A proof of the Bloch theorem for lattice models

Haruki Watanabe[1, *]

[1]Department of Applied Physics, University of Tokyo, Tokyo 113-8656, Japan

The Bloch theorem is a powerful theorem stating the absence of nonzero expectation value of the averaged current operator associated with any conserved U(1) charge in a large quantum system. The theorem applies to the ground state and to the thermal equilibrium at a finite temperature. This work presents a simple yet rigorous proof for general lattice models. Our discussion clarifies the relation to the twist operator widely used in the context of the Lieb-Schultz-Mattis theorem.

I. INTRODUCTION

The Bloch theorem [1] states that the equilibrium state of a thermodynamically large system, in general, does not support non-vanishing expectation value of the averaged current density of any conserved U(1) charge, regardless of the details of the Hamiltonian such as the form of interactions or the size of the excitation gap. Despite its wide applications, the proof of the theorem in the existing literature is mostly for specific continuum models [1–3]. There are also proofs for lattice models [4–6] but the setting considered in these works are not fully general. For example, Ref. [4] is for a concrete spin model with a translation symmetry and the assumption of their discussion is unclear. Ref. [5] assumes an extended-Hubbard type Hamiltonian and their definition of the current operator heavily relies on the specific form of the Kinetic term. Finally, Ref. [6] assumes the uniqueness of the ground state with non-vanishing excitation gap.

In this work, we present a proof for general models defined on a one-dimensional lattice. We clarify assumptions and the statement of the theorem in Sec. II. It followed by a fully general yet concise proof in Sec. III. The argument will then be extended to higher dimensions in Sec. IV.

II. SETUP AND STATEMENT

Let us consider a quantum many-body system defined on a one dimensional lattice. We impose the periodic boundary condition with system size L. The Hamiltonian $\hat{H}$ of the system can be very general. It may contain arbitrary hopping matrices and interactions as far as each term in the Hamiltonian is short-ranged (i.e., the size of its support is finite and does not scale with L) and respects the U(1) symmetry we discuss shortly. In particular, we *do not* put any restriction on the translation symmetry, the ground state degeneracy, or the excitation gap. To simplify the notation we set the lattice constant to be 1 and denote lattice sites by $x \in \{1, 2, \cdots, L\}$.

We assume that the Hamiltonian $\hat{H}$ commutes with the particle number operator

$$\hat{N} = \sum_{x=1}^{L} \hat{n}_x. \tag{1}$$

Here, $\hat{n}_x$ is the local charge density operator at site x. We assume that density operators at different sites commute, $[\hat{n}_x, \hat{n}_{x'}] = 0$. The U(1) symmetry implies the conservation law:

$$i[\hat{H}, \hat{n}_x] + \hat{j}_{x+\frac{1}{2}} - \hat{j}_{x-\frac{1}{2}} = 0, \tag{2}$$

where $\hat{j}_{x+\frac{1}{2}}$ is the local U(1) current operator that measures the net charge transfer across the 'seam' in between x and $x+1$ [Fig. 1 (a)]. We present the precise definition of $\hat{j}_{x+\frac{1}{2}}$ in Sec. III C.

With this setting, the Bloch theorem states that the ground state expectation value of the local current operator vanishes in the limit of the large system size:

$$\lim_{L\to\infty} \langle \mathrm{GS}|\hat{j}_{x+\frac{1}{2}}|\mathrm{GS}\rangle = 0. \tag{3}$$

Here $|\mathrm{GS}\rangle$ is the ground state of $\hat{H}$ with the energy eigenvalue E_{GS}. When there is a ground state degeneracy we arbitrary pick one state. The current conservation law in Eq. (2), together with $\hat{H}|\mathrm{GS}\rangle = E_{\mathrm{GS}}|\mathrm{GS}\rangle$, implies

$$\langle \mathrm{GS}|\hat{j}_{x+\frac{1}{2}}|\mathrm{GS}\rangle = \langle \mathrm{GS}|\hat{j}_{x-\frac{1}{2}}|\mathrm{GS}\rangle \tag{4}$$

for all $x = 1, 2, \cdots, L$, meaning that the expectation value is independent of the position. Therefore, we can equally states the Bloch theorem in terms of the averaged current operator

$$\hat{\bar{j}} \equiv \frac{1}{L}\sum_{x=1}^{L} \hat{j}_{x+\frac{1}{2}}, \tag{5}$$

$$\lim_{L\to\infty} \langle \mathrm{GS}|\hat{\bar{j}}|\mathrm{GS}\rangle = 0. \tag{6}$$

