CES Technology and Business Cycle Fluctuations
Online Appendix - Not for Publication*

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A The Augmented SW Model

Here we present the augmented SW model with a wholesale and a retail sector, Calvo prices and wages, CES production function, adjustment costs of investment and variable capital utilization. The model equilibrium conditions are presented in non-linear form.[1]

We set out the model without specifying the form of the utility and production functions in order to obtain a flexible framework in which it will be easy to stick different functional forms as we do in the paper.

A.1 Final Goods

Each final goods firms minimizes the cost $\int_0^1 P_t(f)Y_t(f)df$ of producing the final output $Y_t = \left(\int_0^1 Y_t(f)^{(\zeta-1)/\zeta}df\right)^{\zeta/(\zeta-1)}$. This leads to the standard result for the Dixit-Stigliz aggregator

$$Y_t(f) = \left(\frac{P_t(f)}{P_t}\right)^{-\zeta} Y_t \tag{A.1}$$

$$P_t = \left[\int_0^1 P_t(f)^{1-\zeta}df\right]^{\frac{1}{1-\zeta}} \tag{A.2}$$

$$P_tY_t = \int_0^1 P_t(f)Y_t(f)df \tag{A.3}$$

where P_t is an aggregate price index. Note that (A.1) and (A.3) imply (A.2).

A.2 Intermediate Firms

In the intermediate goods sector each good f is produced by a single firm f using composite labour and capital with a technology:

$$Y_t(f) = F(ZK_t, ZN_t, N_t, U_tK_t) - F \tag{A.4}$$

where F are fixed costs of production and U_t allows for variable capital utilization. The parameter c is pinned down by a free-entry condition that drives profits in the steady state to zero. Given that at this stage we do not specify the form of the production function we allow for all the possible specification of technology shocks. Calling ZK_t capital-augmenting and ZN_t labour-augmenting we are in the case of Hicks neutrality if $ZK_t = ZN_t > 0$, Solow neutrality if $ZK_t > 0$ and $ZN_t = 0$ and Harrod neutrality in the case of $ZK_t = 0$ and $ZN_t > 0$.

[1] See section 2 in the paper for a discussion of similarities and differences with the model of Smets and Wouters (2007).

Intermediate firms can also control the intensity at which capital is utilized in production. As in Christiano *et al.* (2005) and Smets and Wouters (2007) we assume that using the stock of capital with intensity U_t produces a cost of $a(U_t)K_t$ units of the composite final good. The functional form is chosen consistent with the literature:

$$a(U_t) = \gamma_1(U_t - 1) + \frac{\gamma_2}{2}(U_t - 1)^2 \tag{A.5}$$

and satisfies $a(1) = 0$ and $a'(1), a''(1) > 0$. Note that $\frac{\gamma_1}{\gamma_2} = \frac{1-\phi}{\phi}$ in the Smets and Wouters (2007) set-up. In order to compare results we will estimate ϕ.

Then minimizing costs $P_t(r_{K,t}U_t - a(U_t))K_t + W_tN_t$ leads to

$$\frac{W_t}{P_t} \equiv MPL_t = MC_tF_{N,t} \tag{A.6}$$

$$r_{K,t} \equiv MPK_t = MC_tF_{K,t} \tag{A.7}$$

$$r_{K,t} = a'(U_t) \tag{A.8}$$

where MPL_t and MPK_t are the marginal products of labour and capital respectively, $r_{K,t}$ is the real cost of capital. As usual the firm's cost minimizing real marginal costs ($MC_t(f)$) is given by the Lagrange multiplier related to the production function constraint.

Pricing by the firm follows the standard Calvo framework supplemented with indexation. At each period there is a probability of $1 - \xi_p$ that the price is set optimally.[2] The optimal price derives from maximizing discounted profits. For those firms and workers unable to reset, prices are indexed to last period's aggregate inflation, with indexation parameter γ_p. With indexation parameter $\gamma_p \geq 0$, this implies that successive prices with no re-optimization are given by $P_t^0(f)$, $P_t^0(f)\left(\frac{P_t}{P_{t-1}}\right)^{\gamma_p}$, $P_t^0(f)\left(\frac{P_{t+1}}{P_{t-1}}\right)^{\gamma_p}$, ... For each intermediate producer f the objective is at time t to choose $\{P_t^O(f)\}$ to maximize discounted profits

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi_p^k \Lambda_{t,t+k} Y_{t+k}(f) \left[P_t^0(f) \left(\frac{P_{t+k-1}}{P_{t-1}} \right)^{\gamma_p} - P_{t+k}MC_{t+k} \right] \tag{A.9}$$

subject to $Y_{t+k}(f) = \left(\frac{P_t^O(f)}{P_{t+k}} \left(\frac{P_{t+k-1}}{P_{t-1}} \right)^{\gamma_p} \right)^{-\zeta} Y_{t+k}$ (from (A.1)), where $\Lambda_{t,t+k} \equiv \beta \frac{U_{C,t+k}/P_{t+k}}{U_{C,t}/P_t}$, is the nominal stochastic discount factor over the interval $[t, t+k]$ and ζ is the elasticity of substitution across intermediate goods. Since firms are atomistic, the aggregate price index and the discount factor are given in their calculations.

