

Hydrodynamic theory of thermoelectric transport and negative magnetoresistance in Weyl semimetals

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We present a theory of thermoelectric transport in weakly disordered Weyl semimetals where the electron–electron scattering time is faster than the electron–impurity scattering time. Our hydrodynamic theory consists of relativistic fluids at each Weyl node, coupled together by perturbatively small intervalley scattering, and long-range Coulomb interactions. The conductivity matrix of our theory is Onsager reciprocal and positive semidefinite. In addition to the usual axial anomaly, we account for the effects of a distinct, axial–gravitational anomaly expected to be present in Weyl semimetals. Negative thermal magnetoresistance is a sharp, experimentally accessible signature of this axial–gravitational anomaly, even beyond the hydrodynamic limit.

Weyl metals | hydrodynamics | thermoelectric effects | anomalies

The recent theoretical predictions (1–3) and experimental discoveries (4–6) of Weyl semimetals open up an exciting new solid state playground for exploring the physics of anomalous quantum field theories. These anomalies can lead to striking signatures in simple transport measurements. Upon applying a magnetic field $\mathbf{B} = B\hat{z}$ and measuring the electrical conductivity σ_{zz} parallel to \mathbf{B} , one predicts σ_{zz} has a contribution that grows as B^2 (7–9). This longitudinal negative magnetoresistance is a direct signature of the anomaly associated with the Weyl points in momentum space. Similar results have also been predicted for thermal and thermoelectric transport (10, 11). Negative magnetoresistance in σ , with the predicted B^2 dependence, has been observed experimentally in many different materials (12–18).

So far, the theories of this negative magnetoresistance assume two facts about the dynamics of the quasiparticles of the Weyl semimetal. First, it is assumed that the quasiparticles are long lived, and that a kinetic description of their dynamics is valid. Second, it is assumed that the dominant scattering mechanism is between quasiparticles and impurities or phonons. In most simple crystals—including Weyl semimetals—it is likely that this description is reasonable.

However, there are exotic metals in which the quasiparticle–quasiparticle scattering time is much smaller than the quasiparticle–impurity/phonon scattering time. In such a finite temperature metal, the complicated quantum dynamics of quasiparticles reduces to classical hydrodynamics of long-lived quantities—charge, energy, and momentum—on long time and length scales. Most theoretical (19–25) and experimental (26–28) work on such electron fluids studies the dynamics of (weakly interacting) Fermi liquids in ultrapure crystals. As expected, the physics of a hydrodynamic electron fluid is qualitatively different from the kinetic regime where quasiparticle–impurity/phonon scattering dominates, and there are qualitatively distinct signatures to look for in experiments.

Experimental evidence for a strongly interacting quasirelativistic plasma of electrons and holes has recently emerged in graphene (29, 30). The relativistic hydrodynamic theories necessary to understand this plasma are different from ordinary Fermi liquid theory (31), and lead to qualitatively different transport phenomena (32, 33). The hydrodynamics necessary to describe an electron fluid in a Weyl material, when the Fermi energy is close to a Weyl node, is similar to the hydrodynamics of the graphene plasma,

though with additional effects related to anomalies (34, 35). Such a quasirelativistic regime is where negative magnetoresistance is most pronounced (9), and also where interaction effects can be strongest, due to the lack of a large Fermi surface to provide effective screening.

In this paper, we develop a minimal hydrodynamic model for direct current (dc) thermoelectric transport in a disordered, interacting Weyl semimetal, where the Fermi energy is close to the Weyl nodes. The first hydrodynamic approach to transport in a Weyl semimetal may be found in ref. 36 (see also refs. 37, 38). In contrast to these, our approach ensures that the conductivity matrix is positive semidefinite and Onsager reciprocal. We apply an infinitesimal electric field E_i and temperature gradient $\partial_i T$ to a Weyl semimetal, and compute the total charge current J_i and heat current Q_i using hydrodynamics. We then read off the thermoelectric conductivity matrix defined by

$$\begin{pmatrix} J_i \\ Q_i \end{pmatrix} = \begin{pmatrix} \sigma_{ij} & \alpha_{ij} \\ T\bar{\alpha}_{ij} & \bar{\kappa}_{ij} \end{pmatrix} \begin{pmatrix} E_j \\ -\partial_j T \end{pmatrix}. \quad [1]$$

In the limit where disorder, magnetic field, and intervalley scattering are perturbatively weak, we show that all conductivities may be written as the sum of a Drude conductivity for each valley fluid, and a correction due to intervalley scattering: e.g., $\sigma_{ij} = \sigma_{ij}^{\text{Drude}} + \sigma_{ij}^{\text{anom}}$. We present a general formula for the coefficient of B^2 in $\sigma_{zz}^{\text{anom}}$: the quantitative dependence of this coefficient on temperature and electron density can be different from quasiparticle-based methods.

