21
Conformal Field Theory

21.1 Scale and Conformal Invariance in Field Theory

In Chapter 15 we looked at explicit constructions of RG transformations using ideas largely inspired by the theory of phase transitions. As we had anticipated the key concept is that of a fixed point. In particular we identified a class of fixed points at which the physical length scale, the correlation length, diverges. From the point of view of QFT these are the fixed points of interest since their vicinity define a continuum quantum field theory. Notice that, e.g. in Eq.(15.88) the UV cutoff a enters only to provide the necessary units to the correlation length. But, aside from that, in this limit the UV regulator essentially disappears from the theory.

In this sense the fixed points associated with continuous phase transitions express the behavior of the theory in the IR, whereas the fixed points that have vanishing correlation lengths and define the phases of the theory define the behavior of the theory in the UV. Thus, in one phase, the quantum field theory is represented as an RG flow from the UV to the IR in that as a coupling constant is varied the correlation length grows from its microscopic definition to a behavior largely independent of the microscopic physics.

In this section we will discuss general properties of scale-invariant theories. As such, these theories must be defined as quantum field theory as at a fixed point with a divergent correlation length (representing some continuous phase transition). Here we will assume that the theory is scale and rotational invariant (or Lorentz invariant in the Minkowski signature). From a “microscopic” point of view scale and conformal invariance are emergent symmetries of the fixed point theory.

There is a general result (Polchinski, 1988) that shows that, under most circumstances, scale-invariant theories have a much larger symmetry, conformal invariance. The general framework for such theories is known as Conformal Field Theory (CFT). This approach will allow us to describe not only the CFT but also deformed CFTs by the action of relevant operators. (Polyakov, 1974)

Scale invariance alone implies that observables in this theory should obey scaling, i.e. they should transform irreducibly under scale transformations, dilatations of the form $x' = \lambda x$. Thus, expectation values of a physical observable $F(x)$ must transform *homogeneously* under dilatations as $F(\lambda x) = \lambda^k F(x)$, where k is called the degree of the homogeneous function. Scale transformations are a subgroup of more general transformations known as *conformal transformations*. Conformal transformations are coordinate transformations that preserve the angles (i.e. scalar products) between vectors in a space (or space-time).

Several excellent treatments of Conformal Field Theory in the literature which have inspired the presentation of this topic in this book. A modern general approach to conformal field theory are the 2015 TASI Lectures by David Simmons-Duffin (Simmons-Duffin, 2017), the 1988 Les Houches Summer School lectures by Paul Ginsparg on Applied Conformal Field Theory (Ginsparg, 1989), and by John Cardy on Conformal Invariance and Statistical Mechanics (Cardy, 1996). Other excellent presentations are found in the book Conformal Field Theory by Philippe Di Francesco, Paul Mathieu and Daniel Sénéchal (Di Francesco et al., 1997), and in the two-volume book String Theory by Joseph Polchinski (Polchinski, 1998).

21.2 The Conformal Group in D dimensions

In this section we discuss the general consequences of conformal invariance in a field theory (Ginsparg, 1989; Simmons-Duffin, 2017). Let us consider a local field theory in a flat D-dimensional space-time which will be regarded as $\mathbb{R}^D$. The flat metric will be $g_{\mu\nu} = \eta_{\mu\nu}$, with signature (p, q). The line element is, as usual

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{21.1}$$

The change of the metric tensor under a change of coordinates $x_\mu \mapsto x'_\mu$ is

$$g_{\mu\nu} \mapsto g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \tag{21.2}$$

Conformal transformations are a subgroup of diffeomorphisms (i.e. differentiable coordinate transformations) that leave the metric tensor invariant up to a local change of scale, i.e.

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \tag{21.3}$$

where $\Omega(x)$ is the conformal factor. In this case, if v^μ and w^μ are two vectors whose scalar product is $v \cdot w = g_{\mu\nu} v^\mu w^\nu$, then the quantity

$$\frac{v \cdot w}{\sqrt{|v|^2 |w|^2}} \tag{21.4}$$

is invariant under conformal transformations which, therefore, preserve angles. Here $|v|^2 = g_{\mu\nu} v^\mu v^\nu$ is the norm of the vector v_μ (an example is shown in Fig.21.1). Conformal transformations for a group, the Conformal Group. The Poincaré group, consisting of space-time translations and Lorentz transformations, is a subgroup of the conformal group.

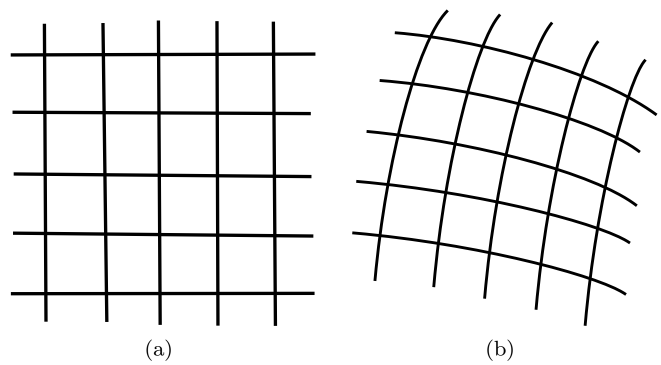

Figure 21.1 A conformal mapping transforms a Cartesian coordinates (a) into curvilinear coordinates (b) while preserving angles.

Let us first look at the generators of infinitesimal conformal transformations. Under an infinitesimal transformation, $x_\mu \mapsto x'_\mu = x_\mu + \varepsilon_\mu$, the line element transforms as

$$ds^2 \mapsto ds^2 + \left(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\right) dx^\mu dx^\nu \tag{21.5}$$

For this mapping to be conformal we require that

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{D}\, \partial \cdot \varepsilon\, \eta_{\mu\nu} \tag{21.6}$$

which implies that for an infinitesimal conformal transformation the conformal factor $\Omega(x)$ is

$$\Omega(x) = 1 + \frac{2}{D}\, \partial \cdot \varepsilon \tag{21.7}$$

Eq.(21.6) implies that $\partial \cdot \varepsilon$ is the solution of the partial differential equation

$$\left(\eta_{\mu\nu} \partial^2 + (D-2) \partial_\mu \partial_\nu\right) (\partial \cdot \varepsilon) = 0 \tag{21.8}$$

The case $D = 2$ is special (and specially important). For $D = 2$ and for the Euclidean signature, $\eta_{\mu\nu} = \delta_{\mu\nu}$, Eq.(21.6) implies that the components of ϵ_μ satisfy

$$\partial_1\varepsilon_2 = \partial_2\varepsilon_1, \qquad \partial_2\varepsilon_1 = -\partial_1\varepsilon_2 \tag{21.9}$$

In other terms, the vector field ε_μ satisfies the Cauchy-Riemann equations. Moreover, the function $f(x_1, x_2) = \varepsilon_1 + i\varepsilon_2$ is an *analytic* function of the coordinates. This feature has powerful implications for the special case of $D = 2$ and will be discussed separately.

Returning to the general case $D > 2$, Eq.(21.8) implies that third derivatives of ε must vanish and, hence, it can have at most a quadratic dependence on the coordinates x_μ. In general we have three cases:

i) Zeroth-order in x_μ: $\varepsilon_\mu = a_\mu$, and represents translations,
ii) Linear in x_μ: we have two choices, a) $\varepsilon_\mu = \omega_{\mu\nu}x^\nu$ that represents an infinitesimal rotation (or a Lorentz transformation in the Minkowski signature), and b) $\varepsilon_\mu = \lambda x_\mu$ that represents an infinitesimal scale transformation,
iii) Quadratic order in x_μ: $\varepsilon_\mu = b_\mu x^2 - 2x_\mu b \cdot x$ (or $\frac{x'_\mu}{x'^2} = \frac{x_\mu}{x^2} + b_\mu$), that represents an infinitesimal special conformal transformation.

The infinitesimal generators of conformal transformations are

$$a^\mu\partial_\mu, \qquad (p+q) \text{ translations,} \tag{21.10}$$

$$\omega^{\mu\nu}x_\nu\partial_\mu, \qquad \frac{1}{2}(p+q+1)(p+q-1) \text{ rotations,} \tag{21.11}$$

$$\lambda x^\mu\partial_\mu, \qquad \text{one scale transformation,} \tag{21.12}$$

$$b^\mu(x^2\partial_\mu - 2x^\mu x^\nu\partial_\nu), \qquad (p+q) \text{ special conformal transformations.} \tag{21.13}$$

The total number of generators is $\frac{1}{2}(p+q+1)(p+q+2)$. This algebra is isomorphic to the algebra of the group $SO(p+1, q+1)$.

Finite conformal transformations are

1) Translations: $x'_\mu = x_\mu + a_\mu$ (which has $\Omega = 1$),
2) Rotations: $x'_\mu = \Lambda_{\mu\nu}x^\nu$, with $\Lambda \in SO(p,q)$ (also with $\Omega = 1$),
3) Scale transformations: $x'_\mu = \lambda x_\nu$ (with $\Omega = \lambda^{-2}$),
4) Special conformal transformations: $x'_\mu = (x_\mu + b_\mu x^2)/(1 + 2b \cdot x + b^2x^2)$ (with $\Omega = (1 + 2b \cdot x + b^2x^2)^2$).

In $D = p + q$ dimensions, the Jacobian J of a conformal transformation is

$$J = \left|\frac{\partial x'}{\partial x}\right| = (\det g'_{\mu\nu})^{-1/2} = \Omega^{-D/2} \tag{21.14}$$

Translations and rotations have $J = 1$, dilatations have $J = \lambda^D$ and special conformal transformations have $J = (1 + 2b \cdot x + b^2 x^2)^{-D}$.

21.3 The Energy-Momentum Tensor and Conformal Invariance

In Section 3.8 we discussed the role of the energy-momentum tensor $T^{\mu\nu}$ in the context of classical field theory and, equivalently, the stress tensor in classical Euclidean field theory. We will now see that its counterpart in quantum field theory plays a prominent role in the presence of conformal invariance. There we showed that the energy-momentum tensor can be viewed as the response to an infinitesimal change of the metric, c.f. Eq.(3.198). Furthermore, we showed that in a Poincaré invariant theory, i.e. invariant under translations and Lorentz transformations, the energy-momentum tensor is locally conserved, i.e. $\partial_\mu T^{\mu\nu} = 0$, and can always be made symmetric, $T^{\mu\nu} = T^{\nu\mu}$.

The associated Noether charge to the energy-momentum tensor is the total linear momentum 4-vector P^μ, which in Chapter 3 was defined as the integral of $T^{\mu\nu}$ on a constant time hypersurface. However, since $T^{\mu\nu}$ is conserved, its integral on *any* closed oriented hypersurface Σ, the boundary of a region Ω (i.e. $\Sigma = \partial\Omega$), the quantity defined by the surface integral

$$P^\nu[\Sigma] = -\int_\Sigma dS_\mu T^{\mu\nu} \tag{21.15}$$

does not change under smooth changes of the shape of the boundary Σ or even the size of the enclosed region, provided that no operators are inserted in the bulk of Ω (or become included in Ω as Σ changes).

At the quantum level, the classical conservation law is replaced by the (conformal) Ward Identity

$$\partial_\mu\Big\langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_N(x_N)\Big\rangle = -\sum_{j=1}^{N}\delta(x-x_j)\partial_j^\nu\Big\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_N(x_N)\Big\rangle \tag{21.16}$$

where $\{\mathcal{O}_j(x)\}$ is an arbitrary set of local operators.

Let $\Omega(x_1)$ be a simply connected region of spacetime (which we will take to be Euclidean and flat), and $\Sigma(x_1)$ be its boundary, that encloses only the operator $\mathcal{O}(x_1)$ but not any of the other operators involved in the expectation values of Eq.(21.16). Then, the divergence theorem applied to the left hand side of Eq.(21.16) says that

$$
\begin{aligned}
\int_{\Omega(x_1)} d^D x\, \partial_\mu \Big\langle T^{\mu\nu}(x) \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle = & \\
= \int_{\Sigma(x_1)} dS_\mu \Big\langle T^{\mu\nu}(x) \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle & \\
= - \Big\langle P^\nu [\Sigma(x_1)] \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle & \qquad (21.17)
\end{aligned}
$$

Upon performing the same computation on the right hand side of Eq.(21.16) we obtain

$$
\begin{aligned}
- \int_{\Omega(x_1)} d^D x \sum_{j=1}^{N} \delta(x - x_j) \partial_j^\nu \Big\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle = & \\
= - \, \partial_{x_1}^\nu \Big\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle & \qquad (21.18)
\end{aligned}
$$

Therefore

$$
\Big\langle P^\nu [\Sigma(x_1)] \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle = \partial_{x_1}^\nu \Big\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle \qquad (21.19)
$$

In the operator language the expectation value of a product of operators is interpreted as the vacuum expectation value of the time ordered product of the operators. We can now consider two oppositely oriented surfaces Σ_1 and Σ_2 at two different times t_1 and t_2, and the region Ω whose boundary are both surfaces. If only one operator, say at x_1 is included in Ω, it is clear that we can always deform the region to a ball centered about x_1. Conversely, we can deform the simply connected region into the shell just described. Then, using the fact that Σ_1 and Σ_2 are oppositely oriented, the right hand side of Eq.(21.19) can be rewritten as

$$
\Big\langle \big(P^\mu [\Sigma_2] - P^\mu [\Sigma_1] \big) \mathcal{O}(x) \ldots \Big\rangle = \langle 0 | T \left\{ \big[P^\mu, \mathcal{O}(x) \big] \ldots \right\} | 0 \rangle \qquad (21.20)
$$

where T denotes the Euclidean imaginary time ordered product. Hence, we can make the operator identification

$$
\big[P^\mu, \mathcal{O}(x) \big] = \partial^\mu \mathcal{O}(x) \qquad (21.21)
$$

(we are missing a factor of i since we are in the Euclidean signature).

We saw in the preceding section that if a theory is conformally invariant it has more symmetries aside from translation invariance. Thus, rotational (or Lorentz) invariance implies the existence of a conserved angular momentum tensor $M_{\mu\nu}$ which, at the classical level is

$$
M_{\mu\nu} = - \int_\Sigma dS^\rho (x_\mu T_{\nu\rho} - x_\nu T_{\mu\rho}) \qquad (21.22)
$$

At the quantum level, the action of $M_{\mu\nu}$ on a local operator $\mathcal{O}^a(0)$ at the origin transform in irreducible representations of $SO(d)$ (here a labels the representation of $SO(d)$, again using the Euclidean signature) is

$$\left[M_{\mu\nu}, \mathcal{O}^a(0)\right] = \left[\mathcal{S}_{\mu\nu}\right]^a_b \mathcal{O}^b(0) \tag{21.23}$$

where a, b are the indices of the $SO(d)$ representation and the matrix $\mathcal{S}_{\mu\nu}$ are matrices that obey the same algebra as $M_{\mu\nu}$. On the other hand, they act on an operator at a general location x as

$$\left[M_{\mu\nu}, \mathcal{O}(x)\right] = \left[x_\mu \partial_\nu - x_\nu \partial_\mu + \mathcal{S}_{\mu\nu}\right] \mathcal{O}(x) \tag{21.24}$$

Likewise, the generator of infinitesimal dilatations is expressed in terms of the dilatation operator

$$D = -\int_\Sigma dS_\mu x_\nu T^{\mu\nu} \tag{21.25}$$

and the conserved dilatation current j_D^μ is

$$j_D^\mu = x_\nu T^{\mu\nu} \tag{21.26}$$

An operator $\mathcal{O}(0)$ that transforms irreducibly under dilatations is an eigen-operator of D, i.e.

$$[D, \mathcal{O}(0)] = \Delta \mathcal{O}(0) \tag{21.27}$$

where the eigenvalue Δ is the *scaling dimension* (or dimension) of the operator. We also have the Ward identity

$$[D, \mathcal{O}(x)] = \left(x^\mu \partial_\nu + \Delta\right) \mathcal{O}(x) \tag{21.28}$$

Similarly, the generator of infinitesimal special conformal transformations by

$$K^\mu = -\int_\Sigma dS_\rho \left(2x_\mu x^\nu T^{\nu\rho} - x^2 T^{\mu\rho}\right) \tag{21.29}$$

The operators D and K_μ generate symmetries (and are conserved) provided the energy-momentum tensor is traceless,

$$T^\mu_\mu = 0 \tag{21.30}$$

The operators P^μ, $M^{\mu\nu}$, D and K^μ satisfy the conformal algebra

$$\left[M_{\mu\nu}, P_\rho\right] = \delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu \tag{21.31}$$

$$\left[M_{\mu\nu}, K_\rho\right] = \delta_{nu\rho} K_\mu - \delta_{\mu\rho} K_\nu \tag{21.32}$$

$$\left[M_{\mu\nu}, M_{\rho\sigma}\right] = \delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\rho} M_{\nu\sigma} + \delta_{\nu\sigma} M_{\rho\mu} - \delta_{\mu\sigma} M_{\rho\nu} \tag{21.33}$$

$$\left[D, P_\mu\right] = P_\mu \tag{21.34}$$

$$\left[D, K_\mu\right] = -K_\mu \tag{21.35}$$

$$\left[K_\mu, P_\nu\right] = 2\delta_{\mu\nu} D - 2M_{\mu\nu} \tag{21.36}$$

Finally we note two more Ward identities. The first involves rotations (and Lorentz transformations) and involves the symmetry of the energy-momentum tensor,

$$\begin{aligned} \partial_\mu \Big\langle \Big(T^{\mu\nu}(x) x^\rho - T^{\mu\rho}(x) x^\nu \Big) \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle = \\ = \sum_{i=1}^N \delta(x - x_i) \Big[\Big(x_i^\mu \partial_i^\rho - x_i^\rho \partial_i^\nu + \mathcal{S}^{\nu\rho} \Big) \Big\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle \Big] \end{aligned} \tag{21.37}$$

which, using the conformal Ward identity of Eq.(21.16), becomes

$$\begin{aligned} \Big\langle \Big(T^{\mu\nu}(x) - T^{\nu\mu}(x) \Big) \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle = \\ = -\sum_i \delta(x - x_i) \mathcal{S}_i^{\mu\nu} \Big\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle \end{aligned} \tag{21.38}$$

Hence, the energy-momentum tensor as an operator is generally symmetric away from other operator insertions.

