Breakdown of Diffusion on the Edge

Luca V. Delacrétaz and Paolo Glorioso

Kadanoff Center for Theoretical Physics,
University of Chicago, Chicago, IL 60637, USA

Abstract

We show that dirty Quantum Hall systems exhibit large hydrodynamic fluctuations at their edge that lead to anomalously damped charge excitations in the Kardar-Parisi-Zhang universality class $\omega \simeq ck - i\mathcal{D}k^{3/2}$. The dissipative optical conductivity of the edge is singular at low frequencies $\sigma(\omega) \sim 1/\omega^{1/3}$. These results are direct consequences of the charge continuity relation, the chiral anomaly, and thermalization on the edge – in particular translation invariance is not assumed. Diffusion of heat similarly breaks down, with a universality class that depends on whether the bulk thermal Hall conductivity vanishes. We further establish the theory of fluctuating hydrodynamics for surface chiral metals, where charge fluctuations give logarithmic corrections to transport.

1 Introduction and Results

Quantum Hall (QH) droplets feature gapless excitations at their edge [1]. At temperatures far below the bulk gap $T \ll \Delta$, the bulk essentially remains non-dissipative but the edge is expected to thermalize; thermalization implies that modes not protected by conservation laws should relax. In particular, the plethora of chiral Luttinger liquid channels predicted for certain QH states are Landau damped by disorder and interactions, and only the collective excitations corresponding to charge [2, 3] and heat [4] survive at late times. Early experiments in GaAs [5, 6] indeed observed a single linearly dispersing collective excitation – the edge magnetoplasmon, associated with charge fluctuations – and later experiments found evidence for the neutral heat mode [7, 8]. More recently, these QH edge modes were observed in graphene [9, 10] and cold atoms [11, 12].

Charge propagates ballistically on the edge, in the direction fixed by the sign of the filling $\nu = n/B$. The damping of this mode was first studied systematically at zero temperature by Volkov and Mikhailov long ago [13]. In the hydrodynamic regime, i.e. at finite temperature $T > 0$ and low frequencies $\omega\tau_{\rm th} \ll 1$, the chiral ballistic front is expected to broaden diffusively [3]. The thermalization time $\tau_{\rm th}$ may be controlled by various mechanisms depending on the microscopics of the edge[1] – our central assumption is that it is sufficiently small so that frequencies $\omega \lesssim 1/\tau_{\rm th}$ can be probed experimentally.

Using fluctuating hydrodynamics, we will find that nonlinearities are relevant; large charge fluctuations lead to a breakdown of diffusion and drive the edge to a dissipative fixed point in the Burgers-Kardar-Parisi-Zhang (KPZ) universality class [15, 16], with dynamic critical exponent $z = 3/2$ controlling the broadening of the chiral ballistic front

$$\omega = ck - i\mathcal{D}k^{3/2} + \cdots . \tag{1}$$

KPZ scaling in the IR ties dissipation to thermodynamics

$$\mathcal{D} = \sqrt{\frac{T\chi'^2}{\chi^3}}|\sigma_{xy}| , \tag{2}$$

where the boundary charge susceptibility $\chi \equiv \partial n/\partial\mu$ and $\chi' \equiv \partial\chi/\partial n$ are non-universal but expected to have weak field and temperature dependence, and $\sigma_{xy} = \nu\frac{e^2}{h}$. The linear dependence of damping on filling ν for gapped bulks has been widely observed experimentally, see e.g. [5, 6]. The temperature and wavevector dependence of damping have been less

[1] One possible mechanism for thermalization at the edge of fractional QH bulks is interchannel scattering in the presence of disorder [2, 3] (see e.g. Ref. [14] for other possible mechanisms, still in the context of Luttinger liquids). However even at integer fillings edge electrons feel Coulomb interactions and may thermalize.

systematically reported, although the available results are consistent with Eqs. (1) and (2), see Discussion below.