This leads to the following first-order condition:

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi_p^k \Lambda_{t,t+k} Y_{t+k}(f) \left[P_t^0(f) \left(\frac{P_{t+k-1}}{P_{t-1}} \right)^{\gamma_p} - MS_{p,t}P_{t+k}\mathrm{MC}_{t+k} \right] = 0 \tag{A.10}$$

[2] Thus we can interpret $\frac{1}{1-\xi_p}$ as the average duration for which prices are left unchanged.

where we introduced, as usual in the literature, a time varying mark-up of prices over marginal costs $MS_{p,t} = \frac{\zeta}{(\zeta-1)} eP_t$ with eP_t being the price mark-up shock process. Then by the law of large numbers the evolution of the price index is given by

$$P_{t+1}^{1-\zeta} = \xi_p \left(P_t \left(\frac{P_t}{P_{t-1}} \right)^{\gamma_p} \right)^{1-\zeta} + (1-\xi_p)(P_{t+1}^0(f))^{1-\zeta} \tag{A.11}$$

A.3 Labour Packer

As with final goods firms, the labour packer minimizes the cost $\int_0^1 W_t(j)N_t(j)dj$ of producing the composite labour service $N_t = \left(\int_0^1 N_t(j)^{(\mu-1)/\mu} dj \right)^{\mu/(\mu-1)}$. This leads to the standard result for the Dixit-Stigliz aggregator

$$N_t(j) = \left(\frac{W_t(j)}{W_t} \right)^{-\mu} N_t \tag{A.12}$$

$$W_t = \left[\int_0^1 W_t(j)^{1-\mu} dj \right]^{\frac{1}{1-\mu}} \tag{A.13}$$

$$W_t N_t = \int_0^1 W_t(j) N_t(j) dj \tag{A.14}$$

where W_t is an aggregate wage index. Note that (A.12) and (A.14) imply (A.13).

A.4 Trade-Unions

Wage setting by the trade-union again follows the standard Calvo framework supplemented with indexation. At each period there is a probability $1-\xi_w$ that the wage is set optimally. The optimal wage derives from maximizing discounted profits. For those trade-unions unable to reset, wages are indexed to last period's aggregate inflation, with wage indexation parameter γ_w. Then as for price contracts the wage rate trajectory with no re-optimization is given by $W_t^O(j)$, $W_t^O(j) \left(\frac{P_t}{P_{t-1}} \right)^{\gamma_w}$, $W_t^O(j) \left(\frac{P_{t+1}}{P_{t-1}} \right)^{\gamma_w}$, $\cdots$. The trade union that buys homogeneous labour at a price $W_{h,t}$ and converts it into a differentiated labour service of type j. The trade union time t then chooses $W_t^O(j)$ to maximize

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi_w^k \Lambda_{t,t+k} N_{t+k}(j) \left[W_t^O(j) \left(\frac{P_{t+k-1}}{P_{t-1}} \right)^{\gamma_w} - W_{h,t+k} \right] \tag{A.15}$$

where $N_t(j)$ is given by (A.12) so that $N_{t+k}(j) = \left(\frac{W_t^O(j)}{W_{t+k}} \left(\frac{P_{t+k-1}}{P_{t-1}} \right)^{\gamma_w} \right)^{-\mu} N_{t+k}$ and μ is the elasticity of substitution across labour varieties. By analogy with (A.10) this leads to the following first-order condition

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi_w^k \Lambda_{t,t+k} N_{t+k}(j) \Big[W_t^0(j) \left(\frac{P_{t+k-1}}{P_{t-1}} \right)^{\gamma_w} - MS_{w,t} W_{h,t+k} \Big] = 0 \tag{A.16}$$

where $MS_{w,t} = \frac{\mu}{(\mu-1)} eW_t$ is the time varying wage mark-up with eW_t being the wage mark-up shock process. Then by the law of large numbers the evolution of the wage index is given by

$$W_{t+1}^{1-\mu} = \xi_w \left(W_t \frac{\left(\frac{P_t}{P_{t-1}}\right)^{\gamma_w}}{\frac{P_{t+1}}{P_t}} \right)^{1-\mu} + (1-\xi_w)(W_{t+1}^0(j))^{1-\mu} \tag{A.17}$$

A.5 Representation of Price-Wage Dynamics as Difference Equations

We now proceed to represent the price and wage dynamics as difference equations. This is necessary to set up the model in standard software such as DYNARE and is also convenient when it comes to linearizing the model about a steady state. Both sides of the foc for pricing (A.10) and wage (A.16), are of the form considered in Appendix B. Using the Lemma, first define

$$\Pi_{p,t} \equiv \frac{P_t}{P_{t-1}} = \pi_t + 1 \tag{A.18}$$

$$\frac{P_t^0}{P_t} \equiv \frac{J_{p,t}}{H_{p,t}} \tag{A.19}$$

$$\tilde{\Pi}_{p,t}(\gamma) \equiv \frac{\Pi_{p,t}}{\Pi_{p,t-1}^{\gamma}} \tag{A.20}$$

and then aggregate inflation dynamics are given by

$$H_{p,t} - \xi_p \beta \mathbb{E}_t[\tilde{\Pi}_{p,t+1}(\gamma_p)^{\zeta-1} H_{p,t+1}] = Y_t U_{C,t} \tag{A.21}$$

$$J_{p,t} - \xi_p \beta \mathbb{E}_t[\tilde{\Pi}_{p,t+1}(\gamma_p)^{\zeta} J_{p,t+1}] = MS_{p,t} Y_t MC_t U_{C,t} \tag{A.22}$$

$$1 = \xi_p \tilde{\Pi}_{p,t}^{\zeta-1} + (1-\xi_p) \left(\frac{J_{p,t}}{H_{p,t}} \right)^{1-\zeta} \tag{A.23}$$

$$MS_{p,t} \equiv \frac{\zeta}{\zeta-1} eP_t \tag{A.24}$$

For staggered wage setting, symmetrically, wage dynamics are given by defining:

$$\Pi_{w,t} \equiv \frac{W_t}{W_{t-1}} \tag{A.25}$$

$$\tilde{\Pi}_{w,t} \equiv \frac{\Pi_{w,t}}{\Pi_{p,t-1}^{\gamma_w}} \tag{A.26}$$

$$MS_{w,t} \equiv \frac{\mu}{\mu-1} eW_t \tag{A.27}$$

In (A.24) and (A.27), eP_t and eW_t are shock processes to marginal costs and the marginal rate of substitution.

$$\frac{W_t^0}{P_t} = -\frac{\mathbb{E}_t \sum_{k=0}^{\infty} (\xi_w \beta)^k U_{N,t+k} \tilde{\Pi}^{\mu}_{w,t,t+k} N_{t+k} MS_{w,t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} (\xi_w \beta)^k U_{C,t+k} \frac{\tilde{\Pi}^{\mu}_{w,t,t+k}}{\tilde{\Pi}_{p,t,t+k}(\gamma_w)} N_{t+k}} \equiv \frac{J_{w,t}}{H_{w,t}}$$