Although the qualitative form of our results (e.g., $\sigma_{ij}^{\text{anom}} \sim B_i B_j$) is very similar to that found using kinetic theory approaches (8–11),

Significance

Weyl semimetals are exotic materials with negative electrical magnetoresistance: when an electric and magnetic field are applied in parallel, the induced electrical current increases upon increasing magnetic field strength. This is due to an emergent axial quantum anomaly in Weyl semimetals. We present a universal description of thermoelectric transport in weakly disordered Weyl semimetals where electron–electron interactions are faster than electron–impurity scattering. We predict negative thermal magnetoresistance: upon applying a parallel temperature gradient and magnetic field, the induced heat current increases with increasing magnetic field strength. This is caused by a distinct emergent quantum anomaly—the axial–gravitational anomaly. Measuring this effect may be the most practical route to experimentally observing this anomaly in any branch of physics.

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we strongly emphasize that the physical interpretations are often quite different. For example, the emergence of Drude conductivities in our model is not due to the existence of long-lived quasiparticles, but due to the fact that momentum relaxation is a perturbatively slow process (31, 39). Furthermore, distinct anomalies are responsible for the negative magnetoresistance in electrical vs. thermal transport. This remains true even beyond our strict hydrodynamic limit.

In this paper, we work in units where $\hbar = k_B = e = 1$. We will also generally set the Fermi velocity $v_F = 1$. In our relativistic formalism, the effective speed of light is set by v_F .

Weyl Hydrodynamics

We begin by developing our hydrodynamic treatment of the electron fluid, assuming the chemical potential lies close to the charge neutrality point for every node. For simplicity, we assume that the Weyl nodes are locally isotropic to reduce the number of effective parameters. It is likely straightforward, although tedious, to generalize and study anisotropic systems.

We will firstly review the hydrodynamic theory of a chiral fluid with an anomalous axial $U(1)$ symmetry, derived in refs. 34, 35. Neglecting intervalley scattering, this theory describes the dynamics near one Weyl node. The equations of relativistic chiral hydrodynamics are the conservation laws for charge, energy, and momentum, modified by the external electromagnetic fields, which we denote with $F_{\mu\nu}$. On a curved space with Riemann tensor $R_{\alpha\beta\gamma\delta}$, they read

$$\nabla_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu - \frac{G}{16\pi^2} \nabla_\mu [\epsilon^{\rho\sigma\alpha\beta} F_{\rho\sigma} R^\nu_{\alpha\beta}], \quad [2a]$$

$$\nabla_\mu J^\mu = -\frac{C}{8} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} - \frac{G}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} R^\alpha_{\beta\mu\nu} R^\beta_{\alpha\rho\sigma}, \quad [2b]$$

where C is a coefficient related to the standard axial anomaly and G is a coefficient related to an axial-gravitational anomaly (40). For a Weyl fermion

$$C = \frac{k}{4\pi^2}, \quad G = \frac{k}{24}, \quad [3]$$

with $k \in \mathbb{Z}$ the Berry flux associated with the Weyl node (41). J^μ and the energy-momentum tensor $T^{\mu\nu}$ are related to the hydrodynamic variables of chemical potential μ , temperature T , and velocity u^μ in a tightly constrained way (34, 35), which we review in the *SI Appendix*. We will take the background electromagnetic field to be

$$F = B dx \wedge dy + \partial_i \mu_0 dx_i \wedge dt, \quad [4]$$

with B as a constant. Constant B is required by Maxwell's equations for the external electromagnetic field in equilibrium, at leading order.

A single chiral fluid cannot exist in a Weyl material. Instead, enough Weyl nodes must exist so that the “net” C for the material vanishes. This follows mathematically from the fact that the Brillouin zone of a crystal is necessarily a compact manifold and so the sum of the Berry fluxes associated with each node must vanish—this is the content of the Nielsen–Ninomiya theorem (7). Hence, we must consider the response of multiple chiral fluids when developing our theory of transport.