Breakdown of diffusion leads to a failure of the Einstein relation, and the optical conductivity is singular at low-frequency

$$\sigma(\omega) = a\frac{\chi\mathcal{D}^{4/3}}{\omega^{1/3}} + \cdots , \tag{3}$$

with $a \approx 0.418$. Singular low frequency transport is a hallmark of large hydrodynamic fluctuations [15, 17]: when hydrodynamic interactions are instead irrelevant, response functions at the lowest frequencies are analytic and the interesting physics is instead hidden e.g. in the temperature dependence of transport parameters. We stress that momentum conservation is not assumed – disorder therefore does not have to be introduced by hand, and does not regulate the singularity (3) which is only cut off by finite system size.

Without momentum conservation, hydrodynamic fluctuations are usually irrelevant and give small 'long-time tail' corrections to diffusive transport [18, 19] $\sigma(\omega) = \chi D + |\omega|^{d/2}$, where D is the diffusion constant and d the spatial dimension. The difference here stems from the fact that the $U(1)$ symmetry has a chiral anomaly. The interplay of anomalies and hydrodynamics has been appreciated since the work of Son and Surowka [20]. Although anomalies often only lead to subtle effects on transport, we show that the (1+1)d chiral anomaly has dramatic consequences, with ballistic propagation and large hydrodynamic fluctuations.

The connection between hydrodynamics and the KPZ universality class has been long known [15]. Recently, it has been shown that the hydrodynamics of a non-integrable spin chain, despite the lack momentum conservation, shows KPZ scaling at not too low energies [21]. The system we are interested in will turn out to display KPZ scaling all the way to arbitrarily low energies, independently of momentum conservation.

Our main results quoted above are obtained in Section 2; heat is then discussed in Section 3. In Section 4, we show that the KPZ fixed point describing the chiral edge can be accessed perturbatively from the upper critical dimension $d_c = 2$, where interactions are marginally irrelevant. Incidentally, the theory at $d = d_c$ is interesting in its own right because it describes the hydrodynamics of surface chiral metals [22, 23], i.e. coupled layered edge states. This generalization to higher dimensions is distinct from the one natural for the KPZ equation, where interactions are instead marginally relevant in $d = 2$.

2 Charge fluctuations on the edge

We study systems in one spatial dimension with a single $U(1)$ symmetry, that is anomalous

$$\partial_\mu j^\mu = \frac{\nu}{4\pi}\epsilon^{\mu\rho}F_{\mu\rho}\,. \tag{4}$$

We are working in units where $e^2/\hbar = 1$. Such systems can be thought of as living on the boundary of a gapped bulk. When $\nu \in \mathbb{Z}$, the topological order in the bulk is trivial and the anomaly can be canceled by a properly quantized Chern-Simons term $\frac{\nu}{4\pi}AdA$ for the background field. When $\nu \notin \mathbb{Z}$ as in fractional QH states, the bulk has non-trivial topological order. We make no additional symmetry assumptions – in particular, momentum is not approximately conserved in any limit.

We are interested in the finite temperature properties of the system on the edge, at temperatures small compared to the bulk gap $T \ll \Delta$. We will assume that the edge thermalizes – this implies that physics at the lowest frequencies is governed by hydrodynamics, namely the dynamics of conserved densities: charge $n = j^0$, and heat (or energy). We postpone the treatment of heat to the next section; as we will see the dynamics of charge density alone is already surprisingly rich.

Dissipation in a theory with a non-anomalous $U(1)$ symmetry is described by simple diffusion $\omega \sim -iDk^2$. The goal of this section is to determine how the anomaly $\nu \neq 0$ changes this picture. The hydrodynamic treatment proceeds as follows [24, 25]: at late times, all operators are controlled by their overlaps with hydrodynamic densities, here n. This holds in particular for the current, which can be written in terms of n – or its associated potential μ – in a gradient expansion

$$\langle j_x\rangle = \frac{\nu}{2\pi}\mu - \chi D\partial_x\mu + \cdots\,, \tag{5}$$

where the charge susceptibility χ and diffusivity D are unknown functions of n (or μ), and $\cdots$ denotes higher gradient terms $O(\partial_x^2\mu)$. The anomaly fixes the leading term in the constitutive relation[2]. Combining (4) and (5), one finds the following equation of motion for the charge density

$$0 = \dot{n} + c\partial_x n - \partial_x(D\partial_x n) + \cdots\,, \tag{6}$$

with velocity $c = \nu/(2\pi\chi)$. Linearizing in the fluctuations $n = \bar{n} + \delta n$, this can be solved to

