

Lieb-Robinson Bounds and the Exponential Clustering Theorem^{*}

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Abstract: We give a Lieb-Robinson bound for the group velocity of a large class of discrete quantum systems which can be used to prove that a non-vanishing spectral gap implies exponential clustering in the ground state of such systems.

1. Introduction

One of the folk theorems in quantum lattice models claims the equivalence of the existence of a nonvanishing spectral gap and exponential decay of spatial correlations in the ground state. It has been known for some time that there are exceptions to one direction of this equivalence. There are models with a unique ground state with exponential decay of correlations but without a gap in the excitation spectrum above the ground state. For a simple example see Example 2 in [2, p. 596]. In this paper we provide a rigorous proof of the other implication: a spectral gap implies exponential decay in the ground state.

In relativistic quantum field theory it has been known for a long time that the existence of a mass (energy) gap implies exponential decay of spatial correlations. For example, in [1] Fredenhagen proves a general theorem applicable to arbitrary strictly local theories. It is the strict locality, i.e., the fact that space-like separated observables commute, not the relativistic invariance per se, which plays a crucial role in the proof of exponential decay. Non-relativistic models of statistical mechanics do not have strict locality, but there is a finite speed of propagation up to exponentially small corrections. This was first proven by Lieb and Robinson [13]. It is not a surprise that the Lieb-Robinson bound can replace the strict locality property. In particular, Wreszinski relied on it to prove a Goldstone Theorem in non-relativistic quantum statistical mechanics [9]. Precisely how to apply the Lieb-Robinson result to work around non-locality is not entirely obvious. Only recently, Hastings used it to derive exponential clustering for lattice models with a

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gap [12], and to obtain a generalization to higher dimensions of the Lieb-Schultz-Mattis theorem [10] in [11]. Our proof of exponential clustering is a rigorous version of Hastings' argument.

In the proof by Lieb and Robinson the lattice structure played an essential role through the use of the Fourier transform. This was emphasized in a later version of their proof in [15]. Since lately there has been some interest in models without translation invariance or even without an underlying lattice structure such as spaces of fractal dimension [3–5] or the so-called complex networks [7, 8], we revisited the Lieb-Robinson result and provide here a proof that works for quite arbitrary models defined on a set of vertices with a metric.

2. Main Results

We will consider quantum systems defined on a set of vertices V with a finite dimensional Hilbert space \mathcal{H}_x at each vertex $x \in V$. At first we will assume that V is finite. For $X \subset V$, the Hilbert space associated with X is the tensor product $\mathcal{H}_X = \bigotimes_{x \in V} \mathcal{H}_x$, and the algebra of observables in X is denoted by $\mathcal{A}_X = \mathcal{B}(\mathcal{H}_X)$. An interaction for such a system is a map Φ from the set of subsets of V to \mathcal{A}_V such that $\Phi(X) \in \mathcal{A}_X$ and $\Phi(X) = \Phi(X)^*$ for all $X \subset V$. The Hamiltonian is defined by

$$H = \sum_{X \subset V} \Phi(X).$$

The dynamics of the model is the one-parameter group of automorphisms, $\{\tau_t\}_{t \in \mathbb{R}}$, defined by

$$\tau_t(A) = e^{itH} A e^{-itH}, \quad A \in \mathcal{A}_V.$$

We will assume that V is equipped with a metric d . In the most common cases V is a graph, and the metric is given by the graph distance, $d(x, y)$, which may be the length of the shortest path of edges connecting x and y in the graph. We will not, however, require an underlying graph structure for our results. In terms of d we define the diameter, $D(X)$, of a subset $X \subset V$ by

$$D(X) = \max\{d(x, y) \mid x, y \in X\}.$$

Infinite systems can be introduced by considering a net of finite systems indexed by finite sets V partially ordered by inclusion. The C^* -algebra of observables, \mathcal{A} , is the norm completion of the union of the local observable algebras \mathcal{A}_V . We will assume that there is a uniform bound, N , on the dimension of the single site Hilbert spaces. An interaction Φ is defined as before but it is necessary to impose a boundedness condition in order for the finite-volume dynamics to converge to a strongly continuous one-parameter group of automorphisms on \mathcal{A} . A standard reference for these infinite volume techniques is [15]. The strength of a given interaction will be measured by a norm, $\|\cdot\|_\lambda$, which for $\lambda > 0$ is defined by

$$\|\Phi\|_\lambda := \sup_{x \in V} \sum_{X \ni x} |X| \|\Phi(X)\| N^{2|X|} e^{\lambda D(X)}. \quad (1)$$