One might hope that so long as each chiral fluid has a well-behaved response, then the net conductivities are simply additive. This is not so: the transport problem is ill-posed for a single chiral fluid, once we apply a background magnetic field. To see

this, suppose that we apply an electric field such that $\mathbf{E} \cdot \mathbf{B} \neq 0$. Then, the total charge in the sample obeys

$$\frac{dQ_{\text{tot}}}{dt} = \int d^3\mathbf{x} \partial_\mu J^\mu = C \mathbf{E} \cdot \mathbf{B} V_3, \quad [5]$$

with V_3 the spatial volume of the metal. Even at the linear response level, we see that there is a necessary $\mathcal{O}(E)$ time dependence to any solution to the hydrodynamic equations (with spatial directions periodically identified). If there is no static solution to the equations of motion, then any dc conductivity is an ill-posed quantity to compute. There is also energy production in a uniform temperature gradient, proportional to $G \nabla T \cdot \mathbf{B}$, even when $C = 0$ (*SI Appendix*).

The physically relevant solution to this issue is that multiple Weyl nodes exist in a real material, and this means that we must consider the coupled response of multiple chiral fluids. Rare intervalley processes mediated by phonons and/or impurities couple these chiral fluids together (8) and make the transport problem far richer for Weyl fluids than for simpler quantum critical fluids, including the Dirac fluid (32).

We label each valley fluid quantity with the labels $ab \dots$. For example, u_a^μ is the velocity of valley fluid a . To avoid being completely overwhelmed with free parameters, we only include coefficients at zeroth order in derivatives coupling distinct fluids together. In fact, this will be sufficient to capture the negative magnetoresistance, as we explain in the next section. Accounting for this coupling modifies the conservation equations to

$$\nabla_\mu J_a^\mu = -\frac{C_a}{8} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} - \frac{G_a}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} R^\alpha_{\beta\mu\nu} R^\beta_{\alpha\rho\sigma} - \sum_b [\mathcal{R}_{ab} \nu_b + \mathcal{S}_{ab} \beta_b] \quad [6a]$$

$$\nabla_\mu T_a^{\mu\nu} = F^{\nu\mu} J_{a\mu} - \frac{G_a}{16\pi^2} \nabla_\mu [\epsilon^{\rho\sigma\alpha\beta} F_{\rho\sigma} R^\nu_{\alpha\beta}] + u_a^\nu \sum_b [\mathcal{U}_{ab} \nu_b + \mathcal{V}_{ab} \beta_b], \quad [6b]$$

where we have defined $\beta_a \equiv 1/T_a$ and $\nu_a \equiv \beta_a \mu_a$. The transport problem is well-posed if

$$\sum_a C_a = \sum_a G_a = 0. \quad [7]$$

The new coefficients $\mathcal{R}, \mathcal{S}, \mathcal{U}$, and \mathcal{V} characterize the rate of the intervalley transfer of charge, energy, and momentum due to relative imbalances in chemical potential or temperature. In writing [6], we have chosen the intervalley scattering of energy and momentum to be relativistic. This makes the analysis easier as it preserves Lorentz covariance, but will not play an important role in our results. In particular, the intervalley momentum transfer processes are subleading effects in our theory of transport.

The gradient expansion may be different for each fluid, but we will assume that J_a^μ and $T_a^{\mu\nu}$ depend only on fluid a . We require that

$$\sum_{a \text{ or } b} \mathcal{R}_{ab} = \sum_{a \text{ or } b} \mathcal{S}_{ab} = \sum_{a \text{ or } b} \mathcal{U}_{ab} = \sum_{a \text{ or } b} \mathcal{V}_{ab} = 0. \quad [8]$$

This ensures that globally charge and energy are conserved, as well as that uniform shifts in the background chemical potential and/or temperature, for all fluids simultaneously, are exact zero modes of the equations of motion.

For simplicity in [6], we have implicitly assumed that the Weyl nodes are all at the same chemical potential in equilibrium. This is generally not true for realistic Weyl materials. As nontrivial issues in hydrodynamics already arise without making this

generalization, we will stick to the case where all Weyl nodes are at the same chemical potential in equilibrium in this paper.

For the remainder of this paper, we will be interested in transport in flat spacetimes where $R_{\mu\nu\alpha\beta}=0$. Except where otherwise stated, we will assume Minkowski space from now on. Hence, for most purposes, we write partial derivatives ∂_μ rather than covariant derivatives ∇_μ . However, we will continue to use the covariant derivative ∇_μ at intermediate steps of the calculations where it is necessary.