The other Ward identity involves the dilatation current, j_D^μ, and it is given by

$$\begin{aligned} \partial_\nu \Big\langle T_\nu^\mu(x) x^\nu \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle = \\ = -\sum_i \delta(x - x_i) \Big(x_i^\nu \partial_\nu^i + \Delta_i \Big) \Big\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle \end{aligned} \tag{21.39}$$

Using once again Eq.(21.16), we obtain

$$\begin{aligned} \Big\langle T_\mu^\mu(x) \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle = \\ = -\sum_i \delta(x - x_i) \Delta_i \Big\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_N(x_N) \Big\rangle \end{aligned} \tag{21.40}$$

Thus, the symmetry of the energy-momentum tensor holds in expectation values away from operator insertions. It also implies that at the quantum level the trace of the energy-momentum tensor is the generator of scale transformations. Hence, as an operator we have

$$\left[T_\mu^\mu, \mathcal{O}_i(0)\right] = \Delta_i \mathcal{O}_i(0) \tag{21.41}$$

from which it follows that, as an operator, the trace of the energy-momentum tensor must be the same as the divergence of the dilatation current,

$$T_\mu^\mu = \partial_\mu j_D^\mu \tag{21.42}$$

Therefore, scale invariance requires the energy-momentum tensor to be traceless.

21.4 General Consequences of Conformal Invariance

Let us consider now a theory which is invariant under the action of the generators of the conformal group, Eqs.(21.31)-(21.36). Thus, the vacuum state $|0\rangle$ is, by definition, annihilated by all the generators.

Let us consider now the consequences for the two point function of two operators $\mathcal{O}_i$ and $\mathcal{O}_j$,

$$F_{ij}(x,y) = \langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\rangle = \langle 0|T\left(\mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\right)|0\rangle \tag{21.43}$$

Translation and rotation invariance imply that it must depend only on the distance, $F(x_i, x_j) = F(|x_i - x_j|)$.

To examine the behavior of operators of a theory with conformal invariance under scale transformations we use the condition that the vacuum must be scale invariant

$$D|0\rangle = 0 \tag{21.44}$$

Then, we must also have

$$\begin{aligned} 0 =& \langle 0|\left[D, \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\right]|0\rangle \\ =& \langle 0|\left(\left[D, \mathcal{O}_i(x_i)\right]\mathcal{O}_j(x_j) + \mathcal{O}_i(x_i)\left[D, \mathcal{O}_j(x_j)\right]\right)|0\rangle \\ =& \left(x_i^\mu \partial_\mu^i + \Delta_i + x_j^\mu \partial_\mu^j + \Delta_j\right)\langle 0|\mathcal{O}_i(x_i)\mathcal{O}_j(x_j)|0\rangle \end{aligned} \tag{21.45}$$

The solution of this equation is that the correlator obeys a power law

$$F_{ij}(x_i - x_j) = \frac{C}{|x_i - x_j|^{\Delta_i + \Delta_j}} \tag{21.46}$$

where C is an arbitrary constant.

In addition, the action of special conformal transformations, whose infinitesimal generators are the operators K_μ, further restrict the form of the two-point functions of primary operators obey an "orthogonality condition", i.e. to vanish unless the scaling dimensions are equal,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \begin{cases} \dfrac{C_{12}}{|x_1 - x_2|^{2\Delta}}, & \text{if } \Delta_1 = \Delta_2 = \Delta \\ 0, & \text{otherwise} \end{cases} \tag{21.47}$$

Let us now turn to the three-point function of primary operators. Invariance under translations and rotations, and covariance under dilatations require that the three-point functions of primary operators must have the form

$$\Big\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\Big\rangle == \sum_{a,b,c} \frac{C_{abc}}{|x_1 - x_2|^a |x_2 - x_3|^b |x_3 - x_1|^c} \tag{21.48}$$

where the sum is restricted to values of a, b and c such that $a + b + c = \Delta_1 + \Delta_2 + \Delta_3$. However, covariance under special conformal transformations imply the additional restriction that $a = \Delta_1 + \Delta_2 - \Delta_3$, $b = \Delta_2 + \Delta_3 - \Delta_1$, and $c = \Delta_3 + \Delta_1 - \Delta_2$. Therefore, the three-point function must have the general form (Polyakov, 1970)

$$\Big\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\Big\rangle = \\ = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3} |x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1} |x_3 - x_1|^{\Delta_3+\Delta_1-\Delta_2}} \tag{21.49}$$

where C_{123} is a so-far undetermined constant. In fact, if the operators are normalized such that the coefficient of the two-point function $C_{12} = 1$, then the coefficients C_{123} of the three-point function must be universal numbers. We have used these results already in Sections 15.3 and 15.4 where we introduced the Renormalization Group and the Operator Product Expansion.

Finally, let us consider the implications of conformal invariance for N-point functions, with $N \geq 4$. Here the behavior is more complex. For instance, in the case of the four-point function, the general behavior, already presented in Eq.(15.98), depends also on the cross ratios

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = F\left(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{12}r_{34}}{r_{23}r_{41}}\right) \prod_{i<j} r_{ij}^{-(\Delta_i+\Delta_j)+\Delta/3} \tag{21.50}$$

where $\Delta = \sum_{i=1}^{4} \Delta_i$, and where we used the notation $r_{ij} = |x_i - x_j|$. Again, once the two-point function is normalized as before, the prefactor is a universal function of the two cross ratios.

In summary, conformal invariance of the theory restricts the form of the correlation functions and reveal that there are some quantities such as the scaling dimensions, the coefficient of the three-point functions and the functions of the cross ratios that are not determined, unless additional conditions are imposed. We will see in that imposing unitarity (or, equivalently, reflection positivity) and some additional symmetries allow in some cases to fully determine these quantities. This approach yields powerful results in two dimensions (that we will discuss below), and to some extent in three dimensions (which we will not discuss here).

In a physically sensible theory the correlators must obey cluster decomposition and must decay at long distances. This implies that the scaling dimensions of the operators, Δ_i, must be non-negative real numbers, i.e. $\Delta_i \geq 0$ (for all operators). In addition, using the commutation relation of Eq.(21.35), we find

$$DK_\mu \mathcal{O}(0) = \left(\left[D, K_\mu\right] + K_\mu D\right) \mathcal{O}(0) = (\Delta - 1)\, K_\mu \mathcal{O}(0) \tag{21.51}$$

In other words, the operator K_μ lowers the scaling dimension of the operator. Hence, if we act repeatedly with the operators K_μ on an operator $\mathcal{O}$, one obtains operators of the form $K_{\mu_1} \ldots K_{\mu_N} \mathcal{O}$ which can have an arbitrarily low dimension. Since the allowed dimensions must be non negative, the theory must have a special class of operators such that

$$\left[K_\mu, \mathcal{O}(0)\right] = 0 \tag{21.52}$$

Away from the origin, at a finite location x, the action of K_μ generalizes to

$$\left[K_\mu, \mathcal{O}\right] = \left(2x_\mu x_\nu \partial_\nu - x^2 \partial_\mu + 2x_\mu \Delta - 2x^\nu \mathcal{S}_{\mu\nu}\right) \mathcal{O}(x) \tag{21.53}$$

Operators that obey the condition of Eq.(21.52), i.e. invariant under special conformal transformations, are called *primary fields*. Then, given a primary operator $\mathcal{O}$ of scaling dimension Δ, we can construct an (in principle) infinite tower of em descendant operators $P_{\mu_1} \ldots P_{\mu_N} \mathcal{O}(0)$, of increasing dimension, $\Delta + N$ since the action of one momentum operator increases the dimension of the operator $\mathcal{O}$ by 1. Hence, the scaling dimensions Δ of the primary operators play the role of quantum numbers of representations of the conformal group, and label an infinite tower of descendant operators.

We now state the axioms that a conformal field theory must satisfy.

1) The vacuum state $|0\rangle$ of the theory must be invariant under conformal transformation.
2) The theory has a set of primary operators (or fields) that satisfy:
   i) They are eigen-operators of the dilatation operator D with eigenvalue Δ,

ii) of the angular momentum operator $M_{\mu\nu}$ with eigenvalue $\mathcal{S}_{\mu\nu}$,

ii) commute with the operator K_μ.

3) Under a conformal transformation scalar (spinless) primary fields transform as

$$\mathcal{O}_j(x) \mapsto \left|\frac{\partial x'}{\partial x}\right|^{\Delta_j/D} \mathcal{O}_j(x') \tag{21.54}$$

where, as before (c.f. Eq.(21.14))

$$\left|\frac{\partial x'}{\partial x}\right| = \Omega^{-D/2} \tag{21.55}$$

is the Jacobian of the conformal transformation with scale factor Ω.

4) The theory is covariant under conformal transformations in the sense that the correlators of the primary fields satisfy

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_N(x_N)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/D} \cdots \left|\frac{\partial x'}{\partial x}\right|_{x=x_N}^{\Delta_N/D} \langle \mathcal{O}_1(x_1')\dots\mathcal{O}_N(x_N')\rangle \tag{21.56}$$

5) The correlators must obey unitarity or, equivalently in an Euclidean theory, satisfy reflection positivity.

6) All other fields in the theory can be expressed as linear combinations of primary fields and their descendants.

These axioms extend to the case of operators with spin such as currents. One such example is the energy-momentum tensor which in dimensions $D > 2$ is a primary field with scaling dimension D. We will see shortly that in two dimensions it obeys an anomalous algebra and it is no longer a primary field.

21.5 Conformal Field Theory in two dimensions

We will turn to conformal field theory in two dimensions. This case has been studied in greater detail and it is better understood. It has many physical applications. Originally it was developed to formulate perturbative String Theory. It also has direct application to quantum field theories in 1+1 dimensions and to two-dimensional classical critical phenomena.(Belavin et al., 1984; Friedan et al., 1984; Ginsparg, 1989)

21.5.1 Classical Conformal Invariance in Two Dimensions

Let us consider theories in two-dimensional flat Euclidean spacetime. In Eq.(21.9) we showed that two-dimensional conformal transformations ε_μ obey the Cauchy-Riemann equations. Hence, the conformal transformations are analytic (or anti-analytic) functions. Let us write the Euclidean coordinates as complex coordinates $z = x_1 + ix_2$, $z \in \mathbb{C}$. Define also

$$\partial_z = \frac{1}{2}\left(\partial_1 - i\partial_2\right), \qquad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_1 + i\partial_2\right) \tag{21.57}$$

such that

$$\partial_z z = 1,\ \partial_{\bar{z}}\bar{z} = 1,\ \partial_z\bar{z} = 0,\ \partial_{\bar{z}}z = 0 \tag{21.58}$$

In what follows we will use the notation $\partial_z = \partial$ and $\partial_{\bar{z}} = \bar{\partial}$.

For a general vector field v^a $(a = 1, 2)$ we can also define the complex components

$$v^z = v^1 + iv^2,\ v^{\bar{z}} = v^1 - iv^2,\ v_z = \frac{1}{2}\left(v^1 - iv^2\right),\ v_{\bar{z}} = \frac{1}{2}\left(v^1 + iv^2\right) \tag{21.59}$$

While in Cartesian indices (1, 2) the Euclidean metric is the identity, $g_{ab} = \delta_{ab}$, in complex coordinates the metric tensor is

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2},\ g_{zz} = g_{\bar{z}\bar{z}} = 0,\ g^{z\bar{z}} = g^{\bar{z}z} = 2,\ g^{zz} = g^{\bar{z}\bar{z}} = 0 \tag{21.60}$$

It is natural to write the conformal transformations as $\varepsilon(z) = \varepsilon_1 + i\varepsilon_2$ and $\bar{\varepsilon}(\bar{z}) = \varepsilon_1 - i\varepsilon_2$. Two-dimensional conformal transformation are the analytic (and anti-analytic) coordinate transformations

$$z \mapsto f(z), \quad \bar{z} \mapsto \bar{f}(\bar{z}) \tag{21.61}$$

under which the Euclidean interval transform as

$$ds^2 = dzd\bar{z} \mapsto \left|\frac{\partial f}{\partial z}\right|^2 dzd\bar{z} \tag{21.62}$$

and the Jacobian is

$$\Omega = \left|\frac{\partial f}{\partial z}\right|^2 \tag{21.63}$$

A natural basis for conformal transformations are $\varepsilon_n = z^{n+1}$ and $\bar{\varepsilon}_n = -\bar{z}^{n+1}$ (with $n \in \mathbb{Z}$). The infinitesimal generators of 2D classical conformal transformations are

$$\ell_n = -z^{n+1}\partial_z, \quad \bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}}, \quad (n \in \mathbb{Z}) \tag{21.64}$$

which obey the classical local algebra

$$\left[\ell_n, \ell_m\right] = (n-m)\ell_{n+m}, \quad \left[\bar{\ell}_n, \bar{\ell}_m\right] = (n-m)\ell_{n+m} \tag{21.65}$$

We will see shortly that at the quantum level this algebra is corrected to include a key new term, the conformal anomaly. Hence, we have two independent algebras. This means that, at the formal level, we will be working with z and $\bar{z}$ as independent variables, and hence not just with complex functions but with those on $\mathbb{C}^2$. We will need to project on the "physical subspace" in which $\bar{z} = z^*$.

We have called the classical conformal algebra, Eq.(21.65), *local* since the generators are not all well defined on the Riemann sphere, $S^2 = \mathbb{C} \bigcup \infty$. Holomorphic (analytic) conformal transformations are generated by vector fields $v(z)$,

$$v(z) = \sum_{n \in \mathbb{Z}} a_n z^{n+1} \partial_z \tag{21.66}$$

Such vector fields are generally singular as $z \to 0$. The only exception are conformal transformations generated by $\ell_{-1}, \ell_0, \ell_1$, and $\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1$. From their definitions we see that ℓ_{-1} and $\bar{\ell}_{-1}$ generate translations, $\ell_0 + \bar{\ell}_0$ generates dilatations (i.e. translations on the radial polar coordinate r), $i(\ell_0 - \bar{\ell}_0)$ generates rotations (i.e. translations on the angular polar coordinate θ), and ℓ_1, $\bar{\ell}_{-1}$ are the generators of special conformal transformations. The finite form of these transformations is

$$z \mapsto \frac{az + b}{cz + d}, \quad \bar{z} = \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}} \tag{21.67}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. This is the group $SL(2, \mathbb{C})/\mathbb{Z}_2$ (this quotient says that the transformation is unaffected by a change of sign of a, b, c, d. These are the only globally well defined conformal transformations and the only ones that also exist in dimensions $D > 2$.