Here $|X|$ denotes the cardinality of the finite set X . For finite V , the supremum in this definition is of course a maximum. For infinite systems, finiteness of this norm is stronger

than what is required for the existence of the dynamics, but it is this norm that appears in the Lieb-Robinson bound [13]. We will denote by \mathcal{B}_λ the set of all potentials for the system under consideration such that $\|\Phi\|_\lambda < \infty$. A Lieb-Robinson bound is an estimate for the following quantities:

$$C_{A,B}(x, t) := [\tau_t(A), B], \quad (2)$$

where $x \in V$, $t \in \mathbb{R}$, $A \in \mathcal{A}_x$, $B \in \mathcal{A}$. Due to the automorphism property of τ_t , there is a symmetry in the norm of such commutators: $\|[\tau_t(A), B]\| = \|[A, \tau_{-t}(B)]\|$.

It will be useful to consider

$$C_B(x, t) := \sup_{A \in \mathcal{A}_x} \frac{\|C_{A,B}(x, t)\|}{\|A\|}. \quad (3)$$

In a typical application we would have $B \in \mathcal{A}_Y$, for some $Y \subset V$, and $x \in V \setminus Y$. Then, A and B commute and $C_B(x, 0) = 0$. A Lieb-Robinson bound then aims to show that $C_B(x, t)$ is small for $|t| \leq T$ with T proportional to the distance between x and Y .

Theorem 1 (Lieb-Robinson Bound). *Fix $\lambda > 0$, then for all $\Phi \in \mathcal{B}_\lambda$, $x \in V$, $t \in \mathbb{R}$, and $B \in \mathcal{A}$, we have the bound*

$$C_B(x, t) \leq e^{2|t|\|\Phi\|_\lambda} C_B(x, 0) + \sum_{y \in V: y \neq x} e^{-\lambda d(x, y)} \left(e^{2|t|\|\Phi\|_\lambda} - 1 \right) C_B(y, 0). \quad (4)$$

It is straightforward to derive from Theorem 1 a bound for $\|[\tau_t(A), B]\|$ for general local observables $A \in \mathcal{A}_X$. One gets

$$\|[\tau_t(A), B]\| \leq N^{2|X|} \|A\| \sum_{x \in X} C_B(x, t). \quad (5)$$

For observables with overlapping supports, the Lieb-Robinson bound may not be useful in the sense that the trivial bound $\|[\tau_t(A), B]\| \leq 2\|A\| \|B\|$ may be better. The problem of estimating $\|[\tau_t(A), B]\|$ for large t , in cases where it is expected to decay, is a separate issue that we do not address here.

In the case of observables with widely separated supports the information in (4) is essentially equivalent to

$$C_B(x, t) \leq \sum_{y \in V} e^{2|t|\|\Phi\|_\lambda - \lambda d(x, y)} C_B(y, 0). \quad (6)$$

For strictly local $B \in \mathcal{A}_Y$, we can easily derive a more explicit bound by using that, $C_B(y, 0) \leq 2\|B\| \chi_Y(y)$, where χ_Y is the characteristic function of the set Y . One obtains for all $A \in \mathcal{A}_x$ and $B \in \mathcal{A}_Y$ the bound

$$\|[\tau_t(A), B]\| \leq 2|Y| \|A\| \|B\| e^{2|t|\|\Phi\|_\lambda - \lambda d(x, Y)}. \quad (7)$$

If $x \notin Y$, a stronger inequality holds:

$$\|[\tau_t(A), B]\| \leq 2|Y| \|A\| \|B\| \left(e^{2|t|\|\Phi\|_\lambda} - 1 \right) e^{-\lambda d(x, Y)}. \quad (8)$$

It would be interesting to have an analogue of (8) for complex times. We may apply Theorem 1 with B replaced by $\tau_{ib}(B)$, for any B which is analytic in a disk centered at 0 of radius $r > 0$, e.g., a local B , and $b < r$. This way one can obtain, e.g.,

$$\frac{\| [A, \tau_{t+ib}(B)] \|}{\| A \|} \leq \sum_{y \in V} e^{2|t| \|\Phi\|_\lambda - \lambda d(x, y)} C_{\tau_{ib}(B)}(y, 0). \quad (9)$$

The remaining open question is to obtain good bounds for $C_{\tau_{ib}(B)}(y, 0)$.

