\ \mathfrak{NK}(Contr)}{\Vdash_{\mathfrak{NK}_\beta} ((\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))\mathsf{q})}\ \mathfrak{NK}(\beta)}{\Vdash_{\mathfrak{NK}_\beta} (\mathsf{A} \doteq^\alpha \mathsf{A}) := \Pi^\alpha(\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))}\ \mathfrak{NK}(\Pi I)$$

Derivation of:

$$(\mathsf{A} \doteq^\alpha \mathsf{A})$$

HOL: ND Calculi

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Phi^1 \Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA}) \qquad \dfrac{}{\Phi^1 \Vdash_{\mathfrak{NK}_\beta} \{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))}\,\mathfrak{NK}(Hyp)}{\Phi^1 := \{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_\circ}\,\mathfrak{NK}(\neg E)}{\Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})}\,\mathfrak{NK}(Contr)}{\Vdash_{\mathfrak{NK}_\beta} ((\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))\mathsf{q})}\,\mathfrak{NK}(\beta)}{\Vdash_{\mathfrak{NK}_\beta} (\mathsf{A} \doteq^\alpha \mathsf{A}) := \Pi^\alpha(\lambda \mathsf{P}(\neg(\mathsf{PA}) \vee (\mathsf{PA})))}\,\mathfrak{NK}(\Pi I)
$$

Derivation of:

$$(\mathsf{A} \doteq^\alpha \mathsf{A})$$

HOL: ND Calculi

Derivation of:

$$\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p$$

resp. $\{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{NK}_\beta} \neg(qA) \vee (qA)$

$$\frac{}{\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}}\,\mathfrak{NK}(\vee I_L)$$

Derivation of:

$$\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}$$

resp. $\{\neg(\neg(\mathsf{q}\mathsf{A}) \vee (\mathsf{q}\mathsf{A}))\} \Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{q}\mathsf{A}) \vee (\mathsf{q}\mathsf{A})$

HOL: ND Calculi

$$\dfrac{\dfrac{}{\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p}}\ \mathfrak{NK}(\neg I)}{\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}}\ \mathfrak{NK}(\vee I_L)$$

Derivation of:

$$\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}$$

resp. $\{\neg(\neg(\mathsf{q}\mathsf{A}) \vee (\mathsf{q}\mathsf{A}))\} \Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{q}\mathsf{A}) \vee (\mathsf{q}\mathsf{A})$

HOL: ND Calculi

$$\dfrac{\dfrac{\dfrac{}{\Phi^2 := \{\neg(\neg \mathsf{p} \vee \mathsf{p}), \mathsf{p}\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}\mathfrak{NK}(\neg E)}{\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p}}\mathfrak{NK}(\neg I)}{\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}}\mathfrak{NK}(\vee I_L)$$

Derivation of:

$$\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}$$

resp. $\{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))\} \Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})$

HOL: ND Calculi

$$\dfrac{\dfrac{}{\Phi^2 \Vdash_{\mathfrak{NK}_\beta} \neg(\neg p \vee p)}\mathfrak{NK}(Hyp) \qquad \dfrac{\dfrac{\Phi^2 := \{\neg(\neg p \vee p), p\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p}\mathfrak{NK}(\neg I)}{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p}\mathfrak{NK}(\vee I_L)}{}\mathfrak{NK}(\neg E)$$

Derivation of:

$$\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p$$

resp. $\{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{NK}_\beta} \neg(qA) \vee (qA)$

HOL: ND Calculi

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Phi^2 \Vdash_{\mathfrak{NK}_\beta} \neg(\neg \mathsf{p} \vee \mathsf{p})}\,\mathfrak{NK}(Hyp) \qquad \dfrac{}{\Phi^2 \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}}\,\mathfrak{NK}(\vee I_R)}{\Phi^2 := \{\neg(\neg \mathsf{p} \vee \mathsf{p}), \mathsf{p}\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}\,\mathfrak{NK}(\neg E)}{\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p}}\,\mathfrak{NK}(\neg I)}{\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}}\,\mathfrak{NK}(\vee I_L)
$$

Derivation of:

$$\{\neg(\neg \mathsf{p} \vee \mathsf{p})\} \Vdash_{\mathfrak{NK}_\beta} \neg \mathsf{p} \vee \mathsf{p}$$

resp. $\{\neg(\neg(\mathsf{qA}) \vee (\mathsf{qA}))\} \Vdash_{\mathfrak{NK}_\beta} \neg(\mathsf{qA}) \vee (\mathsf{qA})$

HOL: ND Calculi

$$
\dfrac{\dfrac{}{\Phi^2 \Vdash_{\mathfrak{NK}_\beta} \neg(\neg p \vee p)}\ \mathfrak{NK}(Hyp) \qquad \dfrac{\dfrac{}{\Phi^2 \Vdash_{\mathfrak{NK}_\beta} p}\ \mathfrak{NK}(Hyp)}{\Phi^2 \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p}\ \mathfrak{NK}(\vee I_R)}{\dfrac{\Phi^2 := \{\neg(\neg p \vee p), p\} \Vdash_{\mathfrak{NK}_\beta} \mathbf{F}_o}{\dfrac{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p}{\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p}\ \mathfrak{NK}(\vee I_L)}\ \mathfrak{NK}(\neg I)}\ \mathfrak{NK}(\neg E)
$$

Derivation of:

$$\{\neg(\neg p \vee p)\} \Vdash_{\mathfrak{NK}_\beta} \neg p \vee p$$

resp. $\{\neg(\neg(qA) \vee (qA))\} \Vdash_{\mathfrak{NK}_\beta} \neg(qA) \vee (qA)$

HOL: Sequent Calculi

One-sided sequent calculus $\mathcal{G}_{\beta\mathfrak{f}\mathfrak{b}}$ [BenzmüllerBrownKohlhase, LMCS, 2009]
(Δ: finite sets of β-normal closed formulas, $\Delta * \mathbf{A}$ stands for $\Delta \cup \{\mathbf{A}\}$,
$cwff_\alpha$: set of closed terms of type α, $\doteq$ abbreviates Leibniz equality):

Base Rules

$$\frac{\mathbf{A} \text{ atomic (\& } \beta\text{-normal)}}{\Delta * \mathbf{A} * \neg\mathbf{A}}\,\mathcal{G}(init) \qquad \frac{\Delta * \mathbf{A}}{\Delta * \neg\neg\mathbf{A}}\,\mathcal{G}(\neg) \qquad \frac{\Delta * \neg\mathbf{A} \quad \Delta * \neg\mathbf{B}}{\Delta * \neg(\mathbf{A} \vee \mathbf{B})}\,\mathcal{G}(\vee_-)$$

$$\frac{\Delta * \mathbf{A} * \mathbf{B}}{\Delta * (\mathbf{A} \vee \mathbf{B})}\,\mathcal{G}(\vee_+) \qquad \frac{\Delta * \neg\,(\mathbf{AC})\!\downarrow_\beta \quad \mathbf{C} \in cwff_\alpha}{\Delta * \neg\Pi^\alpha\mathbf{A}}\,\mathcal{G}(\Pi_-^{\mathbf{C}}) \qquad \frac{\Delta * (\mathbf{A}c)\!\downarrow_\beta \quad c_\alpha\, new}{\Delta * \Pi^\alpha\mathbf{A}}\,\mathcal{G}(\Pi_+^c)$$

Full Extensionality

$$\frac{\Delta * (\forall X_\alpha . \mathbf{A}X \doteq^\beta \mathbf{B}X)\!\downarrow_\beta}{\Delta * (\mathbf{A} \doteq^{\alpha\to\beta} \mathbf{B})}\,\mathcal{G}(\mathfrak{f}) \qquad \frac{\Delta * \neg\mathbf{A} * \mathbf{B} \quad \Delta * \neg\mathbf{B} * \mathbf{A}}{\Delta * (\mathbf{A} \doteq^o \mathbf{B})}\,\mathcal{G}(\mathfrak{b})$$

Initial. and Decomp. of Leibniz Equality

$$\frac{\Delta * (\mathbf{A} \doteq^o \mathbf{B}) \quad \mathbf{A},\mathbf{B} \text{ atomic}}{\Delta * \neg\mathbf{A} * \mathbf{B}}\,\mathcal{G}(Init^{\doteq})$$

$$\frac{\Delta * (\mathbf{A}^1 \doteq^{\alpha_1} \mathbf{B}^1) \cdots \Delta * (\mathbf{A}^n \doteq^{\alpha_n} \mathbf{B}^n) \quad n \geq 1, \beta \in \{o, \iota\}, h_{\overline{\alpha^n}\to\beta} \in \Sigma}{\Delta * (h\overline{\mathbf{A}^n} \doteq^\beta h\overline{\mathbf{B}^n})}\,\mathcal{G}(d)$$

Soundness, Completeness and Cut-elimination

iff $\quad \mathrm{Ax} \models^{HOL}_{\text{Henkin}} \varphi_{\iota\to o} \quad$ iff $\quad \mathrm{Ax} \vdash^{\text{HOL}}_{\text{cut-free}} \varphi_{\iota\to o}$

Higher-order Modal Logic (HOML)

HOML: Motivation

$$\Box P$$

P is necessary, P is obligatory, P is known,
P is believed, always P ...

$$\Diamond P$$

P is possible, P is permissible, P is epistemically possible,
P is doxastically possible, eventually P ...

$\Box$ and $\Diamond$ are not truth-functional

HOL can be extended by $\Box P$ and $\Diamond P$ to obtain HOML

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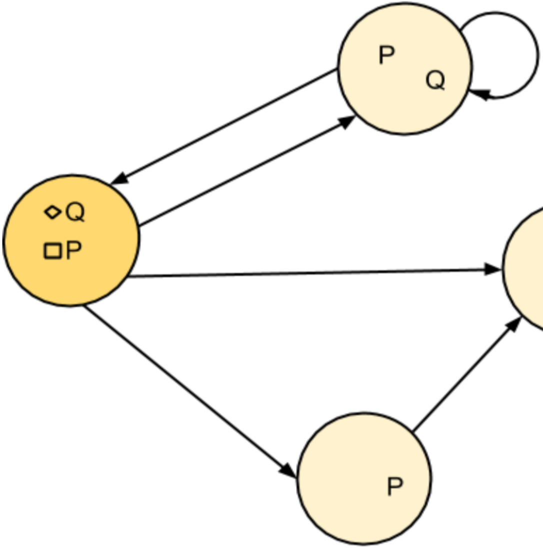
HOML: Motivation
Kripke Semantics - Possible Worlds



HOML: Syntax

Simple Types: $\alpha, \beta \quad ::= \quad \iota \mid o \mid (\alpha \to \beta)$

HOML Language:

$$
\begin{aligned}
s, t \quad ::= \quad & p_\alpha \mid X_\alpha \mid (\lambda X_\alpha\, s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \mid \\
& (\neg_{o\to o}\, s_o) \mid ((\vee_{o\to o\to o}\, s_o)\, t_o) \mid \forall_{(\alpha\to o)\to o}(\lambda X_\alpha\, s_o) \\
& (\Box_{o\to o}\, s_o)
\end{aligned}
$$

constant symbols
variable symbols
lambda abstraction
application
negation
disjunction
universal quantification
modal box operator

Terms of type o: formulas

Other logical connectives can be defined, e.g. $\Diamond s$ stands for $\neg\Box\neg s$

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HOML: Syntax

α-conversion

. . . as before . . .

Substitution

. . . as before . . .

β-reduction and η-reduction

. . . as before . . .

HOML: Semantics

Semantics of HOML investigated in

- [D. Gallin, Instensional and Higher-Order Modal Logic, North Holland, 1975]
- [R. Muskens, Higher Order Modal Logic, Handbook of Modal Logic, 2006]
- [Benzmüller and Woltzenlogel Paleo, Automating Gödel's Ontological Proof . . . , ECAI, 2014]
- our interest: combination of Kripke style models and Henkin semantics

HOML: Semantics

A Frame D

... as before ...

A Model M

for HOML is a quadruple $M = \langle W, R, D, \{I_w\}_{w \in W} \rangle$, where

- W is a set of worlds (or states);
- R is an accessibility relation between the worlds in W;
- D is a frame;
- for each $w \in W$, $\{I_w\}_{w \in W}$ is a family of typed interpretation functions mapping constant symbols p_α to appropriate elements of D_α, called the <u>denotation of p_α in world w</u>;
- the logical connectives $\neg$, $\vee$, $\forall$, and $\Box$ are always given the standard denotations;
- moreover, it is assumed that the domains $D_{\alpha \to \alpha \to o}$ contain the respective identity relations on objects of type α.

HOML: Semantics

Variable Assignment

. . . as before . . .

Interpretation/Value of a HOML term

The value $\|s_\alpha\|^{M,g,w}$ of a HOML term s_α on a model $M = \langle W, R, D, \{I_w\}_{w\in W}\rangle$ in a world $w \in W$ under variable assignment g is an element $d \in D_\alpha$ defined in the following way:

1. $\|p_\alpha\|^{M,g,w} = I_w(p_\alpha)$
2. $\|X_\alpha\|^{M,g,w} = g(X_\alpha)$
3. $\|(s_{\alpha\to\beta}\, t_\alpha)_\beta\|^{M,g,w} = \|s_{\alpha\to\beta}\|^{M,g,w}(\|t_\alpha\|^{M,g,w})$
4. $\|(\lambda X_\alpha\, s_\beta)_{\alpha\to\beta}\|^{M,g,w} =$ the function f from D_α to D_β such that $f(d) = \|s_\beta\|^{M,g[d/X_\alpha],w}$ for all $d \in D_\alpha$
5. $\|(\neg_{o\to o}\, s_o)_o\|^{M,g,w} = T$ iff $\|s_o\|^{M,g,w} = F$
6. $\|((\vee_{o\to o\to o}\, s_o)\, t_o)_o\|^{M,g,w} = T$ iff $\|s_o\|^{M,g,w} = T$ or $\|t_o\|^{M,g,w} = T$
7. $\|(\forall_{(\alpha\to o)\to o}(\lambda X_\alpha\, s_o))_o\|^{M,g,w} = T$ iff for all $d \in D_\alpha$ we have $\|s_o\|^{M,g[d/X_\alpha],w} = T$
8. $\|(\Box_{o\to o}\, s_o)_o\|^{M,g,w} = T$ **iff for all** $v \in W$ **with** wRv **we have** $\|s_o\|^{M,g,v} = T$

Standard and Henkin Models (as before)

In a standard model $M = \langle W, R, D, \{I_w\}_{w \in W} \rangle$ we have

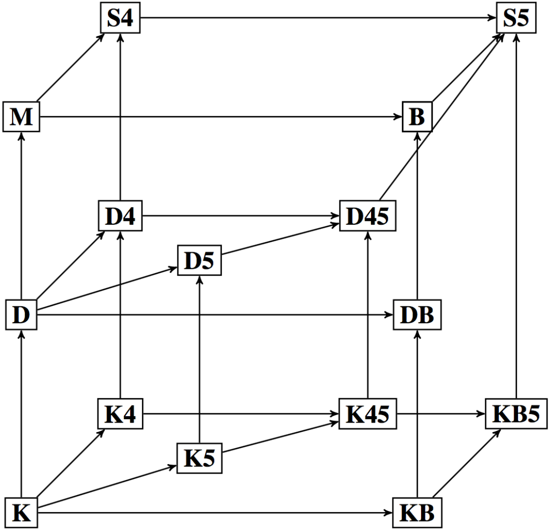
- $D_{\alpha \to \beta} = \{f \mid f : D_\alpha \longrightarrow D_\beta\}$ (for all types α, β)

In a Henkin model $M = \langle W, R, D, \{I_w\}_{w \in W} \rangle$ we only require

- $D_{\alpha \to \beta} \subseteq \{f \mid f : D_\alpha \longrightarrow D_\beta\}$ (for all types α, β)
- the valuation function $\| \cdot \|^{M,g,w}$ from above is total (every term denotes)

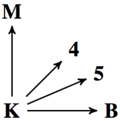
Any standard model is obviously also a Henkin model.
We consider Henkin models in the remainder.

