Jordan-Wigner Transformation for Quantum-Spin Systems in Two Dimensions and Fractional Statistics

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(Received 3 January 1989)

I construct a Jordan-Wigner transformation for spin-one-half quantum systems on two-dimensional lattices. I show that the spin-one-half *XY* model (i.e., a hard-core Bose system) is equivalent (on any two-dimensional Bravais lattice) to a system of spinless fermions and gauge fields satisfying the constraint that the gauge flux on a plaquette must be proportional to the spin (particle) density on site. The constraint is enforced by the addition of a Chern-Simons term of strength θ to the Lagrangian of the theory. For the particular value $\theta = 1/2\pi$, the resulting particles are fermions. In general they are anyons. The implications of these results for quantum spin liquids are briefly discussed.

PACS numbers: 75.10.Jm, 05.30.Jp, 05.50.+q, 11.10.−z

Fermi-Bose mappings have a long history in theoretical physics. They are an essential tool for the study of one-dimensional quantum systems, both on lattices and in the continuum. Perhaps the most famous application is Onsager's solution to the two-dimensional classical Ising model in its transfer-matrix form.[1] Schultz, Mattis, and Lieb[2] showed that a Jordan-Wigner transformation[3] mapped the transfer matrix into a spinless-fermion problem. They[4] showed that such a transformation could be used to solve the one-dimensional spin-one-half *XY* model.

Much effort has been devoted to generalizing this procedure to higher dimensions. It was soon realized that naive extensions of the one-dimensional mapping would generally lead to highly nonlocal interactions. Fradkin, Srednicki, and Susskind[5] and Srednicki[6] found a transformation for the (2+1)-dimensional Ising gauge theory (dual to the 2+1 quantum Ising model) and showed that it could be mapped onto a theory of Fermi fields on the links of the lattice with a plaquette four-fermion interaction.

More recent work on fractional statistics in (2+1)-dimensional relativistic field theories,[7,8] and Laughlin's[9] work on the fractional quantum Hall effect (FQHE) suggest new options. Wilczek and Zee[7] and Wu and Zee[8] have shown that, at least semiclassically, nonlinear σ models with Hopf terms have soliton states (skyrmions) which may exhibit fractional statistics. Laughlin[9] and Halperin[10] have stressed that the elementary excitations of the FQHE state also have fractional statistics. In both cases, the construction of these "anyons" closely resembles the "parafermion" operators (charge plus flux) of two-dimensional classical statistical mechanics introduced by Fradkin and Kadanoff[11] a few years earlier in the context of clock models. However, these constructions were limited by their inherent semiclassical character or by being tied to a particular *Ansatz* for the ground-state wave function.

More recently it has become apparent that Fermi-Bose correspondences in two dimensions are closely related to the properties of Chern-Simons (CS) terms in the underlying Lagrangians. Semenoff[12] has recently introduced a Bose-anyon mapping for a (2+1)-dimensional relativistic scalar field theory minimally coupled to a gauge field with only a CS term. Zhang, Hansson, and Kivelson[13] have used these ideas to develop a rather simple semiclassical approach to the FQHE which yields most of Laughlin's results. However, continuum theories have a number of difficulties associated with their ultraviolet behavior and regularization schemes. The resolution of these problems requires a careful treatment of anomalies which play a central role in both 1+1 dimensions[14] and in 2+1 dimensions.[15] This is particularly pressing in view of recent work by Witten,[16] who has uncovered a number of remarkable connections between non-Abelian Chern-Simons theories, the theory of knots, their Abelian counterparts, and (1+1)-dimensional conformal field theories.