Since the transformation generated by $\ell_{-1}, \ell_0, \ell_1$, and $\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1$ are globally well defined, we will work in the basis of their eigenstates. Thus, we will consider the eigenstates of ℓ_0 and $\bar{\ell}_0$ and we will denote their eigenvalues, known as the conformal weights, the real numbers h and $\bar{h}$ (not the complex conjugate!). Since $\ell_0 + \bar{\ell}_0$ generates dilatations and $i(\ell_0 - \bar{\ell}_0)$ generate rotations, the scaling dimension Δ and the spin s are given by $\Delta = h + \bar{h}$ and $s = h - \bar{h}$.

21.5.2 Quantization

We now turn to the quantum theory. We will work with Euclidean coordinates where σ^1 labels space and σ^0 labels imaginary time. We will assume that the space coordinate is finite and periodic. Hence we will identify $\sigma^1 \cong \sigma^1 + 2\pi$, while the imaginary time coordinate σ^0 can take any real value, positive or negative. In other words, the Euclidean space-time has been compactified to an infinitely long cylinder of circumference 2π, as shown in Fig.21.2a. Let $\zeta = \sigma^2 + i\sigma^1$ be the coordinates on the cylinder, and

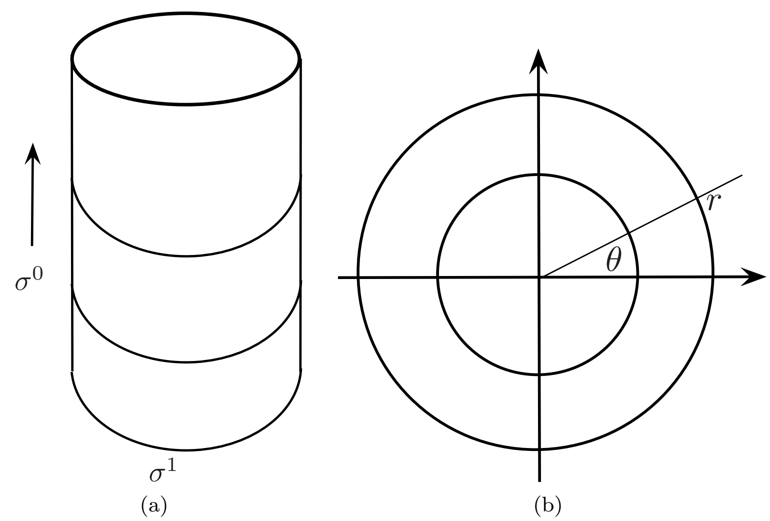

Figure 21.2 A conformal mapping transforms the cylinder to the plane.

consider the conformal mapping to the plane with coordinates $z = x^1 + ix^2$,

$$z = \exp \zeta = \exp\left(\sigma^0 + i\sigma^1\right) \tag{21.68}$$

which maps the cylinder with coordinates ζ to the complex plane with coordinates z. Notice that the infinite past on the cylinder, $\sigma^0 \to -\infty$, maps to the origin, $z = 0$, on the plane, and that the infinite future on the cylinder, $\sigma^0 \to +\infty$, maps to the point of infinity, $z \to \infty$, on the complex plane. Equal-time surfaces on the cylinder, at σ^0 = constant, map to a circle of constant radius on the z plane.

To develop a quantum theory we need the operators that implement conformal transformations on the plane. Thus, a dilatation $z \mapsto \exp(a\ z)$ on the plane is a time translation on the cylinder, $\sigma^0 \mapsto \sigma^0 + a$. Therefore, the dilatation generator can be regarded as the Hamiltonian H of the system. In other words, on the cylinder we have a Hilbert space of states on an imaginary time surface, σ^0 = constant. On the plane, the Hilbert space is defined on circles of constant radii which relate to each other by the action of the dilatation operator D. Likewise, the linear momentum P on the cylinder becomes the generator of rotations on the plane. For obvious reasons, this way of defining the quantum theory is known as radial quantization (Friedan et al., 1984).

As we saw before symmetry generators are constructed by the Noether's prescription. Thus, given a conserved current j^μ the conserved charge Q is obtained as the integral of the time component of the current on a fixed time surface. The associated symmetry transformations act on a field $\mathcal{O}$ as $\delta_\varepsilon \mathcal{O} = \varepsilon\left[Q, \mathcal{O}\right]$. Here we will be interested in coordinate transformations generated by the energy-momentum tensor $T^{\mu\nu}$. As we saw before, conformal invariance requires that the energy-momentum tensor be conserved, $\partial_\mu T^{\mu\nu}$ and traceless, $T^\mu_\mu = 0$, as operators acting on the Hilbert space.

It will be convenient to have the components of $T^{\mu\nu}$ in complex coordinates. In these frames they are,

$$
\begin{aligned}
T \equiv & T_{zz} = \frac{1}{4}\left(T_{00} - 2iT_{10} - T_{11}\right), \\
\bar{T} \equiv & T_{\bar{z}\bar{z}} = \frac{1}{4}\left(T_{00} + 2iT_{10} - T_{11}\right), \\
T^\mu_\mu \equiv & \Theta = T_{00} + T_{11} = 4T_{z\bar{z}} = 4T_{\bar{z}z}
\end{aligned}
\tag{21.69}
$$

So, in general, it has three components, T, $\bar{T}$, and Θ. The local conservation of the energy-momentum tensor yields the conditions

$$
\bar{\partial} T + \frac{1}{4}\partial\Theta = 0, \quad \partial\bar{T} + \frac{1}{4}\bar{\partial}\Theta = 0 \tag{21.70}
$$

Conformal invariance requires that the energy-momentum tensor be traceless condition, $\Theta = 0$. Then, the conservation laws simply become

$$
\bar{\partial} T = 0, \text{ and } \partial\bar{T}(\bar{z}) = 0 \tag{21.71}
$$

Therefore, the remaining non-vanishing components of $T^{\mu\nu}$ satisfy

$$
T(z) \equiv T_{zz}(z), \quad \bar{T}(\bar{z}) = T_{\bar{z}\bar{z}} \tag{21.72}
$$

are, respectively, holomorphic (analytic) and anti-holomorphic (anti-analytic). We will see that this property means that expectation values of physical observables factorize into analytic and anti-analytic components. In the Minkowski spacetime signature, we will refer to them as the right and left-moving (or chiral and anti-chiral) components of the fields.

Given an infinitesimal conformal transformation $\varepsilon(z)$, the conserved charge Q is

$$
Q(\varepsilon, \bar{\varepsilon}) = \frac{1}{2\pi i}\oint_{C(r)} \left(dz T(z)\varepsilon(z) + \bar{T}(\bar{z})\bar{\varepsilon}(\bar{z})\right) \tag{21.73}
$$

where $C(r)$ is a circle of radius r centered at the origin, $z = 0$. The variation of a field $\Phi(u, \bar{u})$ is the equal-time commutator with the charge

$$\delta_{\epsilon,\bar{\varepsilon}}\Phi(u,\bar{u}) = \frac{1}{2\pi i}\oint_{C(r)} \left[dzT(z)\varepsilon(z), \Phi(u,\bar{u})\right] + \left[dz\bar{T}(\bar{z})\bar{\varepsilon}(\bar{z}), \Phi(u,\bar{u})\right] \tag{21.74}$$

A primary field $\Phi(z, \bar{z})$ transforms under a local conformal transformation as

$$\Phi(z,\bar{z}) \mapsto \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})) \tag{21.75}$$

where h and $\bar{h}$ are the conformal weights. For an infinitesimal transformation this should be

$$\delta_{\epsilon,\bar{\varepsilon}}\Phi(z,\bar{z}) = \left((h\partial\varepsilon + \varepsilon\partial) + (\bar{h}\bar{\partial}\bar{\varepsilon} + \bar{\varepsilon}\bar{\partial})\right)\Phi(z,\bar{z}) \tag{21.76}$$

Since equal-time surfaces on the cylinder map onto circles of fixed radius on the complex plane, we will introduce the concept or *radially ordered product* (analogous to a time-ordered product) for two bosonic operators $A(z)$ and $B(w)$

$$R(A(z)B(w)) = \begin{cases} A(z)B(w), & \text{if } |z| > |w|, \\ B(w)A(z), & \text{if } |z| < |w| \end{cases} \tag{21.77}$$

(with a minus sign for fermions). Then, the equal-time commutator is the contour integral of the radially-ordered product, shown in Fig.21.3,

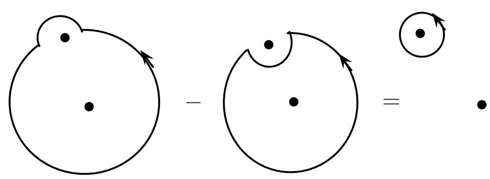

Figure 21.3 Computation of an equal-time commutation relation.

$$\delta_{\epsilon,\bar{\varepsilon}}\Phi(u,\bar{u}) =$$
$$\frac{1}{2\pi i}\left(\oint_{|z|>|u|} - \oint_{|u|>|z|}\right)\left(dz\varepsilon(z)R(T(z),\Phi(u,\bar{u})) + d\bar{z}\bar{\varepsilon}(\bar{z})R(\bar{T}(\bar{z})\Phi(u,\bar{u}))\right) \tag{21.78}$$

$$=\frac{1}{2\pi i}\oint\left(dz\varepsilon(z)R(T(z)\Phi(u,\bar{u})) + d\bar{z}\bar{\varepsilon}(\bar{z})R(\bar{T}(\bar{z})\Phi(u,\bar{u}))\right) \tag{21.79}$$

$$=h\partial\varepsilon(u)\Phi(u,\bar{u}) + \varepsilon(u)\partial\Phi(u,\bar{u}) + \bar{h}\bar{\partial}\bar{\varepsilon}(\bar{u})\Phi(u,\bar{u}) \tag{21.80}$$

where we imposed consistency with Eq.(21.76). This implies that for the last two lines of Eq.(21.80) to be consistent with each other, the the product of $T(z)$ and $\bar{T}(z)$ with $\Phi(u,\bar{u})$ should have short-distance singularities and obey the OPEs

$$T(z)\Phi(u,\bar{u}) = \frac{h}{(z-u)^2}\Phi(u,\bar{u}) + \frac{1}{z-u}\partial_u\Phi(u,\bar{u}) + \ldots \tag{21.81}$$

$$\bar{T}(\bar{z})\Phi(u,\bar{u})) = \frac{\bar{h}}{(\bar{z}-\bar{u})^2}\Phi(u,\bar{u}) + \frac{1}{\bar{z}-\bar{u}}\partial_{\bar{u}}\Phi(u,\bar{u}) + \ldots \tag{21.82}$$

In other terms, a field $\Phi(z,\bar{z})$ is a primary field if it has an operator product expansion with the holomorphic T and anti-holomorphic $\bar{T}$ components of the energy-momentum tensor of the form of Eq.(21.82). Here, the ellipsis represents non-singular contributions which drop out of the contour integrals, and hence from the commutators. Eq.(21.82) has to be understood as an operator identity. A more precise way to state this is the following conformal ward identity

$$\begin{aligned}&\Big\langle T(z)\mathcal{O}_1(u_1,\bar{u}_1)\ldots\mathcal{O}_N(u_N,\bar{u}_N)\Big\rangle\\ &\qquad= \sum_{j=1}^{N}\Big(\frac{h_j}{(z-u_j)^2} + \frac{1}{z-u_j}\partial_j\Big\langle\mathcal{O}_1(u_1,\bar{u}_1)\ldots\mathcal{O}_N(u_N,\bar{u}_N)\Big\rangle\end{aligned} \tag{21.83}$$

This identity requires that the correlation functions be meromorphic functions of z with singularities at the positions of the operators.

In a 2D CFT the correlator of two primary fields has the form

$$\langle\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(u,\bar{u})\rangle = \delta_{ij}\frac{1}{(z-u)^{2h_i}(\bar{z}-\bar{u})^{2\bar{h}_i}} \tag{21.84}$$

On the other hand, as we already saw in Section 15.4, the primary fields obey an algebra know as the Operator Product Expansion which has the form

$$\lim_{z\to u,\bar{z}\to\bar{u}}\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(u,\bar{u}) = \sum_k \frac{C_{ijk}}{(z-u)^{h_i+h_j-h_k}(\bar{z}-\bar{u})^{\bar{h}_i+\bar{h}_j-\bar{h}_k}}\mathcal{O}_k(u,\bar{u}) \tag{21.85}$$

21.5.3 The Virasoro Algebra

In the previous subsection we worked out the form of the OPE between the energy-momentum tensor an a primary field. All operators (or fields) in the theory can be classified into families each labeled by a primary field. The other members of each family are called the descendants of the primary. In this sense the primary field is the highest weight of the representation.

However we have not said anything about the energy-momentum tensor and of its OPE with itself. We know that in dimensions $D > 2$ the energy-momentum tensor is a primary field. However, this is not the case in $D = 2$ dimensions. The energy-momentum tensor $T(z)$ is holomorphic and has dimension 2. Thus it must have conformal weight $(h, \bar{h}) = (2, 0)$. By performing two conformal transformations in sequence, we see that the OPE of the energy-momentum tensor with itself must have the following form,

$$T(z)T(u) = \frac{c/2}{(z-u)^4} + \frac{2}{(z-u)^2}T(u) + \frac{1}{z-u}\partial T(u) \tag{21.86}$$

The first term, proportional to the identity field, is allowed by analyticity, Bose symmetry, and scale invariance. The coefficient c is known as the *central charge* of the CFT.

In left-moving (light-cone) coordinates, x_-, of Minkowski space-time, the Virasoro algebra is

$$-i[T(x_-), T(x'_-)] = \delta(x_- - x'_-)\partial_- T(x_-) - 2\partial_- \delta\delta(x_- - x'_-) + \frac{c}{24\pi}\partial_-^3 \delta(x_- - x'_-) \tag{21.87}$$

The last term of the right hand side of the algebra, a Schwinger term, is the central extension of the Virasoro algebra.. In this case is an anomaly of the trace of the energy momentum tensor.

The OPE implies that the correlator of the energy-momentum tensor must be

$$\langle T(z)T(u)\rangle = \frac{c/2}{(z-u)^4} \tag{21.88}$$

For $\bar{T}$ we have instead

$$\bar{T}(\bar{z})\bar{T}(\bar{u}) = \frac{\bar{c}/2}{(\bar{z}-\bar{u})^4} + \frac{2}{(\bar{z}-\bar{u})^2}T(u) + \frac{1}{\bar{z}-\bar{u}}\partial T(u) \tag{21.89}$$

and the correlator is, instead,

$$\langle \bar{T}(\bar{z})\bar{T}(\bar{u})\rangle = \frac{\bar{c}/2}{(\bar{z}-\bar{u})^4} \tag{21.90}$$

In principle the central charges c and $\bar{c}$ can be different. If that is the case the CFT is chiral. If an additional symmetry, called modular invariance, is imposed, then the the two central charges must be equal, $c = \bar{c}$. This condition follows from the requirement that the gravitational anomaly cancels and it is normally required. However, there are physical systems, such as the edge states of the fractional quantum Hall fluids, which are chiral theories and this condition is violated. This is possible since in that case these states cannot exist on themselves but as boundaries of a higher dimensional system.