This statement can be directly generalized to a finite temperature $T > 0$ [2, 5] described by the Gibbs state (we set $k_B = 1$):

$$\hat{\rho}_0 \equiv \frac{1}{Z} e^{-\hat{H}/T}, \quad Z \equiv \mathrm{tr}\big(e^{-\hat{H}/T}\big), \tag{7}$$

$$\lim_{L\to\infty} \mathrm{tr}\big(\hat{\rho}_0 \hat{j}_{x+\frac{1}{2}}\big) = \lim_{L\to\infty} \mathrm{tr}\big(\hat{\rho}_0 \hat{\bar{j}}\big) = 0. \tag{8}$$

III. PROOF OF THE BLOCH THEOREM

A. Variational principle

Our proof of the theorem makes use of the twist operator introduced by Ref. [7], which reads

$$\hat{U}_m \equiv e^{\frac{2\pi i m}{L}\sum_{x=1}^{L} x\hat{n}_x}, \quad m \in \mathbb{Z}. \tag{9}$$

* haruki.watanabe@ap.t.u-tokyo.ac.jp

This unitary operator is consistent with the periodic boundary condition since replacing x with $x+L$ in the exponent does not affect $\hat{U}_m$ as $e^{2\pi i m \hat{N}} = 1$. The key observation of the proof is the following Taylor expansion in the power series of L^{-1}, which we show in Sec. III C:

$$\hat{U}_m^\dagger \hat{H} \hat{U}_m = \hat{H} + 2\pi m \hat{\bar{j}} + O(L^{-1}). \tag{10}$$

Taking the ground state expectation value of this equation, we find the following relation for the energy expectation value of the variational state $|\Phi_m\rangle \equiv \hat{U}_m|\mathrm{GS}\rangle$:

$$\langle \Phi_m|\hat{H}|\Phi_m\rangle = E_{\mathrm{GS}} + 2\pi m \langle \mathrm{GS}|\hat{\bar{j}}|\mathrm{GS}\rangle + O(L^{-1}). \tag{11}$$

Suppose first that $\langle \mathrm{GS}|\hat{\bar{j}}|\mathrm{GS}\rangle > 0$. Then we find that $\langle \Phi_m|\hat{H}|\Phi_m\rangle$ with $m<0$ is lower than the ground state energy for a large L, which contradicts with the variational principle. If $\langle \mathrm{GS}|\hat{\bar{j}}|\mathrm{GS}\rangle < 0$, $|\Phi_m\rangle$ with $m>0$ does the same job. Hence, $\langle \mathrm{GS}|\hat{\bar{j}}|\mathrm{GS}\rangle$ cannot remain nonzero as $L\to\infty$ and must be smaller than or equal to L^{-1}. This variational argument is common among the majority of proofs in the literature [1–4].

B. Finite temperature

The proof of the Bloch theorem for a finite temperature is almost identical to that for the ground state. Given the Hamiltonian $\hat{H}$ and a density operator $\hat{\rho}$, in general, the free energy at $T>0$ is given by

$$F(\hat{\rho}) = \mathrm{tr}\big(\hat{\rho}\hat{H} + T\hat{\rho}\ln\hat{\rho}\big). \tag{12}$$

This is minimized by the Gibbs state in Eq. (7) with the minimum value $F(\hat{\rho}_0) = -k_B T \ln Z$ [8, 9]. Using Eq. (10), we find

$$\begin{aligned} F(\hat{U}_m\hat{\rho}_0\hat{U}_m^\dagger) &= \mathrm{tr}\left[\hat{\rho}_0(\hat{U}_m^\dagger\hat{H}\hat{U}_m) + T\hat{\rho}_0\ln\hat{\rho}_0\right] \\ &= F(\hat{\rho}_0) - 2\pi m\,\mathrm{tr}\big(\hat{\rho}_0\hat{\bar{j}}\big) + O(L^{-1}). \end{aligned} \tag{13}$$

If the magnitude of the current expectation value is bigger than $O(L^{-1})$, we get $F(\hat{\rho}_m) < F(\hat{\rho}_0)$ for either $m=\pm1$, which is a contradiction.

