



# The chiral magnetic effect in hydrodynamical approach

M.V. Isachenkov, A.V. Sadofyev\*

*Institute for Theoretical and Experimental Physics, Moscow, Russian Federation*

## ARTICLE INFO

### Article history:

Received 30 November 2010  
Received in revised form 10 February 2011  
Accepted 14 February 2011  
Available online 19 February 2011  
Editor: J.-P. Blaizot

### Keywords:

Chiral hydrodynamics  
Chiral magnetic effect  
Quark–gluon plasma

## ABSTRACT

We consider a model with two conserved currents, vector and axial, and two associated chemical potentials,  $\mu$  and  $\mu_5$ . In the presence of external magnetic and electric fields the axial current is anomalous. Generalizing recent results and using thermodynamic relations alone we demonstrate that one can evaluate the chiral magnetic effect (ChME), that is the electric current flowing in the direction of the external magnetic field. In the linear in the chemical potential approximation the current is the same as for non-interacting fermions. In other words, there exists a hydrodynamic “non-renormalization theorem for the ChME”.

© 2011 Elsevier B.V. Open access under CC BY license.

## 1. Introduction

The chiral magnetic effect (for review and references see [1]) gradually becomes a litmus-paper for quark–gluon plasma physics. So much the more interesting is the task of its explanation and study. The effect itself is the generation of electric current by applying an external magnetic field  $\mathbf{H}$  to a chiral medium:

$$\mathbf{j}_{el} = c_{ch} \mathbf{H}, \quad (1)$$

where  $c_{ch}$  is a constant. The problem is to evaluate the coefficient  $c_{ch}$ . In particular, for fermions interacting with the magnetic field but not among themselves one gets [1,2]:

$$c_{ch} = N_c \frac{e^2 \mu_5}{2\pi^2}, \quad (2)$$

where  $N_c$  is the number of colors,  $e^2$  is the electric charge squared and  $\mu_5$  is the chemical potential conjugated to the difference in the number of left- and right-handed fermions. Taking into account the interaction between the fermions is a challenge.

In our considerations, we start from the hydrodynamic approximation which is, in a way, opposite to the limit of free fermions. Indeed, in the hydrodynamic approximation one considers distances  $l$  much larger than the free path of the particles of the medium,

$$l \gg l_{\text{free path}}.$$

\* Corresponding author.

E-mail addresses: m.isachenkov@gmail.com (M.V. Isachenkov), sadofyev@itep.ru (A.V. Sadofyev).

The hydrodynamic approach to chiral media is pioneered in [3,4]. In [3] it is demonstrated that the general thermodynamic identities are powerful enough to fix some transport coefficients. In the model of [3] all the currents are chiral. We generalize these considerations to the case of vector and axial currents separately conserved (at least in the absence of external electromagnetic fields). The two-current set-up (see also [5]) enables physical quantities to have their proper parity.

With all the calculations completed, we'll obtain the first term in the expansion of the coefficient  $c_{ch}$  in the axial chemical potential  $\mu_5$ , whereas in coupling constant it includes all orders due to our use of the hydrodynamical approximation. The result coincides with (2). The fact that calculated coefficient coincides with those derived in linear approximation permits us to formulate a “non-renormalization theorem” for ChME that seems to be analogous to already existing non-renormalization theorems [6]: the ChME coefficient does not receive corrections in coupling constant, but only in chemical potentials of right- and left-handed particles. In fact, this theorem is a consequence of the second law of thermodynamics.

Through the paper we use the following conventions: the metric tensor has  $(-, +, +, +)$  signature, 4-velocity of an element of the liquid satisfies  $u_\mu u^\mu = -1$ , the quantities which correspond to the electric and magnetic fields in the rest frame are  $E^\mu = F^{\mu\nu} u_\nu$  and  $B_\mu = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} u^\nu F^{\alpha\beta}$ . We also introduce the following notation:

$$\omega_\mu = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} u^\nu \partial^\alpha u^\beta.$$

For simplicity we consider an Abelian case henceforth ( $N_c = 1$ ).

## 2. Hydrodynamical approximation

We discuss a hydrodynamical model of the ideal chiral liquid in the presence of the chiral anomaly. To describe a nonzero chirality we introduce independent densities of the right- and left-handed particles or, equivalently, density  $n$  and “axial” density  $n_5$  (which phenomenologically correspond to the QCD currents  $J^\mu = \bar{q}\gamma^\mu q$  and  $J_5^\mu = \bar{q}\gamma^\mu\gamma^5 q$ ). We also introduce their chemical potentials  $\mu, \mu_5$ . This model is described by following equations:

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= F^{\nu\lambda} j_\lambda, \\ \partial_\mu j^\mu &= 0, \\ \partial_\mu j_5^\mu &= CE^\mu B_\mu,\end{aligned}\quad (3)$$

where the stress-energy tensor, vector and axial current are

$$\begin{aligned}T^{\mu\nu} &= wu^\mu u^\nu + pg^{\mu\nu}, \\ j^\mu &= nu^\mu + v^\mu, \\ j_5^\mu &= n_5 u^\mu + v_5^\mu.\end{aligned}\quad (4)$$