Aggregate wage dynamics are then given by

$$H_{w,t} - \xi_w \beta \mathbb{E}_t \left[\tilde{\Pi}^{\mu}_{w,t+1} \frac{\tilde{\Pi}^{\mu}_{w,t,t+1}}{\tilde{\Pi}_{p,t,t+1}(\gamma_w)} \right] H_{w,t+1} = N_t U_{C,t} \tag{A.28}$$

$$J_{w,t} - \xi_w \beta \mathbb{E}_t \left[\tilde{\Pi}^{\mu}_{w,t+1} \right] = -MS_{w,t} N_t U_{N,t} \tag{A.29}$$

$$\begin{aligned} \left(\frac{W_t}{P_t}\right)^{1-\mu} &= \xi_w \left(\left(\frac{W_{t-1}}{P_{t-1}}\right) \frac{1}{\Pi_{p,t}} \Pi^{\gamma_w}_{p,t-1} \right)^{1-\mu} + (1-\xi_w)(\left(\frac{W_t^O(j)}{P_t}\right))^{1-\mu} \\ &= \xi_w \left(\left(\frac{W_{t-1}}{P_{t-1}}\right) \frac{1}{\tilde{\Pi}_{p,t}(\gamma_w)} \right)^{1-\mu} + (1-\xi_w)(\left(\frac{W_t^O(j)}{P_t}\right))^{1-\mu} \end{aligned} \tag{A.30}$$

The set-up is completed with

$$\left(\frac{W_t}{P_t}\right) = \frac{\Pi_{w,t}}{\Pi_{p,t}} \left(\frac{W_{t-1}}{P_{t-1}}\right) \tag{A.31}$$

A.6 Capital Producers

Capital producing firms convert I_t of output into $(1 - S(X_t))I_t$ of new capital sold at a real price Q_t. They then maximize expected discounted profits

$$\mathbb{E}_t \sum_{k=0}^{\infty} \Lambda^r_{t,t+k} \left[Q_{t+k} ZI_{t+k} (1 - S\left(I_{t+k}/I_{t+k-1}\right)) I_{t+k} - I_{t+k} \right]$$

where $\Lambda^r_{t,t+k} \equiv \beta \frac{U_{C,t+k}}{U_{C,t}}$ is the *real* stochastic discount factor over the interval $[t, t+k]$. This results in the first-order condition

$$Q_t ZI_t (1 - S(X_t) - X_t S'(X_t)) + \mathbb{E}_t \left[\Lambda^r_{t,t+1} Q_{t+1} ZI_{t+1} S'(X_{t+1}) \frac{I^2_{t+1}}{I^2_t} \right] = 1 \tag{A.32}$$

Capital accumulation is given by

$$K_{t+1} = (1-\delta) K_t + (1 - S(X_t)) I_t ZI_t; \tag{A.33}$$

where δ is the depreciation rate, ZI_t is the investment specific shock, $X_t = \frac{I_t}{I_{t-1}}$ and $S()$ satisfies S', $S'' \geq 0$; $S(1+g) = S'(1+g) = 0$.

Demand for capital by risk-neutral firms firms must satisfy the arbitrage condition

$$\mathbb{E}_t[R_{t+1}] = \frac{\mathbb{E}_t\left[r_{K,t+1}U_{t+1} + (1-\delta)Q_{t+1} - a(U_{t+1})\right]}{Q_t} \tag{A.34}$$

In (A.34) the right-hand-side is the gross return to holding a unit of capital in from t to $t+1$. The left-hand-side is the gross return from holding bonds, the opportunity cost of capital. We complete this set-up with the functional form

$$S(X) = \phi_X(X_t - (1+g))^2 \tag{A.35}$$

where g is the balanced growth rate.

A.7 The Household Problem

The Household problem is standard and can be summarized by:

$$\begin{aligned}
\text{Utility}: U_t &= U(C_t, L_t) && \text{(A.36)}\\
\text{Euler}: U_{C,t} &= \beta\mathbb{E}_t\left[R_{t+1}U_{C,t+1}\right] && \text{(A.37)}\\
\text{Labour Supply}: \frac{U_{N,t}}{U_{C,t}} &= -MRS_t \equiv -\frac{W_{h,t}}{P_t} && \text{(A.38)}\\
\text{Leisure}: L_t &\equiv 1 - N_t && \text{(A.39)}
\end{aligned}$$

A.8 Monetary Authority, Aggregation and Equilibrium

The nominal and real interest rates are related by

$$R_t = \frac{R_{n,t-1}}{\Pi_t} \tag{A.40}$$

where the nominal gross interest rate $R_{n,t}$ is a policy variable, typically given by a simple Taylor-type rule:

$$\log\left(\frac{R_{n,t}}{R_n}\right) = \alpha_R \log\left(\frac{R_{n,t-1}}{R_n}\right) + \alpha_\pi \log\left(\frac{\Pi_t}{\Pi}\right) + \alpha_Y \log\left(\frac{Y_t}{Y}\right) + \epsilon_{m,t} \tag{A.41}$$

where, we define the output gap as the deviation between the output and its steady-state value.