HOML: The Modal Logic Cube



$\equiv$ **M5** $\equiv$ **MB5** $\equiv$ **M4B5** $\equiv$ **M45** $\equiv$ **M4B** $\equiv$ **D4B** $\equiv$ **D4B5** $\equiv$ **DB5**

M: $\Box P \rightarrow P$
B: $P \rightarrow \Box\Diamond P$
D: $\Box P \rightarrow \Diamond P$
4: $\Box P \rightarrow \Box\Box P$
5: $\Diamond P \rightarrow \Box\Diamond P$



$\equiv$ **K4B5** $\equiv$ **K4B**

How to automate HOML?

Embedding HOML in HOL

Challenge: No provers for Higher-order Modal Logic (HOML)

Church's Simple Type Theory

Our solution: Embedding in **Higher-order Classical Logic** (HOL)

Then use existing HOL theorem provers for reasoning in HOML

[BenzmüllerPaulson, Logica Universalis, 2013]

Assumption: Henkin semantics for both HOML and HOL

Previous empirical findings:

Embedding of First-order Modal Logic in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[Benzmüller, LPAR, 2013]

Embedding HOML in HOL

HOML $\quad \varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

HOL $\quad s, t \quad ::= \quad C_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \mid \neg s_o \mid s_o \vee t_o \mid \forall x_\alpha\, t_o$

HOML in HOL: HOML formulas φ are mapped to HOL predicates $\varphi_{\iota\rightarrow o}$ (explicit representation of labelled formulas)

$$
\begin{array}{lcl}
\neg & = & \lambda\varphi_{\iota\rightarrow o}\lambda w_\iota \neg\varphi w \\
\wedge & = & \lambda\varphi_{\iota\rightarrow o}\lambda\psi_{\iota\rightarrow o}\lambda w_\iota(\varphi w \wedge \psi w) \\
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Ax (polymorphic over γ)

The equations in Ax are given as axioms to the HOL provers!

Embedding HOML in HOL

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Embedding HOML in HOL

Example

HOML formula $\Diamond \exists x G(x)$

HOML formula in HOL valid $(\Diamond \exists x G(x))_{\iota \to o}$

expansion, $\beta\eta$-conversion $\forall w_\iota (\Diamond \exists x G(x))_{\iota \to o}\, w$

expansion, $\beta\eta$-conversion $\forall w_\iota \exists u_\iota (rwu \wedge (\exists x G(x))_{\iota \to o} u)$

expansion, $\beta\eta$-conversion $\forall w_\iota \exists u_\iota (rwu \wedge \exists x Gxu)$

Expansion: user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid $\varphi_{\iota \to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

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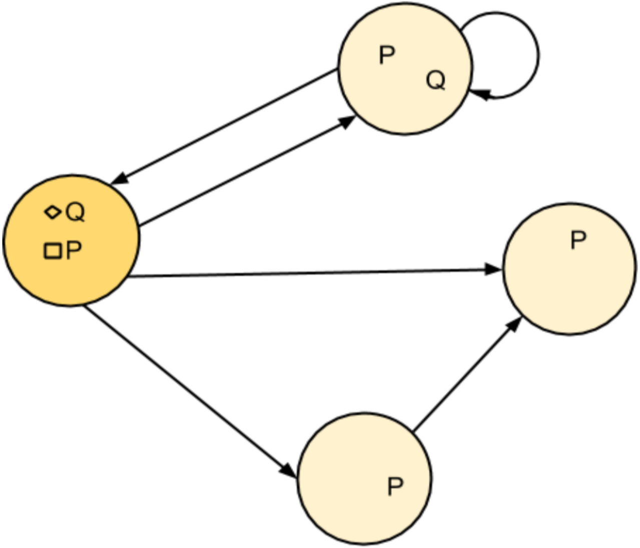
Modal logic axioms	Semantical conditions
M: valid $\forall\varphi(\Box\varphi \rightarrow \varphi)$	$\forall x(rxy)$
B: valid $\forall\varphi(\varphi \rightarrow \Box\Diamond\varphi)$	$\forall x\forall y(rxy \rightarrow ryx)$
D: valid $\forall\varphi(\Box\varphi \rightarrow \Diamond\varphi)$	$\forall x\exists y(rxy)$
4: valid $\forall\varphi(\Box\varphi \rightarrow \Box\Box\varphi)$	$\forall x\forall y\forall z(rxy \wedge ryz \rightarrow rxz)$
5: valid $\forall\varphi(\Diamond\varphi \rightarrow \Box\Diamond\varphi)$	$\forall x\forall y\forall z(rxy \wedge rxz \rightarrow ryz)$

HOML: Motivation

Kripke Semantics - Possible Worlds

Possibilist vs. Actualist Quantification



Embedding HOML in HOL: Possibilist vs. Actualist Quantification

$\forall \;\; = \lambda h_{\gamma\to(\iota\to o)} \lambda w_\iota \forall d_\gamma \, hdw$ (possibilist / constant dom.)

becomes

$\forall^{va} = \lambda h_{\gamma\to(\iota\to o)} \lambda w_\iota \forall d_\gamma \, (\textbf{ExInW} dw \to hdw)$ (actualist / varying dom.)

where **ExInW** is an existence predicate.

Additional axioms (optional):

- domains are non-empty $\forall w_\iota \exists x_\mu \textbf{exInW} xw$
- denotation (constants & functions) $\forall w_\iota \textbf{exInW} cw$
  $\forall w_\iota (\textbf{exInW} t^1 w \wedge \ldots \wedge \textbf{exInW} t^n w \supset \textbf{exInW}(f\, t^1 \ldots t^n) w)$

Cumulative domains: $\forall x \forall v \forall w (\textbf{exInW} xv \wedge rvw \supset \textbf{exInW} xw)$

Embedding HOML in HOL: Theoretical Results

Soundness and Completeness

$$\models^{HOML}_{\text{Henkin}} s_o \quad \text{iff} \quad \text{Ax} \models^{HOL}_{\text{Henkin}} \mathit{valid}\; s_{\iota\rightarrow o} \quad (\text{iff} \quad \text{Ax} \vdash^{\text{HOL}}_{\text{cut-free}} \mathit{valid}\; \varphi_{\iota\rightarrow o})$$

Embedding HOML in HOL: Theoretical Results

Soundness and Completeness

$$\models^{HOML}_{\text{Henkin}} s_o \quad \text{iff} \quad \mathsf{Ax} \models^{HOL}_{\text{Henkin}} \mathit{valid}\ s_{\iota\to o}$$

Proof sketch (adapts [Benzmüller and Paulson, Logica Universalis, 2013]):
By contraposition it is sufficient to show

$$\not\models^{HOML}_{\text{Henkin}} s_o \text{ iff } \mathsf{Ax} \not\models^{HOL}_{\text{Henkin}} \mathit{valid}\ s_{\iota\to o}$$

One easily gets the proof by choosing the obvious correspondences between D and $\boldsymbol{D}$, W and $\boldsymbol{D_\iota}$, I and $\boldsymbol{I}$, g and $\boldsymbol{g}$, R and $r_{\iota\to\iota\to o}$, and w and $\boldsymbol{w}$. $\square$

Embedding HOML in HOL: Theoretical Results

Soundness and Completeness (and Cut-elimination)

$\models^{HOML}_{\mathrm{Henkin}} s_o$ iff $Ax \models^{HOL}_{\mathrm{Henkin}} \mathit{valid}\ s_{\iota\to o}$

Proof sketch (adapts [Benzmüller and Paulson, Logica Universalis, 2013]):
By contraposition it is sufficient to show

$$\not\models^{HOML}_{\mathrm{Henkin}} s_o \text{ iff } Ax \not\models^{HOL}_{\mathrm{Henkin}} \mathit{valid}\ s_{\iota\to o}$$

One easily gets the proof by choosing the obvious correspondences between D and $\boldsymbol{D}$, W and $\boldsymbol{D}_{\boldsymbol{\iota}}$, I and $\boldsymbol{I}$, g and $\boldsymbol{g}$, R and $r_{\iota\to\iota\to o}$, and w and $\boldsymbol{w}$. $\square$

Embedding of Other Logics in HOL: Theoretical Results

Soundness and Completeness (and Cut-elimination)

$$\models^{L} s_o \quad \text{iff} \quad \mathsf{Ax} \models^{HOL}_{\mathsf{Henkin}} \mathit{valid}\, s_{\iota\to o} \quad (\text{iff} \quad \mathsf{Ax} \vdash^{\mathsf{HOL}}_{\mathsf{cut\text{-}free}} \mathit{valid}\, \varphi_{\iota\to o})$$

Logic L:

- Higher-order Modal Logics
- First-order Multimodal Logics
- Propositional Multimodal Logics
- Quantified Conditional Logics
- Propositional Conditional Logics
- Intuitionistic Logics
- Access Control Logics
- Logic Combinations
- ...more is on the way ... including:
  - Description Logics
  - Nominal Logics
  - Multivalued Logics (SIXTEEN)
  - Logics based on Neighborhood Semantics
  - (Mathematical) Fuzzy Logics
  - Paraconsistent Logics

Embedding HOML in HOL: TPTP THF

```
%----The base type $i (already built-in) stands here for worlds and
%----mu for individuals; $o (also built-in) is the type of Booleans
thf(mu_type,type,(mu:$tType)).
%----Reserved constant r for accessibility relation
thf(r,type,(r:$i>$i>$o)).
%----Modal logic operators not, or, and, implies, box, diamond
thf(mnot_type,type,(mnot:($i>$o)>$i>$o)).
thf(mnot,definition,(mnot = (^[A:$i>$o,W:$i]:~(A@W)))).
thf(mor_type,type,(mor:($i>$o)>($i>$o)>$i>$o)).
thf(mor,definition,(mor = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)|(Psi@W))))).
thf(mand_type,type,(mand:($i>$o)>($i>$o)>$i>$o)).
thf(mand,definition,(mand = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)&(Psi@W))))).
thf(mimplies_type,type,(mimplies:($i>$o)>($i>$o)>$i>$o)).
thf(mimplies,definition,(mimplies = (^[A:$i>$o,Psi:$i>$o,W:$i]:((A@W)&(Psi@W))))).
thf(mbox_type,type,(mbox:($i>$o)>$i>$o)).
thf(mbox,definition,(mbox = (^[A:$i>$o,W:$i]:![V:$i]:(~(r@W@V)|(A@V))))).
thf(mdia_type,type,(mdia:($i>$o)>$i>$o)).
thf(mdia,definition,(mdia = (^[A:$i>$o,W:$i]:?[V:$i]:((r@W@V)&(A@V))))).
%----Quantifiers (constant domains) for individuals and propositions
thf(mforall_ind_type,type,(mforall_ind:(mu>$i>$o)>$i>$o)).
thf(mforall_ind,definition,(mforall_ind = (^[A:mu>$i>$o,W:$i]:![X:mu]:(A@X@W)))).
thf(mforall_indset_type,type,(mforall_indset:((mu>$i>$o)>$i>$o)>$i>$o)).
thf(mforall_indset,definition,(mforall_indset = (^[A:(mu>$i>$o)>$i>$o,W:$i]:![X:mu>$i>$o]:(A@X@W)))).
thf(mexists_ind_type,type,(mexists_ind:(mu>$i>$o)>$i>$o)).
thf(mexists_ind,definition,(mexists_ind = (^[A:mu>$i>$o,W:$i]:?[X:mu]:(A@X@W)))).
thf(mexists_indset_type,type,(mexists_indset:((mu>$i>$o)>$i>$o)>$i>$o)).
thf(mexists_indset,definition,(mexists_indset = (^[A:(mu>$i>$o)>$i>$o,W:$i]:?[X:mu>$i>$o]:(A@X@W)))).
%----Definition of validity (grounding of lifted modal formulas)
thf(v_type,type,(v:($i>$o)>$o)).
thf(mvalid,definition,(v = (^[A:$i>$o]:![W:$i]:(A@W)))).
%----Properties of accessibility relations: symmetry
thf(msymmetric_type,type,(msymmetric:($i>$i>$o)>$o)).
thf(msymmetric,definition,(msymmetric = (^[R:$i>$i>$o]:![S:$i,T:$i]:((R@S@T)=>(R@T@S))))).
%----Here we work with logic KB, i.e., we postulate symmetry for r
thf(sym,axiom,(msymmetric@r)).
```

Reading on THF0 syntax: [SutcliffeBenzmüller, J.Formalized Reasoning, 2010]

Proof Automation with LEO-II

```
>
>
> Beweise-mit-Leo2 Notwendigerweise-existiert-Gott.p

Leo-II tries to prove
=====================

  Goedel's Theorem T3: "Necessarily, God exists"
  thf(thmT3,conjecture,
      ( v
      @ ( mbox
        @ ( mexists_ind
          @ ^ [X: mu] :
              ( g @ X ) ) ) )).

  Assumptions: D1, C, T2, D3, A5

 . searching for proof ..

 ********************
 *   Proof found    *
 ********************
 % SZS status Theorem for Notwendigerweise-existiert-Gott.p

 . generating proof object
```

Provers can be called remotely in Miami — no local installation needed!

Download our experiments from

https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/THF

Embedding HOML in HOL: Evaluation — How good is the approach?