Terms  $v, v_5$  are higher-order corrections in velocities and their derivatives, which corresponds, in particular, to dissipative effects. There is no a dissipative force in the rest frame of the element of a liquid. Therefore  $v, v_5$  satisfy conditions (see [7]):

$$\begin{aligned}v_\mu u^\mu &= 0, \\ v_{5,\mu} u^\mu &= 0.\end{aligned}\quad (5)$$

Following the pattern of [3], let us consider consequences from the second law of thermodynamics. Transforming  $u_\nu \partial_\mu T^{\mu\nu} + \partial_\mu j^\mu + \partial_\mu j_5^\mu$  we get:

$$\begin{aligned}\partial_\mu \left( su^\mu - \frac{\mu}{T} v^\mu - \frac{\mu_5}{T} v_5^\mu \right) \\ = -v^\mu \partial_\mu \frac{\mu}{T} - v_5^\mu \partial_\mu \frac{\mu_5}{T} + \frac{E^\lambda}{T} v_\lambda - \mu_5 \frac{C}{T} EB.\end{aligned}\quad (6)$$

The second law of thermodynamics implies that the entropy is non-decreasing and the entropy current divergence is non-negative. The entropy current is usually introduced as:

$$s^\mu = su^\mu - \frac{\mu}{T} v^\mu - \frac{\mu_5}{T} v_5^\mu. \quad (7)$$

For the ideal liquid  $v$  and  $v_5$  are zero. But in the presence of the anomaly such a choice does not guarantee fulfillment of the second law of thermodynamics since the  $CE \cdot B$  term can have both signs. To compensate for the contribution of this term we can introduce in  $v$ , in  $v_5$  and in  $s_\mu$  further terms, not forbidden by the second law of thermodynamics in the presence of the anomaly (see [3,4]). We choose such terms in the accordance with [3]:

$$s^\mu = su^\mu - \frac{\mu}{T} v^\mu - \frac{\mu_5}{T} v_5^\mu + D\omega^\mu + D_B B^\mu, \quad (8)$$

$$\begin{aligned}v^\mu &= \kappa\omega^\mu + \kappa_B B^\mu, \\ v_5^\mu &= \xi\omega^\mu + \xi_B B^\mu.\end{aligned}\quad (9)$$

Coefficients  $\xi, \xi_B, \kappa, \kappa_B, D, D_B$  are functions of the thermodynamical variables. The coefficient in front of  $B^\mu$  in the vector current corresponds to the ChME in hydrodynamical approach. To impose the second law of thermodynamics we demand the contributions to the divergence of the new terms to be compensated by the chiral anomaly. When  $B^\mu, \omega^\mu, E\omega, EB$  are arbitrary, the following set of equations takes place:

$$\left( \partial_\mu D - 2 \frac{\partial_\mu P}{w} D - \kappa \partial_\mu \frac{\mu}{T} - \xi \partial_\mu \frac{\mu_5}{T} \right) \cdot \omega^\mu = 0, \quad (10)$$

$$\left( \partial_\mu D_B - \frac{\partial_\mu P}{w} D_B - \kappa_B \partial_\mu \frac{\mu}{T} - \xi_B \partial_\mu \frac{\mu_5}{T} \right) \cdot B^\mu = 0, \quad (11)$$

$$\left( \frac{2nD}{w} - 2D_B + \frac{\kappa}{T} \right) \cdot (E_\mu \omega^\mu) = 0, \quad (12)$$

$$\left( \frac{nD_B}{w} + \frac{\kappa_B}{T} - \mu_5 \frac{C}{T} \right) \cdot (E_\mu B^\mu) = 0. \quad (13)$$

We don't write here the terms proportional to  $E_\mu$ , because in such case one cannot solve the resulting system (10)–(13) in a self-consistent way. Namely, electro- and thermal-conductivities are tied and appear together already in the absence of the anomaly, the form of these terms being governed by the second law of thermodynamics. Prima facie, this circumstance allows to reduce analogous case in the presence of the anomaly to the case considered in our work. However, the relations that express gradients of fields and vorticity through thermodynamical variables become more complicated. Therefore, to incorporate the dissipative and conductive (both thermal- and electro-) effects in our framework one should find a way of self-consistent solving a system like (10)–(13) in such a case. In our work we restrict us to the ideal case, as in [3].