Thermodynamic Constraints

We will now derive the constraints on our hydrodynamic parameters which are imposed by demanding that the second law of thermodynamics is obeyed locally. Without intervalley coupling processes, and at the ideal fluid level (derivative corrections, including $F_{\mu\nu}$, are neglected), the second law of thermodynamics implies that the total entropy current s^μ (where s_a is the entropy density of fluid a) obeys (42, 43)

$$\partial_\mu s^\mu = \partial_\mu \left(\sum_a s_a u_a^\mu \right) = 0. \quad [9]$$

In the more generic, nonideal, case the right-hand side of [9] must be nonnegative. In our theory of coupled chiral fluids, the right-hand side of [9] does not vanish already at the ideal fluid level:

$$\partial_\mu s^\mu = \sum_{ab} (\beta_a [\mathcal{U}_{ab} \nu_b + \mathcal{V}_{ab} \beta_b] + \nu_a [\mathcal{R}_{ab} \nu_b + \mathcal{S}_{ab} \beta_b]) \geq 0. \quad [10]$$

There is no possible change we can make to the entropy current that is local and that can subtract off the right-hand side of [10]. Hence, we demand that the matrix

$$A \equiv \begin{pmatrix} \mathcal{R} & -\mathcal{S} \\ -\mathcal{U} & \mathcal{V} \end{pmatrix} \quad [11]$$

is positive semidefinite.

Using standard arguments for Onsager reciprocity in statistical mechanics (44), one can show that $A=A^\top$. In the [SI Appendix](#), we will show using the memory matrix formalism (39, 45) that whenever the quantum mechanical operators n_a and ϵ_a are naturally defined:

$$A_{IJ} = T \lim_{\omega \rightarrow 0} \frac{\text{Im}(G_{x_I x_J}^R(\omega))}{\omega}, \quad [12]$$

where x_I denotes (n_a, ϵ_a) and dots denote time derivatives. We also prove [8], and the symmetry and positive semidefiniteness of A through the memory matrix formalism, at the quantum mechanical level.

Equilibrium Fluid Flow

We now find an equilibrium solution to [6]. Beginning with the simple case of $B=0$, it is straightforward to see following ref. 46 that an equilibrium solution is

$$\mu_a = \mu_0(\mathbf{x}), \quad [13a]$$

$$T_a = T_0 = \text{constant}, \quad [13b]$$

$$u_a^\mu = (1, 0). \quad [13c]$$

Indeed as pointed out in refs. 32, 46, this exactly satisfies [6] neglecting the intervalley and anomalous terms. Using [8] it is straightforward to see that the intervalley terms also vanish on this solution. If $B=0$ then $C \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} = 0$, and hence this is an

exact solution to the hydrodynamic equations. We define the parameter ξ as the typical correlation length of $\mu(\mathbf{x})$: roughly speaking, $\xi \sim |\mu_0|/|\partial_x \mu_0|$.

Following ref. 43, we can perturbatively construct a solution to the equations of motion when $B \neq 0$, assuming that

$$B \ll T^2 \text{ and } 1 \ll \xi T. \quad [14]$$

Both of these assumptions are necessary for our hydrodynamic formalism to be physically sensible. Using these assumptions, it is consistent at leading order to only change $v_{ia} \neq 0$, but to keep μ_a and T_a the same:

$$v_{za} = \frac{C_a \mu_0^2 B}{2(\epsilon_a + P_a)} + \frac{G_a T_0^2 B}{\epsilon_a + P_a} \equiv v(\mu(x), T, B_i). \quad [15]$$

It may seem surprising that in a single chiral fluid, there would be a nonvanishing charge current. This is a well-known phenomenon called the chiral magnetic effect (for a recent review, see ref. 47). In our model, the net current flow is the sum of the valley contributions:

$$J^z = \sum_a J_a^z = \sum_a C_a \mu_0 B = 0, \quad [16]$$

and so indeed, this complies with the expectation that the net current in a solid-state system will vanish in equilibrium, as discussed (in more generality) in refs. 41, 48.