The output and labour market clearing conditions must take into account relative price dispersion across varieties and wage dispersion across firms. Total demand

Integrating across all firms, taking into account that the capital-labour ratio is common across firms and that the wholesale sector is separated from the retail sector we obtain aggregate demand

$$Y_t = (C_t + I_t + G_t + a(U_t)K_{t-1}) \int_0^1 \left(\frac{P_t(f)}{P_t} \right)^{-\zeta} df \tag{A.42}$$

and aggregate supply as

$$Y_t = F(Z_t, N_t^d, U_t K_{t-1}) - F \tag{A.43}$$

where labour market clearing gives total demand for labour, N_t^d, as

$$N_t = \int_0^1 N_t(j) dj = \int_0^1 \left(\frac{W_t(j)}{W_t} \right)^{-\mu} dj N_t^d = \Delta_{w,t} N_t^d \tag{A.44}$$

where the price dispersion is given by $\Delta_{p,t} = \int_0^1 \left(\frac{P_t(f)}{P_t} \right)^{-\zeta} df$ and wage dispersion is given by $\Delta_{w,t} = \int_0^1 \left(\frac{W_t(j)}{W_t} \right)^{-\mu} dj$. As shown in Appendix C:

$$\Delta_{p,t} = \xi \tilde{\Pi}_t^{\zeta} \Delta_{p,t-1} + (1-\xi) \left(\frac{P_t^O}{P_t} \right)^{-\zeta} \tag{A.45}$$

$$\Delta_{w,t} = \xi_w \tilde{\Pi}_{w,t}^{\mu} \Delta_{w,t-1} + (1-\xi_w) \left(\frac{W_t^O}{W_t} \right)^{-\mu} \tag{A.46}$$

$$\tag{A.47}$$

A.9 Shock Processes

To close the model we need to specify the law of motion of the shock processes

$$\begin{aligned}
\log ZK_t - \log ZK &= \rho_{ZK}(\log ZK_{t-1} - \log ZK) + \epsilon_{ZK,t} && \text{(A.48)}\\
\log ZN_t - \log \overline{ZN}_t &= \rho_{ZN}(\log ZN_{t-1} - \log \overline{ZN}_t) + \epsilon_{ZN,t} && \text{(A.49)}\\
\log ZI_t - \log ZI &= \rho_{ZI}(\log ZI_{t-1} - \log ZI) + \epsilon_{ZI,t} && \text{(A.50)}\\
\log G_t - \log \overline{G}_t &= \rho_G(G_{t-1} - \overline{G}_t) + \epsilon_{G,t} && \text{(A.51)}\\
\log eP_t - \log eP &= \rho_P(eP_{t-1} - eP) + \epsilon_{P,t} && \text{(A.52)}\\
\log eW_t - \log eW &= \rho_W(eW_{t-1} - eW) + \epsilon_{W,t} && \text{(A.53)}\\
\log eB_t - \log eB &= \rho_W(eB_{t-1} - eB) + \epsilon_{B,t} && \text{(A.54)}
\end{aligned}$$

In total the model has these 7 AR(1) shocks plus the shock to the monetary policy rule.

B Expressing Summations as Difference Equations

In the first order conditions for Calvo contracts and expressions for value functions we are confronted with expected discounted sums of the general form

$$\Omega_t = \mathbb{E}_t \left[\sum_{k=0}^{\infty} \beta^k X_{t,t+k} Y_{t+k} \right] \tag{B.55}$$

where $X_{t,t+k}$ has the property $X_{t,t+k} = X_{t,t+1} X_{t+1,t+k}$ (for example an inflation, interest or discount rate over the interval $[t, t+k]$).

Lemma

Ω_t can be expressed as

$$\Omega_t = X_{t,t} Y_t + \beta \mathbb{E}_t \left[X_{t,t+1} \Omega_{t+1} \right] \tag{B.56}$$

Proof

$$\begin{aligned}
\Omega_t &= X_{t,t} Y_t + \mathbb{E}_t \left[\sum_{k=1}^{\infty} \beta^k X_{t,t+k} Y_{t+k} \right] \\
&= X_{t,t} Y_t + \mathbb{E}_t \left[\sum_{k'=0}^{\infty} \beta^{k'+1} X_{t,t+k'+1} Y_{t+k'+1} \right] \\
&= X_{t,t} Y_t + \beta \mathbb{E}_t \left[\sum_{k'=0}^{\infty} \beta^{k'} X_{t,t+1} X_{t+1,t+k'+1} Y_{t+k'+1} \right] \\
&= X_{t,t} Y_t + \beta \mathbb{E}_t \left[X_{t,t+1} \Omega_{t+1} \right]
\end{aligned}$$

C Proof of Price and Wage Dispersion Results

For prices and without indexation, in the next period, ξ_p of these firms will keep their old prices, and $(1 - \xi_p)$ will change their prices to P^O_{t+1}. By the law of large numbers, we assume that the distribution of prices among those firms that do not change their prices is the same as the overall distribution in period t. It follows that we may write

$$\begin{aligned}
\Delta_{p,t+1} &= \xi_p \int_{m\ no\ change} \left(\frac{P_t(m)}{P_{t+1}} \right)^{-\zeta} + (1 - \xi_p) \left(\frac{P^0_{t+1}}{P_{t+1}} \right)^{-\zeta} \\
&= \xi_p \left(\frac{P_t}{P_{t+1}} \right)^{-\zeta} \int_{m\ no\ change} \left(\frac{P_t(m)}{P_t} \right)^{-\zeta} dm + (1 - \xi_p) \left(\frac{P^0_{t+1}}{P_{t+1}} \right)^{-\zeta} \\
&= \xi_p \left(\frac{P_t}{P_{t+1}} \right)^{-\zeta} \int_m \left(\frac{P_t(m)}{P_t} \right)^{-\zeta} dm + (1 - \xi_p) \left(\frac{P^0_{t+1}}{P_{t+1}} \right)^{-\zeta} \\
&= \xi_p \Pi^{\zeta}_{t+1} \Delta_{p,t} + (1 - \xi_p) \left(\frac{P^0_{t+1}}{P_{t+1}} \right)^{-\zeta}
\end{aligned} \tag{C.57}$$

The generalization to indexation is straightforward.