Eq.(21.86) implies that under an infinitesimal conformal transformation, $\varepsilon(z)$, the energy momentum tensor changes as

$$\delta_\varepsilon T(z) = \varepsilon(z)\partial T(z) + 2\partial\varepsilon(z)T(z) + \frac{c}{12}\partial^3\varepsilon(z) \tag{21.91}$$

For a conformal finite transformation $z \mapsto z' = f(z)$ the energy-momentum tensor transforms as follows

$$T'(z') = (f'(z))^2 T(z') + \frac{c}{12}\{z'; z\} \tag{21.92}$$

and similarly for $\bar{T}$ (upon replacing $c \to \bar{c}$). Here $f'(z) = \frac{df}{dz}$ and $\{u; z\}$ is the Schwartzian derivative

$$\{z'; z\} = \frac{f''' f' - \frac{3}{2} f''^2}{f'^2} \tag{21.93}$$

The second term in the transformation of Eq.(21.92) is known as the *conformal anomaly*. It is an anomaly in the sense that it is absent in the classical theory, where the energy momentum-tensor transforms *homogeneously* (with rank 2), as shown in the first term of the transformation. For this reason in a 2D CFT the energy-momentum tensor is not a primary field. The conformal anomaly is a quantum effect which violates the naive homogeneous transformation law.

It is useful to write the Laurent expansion of the energy-momentum tensor

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}, \quad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \frac{\bar{L}_n}{\bar{z}^{n+2}} \tag{21.94}$$

in terms of the operators L_n and $\bar{L}_n$ (the "modes"), which have scaling dimension n which satisfy the hermiticity condition

$$L_n^\dagger = L_{-n} \tag{21.95}$$

The series expansion of Eq.(21.94) can be inverted using contour integrals

$$L_n = \frac{1}{2\pi i}\oint_C dz\, z^{n+1} T(z), \quad \bar{L}_n = \frac{1}{2\pi i}\oint_C d\bar{z}\, \bar{z}^{n+1} \bar{T}(\bar{z}) \tag{21.96}$$

where there are no other operator insertions inside these contours.

The commutation relations for the operators L_n (and similarly for $\bar{L}_n$) are obtained using the OPE for the energy-momentum tensor. The result is

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \tag{21.97}$$

This is the Virasoro algebra. Similarly, for the mode expansion of $\bar{T}$ we get

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12}(n^3-n)\delta_{n+m,0} \tag{21.98}$$

Finally, since the OPE of $T(z)$ and $\bar{T}(\bar{z})$ have no singularities, the generators of the two Virasoro algebras must commute

$$[L_n, \bar{L}_m] = 0 \tag{21.99}$$

The algebra of Eq.(21.97), the Virasoro algebra, differs from the classical conformal algebra of Eq.(21.65) by the conformal anomaly term, the term proportional to the identity operator on the right hand side of Eq.(21.97). This term is known as the central extension of the Virasoro algebra. As we will see below, in quantum field theory it arises as a Schwinger term that reflects the short-distance singularities of the theory.

The central extension is absent if $n = 0, \pm 1$ for which the operators L_0, L_1 and L_{-1} satisfy a closed subalgebra without a central extension,

$$[L_1, L_{-1}] = 2L_0, \quad [L_{\pm 1}, L_0] = \pm L_{\pm 1} \tag{21.100}$$

These operators generate the global conformal group $SL(2,\mathbb{C})$.

The action of the Virasoro generators on the primary fields is derived from Eq.(21.82),

$$\begin{aligned}[L_n, \mathcal{O}] &= z^{n+1}\partial\mathcal{O} + h(n+1)z^n\mathcal{O} \\ [\bar{L}_n, \mathcal{O}] &= \bar{z}^{n+1}\bar{\partial}\mathcal{O} + \bar{h}(n+1)\bar{z}^n\mathcal{O}\end{aligned} \tag{21.101}$$

where, as before, $\Delta = h + \bar{h}$ is the scaling dimension and $s = h - \bar{h}$ is the spin.

For a theory is quantized on a cylinder, c.f. Fig.21.2a, we will regard the operator $H = L_0 + \bar{L}_0$ as the quantum Hamiltonian and $P = L_0 - \bar{L}_0$ with the linear momentum. On the other hand, in radial quantization, c.f. Fig.21.2b, $L_0 + \bar{L}_0$ is the dilatation operator of the Euclidean theory and $L_0 - \bar{L}_0$ is the angular momentum. Let $|0\rangle$ be the vacuum state of a two-dimensional CFT, defined as the state with the lowest (zero) eigenvalue of L_0

$$L_0|0\rangle = \bar{L}_0|0\rangle = 0 \tag{21.102}$$

The lowering operators of L_0 ($\bar{L}_0$) are L_{-n} ($\bar{L}_{-n}$) with $n > 0$.

Just as in the theory of angular momentum, a state annihilated by the lowering operators is said to be a highest weight state.The vacuum is a highest weight state because it has the lowest eigenvalue of $L_0+\bar{L}_0$. Similarly, the state $\mathcal{O}|0\rangle$ is an eigenstate of L_0 ($\bar{L}_0$) with eigenvalue h ($\bar{h}$) and is also a highest weight state. Given a highest weight state an (in principle infinite) tower of states, the descendants, is constructed by acting on the highest weight state with lowering operators. The space of states of a CFT is then a sum of irreducible representations of the algebra of L's and $\bar{L}$'s each generated from a highest weight state,

$$\mathcal{O}|0\rangle \equiv |h, \bar{h}\rangle \tag{21.103}$$

resulting from the action of a primary field on the vacuum state. The L_0 and $\bar{L}_0$ eigenvalues of the highest weight state are

$$L_0\mathcal{O}|0\rangle = h\mathcal{O}|0\rangle, \qquad \bar{L}_0\mathcal{O}|0\rangle = \bar{h}\mathcal{O}|0\rangle \tag{21.104}$$

A representation of the Virasoro algebra is built from the highest weight state by the action of the lowering operators, L_{-n} (with $n \geq 1$). A state is in the nth level if the L_0 eigenvalue is $h+n$. The nth level is spanned by the states $L_{-k_1}\dots L_{-k_n}\mathcal{O}|0\rangle$, with $k_1 \geq \dots \geq k_n \geq 0$ and $\sum_j k_j = n$. There are $P(n)$ such states, where $P(n)$ is the number of ways of writing n as a sum of positive integers (partitions). The higher level states correspond to operators of increasingly higher dimensions obtained by applying products of energy-momentum tensors on the highest weight state.

In summary, a two-dimensional CFT is characterized the following data: the a central charge c, the conformal weights h of the primary fields $\mathcal{O}$, and the coefficients of the OPEs of the primary fields, i.e. their fusion rules. We will discuss below how this works in a few examples of 2D CFT's of interest. However several questions arise. One is whether there are additional constrains, aside form conformal invariance, that may restrict (or even specify) what 2D CFTs are allowed.

One constraint, natural from the point of view of quantum field theory is unitarity (or, its Euclidean version, reflection positivity). We will see that this is a powerful constraint. We should note that there are many examples of systems in classical statistical mechanics which are conformally invariant but not unitary. So far as what we have discussed is concerned the number of primaries may well be infinite. It is natural to ask if there are constraints that will restrict the primaries to a finite number. Such theories are known as rational CFTs. Another way to further restrict the CFTs is to impose the condition that the theory may also have global continuous symmetries and the associated conserved currents. In this case the algebra of the currents, generally known as a Kac-Moody algebra, combined with the Virasoro algebra, provides a framework to construct CFTs.

21.5.4 Physical Meaning of the Central Charge

The central charge plays such a crucial role in a two-dimensional CFT. We will now see that it has a direct physical meaning (Affleck, 1986b; Blöte et al., 1986)

Let Z be the partition function of the theory, which we will regard as either a quantum field theory in $D = 2$ Euclidean spacetime of a classical statistical mechanical system in $D = 2$ dimensions. The free energy of the system is $F = -\ln Z$. Let us consider a system of linear size L. Then, in a local theory, we expect the system to have a well defined thermodynamic limit, $L \to \infty$. In this limit the free energy can be expressed in terms of finite densities as

$$F = AL^2 + BL + \dots \tag{21.105}$$

The coefficient of the leading term, f_0, is the free energy density. It is independent of the boundary conditions and in general it is not universal. the second term is also generally not universal and it appears if the system has a boundary. Among the correction terms there is a term $O(L^0)$ which, as we will see, is universal if the system has no boundaries (i.e. it is defined on a sphere, a torus, etc) is conformally invariant. In addition, if the system has universal corrections proportional to $\ln L$.

To understand the origin of these universal corrections we will consider (for simplicity) the case of a conformal field theory on a manifold $\mathcal{M}$ without boundaries. Such a manifold is in general curved. Under a global infinitesimal dilatation, $x^\mu \mapsto (1+\varepsilon)x^\mu$, the action changes by an amount determined by the trace of the energy-momentum tensor, Θ,

$$\delta S = -\frac{\varepsilon}{2\pi} \int_{\mathcal{M}} \Theta(x)\sqrt{g} d^2x \tag{21.106}$$

where $g^{\mu\nu}$ is the metric of the manifold $\mathcal{M}$ and $g = \left|\det g^{\mu\nu}\right|$ is the determinant. The factor of $1/(2\pi)$ is introduced for later convenience.

In the renormalization group, the total partition function must be invariant under such a rescaling. Thus is must be true that

$$Z = \exp(-F(L)) = \exp(-F(L+\delta L) - \langle \delta S \rangle) \tag{21.107}$$

so that the change in the free energy is the *negative* of $\langle \delta S \rangle$. Therefore

$$L \frac{\partial F}{\partial L} = \frac{1}{2\pi} \int_{\mathcal{M}} \langle \Theta \rangle \sqrt{g} d^2x \tag{21.108}$$

We have seen that conformal invariance implies that the energy-momentum tensor be traceless, $\Theta = 0$. This is true only in flat spacetime. If the manifold $\mathcal{M}$ has a scalar curvature, then there is a scale and the energy-momentum tensor is not traceless. The expectation value of the trace of the energy-momentum tensor is derived by considering a weakly curved system such an infinitesimal coordinate transformation is represented by the infinitesimal change in the metric $\delta g_{\mu\nu} = \partial_\nu \varepsilon_\mu + \partial_\mu \varepsilon_\nu$, so that the change in the action is

$$\delta S = -\frac{1}{4\pi} \int T^{\mu\nu} \delta g_{\mu\nu} d^2x \tag{21.109}$$

However this expression also apply to changes in the metric due to changes in the geometry, not just to coordinate changes. This means that the invariance of the partition function implies

$$\operatorname{Tr} \exp\left(-S + \frac{1}{4\pi} \int T^{\mu\nu} \delta g_{\mu\nu} d^2x \right) \Big|_{\text{new geometry}} = \operatorname{Tr} \exp(-S) \Big|_{\text{old geometry}} \tag{21.110}$$

The computation of the changes of the expectation values of all three components of the energy-momentum tensor. This requires to compute integrals of the correlators of energy-momentum tensors which, in turn, need a short-distance regularization. The origin trace anomaly can be traced back to the violation of conformal invariance by the regularization. This calculation leads to the important result, for closed manifolds,

$$\langle \Theta(x) \rangle = \frac{c}{12} R(x) \tag{21.111}$$

This relation, called the *trace anomaly*, measures the response to a weak deformation of the geometry. Here $R(x)$ is the scalar curvature of the two-dimensional space. Notice that $\langle \Theta \rangle \neq 0$ in flat spacetime if the system is not at a fixed point. Although the term trace anomaly is also used in such systems it is clear that there is nothing anomalous about them.

Another interesting interpretation of the central charge is obtained by considering conformally invariant theories defined on an infinite cylinder, i.e. a strip with coordinates $-\infty < u < \infty$ and a periodic coordinate $0 \leq v \leq \ell$. We can think of this geometry in two different ways. One is a 1+1-dimensional conformal field theory at finite temperature $T = 1/\ell$ (in units in which the Boltzmann constant $k_B = 1$). The other interpretation is that we are considering a conformally field theory on a circle with circumference ℓ.

In both cases the theory on the cylinder is related to the theory on the infinite flat plane by the conformal mapping

$$w = u + iv = \frac{\ell}{2\pi} \ln z \tag{21.112}$$

The energy momentum tensor on the cylinder and on the plane are related by the transformation law of Eq.(21.92) which, for this specific conformal mapping yields

$$T(w)_{\text{cylinder}} = \left(\frac{2\pi}{\ell}\right)^2 \left(z^2 T(z)_{\text{plane}} - \frac{c}{24}\right) \tag{21.113}$$

By translation and rotational invariance the expectation value of the energy-momentum tensor on an infinite plane must vanish. Hence

$$\langle T(w)_{\text{cylinder}} \rangle = -\frac{c\pi^2}{6\ell^2} \tag{21.114}$$

Let us consider first the case that we are considering the case of a conformal field theory on a finite spatial interval of length ℓ. In this case, the coordinate u is imaginary time and the periodic coordinate v is space. In this case the Hamiltonian is

$$\begin{aligned} H =& \frac{1}{2\pi} \int_0^\ell T_{uu} dv = \frac{1}{2\pi} \int_0^\ell \left(T(v) + \bar{T}(v)\right) dv \\ =& \frac{2\pi}{\ell} \left(L_0 + \bar{L}_0\right) - \frac{\pi c}{6\ell} \end{aligned} \tag{21.115}$$

where we used Eq.(21.114) and that

$$L_0 = \frac{1}{2\pi i} \oint zT(z) dz, \qquad \bar{L}_0 = -\frac{1}{2\pi i} \oint \bar{z}\bar{T} d\bar{z} \tag{21.116}$$

Similarly, the momentum operator P is

$$P = \frac{1}{2\pi} \int_0^\ell T_{uv} dv = \frac{2\pi}{\ell} (L_0 - \bar{L}_0) \tag{21.117}$$

Eq.(21.115) implies that the energy of the ground state, defined by $L_0|0\rangle = \bar{L}_0|0\rangle = 0$, is

$$E_{\text{gnd}} = -\frac{\pi c}{6\ell} \tag{21.118}$$

where we have set to zero the extensive (and non-universal) part of the ground state energy.

Eq(21.118) is the Casimir effect, which we have already discussed in Chapter 8 for a free massless scalar field. Here we see that it is a general result of a CFT. An important observation is that it is proportional to the central charge c. In this, sense the central charge "counts" the number of degrees of freedom (even though c generally is not an integer!).

In addition, we see that the energy E and momentum P eigenvalues of the highest weight state $|h, \bar{h}\rangle$ are

$$E = E_{\text{gnd}} + \frac{2\pi\Delta}{\ell}, \qquad P = \frac{2\pi s}{\ell} \tag{21.119}$$

where $\Delta = h + \bar{h}$ is the scaling dimension and $s = h - \bar{h}$ is the conformal spin of the primary field.

In the other interpretation of the theory on the cylinder, the space coordinate has infinite extent, $-\infty \le u \le \infty$, while the periodic coordinate $0 \le v \le \ell$ is interpreted as imaginary time with length $\ell = 1/T$, where T is the temperature. Now Eq.(21.118) becomes the thermal contribution to the free energy density (i.e per unit length) f of a CFT at temperature T,

$$f = \varepsilon_{\text{gnd}} - \frac{\pi}{6} c T^2 \tag{21.120}$$

where the extra factor of T comes from the definition of the free energy and $\varepsilon_{\text{gnd}} = \lim_{\ell\to\infty} E_{\text{gnd}}/\ell$. Here we have assumed that the speed of the excitations, the speed of light, was set to 1.

The last term of Eq.(21.120) is a generalization of the Stefan-Boltzmann law for blackbody radiation. From here it follows that the specific heat $C(T)\ell$ of a CFT is

$$\lim_{\ell\to\infty} \frac{C(T)}{\ell} = \frac{\pi}{3} c T \tag{21.121}$$

This result, again, suggests the interpretation of the central charge as a measure of the number of degrees of freedom.

21.5.5 The C Theorem

So far we have discussed general properties of conformal field theories. However, in general a theory may not be conformal. We have seen this in detail in our discussion of the Renormalization Group (RG) in Chapter 15 where we considered a theory at some fixed point perturbed by a set of interactions labeled by some coupling constants. In this case we found that there is a flow in coupling constant space induced by the RG. We will now revisit the RG flows in the case of two-dimensional CFTs.

The C-theorem is a deceptively simple but yet profound result due to A.