- There are no other provers for HOML
- There are some provers for first-order modal logic (FML)
- Comparative evaluation done in 2014 for the proof problems in the QMLTP (v1.1) library [Otten and Raths, `http://www.iltp.de/qmltp/`]
- This library contains 580 FML problems (in fact, 5 x 3 x 580 = 8700 problems).
- Metaprover HOL-P: sequentially schedules LEO-II—1.6.2, Satallax—2.7, Isabelle—2013, Nitrox—2013, agsyHOl—1.0
- Timeout for each HOL prover 120sec of CPU time (HOL-P: 600sec)
- Timeout for competitor systems: 600sec

HOML in HOL: Evaluation — FML's (D — constant/varying/cumulative)

No. of solved problems in the QMLTP library

	MleanSeP labelled sequents	**MleanTAP** labelled tableaux	**f2p-MSPASS** instant. & transform.	**MleanCoP** labelled connections	**HOL-P**
Logic D, constant domains					
Theorem	135	134	76	**217**	208
Non-Theorem	1	4	107	209	**250**
Solved	136	138	183	426	**458**
Logic D, cumulative domains					
Theorem	130	120	79	**200**	184
Non-Theorem	4	4	108	224	**269**
Solved	134	124	187	424	**453**
Logic D, varying domains					
Theorem	-	100	-	**170**	163
Non-Theorem	-	4	-	243	**295**
Solved	-	104	-	413	**458**

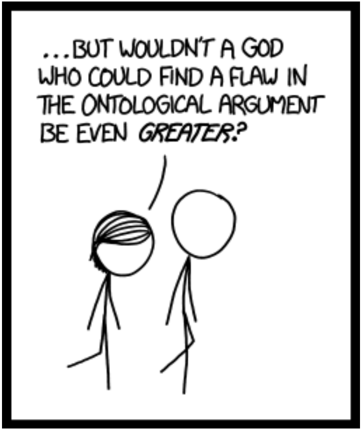
HOML in HOL: Evaluation — FML's (S4 — constant/varying/cumulative)

No. of solved problems in the QMLTP library

	MleanSeP labelled sequents	**MleanTAP** labelled tableaux	**f2p-MSPASS** instant. & transform.	**MleanCoP** labelled connections	**HOL-P**
		Logic S4, constant domains			
Theorem	197	220	111	**352**	300
Non-Theorem	1	4	36	82	**132**
Solved	198	224	147	**434**	432
		Logic S4, cumulative domains			
Theorem	197	205	121	**338**	278
Non-Theorem	4	4	41	94	**146**
Solved	201	209	162	**432**	424
		Logic S4, varying domains			
Theorem	-	169	-	**274**	245
Non-Theorem	-	4	-	119	**184**
Solved	-	173	-	393	**429**

The above results are meanwhile outdated; see more recent experiments in:
Theorem Provers for Every Normal Modal Logic (Tobias Gleißner, Alexander Steen, Christoph Benzmüller), In LPAR-21. 21st International Conference on Logic for Programming, Artificial Intelligence and Reasoning (Thomas Eiter, David Sands, eds.), EasyChair, EPiC Series in Computing, volume 46, pp. 14-30, 2017. `http://dx.doi.org/10.29007/jsb9`

The remaining slides are optional!
We will use others in the upcoming lectures.



The Ontological Argument

SCIENCE NEWS

HOME / SCIENCE NEWS / RESEARCHERS SAY THEY USED MACBOOK TO PROVE GOEDEL'S GOD THEOREM

Researchers say they used MacBook to prove Goedel's God theorem

Oct. 23, 2013 | 8:14 PM | 1 comments

See more serious and funny news links at
`https://github.com/FormalTheology/GoedelGod/tree/master/Press`

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Nachrichten > Wissenschaft > Mensch > Mathematik > Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis jahrzehntelang geheim

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

Montag, 09.09.2013 – 12:03 Uhr

Drucken | Versenden | Merken

Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

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Front Page World Europe Germany Business Zeitgeist Newsletter

English Site > Germany > Science > Scientists Use Computer to Mathematically Prove Gödel God Theorem

Holy Logic: Computer Scientists 'Prove' God Exists

By David Knight

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Austrian mathematician Kurt Gödel kept his proof of God's existence a secret for decades. Now two scientists say they have proven it mathematically using a computer.

Two scientists have formalized a theorem regarding the existence of God penned by mathematician Kurt Gödel. But the God angle is somewhat of a red herring -- the real step forward is the example it sets of how computers can make scientific progress simpler.

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A Long History of Ontological Arguments

pros and cons

... Anselm v. C. Gaunilo ... Th. Aquinas ...... Descartes Spinoza Leibniz ... Hume Kant ... Hegel ... Frege ... Hartshorne Malcolm Lewis Plantinga Gödel ...

Anselm's notion of God (Proslogion, 1078):

"God is that, than which nothing greater can be conceived."

Gödel's notion of God:

"A God-like being possesses all 'positive' properties."

To show by logical, deductive reasoning:

"God exists."

$$\exists x G(x)$$

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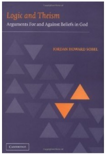
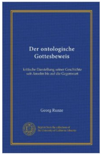
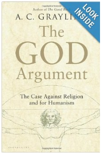
"Necessarily, God exists."

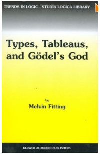
$$\Box \exists x G(x)$$

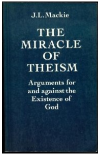
The Ontological Proof Today

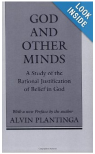


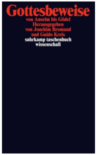


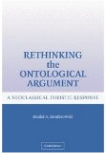


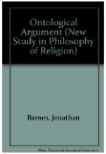












See more recent papers at my website.

Various Different Interests in Ontological Arguments

- Philosophical: Boundaries of Metaphysics & Epistemology
- Theistic: Successful argument could convince atheists?
- Ours: Can computers (theorem provers) be used ...

  ...to formalize the definitions, axioms and theorems?
  ...to verify/falsify the arguments step-by-step?
  ...to automate (sub-)arguments?
  ...to discover new philosophical knowledge?

Vision of Leibniz (1646–1716): *Calculemus!*

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.
(Translation by Russell)

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo . . . dicere: calculemus. (Leibniz, 1684)

Required:
characteristica universalis and **calculus ratiocinator**

Gödel's Manuscript: 1930's, 1941, 1946-1955, 1970

Ontologischer Beweis   Feb 10, 1970

$P(\varphi)$ φ is positive   (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess.} x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q \;=\; N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$, $\sim P(\varphi) \supset N \sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess.} x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess} x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x)G(x) \supset N(\exists y)\, G(y)$

" $M(\exists x)G(x) \supset MN(\exists y)\, G(y)$ M = possibility

" $\supset N(\exists y)\, G(y)$

any two essences of x are nec. equivalent

exclusive or   * and for any number of summands

$M(\exists x)G(x)$ means the system of all pos. props. is compatible. This is true because of:

At 4: $P(\varphi).\, \varphi \supset_N \psi :\supset P(\psi)$ which impl

~~true~~ { $x=x$ is positive
~~false~~ { $x \neq x$ is negative

But if a system S of pos. props. were incom

It would mean, that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth. sense (independently of the accidental structure of the world). Only then the ax. true. It may also mean pure "attribution"* as opposed to "privation" (or containing privation). This interpret. simpler proof

If φ pos. not: $(x)\, N \sim \varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive not $x = x$ contrary Ax or the exist. of pos. prop.

* i.e. the disj. normal form in terms of elem. prop. contains a member without negation.

Proof Overview

T3: $\Box\exists x.G(x)$

Proof Overview

$$\frac{\textbf{C1: } \Diamond \exists z. G(z)}{\textbf{T3: } \Box \exists x. G(x)}$$

Proof Overview

$$\frac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

L2: $\Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)$

$$\frac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\dfrac{\dfrac{\textbf{S5}}{\forall \xi.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\dfrac{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x) \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\dfrac{\dfrac{\textbf{L1: } \exists z.G(z) \to \Box\exists x.G(x)}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi\,.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

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Proof Overview

$$\textbf{D1: } G(x) \equiv \forall \varphi . [P(\varphi) \longrightarrow \varphi(x)]$$

$$\dfrac{\dfrac{\textbf{L1: } \exists z . G(z) \to \Box \exists x . G(x)}{\Diamond \exists z . G(z) \to \Diamond \Box \exists x . G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi . [\Diamond \Box \xi \to \Box \xi]}}{\textbf{L2: } \Diamond \exists z . G(z) \to \Box \exists x . G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond \exists z . G(z) \qquad \textbf{L2: } \Diamond \exists z . G(z) \to \Box \exists x . G(x)}{\textbf{T3: } \Box \exists x . G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall \varphi.[P(\varphi) \longrightarrow \varphi(x)]$$

$$\textbf{D3*: } E(x) \equiv \Box \exists y.G(y)$$

$$\dfrac{\dfrac{\dfrac{P(E)}{\textbf{L1: } \exists z.G(z) \to \Box \exists x.G(x)}}{\Diamond \exists z.G(z) \to \Diamond \Box \exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi.[\Diamond \Box \xi \to \Box \xi]}}{\textbf{L2: } \Diamond \exists z.G(z) \to \Box \exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond \exists z.G(z) \qquad \textbf{L2: } \Diamond \exists z.G(z) \to \Box \exists x.G(x)}{\textbf{T3: } \Box \exists x.G(x)}$$

Proof Overview

D1: $G(x) \equiv \forall \varphi . [P(\varphi) \longrightarrow \varphi(x)]$

D3*: $E(x) \equiv \Box \exists y . G(y)$ (cheating!)

$$\dfrac{\dfrac{\dfrac{P(E)}{\textbf{L1: } \exists z . G(z) \rightarrow \Box \exists x . G(x)}}{\Diamond \exists z . G(z) \rightarrow \Diamond \Box \exists x . G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi . [\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2: } \Diamond \exists z . G(z) \rightarrow \Box \exists x . G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond \exists z . G(z) \qquad \textbf{L2: } \Diamond \exists z . G(z) \rightarrow \Box \exists x . G(x)}{\textbf{T3: } \Box \exists x . G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \longrightarrow \varphi(x)]$$

$$\textbf{D3*: } E(x) \equiv \Box\exists y.G(y) \qquad\qquad \textbf{D3: } E(x) \equiv \forall\varphi\,.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\dfrac{\textbf{T2: } \forall y\,.[G(y) \to G\ ess\ y] \qquad\qquad P(E)}{\textbf{L1: } \exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi\,.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall \varphi.[P(\varphi) \longrightarrow \varphi(x)]$$

$$\textbf{D3*: } E(x) \equiv \Box\exists y.G(y) \qquad \textbf{D3: } E(x) \equiv \forall \varphi\,.[\varphi \; ess \; x \rightarrow \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\dfrac{\textbf{T2: } \forall y\,.[G(y) \rightarrow G \; ess \; y] \qquad \dfrac{\textbf{A5}}{P(E)}}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi\,.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \longrightarrow \varphi(x)]$$

$$\textbf{D3*: } E(x) \equiv \Box\exists y.G(y) \qquad\qquad \textbf{D3: } E(x) \equiv \forall\varphi\,.[\varphi\ ess\ x \rightarrow \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\dfrac{\textbf{T2: } \forall y\,.[G(y) \rightarrow G\ ess\ y] \qquad \dfrac{\textbf{A5}}{P(E)}}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi\,.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \longrightarrow \varphi(x)]$$

$$\textbf{D2: } \varphi \; ess \; x \equiv \varphi(x) \wedge \forall\psi\,.(\psi(x) \rightarrow \Box\forall x\,.(\varphi(x) \rightarrow \psi(x)))$$

$$\textbf{D3*: } E(x) \equiv \Box\exists y.G(y) \qquad \textbf{D3: } E(x) \equiv \forall\varphi\,.[\varphi \; ess \; x \rightarrow \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\textbf{A1b}}{\forall\varphi\,.[\neg P(\varphi) \rightarrow P(\neg\varphi)]} \quad \overset{\textbf{A4}}{\forall\varphi\,.[P(\varphi) \longrightarrow \Box\, P(\varphi)]}}{\textbf{T2: } \forall y\,.[G(y) \rightarrow G \; ess \; y]} \quad \overset{\textbf{A5}}{P(E)}}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \quad \overset{\textbf{S5}}{\forall\xi\,.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall \varphi.[P(\varphi) \longrightarrow \varphi(x)]$$

$$\textbf{D2: } \varphi \ ess \ x \equiv \varphi(x) \wedge \forall \psi\,.(\psi(x) \rightarrow \Box \forall x\,.(\varphi(x) \rightarrow \psi(x)))$$

$$\textbf{D3*: } E(x) \equiv \Box \exists y.G(y) \qquad \textbf{D3: } E(x) \equiv \forall \varphi\,.[\varphi \ ess \ x \rightarrow \Box \exists y.\varphi(y)]$$

$$\textbf{C1: } \Diamond \exists z.G(z)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\textbf{A1b}}{\forall \varphi\,.[\neg P(\varphi) \rightarrow P(\neg \varphi)]} \qquad \overset{\textbf{A4}}{\forall \varphi\,.[P(\varphi) \longrightarrow \Box\, P(\varphi)]}}{\textbf{T2: } \forall y\,.[G(y) \rightarrow G \ ess \ y]} \qquad \overset{\textbf{A5}}{P(E)}}{\textbf{L1: } \exists z.G(z) \rightarrow \Box \exists x.G(x)}}{\Diamond \exists z.G(z) \rightarrow \Diamond \Box \exists x.G(x)} \qquad \overset{\textbf{S5}}{\forall \xi\,.[\Diamond \Box \xi \rightarrow \Box \xi]}}{\textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond \exists z.G(z) \qquad \textbf{L2: } \Diamond \exists z.G(z) \rightarrow \Box \exists x.G(x)}{\textbf{T3: } \Box \exists x.G(x)}$$

Proof Overview

$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \longrightarrow \varphi(x)]$$

$$\textbf{D2:}\ \varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi\,.(\psi(x) \rightarrow \Box\forall x\,.(\varphi(x) \rightarrow \psi(x)))$$

$$\textbf{D3*:}\ E(x) \equiv \Box\exists y.G(y) \qquad\qquad \textbf{D3:}\ E(x) \equiv \forall\varphi\,.[\varphi\ ess\ x \rightarrow \Box\exists y.\varphi(y)]$$

$$\frac{P(G)}{\textbf{C1:}\ \Diamond\exists z.G(z)}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\overset{\textbf{A1b}}{\forall\varphi\,.[\neg P(\varphi) \rightarrow P(\neg\varphi)]} \qquad \overset{\textbf{A4}}{\forall\varphi\,.[P(\varphi) \longrightarrow \Box\, P(\varphi)]}}{\textbf{T2:}\ \forall y\,.[G(y) \rightarrow G\ ess\ y]} \qquad \overset{\textbf{A5}}{P(E)}}{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \overset{\textbf{S5}}{\forall\xi\,.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\frac{\textbf{C1:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \longrightarrow \varphi(x)]$$

$$\textbf{D2:}\ \varphi\ ess\ x \equiv \varphi(x) \land \forall\psi\,.(\psi(x) \to \Box\forall x\,.(\varphi(x) \to \psi(x)))$$