D Posterior Distribution

Parameter	Prior mean	Post. mean CD (SW07)	Post. mean CES	5% CES	95% CES	Prior	Prior s.d.
ρ_{ZN}	0.5	0.9600 (0.95)	0.9443	0.9009	0.9924	beta	0.2
ρ_{ZK}	0.5	N/A (N/A)	**0.4441**	0.1134	0.7595	beta	0.2
ρ_G	0.5	0.9509 (0.97*)	0.9631	0.9449	0.9829	beta	0.2
ρ_{ZI}	0.5	0.6785 (0.71)	0.7429	0.6232	0.8629	beta	0.2
ρ_P	0.5	0.6877 (0.89*)	**0.9744**	0.9527	0.9965	beta	0.2
ρ_W	0.5	0.9124 (0.96*)	0.9656	0.9343	0.9965	beta	0.2
ρ_B	0.5	0.3765 (N/A)	**0.9311**	0.0929	0.9623	beta	0.2
ε_{ZN}	0.1	0.7258 (0.45)	0.6833	0.5894	0.7718	invg	2.0
ε_{ZK}	0.1	N/A (N/A)	**0.0744**	0.0247	0.1356	invg	2.0
ε_G	0.5	2.1599 (0.53*)	1.9904	1.7479	2.2286	invg	2.0
ε_{ZI}	0.1	4.1634 (0.45)	**3.0647**	1.7292	4.3950	invg	2.0
ε_P	0.1	0.6097 (0.14*)	**0.3756**	0.3092	0.4386	invg	2.0
ε_W	0.1	1.1304 (0.24*)	**0.9482**	0.8004	1.0957	invg	2.0
ε_M	0.1	0.1505 (0.24)	0.1579	0.1360	0.1809	invg	2.0
ε_B	0.1	1.3639 (N/A)	1.4997	1.0848	1.9270	invg	2.0

Table D.1: Posterior Results for the Exogenous Shocks

Parameter	Prior mean	Post. mean CD (SW07)	Post. mean CES	5% CES	95% CES	Prior	Prior s.d.
σ	1	1 (1)	**0.1542**	0.0603	0.2434	gamma	1
N	0.4	0.5970 (N/A)	0.5136	0.3741	0.6509	beta	0.1
ϕ	0.5	0.8832 (0.54)	0.7860	0.7145	0.8625	beta	0.15
ϕ^X	2	2.6754 (2.87)	**1.9210**	0.8640	2.9677	norm	1.5
σ_c	1.5	2.0932 (1.38*)	**1.1571**	0.5539	1.7211	norm	0.375
χ	0.7	0.5553 (0.71*)	**0.3445**	0.2191	0.4620	beta	0.1
ξ_w	0.5	0.6016 (0.7)	**0.4577**	0.3606	0.5537	beta	0.1
ξ_p	0.5	0.7770 (0.66)	**0.5677**	0.4808	0.6484	beta	0.1
γ_w	0.5	0.4340 (0.58)	0.4489	0.2047	0.6855	beta	0.15
γ_p	0.5	0.3062 (0.24)	0.3512	0.1325	0.5577	beta	0.15
α	0.3	0.2052 (0.19)	**0.3553**	0.2721	0.4367	norm	0.05
α_π	1.5	2.2379 (2.04)	2.3771	2.0900	2.6506	norm	0.25
α_r	0.75	0.8227 (0.81)	0.7911	0.7523	0.8301	beta	0.1
α_y	0.125	0.050 (0.08)	0.0667	0.0331	0.1007	norm	0.05
conspie	0.625	0.5502 (0.78)	0.5732	0.5069	0.6371	gamma	0.1
ctrend	0.4	0.4640 (0.43)	0.4975	0.4584	0.5379	norm	0.1

Table D.2: Posteriors Results for Model Parameters

E Figures of Priors and Posteriors

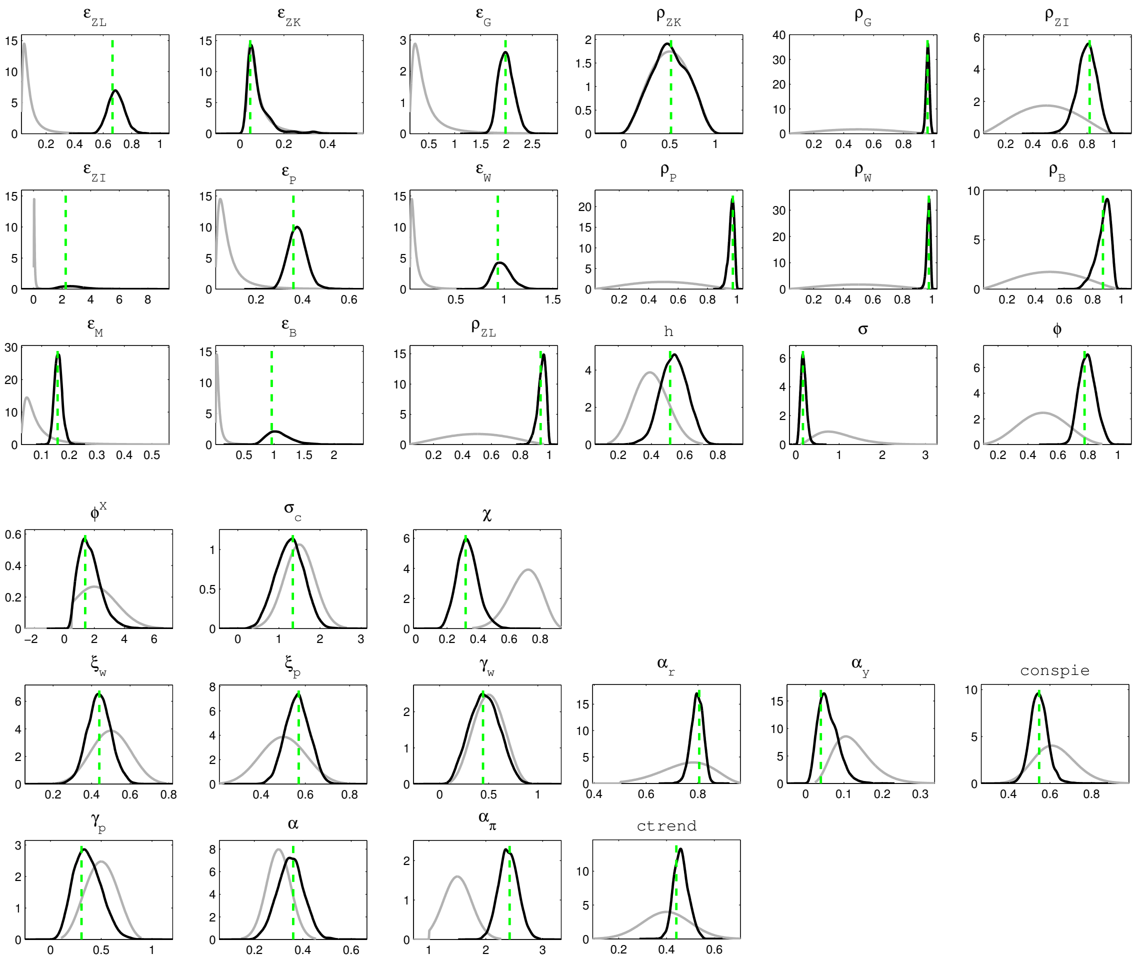

Figure E.1: Prior and Posterior Distributions (Model CES2II)