Thermoelectric Conductivity

We now linearize the hydrodynamic equations around this equilibrium solution, applying infinitesimally small external electric fields \tilde{E}_i , and temperature gradients $\tilde{\zeta}_i = -\partial_i \log T$ to the fluid. Although we have placed an equal sign in this equation, we stress that we will apply $\tilde{\zeta}_i$ in such a way we may apply a constant temperature gradient on a compact space (with periodic boundary conditions). Applying a constant \tilde{E}_i is simple, and corresponds to turning on an external electric field in $F_{\mu\nu}$. Applying a constant $\tilde{\zeta}_i$ is more subtle, and can be done by changing the spacetime metric to (49):

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu - 2 \frac{e^{-i\omega t}}{-i\omega} \tilde{\zeta}_i dx_i dt, \quad [17]$$

ω is a regulator, which we take to 0 at the end of the calculation. This spacetime is flat ($R_{\alpha\beta\gamma\delta}=0$). To account for both \tilde{E}_i and $\tilde{\zeta}_i$, the external gauge field is modified to $A + \tilde{A}$, where

$$\tilde{A}_i = -(\tilde{E}_i - \mu_0(\mathbf{x}) \tilde{\zeta}_i) \frac{e^{-i\omega t}}{-i\omega}. \quad [18]$$

The hydrodynamic Eq. 6 must then be solved in this modified background. In linear response, the hydrodynamic variables become

$$\mu_a = \mu_0(\mathbf{x}) + \tilde{\mu}_a(\mathbf{x}), \quad [19a]$$

$$T_a = T_0 + \tilde{T}_a(\mathbf{x}), \quad [19b]$$

$$u_a^\mu \approx \left(1 + v_{ia} \frac{\tilde{\zeta}_i e^{-i\omega t}}{i\omega}, v_{ia} + \tilde{v}_{ia} \right). \quad [19c]$$

Note that tilded variables represent objects that are first order in linear response. The correction to u_a^μ is necessary to ensure that $u^\mu u_\mu = -1$ is maintained. In general we cannot solve the

linearized equations analytically, except in the limit of perturbatively weak disorder and magnetic field strength.

We assume that the inhomogeneity in the chemical potential is small:

$$\mu_0 = \bar{\mu}_0 + u\hat{\mu}_0(\mathbf{x}), \quad [20]$$

with $u \ll \bar{\mu}_0$ and T ; u is our perturbative parameter, and we assume that $\hat{\mu}_0$ is a zero-mean random function with unit variance. We assume the scalings

$$B \sim u^2 \text{ and } \mathcal{R}, \mathcal{S}, \mathcal{U}, \mathcal{V} \sim u^6. \quad [21]$$

The hydrodynamic equations can be solved perturbatively, and the charge and heat currents may be spatially averaged on this perturbative solution. The computation is presented in the [SI Appendix](#), and we present highlights here. At leading order, the linearized hydrodynamic equations reduce to

$$\begin{aligned} \partial_i \left[n_a \tilde{w}_{ia} + \sigma_{Qa} \left(\tilde{E}_i - \partial_i \tilde{\mu}_a - \frac{\mu_0}{T} (T \tilde{\zeta}_i - \partial_i \tilde{T}_a) \right) \right] \\ = C_a \tilde{E}_i B_i - \sum_b [\mathcal{R}_{ab} \tilde{\nu}_b + \mathcal{S}_{ab} \tilde{\beta}_b] \end{aligned} \quad [22a]$$

$$\begin{aligned} \partial_i \left[T s_a \tilde{w}_{ia} - \mu_0 \sigma_{Qa} \left(\tilde{E}_i - \partial_i \tilde{\mu}_a - \frac{\mu_0}{T} (T \tilde{\zeta}_i - \partial_i \tilde{T}_a) \right) \right] \\ = 2G_a T_0^2 \tilde{\zeta}_i B_i + \sum_b [(\mathcal{R}_{ab} \mu_0 + \mathcal{U}_{ab}) \tilde{\nu}_b + (\mathcal{S}_{ab} \mu_0 + \mathcal{V}_{ab}) \tilde{\beta}_b] \end{aligned} \quad [22b]$$

$$n_a (\partial_i \tilde{\mu}_a - \tilde{E}_i) + s_a (\partial_i \tilde{T}_a - T \tilde{\zeta}_i) = \varepsilon_{ijk} \tilde{w}_{ja} n_a B_k. \quad [22c]$$

We have defined

$$\tilde{w}_{ia} = \tilde{w}_{ia} + \frac{\partial v_i}{\partial \mu} \tilde{\mu}_a + \frac{\partial v_i}{\partial T} \tilde{T}_a. \quad [23]$$

Here, \tilde{w}_{ia} represents the fluid velocity after subtracting the contribution coming from [15] in local thermal equilibrium.