[2]This is analogous to how the chiral anomaly in (3+1)d hydrodynamics fixes terms in the constitutive relation that are first order in gradients [20]. The anomaly has a more important role here, as it enters at zeroth order in the expansion. A similar zeroth order anomaly fixes the speed of sound of superfluids [26].

yield the retarded Green's function

$$G^R_{nn}(\omega,k)=\chi\frac{ick+Dk^2}{-i(\omega-ck)+Dk^2}+\cdots\,, \tag{7}$$

where the corrections $\cdots$ are non-singular as $\omega,k\to 0$. Here and in the following, functions of n such as c, D, χ are evaluated on the background density $\bar{n}$. In the absence of an anomaly, the velocity c vanishes and one obtains a diffusive Green's function as expected. The linear analysis suggests that the anomaly $\nu\neq 0$ leads to a right-moving ballistic front at velocity $c=\nu/(2\pi\chi)$, with diffusive spreading around the front [2, 3]. We will see that this conclusion is incorrect. The chiral ballistic front controls the bulk Hall conductivity

$$\sigma^{\rm bulk}_{xy}=\lim_{\omega\to 0}G^R_{j_x n}(\omega,0)=\chi c=\frac{\nu}{2\pi}\,, \tag{8}$$

and is a robust consequence of the anomaly. However, dissipation does not lead to diffusive spreading around the chiral front, because of a breakdown of the perturbative expansion in dissipative hydrodynamics. This can be seen by expanding the equation of motion (6) beyond leading order in δn (which we denote as n in the following for simplicity)

$$\partial_x\eta_x=\dot{n}+c\partial_x n+\frac{1}{2}c'\partial_x n^2-D\partial_x^2 n+\cdots\,, \tag{9}$$

where $c'\equiv\partial c/\partial n=-\frac{\nu}{2\pi}\frac{\chi'}{\chi^2}$. In the absence of additional symmetries, there is no reason for c' to vanish and nonlinearities are generically expected, see e.g. Ref. [27]. Less relevant nonlinear terms coming from the n dependence of the diffusivity D are omitted. We included a noise term η_x in the constitutive relation, whose symmetric Green's function is constrained by the fluctuation-dissipation theorem at leading order in gradients to be $\langle\eta_x(t,x)\eta_x\rangle = 2D\chi T\delta(x)\delta(t)+\cdots$. Scaling $\omega\sim k$ to leading order, one finds that $\eta\sim k$ and $n\sim k$, so that both the diffusive term D and nonlinearity c' are irrelevant corrections to the chiral ballistic front. This guarantees, from the edge perspective, the robustness of the quantization of the bulk Hall conductivity (8) at finite temperature.

To establish the leading correction to ballistic propagation, it is convenient to work in the frame of the chiral front $x'=x-ct$, $t'=t$ (or equivalently $\omega'=\omega+ck$, $k'=k$). In these coordinates the equation of motion

$$\partial_x\eta_x=\partial_{t'}n+\frac{1}{2}c'\partial_x n^2-D\partial_x^2 n+\cdots\,, \tag{10}$$

leads to a scaling $\omega'\sim k^2$, so that $\eta\sim k^{3/2}$ and $n\sim k^{1/2}$. One then finds that the interaction term c' is *relevant*, and drives the system to a new dissipative fixed point that is not described by diffusive spreading around the chiral front (7). In terms of the original coordinates, we expect $\omega - ck \sim k^z$, with $z < 2$ at the stable fixed point. In fact, (10) is nothing but the KPZ equation, with charge mapping to the slope of the interface $n = \partial_x h$, and the system is described by Burgers–KPZ [15, 16] universality with $z = 3/2$. The symmetric Green's function is given by

$$G_{nn}(\omega, k) = \frac{\chi T}{\mathcal{D}k^z} g_{\text{KPZ}}\left(\frac{\omega - ck}{\mathcal{D}k^z}\right) + \cdots , \tag{11}$$