C. Local current operator

It remains to verify Eq. (10). This requires a precise formulation of the local current operator. To this end, let us temporary introduce the *twisted* boundary condition. We place the position of the 'seam' to be somewhere in between x and $x+1$, which we denote by $\bar{x} \equiv x + \frac{1}{2}$ [Fig. 1 (a)]. Let $\theta_{\bar{x}}$ be the angle of the twist. Later we will set $\theta_{\bar{x}}=0$, as, after all, we are interested in the original system under the periodic boundary condition.

The Hamiltonian $\hat{H}^{\theta_{\bar{x}}}$ under the twisted boundary condition has $\theta_{\bar{x}}$-dependence localized around the seam. This is because every term in the original Hamiltonian $\hat{H}$ that goes across the seam acquires a phase $e^{i\ell_{\bar{x}}\theta_{\bar{x}}}$. For example, the hopping term $tc_{x+1}^\dagger c_x + t^* c_x^\dagger c_{x+1}$ becomes $te^{-i\theta_{\bar{x}}}c_{x+1}^\dagger c_x + t^* e^{i\theta_{\bar{x}}} c_x^\dagger c_{x+1}$. More generally, a term in $\hat{H}$ is multiplied by the factor $e^{i(n_{\bar{x}}^a - n_{\bar{x}}^c)\theta_{\bar{x}}}$ where $n_{\bar{x}}^a$ ($n_{\bar{x}}^c$) is the number of annihilation (creation) operators in the right side of the seam in the term. Other terms in $\hat{H}$ that reside either one side of the seam remain unchanged. The local current operator across the seam under the *periodic* boundary condition is given by

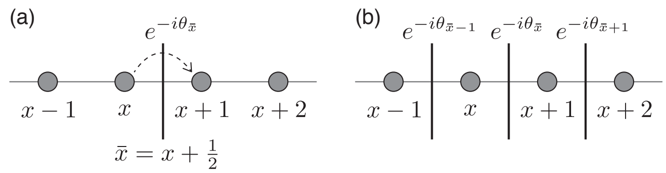

FIG. 1. (a) Twisted boundary condition with the U(1) phase $e^{-i\theta_{\bar{x}}}$ at the seam $\bar{x} = x + \frac{1}{2}$. (b) Generalized twisted boundary condition with a seam at every $\bar{x}$ for $x = 1, 2, \cdots, L$.

$$\hat{j}_{\bar{x}} \equiv \partial_{\theta_{\bar{x}}}\hat{H}^{\theta_{\bar{x}}}\big|_{\theta_{\bar{x}}=0}. \tag{14}$$

The current operator defined this way satisfies the conservation law in Eq. (2). To see this explicitly, let us introduce a seam for every $\bar{x} = x + \frac{1}{2}$ ($x = 1, 2, \cdots, L$) and denote the twisted Hamiltonian by $\hat{H}^{(\theta_{\bar{1}},\theta_{\bar{2}},\cdots,\theta_{\bar{L}})}$ [Fig. 1 (b)]. It satisfies

$$\hat{H} = \hat{H}^{(\theta_{\bar{1}},\theta_{\bar{2}},\cdots,\theta_{\bar{L}})}\big|_{\theta_{\bar{1}}=\theta_{\bar{2}}=\cdots=\theta_{\bar{L}}=0}, \tag{15}$$

$$\hat{j}_{\bar{x}} = \partial_{\theta_{\bar{x}}}\hat{H}^{(\theta_{\bar{1}},\theta_{\bar{2}},\cdots,\theta_{\bar{L}})}\big|_{\theta_{\bar{1}}=\theta_{\bar{2}}=\cdots=\theta_{\bar{L}}=0} \tag{16}$$

and

$$\begin{aligned} &e^{i\epsilon\hat{n}_x}\hat{H}^{(\theta_{\bar{1}},\theta_{\bar{2}},\cdots,\theta_{\bar{L}})}e^{-i\epsilon\hat{n}_x} \\ &= \hat{H}^{(\theta_{\bar{1}},\cdots,\theta_{\bar{x}-2},\theta_{\bar{x}-1}-\epsilon,\theta_{\bar{x}}+\epsilon,\theta_{\bar{x}+1},\cdots,\theta_{\bar{L}})}. \end{aligned} \tag{17}$$