Eq. 22 depends on G_a , despite the fact that our spacetime is flat. This follows from the subtle fact that thermodynamic consistency of the anomalous quantum field theory on curved spacetimes requires that the axial-gravitational anomaly alters the thermodynamics of fluids on flat spacetimes (40).

The total charge current is $\tilde{J}^i = \sum_a \tilde{J}_a^i$, and the total heat current is $\tilde{Q}^i = \sum_a \tilde{T}_a^i - \mu_0 \tilde{J}^i$. At leading order in perturbation theory, we find that the charge current in each valley fluid may be written as

$$\tilde{J}_a^i = n_a \tilde{V}_{ia} + C_a \tilde{M}_a B_i, \quad [24]$$

and the heat current per valley, $\tilde{T}_a^i - \mu_0 \tilde{J}_a^i$, may be written as

$$\tilde{Q}_a^i = T s_a \tilde{V}_{ia} + 2G_a T_0 \tilde{T}_a B_i. \quad [25]$$

In the above expressions \tilde{V}_{ia} is a homogeneous $\mathcal{O}(u^{-2})$ contribution to \tilde{w}_{ia} , and \tilde{M}_a and \tilde{T}_a are $\mathcal{O}(u^{-4})$ homogeneous contributions to $\tilde{\mu}_a$ and \tilde{T}_a , respectively.

The thermoelectric conductivity matrix is:

$$\sigma_{xx} = \sigma_{yy} = \sum_a \frac{n_a^2 \Gamma_a}{\Gamma_a^2 + B^2 n_a^2}, \quad [26a]$$

$$\sigma_{xy} = \sum_a \frac{B n_a^3}{\Gamma_a^2 + B^2 n_a^2}, \quad [26b]$$

$$\sigma_{zz} = \sum_a \frac{n_a^2}{\Gamma_a} + s B^2, \quad [26c]$$

$$\bar{\kappa}_{xx} = \bar{\kappa}_{yy} = \sum_a \frac{T s_a^2 \Gamma_a}{\Gamma_a^2 + B^2 n_a^2}, \quad [26d]$$

$$\bar{\kappa}_{xy} = \sum_a \frac{B T n_a s_a^2}{\Gamma_a^2 + B^2 n_a^2}, \quad [26e]$$

$$\bar{\kappa}_{zz} = \sum_a \frac{T s_a^2}{\Gamma_a} + \mathfrak{h} B^2, \quad [26f]$$

$$\alpha_{xx} = \alpha_{yy} = \sum_a \frac{n_a s_a \Gamma_a}{\Gamma_a^2 + B^2 n_a^2}, \quad [26g]$$

$$\alpha_{xy} = -\alpha_{yx} = \sum_a \frac{B n_a^2 s_a}{\Gamma_a^2 + B^2 n_a^2}, \quad [26h]$$

$$\alpha_{zz} = \sum_a \frac{n_a s_a}{\Gamma_a} + \mathfrak{a} B^2, \quad [26i]$$

where we have defined the four parameters

$$\mathfrak{s} \equiv T(C_a \quad C_a \mu) \begin{pmatrix} \mathcal{R}_{ab} & -\mathcal{S}_{ab} \\ -\mathcal{U}_{ab} & \mathcal{V}_{ab} \end{pmatrix}^{-1} \begin{pmatrix} C_b \\ C_b \mu \end{pmatrix}, \quad [27a]$$

$$\mathfrak{h} \equiv 4T^4(0 \quad G_a) \begin{pmatrix} \mathcal{R}_{ab} & -\mathcal{S}_{ab} \\ -\mathcal{U}_{ab} & \mathcal{V}_{ab} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ G_b \end{pmatrix}, \quad [27b]$$

$$\mathfrak{a} \equiv 2T^2(0 \quad G_a) \begin{pmatrix} \mathcal{R}_{ab} & -\mathcal{S}_{ab} \\ -\mathcal{U}_{ab} & \mathcal{V}_{ab} \end{pmatrix}^{-1} \begin{pmatrix} C_b \\ C_b \mu \end{pmatrix}, \quad [27c]$$

$$\Gamma_a \equiv \frac{T_0^2 (s_a (\partial n_a / \partial \mu) - n_a (\partial s_a / \partial \mu))^2}{3\sigma_{Qa} (\epsilon_a + P_a)^2} u^2. \quad [27d]$$

In these expressions, sums over valley indices are implicit. Coefficients odd under $z \rightarrow -z$ (such as σ_{xz}) vanish. Note that all of the contributions to the conductivities listed above are of the same order $\mathcal{O}(u^{-2})$ in our perturbative expansion, explaining the particular scaling limit (21) in u that was taken.