The next theorem provides an estimate on the spatial decay of correlations in states with a spectral gap. To state the gap condition precisely, we consider a representation of the system on a Hilbert space \mathcal{H} . This means that there is a representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, and a self-adjoint operator H on \mathcal{H} such that

$$\pi(\tau_t(A)) = e^{itH} \pi(A) e^{-itH}, \quad A \in \mathcal{A}.$$

In the finite-system setting using a representation is, of course, merely for convenience and not essential. We assume that $H \geq 0$ and that $\Omega \in \mathcal{H}$ is a vector state such that $H\Omega = 0$. We say that the system has a spectral gap in the representation if there exists $\delta > 0$ such that $\text{spec}(H) \cap (0, \delta) = \emptyset$ and in that case the spectral gap, γ , is defined by

$$\gamma = \sup\{\delta > 0 \mid \text{spec}(H) \cap (0, \delta) = \emptyset\}.$$

Let P_0 denote the orthogonal projection onto $\ker H$. From now on, we will work in this representation and simply write A instead of $\pi(A)$.

We will derive a bound for ground state correlations of the form

$$\langle \Omega, A \tau_{ib}(B) \Omega \rangle, \quad (10)$$

where $b \geq 0$ and A and B are local observables. The case $b = 0$ is the standard (equal-time) correlation function.

Theorem 2 (Exponential Clustering). *There exists $\mu > 0$ such that for all $A \in \mathcal{A}_x$, $B \in \mathcal{A}_Y$, $x \notin Y$, for which $P_0 B \Omega = P_0 B^* \Omega = 0$, and b sufficiently small, there is a constant $c(A, B)$ such that*

$$|\langle \Omega, A \tau_{ib}(B) \Omega \rangle| \leq c(A, B) e^{-\mu d(x, Y)} \left(1 + \frac{\gamma^2 b^2}{4\mu^2 d(x, Y)^2} \right). \quad (11)$$

One can take

$$c(A, B) = \|A\| \|B\| \left(1 + \frac{2|Y|}{\pi} + \frac{1}{\sqrt{\pi \mu d(x, Y)}} \right), \quad (12)$$

and

$$\mu = \frac{\gamma \lambda}{4 \|\Phi\|_\lambda + \gamma}. \quad (13)$$

The bound is valid for $0 \leq \gamma b \leq 2\mu d(x, Y)$.

Usually, the distance between x and Y will be bounded below by some $a > 0$, so $c(A, B)$ in (12) can be replaced by a constant which depends only on the norms of A and B and the size of their supports, but not the distance between them.

Note that in the case of a non-degenerate ground state, the condition on B is equivalent to $\langle \Omega, B\Omega \rangle = 0$. In that case, the theorem with $b = 0$ becomes

$$|\langle \Omega, AB\Omega \rangle - \langle \Omega, A\Omega \rangle \langle \Omega, B\Omega \rangle| \leq c(A, B) e^{-\mu d(x, Y)}. \quad (14)$$

For $b > 0$ large, there is a trivial bound

$$|\langle \Omega, A\tau_{ib}(B)\Omega \rangle| \leq \|A\| \|B\| e^{-\gamma b}. \quad (15)$$

For small $b > 0$, the estimate (11) can be viewed as a perturbation of (14). Often, the important observation is that the decay estimate (11) is uniform in the imaginary time ib , for b in some interval whose length, however, depends on $d(x, Y)$.

3. Proofs

3.1. Lieb-Robinson bound. Proof of Theorem 1. Our argument closely follows the proof of [15, Prop. 6.2.9], but we avoid the use of the Fourier transform in order to obtain a generalization to arbitrary sets of vertices with a metric.

Let $A \in \mathcal{A}_x$ and $B \in \mathcal{A}$, and consider the quantities $C_{A,B}(x, t)$ and $C_B(x, t)$, defined in (2) and (3), respectively.