This relation implies that $\theta_{\bar{x}}$ can be identified with the background U(1) gauge field $A_{\bar{x}}$. When Eq. (17) for $\theta_{\bar{1}} = \theta_{\bar{2}} = \cdots = \theta_{\bar{L}} = 0$ is expanded in the power series of ϵ, the $O(\epsilon)$-term reproduces the conservation law (2). It also follows by using Eq. (17) repeatedly that

$$\hat{U}_m^\dagger\hat{H}\hat{U}_m = \hat{H}^{(\frac{2\pi m}{L},\cdots,\frac{2\pi m}{L})}. \tag{18}$$

The Taylor series of the right-hand side reads

$$\hat{H}^{(\frac{2\pi m}{L},\cdots,\frac{2\pi m}{L})} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2\pi m}{L}\right)^n\hat{H}^{(n)}, \tag{19}$$

where $\hat{H}^{(\ell)}$ ($\ell = 0, 1, 2, \cdots$) is defined by

$$\sum_{x_1,x_2,\cdots,x_\ell=1}^{L}\partial_{\theta_{\bar{x}_1}}\partial_{\theta_{\bar{x}_2}}\cdots\partial_{\theta_{\bar{x}_\ell}}\hat{H}^{(\theta_{\bar{1}},\cdots,\theta_{\bar{L}})}\Big|_{\theta_{\bar{1}}=\cdots=\theta_{\bar{L}}=0} \tag{20}$$

For example, $\hat{H}^{(0)} = \hat{H}$ and

$$\hat{H}^{(1)} = \sum_{x=1}^{L}\partial_{\theta_{\bar{x}}}\hat{H}^{(\theta_{\bar{1}},\cdots,\theta_{\bar{L}})}\Big|_{\theta_{\bar{1}}=\cdots=\theta_{\bar{L}}=0} = L\hat{\bar{j}}. \tag{21}$$

For short-ranged Hamiltonians, each $\hat{H}^{(\ell)}$ is at most the order of L at least for $\ell = O(L^0)$. Eqs. (18)–(21) altogether verify Eq. (10) and the proof is completed.

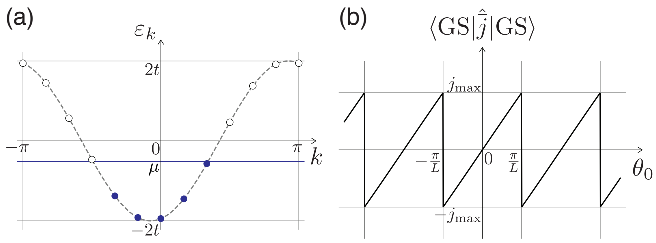

FIG. 2. (a) The band structure of the tight-binding model for $L = 12$ and $\theta_0 = 0.9\pi/L$. Blue (white) dots represents occupied (unoccupied) states at $\mu = -2t\cos(\frac{\pi N}{L})$ with $N = 5$. (b) The current expectation value of the ground state as a function of θ_0 for $L = 12$ and $N = 5$.

IV. DISCUSSIONS

A. The Lieb-Schultz-Mattis theorem

The conclusion in Sec. III A immediately implies that the variational state $|\Phi_m\rangle = \hat{U}_m|\mathrm{GS}\rangle$ is a low-energy state whose excitation energy $|\langle\Phi_m|\hat{H}|\Phi_m\rangle - E_{\mathrm{GS}}|$ is bounded by $O(L^{-1})$. Further assuming the translation symmetry $\hat{T}_1$ with $\hat{T}_1\hat{n}_x\hat{T}_1^\dagger = \hat{n}_{x+1}$, we find [7, 10, 11]

$$\hat{T}_1\hat{U}_m\hat{T}_1^\dagger = \hat{U}_m e^{-2\pi m i\frac{\hat{N}}{L}}. \tag{22}$$