$\textbf{D3*:}\ E(x) \equiv \Box\exists y.G(y)$ $\qquad$ $\textbf{D3:}\ E(x) \equiv \forall\varphi\,.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$

$$\dfrac{\dfrac{\textbf{A3}}{P(G)}}{\textbf{C1:}\ \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi\,.[\neg P(\varphi) \to P(\neg\varphi)]} \quad \dfrac{\textbf{A4}}{\forall\varphi\,.[P(\varphi) \longrightarrow \Box\, P(\varphi)]}}{\textbf{T2:}\ \forall y\,.[G(y) \to G\ ess\ y]} \quad \dfrac{\textbf{A5}}{P(E)}}{\dfrac{\textbf{L1:}\ \exists z.G(z) \to \Box\exists x.G(x)}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)}} \quad \dfrac{\textbf{S5}}{\forall\xi\,.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \longrightarrow \varphi(x)]$$

$$\textbf{D2: } \varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi\,.(\psi(x) \rightarrow \Box\forall x\,.(\varphi(x) \rightarrow \psi(x)))$$

$$\textbf{D3*: } E(x) \equiv \Box\exists y.G(y) \qquad\qquad \textbf{D3: } E(x) \equiv \forall\varphi\,.[\varphi\ ess\ x \rightarrow \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\textbf{A3}}{P(G)} \qquad \textbf{T1: } \forall\varphi\,.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}{\textbf{C1: } \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi\,.[\neg P(\varphi) \rightarrow P(\neg\varphi)]} \qquad \dfrac{\textbf{A4}}{\forall\varphi\,.[P(\varphi) \longrightarrow \Box\, P(\varphi)]}}{\textbf{T2: } \forall y\,.[G(y) \rightarrow G\ ess\ y]} \qquad \dfrac{\textbf{A5}}{P(E)}}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi\,.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \longrightarrow \varphi(x)]$$

$$\textbf{D2:}\ \varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi\,.(\psi(x) \rightarrow \Box\forall x\,.(\varphi(x) \rightarrow \psi(x)))$$

$$\textbf{D3*:}\ E(x) \equiv \Box\exists y.G(y) \qquad\qquad \textbf{D3:}\ E(x) \equiv \forall\varphi\,.[\varphi\ ess\ x \rightarrow \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\textbf{A3}}{P(G)} \qquad \dfrac{\dfrac{\textbf{A2}}{\forall\varphi\,.\forall\psi\,.[(P(\varphi) \wedge \Box\forall x\,.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]} \qquad \dfrac{\textbf{A1a}}{\forall\varphi\,.[P(\neg\varphi) \rightarrow \neg P(\varphi)]}}{\textbf{T1:}\ \forall\varphi\,.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}}{\textbf{C1:}\ \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi\,.[\neg P(\varphi) \rightarrow P(\neg\varphi)]} \qquad \dfrac{\textbf{A4}}{\forall\varphi\,.[P(\varphi) \longrightarrow \Box\ P(\varphi)]}}{\textbf{T2:}\ \forall y\,.[G(y) \rightarrow G\ ess\ y]} \qquad \dfrac{\textbf{A5}}{P(E)}}{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi\,.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \longrightarrow \varphi(x)]$$

$$\textbf{D2: } \varphi\ ess\ x \equiv \varphi(x) \land \forall\psi\,.(\psi(x) \to \Box\forall x\,.(\varphi(x) \to \psi(x)))$$

$$\textbf{D3: } NE(x) \equiv \forall\varphi\,.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\textbf{A3}}{P(G)} \qquad \dfrac{\dfrac{\textbf{A2}}{\forall\varphi\,.\forall\psi\,.[(P(\varphi) \land \Box\forall x\,.[\varphi(x) \to \psi(x)]) \to P(\psi)]} \qquad \dfrac{\textbf{A1a}}{\forall\varphi\,.[P(\neg\varphi) \to \neg P(\varphi)]}}{\textbf{T1: } \forall\varphi\,.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}}{\textbf{C1: } \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi\,.[\neg P(\varphi) \to P(\neg\varphi)]} \qquad \dfrac{\textbf{A4}}{\forall\varphi\,.[P(\varphi) \longrightarrow \Box\ P(\varphi)]}}{\textbf{T2: } \forall y\,.[G(y) \to G\ ess\ y]} \qquad \dfrac{\textbf{A5}}{P(NE)}}{\textbf{L1: } \exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi\,.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Scott's Version of Gödel's Axioms, Definitions and Theorems

Axiom A1 Either a property or its negation is positive, but not both: $\forall\varphi[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$

Axiom A2 A property necessarily implied by a positive property is positive:
$$\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

Thm. T1 Positive properties are possibly exemplified: $\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$

Def. D1 A <u>God-like</u> being possesses all positive properties: $G(x) \leftrightarrow \forall\varphi[P(\varphi) \rightarrow \varphi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists x G(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\varphi[P(\varphi) \rightarrow \Box P(\varphi)]$

Def. D2 An <u>essence</u> of an individual is a property possessed by it and necessarily implying any of its properties: $\varphi\ ess\ x \leftrightarrow \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G\ ess\ x]$

Def. D3 <u>Necessary existence</u> of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\varphi[\varphi\ ess\ x \rightarrow \Box\exists y\varphi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box\exists x G(x)$

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Difference to Gödel (who omits this conjunct)

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Thm. T1 Positive properties are possibly exemplified: $\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$

Def. D1 A <u>God-like</u> being possesses all positive properties: $G(x) \leftrightarrow \forall\varphi[P(\varphi) \rightarrow \varphi(x)]$

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Cor. C Possibly, God exists: $\boxed{\Diamond\exists}xG(x)$

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Def. D2 An <u>essence</u> of an individual is a property possessed by it and necessarily implying any of its properties: $\varphi\ ess\ x \leftrightarrow \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$

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Def. D3 <u>Necessary existence</u> of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\varphi[\varphi\ ess\ x \rightarrow \Box\exists y\varphi(y)]$

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Thm. T3 Necessarily, God exists: $\Box\exists xG(x)$

Modal operators are used

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Def. D2 An essence of an individual is a property possessed by it and necessarily implying any of its properties: $\varphi \; ess \; x \leftrightarrow \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G \; ess \; x]$

Def. D3 Necessary existence of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\varphi[\varphi \; ess \; x \rightarrow \Box\exists y\varphi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box\exists xG(x)$

second-order quantifiers

The Ontological Argument in TPTP THF0

```
%------------------------------------------------------------------------------
%----Axioms for Quantified Modal Logic KB.
include('Quantified_KB.ax').
%------------------------------------------------------------------------------
%----constant symbol for positive (p), God-like (g), essence (ess), necessary existence (ne)
thf(p_tp,type,(p:(mu>$i>$o)>$i>$o)).
thf(g_tp,type,(g:mu>$i>$o)).
thf(ess_tp,type,(ess:(mu>$i>$o)>mu>$i>$o)).
thf(ne_tp,type,(ne:mu>$i>$o)).
%----D1:A God-like being possesses all positive properties.
thf(defD1,definition,(g = (^[X:mu]:(mforall_indset@^[Phi:mu>$i>$o]:(mimplies@(p@Phi)@(Phi@X)))))).
%----C: Possibly, God exists. (Proved in C.p)
thf(corC,axiom,(v@(mdia@(mexists_ind@^[X:mu]:(g@X))))).
%----T2: Being God-like is an essence of any God-like being. (Proved in T2.p)
thf(thmT2,axiom,(v@(mforall_ind@^[X:mu]:(mimplies@(g@X)@(ess@g@X))))).
%----D3: Necessary existence of an individual is the necessary exemplification of all its essences
thf(defD3,definition,(ne = (^[X:mu]:(mforall_indset@^[Phi:mu>$i>$o]:
     (mimplies@(ess@Phi@X)@(mbox@(mexists_ind@^[Y:mu]:(Phi@Y)))))))).
%----A5:Necessary existence is positive.
thf(axA5,axiom,(v@(p@ne))).
%----T3: Necessarily God exists.
thf(thmT3,conjecture,(v@(mbox@(mexists_ind@^[X:mu]:(g@X))))).
```

The Ontological Argument in Isabelle/HOL

GoedelGod.thy (~/Varia/Talks/2015-ReasoningWeb/DemoMaterial/DEMO-Isabelle/)

```
text {* QML formulas are translated as HOL terms of type @{typ "i ⇒ bool"}.
This type is abbreviated as @{text "σ"}. *}

  type_synonym σ = "(i ⇒ bool)"

text {* The classical connectives $\neg, \wedge, \rightarrow$, and $\forall$
(over individuals and over sets of individuals) and $\exists$ (over individuals) are
lifted to type $\sigma$. The lifted connectives are @{text "m¬"}, @{text "m∧"}, @{text "m→"},
@{text "∀"}, and @{text "∃"} (the latter two are modeled as constant symbols).
Other connectives can be introduced analogously. We exemplarily do this for @{text "m∨"} ,
@{text "m≡"}, and @{text "mL="} (Leibniz equality on individuals). Moreover, the modal
operators @{text "□"} and @{text "◇"}  are introduced. Definitions could be used instead of
abbreviations. *}

  abbreviation mnot :: "σ ⇒ σ" ("m¬") where "m¬ φ ≡ (λw. ¬ φ w)"
  abbreviation mand :: "σ ⇒ σ ⇒ σ" (infixr "m∧" 51) where "φ m∧ ψ ≡ (λw. φ w ∧ ψ w)"
  abbreviation mor :: "σ ⇒ σ ⇒ σ" (infixr "m∨" 50) where "φ m∨ ψ ≡ (λw. φ w ∨ ψ w)"
  abbreviation mimplies :: "σ ⇒ σ ⇒ σ" (infixr "m→" 49) where "φ m→ ψ ≡ (λw. φ w ⟶ ψ w)"
  abbreviation mequiv:: "σ ⇒ σ ⇒ σ" (infixr "m≡" 48) where "φ m≡ ψ ≡ (λw. φ w ⟷ ψ w)"
  abbreviation mforall :: "('a ⇒ σ) ⇒ σ" ("∀") where "∀ Φ ≡ (λw. ∀x. Φ x w)"
  abbreviation mexists :: "('a ⇒ σ) ⇒ σ" ("∃") where "∃ Φ ≡ (λw. ∃x. Φ x w)"
  abbreviation mLeibeq :: "μ ⇒ μ ⇒ σ" (infixr "mL=" 52) where "x mL= y ≡ ∀(λφ. (φ x m→ φ y))"
  abbreviation mbox :: "σ ⇒ σ" ("□") where "□ φ ≡ (λw. ∀v.  w r v ⟶ φ v)"
  abbreviation mdia :: "σ ⇒ σ" ("◇") where "◇ φ ≡ (λw. ∃v. w r v ∧ φ v)"

text {* For grounding lifted formulas, the meta-predicate @{text "valid"} is introduced. *}

  (*<*) no_syntax "_list" :: "args ⇒ 'a list" ("[(_)]") (*>*)
  abbreviation valid :: "σ ⇒ bool" ("[_]") where "[p] ≡ ∀w. p w"
```

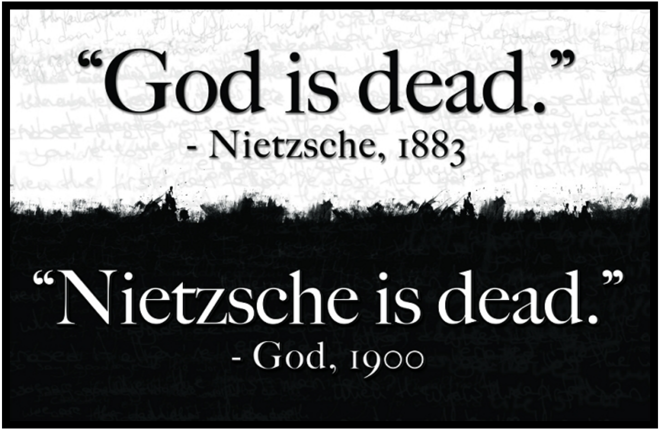
100% Auto update Update Detach

Output README Symbols

41,7 (1708/9125) (isabelle,sidekick,UTF-8-Isabelle) UG 68/124MB 6:28 PM

See verifiable Isabelle/HOL document (Archive of Formal Proofs) at:
`http://afp.sourceforge.net/entries/GoedelGod.shtml`

Findings from Our Experiments



Experimental Results and Philosophical Discoveries

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \doteq \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \mathbin{\dot\wedge} \dot\Box\dot\forall X_\mu.\, (\phi X \mathbin{\dot\supset} \psi X)) \mathbin{\dot\supset} p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \mathbin{\dot\supset} \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1($\supset$), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \mathbin{\dot\supset} \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \mathbin{\dot\supset} \dot\Box p\phi]$						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \mathbin{\dot\wedge} \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \mathbin{\dot\supset} \dot\Box\dot\forall Y_\mu.\, (\phi Y \mathbin{\dot\supset} \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \mathbin{\dot\supset} (\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathbf{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathbf{ess}\,\phi X \mathbin{\dot\supset} \dot\Box\dot\exists Y_\mu.\, \phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathbf{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \mathbin{\dot\supset} \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \mathbin{\dot\supset} (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \mathbin{\dot\supset} \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \mathbin{\dot\supset} (g_{\mu\to\sigma}Y \mathbin{\dot\supset} X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \mathbin{\dot\supset} \dot\Box\dot\forall Y_\mu.\, (\phi Y \mathbin{\dot\supset} \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1($\supset$), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\,\dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\,\dot\forall\psi_{\mu\to\sigma}.\,(p_{(\mu\to\sigma)\to\sigma}\phi \mathbin{\dot\wedge} \dot\Box\dot\forall X_\mu.\,(\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\,\phi X]$	A1($\supset$), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\,\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\,\lambda X_\mu.\,\phi X \mathbin{\dot\wedge} \dot\forall\psi_{\mu\to\sigma}.\,(\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\,(\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\text{NE}_{\mu\to\sigma} = \lambda X_\mu.\,\dot\forall\phi_{\mu\to\sigma}.\,(\text{ess}\,\phi X \dot\supset \dot\Box\dot\exists Y_\mu.\,\phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\text{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\,\dot\forall X_\mu.\,(g_{\mu\to\sigma}X \dot\supset (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\,\dot\forall Y_\mu.\,(g_{\mu\to\sigma}X \dot\supset (g_{\mu\to\sigma}Y \dot\supset X \dot= Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2’	$\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\,\lambda X_\mu.\,\dot\forall\psi_{\mu\to\sigma}.\,(\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\,(\phi Y \dot\supset \psi Y))$						
CO’	$\emptyset$ (no goal, check for consistency)	A1($\supset$), A2, D2’, D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2’, D3, A5	KB	UNS	—/—	—/—	—/—

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1($\supset$), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathrm{es}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma} X \dot\supset ($						
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \dot\supset (g_{\mu\to}$						
CO	$\emptyset$ (no goal, check for cons						
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
CO'	$\emptyset$ (no goal, check for cons						

Automating Scott's proof script

T1: "Positive properties are possibly exemplified" proved by LEO-II and Satallax

- in logic: K
- from assumptions:
  - A1 and A2
  - A1($\supset$) and A2
- notion of quantification
  - possibilist quantifiers (constant dom.)
  - actualist quantifiers for individuals (varying dom.)