From the fundamental theorem of calculus, one has that

$$C_{A,B}(x, t) = C_{A,B}(x, 0) + i \sum_{X \ni x} \int_0^t [\tau_s([\Phi(X), A]), B] ds. \quad (16)$$

Several applications of the triangle inequality to (16) yields

$$C_B(x, t) \leq C_B(x, 0) + \sum_{X \ni x} \int_0^{|t|} \sup_{A \in \mathcal{A}_x} \frac{\|[\tau_s([\Phi(X), A]), B]\|}{\|A\|} ds. \quad (17)$$

Now, for any finite $X \subset V$ with $x \in X$, one may write

$$[\Phi(X), A] = \sum_{i=1}^{|X|} \sum_{j_{x_i}=1, \dots, N_{x_i}^2} C(j_{x_1}, \dots, j_{x_{|X|}}) \prod_{i=1}^{|X|} e(j_{x_i}), \quad (18)$$

where each $x_i \in X$ and $e(j_{x_i})$ is a matrix unit for the algebra \mathcal{A}_{x_i} . For each term in (18), expanding the commutator of the product yields

$$\left\| \left[\prod_{i=1}^{|X|} \tau_s(e(j_{x_i})), B \right] \right\| \leq \sum_{y \in X} C_B(y, s). \quad (19)$$

Combining the linearity of τ_s , the basic estimate (19), and the fact that the coefficients in the above expansion (18) satisfy

$$|C(j_{x_1}, \dots, j_{x_{|X|}})| \leq 2 \|\Phi(X)\| \|A\|, \quad (20)$$

we arrive at the inequality

$$C_B(x, t) \leq C_B(x, 0) + 2 \sum_{X \ni x} \|\Phi(X)\| N^{2|X|} \int_0^{|t|} \sum_{y \in X} C_B(y, s) ds. \quad (21)$$

Motivated by the expression above, we define the quantity

$$\epsilon(x, y) := \sum_{X \ni x, y} \|\Phi(X)\| N^{2|X|}, \quad (22)$$

and rewrite (21) as

$$C_B(x, t) \leq C_B(x, 0) + 2 \int_0^{|t|} \sum_{y \in V} \epsilon(x, y) C_B(y, s) ds. \quad (23)$$

Iteration of (23) yields

$$\begin{aligned} C_B(x, t) &\leq C_B(x, 0) + 2|t| \sum_{y \in V} \epsilon(x, y) C_B(y, 0) \\ &\quad + \frac{(2|t|)^2}{2} \sum_{y \in V} \epsilon(x, y) \sum_{y' \in V} \epsilon(y, y') C_B(y', 0) \\ &\quad + \frac{(2|t|)^3}{3!} \sum_{y \in V} \epsilon(x, y) \sum_{y' \in V} \epsilon(y, y') \sum_{y'' \in V} \epsilon(y', y'') C_B(y'', 0) + \cdots \end{aligned}$$

Recall that we assumed that the interaction satisfies a bound of the form

$$\|\Phi\|_\lambda = \sup_{x \in V} \sum_{X \ni x} |X| \|\Phi(X)\| N^{2|X|} e^{\lambda D(X)} < \infty, \quad (24)$$

for some $\lambda > 0$. We wish to prove an exponential bound on the quantity $C_B(x, t)$. To make this bound explicit, we set $\epsilon_\lambda(x, y) := e^{\lambda d(x, y)} \epsilon(x, y)$, and observe that

$$\begin{aligned} \sum_{y \in V} \epsilon_\lambda(x, y) &= \sum_{y \in V} \sum_{X \ni x, y} \|\Phi(X)\| N^{2|X|} e^{\lambda d(x, y)} \\ &\leq \sum_{X \ni x} \sum_{y \in X} \|\Phi(X)\| N^{2|X|} e^{\lambda D(X)} = \|\Phi\|_\lambda. \end{aligned} \quad (25)$$

Now, returning to the iterated version of (23) above, we may resum the upper bound, which is allowable as all the terms are non-negative. Using the triangle inequality often, one derives

$$C_B(x, t) \leq \sum_{y \in L} e^{-\lambda d(x, y)} f(x, y) C_B(y, 0), \quad (26)$$

where

$$\begin{aligned} f(x, y) &= \delta_{x, y} + 2|t| \epsilon_\lambda(x, y) + \frac{(2|t|)^2}{2} \sum_{y' \in V} \epsilon_\lambda(x, y') \epsilon_\lambda(y', y) \\ &\quad + \frac{(2|t|)^3}{3!} \sum_{y'' \in V} \sum_{y' \in V} \epsilon_\lambda(x, y'') \epsilon_\lambda(y'', y') \epsilon_\lambda(y', y) + \cdots \end{aligned}$$