where $\cdots$ are terms that are subleading in the scaling $\frac{\omega - ck}{k^z} \sim 1$, and g_{KPZ} is the KPZ scaling function which is known numerically to high precision [28] (some of its properties are reviewed in Appendix A). $G_{nn}(\omega, k)$ is sharply peaked around $\omega = ck$ with a width of order $\mathcal{D}k^z$ – it is in this sense that the dispersion relation Eq. (1) holds. The dimensionful constant $\mathcal{D} \sim \text{length}^z/\text{time}$ is fixed in terms of parameters in the equation of motion by dimensional analysis

$$\mathcal{D} = \sqrt{\frac{T}{\chi^3}\frac{|\nu|}{2\pi}}|\chi'| \, . \tag{12}$$

This expression makes manifest three crucial ingredients that led to KPZ universality around the chiral front: finite temperature T, the anomaly ν, and hydrodynamic interactions through $\chi' = \partial\chi/\partial n$. It is striking that the scaling function in the dissipative fixed point involves bare parameters from the equation of motion (9). This is possible because ν, χ and χ' do not get renormalized [15].

Eq. (11) leads to a universal prediction for transport on the edge: the symmetric Green's function controls the dissipative optical conductivity at low frequencies $\omega\tau_{\text{th}} \ll 1$ by the fluctuation-dissipation theorem

$$\sigma(\omega) = \lim_{k\to 0} \frac{\omega}{k^2} \operatorname{Im} G^R_{nn}(\omega, k) = \lim_{k\to 0} \frac{\omega^2}{2Tk^2} G_{nn}(\omega, k) \, , \tag{13}$$

so that using (11)

$$\sigma(\omega) \simeq \lim_{k\to 0} \frac{\chi\omega^2}{2\mathcal{D}k^{7/2}} g_{\text{KPZ}}(\omega/\mathcal{D}k^{3/2}) = a\frac{\chi\mathcal{D}^{4/3}}{\omega^{1/3}} \, , \tag{14}$$

with $a \approx 0.417816$ (see Appendix A for the limiting behavior of g_{KPZ}). This singular conductivity as $\omega \to 0$ will be regulated in a system of finite length L, see Ref. [29] for a discussion on subtleties with the Kubo formula in this situation. Although $\lim_{\omega\to 0} \sigma(\omega, k)$ vanishes for $k \neq 0$, the relevant observable may be $\sigma(\omega, k)$ at $\omega \sim ck \sim c/L$ [17, 29], in which case one finds $\sigma_{\text{dc}} \sim \chi\mathcal{D}^{4/3}(L/c)^{1/3}$. This could be probed by shot noise experiments in QH systems, which are usually used to study the fractional charge of the edge carriers [30, 31, 32]. Our results imply that the current noise receives an additional contribution that scales in system size as $S = \sigma_{\text{dc}}/L \sim L^{-2/3}$, which vanishes more slowly than the standard thermal contribution $S \sim L^{-1}$.

Non-dissipative response such as the bulk Hall conductivity $\sigma_{xy}^{\text{bulk}}$ is controlled instead by the real part of the retarded Green's function $\mathrm{Re}\, G^R_{nn}$, which can be obtained from $\mathrm{Im}\, G^R_{nn}$ by analyticity. This is done in Appendix A, where we show that the quantized bulk Hall conductivity (8) is unchanged. Finally, long-range Coulomb interactions can be taken into account as usual in the random phase approximation by resumming a geometric series of diagrams involving photons – this does not qualitatively change the dispersion relation, which simply receives logarithmic corrections, see e.g. [13]. The screened charge excitation is called the edge magnetoplasmon and has been observed in a number of experiments, see Discussion below.

3 The fate of heat

We found that the large hydrodynamic fluctuations of the charge density lead to anomalous dissipation on chiral edges. Another collective excitation that is long lived is heat, or energy. Heat has a less privileged a status than charge, since it can leak out of the edge through phonons and will therefore only be approximately conserved. However, the time scale for heat loss may be parametrically longer than the thermalization time as it is controlled by different physics – a possibility affirmed by the experimental observation of the collective heat mode [7]. Ref. [4] studied heat transport in the context of QH edge states, neglecting thermoelectric effects. In this case a nonzero bulk thermal Hall conductivity κ_{xy} gives heat a finite chiral speed of sound $c_{\text{heat}} = \frac{\kappa_{xy}}{c_V}$, where c_V is the specific heat, and the analysis in the previous section holds with heat replacing charge. This result is largely unaffected by coupling between charge n and energy ε – one then faces a system of KPZ equations