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}\centerdot p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu\centerdot \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}\centerdot\dot\forall\psi_{\mu\to\sigma}\centerdot(p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu\centerdot(\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}\centerdot p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu\centerdot\phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu\centerdot\dot\forall\phi_{\mu\to\sigma}\centerdot p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu\centerdot g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}\centerdot p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}\centerdot\lambda X_\mu\centerdot\phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}\centerdot(\psi X \dot\supset \dot\Box\dot\forall Y_\mu\centerdot(\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu\centerdot g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}gX)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu\centerdot\dot\forall\phi_{\mu\to\sigma}\centerdot(\mathrm{es}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu\centerdot g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}\centerdot\dot\forall X_\mu\centerdot(g_{\mu\to\sigma}X \dot\supset ($						
MT	$[\dot\forall X_\mu\centerdot\dot\forall Y_\mu\centerdot(g_{\mu\to\sigma}X \dot\supset (g_{\mu}$						
CO	∅ (no goal, check for cons						
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}\centerdot\lambda$						
CO'	∅ (no goal, check for cons						

Automating Scott's proof script

C: "Possibly, God exists" proved by LEO-II and Satallax

- **in logic: K**
- **from assumptions:**
  - **T1, D1, A3**
- **for domain conditions:**
  - **possibilist quantifiers (constant dom.)**
  - **actualist quantifiers for individuals (varying dom.)**

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_\mu.\, (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_\mu.\, \phi X]$	A1($\supset$), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \,\dot{\wedge}\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$						
T2	$[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \dot{\supset} (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, gX)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (\mathrm{es}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$						
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g$ ... $X \dot{\supset} ($						
MT	$[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (g_\mu$						
CO	$\emptyset$ (no goal, check for cons						
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
CO'	$\emptyset$ (no goal, check for cons						

Automating Scott's proof script

T2: "Being God-like is an ess. of any God-like being" proved by LEO-II and Satallax

- **in logic: K**
- **from assumptions:**
  - **A1, D1, A4, D2**
- **for domain conditions:**
  - **possibilist quantifiers (constant dom.)**
  - **actualist quantifiers for individuals (varying dom.)**

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}\centerdot\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu\centerdot\, \dot{\neg}(\phi X)) \,\dot{\equiv}\, \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}\centerdot\, \dot{\forall}\psi_{\mu\to\sigma}\centerdot\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_\mu\centerdot\, (\phi X \,\dot{\supset}\, \psi X)) \,\dot{\supset}\, p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}\centerdot\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\supset}\, \dot{\Diamond}\dot{\exists}X_\mu\centerdot\, \phi X]$	A1($\supset$), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu\centerdot\, \dot{\forall}\phi_{\mu\to\sigma}\centerdot\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\supset}\, \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu\centerdot\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall}\phi_{\mu\to\sigma}\centerdot\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\supset}\, \dot{\Box}p\phi]$						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}\centerdot\, \lambda X_\mu\centerdot\, \phi X \,\dot{\wedge}\, \dot{\forall}\psi_{\mu\to\sigma}\centerdot\, (\psi X \,\dot{\supset}\, \dot{\Box}\dot{\forall}Y_\mu\centerdot\, (\phi Y \,\dot{\supset}\, \psi Y))$						
T2	$[\dot{\forall}X_\mu\centerdot\, g_{\mu\to\sigma}X \,\dot{\supset}\, (\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\mathbf{NE}_{\mu\to\sigma} = \lambda X_\mu\centerdot\, \dot{\forall}\phi_{\mu\to\sigma}\centerdot\, ($es [illegible]						
A5	$[p_{(\mu\to\sigma)\to\sigma}\, \mathbf{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu\centerdot\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \,\dot{\supset}\, \dot{\Box}s_\sigma]$						
FG	$[\dot{\forall}\phi_{\mu\to\sigma}\centerdot\, \dot{\forall}X_\mu\centerdot\, (g_{\mu\to\sigma}X \,\dot{\supset}\, ($ [illegible]						
MT	$[\dot{\forall}X_\mu\centerdot\, \dot{\forall}Y_\mu\centerdot\, (g_{\mu\to\sigma}X \,\dot{\supset}\, (g_{\mu}$ [illegible]						
CO	$\emptyset$ (no goal, check for cons [illegible]						
D2'	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}\centerdot\, \lambda$ [illegible]						
CO'	$\emptyset$ (no goal, check for cons [illegible]						

Automating Scott's proof script

T3: "Necessarily, God exists" proved by LEO-II and Satallax

- **in logic: KB**
- **from assumptions:**
  - **D1, C, T2, D3, A5**
- **for domain conditions:**
  - **possibilist quantifiers (constant dom.)**
  - **actualist quantifiers for individuals (varying dom.)**

For logic K we got a countermodel by Nitpick

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \dot{\wedge} \dot{\Box}\dot{\forall}X_\mu.\, (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_\mu.\, \phi X]$	A1($\supset$), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p\phi]$						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \dot{\wedge} \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$						
T2	$[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \dot{\supset} (\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (\mathbf{es}$ …						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$						
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g$ … $X \dot{\supset} ($ …						
MT	$[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (g_\mu$ …						
CO	$\emptyset$ (no goal, check for cons…						
D2'	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$ …						
CO'	$\emptyset$ (no goal, check for cons…						

Automating Scott's proof script

Summary

- proof verified and automated
- KB is sufficient (criticized logic S5 not needed!)
- possibilist and actualist quantifiers (individuals)
- exact dependencies determined experimentally
- ATPs have found alternative proofs
  e.g. self-identity $\lambda x(x = x)$ is not needed

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\wedge\, \dot\Box\dot\forall X_\mu.\, (\phi X \,\dot\supset\, \psi X)) \,\dot\supset\, p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to}$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Box p\phi$						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \,\dot\supset\, (\mathbf{ess}_{(\mu\to\sigma)\to}$						
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathbf{es}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \,\dot\supset\, \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \,\dot\supset\, (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \,\dot\supset\, \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \,\dot\supset\, (g_{\mu\to\sigma}Y \,\dot\supset\, X \dot= Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2’	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \,\dot\supset\, \dot\Box\dot\forall Y_\mu.\, (\phi Y \,\dot\supset\, \psi Y))$						
CO’	$\emptyset$ (no goal, check for consistency)	A1(⊃), A2, D2’, D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2’, D3, A5	KB	UNS	—/—	—/—	—/—

Consistency check: Gödel vs. Scott

- **Scott’s assumptions are consistent; shown by Nitpick**
- **Gödel’s assumptions are inconsistent; shown by LEO-II (new philosophical result?)**

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \,\dot\equiv\, \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\wedge\, \dot\Box\dot\forall X_\mu.\, (\phi X \,\dot\supset\, \psi X)) \,\dot\supset\, p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to}$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Box p\phi$						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \,\dot\supset\, (\mathbf{ess}_{(\mu\to\sigma)\to}$						
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathbf{es}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \,\dot\supset\, \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \,\dot\supset\, (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \,\dot\supset\, \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \,\dot\supset\, (g_{\mu\to\sigma}Y \,\dot\supset\, X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \,\dot\supset\, \dot\Box\dot\forall Y_\mu.\, (\phi Y \,\dot\supset\, \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1(⊃), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Further Results

- Monotheism holds
- God is flawless

Main Findings

Modal Collapse (Sobel)

$$\forall \varphi(\varphi \supset \Box \varphi)$$

- proved by LEO-II and Satallax
- for possibilist and actualist quantification (ind.)

Main critique on Gödel's ontological proof:

- there are no contingent truths
- everything is determined / no free will

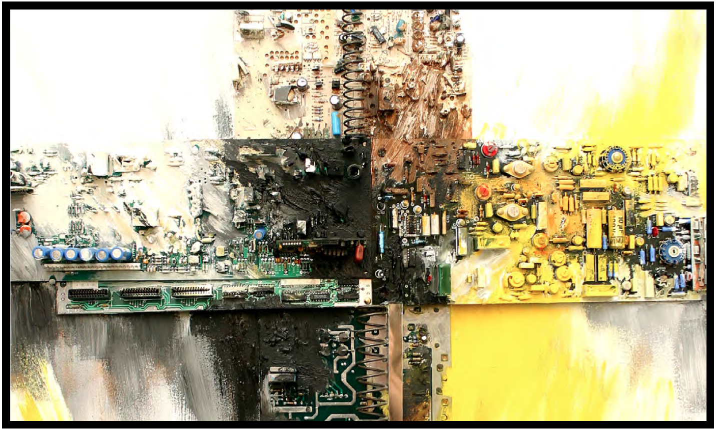
	HOL encoding						
A1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}($						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\supset}\, \dot{\Diamond}\dot{\exists}X$						
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to}$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$						
A4	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\supset}\, \dot{\Box}p\phi$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
T2	$[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \,\dot{\supset}\, (\mathrm{ess}_{(\mu\to\sigma)\to}$						
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\,(\mathrm{es}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \,\dot{\supset}\, \dot{\Box}s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g_{\mu\to\sigma}X \,\dot{\supset}\, (\dot{\neg}(p_{(\mu\to\sigma)\to\sigma}\phi) \,\dot{\supset}\, \dot{\neg}(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \,\dot{\supset}\, (g_{\mu\to\sigma}Y \,\dot{\supset}\, X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \,\dot{\supset}\, \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \,\dot{\supset}\, \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1($\supset$), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathrm{ess}\,\phi X \dot\supset \dot\Box\dot\exists Y_\mu.\, \phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
			KB	THM	17.9/—	3.3/3.2	—/—
		3, A5	KB	THM	—/—	—/—	—/—
			KB	THM	16.5/—	0.0/0.0	—/—
		3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
			KB	THM	—/—	0.0/3.3	—/—
		3, A5	KB	THM	—/—	—/—	—/—
		3, A5	KB	SAT	—/—	—/—	7.3/7.4
			KB	UNS	7.5/7.8	—/—	—/—
		)3, A5	KB	UNS	—/—	—/—	—/—

Observation

- good performance of ATPs
- excellent match between argumentation granularity in papers and the reasoning strength of the ATPs



Reconstruction of the Inconsistency of Gödel's Axioms

Inconsistency (Gödel): Proof by LEO-II in KB

DemoMaterial — bash — 166×52

```
@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: $false))@SV3)=$true))),inference(prim_subst,[status(thm)],[66:[bind(SV11,$thf(^[SV23:mu,SV24:$i]: $false))]])).
 thf(84,plain,(![SV22:(mu>($i>$o)),SV3:$i,SV8:(mu>($i>$o))]: ((((SV8@(((sK2_SY33@SV3)@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1))))@SV8))@(((sK1_SY31@(^[SX0:mu,SX1:$i]: (
~ ((SV22@SX0)@SX1))))@SV8)@SV3))=$true) | (((p@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1))))@SV3)=$true))),inference(prim_subst,[status(thm)],[66
:[bind(SV11,$thf(^[SV20:mu,SV21:$i]: (~ ((SV22@SV20)@SV21))))]])).
 thf(85,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)=$false) | ((((p@SV9)@SV4) = ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)
@SY28))))@SV4))=$false))),inference(fac_restr,[status(thm)],[56])).
 thf(86,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)=$true) | ((((p@SV9)@SV4) = ((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@
SY30))))@SV4))=$false))),inference(fac_restr,[status(thm)],[57])).
 thf(87,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((~ (((p@SV9)@SV4) | ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4))) | (~ ((~ ((p@SV9)@SV4)) | (~ ((p@(^[SY27:mu,
SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)))))=$false) | (((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_equal_neg,[status(thm)],[85])).
 thf(89,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((~ (((p@SV9)@SV4) | ((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4))) | (~ ((~ ((p@SV9)@SV4)) | (~ ((p@(^[SY29:mu,
SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)))))=$false) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)=$true))),inference(extcnf_equal_neg,[status(thm)],[86])).
 thf(92,plain,(![SV4:$i,SV9:(mu>($i>$o))]: (((~ ((~ ((p@SV9)@SV4)) | (~ ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4))))=$false) | (((p@(^[SY27:mu,SY28:$i]: (
~ ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_or_neg,[status(thm)],[87])).
 thf(93,plain,(![SV4:$i,SV9:(mu>($i>$o))]: (((~ (((p@SV9)@SV4) | ((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)))=$false) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@
SY29)@SY30))))@SV4)=$true))),inference(extcnf_or_neg,[status(thm)],[89])).
 thf(96,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((~ ((p@SV9)@SV4)) | (~ ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)))=$true) | (((p@(^[SY27:mu,SY28:$i]: (~ ((S
V9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_not_neg,[status(thm)],[92])).
 thf(97,plain,(![SV4:$i,SV9:(mu>($i>$o))]: (((((p@SV9)@SV4) | ((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4))=$true) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)
@SY30))))@SV4)=$true))),inference(extcnf_not_neg,[status(thm)],[93])).
 thf(100,plain,(![SV4:$i,SV9:(mu>($i>$o))]: (((~ ((p@SV9)@SV4))=$true) | ((~ ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4))=$true) | (((p@(^[SY27:mu,SY28:$i]:
 (~ ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_or_pos,[status(thm)],[96])).
 thf(101,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@SV9)@SV4)=$true) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)=$true) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV
9@SY29)@SY30))))@SV4)=$true))),inference(extcnf_or_pos,[status(thm)],[97])).
 thf(103,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@SV9)@SV4)=$false) | ((~ ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4))=$true) | (((p@(^[SY27:mu,SY28:$i]: (~
 ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_not_pos,[status(thm)],[100])).
 thf(105,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)=$false) | (((p@SV9)@SV4)=$false) | (((p@(^[SY27:mu,SY28:$i]: (~ ((
SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_not_pos,[status(thm)],[103])).
 thf(107,plain,(![SV8:(mu>($i>$o)),SV3:$i,SV22:(mu>($i>$o))]: ((((SV22@(((sK2_SY33@SV3)@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1))))@SV8))@(((sK1_SY31@(^[SX0:mu,SX1:$i]:
 (~ ((SV22@SX0)@SX1))))@SV8)@SV3))=$true) | (((p@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1))))@SV3)=$true))),inference(extcnf_not_neg,[status(thm
)],[78])).
 thf(108,plain,(![SV11:(mu>($i>$o)),SV3:$i,SV15:(mu>($i>$o))]: ((((SV15@(((sK2_SY33@SV3)@SV11)@(^[SX0:mu,SX1:$i]: (~ ((SV15@SX0)@SX1)))))@(((sK1_SY31@SV11)@(^[SX0:mu,
SX1:$i]: (~ ((SV15@SX0)@SX1))))@SV3))=$false) | (((p@(^[SX0:mu,SX1:$i]: (~ ((SV15@SX0)@SX1))))@SV3)=$false) | (((p@SV11)@SV3)=$true))),inference(extcnf_not_pos,[statu
s(thm)],[81])).
 thf(109,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)=$false) | (((p@SV9)@SV4)=$false))),inference(sim,[status(thm)],[10
5])).
 thf(110,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@SV9)@SV4)=$true) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)=$true))),inference(sim,[status(thm)],[101]
)).
 thf(111,plain,(![SV3:$i,SV8:(mu>($i>$o))]: ((((p@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: $true))@SV3)=$true))),inference(sim,[status(thm)],[76])).
 thf(112,plain,(![SV11:(mu>($i>$o)),SV3:$i]: ((((p@(^[SX0:mu,SX1:$i]: $false))@SV3)=$false) | (((p@SV11)@SV3)=$true))),inference(sim,[status(thm)],[80])).
 thf(113,plain,((($false)=$true)),inference(fo_atp_e,[status(thm)],[25,112,111,110,109,108,107,84,83,82,75,74,73,72,71,70,69,68,67,66,65,62,57,56,51,42,29])).
 thf(114,plain,($false),inference(solved_all_splits,[solved_all_splits(join,[])],[113])).
% SZS output end CNFRefutation

%***** End of derivation protocol ****
%***** no. of clauses in derivation: 97 ****
%***** clause counter: 113 ****

% SZS status Unsatisfiable for ConsistencyWithoutFirstConjunctinD2.p : (rf:0,axioms:6,ps:3,u:6,ude:false,rLeibEQ:true,rAndEQ:true,use_choice:true,use_extuni:true,use_
extcnf_combined:true,expand_extuni:false,foatp:e,atp_timeout:25,atp_calls_frequency:10,ordering:none,proof_output:1,clause_count:113,loop_count:0,foatp_calls:2,transl
ation:fof_full)
ontoleo:DemoMaterial cbenzmueller$
```