B. Zamolodchikov (Zamolodchikov, 1986; Ludwig and Cardy, 1987). Let us consider as set of 2D (Euclidean) conformal field theories defined in terms of a set of coupling constants $\{g_i\}$, representing the couplings of a set of primary fields $\mathcal{O}_i$. We will consider continuum field theories only. We will assume that in all the possible fixed points are critical fixed points (hence with a divergent correlation length) and are represented by two-dimensional CFTs. We will further assume that in this space of coupling constants that all the irrelevant operators introduced by the regularization, i.e. lattice effects, have already flown to zero. In this framework, the RG induces a flow in the space of coupling constants linking different CFTs.

The C-theorem is stated as follows: there exists a function C of the coupling constants which is non-increasing along the RG flows and it is stationary only at the fixed points. Moreover, at the fixed points the C function is equal to the central charge of the CFT of the fixed point.

Before we present the proof it is worth to discuss its meaning an implications. Intuitively this result makes a lot sense. Consider to CFTs related by the RG flow. Since the flow is always from the UV to the IR, what the RG describes is the gapping-out of a set of degrees present in theory I but absent in theory II by turning on some relevant operators. In this sense theory II has fewer degrees of freedom than theory I. Therefor we expect that the central charges of theory I be larger than the central charge of theory II, $c_I \geq c_{II}$.

The proof goes as follows. We will assume that the theories are invariant under translations and Euclidean rotations, and hence have a conserved energy-momentum tensor $T^{\mu\nu}$. We will also assume that the theories obey reflection positivity. Away from the fixed points, the energy-momentum tensor has components, $T = T_{zz}$, $\bar{T} = T_{\bar{z}\bar{z}}$ and a non-vanishing trace $\Theta = T_z^z + T_{\bar{z}}^{\bar{z}} = 4T_{z\bar{z}}$. Under rotations T, Θ, and $\bar{T}$ have spins $s = 2, 0, -2$ respectively. Their correlation functions have the form

$$\begin{aligned}
\langle T(z,\bar{z})T(0,0)\rangle &= \frac{F(a\bar{z})}{z^4} \\
\langle \Theta(z,\bar{z})T(0,0)\rangle &= \langle T(z,\bar{z})\Theta(0,0)\rangle = \frac{G(z\bar{z})}{z^3\bar{z}} \\
\langle \Theta(z,\bar{z})\Theta(0,0)\rangle &= \frac{H(z\bar{z})}{z^2\bar{z}^2}
\end{aligned} \tag{21.122}$$

where F, G and H are non-trivial scalar functions. Conservation of the energy-momentum tensor, $\partial_\mu T^{\mu\nu} = 0$, in complex coordinates implies

$$\bar{\partial} T + \frac{1}{4}\partial\Theta = 0 \tag{21.123}$$

Taking correlation functions with $T(0,0)$ and $\Theta(0,0)$ we obtain

$$\begin{aligned}\dot{F} + \frac{1}{4}\left(\dot{G} - 3G\right) =& 0 \\ \dot{G} - G + \frac{1}{4}\left(\dot{H} - 2H\right) =& 0\end{aligned} \qquad (21.124)$$

where

$$\dot{F} \equiv z\bar{z}F'(z\bar{z}), \quad \dot{G} \equiv z^3\bar{z}G'(z\bar{z}), \quad \dot{H} \equiv z^2\bar{z}^2H'(z\bar{z}) \qquad (21.125)$$

Upon eliminating the function G using the conservation laws, and defining the function C

$$C \equiv 2F - G - \frac{3}{8}H \qquad (21.126)$$

we obtain

$$\dot{C} = -\frac{3}{4}H \qquad (21.127)$$

Now, reflection positivity requires that

$$\langle\Theta(z,\bar{z})\Theta(0,0)\rangle \geq 0 \qquad (21.128)$$

which implies that $H \geq 0$. Therefore, C is a non-increasing function of $R = (z\bar{z})^{1/2}$ and it is stationary, i.e. $\dot{C} = 0$, only if $H = 0$.

Under an RG transformation, the short-distance cutoff is scaled as $a \to a(1+\delta\ell)$. Since the quantity C is a dimensionless function of R and of the coupling constants $\{g_i\}$, a change in the UV scale is equivalent to sending $R \to R(1-\delta\ell)$ and to a new set of couplings to $\{g_i'\}$ according to the RG equations defined by the beta-functions, $\{\beta_i\}$.Therefore, the function $C(R,\{g_i\})$ satisfies the Callan-Symanzik RG equation

$$\left(R\frac{\partial}{\partial R} + \sum_i \beta_i(\{g\})\frac{\partial}{\partial g_i}\right)C(R,\{g\}) = 0 \qquad (21.129)$$

where

$$\frac{dC}{d\ell} = -\sum_i \beta_i(\{g\})\frac{\partial C}{\partial g_i} \qquad (21.130)$$

is the rate of change of C along the RG trajectory, at fixed R. This result implies that if we define

$$C(\{g\}) \equiv C(1,\{g\}) \qquad (21.131)$$

then this quantity is non-increasing under the RG. Furthermore, it vanishes only if $H = 0$ which, reflection positivity, means $\Theta = 0$ and the theory is scale invariant (and hence is a fixed point). In addition at the fixed point

$G = H = 0$ and $F = c/2$ (where c is the central charge). Hence, at the stationary points the function C is equal to the central charge of the fixed point, $C = c$.

The function$C$ can be computed using the perturbative RG discussed in Section 15.6. There we showed that the beta functions have the form

$$\frac{dg_i}{d\ell} = (2 - \Delta_i)g_i - \frac{1}{2}\sum_{k,l} c_{ijk}g_j g_k + \dots \tag{21.132}$$

(no summation in the first term) where c_{ijk} are the coefficients of the OPEs of the primary fields $\{\mathcal{O}_i\}$. This equation implies that, to this order, the RG equations describe gradient flows, i.e.

$$\frac{dg_i}{d\ell} = \frac{\partial}{\partial g_i}\widetilde{C}(\{g\}) \tag{21.133}$$

where

$$\widetilde{C}(\{g\}) = \frac{1}{2}\sum_k (2 - \Delta_k)g_k + \frac{1}{6}\sum_{ijk} c_{ijk}g_i g_j g_k \tag{21.134}$$

We can now use this function $\widetilde{C}$ to compute the Zamolodchikov's C-function. Since it must have the same stationary points as $\widetilde{C}$, to this order they must be proportional

$$C(\{g\}) = c + \alpha\widetilde{C}(\{g\}) + O(g^4) \tag{21.135}$$

where α is a constant to be determined. This can be done using perturbation theory.

To this end consider the case of a CFT that is perturbed by a single relevant operator, a primary field $\mathcal{O}$ of conformal weight (h, h), scaling dimension $\Delta = 2h$ and spin $s = 0$. The perturbed action is

$$S = S_{CFT} - \lambda \int d^2z\mathcal{O}(z, \bar{z}) \tag{21.136}$$

For this perturbation to be relevant we must have $\Delta < 2$, and, hence, $h < 1$. The coupling constant λ has dimension $(1 - h, 1 - h)$.

Let us compute the changes of the correlators with insertions of the energy-momentum tensor. To first order in λ we get

$$\langle T(z, \bar{z}) \dots\rangle = \langle T(z, \bar{z} \dots\rangle_{CFT} + \lambda \int \langle T(z)\mathcal{O}(w, \bar{w})\rangle_{CFT} d^2w + \dots \tag{21.137}$$

From the OPE

$$
\begin{aligned}
T(z)\mathcal{O}(w,\bar{w}) =& \frac{h}{(z-w)^2}\mathcal{O}(w,\bar{w}) + \frac{1}{z-w}\partial\mathcal{O}(w,\bar{w}) + \dots \\
=& \frac{h}{(z-w)^2}\mathcal{O}(z,\bar{z}) + \frac{1-h}{z-w}\partial\mathcal{O}(z,\bar{z}) + \dots
\end{aligned}
\tag{21.138}
$$

This implies that the integral in Eq.(21.137) is divergent and, hence, it requires that we introduce a short-distance cutoff, e.g a cutoff step function that exludes a circle of radius a from the integral However, the cutoff violates conformal invariance (as before!) and, as a result $\bar{\partial}T \neq 0$,

$$
\bar{\partial}T = \lambda \int d^2w \frac{1-h}{z-w}(z-w)\partial\mathcal{O}(z,\bar{z})\delta(|z-w|^2 - a^2) + \dots \tag{21.139}
$$

Since the energy-momentum tensor must still be conserved, $\bar{\partial} + \frac{1}{4}\partial\Theta = 0$, we see that the energy-momentum tensor now has a non-vanishing trace,

$$
\Theta(z,\bar{z}) = -4\pi\lambda(1-h)\mathcal{O}(z,\bar{z}) \tag{21.140}
$$

Using Eq.(21.127) we find that the C -unction is

$$
C(\{g_i\}) = c(g^*) - 3(2-\Delta_j)g^j g^i + 2c_{ijk}g^i g^j g^k + O(g^4) \tag{21.141}
$$

where $g^* = \{g_i^*\}$ is the fixed point.

In the simple case of a single relevant perturbation this result implies that the central charge at a nearby fixed point $g^* = -(2-\Delta)^2/c_{111}$ is $c' = c - (2-\Delta)^3/c_{111}^2 + O((2-\Delta)^4)$.

The generalization of the C-theorem to dimensions $D > 2$ turned out to be quite subtle and to require concepts of quantum information theory such as the entanglement entropy. This is the subject of intense research at the time these pages are being written, and will not discuss them here.

21.6 Examples of two-dimensional CFTs

21.6.1 The free compactified boson

Consider a free massless scalar field $\phi(x)$ in two spacetime dimensions. We will work with the Euclidean signature. We have briefly discussed this theory in Section 19.9.

The action of the free massless scalar field is

$$
S = \frac{1}{8\pi}\int d^2x\ \partial_\mu\phi\partial^\mu\phi \tag{21.142}
$$

The two-point function of the field ϕ is

$$\langle \phi(\boldsymbol{x})\phi(\boldsymbol{y})\rangle = -\ln(\boldsymbol{x}-\boldsymbol{y})^2 + \text{const.} \tag{21.143}$$

(which requires a short distance subtraction that determines the value of the additive constant). In complex coordinates it becomes

$$\langle \phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\left[\ln(z-w) + \ln(\bar{z}-\bar{w})\right] + \text{const.} \tag{21.144}$$

The holomorphic and anti-holomorphic components are split if we look at the correlators of gradients of the field

$$\begin{aligned}\langle \partial_z\phi(z,\bar{z})\partial_w\phi(w,\bar{w})\rangle &= -\frac{1}{(z-w)^2}\\ \langle \partial_{\bar{z}}\phi(z,\bar{z})\partial_{\bar{w}}\phi(w,\bar{w})\rangle &= -\frac{1}{(\bar{z}-\bar{w})^2}\end{aligned} \tag{21.145}$$

This implies that the field $\partial\phi$ obeys the OPE

$$\partial\phi(z)\partial\phi(w) = -\frac{1}{(z-w)^2} \tag{21.146}$$

The quantum energy-momentum tensor in complex coordinates is the normal-ordered operator

$$T(z) = -\frac{1}{2} :(\partial\phi(z))^2: \tag{21.147}$$

Here normal-ordering means the limit

$$T(z) = -\frac{1}{2}\lim_{w\to z}\left(\partial\phi(z)\partial\phi(w) - \langle\partial\phi(z)\partial\phi(w)\rangle\right) \tag{21.148}$$

The OPE of $T(z)$ with $\partial\phi(z)$ can be found using Wick's theorem

$$\begin{aligned}T(z)\partial\phi(w) &= -\frac{1}{2} :(\partial\phi(z))^2: \partial\phi(w)\\ &= - :\partial\phi(z)\partial\phi(z): \partial\phi(w) \qquad (21.149)\\ &= \frac{1}{(z-w)^2}\partial\phi(z) \qquad (21.150)\end{aligned}$$

Hence, the OPE between $T(z)$ and $\partial\phi$ is

$$T(z)\partial\phi(w) = \frac{1}{(z-w)^2}\partial\phi(w) + \frac{1}{z-w}\partial_w^2\phi(w) \tag{21.151}$$

This means that the holomorphic operator $\partial\phi$ has conformal weight $h = 1$.

Another interesting operator is the vertex operator $V_\alpha(z) =: \exp(i\alpha\phi(z)):$, where $\alpha \in \mathbb{R}$. Here, as before, normal ordering means not to contract the operators inside the exponential when it is expanded in powers. This rule is equivalent to a multiplicative wave function renormalization of the operator. The OPE of the energy-momentum tensor with the vertex operator is

$$T(z)V_\alpha(w) = \frac{h_\alpha}{(z-w)^2}\, V_\alpha(w) + \frac{1}{z-w}\partial V_\alpha(w) \tag{21.152}$$

We find that the vertex operator is a primary field of conformal weight h_α

$$h_\alpha = \frac{\alpha^2}{2} \tag{21.153}$$

We could have deduced this result by computing the two point function of the vertex operator

$$\langle V_\alpha(z)V_{-\alpha}(w)\rangle = \exp\left(\alpha^2\langle\phi(z)\phi(w)\rangle\right) = \frac{1}{(z-w)^{2h_\alpha}} \tag{21.154}$$

where the conformal weight h_α is given in Eq.(21.153). Notice that in this case the conformal weight also depends on the parameter g. In this theory we could have formally rescaled the field and and the g dependence will amount to a rescaling of α.

Similarly, we can also use Wick's theorem to calculate the OPE of the energy-momentum tensor with itself

$$\begin{aligned} T(z)T(w) =& \frac{1}{4} :(\partial\phi(z))^2 :: (\partial\phi(w))^2 : \\ =& \frac{1/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) \end{aligned} \tag{21.155}$$

This equation implies that the holomorphic component of the energy-momentum tensor $T(z)$ has conformal weight $h = 2$ and that the central charge of the free massless scalar field is $c = 1$.

In Chapter 11 we discussed the concept of spontaneous symmetry breaking in theories with a global symmetry. The massless scalar field ϕ arises formally as the phase field, i.e. the Goldstone boson, of an order parameter field with a global $U(1) \simeq O(2)$ symmetry in two dimensions. The action of the field ϕ, Eq.(21.142), has a global shift symmetry $\phi(x) \to \phi(x) + a$, where a is a real number. As we see, the propagator of the field ϕ breaks cluster decomposition since it grows at long distances (as well as at short distances!). In this sense it is not a physical field.

However the operator $\partial_\mu\phi$ is invariant under such shifts. Likewise, the vertex operators, which exhibit power law behavior, transform non trivially under the shift symmetry and are physical. Moreover, since it is a phase field, the physical operators must be invariant under *periodic* shifts

$$\phi(z,\bar{z}) \mapsto \phi(z,\bar{z}) + 2\pi R \tag{21.156}$$

where R is the compactification radius. In other terms, the field is *compactified*. This terminology comes from String theory in which the field ϕ is the coordinate of a string on a compactified space of radius R. This condition implies that the allowed operators must be *invariant* under the periodic shifts of Eq.(21.156). The compactification condition imposes the strong restrictions on the allowed primary fields.

One operator that is always allowed is the "current" operator $J = i\partial\phi(z)$ since it is automatically invariant under shifts. However, the only vertex operators allowed must be such that $R\alpha \in \mathbb{Z}$. Thus the allowed vertex operators are

$$V_n(z) =: \exp(in\phi(z)/R): \tag{21.157}$$

with n integer, with conformal weight

$$h_n = \frac{n^2}{2R^2} \tag{21.158}$$

An interesting case has compactification radius $R = 1/\sqrt{2}$. In this case the operators $J^3(z) = i\partial\phi(z)$, and $J^{\pm}(z) = \exp(\pm i\sqrt{2}\phi(z))$ have conformal weight $(1,0)$. These operators obey the OPEs

$$\begin{aligned} J^+(z)J^-(w) =& \frac{1}{(z-w)^2} + \frac{\sqrt{2}}{z-w}J^3(w) \\ J^3 J^{\pm}(w) =& \frac{\sqrt{2}}{z-w}J^{\pm}(w) \end{aligned} \tag{21.159}$$

and similarly for $\bar{J}^3$ and $\bar{J}^{\pm}$.