Suppose that the ground state $|\mathrm{GS}\rangle$ is an eigenstate of $\hat{T}_1$ and $\hat{N}$ and that the filling fraction $\nu \equiv \langle\mathrm{GS}|\hat{N}|\mathrm{GS}\rangle/L$ is not an integer. Then the variational state $|\Phi_m\rangle$ and the ground state $|\mathrm{GS}\rangle$ have inequivalent eigenvalues of $\hat{T}_1$ and hence are orthogonal to each other. This implies the Lieb-Schultz-Mattis theorem for translation invariant one-dimensional systems which suggests the presence of either a ground state degeneracy or a gapless excitation when $\nu \notin \mathbb{Z}$ [7, 10, 11]. Note that we did not assume any additional symmetry such as the spatial inversion or the time-reversal symmetry unlike the original argument [7, 10, 11]. The same conclusion was derived in Ref. [12] from an alternative argument.

B. Persistent current in a finite system

The Bloch theorem allows a persistent current of the order $O(L^{-1})$ in a finite system. For a later purpose, let us consider a concrete tight-binding model with the nearest neighbor hopping $t > 0$.

$$\hat{H} = -te^{-i\theta_0}\sum_{x=1}^{L}\hat{c}_{x+1}^\dagger\hat{c}_x + \text{h.c.} \tag{23}$$

We introduced a phase $e^{-i\theta_0}$ to break the time-reversal symmetry. Introducing the Fourier transform $\hat{c}_k^\dagger = \frac{1}{\sqrt{L}}\sum_{x=1}^{L}\hat{c}_x^\dagger e^{ikx}$ for $k = \frac{2\pi q}{L}$ ($q = 1, 2, \cdots L$), we find [13]

$$\hat{H} = \sum_k \varepsilon_k\hat{c}_k^\dagger\hat{c}_k, \quad \hat{\bar{j}} = \frac{1}{L}\sum_k \partial_k\varepsilon_k\hat{c}_k^\dagger\hat{c}_k \tag{24}$$

with $\varepsilon_k = -2t\cos(k + \theta_0)$ [Fig. 2 (a)].

For concreteness, let us set the Fermi energy to be $\mu = -2t\cos(\frac{\pi N}{L})$ for an odd number of particles N, and consider the Fermi sea $|\mathrm{GS}\rangle = \prod_{k,\varepsilon_k<\mu} c_k^\dagger|0\rangle$ [Fig. 2 (a)]. The current expectation value $\langle\mathrm{GS}|\hat{\bar{j}}|\mathrm{GS}\rangle = \frac{1}{L}\sum_{k,\varepsilon_k<\mu}\partial_k\varepsilon_k$ shows the periodicity in θ_0 with the period $2\pi/L$ [Fig. 2 (b)]. Its maximum value is given by

$$j_{\max} \equiv \lim_{\theta_0\uparrow\frac{\pi}{L}}\langle\mathrm{GS}|\hat{\bar{j}}|\mathrm{GS}\rangle = \tfrac{2t}{L}\sin(\tfrac{\pi N}{L}) = O(L^{-1}). \tag{25}$$

These results are consistent with the previous studies, for example, in Ref. [13].

C. Continuum models

The generalization of the Bloch theorem to continuum models can be done simply by replacing $\sum_{x=1}^{L}$ with $\int_0^L dx$, for example. For continuum models, the Noether theorem provides the definition of the conserved U(1) current. The key relation Eq. (10) remains unchanged.

D. Long-range interactions

The assumption on the range of hopping matrices and interactions can be slightly relaxed. They are not necessarily strictly finite and exponentially decaying interactions, for example, are allowed. However, when the Hamiltonian contains long-range interactions, the order estimate of the series expansion in Eq. (10) would be spoiled and the theorem might be violated.

E. Higher dimensions

Models in higher dimensions can be reduced to one dimension either by compactifying all other directions by the periodic boundary condition or by taking a finite-width strip with the open boundary conditions (we still impose the periodic boundary condition in x). All quantities then contain an additional summation over transverse directions.