$$\partial_t n_a + C_{ab}\nabla n_b + D_{ab}\nabla^2 n_b + \lambda_{abc} n_b \nabla n_c + \cdots = 0\,, \tag{15}$$

with $n_1 = n$ and $n_2 = \varepsilon$. As long as the velocity eigenvalues are distinct, going into the rest frame of any eigenmode one finds that interactions with the other eigenmodes are irrelevant (intuitively, other modes are far off-shell). One therefore expects two independent KPZ modes around each chiral ballistic front – this is indeed what is observed numerically[3] [34].

One important exception is when the thermal Hall conductivity vanishes $\kappa_{xy} = 0$, so that the heat mode does not propagate ballistically [4] (this happens e.g. for $\nu = 2/3$). Although a linearized analysis would suggest that heat then diffuses, its nonlinear coupling to the fluctuating charge mode also leads to a breakdown of diffusion in this case. This

[3] The case where both velocity eigenvalues are equal is more subtle and has been explored in [33].

nonlinear coupling comes from the fact that in a background field $F_{0x} = E_x$, the energy continuity relation is changed to $\dot{\varepsilon} + \partial_x j^\varepsilon_x = E_x j_x$, which using (5) fixes the leading term in the constitutive relation for the energy current[4] $j^\varepsilon_x = \frac{\nu}{4\pi}\mu^2 + \cdots$. The two modes that diagonalize the C matrix in (15) are now $\delta\mu = \chi^{-1}_{nn}\delta n + \chi^{-1}_{n\varepsilon}\delta\varepsilon$ and $\delta s = \frac{1}{T}(\delta\varepsilon - \mu\delta n)$. The former is still described by KPZ universality, with a correlator of the form (11). Instead when $\kappa_{xy} = 0$, entropy fluctuations have a vanishing speed and self-coupling $\lambda_{sss} = 0$. However they do couple to the KPZ mode $\lambda_{s\mu\mu} \neq 0$ – the resulting infrared behavior has been extensively studied, but remains an open problem (see Refs. [29, 37] for reviews). A 'mode coupling' approximation yields $z_{\text{heat}} = 5/3$ – heat transport would then also be singular at low frequencies $\bar{\kappa}(\omega) \sim 1/\omega^{1/5}$.

4 Higher dimension: Surface chiral metals and chiral magnetic effect

Two reasons drive us to generalize the theory of Section 2 to higher dimensions: first, we will find that the KPZ fixed point can be accessed perturbatively from the upper critical dimension $d_c = 2$; second, chiral systems with diffusive broadening naturally occur in higher dimensions as well. In $d = 2$ the theory we consider furnishes the low-energy description of surface chiral metals [22, 23]. These are boundaries of 3-dimensional materials made from layered QH systems – they exhibit propagation of a chiral diffusive front in the direction of the layer and regular diffusion in the transverse direction, which was shown to be stable against localization [22]. In $d = 3$, the theory describes the hydrodynamics of a charge current subject to the chiral magnetic effect in the presence of a background magnetic field (decoupling momentum and energy fluctuations). The chiral magnetic effect [38] corresponds to a nonvanishing equilibrium value of the charge current in the presence of a magnetic field, and is due to the chiral anomaly. This effect arises in condensed matter systems such as Weyl semimetals [39, 40], in heavy ion physics [41] and astrophysics [42].