To solve the set of Eqs. (10)–(13) one should use initial conditions. For this purpose it is convenient to change a thermodynamical variables from  $(\mu, \mu_5, T)$  to  $(\bar{\mu} = \frac{\mu}{T}, \bar{\mu}_5 = \frac{\mu_5}{T}, P)$ . Then

$$dT = \frac{T}{w} dP - \frac{nT^2}{w} d\bar{\mu} - \frac{n_5 T^2}{w} d\bar{\mu}_5. \quad (14)$$

We can set the gradients of thermodynamical variables  $(\partial P, \partial \frac{\mu}{T}, \partial \frac{\mu_5}{T})$  arbitrary at initial time. Then coefficients in front of them must be zero at arbitrary time to make (10)–(13) valid:

$$\frac{\partial D}{\partial p} - \frac{2D}{w} = 0, \quad \frac{\partial D_B}{\partial p} - \frac{D_B}{w} = 0, \quad (15)$$

$$\frac{\partial D}{\partial \bar{\mu}} = \kappa, \quad \frac{\partial D_B}{\partial \bar{\mu}} = \kappa_B, \quad (16)$$

$$\frac{\partial D}{\partial \bar{\mu}_5} = \xi, \quad \frac{\partial D_B}{\partial \bar{\mu}_5} = \xi_B. \quad (17)$$

We search for a solution in the form:

$$\begin{aligned}D &= T^2 d(\bar{\mu}, \bar{\mu}_5), & D_B &= T d(\bar{\mu}, \bar{\mu}_5), \\ \kappa &= \frac{\partial}{\partial \bar{\mu}} (T^2 d), & \xi &= \frac{\partial}{\partial \bar{\mu}_5} (T^2 d), \\ \kappa_B &= \frac{\partial}{\partial \bar{\mu}} (T d_B), & \xi_B &= \frac{\partial}{\partial \bar{\mu}_5} (T d_B),\end{aligned}\quad (18)$$

where  $d$  and  $d_B$  are arbitrary functions. One finds after straightforward calculations:

$$\begin{aligned}d_B &= C \bar{\mu}_5 \bar{\mu} + f(\bar{\mu}_5), \\ d &= C \bar{\mu}_5 \bar{\mu}^2 + \bar{\mu} f(\bar{\mu}_5) + F(\bar{\mu}_5),\end{aligned}\quad (19)$$

where  $f$  and  $F$  are arbitrary functions.

We are interested only in the ChME term which corresponds to the coefficient in front of  $B_\mu$ . For this coefficient  $\kappa_B$  in the lowest order in chemical potentials one gets:

$$\kappa_B = C\mu_5 + O(\mu_5^2, \mu\mu_5, \mu^2). \quad (20)$$

Higher-order terms in  $\kappa_B$  are not fixed by this calculation. It's worth pointing out that the coefficients in front of  $\omega^\mu$  have uncertainty even in the lowest order while in [3] coefficient before  $\omega^\mu$  in the chiral current is uniquely fixed.

It is readily seen that there is a term proportional to the magnetic field in the vector current. This term corresponds to the ChME in the hydrodynamical model. Substituting the anomalous coefficient in front of  $(E \cdot B)$  by  $\frac{e^2}{2\pi^2}$  we finally get in the lowest order:

$$j_{\text{ChME}}^\mu = \frac{e^2 \mu_5}{2\pi^2} B^\mu. \quad (21)$$

The answer for an arbitrary  $N_c$  is obtained by simple multiplication.

### 3. Conclusion

To summarize, it follows from the consideration of the entropy that the chiral magnetic effect is not vanishing in the hydrodynamical approximation. The corresponding coefficient  $c_{ch}$ , see (2), is expandable in the chemical potentials. The first, linear term is uniquely fixed and coincides with the result for non-interacting particles. Unlike the free-particle case, further terms in the expansion are not vanishing, generally speaking. They seem to be not fixed, however, by the general thermodynamic relations. The hydrodynamical approximation addresses physics in the infrared region of energies and is sensitive, generally speaking, to all orders

of perturbation theory. Our result suggests, therefore, that there exists a new non-renormalization theorem for the coefficient  $c_{ch}$  in the linear in the chemical potential approximation. Besides the result itself which looks quite non-presumable, one could wonder about power of pure thermodynamics in deriving of highly non-trivial relations in the field theory.

### Acknowledgements

We are grateful to E.T. Akhmedov, F. Gubarev and V.I. Zakharov for valuable discussions. The work of A.V. Sadofyev was partly supported by the grant for the Leading Scientific School NSh-6260.2010.2, DAAD Leonhard-Euler-Stipendium 2010–2011 and Ministry of Education and Science of Russian Federation under contract 14.740.11.0347.

### References

- [1] K. Fukushima, D.E. Kharzeev, H.J. Warringa, Phys. Rev. D 78 (2008) 074033.
- [2] D.E. Kharzeev, H.J. Warringa, Phys. Rev. D 80 (2009) 034028.
- [3] D.T. Son, P. Surowka, Phys. Rev. Lett. 103 (2009) 191601.
- [4] J. Erdmenger, M. Haack, M. Kaminski, A. Yarom, JHEP 0901 (2009) 055.
- [5] S. Pu, J. Gao, Q. Wang, arXiv:1008.2418v2 [nucl-th].
- [6] S.L. Adler, W.A. Bardeen, Phys. Rev. 182 (1969).
- [7] L.D. Landau, E.M. Lifshitz, Fluid Mechanics, Pergamon, New York, 1959.