F Labour Market Parameters and CES: Identification and Sensitivity

Here we present all the details of the identification analysis presented in section 4.3 of the paper. In this section we address the following two questions: (a) How well identified are those parameters related to the labour market? (b) How sensitive are the results to the value of the capital/labour elasticity of substitution? In order to answer them we first check the identification and sensitivity of the parameters linked to the labour market, wage setting behaviour. The literature provides a wide range of estimates for these parameters hence we want to check if these parameters are identified in our model estimation and how sensible our results are to their estimated values. Second we want to check the sensitivity of our results to the 'key' parameter of our 'best' model, the elasticity of substitution between capital and labour.

F.1 Identification Analysis

We begin with testing whether the labour market parameters are identified from the model and from the data. As mentioned our results show how the CES production better captures the labour share dynamics so we focus on the three key parameters characterising the labour market and the wage setting behaviour and on the elasticity substitution of inputs in production: the labour elasticity of substitution that enters in the wage mark-up μ, wage indexation γ_w and the Calvo coefficient for wage contracts ξ_w. The identification and robustness exercise in this section consists of three parts: i) a formal identification diagnostic using the procedure suggested in Iskrev (2010b); ii) a robustness check on the choice of priors for the labour market parameters; iii) a test of significance of an alternative parameterization in the estimation for these parameters.

The procedure in Iskrev (2010b) reveals whether or not there are identification problems (i.e. flat likelihood surface), stemming from the Jacobian matrix of the mapping from the structure parameters, θ, to the moments of the observation. A condition for identification is that distinct values of θ imply distinct values of the probability density function of the data, as the latter contains all available sample information about the value of the parameter vector of interest θ. Usually, the distribution of the data is unknown or assumed to be Gaussian and estimation of θ is based on the first two moments of the data. Define $\mathbf{m_T} := [\mu', \sigma'_T]$ as the vector collecting the first and second order moments of the observable variables. Identification based on the mean and the variance of the data is only sufficient but not necessary for identification with the complete distribution, so that the mapping from the population moments of the sample, $\mathbf{m_T}$, to θ is unique. Local identification can be verified by means of a rank condition of the Jacobian matrix $J_T = \frac{\delta m_T}{\delta \theta'}$. Indeed, θ_0 is said to be locally identified if J_T has full column rank when evaluated at θ_0, although this does not guarantee that the model is locally identified everywhere in the parameter space.

Iskrev (2010b) proposes using analytical derivatives, employing implicit derivation, breaking down the mapping from θ to $\mathbf{m_T}$ in two steps: i) a transformation of θ to τ, ii) a transformation from τ to $\mathbf{m_T}$. The Jacobian can then be expressed as

$$J_T = \frac{\delta m_T}{\delta \tau'} \frac{\delta \tau'}{\delta \theta'} \tag{F.58}$$

where τ is the vector collecting all the reduced-form coefficients from the DGSE model. Note that the second term, $H_T = \frac{\delta \tau'}{\delta \theta'}$, does not depend on the data, thus implying that it is possible to detect lack of identification, inherent to the structure of the DSGE model, before taking the model to the data.

The above suggests a procedure based on Monte Carlo exploration of the parameter space Θ. The local analysis is performed in turn using the Monte Carlo realisations. This provides a 'global' exploration of point identification properties of DSGE models. One starts by constructing a sample drawn from Θ (discarding values that do not ensure stability and determinacy). This step can be guided by use of priors, specifying a theoretically admissible range and/or a particular distribution for θ. The identifiability of each draw for θ_j is then established by studying the rank of J_T and H_T, resorting to the necessary and sufficient conditions enumerated in Iskrev (2010b).[3]

We first carry out the local identification check for the whole set of the estimated parameters in the model at a chosen central tendency measure at the defined priors including setting the mean of $\mu = 7$ in a normal distribution with a large standard deviation (s.d.=1), so that the parameter lies well within the range of values reported in the RBC and labour literature. For the 'global' check we conduct the Monte Carlo exploration in the entire prior space by drawing 10000 sets of parameters to evaluate the Jacobian matrix J_T. These tests reveal that the Jacobian matrix is of full rank everywhere in the space of the prior range so all these parameters including our key parameter for the CES setting, σ, are well-identified within a theoretically admissible range defined by the prior distribution.[4]

Now we examine more carefully the identification of γ_w, ξ_w, μ and σ at the means of the prior and posterior distributions and using the prior uncertainty. Again we follow Iskrev (2010a)'s procedures and measure the identification strength, based on the information matrix, as sensitivity of the information derived from the likelihood to the parameters and collinearity between the effects of different parameters on the likelihood. The procedures are based on either the asymptotic or a moment information matrix. The first can be obtained given a sample of size T, whereas the second can be computed based on Monte Carlo simulations for samples of size T, from which sample moments of the observed variables are computed, forming a sample of N replicas of simulated moments. The corresponding information matrix is then obtained as $I_T(\theta|\mathbf{m_T}) = H_T \Sigma_{\mathbf{m_T}} H_T$, where $\Sigma_{\mathbf{m_T}}$ is the covariance matrix of simulated moments.

The 'strength' of identification can be decomposed into a 'sensitivity' and 'correlation' component. The first referring to the case when weak identification arises when the moments do not change with θ_i and the second when collinearity dampens the effect of θ_i. The former is defined as

$$\Delta_i = \sqrt{\theta_i^2 \cdot I_T(\theta)_{(i,i)}} \tag{F.59}$$

[3]See, Iskrev (2010a) and Iskrev (2010b), for more details of the procedure.