The common feature to all such systems is the presence of a chiral front with diffusive broadening along a given direction, which we label with x, and of ordinary diffusion in the orthogonal directions, which we label with y^A, where $A = 2, \ldots, d$. The constitutive relations for the current are

$$j^x = \frac{\nu}{2\pi}\mu - \chi D_x \partial_x \mu, \qquad j^A = -\chi D_\perp \partial^A \mu \ , \tag{16}$$

[4] This equation was discussed in a relativistic context in [35, 36]; we stress that it is derived here using only the continuity relations for charge and energy.

where the chemical potential μ is an arbitrary function of the charge density n. Following the discussion around Eq. (10), we go in the frame of the chiral front $x' = x - ct$, $y' = y$, $t' = t$ – the conservation equation for (16) is then

$$\partial_{t'} n + \frac{1}{2} c' \partial_x n^2 - D_x \partial_x^2 n - D_\perp \partial_A^2 n = \partial_x \eta^x + \partial_A \eta^A \ , \tag{17}$$

where we included the noise current (η^x, η^A), whose two-point functions are again fixed by thermal equilibrium:[5]

$$\langle \eta_i(t,x,y) \eta_j \rangle = 2 D_{ij} \chi T \delta(t) \delta(x) \delta^{(d-1)}(y) \, , \tag{18}$$

with $D_{ij} = \mathrm{diag}(D_x, D_\perp, \ldots, D_\perp)$. In this frame, $\omega' \sim k^2$, implying that $\eta^x, \eta^A \sim k^{\frac{d}{2}+1}$, $n \sim k^{\frac{d}{2}}$, and thus c' scales as $k^{\frac{2-d}{2}}$, i.e. it becomes marginal in $d = 2$ and irrelevant in $d > 2$. To study the effects of fluctuations and determine the RG fate in $d = 2$ we implement the effective field theory approach to hydrodynamics [44, 45], reviewed in Appendix B. This framework allows to perform a dynamical RG analysis keeping all the symmetries manifest, and appropriately capturing contact terms in correlation functions. The central object is the path integral

$$Z = \int D n D \varphi_a e^{i S[n, \varphi_a]} \ , \tag{19}$$

where $\varphi_a(t, x, y^A)$ is an auxiliary field, and can be related to the noise currents in (17) following the Martin-Siggia-Rose formalism [46]. The action associated to the stochastic equation (17) is

$$\begin{aligned} S = \int dt dx d^{d-1} y \Big[& n \partial_{t'} \varphi_a + \frac{1}{2} c' n^2 \partial_x \varphi_a - D_x \partial_x n \partial_x \varphi_a - D_\perp \partial_A n \partial_A \varphi_a \\ & + i T \chi D_x (\partial_x \varphi_a)^2 + i \chi T D_\perp (\partial_A \varphi_a)^2 + \cdots \Big] \, , \end{aligned} \tag{20}$$

see Appendix B for details. To study renormalization, we integrate over a momentum shell $\mu \leq |\vec{k}| \leq \Lambda$ and over all values of ω. We find that $\chi, c', D_\perp$, as well as the coefficient in the first term in (20) do not renormalize, while

$$\frac{\partial D_x}{\partial \log \mu} = \varepsilon D_x - \frac{\chi T c'^2}{8\pi \sqrt{D_x D_\perp}} \equiv \beta_{D_x} \ , \tag{21}$$

where $\varepsilon = 2 - d$. By rescaling $\partial_x \to \partial_x / \sqrt{D_x}$, $\partial_A \to \partial_A / \sqrt{D_\perp}$, $\varphi_a \to \varphi_a / \sqrt{T\chi}$, $n \to n\sqrt{T\chi}$ to canonically normalize the fields in the action (20), the cubic coupling becomes $\lambda = c' \sqrt{\frac{\chi T}{D_x}}$. The physical β-function is then fixed by that of D_x and one has

$$\beta_\lambda = -\frac{\varepsilon}{2} \lambda + \frac{\lambda^4}{32\pi c' \sqrt{D_\perp \chi T}} \ , \tag{22}$$

[5] An equation similar to (17) appeared also in the context of "Malthusian flocks" [43]. The noise there is different, due to the fact that the system is not in equilibrium.

which shows that the coupling is marginally irrelevant in $d = 2$. For $d = 2 - \varepsilon$, the diffusive fixed point with $\lambda = 0$ is unstable, and the stable fixed point can be accessed perturbatively

$$\lambda^{*3} = 16\pi\varepsilon c' \sqrt{D_\perp \chi T} \; . \tag{23}$$

Although this perturbative scheme breaks down for $\varepsilon \to 1$, we expect that in this limit the stable fixed point we found will approach the KPZ universality class. For $d > 2$, $\lambda = 0$ becomes stable and is the only fixed point[6]. This is summarized in Fig.1.