Moreover, summing over all but the first term appearing in f and using (25), we see that

$$f(x, y) \leq \delta_{x,y} + \left(e^{2|t| \|\Phi\|_\lambda} - 1 \right). \quad (27)$$

Therefore, we have proven the estimate

$$C_B(x, t) \leq e^{2|t| \|\Phi\|_\lambda} C_B(x, 0) + \sum_{\substack{y \in V: \\ y \neq x}} e^{-\lambda d(x,y)} \left(e^{2|t| \|\Phi\|_\lambda} - 1 \right) C_B(y, 0), \quad (28)$$

as claimed. \square

3.2. Decay of correlations. Proof of Theorem 2. It is well-known, see e.g. [15, 16], that for interactions $\Phi \in \mathcal{B}_\lambda$ and B a local observable the function $t \mapsto \tau_t(B)$ is analytic in a disk centered at the origin.

Part of our argument requires analyticity for arbitrarily large $|t|$. Therefore, we consider entire analytic elements $B(a)$, $a > 0$, (see [14, Prop. 2.5.22]), with the property that $\|B(a)\| \leq \|B\|$ and $\lim_{a \rightarrow 0} \|B(a) - B\| = 0$. In this case, the function

$$f_a(z) := \langle \Omega, A \tau_z(B(a)) \Omega \rangle \quad (29)$$

is entire. Observe that the quantity of interest, i.e. that which appears on the left hand side of (11), corresponds to $f_0(ib)$, where f_0 is as in (29) above with $B(a) = B$ and $b > 0$. The case $b = 0$ follows by a limiting argument.

Take $T > 2b$ and denote by Γ_T the contour in the complex plane which joins $-T$ to T along the real axis and T to $-T$ via the semi-circle $T e^{i\theta}$ for $\theta \in [0, \pi]$. By analyticity, we have that

$$f_a(ib) = \frac{1}{2\pi i} \int_{\Gamma_T} \frac{f_a(z)}{z - ib} dz. \quad (30)$$

Using the existence of a gap $\gamma > 0$ and the assumption $P_0 B \Omega = 0$, one easily finds that, for $\theta \in [0, \pi]$,

$$|f_a(T e^{i\theta})| = \left| \langle \Omega, A e^{iT e^{i\theta} H} B(a) \Omega \rangle \right| \leq \|A\| \|B\| e^{-T \gamma \sin \theta}. \quad (31)$$

From this, we conclude that

$$|f_0(ib)| = \lim_{a \rightarrow 0} |f_a(ib)| \leq \left| \frac{1}{2\pi i} \int_{-T}^T \frac{f_0(t)}{t - ib} dt \right| + \frac{\|A\| \|B\|}{\pi} \int_0^\pi e^{-T \gamma \sin(\theta)} d\theta, \quad (32)$$

where we have used that $\|B(a) - B\| \rightarrow 0$ as $a \rightarrow 0$. Since the second term vanishes in the limit $T \rightarrow \infty$, we have shown that

$$|\langle \Omega, A \tau_{ib}(B) \Omega \rangle| \leq \limsup_{T \rightarrow \infty} \left| \frac{1}{2\pi i} \int_{-T}^T \frac{f_0(t)}{t - ib} dt \right|, \quad (33)$$

and thus, the proof has been reduced to estimating this integral over the real axis.

Our estimate follows by splitting the integrand into three terms. Let $\alpha > 0$ and write

$$f_0(t)e^{\alpha b^2} = f_0(t)e^{-\alpha t^2} + f_0(t)(e^{\alpha b^2} - e^{-\alpha t^2}). \quad (34)$$

The first term above may be written so that the commutator explicitly appears, i.e.,

$$f_0(t)e^{-\alpha t^2} = \langle \Omega, \tau_t(B)A\Omega \rangle e^{-\alpha t^2} + \langle \Omega, [A, \tau_t(B)]\Omega \rangle e^{-\alpha t^2}. \quad (35)$$

Using (34) and (35), we arrive at a bound of the form:

$$\left| \frac{1}{2\pi i} \int_{-T}^T \frac{f_0(t)}{t - ib} dt \right| \leq I + II + III, \quad (36)$$

where I corresponds to the integral containing the first term of (35), II contains the second term of (35), and III the second term of (34). Each of these terms will be bounded separately in the limit $T \rightarrow \infty$.