If we define $J^{\pm} = \frac{1}{\sqrt{2}}(J^1 \pm iJ^2)$, then this current algebra can also be written as

$$J^i(z)J^j(w) = \frac{\delta^{ij}}{(z-w)^2} + \frac{i\sqrt{2}\epsilon^{ijk}}{z-w}J^k(w) \tag{21.160}$$

This defines an $SU(2)_1$ Kac-Moody algebra of the $SU(2)$ currents. The first term on the right hand side represents the central extension of the $SU(2)$ algebra at "level" 1 (the prefactor of the Kronecker delta). In terms of the mode expansions

$$J^i(z) = \sum_{n\in\mathbb{Z}} \frac{J_n^i}{z^{n+1}}, \quad \text{where} J_n^i = \oint \frac{dz}{2\pi i} z^n J^i(z) \tag{21.161}$$

the modes obey the commutation relations

$$\left[J_b^i, J_m^j\right] = i\sqrt{2}\epsilon^{ijk}J_{n+m}^k + n\delta^{ij}\delta_{n+m,0} \tag{21.162}$$

On the other hand, the vertex operators $V_{\pm 1}(z) = \exp(\pm i\phi(z)/\sqrt{2})$ have conformal weight $(1/4, 0)$ and their two-point functions are

$$\langle V_1(z) V_{-1}(w) \rangle = \frac{1}{(z-w)^{1/2}} \tag{21.163}$$

and are double-valued. It can be shown that these fields constitute $j = 1/2$ and $m = \pm 1/2$ (spinor) representation of the $SU(2)$ current algebra.

A more general case of this type occurs when the compactification radius is $R = \sqrt{m}$ with $m \in \mathbb{Z}$. In this case the conformal weights of the vertex operators are $h_n = n^2/(2m)$. Their two point functions are

$$\langle V_n(z) V_{-n}(w) \rangle = \frac{1}{z^{n^2/m}} \tag{21.164}$$

In general these two point functions are multivalued since under a rotation by 2π the argument changes by $2\pi n^2/m$. In other words these chiral fields represent are anyons (or parafermions). Now, among this infinite list there are some operators that remain local in the sense that they are single valued. This happens for $n = \sqrt{mp}$ (where p is an integer). However, the non-local part of these operators can be classified into $m - 1$ sectors, with $n = 0, 1, \ldots, m-1$. Each sector is labelled by a primary field, the vertex operator $V_n(z) = \exp(in\phi(z)/\sqrt{m})$. hence, instead of having infinitely many primaries, these compactified bosons have a finite number (m) of primaries. CFTs with a finite number of primaries are called rational CFTs (Ginsparg, 1989).

We close by noting a case of special interest, $R = 1$. In this case there are only two sectors, one labelled by the identity $V_0 = I$ and the other by the primary $V_1(z) = \exp(i\phi(z))$. In this case, the two-point function is

$$\langle V_1(z) V_{-1}(w) \rangle = \frac{1}{z-w} \tag{21.165}$$

which is odd under the exchange $z \leftrightarrow w$, a property that we expect in a *fermion*. We will see below that this is indeed the Dirac fermion.

21.6.2 The free massless fermion CFT

Let us consider now a free massless relativistic fermions in two dimensions. As before we will set the speed of light to unity. We will work in the Euclidean signature. A relativistic fermion in $D = 2$ Euclidean dimensions is a two-component spinor. We will work in the chiral basis in which the upper component of the spinor is a right-moving field and the lower component is a left moving field. In the Euclidean signature the upper component of the spinor is holomorphic and the lower component is anti-holomorphic. A chiral (holomorphic) fermion is the $D = 2$ version of a Weyl fermion. In addition the fermionic fields may be complex (Dirac) or real (Majorana).

A: Majorana Fermions

Let ψ_R denote a real (Majorana) chiral (right-moving) Majorana fermion and ψ_L an anti-chiral (left-moving) Majorana fermion. (We are changing the notation here to avoid confusion with the standard notation for relativistic fermions.) We already encountered a theory of Majorana fermions in the context of the solution of the 2D Ising model in Chapter 14. We will see shortly that the CFT of the Majorana fermion is closely related to the Ising CFT.

The action for a massless Majorana spinor in the Euclidean signature is (including both chiralities)

$$S = \frac{1}{8\pi} \int d^2x \left(\psi_R \bar{\partial} \psi_R + \psi_L \partial \psi_L \right) \tag{21.166}$$

Here we used that in $D = 2$ Euclidean dimensions the Dirac operator can be represented as

$$\not{\partial} = \sigma_1 \partial_1 + \sigma_2 \partial_2 = \begin{pmatrix} 0 & \partial \\ \bar{\partial} & 0 \end{pmatrix} \tag{21.167}$$

The equations of motion of of this theory are

$$\bar{\partial} \psi_R = 0, \quad \partial \psi_L = 0 \tag{21.168}$$

Hence, ψ_R is only a function of z and is a holomorphic field, and ψ_L is only a function of $\bar{z}$ and is an anti-holomorphic field.

With the normalization used in Eq.(21.166), the OPEs of the Majorana Fermi fields is

$$\psi_R(z)\psi_R(w) = -\frac{1}{z-w}, \quad \psi_L(\bar{z})\psi_L(\bar{w}) = -\frac{1}{\bar{z}-\bar{w}} \tag{21.169}$$

which also specify the propagators. In this theory, the holomorphic and anti-holomorphic components of the energy-momentum tensor are

$$T(z) = \frac{1}{2} : \psi_R(z) \partial \psi_R(z) :, \quad \bar{T}(\bar{z}) = \frac{1}{2} : \psi_L(\bar{z}) \bar{\partial} \psi_L(\bar{z}) : \tag{21.170}$$

It follows that the OPE of the above energy-momentum tensor $T(z)$ with the chiral Majorana spinor is

$$T(z)\psi_R(w) = \frac{1/2}{(z-w)^2} \psi_R(w) + \frac{1}{z-w} \partial \psi_R(w) \tag{21.171}$$

Hence the chiral majorana spinor ψ_R has conformal weight $(h, \bar{h}) = (1/2, 0)$, as can also be read off Eq.(21.169). Thus, the chiral Majorana fermion has scaling dimension $\Delta = 1/2$ and spin $s = 1/2$ (as it should!). Similarly, one finds that the anti-chiral field ψ_L has conformal weight $(h, \bar{h}) = (0, 1/2)$, and has scaling dimension $\Delta = 1/2$ and (conformal) spin $-1/2$.

Finally, the OPE of the energy-momentum tensors can be computed using Wick's theorem to yield

$$T(z)T(w) = \frac{1/4}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) \tag{21.172}$$

and similarly for $\bar{T}$. This result implies that the central charge of a chiral Majorana spinor is $c = 1/2$.

Another primary field that can be constructed is the Majorana mass operator $\psi_R(z)\psi_L(\bar{z})$ that mixes the right and left moving sectors. It is straightforward to see that this is a primary field with conformal weight $(1/2, 1/2)$ and, hence, scaling dimension $\Delta = 1$ and (conformal) spin $s = 0$.

B: Dirac Fermions

A Dirac (complex) chiral fermion has the form $\psi_R = \eta_R + i\chi_R$, where η_R and χ_R are chiral Majorana fermions. The Euclidean action of the Dirac fermion is

$$S = \frac{1}{4\pi}\int d^2x \bar{\psi}\partial\!\!\!/\psi \tag{21.173}$$

which is just the sum of the action of the Majorana fermions η and χ, each of the form of Eq.(21.166). It then follows that, since each Majorana field has central charge $c = 1/2$, the central charge of the complex (Dirac) fermion is $c = 1$. The energy-momentum tensor of the Dirac field is just the sum of the energy-momentum tensors of the two Majorana fermions.

Just as in the Majorana case, the Dirac fermions ψ_R and ψ_L have conformal weights $(1/2, 0)$ and $(0, 1/2)$, respectively. Hence their two-point functions are

$$\langle \psi_R^\dagger(z)\psi_R(w)\rangle = -\frac{1}{z-w}, \qquad \langle \psi_L^\dagger(\bar{z})\psi_L(\bar{w})\rangle = -\frac{1}{\bar{z}-\bar{w}} \tag{21.174}$$

The Dirac fermion is a complex field and has a global $U(1)$ symmetry. Consequently, it has a locally conserved current $J^\mu = (J_0, J_1) = \bar{\psi}\gamma^\mu\psi$, which can be decomposed into right and left moving (holomorphic and anti-holomorphic) components, $J_R = \psi_R^\dagger\psi_R$ and $J_L = \psi_L^\dagger\psi_L$, which have conformal weights $(1, 0)$ and $(0, 1)$ respectively. Their two-point functions are

$$\langle J_R(z)J_R(w)\rangle = \frac{1}{(z-w)^2}, \qquad \langle J_L(\bar{z})J_L(\bar{w})\rangle = \frac{1}{(\bar{z}-\bar{w})^2} \tag{21.175}$$

The $U(1)$ currents J_R and J_L satisfy an OPE of the form

$$J_R(z)J_R(w) = \frac{1}{(z-w)^2}, \quad J_L(\bar{z})J_L(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^2} \tag{21.176}$$

The singular term of the right hand side is known as the Schwinger term of the $U(1)$ currents.

In terms of the mode expansions we find a $U(1)$ Kac-Moody algebra

$$\left[J_n^R, J_m^R\right] = n\delta_{n+m,0} \tag{21.177}$$

and similarly for the modes of the left moving currents. The right and left moving currents commute with each other.

The expressions of the fermion two-point functions and the Dirac current algebra with the results we obtained for a compactified free massless boson with compactification radius $R = 1$ suggests that there is a direct connection. This mapping is known as *bosonization.* To see how it works we assert that the mapping is the following set of relations between the two theories

$$\begin{aligned} \psi_R(z) &\leftrightarrow e^{i\phi(z)}, \qquad J_R(z) \leftrightarrow -i\partial\phi(z) \\ \psi_L(\bar{z}) &\leftrightarrow e^{-i\bar{\phi}(\bar{z})}, \qquad J_L(\bar{z}) \leftrightarrow i\bar{\partial}\bar{\phi}(\bar{z}) \end{aligned} \tag{21.178}$$

where $\bar{\phi}$ is the left moving (antiholomorphic) component of the field $\phi(z,\bar{z}) = \phi(z) - \bar{\phi}(\bar{z})$.

A more complete proof of the bosonization mapping requires showing that the two theories have identical spectrum. This involves computing that the partition functions on a torus (with suitable boundary conditions) for both theories and showing that they agree. This is a technical argument that we will not be pursued here.

21.6.3 Minimal models and 2D Ising Model CFT

In chapter 14 we looked at the statistical mechanics of the two-dimensional classical Ising model (or, equivalently, the one-dimensional quantum Ising model). There we saw that this is secretly a theory of free relativistic Majorana fermions that become massless at the critical point. Here we will see that this is a special case of a large class of conformal field theories known as the Minimal Models.

In Section 21.5.3 we introduced the Virasoro algebra, Eq.(21.97), of the generators L_n and $\bar{L}_n$ of conformal transformations in two Euclidean dimensions. There we showed that the vacuum state of a CFT, $|0\rangle$ is annihilated by both L_0 and $\bar{L}_0$. We also showed that a primary field $\mathcal{O}$ defines a highest-weight state of the Virasoro algebra, $\mathcal{O}|0\rangle = |h, \bar{h}\rangle$, where h ($\bar{h}$) is the $L_)$ ($\bar{L}_0$) eigenvalue of the highest-weight state. We also argued that given a highest-weight state, an infinite number of descendant states can be constructed by acting repeatedly with the lowering operators L_n ($\bar{L}_{-n}$) with $n > 0$, on the highest-weight state. However, for this scheme to be consistent all the descendants must be linearly independent. A linear combination of descendants that vanishes defines a null state, which implies that the states are linearly dependent. A representation is constructed by removing all the null states.

This scheme follows closely the construction of the representations of the angular momentum representations of the group of rotations. In the case of the group of rotations, the requirement that the representations be unitary, i.e. that the norm of the states be positive, leads to the quantization of the angular momentum eigenvalues. We will now sketch an argument, due to Belavin, Polyakov and Zamolodchikov (Belavin et al., 1984) that the requirement of unitarity similarly leads to powerful restrictions on the values of the representations of the conformal group, i.e. to the allowed values of the central charge c, of the conformal weights $(h, \bar{h})$ and to the fusion rules of the allowed primary fields, encoded in their fusion rules.

The null states are found in a straightforward way. For example, at level 1 the only possibility is $L_{-1}|h\rangle = 0$. But this means that $h = 0$, and $|h\rangle = |0\rangle$, the vacuum state. At level two it may happen that

$$L_{-2}|h\rangle + aL_{-1}^2|h\rangle = 0 \tag{21.179}$$

for some value of a. Acting with L_1 on this equation we find the consistency condition

$$\begin{aligned}[L_1, L_{-2}]|h\rangle + a[L_1, L_{-1}^2]|h\rangle =& 3L_{-1}|h\rangle + a\left(2\{L_{-1}, L_0\}\right)|h\rangle \\ =& \left(3 + 2a(2h+1)\right)L_{-1}|h\rangle = 0\end{aligned} \tag{21.180}$$

which can only happen if $a = -3/2(2h+1)$. If now we act with L_2 we find the condition

$$\begin{aligned}[L_2, L_{-2}]|h\rangle + a[L_2, L_{-1}^2]|h\rangle =& \left(4L_0 + \frac{c}{2}\right)|h\rangle + 3aL_1L_{-1}|h\rangle \\ =& (4h + \frac{c}{2} + 6ah)|h\rangle = 0\end{aligned} \tag{21.181}$$

so that the central charge must satisfy $c = 2((-6ah - 4h) = 2h(5-8h)/(2h+1)$. Thus, this will work if the highest-weight state $|h\rangle$ at this value of c satisfies

$$\left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)|h\rangle = 0 \tag{21.182}$$

Such a state, with a null descendant at level 2, is said to be degenerate at level 2. It can be shown that this equation implies that the N-point functions of primary fields satisfy a set of differential equations. In the case of the four-point functions they can be expressed in terms of hypergeometric functions.

Unitarity means that the inner product in the space of states is positive definite. The inner products of any two descendant states can be computed using the Virasoro algebra. A state $|\psi\rangle$ that has negative norm, $\langle\psi|\psi\rangle < 0$, is called a “ghost”. In an unitary theory a ghost should not be found at any level of the representation. Given the collection of descendants of a highest-weight state, one can define a matrix whose matrix elements are the inner products,. The determinant of this matrix is known as the Kac determinant. A zero eigenvector of this matrix implies that there are null vectors. Therefore it will be sufficient to look for zeros of the Kac determinant.

At level two we have the two-component basis $L_{-2}|h\rangle$ and $L_{-1}^2|h\rangle$. The matrix is

$$\begin{pmatrix} \langle h|L_2 L_{-2}|h\rangle & \langle h|L_1^2 L_{-2}|h\rangle \\ \langle h|L_2 L_{-1}^2|h\rangle & \langle h|L_1^2 L_{-1}^2|h\rangle \end{pmatrix} = \begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 4h(2h+1) \end{pmatrix} \tag{21.183}$$

The determinant of this matrix can be written as

$$2(16h^3 + 2h^2 c = hc) = 32(h - h_{1,1}(c))(h - h_{1,2}(c))(h - h_{2,1}(c)) \tag{21.184}$$

where

$$h_{1,1}(c) = 0, \qquad h_{1,2}(c) = h_{2,1}(c) = \frac{1}{16}(5-c) \mp \sqrt{(1-c)(25-c)} \tag{21.185}$$

The $h = 0$ root is actually due to a null state at level 1, $L_{-1}|0\rangle = 0$, which implies that $L_{-1}^2|0\rangle = 0$ as well. This feature is repeated at all orders.