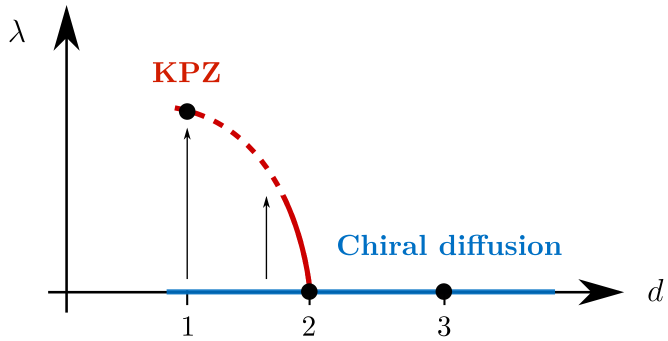

Figure 1: Fixed points λ^* as a function of spatial dimension d.

Transport in these systems is described in more detail in Appendix B. For surface chiral metals in $d = 2$ where interactions are marginally irrelevant, the chiral diffusive fixed point is approached slowly and transport parameters run logarithmically. In this case one can solve the RG flow equation (21) and find the conductivity at low frequencies

$$\sigma_{xx}(\omega) = \chi D_x(\omega) \simeq \chi \left[\frac{3\chi T c'^2}{32\pi\sqrt{D_\perp}} \log \frac{1}{\omega} \right]^{2/3} , \tag{24}$$

In $d = 3$, i.e. for systems displaying chiral magnetic effect, the coupling λ is irrelevant and the conductivity σ_{xx} in the direction of the magnetic field approaches a constant in the $\omega \to 0$ limit.

5 Discussion

We have showed that hydrodynamic fluctuations on edges realizing the chiral anomaly lead to a breakdown of diffusion, giving rise to singular low frequency transport and anomalous damping of edge modes. In particular, edge magnetoplasmons are predicted to be anomalously damped, Eqs. (1) and (2). The weak temperature dependence of damping observed in graphene in Ref. [10] is consistent with (2). To our knowledge only Ref. [6] studied

[6]Note that λ has the same sign as c', so $\lambda = 0$ is the only fixed point for $\varepsilon < 0$.

wavevector dependence; the dependence they observe is between linear to quadratic, which would neatly agree with (1). Moreover, the overall size of damping observed is consistent with Eqs. (1, 2) : estimating $\chi' \sim \omega_c$ to be of order the bulk gap and $\chi \sim 1/c$ (see Section 2), one finds a quality factor $Q \sim \sqrt{\hbar\omega T}/\omega_c \sim 100$ at $\omega \sim 20$ MHz, consistent with Ref. [6]. However more thorough investigation – which is well within experimental reach – is needed to unequivocally confirm our prediction.

In the presence of edge reconstruction [47] with weak interedge interactions, our results can in principle apply to each edge (and perhaps most usefully to the outermost one), with the appropriate anomaly. If interedge interactions are strong enough, we instead expect only a single collective charge and heat mode to be long lived.

Nonlinear charge fluctuations have been argued to lead to a breakdown of the linearized edge picture even at $T = 0$ in the absence of dissipation, where the Burger's equation is stabilized by adding a non-dissipative two-derivative term (the Benjamin-Ono equation) [27], whose coefficient is related to the bulk Hall viscosity. This term leads to a non-analytic correction to the dispersion relation $\omega - ck \sim k|k|$ which seemingly enters at the same order in derivatives as the diffusive term considered in (6). However, we believe that the non-analyticity of the Benjamin-Ono term gets softened at finite temperature, making it less RG relevant than the diffusive term.