In this paper I discuss a Jordan-Wigner transformation for two-dimensional spin-one-half systems (i.e., hard-core bosons) on a lattice. I show that it can be mapped exactly, as an operator identity, into a system of spinless fermions on the same lattice minimally coupled to an Abelian gauge field with a CS term. This term imposes a constraint between charge and flux. For arbitrary values of the CS coupling, the fermions turn into anyons. In one dimension the mapping is possible because there is a natural ordering of particles on a straight line. In two dimensions the mapping becomes possible only if fluxes, (i.e., branch cuts) are attached to the particles. Consider a system of spinless fermions $a(x)$ on the sites of a two-dimensional Bravais lattice (square for simplicity, but it works for any lattice) and gauge field $A_j(x)$ on the links of the same lattice. The Hamiltonian of the system is

$$H_F = t \sum_{\mathbf{x}, j=1,2} [a^\dagger(\mathbf{x}) e^{iA_j(\mathbf{x})} a(\mathbf{x}+\hat{\mathbf{e}}_j) + \text{H.c.}] \,, \qquad (1)$$

with the constraint

$$j_0(\mathbf{x}) - \theta B(\mathbf{r}) = 0 \,, \qquad (2)$$

where the fermion density is

$$j_0(\mathbf{x}) = a^\dagger(\mathbf{x}) a(\mathbf{x})$$

and the flux $B(\mathbf{r})$ is given by

$$B(\mathbf{r}) = \epsilon_{ij} \Delta_i A_j(x) , \tag{3a}$$

$$\Delta_i A_j(x) \equiv A_j(\mathbf{x}+\hat{\mathbf{e}}_i) - A_j(\mathbf{x}) . \tag{3b}$$

The parameter θ will remain arbitrary for the moment. The constraint implies that if there is a fermion at $\mathbf{x}$ there is a flux equal to $1/\theta$ at the adjoining dual site $\mathbf{r}$.

Consider the Lagrangian for spinless fermions on a lattice with a CS term,

$$L = \sum_{\mathbf{x}} a^\dagger(\mathbf{x}) i D_0 a(x) - \sum_{\mathbf{x}, j=1,2} a^\dagger(\mathbf{x}) e^{iA_j(\mathbf{x})} a(\mathbf{x}+\hat{\mathbf{e}}_j) + \text{H.c.} - \frac{\theta}{4} \sum_{\mathbf{x}} \epsilon_{\mu\nu\lambda} A^\mu(\mathbf{x}) F^{\nu\lambda}(\mathbf{x}) , \tag{4}$$

where $\mu,\nu,\lambda = 0,1,2$ $(x_0 \equiv t)$. The covariant time derivative D_0 and the field strengths $F^{\mu\nu}$ are given by

$$D_0 = \partial_0 - iA_0 , \tag{5a}$$

$$F_{ij} = \Delta_i A_j(\mathbf{x}) - \Delta_j A_i(\mathbf{x}) , \tag{5b}$$

$$F_{0i} = \partial_0 A_i(\mathbf{x}) - \Delta_i A_0(\mathbf{x}) . \tag{5c}$$

Let us discuss the canonical quantization of (4). As usual the field A_0 remains classical since the canonical momentum $\Pi_0 = 0$ and A_0 is a Lagrange multiplier field. Thus, it is natural to discuss quantization in the gauge $A_0 = 0$ and to impose the classical equation $\delta L / \delta A_0 = 0$ as a constraint which defines the space of physical states, i.e., Gauss' law.[17] The generator of time-independent gauge transformations $\hat{Q}(x)$, given by

$$\hat{Q}(x) = \frac{\delta L}{\delta A_0(x)} = j_0(\mathbf{x}) - \theta \epsilon_{ij} \Delta_i A_j(\mathbf{x}) \tag{6}$$

commutes with the Hamiltonian (4) and annihilates the physical states. In the gauge $A_0 = 0$, the equal-time canonical (anti)commutation relations are

$$\{a^\dagger(\mathbf{x}), a(\mathbf{y})\} = \delta_{\mathbf{x},\mathbf{y}} \tag{7a}$$

and

$$[A_j(\mathbf{x}), A_k(\mathbf{y})] = i(2/\theta) \epsilon_{jk} \delta_{\mathbf{x},\mathbf{y}} \tag{7b}$$

with the canonical momentum $\Pi_j(\mathbf{y})$ defined by[18]

$$\Pi_j(\mathbf{y}) = (\theta/2) \epsilon_{kj} A_k(\mathbf{y}) . \tag{7c}$$

I now show that the gauge fields can be eliminated at the expense of a change in the commutation relations of the matter fields.