The first term. The first term appearing in the bound on the right-hand side of (36) is

$$I := \left| \frac{1}{2\pi i} \int_{-T}^T \frac{\langle \Omega, \tau_t(B)A\Omega \rangle e^{-\alpha t^2}}{t - ib} dt \right|. \quad (37)$$

Using the spectral theorem, we may write

$$\langle \Omega, \tau_t(B)A\Omega \rangle = \langle e^{itH} B^* \Omega, A\Omega \rangle = \int_{\gamma}^{\infty} e^{-itE} d\langle P_E B^* \Omega, A\Omega \rangle, \quad (38)$$

where P_E is the projection valued spectral measure corresponding to H , and we have used that $P_0 B^* \Omega = 0$. One may now write

$$\int_{-T}^T \frac{\langle \Omega, \tau_t(B)A\Omega \rangle e^{-\alpha t^2}}{t - ib} dt = \int_{\gamma}^{\infty} \int_{-T}^T \frac{e^{-itE} e^{-\alpha t^2}}{t - ib} dt d\langle P_E B^* \Omega, A\Omega \rangle. \quad (39)$$

The inner integral can be estimated using Lemma 1, proven below, from which we have

$$\limsup_{T \rightarrow \infty} I \leq \frac{\|A\| \|B\|}{2} e^{-\frac{\gamma^2}{4\alpha}}, \quad (40)$$

independent of b .

The second term. The second term is the one for which we shall apply the Lieb-Robinson estimate in the form of (8):

$$II := \left| \frac{1}{2\pi i} \int_{-T}^T \frac{\langle \Omega, [A, \tau_t(B)]\Omega \rangle e^{-\alpha t^2}}{t - ib} dt \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\|[A, \tau_t(B)]\|}{|t|} e^{-\alpha t^2} dt. \quad (41)$$

Here it is crucial that the “time”, ib , is purely imaginary. We will now break the integral up into two regions: $\{t \in \mathbb{R} : |t| \leq s\}$, where the Lieb-Robinson bound is useful and $\{t \in \mathbb{R} : |t| \geq s\}$ in which we will use a norm estimate:

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\|[A, \tau_t(B)]\|}{|t|} e^{-\alpha t^2} dt \\ & \leq \frac{2\|Y\| \|A\| \|B\|}{\pi} e^{2s\|\Phi\|_{\lambda} - \lambda d(x, Y)} + \frac{\|A\| \|B\|}{s\sqrt{\pi\alpha}} e^{-\alpha s^2}. \end{aligned} \quad (42)$$

The third term. The final term is

$$III := \left| \frac{1}{2\pi i} \int_{-T}^T \frac{\langle \Omega, A \tau_t(B) \Omega \rangle (e^{\alpha b^2} - e^{-\alpha t^2})}{t - ib} dt \right|, \quad (43)$$

which represents the penalty we incur for introducing the gaussian factor $e^{-\alpha t^2}$ and the scaling $e^{\alpha b^2}$. This term can be made small by a judicious choice of parameters. Using the spectral theorem again, we have

$$III = \left| \int_{\gamma}^{\infty} \frac{1}{2\pi i} \int_{-T}^T \frac{e^{itE} (e^{\alpha b^2} - e^{-\alpha t^2})}{t - ib} dt d\langle A^* \Omega, P_E B \Omega \rangle \right|. \quad (44)$$

By adding and subtracting the appropriate limiting quantities, which are shown to exist in Lemma 1, the integrand above can be rewritten as

$$\left(e^{\alpha b^2} e^{-Eb} - \frac{1}{2\sqrt{\pi\alpha}} \int_0^{\infty} e^{-bw} e^{-\frac{(w-E)^2}{4\alpha}} dw \right) e^{\alpha b^2} (F_{ib,T}(E) - e^{-Eb}) + R_1 + R_2. \quad (45)$$