Inconsistency (Gödel): Reconstruction of Informal Argument (KB)

(special thanks to Chad Brown for a fruitful discussion)

Axiom A1($\supset$) $\forall\varphi[P(\neg\varphi) \rightarrow \neg P(\varphi)]$

Axiom A2 $\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

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by A1($\supset$), A2

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Def. D2* $\varphi \; ess \; x \leftrightarrow \cancel{\varphi(x) \wedge} \; \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$

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by D2*

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Lemma 2 Exemplification of necessary existence is not possible. $\neg\Diamond\exists x\, NE(x)$

by B, D3, Lemma1

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Axiom A5 $P(NE)$

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by A1($\supset$), A2

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Def. D3 $NE(x) \leftrightarrow \forall\varphi[\varphi\ ess\ x \rightarrow \Box\exists y\varphi(y)]$

Axiom B $\forall\varphi(\varphi \rightarrow \Box\Diamond\varphi)$ (resp. $\forall x\forall y(rxy \rightarrow ryx)$)

Lemma 2 Exemplification of necessary existence is not possible. $\neg\Diamond\exists x\,NE(x)$

by B, D3, Lemma1

Axiom A5 $P(NE)$

Inconsistency $\bot$

by A5, T1, Lemma2

Inconsistency (Gödel): Verification in Isabelle/HOL (KB)

GoedelGodWithoutConjunctInEss_KB.thy

GoedelGodWithoutConjunctInEss_KB.thy (~/GoedelGod/Talks/2015/LogicAndReligion/DemoMaterial/)

```
theory GoedelGodWithoutConjunctInEss_KB imports QML
begin
  consts P :: "(μ ⇒ σ) ⇒ σ"
  axiomatization where A1a: "[∀(λΦ. P (λx. m¬ (Φ x)) m→ m¬ (P Φ))]"
                   and A2:  "[∀(λΦ. ∀(λΨ. (P Φ m∧ □ (∀(λx. Φ x m→ Ψ x))) m→ P Ψ))]"

-- {* Positive properties are possibly exemplified. *}
  theorem T1: "[∀(λΦ. P Φ m→ ◇ (∃ Φ))]"                          by (metis A1a A2)

  definition ess (infixr "ess" 85) where "Φ ess x = ∀(λΨ. Ψ x m→ □ (∀(λy. Φ y m→ Ψ y)))"

-- {* The empty property is an essence of every individual. *}
  lemma Lemma1: "[(∀(λx.( (λy.λw. False) ess x)))]"                 by (metis ess_def)

  definition NE  where "NE x = ∀(λΦ. Φ ess x m→ □ (∃ Φ))"
  axiomatization where sym: "x r y ⟶ y r x"

-- {* Exemplification of necessary existence is not possible. *}
  lemma Lemma2: "[m¬ (◇ (∃ NE))]"                          by (metis sym Lemma1 NE_def)

  axiomatization where A5:  "[P NE]"

-- {* Now the inconsistency follows from A5, T1 and Lemma2 *}
  lemma False                                                by (metis A5 T1 Lemma2)
end
```

Documentation Sidekick Theories

Output Query Sledgehammer Symbols

11,1 (477/1095) (isabelle,sidekick,UTF-8-Isabelle)Nmro UG 263/347MB 17:18

Inconsistency (Gödel): Verification in Isabelle/HOL (K)

GoedelGodWithoutConjunctInEss_K.thy

GoedelGodWithoutConjunctInEss_K.thy (~/GoedelGod/Talks/2015/LogicAndReligion/DemoMaterial/)

```
theory GoedelGodWithoutConjunctInEss_K imports QML
begin
  consts P :: "(μ ⇒ σ) ⇒ σ"
  definition ess (infixr "ess" 85) where "Φ ess x = ∀(λΨ. Ψ x m→ □ (∀(λy. Φ y m→ Ψ y)))"
  definition NE  where                   "NE x     = ∀(λΦ. Φ ess x m→ □ (∃ Φ))"
  axiomatization where A1a:  "[∀(λΦ. P (λx. m¬ (Φ x)) m→ m¬ (P Φ))]"
                   and A2:   "[∀(λΦ. ∀(λΨ. (P Φ m∧ □ (∀(λx. Φ x m→ Ψ x))) m→ P Ψ))]"

-- {* Positive properties are possibly exemplified. *}
  theorem T1: "[∀(λΦ. P Φ m→ ◇ (∃ Φ))]"                          by (metis A1a A2)

-- {* The empty property is an essence of every individual. *}
  lemma Lemma1: "[(∀(λx.((λy.λw. False) ess x)))]"              by (metis ess_def)

  axiomatization where A5:  "[P NE]"

-- {* Now the inconsistency follows from A5, Lemma1, NE_def and T1 *}
  lemma False
  -- {* sledgehammer [remote_leo2] *}
                                                   by (metis A5 Lemma1 NE_def T1)
end
```

Output Query Sledgehammer Symbols

21,7 (980/982) (isabelle,sidekick,UTF-8-Isabelle)Nmro UG 302/343MB 11:37

Summary: Results of Experiments

GÃűdel's version	K		KB		S5	
	constant	varying	constant	varying	constant	varying
Consistency	×	×	×	×	×	×
T1	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
C	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
T2	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
T3	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
Flawless God	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
Monotheism	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
Modal Collapse	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete

Further logic details

- Henkin semantics
- full comprehension
- rigid constant symbols

Question: **Has this inconsistency been reported before?**

If not, then LEO-II deserves (part of) the credit!

Summary: Results of Experiments

Scott's version	K		KB		S5	
	constant	varying	constant	varying	constant	varying
Consistency	✓	✓	✓	✓	✓	✓
T1	✓	✓	✓	✓	✓	✓
C	✓	✓	✓	✓	✓	✓
T2	✓	✓	✓	✓	✓	✓
T3	×	×	✓	✓	✓	✓
Flawless God	✓	✓	✓	✓	✓	✓
Monotheism	✓	✓	✓	✓	✓	✓
Modal Collapse	✓	✓	✓	✓	✓	✓

Further logic details

- Henkin semantics
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Avoiding the Modal Collapse: Recent Variants

C. Anthony Anderson

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

Magari and others on Gödel's ontological proof

Petr Hájek
Institute of Computer Science, Academy of Sciences
182 07 Prague, Czech Republic
e-mail: hajek@uivt.cas.cz

1 Introduction

This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variant by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe

Der Mathematiker und die Frage der Existenz Gottes
(betreffend Gödels ontologischen Beweis)

Es ist gut, daß wir nicht wissen, sondern glauben, daß ein Gott sei.
(Kant, Nachlaß)

1. Einführung

Gödels zu Lebzeiten unveröffentlichter Beweis für die notwendige Existenz eines Gott-ähnlichen Wesens hat sowohl philosophisches als auch mathematisches Interesse geweckt. Zweck der vorliegenden Arbeit ist es, zu einer Deutung des Gödelschen Textes beizutragen, 1. durch Kommentierung der einschlägigen Literatur und 2. durch Bereitstellung von etwas Modelltheorie. Die Arbeit enthält keinen philosophischen Beitrag. Während der letzten Jahre habe ich etliche Male über Gödels Gottesbeweis vorgetragen, insbesondere auf dem Symposium zur Feier von Professor Gert Müller (Heidelberg, Jänner 1991), doch habe ich niemals beabsichtigt, eine Veröffentlichung über das Thema zu machen. Da ich wiederholt um eine schriftliche Version gebeten wurde, entschloß ich mich, schnell eine „erweiterte Kurzfassung"[1] zu schreiben, ohne aus ihr einen

Understanding Gödel's Ontological Argument

FRODE BJØRDAL

In 1970 Kurt Gödel, in a hand-written note entitled "Ontologischer Beweis", put forward an ontological argument for the existence of God, making use of second-order modal logical principles. Let the second-order formula P(F) stand for "the property F is positive", and let "God" signify the property of being God-like. Gödel presupposes the following definitions:

Gödel's version of the modal ontological argument for the existence of God has been criticized by J. Howard Sobel [5] and modified by C. Anthony Anderson [1]. In the present paper we consider the extent to which Anderson's emendation is defeated by the type of objection first offered by the Monk Gaunilo to St. Anselm's original Ontological Argument. And we try to push the analysis of this Gödelian argument a bit further to bring it into closer agreement with the details of Gödel's own formulation. Finally, we indicate what seems to be the main weakness of this emendation of Gödel's attempted proof.

PETR HÁJEK

A New Small Emendation of Gödel's Ontological Proof

Keywords: Ontological proof, Gödel, modal logic, comprehension, positive properties.

1. Introduction

Gödel's ontological proof of necessary existence of a godlike being was finally published in the third volume of Gödel's collected works [7]; but it became known in 1970 when Gödel showed the proof to Dana Scott and Scott presented it (in fact a variant of it) at a seminar at Princeton. Detailed history is found in Adams' introductory remarks to the ontological proof in [7]. The proof uses modal logic and its analysis is an exciting exercise in systems of formal modal logic. Needles to say, formal modal logic has found several

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Computer-supported Clarification of Controversy

Results Obtained with Fully Automated Reasoners

A controversy between Magari, Hájek and Anderson regarding the redundancy of some axioms

Proof	D1'	A:A1'	A2'	A3'	A4	A4'	H:A4	A5	A5'	H:A5	T3	T3'	MC
Scott (const)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Scott (var)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Anderson (const)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (var)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (mix)	-	-	-	-	R (K4B)	-	-	-	I	-	-	CS	CS
Hájek AOE' (var)	-	-	CS		S/I	-	-	-	S/I	-	-	P (KB)	CS
Hájek AOE'_0 (var)	-	-	-	CS	R	-	-	-	S/U	-	-	P (KB)	CS
Hájek AOE'' (var)	-	-			-	-	S/I	-	-	S/I	-	P (KB)	CS
Anderson (simp) (var)	-	R	R			R (K4B)	-	-		-	-		
Bjørdal (const)	R (K4)	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS
Bjørdal (var)	CS	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS

Results Obtained with Fully Automated Reasoners

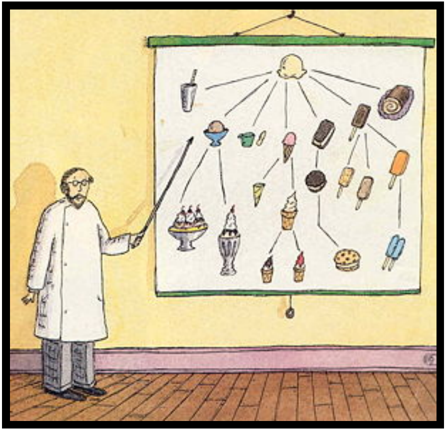
A controversy between Magari, Hájek and Anderson regarding the redundancy of some axioms

Proof	**D1'**	**A:A1'**	**A2'**	**A3'**	**A4**	**A4'**	**H:A4**	**A5**	**A5'**	**H:A5**	**T3**	**T3'**	**MC**
Scott (const)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Scott (var)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Anderson (const)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (var)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (mix)	-	-	-	-	R (K4B)	-	-	-	I	-	-	CS	CS
Hájek AOE' (var)	-	-	CS		S/I	-	-	-	S/I	-	-	P (KB)	CS
Hájek AOE'_0 (var)	-	-	-	CS	R	-	-	-	S/U	-	-	P (KB)	CS
Hájek AOE'' (var)	-	-			-	-	S/I	-	-	S/I	-	P (KB)	CS
Anderson (simp) (var)	-	R	R			R (K4B)	-	-		-	-		
Bjørdal (const)	R (K4)	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS
Bjørdal (var)	CS	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS

Leibniz (1646–1716)

characteristica universalis and **calculus ratiocinator**

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.



SUMO Ontology and HOML

SUMO Ontology and HOML

Context

- prominent challenge in AI (CS, Philosophy)
- McCarthy: modeling of contexts as first-class objects

```
ist(context_of("Ben's Knowledge),likes(Sue,Bill))

ist(context_of("Ben's Knowledge),
    ist(context_of(...),...))
```

- McCarthy's [McCarthy, Comm.ACM 1987] [McCarthy, IJCAI 1993] approach has been followed by many others
- Giunchiglia's contextual reasoning [Giunchiglia, Epistemologia 1993] emphasizes the locality aspect; structured knowledge
- McCarthy and Giunchiglia avoid modal logics
- they also avoid a HOL perspective

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- McCarthy and Giunchiglia avoid modal logics
- they also avoid a HOL perspective

Our approach is complementary:
takes a HOL perspective and integrates modal logics

SUMO Ontology and HOML

Expressive ontologies

- SUMO and Cyc
- modeling of contexts:

```
(holdsDuring (yearFn 2009) (loves Bill Mary))

   (believes Bill
     (knows Ben
       (forall (?X)
         ((woman ?X) => (loves Bill ?X)))))
```

- relation to McCarthy's approach is obvious
- often a questionable semantics assumed for embedded formulas and modal predicates (also in Common Logic)

SUMO Ontology and HOML

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```

- relation to McCarthy's approach is obvious
- often a questionable semantics assumed for embedded formulas and modal predicates (also in Common Logic)

Our approach:
HOL-based semantics, but holdsDuring, believes, knows and alike are associated with modal logic connectives

Combining logics

- prominent challenge in AI (CS, Philosophy)
- epistemic, deontic, temporal, intuitionistic, relevant, linear, conditional, security ...
- wide literature—few implementations
- some propositional systems exists: Logic Workbench, LoTREC, Tableaux Workbench, FaCT, ileanCoP, MSPASS
- no implemented systems for combinations of first-order logics
- combination is typically approached bottom-up

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- no implemented systems for combinations of first-order logics
- combination is typically approached bottom-up

Our approach is complementary:
works top-down starting from classical higher-order logic (HOL)

SUMO Ontology and HOML

SUMO — Suggested Upper Merged Ontology

[NilesPease FOIS 2001, Pease 2011]

- open source, formal ontology: `www.ontologyportal.org`
- has been extended for a number of domain specific ontologies
- altogether approx. 20,000 terms and 70,000 axioms
- employs the SUO-KIF representation language, a simplification of Genesereth's original Knowledge Interchange Format (KIF)

Sigma [PeaseBenzmüller AI Comm. 2013]

- browsing and inference system for ontology development
- integrates KIF-Vampire and SystemOnTPTP

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SUMO (and similarly Cyc) contains higher-order representations, but there is only very limited automation support so far

Higher-Order Aspects in SUO-KIF and SUMO: Examples

- Embedded formulas

$$term ::= variable|word|string|funterm|number|sentence$$

```
(holdsDuring (YearFn 2009) (likes Mary Bill))
```

Higher-Order Aspects in SUO-KIF and SUMO: Examples

- Embedded formulas
- …often in combination with modal operators such as `holdsDuring`, `knows`, `believes`, etc.