We have not listed the full set of transport coefficients. The unlisted transport coefficients are related to those in ref. 26 by Onsager reciprocity:

$$\sigma_{ij}(B) = \sigma_{ji}(-B), \quad [28a]$$

$$\bar{\kappa}_{ij}(B) = \bar{\kappa}_{ji}(-B), \quad [28b]$$

$$\alpha_{ij}(B) = \alpha_{ji}(-B). \quad [28c]$$

The symmetry of \mathbf{A} is crucial in order for the final conductivity matrix to obey [28].

Evidently, the conductivities perpendicular to the magnetic field are Drude-like. This follows from principles, which are by now very well understood (31, 39). In these weakly disordered fluids, the transport coefficients are only limited by the rate at which momentum relaxes due to the disordered chemical potential $\Gamma_a / (\epsilon_a + P_a)$, and/or the rate at which the magnetic field

relaxes momentum (by “rotating” it in the xy plane), $Bn_a/(\epsilon_a + P_a)$. This latter energy scale is the hydrodynamic cyclotron frequency (31). In our hydrodynamic theory, we can see this momentum “bottleneck” through the fact that the components of the charge and heat currents in [24] and [25], perpendicular to B_i , are proportional to the same fluid velocity $\tilde{v}_{ia} \sim u^{-2}$ at leading order. The transport of the fluid is dominated by the slow rate at which this large velocity can relax. Because the heat current and charge current are proportional to this velocity field, the contribution of each valley fluid to σ , α , and $\bar{\kappa}$ are all proportional to one another in the xy plane.

The remaining nonvanishing transport coefficients are σ_{zz} , α_{zz} , and $\bar{\kappa}_{zz}$. From [26], we see that these conductivities are a sum of a Drude-like contribution (because this is transport parallel to the magnetic field, there is no magnetic momentum relaxation) from each valley, as before, and a new “anomalous” contribution that couples the valley fluids together. This anomalous contribution has a qualitatively similar origin as that discovered in refs. 8, 36. It can crudely be understood as follows: the chemical potential and temperature imbalances \tilde{M} and \tilde{T} are proportional to B and inversely proportional to A , as the homogeneous contributions to the right-hand side of [22] cancel. Such thermodynamic imbalances lead to corrections to valley fluid charge and heat currents, analogous to the chiral magnetic effect—these are the linear in B terms in [24] and [25]. Combining these scalings together immediately gives us the qualitative form of the anomalous contributions to the conductivity matrix.

The positive semidefiniteness of the thermoelectric conductivity matrix is guaranteed. Thinking of the conductivity matrices as a sum of the anomalous contribution and Drude contributions for each valley, it suffices to show each piece is positive-definite individually. The Drude pieces are manifestly positive definite, as is well known (it is an elementary exercise in linear algebra to confirm). To show the anomalous pieces are positive semidefinite, it suffices to show that $s\hbar \geq T\alpha^2$. This follows from [27], and the Cauchy–Schwarz inequality $(\mathbf{v}_1^T A \mathbf{v}_1)(\mathbf{v}_2^T A \mathbf{v}_2) \geq (\mathbf{v}_1^T A \mathbf{v}_2)^2$ for any vectors $\mathbf{v}_{1,2}$, and a symmetric, positive-semidefinite matrix A . These arguments also guarantee $s, \hbar > 0$.

Our expression for the conductivity may seem ill posed—it explicitly depends on the matrix inverse A^{-1} , but A is not invertible. In fact, the kernel of A has two linearly independent vectors: $(1_a, 1_a)$ and $(1_a, -1_a)$, with 1_a denoting a vector with valley indices with each entry equal to 1. However, in the final formula for the thermoelectric conductivities, which sums over all valley fluids, we see that A^{-1} is contracted with vectors that are orthogonal to the kernel of A due to [7]. The expression for the conductivities is therefore finite and unique.

We present a simple example of our theory for a fluid with two identical Weyl nodes of opposite chirality in the *SI Appendix*, along with a demonstration that the equations of motion are unchanged when we account for long-range Coulomb interactions, or impose electric fields and temperature gradients through boundary conditions in a finite domain. Hence, the transport coefficients we have computed above are in fact those which will be measured in experiment.