For example, in two dimensions, we have

$$\partial_{\theta_{\bar{x}}}\hat{H}^{\theta_{\bar{x}}}\big|_{\theta_{\bar{x}}=0} = \sum_{y=1}^{L_y}\hat{j}^x_{(\bar{x},y)}. \tag{26}$$

Correspondingly, Eqs. (5), (9) and (10) become [Fig. 3 (a)]

$$\hat{\bar{j}}^x \equiv \frac{1}{L_xL_y}\sum_{x=1}^{L_x}\sum_{y=1}^{L_y}\hat{j}^x_{(\bar{x},y)}, \tag{27}$$

$$\hat{U}_m \equiv e^{\frac{2\pi im}{L_x}\sum_{x=1}^{L_x}\sum_{y=1}^{L_y}x\hat{n}_{(x,y)}}, \tag{28}$$

$$\hat{U}_m^\dagger\hat{H}\hat{U}_m = \hat{H} + 2\pi mL_y\hat{\bar{j}}^x + O(L_x^{-1}L_y). \tag{29}$$

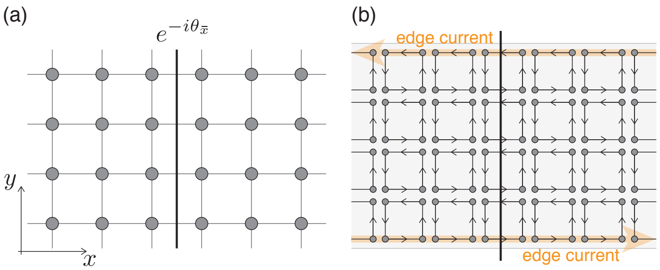

FIG. 3. (a) Two dimensional counterpart of Fig. 1. (b) Example of an insulator with a nonzero edge current.

In this case, the Bloch theorem states that the expectation value of the averaged current density vanishes in the ground state or in the thermal equilibrium at $T > 0$:

$$\lim_{L_x\to\infty}\langle\mathrm{GS}|\hat{\bar{j}}^x|\mathrm{GS}\rangle = \lim_{L_x\to\infty}\mathrm{tr}\big(\hat{\rho}_0\hat{\bar{j}}^x\big) = 0. \tag{30}$$

It is important to note that both

- the total current integrated over transverse directions ($L_y\hat{\bar{j}}^x$ in two dimensions)
- the local current density ($\hat{j}^x_{(\bar{x},y)}$ in two dimensions)

may have a non-vanishing expectation value in the limit of large L_x. An example of the former is given by the L_y copies of decoupled 1D chains described by the tight-binding model in Sec. IV B. As each chain supports a persistent current of $O(L_x^{-1})$, we find

$$L_y j^x_{\max} = \tfrac{L_y}{L_x}2t\sin\big(\tfrac{\pi N}{L_xL_y}\big) = O(L_y/L_x). \tag{31}$$

We discuss a simple example of the latter below.

F. Edge current

The multi-dimensional Bloch theorem allows, for example, a nonzero edge current that flows in the opposite direction at the two edges as illustrated in Fig. 3 (b). As a concrete example, let us consider a two-dimensional periodic array of decoupled 1D rings and impose the open boundary condition in y [Fig. 3 (b)]. Each 1D ring is formed by the tight-binding model considered in Sec. IV B with $L = 4$ and $N = 1$. Every ring supports a loop current

$$j_{\text{loop}} = \tfrac{t}{2}\sin\phi_0, \quad \phi_0 \in (-\tfrac{\pi}{4}, \tfrac{\pi}{4}). \tag{32}$$

This is an $O(1)$ quantity, independent of L_x or L_y. In the bulk, contributions from neighboring loops cancel and the local current density may vanish. However, at the edge, there is a residual contribution that flows along the edge as illustrated in Fig. 3 (b) implying the nonzero expectation value of $\hat{j}^x_{(\bar{x},y)}$ at the edge. This is nothing but the magnetization current $\vec{\nabla}\times\vec{m}$ originating from the orbital magnetization $\vec{m}$ produced by the loop currents [14].

Of course, a similar situation occurs for Chern insulators but our model is cleaner as gapless chiral edge modes are absent.

G. Other current densities

The argument in this work coherently applies to any conserved current density associated with an internal U(1) symmetry. For example, when the z-component of the total spin is conserved in a spin model with spin S $(= 1/2, 1, 3/2. \cdots)$ on each site, we can set $\hat{n}_x = \hat{S}^z_x - S$ [15] in our discussion above to prove the absence of the equilibrium spin current. However, our argument is not applicable, for example, to the energy current density as there does not exist the corresponding twist operator. Recently, a completely new argument for the energy current has been developed in Ref. [16].

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