Consider first a *classical* solution to the constraint equation (2). Up to boundary conditions, I can always find vector potentials $A_j(x)$ of the form

$$A_j(\mathbf{x}) = \epsilon_{jk} \Delta_k \Phi(\mathbf{r}) , \tag{8}$$

where $\Phi(\mathbf{r})$ is a scalar field defined on sites $\mathbf{r}$ of the dual lattice. Upon substitution, I get

$$j_0(\mathbf{x}) = \theta \Delta^2 \Phi(\mathbf{r}) , \tag{9}$$

where Δ^2 is the lattice Laplacian operator.

The solution is

$$\Phi(\mathbf{r}) = \frac{1}{\theta} \sum_{\mathbf{r}} G(\mathbf{r},\mathbf{r}') j_0(x') , \tag{10}$$

where $G(\mathbf{r},\mathbf{r}')$ is the lattice Green function, and $(\mathbf{x},\mathbf{r})$ are dual pairs. Thus I get

$$A_j(x) = \frac{1}{\theta} \epsilon_{jk} \Delta_k \sum_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}') j_0(\mathbf{x}') . \tag{11}$$

I now make use of the lattice "Cauchy-Riemann" equations,

$$\Delta_i G(\mathbf{r},\mathbf{r}') = \epsilon_{ij} \Delta_j \Theta(\mathbf{x},\mathbf{r}') , \tag{12}$$

to introduce the multivalued function $\Theta(\mathbf{x},\mathbf{r}')$, where $\mathbf{x}$ and $\mathbf{r}'$ lie in the direct and dual lattices, respectively. Θ is uniquely specified by Eq. (11) and the condition

$$\Delta\Theta = +1 \tag{13}$$

for the total change of Θ on any closed loop on the direct lattice including the dual site $\mathbf{r}$. The function Θ is defined to have a jump across a "branch cut" running from $\mathbf{r}$ off to infinity. The Cauchy-Riemann equation (12) can be integrated along a path on the direct lattice going from $\mathbf{x}'$ to $\mathbf{x}$ and leaving the cut always to its right. The resulting function $\Theta(\mathbf{x},\mathbf{r}')$ is now unique [up to an arbitrary initial phase $\Theta(\mathbf{x}',\mathbf{r}')$ which is of no physical consequence] and has a constant jump $(+1)$ across the cut. The vector potentials $A_j(\mathbf{x})$ are thus gradients of the multivalued function $\phi(\mathbf{x})$,

$$\phi(\mathbf{x}) = \frac{1}{\theta} \sum_{\mathbf{x}'} \Theta(\mathbf{x},\mathbf{r}') j_0(\mathbf{x}') , \tag{14}$$

with $A_j(\mathbf{x}) = \Delta_j \phi(\mathbf{x})$.

I can now *define* the *operator* $e^{i\phi(\mathbf{x})}$. It creates a coherent state of vector potentials representing a fraction $1/\theta$ for a fluxoid attached to each particle. The Jordan-Wigner transformation is defined by the operator identity

$$\tilde{a}(\mathbf{x}) = e^{i\phi(\mathbf{x})} a(\mathbf{x}), \tag{15a}$$

$$\tilde{a}^\dagger(\mathbf{x}) = a^\dagger(\mathbf{x}) e^{-i\phi(\mathbf{x})}. \tag{15b}$$

The nonlocal operator $e^{i\phi(\mathbf{x})}$ is a disorder operator.[19-21] The Jordan-Wigner operators $\tilde{a}(\mathbf{x}')$ and $\tilde{a}^\dagger(\mathbf{x})$ obey local equal-time generalized commutation relations which exhibit fractional statistics,[8,11,22]

$$\tilde{a}(\mathbf{x}')\tilde{a}^\dagger(\mathbf{x}) = \delta_{\mathbf{x}',\mathbf{x}'} - e^{i\delta}\tilde{a}^\dagger(\mathbf{x})\tilde{a}(\mathbf{x}'), \tag{16}$$

where δ is a phase given by

$$\delta = \frac{1}{\theta}[\Theta(\mathbf{x},\mathbf{r}') - \Theta(\mathbf{x}',\mathbf{r})] \equiv \frac{1}{2\theta}. \tag{17}$$