At level N, the Hilbert space are states of the form

$$\sum_{n_i} a_{n_1 \ldots n_k} L_{-n_1} \ldots L_{-n_k} |h\rangle \tag{21.186}$$

with $\sum_i n_i = N$. At level N we need the determinant of the $P(N) \times P(N)$ matrix of inner products of the form

$$M_N(c,h) = \langle h|L_{m_l} \ldots L_{m_1} L_{-n_1} \ldots L_{-n_k}|h\rangle \tag{21.187}$$

If the determinant vanishes, $\det M_N(c,h) = 0$, then there are null states for each c and h. If negative, the determinant has an odd number of negative eigenvalues (at least one). The representation of the Virasoro algebra at those values of c and h is not unitary.

Kac gave an explicit expression for the determinant

$$\det M_N(g,h) = \alpha_N \prod_{pq\leq N} (h - h_{p,q}(c))^{P(N-pq)} \tag{21.188}$$

Upon defining

$$m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}} \tag{21.189}$$

or, equivalently

$$c = 1 - \frac{6}{m(m+1)}, \tag{21.190}$$

the quantity $h_{p,q}$ becomes

$$h_{p,q}(m) = \frac{\left[(m+1)p - mq\right]^2 - 1}{4m(m+1)} \tag{21.191}$$

The Kac determinant does not have zeros at any level if $c > 1$ and $h \geq 0$. For $c = 1$ the determinant vanishes if $h = n^2/4$, but it does not become negative. Thus, there is not obstacle, in principle, to having unitary representations for $c \geq 1$ and $h \geq 0$.

On the other hand, for $0 < c < 1$ and $h > 0$ Friedan, Qiu and Shenker (Friedan et al., 1984) showed that unitary theories only exist for the set of discrete values of the central charge given in Eq.(21.190) for integer values $m = 3, 4, \ldots$. For each value of c there are $m(m-1)/2$ allowed values of h given by Eq.(21.191), where p and q are integers such that $1 \leq p \leq m-1$ and $1 \leq q \leq p$. The set of CFTs that satisfy these conditions is known as the Minimal Models, and the set defined by Eq.(21.190) are their central charges.

We will now consider the first Minimal Model, with $m = 3$. These results implies that is has central charge $c = \frac{1}{2}$. Let us label the primary fields by $\mathcal{O}_{p,q}(z,\bar{z}) = \phi_{p,q}(z)\bar{\phi}_{p,q}(\bar{z})$ with conformal weights $(h,\bar{h})$. For $m = 3$ the allowed values are $h = 0, \frac{1}{2}, \frac{1}{16}$, with primaries

$$\mathcal{O}_{1,1} : (0,0), \qquad \mathcal{O}_{2,1} : \left(\frac{1}{2}, \frac{1}{2}\right), \qquad \mathcal{O}_{2,1} : \left(\frac{1}{16}, \frac{1}{16}\right) \tag{21.192}$$

This is telling us that we have a theory whose non-chiral primaries have scaling dimensions $\Delta = 0, 1, \frac{1}{8}$. It also has a chiral primary with scaling dimension $\frac{1}{2}$ and spin $s = \frac{1}{2}$.

In Chapter 14 we presented the solution to the 2D Ising Model. In Section 14.7 we showed that the critical behavior is described by a theory of Majorana fermions which become massless at the critical point. We see that this the minimal model with $m = 3$ is consistent with these results. indeed it has central charge $\frac{1}{2}$, as a massless Majorana fermion does. It also has a chiral primary field $(\frac{1}{2}, 0)$ which is a fermion with dimension $\frac{1}{2}$ and (conformal) spin $\frac{1}{2}$ (and similarly for the anti-chiral field). The Majorana fermion mass $\varepsilon = \psi_R \psi_L$ is a field with conformal weight $(\frac{1}{2}, \frac{1}{2})$, scaling dimension $\Delta_\varepsilon = 1$, and (conformal) spin $s = 0$.

We also find another field, $\mathcal{O}_{2,1}$, which has conformal weight $(\frac{1}{16}, \frac{1}{16})$. Hence, it has scaling dimension $\frac{1}{8}$ and (conformal) spin $s = 0$. In Chapter 14 we did not compute the spin-spin correlation function. However, it is known that this correlator has a power-law behavior with exponent $\eta = \frac{1}{4}$. This implies that the scaling dimension of the spin operator, σ, the order parameter of the Ising model, is $\Delta = \frac{1}{8}$! Thus, we conclude that the CFT of the 2D Ising model is indeed the $m = 3$ minimal model.

We close by noting that the fusion rules, i.e. the OPEs, of the primaries are also determined. In the case of the Ising model we have three conformal families, each labeled by a primary field, 1, $[\varepsilon]$ and $[\sigma]$, and obey the fusion rules

$$[\sigma][\sigma] = 1 + [\varepsilon], \quad [\sigma][\varepsilon] = [\sigma], \quad [\varepsilon][\varepsilon] = 1 \tag{21.193}$$

This approach also allows to compute the four-point function of the σ field, the order parameter of the Ising model. This a beautiful but quite technical subject and we will not pursue it here.

21.6.4 The Wess-Zumino-Witten CFT

The Wess-Zumino-Witten (WZW) model is a 1+1-dimensional non-linear sigma model whose degree of freedom is an element of a Lie group G. We already discussed this model briefly in Section 20.9.3. The action of this model is

$$S[g] = \frac{1}{4\lambda^2} \int_{S^2_{\text{base}}} d^2x \, \text{tr}\left(\partial_\mu g \partial^\mu g^{-1}\right) + \frac{k}{24\pi} \int_B \epsilon^{\mu\nu\lambda} \text{tr}\left(g^{-1}\partial_\mu g \, g^{-1}\partial_\nu g \, g^{-1}\partial_\lambda g\right) \tag{21.194}$$

In the second term of the action, the Wess-Zumino-Witten term, the field $g(x)$ whose base space is S^2_{base}, is extended to the interior of a three-dimensional ball $B \subset S^3$) whose boundary is S^2_{base}, subject to the condition that $g(x) = 1$ (the identity) at the center of the ball B. The extension to the interior of the ball B is arbitrary. However, not all the extensions are equivalent since the mappings of the ball, S^3, to the group manifold G ($SU(N)$ in this case) are classified by the homotopy group $\pi_3(SU(N)) = \mathbb{Z}$, with topological charge

$$Q = \frac{1}{24\pi^2} \int_{S^3} \epsilon^{\mu\nu\lambda} \mathrm{tr}\left(g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g\, g^{-1}\partial_\lambda g\right) \tag{21.195}$$

Thus, the second term of the WZW action has an ambiguity equal to $2\pi kQ$, which is unobservable if $k \in \mathbb{Z}$. We recognize that that is essentially the same argument that led to the quantization of spin in the coherent-state path integral for spin in Section 8.9.

The WZW term was introduced by Witten in his work on current algebras and non-abelian bosonization (Witten, 1984), and by Polyakov and Wiegmann as an effective action of 1+1-dimensional free fermionic theories (Polyakov and Wiegmann, 1983). This theory has an IR stable fixed point at $\lambda_c^2 = 4\pi/k$ where the theory has full conformal invariance, where it describes a conformal field theory. The WZW CFT was solved by Knizhnik and Zamolodchikov (Knizhnik and Zamolodchikov, 1984).

To motivate the structure of the WZW CFT we will first us consider a theory of N massless Dirac fermions in 1+1 dimensions with $U(N)$ global symmetry. Witten worked with a theory of free Majorana fermions and the $O(N)$ group. Here we are following later results by Affleck (Affleck, 1986a). The Lagrangian for the $U(N)$ group is

$$\mathcal{L} = \bar{\psi}_j i \not{\partial} \psi_j \tag{21.196}$$

Here $j = 1, \ldots, N$ labels the species of Dirac fermions (not their Dirac components!). We will work in the chiral basis in which, in terms of the Pauli matrices, the Dirac matrices are $\gamma_0 = \sigma_1$, $\gamma_1 = i\sigma_2$, and $\gamma_5 = \sigma_3$. In this basis, the Dirac components of the spinor field ψ_i are, respectively, the right-moving component $\psi_{R,i}$ (with chirality +1), and the left-moving component $\psi_{L,i}$ (with chirality −1). This Lagrangian is invariant under global $U(N) \times U(N)$ transformations of the right and left moving fields. Each $U(N)$ can be regarded as a direct product of a $U(1)$ group, generated by the identity matrix of the $U(N)$ algebra, and the $SU(N)$ subgroup of $U(N)$. We have a set of right and left moving (formally) separately conserved currents

$$J_R = \psi_{R,i}^\dagger \psi_{R,i}, \qquad J_L = \psi_{L,i}^\dagger \psi_{L,i} \tag{21.197}$$

and

$$J_R^a = \psi_{R,i}^\dagger t_{ij}^a \psi_{R,j} \qquad J_L^a = \psi_{L,i}^\dagger t_{ij}^a \psi_{L,j} \tag{21.198}$$

for the $SU(N)$ currents. Here, the $N^2 - 1$ matrices t_{ij}^a are the generators of the group $SU(N)$. The generators are normalized such that $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta_{ab}$, and obey the $SU(N)$ algebra $[t^a, t^b] = i f_{abc} t_c$, where f_{abc} are the structure constants of $SU(N)$.

As in the case of the abelian theory, discussed in section 20.3, these currents are affected by the $U(1)$ chiral anomaly, and their commutators have Schwinger terms. The non-abelian currents also have a (non-abelian) chiral anomaly. A careful calculation, using the point-splitting procedure that we already used in section 20.3 (c.f. Eq.(20.15), which preserves gauge-invariance, one finds that, at equal times, the right and left moving currents obey the algebras

$$[J_R(x), J_R(y)] = i\frac{N}{2\pi}\delta_{ab}\partial_x\delta(x-y), \quad [J_L(x), J_L(y)] = -i\frac{N}{4\pi}\delta_{ab}\partial_x\delta(x-y) \tag{21.199}$$

for the abelian currents. This is the $U(1)_N$ Kac-Moody algebra. Similarly, for the non-abelian currents we find

$$\begin{aligned}
[J_R^a(x), J_R^b(y)] =& i f_{abc} J_R^c(x)\delta(x-y) + i\frac{k}{4\pi}\delta_{ab}\partial_x\delta(x-y) \\
[J_L^a(x), J_L^b(y)] =& i f_{abc} J_L^c(x)\delta(x-y) - i\frac{k}{4\pi}\delta_{ab}\partial_x\delta(x-y)
\end{aligned} \tag{21.200}$$

This is the $SU(N)_k$ Kac-Moody algebra. The Schwinger term in Eq.(21.200) is the central extension of the $SU(N)_k$ Kac-Moody algebra.

We recognize the second terms of the right hand side of both sets of equations as Schwinger terms, analogous to the ones we discussed in the abelian theory. Mathematically the Schwinger terms are called "central extensions", and the parameter k, is the level of the Kac-Moody algebra, or its central extension. We already encountered a central extension in the Virasoro algebra, Eq.(21.87). In the free Dirac theory that we are considering the parameter $k = 1$. Furthermore, it is known form the mathematical literature that this current algebra has unitary representations only if $k \in \mathbb{Z}$. Therefore, the level k cannot be an arbitrary real number and is quantized.

We will use light-cone components (i.e. left and right moving), $x_\pm = \frac{1}{2}(x_0 \mp x_1)$, and the notation $\partial_\pm = \partial/\partial_{x_\pm}$. The light-cone components of the energy-momentum tensor, T and $\bar{T}$ for the left and right moving components respectively, are of the energy momentum tensor are

$$T = \frac{1}{2}(H-P) = i :\psi_{L,i}^\dagger \partial_- \psi_{L,i}:, \qquad \bar{T} = \frac{1}{2}(H+P) = i :\psi_{R,i}^\dagger \partial_+ \psi_{R,i}: \tag{21.201}$$

where H and P are, respectively, the (normal-ordered) Hamiltonian and linear momentum. A straightforward (but lengthy) calculation leads to the result that the normal-ordered left and right moving components of the energy momentum tensor, denoted by T_- and $\bar{T}$ respectively, are

$$T = \frac{\pi}{N} J_L J_L + \frac{2\pi}{N+1} J_L^a J_L^a, \qquad \bar{T} = \frac{\pi}{N} J_R J_R + \frac{2\pi}{N+1} J_R^a J_R^a \tag{21.202}$$

The $U(1)$ currents are given in abelian bosonization by

$$J_L = \sqrt{\frac{N}{4\pi}} \partial_- \phi, \qquad J_R = -\sqrt{\frac{N}{4\pi}} \partial_+ \phi \tag{21.203}$$

where ϕ is a boson (scalar field) with compactification radius

$$R = \frac{1}{\sqrt{4\pi N}} \tag{21.204}$$

Let us now focus on the $SU(N)$ currents. Their right and left moving currents obey the conservation laws

$$\partial_- J_R^{ij} = 0, \qquad \partial_+ J_L^{ij} = 0 \tag{21.205}$$

Witten showed that these equations can be solved in terms of a group-valued field, a matrix $g(x_-) \in SU(N)$, in terms of which the chiral currents are (suppressing the group indices)

$$J_R^a(x) = \frac{i}{2\pi} \operatorname{tr}\left(g^{-1}(x) \partial_+ g(x) t^a \right), \qquad J_L^a(x) = -\frac{i}{2\pi} \operatorname{tr}\left((\partial_- g(x)) g^{-1}(x) t^a \right) \tag{21.206}$$

The conservation of the currents imply that

$$\partial_-(g^{-1}\partial_+ g) = 0, \qquad \partial_+((\partial_- g) g^{-1}) = 0 \tag{21.207}$$

The results of Eq.(21.206) provide an operator identification of the non-abelian chiral fermionic currents in terms of the currents of the WZW model. These identifications are the non-abelian version of the identification of the currents in the abelian theory in terms of the compactified boson ϕ.

The next task is to find a local Lagrangian for the matrix-valued field $g(x)$. Since $g(x) \in SU(N)$, it is not a free field. Indeed, we recognize that this is a principal chiral field. Its natural action is

$$\mathcal{L} = \frac{1}{4\lambda^2} \int d^2x \, \operatorname{tr}(\partial^\mu g \partial_\mu g^{-1}) \tag{21.208}$$

This, however, cannot be the correct answer since, as we saw, the non-linear sigma model is an asymptotically free with a positive beta function (using the high-energy physics sign convention), while the theory of free massless fermions is a fixed point theory and, as such, it is scale (and conformally) invariant. In addition, the non-linear sigma model is not compatible with the conservation equations of Eq.(21.207).

So, what operators can be added to the action of the non-linear sigma model to drive the theory to fixed point and to make it consistent with the conservation laws? It is easy to see that all additional $SU(N)$ invariant local operators are irrelevant so they cannot do the job. In fact, the only way to drive this theory of a non-trivial fixed point is to add to the action a Wess-Zumino-Witten term

$$S_{\text{WZW}}[g] = \frac{k}{24\pi}\int_B \epsilon^{\mu\nu\lambda}\text{tr}\left(g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g\, g^{-1}\partial_\lambda g\right) \tag{21.209}$$

which we discussed briefly in section 20.9.3, Eq.(20.142). We will see that we will also obtain a consistent representation of the conservation laws.

Indeed, the conservation encoded in the full WZW action, Eq.(21.194), are

$$\begin{aligned} 0 =& \frac{1}{2\lambda^2}\partial_\mu(g^{-1}\partial^\mu g) - \frac{k}{8\pi}\epsilon^{\mu\nu}\partial_\mu(g^{-1}\partial_\nu g) \\ =& \left(\frac{1}{2\lambda^2} + \frac{k}{8\pi}\right)\partial_-(g^{-1}\partial_+ g) + \left(\frac{1}{2\lambda^2} - \frac{k}{8\pi}\right)\partial_+(g^{-1}\partial_- g) \end{aligned} \tag{21.210}$$

Therefore, in order to the conservation law of Eq.(21.207) to be satisfied, we must require that the coupling constant has to be at a particular value

$$\lambda_c^2 = \frac{4\pi}{k} \tag{21.211}$$

for $k > 0$ (and the negative for $k < 0$). In addition, if the conservation law holds, then it is satisfied by $g(x_+, x_-) = A(x_-)B(x_+)$ (which are arbitrary $SU(N)$ matrices). Then, we see that at this particular value of the coupling, at least classically, the right and left moving waves decouple. This is reminiscent of the behavior of free massless fermions. Furthermore, Witten carried out the program of canonical quantization for the WZW theory and showed that at the quantum level, the currents of the WZW model obey the $O(N)_k$ Kac-Moody algebra, provided the theory is at the value of the coupling constant λ_c. The same result holds for other groups, including $SU(N)_k$.