[4]To completely rule out a flat likelihood we also check collinearity between the effects of different parameters on the likelihood. If there exists an exact linear dependence between a pair and among all possible combinations their effects on the moments are not distinct and the violation of this condition must indicate a flat likelihood and lack of identification (i.e. the columns of J_T are not linearly independent). From high correlations to near-exact collinearity one may suspect some weak identification. Our collinearity results reveal the most closely related parameters with ξ_w, γ_w, μ and σ in terms of explaining the likelihood function and that some similarities with other parameters are detected but no exact-mutilcollinearity is found among all parameters. The highest pair-wise correlation is between ξ_w and h (0.8172136). For γ_w and ρ_B it is 0.4664041, μ and χ 0.7173176, and σ and ψ 0.6603774. Table F.1 below summarises the details.

which can also be normalised relative to the prior standard deviation for θ_i: $\sigma(\theta_i)$, weighting the information matrix using the prior uncertainty:

$$\Delta_i^{prior} = \sigma(\theta_i) \cdot \sqrt{I_T(\theta)_{(i,i)}} \tag{F.60}$$

It is possible to show the standard error of a parameter:

$$s.e.(\theta_i) = \frac{1}{\Delta_i} \frac{1}{\sqrt{1 - \varrho_i^2}} \tag{F.61}$$

where ϱ_i denotes collinearity between the effects of different parameters so that lack of identification and a flat likelihood may be due to either $\Delta_i = 0$ or $\varrho_i = 1$.

	Sensitivity			Collinearity			
	Prior Mean		Posterior Mean	Prior Mean		Posterior Mean	
θ_i	$\Delta_i \theta_i$	$\Delta_i^{prior} \theta_i$	$\Delta_i \theta_i$	ϱ_i	θ_j	ϱ_i	θ_j
σ	16.5929	8.2964	3.5725	0.6604	ψ	0.9376	ψ
ξ_w	59.3180	11.8636	12.2787	0.8172	N	0.9995	N
γ_w	8.3733	2.5120	0.9743	0.4664	ρ_B	0.9845	ξ_w
μ	0.7182	0.1026	0.0874	0.7173	χ	0.9260	γ_w

Table F.1: Identification at Priors and Posteriors

Table F.1 reports the sensitivity measure and collinearity results for the parameters evaluated at the prior mean, relative to the prior standard deviation and at the posterior mean obtained using the estimated benchmark CES model. Note that all three parameters are sensitive in affecting the likelihood through their effects on the moments of the observed variables. ξ_w, in the prior space or estimated, has the strongest effects and μ is the weakest on the moments among them. From the pair-wise correlation coefficients, we find that the effects of ξ_w on the observed likelihood can be approximated very closely with the effects of N – with the highest correlation detected we may suspect potential weak identification of these parameters and their similarity with other parameters in terms of empirical relevance. Although some similarities are detected it is important to confirm that no linear dependance between the columns of J_T (non-identification) is found across the estimated parameters in our model.

Second we check if the parameters are not fixed at a consistent estimate. Canova and Sala (2009) argue that imposing some of the parameters at arbitrary values may induce distortions in the distribution of parameter estimates, leading to serious biases in estimates. For example if there is some partial identification (i.e. parameters cannot be recovered separately when they enter the objective function) this may be problematic when fixing parameters. In other words, when two parameters enter the objective function proportionally, fixing one of them is the standard practice as long as the objective function has enough curvature in the dimension of the other parameter but the optimum may be shifted to the wrong value and give a very precise estimate if the fixed parameter is not at the true value (Canova and Sala (2009)). In our benchmark estimation $\mu = 7$ and the local identification check reveals some correlation between μ and χ ($\varrho_i = 0.7173176$). For this reason we carry out a robustness check on the priors and calibration of all three labour market parameters. As mentioned we give a less informative prior to μ instead of the calibrated value and loosen the standard deviations of both γ_w and ξ_w within the open unit interval (their 95% probability interval is $[0.229, 0.733]$ with a standard deviation of 0.2), which gives larger intervals from which to draw these parameters, to account for a wider range of empirical values for them. This is also a useful test for checking their identifiability in the objective function as potential under-identification can remain hidden and improper use of informative priors may increase the curvature of the likelihood surface so that the posterior distribution can be well defined as long as the joint prior distribution is proper. First the results from Iskrev (2010b)'s test show that this does not lead to any identification problems within the larger prior space. The estimates experimenting with the AI set are precise and largely unchanged compared with those reported in the paper (See Table F.2). The estimated μ is 7.0035 at the posterior optimum, in line with the value we impose in the benchmark estimation. The dynamics of labour market variables including the labour share are not altered, confirmed by various model-generated moments, their comovements with output and the estimated IRFs (not reported).

The final part of the exercise is to check whether our results with CES are sensitive to an alternative parameterization. Using the Smets and Wouters (2007) estimates of the wage dynamics parameters (in Smets and Wouters (2007), $\hat{\gamma}_w = 0.58$ and $\hat{\xi}_w = 0.7$) we re-maximise with the remaining parameters under the preferred II set. The most significant difference we find is that the log marginal likelihood is much worsened, -537.63, suggesting a substantial Bayes Factor difference (implied by $e^{13.04} \approx 4.6047e + 05$) and decisive evidence of rejecting the model. It is also interesting to compare the second moments generated from this model (Table F.3) with the earlier results in Table 6 in the paper.

F.2 Capital/Labour Elasticity of Substitution

As already noted, our estimate of the capital/labour elasticity of substitution is almost at the lower bound of the previous available estimates in the literature.[5] Hence we ask here if a higher elasticity of substitution with an alternative set of parameterization of the labour market parameters discussed in the previous section can deliver similar results of our estimated models.