Equation (17) is a consequence of the definition of Θ and of translation invariance, which here means[23]

$$\Theta(\mathbf{x},\mathbf{r}+\mathbf{R}) = \Theta(\mathbf{x}-\mathbf{R},\mathbf{r}). \tag{18}$$

The Pauli Principle for fermions $(a^\dagger)^2 = a^2 = 0$ implies a "hard-core" condition for the anyon fields $\tilde{a}(\mathbf{x}),\tilde{a}^\dagger(\mathbf{x})$,

$$[\tilde{a}(\mathbf{x})]^2 = [\tilde{a}^\dagger(\mathbf{x})]^2 = 0. \tag{19}$$

In particular, if I choose $1/2\theta$ to be an odd multiple of π, the anyons become bosons with hard cores. Notice that in this case the fluxes obey a Dirac quantization condition. The Hamiltonian H_b of the bosons is

$$H_b = t \sum_{\mathbf{x},j=1,2} \tilde{a}^\dagger(\mathbf{x})\tilde{a}(\mathbf{x}+\hat{\mathbf{e}}_j) + \text{H.c.}, \tag{20}$$

which we recognize as the Hamiltonian for the spin-one-half XY model on a two-dimensional Bravais lattice with exchange constant $J = t/2$ and correspondences $\sigma^\dagger(\mathbf{x}) = \tilde{a}^\dagger(\mathbf{x})$ and $\sigma_3(\mathbf{x}) = 1 - 2j_0(\mathbf{x})$. Thus, the XY model is equivalent to a CS theory coupled to fermions with $\theta = 1/2\pi(2k+1)$. The $S_z = 0$ sector of the spin system maps onto the half-filled sectors of the hard-core Bose and CS-Fermi systems. Thus, the Fermi system has an *average* flux of one-half of the quantum per plaquette. This is quite reminiscent of the flux lattices[24] of resonating-valence-bond mean-field theories[25,26] of high-temperature superconductors. It is not clear, however, under which circumstances it may be possible to liberate the fermions from their fluxes and to use a flux-phase picture.

In summary, I have constructed a Jordan-Wigner transformation which maps a system of spinless fermions and gauge fields with a Chern-Simons term of strength θ on a Bravais lattice into a system of hard-core anyons. The effective Hamiltonian is always bilinear in anyons, just as in the case of clock models. Here too, the equations of motion for the anyon field are not linear because of the hard-core condition. For $\theta = 1/2\pi(2k+1)$ the anyons become hard-core bosons which in turn is equivalent to a spin-one-half quantum XY model. The generalization to more interesting spin Hamiltonians offers no special difficulty. It would be interesting to use these techniques to understand the possible connections between resonating-valence-bond-like states[27-29] and the more traditional phases of a hard-core Bose gas.

This work was supported in part by NSF Grant No. DMR-84-15063.

Note added.—Anderson *et al.*[21] have discussed an analog of the operator $e^{i\phi}$ which, for the case $\theta = 1/2\pi$, agrees with (15) at long distances. They also noted the connection with a fermionic theory coupled to gauge fields determined by the particle density. However, they rule out the possibility of a theory with $\theta \neq 1/2\pi$. My construction works at all distances on the lattice as well as for arbitrary values of flux.

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metry we have $[\Theta(\mathbf{x}-\mathbf{R},\mathbf{x})-\Theta(\mathbf{x}+\mathbf{R},\mathbf{r})]_{\Gamma_1}=[\Theta(\mathbf{x}+\mathbf{R},\mathbf{x})-\Theta(\mathbf{x}-\mathbf{R},\mathbf{r})]_{\Gamma_2}$. Since the total $[\Delta\Theta]_{\Gamma_1+\Gamma_2}=+1$, we have the result Eq. (17).

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