The quantities $F_{ib,T}$, R_1 , and R_2 , are defined in (52), (58), and (59) respectively. Using (56) for complex t , one may estimate the first term in (45) as

$$\frac{1}{2\sqrt{\pi\alpha}} \int_0^{\infty} e^{-bw} e^{-\frac{(w-E)^2}{4\alpha}} dw \leq \frac{1}{2} e^{-\frac{\gamma^2}{4\alpha}}, \quad (46)$$

for $2\alpha b \leq \gamma$ and $E \geq \gamma$. Using dominated convergence, one sees that the other terms vanish in the limit $T \rightarrow \infty$. This implies the following bound:

$$\limsup_{T \rightarrow \infty} III \leq \frac{1}{2} \|A\| \|B\| e^{-\frac{\gamma^2}{4\alpha}}. \quad (47)$$

The proof is completed by choosing $\alpha = \frac{\gamma}{2s}$ and s so that

$$s \left(2\|\Phi\|_{\lambda} + \frac{\gamma}{2} \right) = \lambda d(x, Y). \quad (48)$$

We have that

$$|\langle \Omega, A \tau_{ib}(B) \Omega \rangle| \leq \|A\| \|B\| \left(1 + \frac{2|Y|}{\pi} + \frac{1}{\sqrt{\pi \mu d(x, Y)}} \right) e^{-\mu d(x, Y) \left(1 + \frac{b^2}{s^2} \right)}, \quad (49)$$

where

$$\mu = \frac{\lambda \gamma}{4\|\Phi\|_{\lambda} + \gamma}. \quad (50)$$

This proves Theorem 2. \square

In the proof above we used the following lemma, which is a variation of Lemma 3.1 in [17].

Lemma 1. *Let $E \in \mathbb{R}$, $\alpha > 0$, and $z \in \mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im}[z] > 0\}$. One has that*

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{-T}^T \frac{e^{iEt} e^{-\alpha t^2}}{t - z} dt = \frac{1}{2\sqrt{\pi\alpha}} \int_0^\infty e^{iwz} e^{-\frac{(w-E)^2}{4\alpha}} dw. \quad (51)$$

Moreover, the convergence is uniform in $z \in \mathbb{C}^+$.

Proof. One may prove this lemma by observing some simple estimates on the following function:

$$F_{z,T}(w) := \frac{1}{2\pi i} \int_{-T}^T \frac{e^{iwt}}{t - z} dt. \quad (52)$$

For $w > 0$, integration over the rectangular contour Γ_T joining $-T \rightarrow T \rightarrow T + iT \rightarrow -T + iT \rightarrow -T$ yields

$$\frac{1}{2\pi i} \int_{\Gamma_T} \frac{e^{iwz'}}{z' - z} dz' = e^{iwz} \quad (53)$$

for any $0 < 2|z| < T$. From this one may conclude that

$$\left| F_{z,T}(w) - e^{iwz} \right| \leq \frac{2}{\pi} \left[\frac{1}{wT} \left(1 - e^{-wT} \right) + e^{-wT} \right]. \quad (54)$$

If $w < 0$, then closing a similar rectangular contour in the lower half plane yields

$$\left| F_{z,T}(w) \right| \leq \frac{2}{\pi} \left[\frac{1}{|w|T} \left(1 - e^{-|w|T} \right) + e^{-|w|T} \right]. \quad (55)$$

Now, as the Fourier Transform of a gaussian is another gaussian, one easily sees that for any $E \in \mathbb{R}$ and $\alpha > 0$,

$$e^{iEt} e^{-\alpha t^2} = \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^\infty e^{iwt} e^{-\frac{(w-E)^2}{4\alpha}} dw. \quad (56)$$

We may now write the prelimit quantity on the left-hand side of (51) as the desired limit and two remainder terms, i.e.,

$$\begin{aligned} \frac{1}{2\pi i} \int_{-T}^T \frac{e^{iEt} e^{-\alpha t^2}}{t - z} dt &= \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^\infty e^{-\frac{(w-E)^2}{4\alpha}} F_{z,T}(w) dw \\ &= \frac{1}{2\sqrt{\pi\alpha}} \int_0^\infty e^{iwz} e^{-\frac{(w-E)^2}{4\alpha}} dw + R_1 + R_2. \end{aligned} \quad (57)$$