To answer this we focus on the models in which the elasticity of substitution σ is estimated. In Section 3.2 we have already compared a model with σ calibrated to 0.4, following the literature as in Cantore *et al.* (2014) and Klump *et al.* (2012), and found that the model performance is in fact worse than the cases when σ is estimated. As a further test here we re-estimate the model under imperfect information, using this calibration, combined with the Smets and Wouters (2007) estimates of the labour market parameters. Can this particular parameterization with a higher elasticity of substitution make any

[5] See Klump *et al.* (2012) for a survey.

Parameter	Prior mean	Prior s.d.	Post. mean (CES2AI)
μ	**7**	**1**	7.0035 (N/A)
σ	1	1	0.1533 (0.1542)
N	0.4	0.1	0.5007 (0.5136)
ϕ	0.5	0.15	0.7875 (0.7860)
ϕ^X	2	1.5	1.9734 (1.9210)
σ_c	1.5	0.375	1.1289 (1.1571)
χ	0.7	0.1	0.3409 (0.3445)
ξ_w	0.5	**0.2**	0.4391 (0.4577)
ξ_p	0.5	0.1	0.5603 (0.5677)
γ_w	0.5	**0.2**	0.4250 (0.4489)
γ_p	0.5	0.15	0.3485 (0.3512)
α	0.3	0.05	0.3597 (0.3553)
α_π	1.5	0.25	2.3768 (2.3771)
α_r	0.75	0.1	0.7871 (0.7911)
α_y	0.125	0.05	0.0666 (0.0667)
$conspie$	0.625	0.1	0.5738 (0.5732)
$ctrend$	0.4	0.1	0.4995 (0.4975)
ρ_{ZN}	0.5	0.2	0.9479 (0.9443)
ρ_{ZK}	0.5	0.2	0.4258 (0.4441)
ρ_G	0.5	0.2	0.9631 (0.9631)
ρ_{ZI}	0.5	0.2	0.7408 (0.7429)
ρ_P	0.5	0.2	0.9778 (0.9744)
ρ_W	0.5	0.2	0.9684 (0.9656)
ρ_B	0.5	0.2	0.9309 (0.9311)
ε_{ZN}	0.1	2.0	0.6857 (0.6833)
ε_{ZK}	0.1	2.0	0.0745 (0.0744)
ε_G	0.5	2.0	1.9911 (1.9904)
ε_{ZI}	0.1	2.0	3.1137 (3.0647)
ε_P	0.1	2.0	0.3741 (0.3756)
ε_W	0.1	2.0	0.9380 (0.9482)
ε_M	0.1	2.0	0.1581 (0.1579)
ε_B	0.1	2.0	1.4931 (1.4997)
Log data density			-528.95 (-528.31)

Table F.2: Posterior Estimates Based on Alternative Priors

significant impact on our main results and on the fit of our model to the data, particularly, to the labour share dynamics? In summary, Table F.3 below reports the model-implied second moments generated from the two test models described in this section and above, respectively: both are estimated under II imposing the SW07 estimates but the former is estimating σ and the latter assumes that $\sigma = 0.4$.

As a further check of the identification of the parameters at the posterior mean we can see from the observation of the likelihood and moment comparisons from the sensitivity checks that the identification of these parameters is strong because their effects on the reduced form solutions and likelihood (through their effects on the moments) are strong and distinct. The evidence of sensitivity in the unconditional moments is revealed in Table F.3. Both models using the alternative parameterization show an improvement in matching the standard deviation of hours worked at the expense of other moments – the

	Standard Deviation							
	Output	Consumption	Investment	Wage	Inflation	Interest rate	Hours	Labour share
Data	0.58	0.53	1.74	0.66	0.24	0.61	2.47	2.07
	(0.50,0.69)	(0.46,0.62)	(1.54,2.01)	(0.57,0.82)	(0.21,0.27)	(0.55,0.70)	(2.09,2.94)	(1.81,2.37)
CES2II	0.69	0.66	1.80	0.72	0.36	0.50	4.09	2.59
CES2II (SW)	0.74	0.69	1.81	0.75	0.48	0.54	3.12	3.15
CES2II (SW, $\sigma = 0.4$)	0.77	0.72	1.91	0.78	0.38	0.40	3.39	1.27
	Cross-correlation with Output							
Data	1.00	0.61	0.64	-0.11	-0.12	0.22	-0.25	-0.05
	(-)	(0.46,0.74)	(0.47,0.75)	(-0.40,0.10)	(-0.31,0.10)	(0.02,0.39)	(-0.48,-0.00)	(-0.26,0.16)
CES2II	1.00	0.47	0.63	0.15	-0.06	-0.15	0.06	-0.28
CES2II (SW)	1.00	0.55	0.69	-0.02	-0.09	-0.12	0.07	-0.27
CES2II (SW, $\sigma = 0.4$)	1.00	0.63	0.65	0.25	-0.15	-0.26	0.10	-0.31
	Autocorrelation (Order=1)							
Data	0.28	0.17	0.56	0.17	0.54	0.96	0.93	0.90
	(0.19,0.37)	(0.07,0.27)	(0.46,0.66)	(0.07,0.26)	(0.45,0.64)	(0.87,1.00)	(0.84,1.00)	(0.81,1.00)
CES2II	0.28	0.33	0.63	0.36	0.66	0.91	0.98	0.98
CES2II (SW)	0.34	0.34	0.54	0.53	0.82	0.94	0.97	0.98
CES2II (SW, $\sigma = 0.4$)	0.37	0.42	0.61	0.54	0.68	0.90	0.96	0.97

Table F.3: Second Moment Comparison with Alternative Parameterization

ability of matching almost all other moments in this dimension is distorted, generating much volatility than the data. The performance is even worse when $\sigma = 0.4$ is imposed and in this case the volatility of labour share is under-estimated, lying well below the lower bound of the confidence bands. These findings are in line with the estimated moments of cross-correlation and first-order autocorrelation, except for the comovement of wage with output – the cross-correlation has the correct sign and fits the data better.

From the identification analysis and various sensitivity tests we conclude that, overall, our selected CES model is identified, has a robust set of parameter calibration and prior selection, and provides consistent and plausible estimates under both informational assumptions. The results and analysis we show in this section suggest that the estimation precision of the NK labour market parameters and the elasticity in the CES production should not be a reason of concern given the data.

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