Higher-Order Aspects in SUO-KIF and SUMO: Examples

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- Predicate variables, function variables, propositional variables

funterm ::= (funword arg+)   relsent ::= (relword arg+)

funword, relword ::= initialchar wordchar* | variable

```
(<=>
   (instance ?REL TransitiveRelation)
   (forall (?INST1 ?INST2 ?INST3)
      (=>
         (and
            (?REL ?INST1 ?INST2)
            (?REL ?INST2 ?INST3))
         (?REL ?INST1 ?INST3))))
```

Higher-Order Aspects in SUO-KIF and SUMO: Examples

- Embedded formulas
- …often in combination with modal operators such as `holdsDuring`, `knows`, `believes`, etc.
- Predicate variables, function variables, propositional variables
- Lambda-Abstraction with KappaFN

```
(=>
  (attribute ?X Celebrity)
  (greaterThan
    (CardinalityFn
      (KappaFn ?A
        (knows ?A (exists (?P) (equal ?P ?X)))))
    1000))
```

Higher-Order Aspects in SUO-KIF and SUMO: Examples

- Embedded formulas
- ...often in combination with modal operators such as `holdsDuring`, `knows`, `believes`, etc.
- Predicate variables, function variables, propositional variables
- Lambda-Abstraction with KappaFN

Our focus in the remainder:

embedded formulas and modal operators

Some FO translation 'tricks'

First-order reasoning on a large ontology

[PeaseSutcliffe, CEUR 257, 2007]

- Quoting of embedded formulas

A: `(holdsDuring (YearFn 2009) (likes Mary Bill))`
Q: `(holdsDuring (YearFn ?Y) (likes ?X Bill))`

Some FO translation 'tricks'

First-order reasoning on a large ontology

[PeaseSutcliffe, CEUR 257, 2007]

- Quoting of embedded formulas

A: `(holdsDuring (YearFn 2009) '(likes Mary Bill))`
Q: `(holdsDuring (YearFn ?Y) '(likes ?X Bill))`

Answer with FO-ATPs (`?Y` ← `2009`, `?X` ← `Mary`)

Some FO translation 'tricks'

First-order reasoning on a large ontology

[PeaseSutcliffe, CEUR 257, 2007]

- Quoting of embedded formulas

A: `(holdsDuring (YearFn 2009)`
`'(and (likes Mary Bill) (likes Sue Bill)))`
Q: `(holdsDuring (YearFn ?Y) '(likes ?X Bill))`

Failure with FO-ATP

Some FO translation 'tricks'

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- Quoting of embedded formulas
- Expansion of predicate variables

Some FO translation 'tricks'

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Why not trying higher-order automated theorem proving directly?

Our focus:

embedded formulas and modal operators

The SUO-KIF to TPTP THF0 Translation

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- THF0: TPTP format for simple type theory
  [SutcliffeBenzmüller, J.Formalized Reasoning, 2010]
- THF0 ATPs: LEO-II, TPS, Isabelle, Satallax, ...
  THF0 (counter-)model finders: Nitpick, Satallax

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- THF0 ATPs: LEO-II, TPS, Isabelle, Satallax, ...

  THF0 (counter-)model finders: Nitpick, Satallax

- achieved:

  SUO-KIF $\longrightarrow$ TPTP THF0

  translation mechanism for SUMO as part of Sigma

- so far only exploits base type ι and o in THF0 ($\rightarrow$ improvable)

- generally applicable to SUO-KIF representations

- translation example (for SUMO) available at:

  `http://christoph-benzmueller.de/papers/SUMO.thf`

Main challenge: find consistent typing for untyped SUO-KIF

```
(instance instance BinaryPredicate)
```

The SUO-KIF to TPTP THF Translation

Main challenge: find consistent typing for untyped SUO-KIF

```
(p_instance t_instance BinaryPredicate)
```

Higher-Order Automated Theorem Proving in Ontology Reasoning

Embedded Formulas — An Easy Task for HO-ATP

Example (A: Embedded Formulas)

During 2009 Mary liked Bill and Sue liked Bill. Who liked Bill in 2009?

A: `(holdsDuring (YearFn 2009)`
`(and (likes Mary Bill) (likes Sue Bill)))`

Q: `(holdsDuring (YearFn 2009) (likes ?X Bill))`

Proof by LEO-II in milliseconds

Embedded Formulas — An Easy Task for HO-ATP

Example (B: Embedded Formulas)

During 2009 Mary liked Bill and Sue liked Bill. Who liked Bill in 2009?

A:
```
(holdsDuring (YearFn 2009)
   (not (or (not (likes Mary Bill))
            (not (likes Sue Bill)))))
```
Q:
```
(holdsDuring (YearFn 2009) (likes ?X Bill))
```

Proof by LEO-II in milliseconds

Embedded Formulas — An Easy Task for HO-ATP

Example (C: Embedded Formulas)

At all times Mary likes Bill. During 2009 Sue liked whomever Mary liked. Is there a year in which Sue has liked somebody?

A: `(holdsDuring ?Y (likes Mary Bill))`

B:
```
(holdsDuring (YearFn 2009)
    (forall (?X) (=> (likes Mary ?X) (likes Sue ?X))))
```

Q: `(holdsDuring (YearFn ?Y) (likes Sue ?X))`

Proof by LEO-II in milliseconds

Embedded Formulas — An Easy Task for HO-ATP

In the above examples we have (silently) assumed that the semantics of the logic underlying SUMO is a classical, bivalent logic, meaning that Boolean extensionality is valid:

```
(<=> (<=> ?P ?Q) (equal ?P ?Q))
```

Example (D: Embedded Formulas)

During 2009 Mary liked Bill and Sue liked Bill. Who liked Bill in 2009?

A:
```
(holdsDuring (YearFn 2009)
   (and (likes Mary Bill) (likes Sue Bill)))
```

Q:
```
(holdsDuring (YearFn 2009)
   (and (likes Sue Bill) (likes Mary Bill)))
```

Proof by LEO-II in milliseconds

Boolean extensionality seems fine for the particular temporal contexts of our previous examples.

However, as we will show next, it quickly leads to counterintuitive inferences in other modal contexts.

Problem: Boolean Extensionality versus Modal Operators

Example (E: Embedded Formulas – Epistemic Contexts)

A: `(knows Chris (equal Chris Chris))`
B: `(likes Mary Bill)`
C: `(knows Chris`
`    (forall (?X) (=> (likes Mary ?X) (likes Sue ?X))))`
Q: `(knows Chris (likes Sue Bill))`

Proof by LEO-II in milliseconds

Problem: Boolean Extensionality versus Modal Operators

Example (E: Embedded Formulas – Epistemic Contexts)

A: `(knows Chris (equal Chris Chris))`
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C: `(knows Chris`
`    (forall (?X) (=> (likes Mary ?X) (likes Sue ?X))))`
Q: `(knows Chris (likes Sue Bill))`

Proof by LEO-II in milliseconds

Boolean extensionality is in conflict with (epistemic) modalities!
(Has Boolean extensionality ever been questioned for KIF?)

Proposed Solution: Possible World Semantics for SUMO

SUMO $\longrightarrow$ HOML $\longrightarrow$ TPTP THF

- T-Box like information in SUMO:

  `(instance holdsDuring AsymmetricRelation)` $\longrightarrow$

  $\forall W_\iota$ (instance holdsDuring AsymmetricRelation)$_{\iota\to o}$ W

Proposed Solution: Possible World Semantics for SUMO

SUMO $\longrightarrow$ HOML $\longrightarrow$ TPTP THF

- T-Box like information in SUMO:

  `(instance holdsDuring AsymmetricRelation)` $\longrightarrow$

  $\forall W_\iota$ (instance holdsDuring AsymmetricRelation)$_{\iota\to o}$ W

- A-Box like information as in query problem: current world cw_ι

  (likes Mary Bill) $\longrightarrow$ (likes Mary Bill)$_{\iota\to o}$ cw

  (knows Chris (likes Sue Bill)) $\longrightarrow$

  ($\Box_{\text{Chris}}$ (likes Sue Bill))$_{\iota\to o}$ cw

Challenge: Embedded Formulas — Epistemic Context

Example (F: Embedded Formulas – Epistemic Contexts)

A: $(\Box_{Chris}\,(equal\ Chris\ Chris))\ cw$
B: $(likes\ Mary\ Bill)\ cw$
C: $(\Box_{Chris}\,(\forall^i X_\mu.((likes\ Mary\ X) \supset (likes\ Sue\ X))))\ cw$
Q: $(\Box_{Chris}\,(likes\ Sue\ Bill))\ cw$

Challenge: Embedded Formulas — Epistemic Context

Example (F: Embedded Formulas – Epistemic Contexts)

A: $(\Box_{Chris}\,(equal\ Chris\ Chris))\ cw$
B: $(likes\ Mary\ Bill)\ cw$
C: $(\Box_{Chris}\,(\forall^{i} X_{\mu}.\,((likes\ Mary\ X) \supset (likes\ Sue\ X))))\ cw$
Q: $(\Box_{Chris}\,(likes\ Sue\ Bill))\ cw$

Axioms for $\Box_{Chris}$ can be added:

M: $\forall W_{\iota}\ (\forall^{p} \varphi_{\iota\rightarrow o}.\,\Box_{Chris}\,\varphi \supset \varphi)\ W$
4: $\forall W_{\iota}\ (\forall^{p} \varphi_{\iota\rightarrow o}.\,\Box_{Chris}\,\varphi \supset \Box_{Chris}\,\Box_{Chris}\,\varphi)\ W$
5: $\forall W_{\iota}\ (\forall^{p} \varphi_{\iota\rightarrow o}.\,\Box_{Chris}\,\neg\varphi \supset \Box_{Chris}\,\neg\Box_{Chris}\,\varphi)\ W$

Challenge: Embedded Formulas — Epistemic Context

Example (F: Embedded Formulas – Epistemic Contexts)

A: $(\Box_{Chris}\ (equal\ Chris\ Chris))\ cw$
B: $(likes\ Mary\ Bill)\ cw$
C: $(\Box_{Chris}\ (\forall^i X_{\mu}\blacksquare((likes\ Mary\ X)\ \supset\ (likes\ Sue\ X))))\ cw$
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5: $\forall W_{\iota}\ (\forall^p \varphi_{\iota\to o}\blacksquare \Box_{Chris}\ \neg\ \varphi\ \supset\ \Box_{Chris}\ \neg\ \Box_{Chris}\ \varphi)\ W$

LEO-II cannot solve this problem anymore! Countermodel exists.

Challenge: Embedded Formulas — Epistemic Context

Example (F: Embedded Formulas – Epistemic Contexts)

A: $(\Box_{Chris}\ (equal\ Chris\ Chris))\ cw$
B: $(\Box_{Chris}\ (likes\ Mary\ Bill))\ cw$
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Axioms for $\Box_{Chris}$ can be added:

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5: $\forall W_\iota\ (\forall^p \varphi_{\iota\to o}\blacksquare\Box_{Chris}\,\neg\,\varphi\ \supset\ \Box_{Chris}\,\neg\,\Box_{Chris}\,\varphi)\ W$

But LEO-II(+E) can solve this problem in milliseconds!

Example (F: Embedded Formulas – Epistemic Contexts)

A: $(\Box_{Chris}\ (equal\ Chris\ Chris))\ cw$
B: $(\Box_{fool}\ (likes\ Mary\ Bill))\ cw$
C: $(\Box_{Chris}\ (\forall^{i} X_{\mu}.((likes\ Mary\ X) \supset (likes\ Sue\ X))))\ cw$
Q: $(\Box_{Chris}\ (likes\ Sue\ Bill))\ cw$

Axioms for $\Box_{Chris}$ can be added:

M: $\forall W_{\iota}\ (\forall^{p} \varphi_{\iota\to o}.\ \Box_{Chris}\ \varphi \supset \varphi)\ W$
4: $\forall W_{\iota}\ (\forall^{p} \varphi_{\iota\to o}.\ \Box_{Chris}\ \varphi \supset \Box_{Chris}\ \Box_{Chris}\ \varphi)\ W$
5: $\forall W_{\iota}\ (\forall^{p} \varphi_{\iota\to o}.\ \Box_{Chris}\ \neg\varphi \supset \Box_{Chris}\ \neg\Box_{Chris}\ \varphi)\ W$

Axioms for $\Box_{fool}$ can be added ...
$\forall W_{\iota}\ (\forall^{p} \varphi_{\iota\to o}.\ \Box_{fool}\ \varphi \supset \Box_{Chris}\ \varphi)\ W$
...