Using the estimates (54) and (55), one may apply dominated convergence to conclude that both the remainders

$$R_1 := \frac{1}{2\sqrt{\pi\alpha}} \int_0^\infty e^{-\frac{(w-E)^2}{4\alpha}} \left(F_{z,T}(w) - e^{iwz} \right) dw \quad (58)$$

and

$$R_2 := \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^0 e^{-\frac{(w-E)^2}{4\alpha}} F_{z,T}(w) dw \quad (59)$$

vanish in the limit $T \rightarrow \infty$. \square

4. Applications and Generalizations

For many applications it is of interest to consider the thermodynamic limit. Our clustering theorem allows for two approaches to the thermodynamic limit, each with its own merits. In the first approach, one applies the theorem to finite systems and obtains estimates that are uniform in the size of the systems. This may require a careful choice of boundary conditions. The bounds then carry over automatically to any state which is a thermodynamic limit of finite volume states for which the estimates are obtained. This is the most straightforward way to proceed in cases where the finite volume ground states are unique, weak* convergent and with a uniform lower bound on the spectral gap.

In the second approach one focuses directly on the infinite system. This requires a proof of a spectral gap for the infinite system, but may avoid difficulties associated with boundary states which may obscure the spectral gap for finite volumes.

Another method for dealing with boundary states is to generalize Theorem 2 to systems with quasi-degenerate ground states. As long as the quasi-degenerate states remain separated from the excitation spectrum by a non-vanishing gap, this is a straightforward generalization, but the condition $P_0 B \Omega = 0$, where now P_0 is the spectral projection on all states below the gap, may be a bit tricky to verify.

Another rather obvious generalization of our results is to systems of fermions on (V, d) . For even interactions (only products of an even number of creation and annihilation operators occur), and even observables, all results carry over without change. If A and B are both odd observables, one gets a Lieb-Robinson bound for the anticommutator instead of the commutator.

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References

1. Fredenhagen, K.: A Remark on the Cluster Theorem. *Commun. Math. Phys.* **97**, 461–463 (1985)
2. Nachtergaele, B.: The spectral gap for some quantum spin chains with discrete symmetry breaking. *Commun. Math. Phys.* **175**, 565–606 (1996)
3. Dhar, D.: Lattices of effectively nonintegral dimensionality. *J. Math. Phys.* **18**, 577–585 (1977)
4. Tasaki, H.: Critical phenomena in fractal spin systems. *J. Phys. A: Math. Gen.* **20**, 4521–4529 (1987)
5. Koma, T., Tasaki, H.: Classical XY model in 1.99 dimensions. *Phys. Rev. Lett.* **74**, 3916–3919 (1995)
6. Koma, T.: Spectral Gap and Decay of Correlations in $U(1)$ -Symmetric Lattice Systems in Dimensions $D < 2$. (<http://arxiv.org/list/math-ph/0505022>), 2005
7. Albert, A., Barabási, A.-L.: Statistical Mechanics of Complex Networks. *Rev. Mod. Phys.* **74**, 47–97 (2002)
8. Hastings, M.B.: Mean-field and anomalous behavior on a small-world network. *Phys. Rev. Lett.* **91**, 098701 (2003)
9. Wreszinski, W.F.: Charges and Symmetries in Quantum Theories without Locality. *Fortschr. Phys.* **35**, 379–413 (1987)
10. Lieb, E., Schultz, T., Mattis, D.: Two soluble models of an antiferromagnetic chain. *Ann. Phys. (N.Y.)* **16**, 407–466 (1961)
11. Hastings, M.B.: Lieb-Schultz-Mattis in Higher Dimensions. *Phys. Rev. B* **69**, 104431 (2004)

12. Hastings, M.B.: Locality in Quantum and Markov Dynamics on Lattices and Networks. *Phys. Rev. Lett.* **93**, 140402 (2004)
13. Lieb, E.H., Robinson, D.W.: The Finite Group Velocity of Quantum Spin Systems. *Commun. Math. Phys.* **28**, 251–257 (1972)
14. Bratteli, O., Robinson, D.W.: *Operator Algebras and Quantum Statistical Mechanics 1*. 2nd edn., Berlin-Heidelberg-New York: Springer Verlag, 1987
15. Bratteli, O., Robinson, D.W.: *Operator Algebras and Quantum Statistical Mechanics 2*. 2nd edn., Berlin-Heidelberg-New York: Springer Verlag, 1997
16. Simon, B.: *The Statistical Mechanics of Lattice Gases, Volume I*. Princeton, NJ: Princeton University Press, 1993
17. Hastings, M.B., Koma, T.: Spectral Gap and Exponential Decay of Correlations. <http://arxiv.org/list/math-ph/0507008>, 2005

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