In this paper, we used inhomogeneity in the chemical potential to relax momentum when $B = 0$. By following the hydrodynamic derivation in ref. 46, other mechanisms for disorder likely lead to the same thermoelectric conductivities as reported in ref. 26, but with a different formula for Γ_a .

Violation of the Wiedemann–Franz Law

The thermal conductivity κ_{ij} usually measured in experiments is defined with the boundary conditions $\tilde{J}_i = 0$ (as opposed to $\bar{\kappa}_{ij}$, which is defined with $\tilde{E}_i = 0$). This thermal conductivity is related to the elements of the transport matrix [1] by

$$\kappa_{ij} = \bar{\kappa}_{ij} - T \bar{\alpha}_{ik} \sigma_{kl}^{-1} \alpha_{lj}. \quad [29]$$

In an ordinary metal, the Wiedemann–Franz (WF) law states that (50)

$$\mathcal{L}_{ij} \equiv \frac{\kappa_{ij}}{T\sigma_{ij}} = \frac{\pi^2}{3}. \quad [30]$$

The numerical constant of $\pi^2/3$ comes from the assumption that the quasiparticles are fermions, and that the dominant interactions are between quasiparticles and phonons or impurities, but otherwise is robust to microscopic details.

In general, our model will violate the WF law. Details of this computation are provided in the *SI Appendix*. In general, the WF law is violated by an $\mathcal{O}(1)$ constant, which depends on the magnetic field B . However, in the special case where we have valley fluids of opposite chirality but otherwise identical equations of state, we find that the transverse Lorenz ratios \mathcal{L}_{xx} , \mathcal{L}_{yy} , \mathcal{L}_{zz} are all parametrically smaller than 1 (in fact, they vanish at leading order in perturbation theory). In contrast, we find that $\mathcal{L}_{zz} \sim B^2$ at small B , and saturates to a finite number as B becomes larger (but still $\ll T^2$). This dramatic angular dependence of the WF law would be a sharp experimental test of our formalism in a strongly correlated Weyl material.

If the intervalley scattering rate is almost vanishing, the anomalous conductivities of a weakly interacting Weyl gas are still computable with our formalism. Weak intravalley scattering processes bring the “Fermi liquid” at each Weyl node to thermal equilibrium, and A may be computed via semiclassical kinetic theory. Assuming that intervalley scattering occurs elastically off of pointlike impurities, we compute s , α , and \hbar in the *SI Appendix*. We find that $\mathcal{L}_{zz}^{\text{anom}} < \pi^2/3$, asymptotically approaching the WF law when $\mu \gg T$. This is in contrast to the nonanomalous conductivities of a semiconductor, where under similar assumptions $\mathcal{L}_{zz} > \pi^2/3$ (51). The Mott relation between $\alpha_{zz}^{\text{anom}}$ and $\sigma_{zz}^{\text{anom}}$ differs by an overall sign from the standard relation. These discrepancies occur because we have assumed that elastic intervalley scattering is weaker than intravalley thermalization. In contrast, [11] makes the opposite assumption when $\mu \gg T$, and so recovers all ordinary metallic phenomenology. Even in this limit, negative thermal magnetoresistance is a consequence of nonvanishing G_a .

Outlook

In this paper, we have systematically developed a hydrodynamic theory of thermoelectric transport in a Weyl semimetal where quasiparticle–quasiparticle scattering is faster than quasiparticle–impurity and/or quasiparticle–phonon scattering. We have demonstrated the presence of longitudinal negative magnetoresistance in all thermoelectric conductivities. New phenomenological parameters introduced in our classical model may be directly computed using the memory matrix formalism given a microscopic quantum mechanical model of a Weyl semimetal. Our formalism is directly applicable to microscopic models of interacting Weyl semimetals where all relevant nodes are at the same Fermi energy. Our model should be generalized to the case where different nodes are at different Fermi energies, though our main results about the nature of negative magnetoresistance likely do not change qualitatively.

Previously, exotic proposals had been put forth to measure the axial–gravitational anomaly in an experiment. Measurements involving rotating cylinders of a Weyl semimetal have been proposed in refs. 41, 52, and it is possible that the rotational speed of neutron stars is related to this anomaly (53). A nonvanishing negative magnetoresistance in either α_{zz} or $\bar{\kappa}_{zz}$ is a direct experimental signature of the axial–gravitational anomaly. It is exciting that a relatively mundane transport experiment on a Weyl semimetal is capable of detecting this novel anomaly.

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