SUMO Ontology — Proposal

Redevelop entire SUMO Ontology in HOML (multimodal, eventually other logics)

Should give a proper semantics for SUMO

Employ HOML embedding in HOL to automated reasoning

Experiments with Large Knowledge Bases (SUMO)

Significant Improvements for Large Theories (PAAR-2010)

LEO-II(+E) version v1.1

Ex.	A	B	C	D	E			F	
local	.19	.19	.13	.16	.08	.34	.18	.04	2642.55
SInE	–	–	–	–	–	–	–	–	–
global	–	–	–	–	–	–	–	–	–

global: all SUMO axioms given to LEO-II
SInE: filters SUMO axioms for problem — ˜400 axioms given to LEO-II
local: only handselected axioms given to LEO-II

Further reading and more experiments [BenzmüllerPease J.WebSemantics 2012]

Significant Improvements for Large Theories (PAAR-2010)

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Ex.	A	B	C	D	E			F	
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SInE	–	–	–	–	–	–	–	–	–
global	–	–	–	–	–	–	–	–	–

global: all SUMO axioms given to LEO-II
SInE: filters SUMO axioms for problem — ˜400 axioms given to LEO-II
local: only handselected axioms given to LEO-II

LEO-II(+E) version v1.2.1 (with relevance filtering)

Ex.	A	B	C	D	E			F	
local	.19	.18	.11	.08	.10	.38	.32	.14	.18
SInE	.43	.40	.21	.54	.37	.12	.70	.06	.26
global	2.8	2.7	1.6	4.9	1.4	0.9	4.7	1.3	0.9

Further reading and more experiments [BenzmüllerPease J.WebSemantics 2012]

Meta-Reasoning

Description logic $\mathcal{ALC}$

Syntax	Semantics	Description	Example
A	$A^I \subseteq \Delta^I$	atomic concept	$Human, Female, \ldots$
r	$r^I \subseteq \Delta^I \times \Delta^I$	binary relation	$married, \ldots$
$\bot$	$\emptyset$	empty concept	
$\top$	Δ^I	universal concept	
$\sim A$	$\Delta^I \setminus A^I$	complement	$\sim Female$
$A \sqcup B$	$A^I \cup B^I$	disjunction	$Female \sqcup Male$
$A \sqcap B$	$A^I \cap B^I$	conjunction	$Female \sqcap Human$
$\exists r\ C$	$\{x \mid \exists y.r^I(x,y) \wedge C^I(y)\}$	existential restriction	$\exists married\ Female$
$\forall r\ C$	$\{x \mid \forall y.r^I(x,y) \rightarrow C^I(y)\}$	universal restriction	$\forall married\ Female$

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$\top$	Δ^I	universal concept	
$\sim A$	$\Delta^I \setminus A^I$	complement	$\sim Female$
$A \sqcup B$	$A^I \cup B^I$	disjunction	$Female \sqcup Male$
$A \sqcap B$	$A^I \cap B^I$	conjunction	$Female \sqcap Human$
$\exists r\ C$	$\{x \mid \exists y.r^I(x,y) \wedge C^I(y)\}$	existential restriction	$\exists married\ Female$
$\forall r\ C$	$\{x \mid \forall y.r^I(x,y) \rightarrow C^I(y)\}$	universal restriction	$\forall married\ Female$

Simple exercises (useful lemmata)

$$\top = A \sqcup \sim A \tag{L1}$$

$$\bot = \sim\top \tag{L2}$$

$$A \sqcap B = \sim(\sim A \sqcup \sim B) \tag{L3}$$

$$\forall r\ C = \sim(\exists r \sim C) \tag{L4}$$

Description logic $\mathcal{ALC}$

Syntax	Semantics	Description	Example
A	$A^I \subseteq \Delta^I$	atomic concept	*Human*, *Female*, ...
r	$r^I \subseteq \Delta^I \times \Delta^I$	binary relation	*married*, ...
$\sim A$	$\Delta^I \setminus A^I$	complement	$\sim$*Female*
$A \sqcup B$	$A^I \cup B^I$	disjunction	*Female* $\sqcup$ *Male*
$\exists r\ C$	$\{x \mid \exists y. r^I(x,y) \wedge C^I(y)\}$	existential restriction	$\exists$*married Female*

Simple exercises (useful lemmata)

$$\top = A \sqcup \sim A \qquad (L1)$$

$$\bot = \sim\top \qquad (L2)$$

$$A \sqcap B = \sim(\sim A \sqcup \sim B) \qquad (L3)$$

$$\forall r\ C = \sim(\exists r \sim C) \qquad (L4)$$

Description logic $\mathcal{ALC}$

Syntax	Semantics	Description	Example
$A \sqsubseteq B$	$A^I \subseteq B^I$	B subsumes A	$Doctor \sqsubseteq Human$
$A \doteq B$	$A^I \sqsubseteq B^I$ und $B^I \sqsubseteq A^I$	A defined by B	$Parent \doteq$ $Human \sqcap \exists hasChild\ Human$

Description logic $\mathcal{ALC}$

Syntax	Semantics	Description	Example
$A \sqsubseteq B$	$A^I \subseteq B^I$	B subsumes A	$Doctor \sqsubseteq Human$
$A \doteq B$	$A^I \sqsubseteq B^I$ *und* $B^I \sqsubseteq A^I$	A defined by B	$Parent \doteq$ $Human \sqcap \exists hasChild\ Human$

Simple exercises:

$$A \sqsubseteq B \quad gdw. \quad \exists C.A \doteq C \sqcap B \tag{L5}$$

$$A \sqsubseteq B \quad gdw. \quad (A \sqcap \sim B) \sqsubseteq \bot \tag{L6}$$

$$gdw. \quad \exists x.(A \sqcap \sim B)(x) \text{ unerfüllbar}$$

Knowledge representation in $\mathcal{ALC}$

TBox (terminological knowledge, taxonomy)

Example:

$$
\begin{aligned}
\mathit{HappyMan} &\doteq \mathit{Human} \sqcap \sim\mathit{Female} \sqcap \\
&\quad (\exists \mathit{married}\ \mathit{Doctor}) \sqcap (\forall \mathit{hasChild}(\mathit{Doctor} \sqcup \mathit{Professor})) \\
\mathit{Doctor} &\sqsubseteq \mathit{Human}
\end{aligned}
$$

Knowledge representation in $\mathcal{ALC}$

TBox (terminological knowledge, taxonomy)

Example:

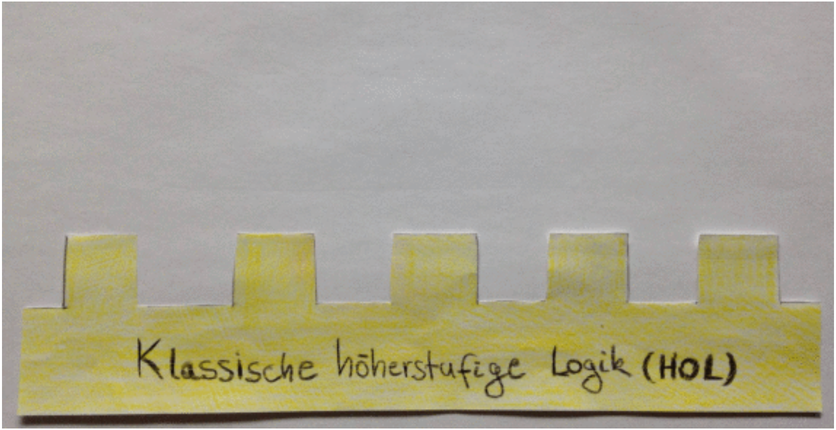
$$\begin{aligned} HappyMan &\doteq Human \ \sqcap \sim Female \sqcap \\ &\quad (\exists married\ Doctor) \sqcap (\forall hasChild(Doctor \sqcup Professor)) \\ Doctor &\sqsubseteq Human \end{aligned}$$

ABox (assertional knowledge, e.g. assumptions on indiviuals)

Example:

$$HappyMan(BOB), \quad hasChild(BOB, MARY), \quad \sim Doctor(MARY)$$

EINSCHUB: Knowledge representation & Inference about ALC in HOL



Animation: Max Benzmüller (5 years)

EINSCHUB: Knowledge representation & Inference about ALC in HOL

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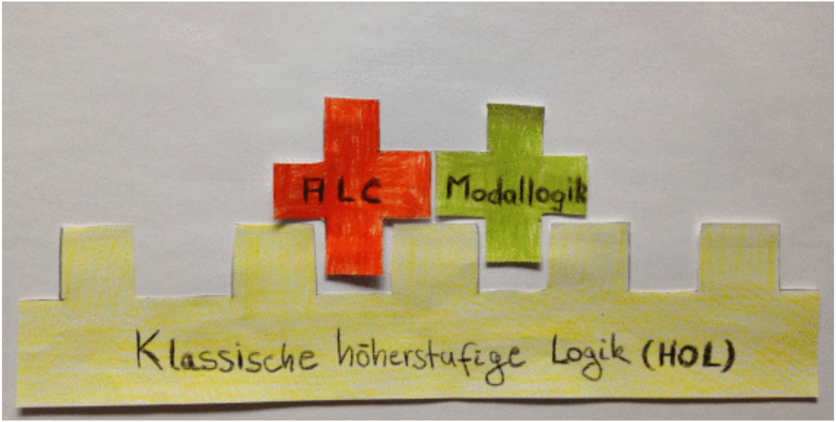
EINSCHUB: Knowledge representation & Inference about ALC in HOL



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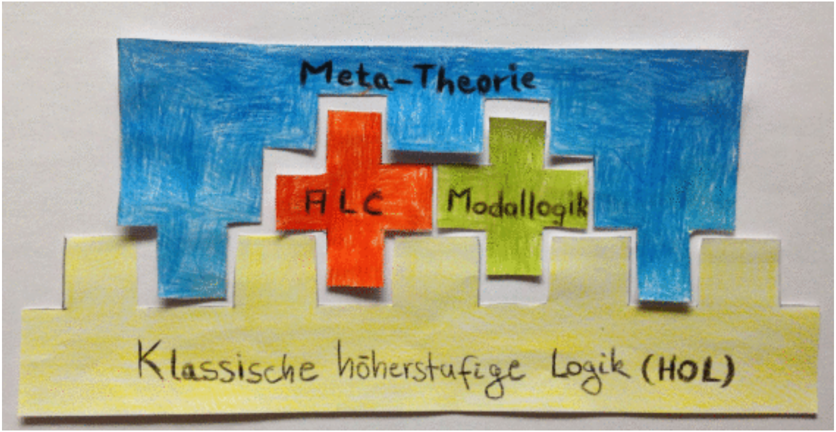
EINSCHUB: Knowledge representation & Inference about ALC in HOL



Animation: Max Benzmüller (5 years)

EINSCHUB: Meta theory of ALC in Isabelle/HOL

ALC.thy

ALC.thy (~/chris/trunk/tex/talks/2015-Erlangen/)

```
theory ALC  imports Main begin

typedecl i  type_synonym τ = "(i ⇒ bool)" type_synonym σ = "(i ⇒ i ⇒ bool)"

abbreviation bot ::"τ"    ("⊥")              where "⊥    ≡ λx. False"
abbreviation top ::"τ"    ("⊤")              where "⊤    ≡ λx. True"
abbreviation neg          ("∼")              where "∼A   ≡ λx. ¬A(x)"
abbreviation disj         (infixr "⊔" 40)    where "A⊔B  ≡ λx. A(x) ∨ B(x)"
abbreviation conj         (infixr "⊓" 41)    where "A⊓B  ≡ λx. A(x) ∧ B(x)"
abbreviation exi_r        ("∃")              where "∃r A  ≡ λx. ∃y. r x y ∧ A(y)"
abbreviation all_r        ("∀")              where "∀r A  ≡ λx. ∀y. r x y ⟶ A(y)"

abbreviation sub (infixr "⊑" 39)             where "A⊑B  ≡ ∀x. A(x) ⟶ B(x)"
abbreviation eq  (infixr "≜" 38)             where "A≜B  ≡ A ⊑ B ∧ B ⊑ A"

(* Einfaches Beispiele für etwas Meta-Theorie *)
lemma "A⊓B ≜ ∼(∼A⊔∼B)" by metis
lemma "∃r C ≜ ∼(∀r (∼C))" by metis  (* sledgehammer [remote_leo2] *)
lemma "A⊓B ≜ A⊔B" nitpick oops
```

Sidekick Theories

100% Auto update Update Detach

```
Nitpicking formula...
Nitpick found a counterexample for card 'a = 2:

  Free variables:
    A = (λx. _)(a₁ := False, a₂ := False)
    B = (λx. _)(a₁ := False, a₂ := True)
```

Output README Symbols

19,22 (939/2329) (isabelle,sidekick,UTF-8-Isabelle)Nmro UG 85/124MB 9:12 PM

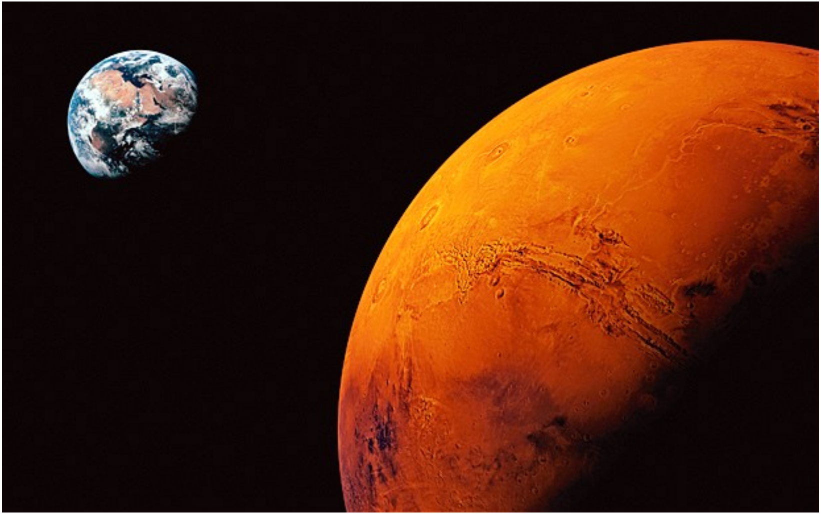
Flexibility in HOML

Many Variations of Higher-Order Modal Logics

- (Meta-)Axioms for the accessibility relation (e.g. reflexivity, transitivity, symmetry)
- Axioms (e.g. $\Box A \to A$, $\Box A \to \Box\Box A$, $A \to \Box\Diamond A$)
- Possibilistic vs. Actualistic Quantifiers
- Constant Domains vs. Varying Domains vs. Cumulative Domains
- Rigidity vs. Flexibility
- Simple Types vs. Dependent Types (e.g. μ vs. $\mu(w)$)
- ...

Flexibility
A Funny Example

$$\Box Blue(sky)$$

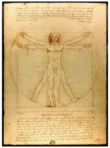


Flexibility
A Funny Example

$$\Box Blue(sky)$$

Flexibility
A Funny Example

$$\Box Blue(sky)$$



Human: “Earth’s sky is blue”

Martian: “Mars’ sky is blue”

Flexibility

$$\Box Blue(sky)$$

- Rigid embedding:

$$\lceil \mu \rceil = \mu \qquad \lceil o \rceil = \iota \to o \qquad \lceil \alpha \to \beta \rceil = \lceil \alpha \rceil \to \lceil \beta \rceil$$

$$\lceil sky_{\mu} \rceil = sky_{\mu}$$
$$\lceil Blue_{\mu \to o} \rceil = Blue_{\mu \to (\iota \to o)}$$

- Flexible embedding:

$$\lceil \mu \rceil = \iota \to \mu \qquad \lceil o \rceil = \iota \to o \qquad \lceil \alpha \to \beta \rceil = \iota \to \lceil \alpha \rceil \to \lceil \beta \rceil$$

$$\lceil sky_{\mu} \rceil = sky_{\iota \to \mu}$$
$$\lceil Blue_{\mu \to o} \rceil = Blue_{\iota \to (\iota \to \mu) \to (\iota \to o)}$$

$$[\Box Blue(sky)]$$

$$\forall w.\forall w'.\ Blue\ w\ (sky\ w)\ w'$$

$$\forall w.\forall w'.\ Blue\ w'\ (sky\ w')\ w'$$

$$\forall w.\forall w'.\ Blue\ w'\ (sky\ w)\ w'$$

$$\forall w.\forall w'.\ Blue\ w\ (sky\ w')\ w'$$

Reasoning with Inconsistencies!

The End