The perturbative renormalization group also offers a hint that the special value λ_c of Eq.(21.211) may be a fixed point. We should first notice that since k is an integer, it cannot flow under the RG, and only the coupling constant can flow. A one loop calculation, which is only accurate in the weak coupling regime, yields a beta function (using the high-energy physics sign convention)

$$\beta(\lambda) = -\lambda^2\left(\frac{N-1}{4\pi}\right)\left[1 - \left(\frac{\lambda^2 k}{4\pi}\right)^2\right] \tag{21.212}$$

which has an IR stable fixed point at λ_c. Of course, this argument cannot be trusted unless k is very large. However, it turns out to be the exact answer.

Witten conjectured a set of bosonization identities for the fermion bilinears, mass terms, that mix the left and right moving sectors. For a theory with $O(N)$ symmetry, i.e. N free massless Majorana fermions, which is identified with the $O(N)_1$ WZW model (and, hence, with level $k = 1$), the identification of the operators

$$-i : \psi_L^i(x)\psi_{j,R}(x) := M g_j^i(x) \tag{21.213}$$

where $g_j^i \in O(N)$, and M is a "mass" whose precise form depends on the normal ordering of the operators on the left hand side of the equation. For free massless Dirac fermions with $U(N)$ symmetry, whose current algebra is $U(1)_N \times SU(N)_1$, they are mapped by bosonization to the $U(1)_N$ free massless compactified boson ϕ, and the $SU(N)_1$ WZW non-linear sigma model whose degree of freedom is $g \in SU(N)$ (again, with level $k = 1$). In the Dirac case the mass fermion bilinears are identified with group elements as (Affleck, 1988)

$$: \psi_L^{i\,\dagger}\psi_{R,j}(x) := M \exp\left(i\sqrt{\frac{4\pi}{N}}\phi(x)\right) g_j^i(x) \tag{21.214}$$

The action of the WZW model at its IR fixed point is

$$\mathcal{L} = \frac{k}{16\pi}\int d^2x\, \mathrm{tr}(\partial^\mu g \partial_\mu g^{-1}) + \frac{k}{24\pi}\int_B \epsilon^{\mu\nu\lambda}\mathrm{tr}\left(g^{-1}\partial_\mu g\; g^{-1}\partial_\nu g\; g^{-1}\partial_\lambda g\right) \tag{21.215}$$

This is the action of the WZW CFT. We will now analyze this theory form the CFT point of view.

Knizhnik and Zamolodchikov solved the WZW CFT for a semi-simple Lie group G (Knizhnik and Zamolodchikov, 1984). They showed that at the special value of the coupling constant λ_c, the left and right moving modes of the WZW model decouple and the theory has an enhanced $G\times G$ symmetry. At this value of λ the theory has full conformal invariance. In the rest of this section we will describe the WZW CFT on the Euclidean complexified plane, with coordinates $z = x_1 + ix_2 \in \mathbb{C}$, as it is standard in two-dimensional CFT (Belavin et al., 1984)

The WZW theory has a Virasoro algebra, generated by its energy-momentum tensor, and a Kac-Moody algebra generated by its chiral currents. Focusing on the left-moving (holomorphic) sector, Knizhnik and Zamolodchikov showed that the generators of the two algebras obey the OPEs

$$T(z)T(z') = \frac{c}{2(z-z')^4} + \frac{2}{(z-z')^2}T(z') + \frac{1}{z-z'}T(z') + \dots \tag{21.216}$$

$$T(z)J_L^a(z') = \frac{1}{(z-z')^2}J_L^a(z') + \frac{1}{z-z'}J_L(z') + \dots \tag{21.217}$$

$$J_L^a(z)J_L^b(z') = \frac{k\delta^{ab}}{(z-z')^2} + \frac{f^{abc}}{(z-z')}J_L^c(z') + \dots \tag{21.218}$$

supplemented with the asymptotic conditions, $T(z) \sim z^{-4}$ and $J_L^a(z) \sim z^{-2}$ as $z \to \infty$. Eq.(21.216) states that $T(z)$ is the generator of a Virasoro algebra with central charge c, and Eq.(21.218) states that $J^a(z)$ are the generators of a Kac-Moody algebra with level k. Eq.(21.217) simply states the the chiral currents $J_L^a(z)$ are fields with dimension $(\Delta, \bar{\Delta}) = (1, 0)$ (see below). The right-moving (anti-holomorphic) components, $\bar{T}(\bar{z})$ and $J_R^a(\bar{z})$ obey similar equations.

The primary fields of this theory, $\phi_l(z, \bar{z})$ have dimensions $(\Delta_l, \bar{\Delta}_l)$ and obey the OPEs

$$T(w)\phi_l(z,\bar{z}) = \frac{\Delta_l}{(w-z)^2}\phi_l(z,\bar{z}) + \frac{1}{w-z}\partial_z\phi_l(z,\bar{z}) + \dots \tag{21.219}$$

$$J_L^a(w)\phi_l(z,\bar{z}) = \frac{t_l^a}{w-z}\phi_l(z,\bar{z}) + \dots \tag{21.220}$$

Eq.(21.219) just states the $\phi_l(z, \bar{z})$ is a Virasoro primary field. In Eq.(21.220), t_l^a are the generators of the group G for the field ϕ_l.

The energy momentum tensor $T(z)$ and the left moving currents $J_L^a(z)$ admit the mode expansions (the Laurent expansion of Eq.(21.94)), which here become

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}, \qquad J_L^a(z) = \sum_{n\in\mathbb{Z}} \frac{J_n^a}{z^{n+1}} \tag{21.221}$$

and similar expressions for the right-moving (anti-holomorphic) components. To alleviate the notation, here we dropped the label L in the WZW current. The WZW primary fields ϕ_l satisfy the following equations (for $n > 0$)

$$\begin{aligned} L_n\phi_l &= 0, & L_0\phi_l &= \Delta_l\phi_l \\ J_n^a\phi_l &= 0, & J_0^a\phi_l &= t_l^a\phi_l \end{aligned} \tag{21.222}$$

The singular terms in the OPEs of Eqs. (21.216), (21.217), and (21.218) imply that the operators L_n and J_n^a obey the algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}c(n^3 - n)\delta_{n+m,0} \tag{21.223}$$

$$[L_n, J_m^a] = -\, mJ_{n+m}^a \tag{21.224}$$

$$[J_n^a, J_m^b] = f^{abc}J_{n+m}^c + \frac{1}{2}k\delta^{ab}\delta_{n+m,0} \tag{21.225}$$

where Eq.(21.223) is the Virasoro algebra and Eq.(21.225) is the Kac-Moody algebra.

The complete set of local fields includes, in addition to the primary fields $\{\phi_l\}$, their descendants under the action of both the Virasoro and Kac-Moody operators, $\{L_{-n}\}$ and $\{J_{-m}^a\}$ (and their right-moving counterparts). The descendant fields constitute the Verma modulus of the primary field ϕ_l, and have dimensions $\Delta_l^{\{n,m\}} = \Delta_l + \sum_{i=1}^N n_i + \sum_{j=1}^M m_j$, and similarly for their anti-holomorphic components. In this sense, neither $T(z)$ nor $J_L^a(z)$ are primary fields since $T(z) = L_{-1}\ I$ and $J_L^a(z) = J_{-1}^a\ I$, where here I is the identity field.

What we described above applies to any conformal field theory with a Virasoro and and a Kac-Moody algebra. We will now apply this formalism to the WZW models, following closely the work (and notation) of Knizhnik and Zamolodchikov. The WZW model is a non-linera sigma model with a field g that takes values on the group G.

The key property of the WZW model are its conserved chiral currents of Eq.(21.206). In complex coordinates, the conserved currents of the WZW model will be denoted as $J(z) = J_a(z)T^a$ and $\bar{J}(\bar{z}) = \bar{J}_a(\bar{z})t^a$, and are given by

$$J(z) = -\frac{k}{2}(\partial_z g(z,\bar{z}))g^{-1}(z,\bar{z}), \quad J(z) = -\frac{k}{2}g^{-1}(z,\bar{z})(\partial_{\bar{z}} g(z,\bar{z})) \tag{21.226}$$

and satisfy the conservation laws, $\partial_{\bar{z}} J = 0$ and $\partial_z \bar{J} = 0$. Notice that we have rescaled the chiral currents relative to the expressions given in Eq.(21.206).

The main assumption of Knizhnik and zamolodchikov is that the set of fields of the WZW theory contains a *primary* field $g(z,\bar{z})$ whose conformal weights (dimensions) are $\Delta_g = \bar{\Delta}_g = \Delta$, and that it satisfies the equations

$$\kappa\partial_z g(z,\bar{z}) =: J_a(z)t^a\ g(z,\bar{z}) :, \qquad \kappa\partial_{\bar{z}} g(z,\bar{z}) =: \bar{J}_a(\bar{z})t^a\ g(z,\bar{z}) : \tag{21.227}$$

where κ will be given below. Since g is a primary field, we expect that it will have an OPE with the chiral (Kac-Moody) currents of the form

$$J_a(w)t^a g(z,\bar{z}) = \frac{c_g}{w-z}c(z,\bar{z}) + \kappa\partial_z g(z,\bar{z}) + \dots \tag{21.228}$$

where c_g, defined by $t^a t^a = c_g I$, is equal to $c_g = 2C(g)$ where $C(g)$ is the quadratic Casimir of the representation, and the ellipsis indicates regular terms as $w \to z$. The, the normal ordered product of Eq.(21.227) is defined as the limit

$$: J_a(z)t^a \; g(z, \bar{z}) := \lim_{w \to z} \left(J^a(w) - \frac{t^a}{w - z} \right) t^a g(z, \bar{z}) \tag{21.229}$$

The OPE of Eq.(21.228) implies that there is a field in this CFT,

$$\chi \equiv (J^a_{-1} t^a - \kappa L_{-1}) g = 0 \tag{21.230}$$

which is a null field. Here we used that $\partial_z g = L_{-1} g$. Therefore, χ is a null state and the representation is degenerate. Consistency then requires that χ be a primary field that satisfies

$$\begin{aligned} L_0 \chi =& (\Delta + 1)\chi, \qquad J^a_0 \chi = t^a \chi \\ L_n \chi =& J^a_n \chi = 0, \quad \text{for } n > 0 \end{aligned} \tag{21.231}$$

While the first equation is automatically satisfied, the second equation holds provided

$$c_g + 2\Delta\kappa = 0, \qquad c_V + k + 2\kappa = 0 \tag{21.232}$$

where c_V, defined by

$$f^{acd} f^{bcd} = c_V \delta_{ab} \tag{21.233}$$

is the quadratic Casimir of the adjoint representation. These conditions imply that the scaling dimension Δ of the primary field g must be

$$\Delta = \frac{2C(g)}{c_V + k} \tag{21.234}$$

and that the parameter κ is

$$\kappa = -\frac{1}{2}(c_V + k) \tag{21.235}$$

Another way to reach the same conclusions is to construct the energy-momentum tensor of the quantized theory. We will assume that the energy-momentum tensor has the Sugawara form,

$$T(z) = \frac{1}{2\kappa} : J^a(z) J^a(z) :, \qquad \bar{T}(z) = \frac{1}{2\kappa} : \bar{J}^a(z) \bar{J}^a(z) : \tag{21.236}$$

with the same constant κ used above. This structure of the energy-momentum tensor means that the OPE of the currents should be

$$J^a(z) J^a(z') = \frac{kD}{(z - z')^2} + 2\kappa T(z) + \dots \tag{21.237}$$

where D is the dimensions of the group G (i.e. the number of generators). Since the energy-momentum tensor $T(z)$ and the currents $J^a(z)$ must also satisfy the Virasoro and Kac-Moody algebras, Eqs.(21.216), (21.217), and (21.218), the above equation will be satisfied only if the central charge c is given by

$$c = \frac{kD}{c_V + k} \tag{21.238}$$

and the constant κ is given by Eq.(21.235).

These results also imply that the generators of the two algebras must be related to each other through the expression

$$L_n = \frac{1}{2\kappa} \sum_{m \in \mathbb{Z}} : J_m^a J_{n-m}^a : \tag{21.239}$$

where normal ordering here means that J_n with $n < 0$ are placed to the left of J_m with $m > 0$. Then, if we use this definition for L_{-1} and have it act on the primary field g, we obtain the null state χ of Eq.(21.230).

Furthermore, this line of reasoning actually applies to *all* the primary fields in the theory, and not just the WZW field g. We then conclude that the scaling dimensions of all primary fields are given by

$$\Delta = \frac{2C(\phi_l)}{c_V + k}, \qquad \bar{\Delta} = \frac{2\bar{C}(\phi_l)}{c_V + k} \tag{21.240}$$

where $C(\phi_l)$ is the quadratic Casimir of the representation associated with the primary field ϕ_l. One result that also follows is that the slope of the beta function at the WZW fixed point is

$$\left.\frac{d\beta(\lambda^2, k)}{d\lambda^2}\right|_{\lambda^2 = 4\pi/k} = \frac{2c_V}{c_V + k} \tag{21.241}$$

which agrees with Witten's one-loop result.

Knizhnik and Zamolodchikov also showed that the correlators of the g field satisfy the differential equations

$$\left[\kappa \frac{\partial}{\partial_z - i} - \sum_{j=1}^{N} \frac{t_i^a t_j^a}{z_i - z_j}\right] \langle g(z_1, \bar{z}_1) \dots g(z_N, \bar{z}_N) \rangle = 0 \tag{21.242}$$

which is known as the Knizhnik-Zamolodchikov equation. These equations, and their generalizations for other primary fields, can then be used to obtain the correlation functions of the WZW theory.

We end our discussion of the WZW CFT by applying these results for the case of the group $G = SU(N)$. In this case, the dimension of the group is $D = N^2 - 1$, the Casimir of the adjoint representation is $c_V = N$. This implies that for $SU(N)_k$ WZW CFT, the energy-tensor is

$$T(z) = -\frac{1}{(N+k)} : J^a(z) J^a(z) : \tag{21.243}$$

The central charge is

$$c = \frac{k(N^2-1)}{N+k} \tag{21.244}$$

and that the scaling dimensions are

$$\Delta_l = \frac{2C_l}{N+k} \tag{21.245}$$

In particular, the dimension of the WZW field g is

$$\Delta = \frac{N^2-1}{2N(N+k)} \tag{21.246}$$

We should not that the central charges for the WZW theory are, in general, fractional numbers. This implies that these fixed points are, in general, not free field theories. However, level $k = 1$ theories can represent free fields.

For instance, in the case of $SU(N)_1$, we readily obtain that $c = 1.$ and that $\Delta = \bar{\Delta} = (N-1)/2N$. Both results are consistent with the non-abelian bosonization identification of the free fermion mass terms in Eq.(21.214) with a product of the $U(1)_N$ vertex operator $\exp(i\sqrt{4\pi/N}\phi)$ and the WZW field g. Indeed, one can see that the dimensions of these operators add up to $1/2$. Hence, the mass terms have dimension 1 and conformal spin 0, as they should. Moreover the central charges also add up to the correct value, $c(U(1)_N) + c(SU(N)_1) = N$, the value for N free Dirac fields. Moreover, the energy momentum tensor of the free fermions becomes the sum

$$T_{\text{Dirac}} = \sum_{i=1}^{N} : \psi_i^\dagger(z) \partial_z \psi_i(z) : \mapsto -\frac{1}{2N} : J(z) J(z) : -\frac{1}{N+1} : J^a(z) J^a(z) : \tag{21.247}$$

In other words, the abelian $U(1)$ sector and the non-abelian $SU(N)$ sector factorize, with the central charges and scaling dimensions adding up to their free field values, and the full Hilbert space decomposes into the tensor product of the individual Hilbert spaces.