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A Response functions

We work with two types of charge response functions in the main text – the retarded Green's function and symmetric Greens function

$$G^R_{nn}(t,x) \equiv i\theta(t)\langle[n(x,t),n(0,0)]\rangle_\beta\,, \qquad G_{nn}(t,x) \equiv \frac{1}{2}\langle\{n(x,t),n(0,0)\}\rangle_\beta\,, \tag{25}$$

where $\langle\cdot\rangle_\beta \equiv \mathrm{Tr}(e^{-\beta H}\cdot)/\,\mathrm{Tr}(e^{-\beta H})$. Their Fourier transforms are related by the fluctuation-dissipation theorem: in the hydrodynamic limit ($\omega \ll \frac{1}{\tau_{\rm th}} \lesssim T$)

$$\mathrm{Im}\, G^R_{nn}(\omega,k) = \tanh\frac{\omega}{2T} G_{nn}(\omega,k) \simeq \frac{2T}{\omega} G_{nn}(\omega,k)\,. \tag{26}$$

A.1 Conventions for the KPZ scaling function

Charge density on the edge maps to the KPZ slope $n \leftrightarrow \partial_x h$. In this appendix, we will translate between our conventions for the scaling function $g_{\rm KPZ}$ in Eq. (11) and those in Ref. [28], where the scaling function was studied to high precision. From the equation of motion in the 'pulse frame'

$$\eta = \dot{n} + \frac{1}{2}c'\partial_x n^2 - D\partial_x^2 n + \cdots\,, \tag{27}$$

we see that the standard KPZ coupling [16, 28] is $\lambda = -c' = \frac{\nu}{2\pi}\frac{\chi'}{\chi^2}$. Furthermore, the sum rule relates our susceptibility χT to Eq. (1.4) of [28] as $A = \chi T$. The dimensionful constant chosen there (in Eq. (1.6) of [28]) is then related to ours (12) as

$$\widetilde{\mathcal{D}} = \sqrt{2\lambda^2 A} = \sqrt{2}\sqrt{\frac{T\chi'^2}{\chi^3}}\frac{|\nu|}{2\pi} = \sqrt{2}\mathcal{D}\,. \tag{28}$$

The symmetric Green's function ($S(t,x)$ in the notation of [28]) is related to the scaling function f (Eq. (1.8) in [28]) as

$$G_{nn}(t,x) = S(t,x) = \frac{\chi T}{(\widetilde{\mathcal{D}}t)^{2/3}} f\left(\frac{x}{(\widetilde{\mathcal{D}}t)^{2/3}}\right)\,. \tag{29}$$

This function has the following asymptotics

$$f(0) = \mathrm{const}\,, \qquad \lim_{y\to\infty} f(y) \approx e^{-\mathrm{const}' y^3}\,. \tag{30}$$

Fourier transforming in space gives

$$G_{nn}(t,k) \equiv \int dx\, G_{nn}(t,x)e^{-ikx} = \chi T \hat{f}\left(k(\widetilde{\mathcal{D}}t)^{2/3}\right)\,, \tag{31}$$

where the scaling function $\hat{f}$ is defined in Eq. (5.3) of [28]. It has the properties

$$\hat{f}(0) = 1 \quad (\text{sum rule})\,, \qquad \lim_{\kappa\to\infty}\hat{f}(\kappa) \approx \mathrm{const}\cdot\kappa^{-9/4}e^{-\frac{1+i}{2}\kappa^{3/2}} + \mathrm{c.c.}\,. \tag{32}$$

Finally, let us perform the time Fourier transform:

$$G_{nn}(\omega, k) = \int dt\, e^{i\omega t} G_{nn}(t, k) = \frac{\chi T}{\widetilde{\mathcal{D}} k^{3/2}}\, \mathring{f}\left(\frac{\omega}{\widetilde{\mathcal{D}} k^{3/2}}\right), \tag{33}$$

where $\mathring{f}$ is the scaling function defined in Eq. (5.9) of [28]. Now given the definition of our scaling function g_{KPZ} in Eq. (11), we see that our scaling function is related to $\mathring{f}$ by

$$g_{\mathrm{KPZ}}(w) = \frac{1}{\sqrt{2}}\, \mathring{f}\left(\frac{w}{\sqrt{2}}\right). \tag{34}$$

Eq. (5.11) in [28] implies that it has the properties

$$\int \frac{dw}{2\pi} g_{\mathrm{KPZ}}(w) = 1\,, \qquad \lim_{w\to\infty} g_{\mathrm{KPZ}}(w) = \frac{2a}{w^{7/3}}\,, \qquad g_{\mathrm{KPZ}}(0) = b\,, \tag{35}$$

with $a \approx 0.417816$ and $b \approx 3.43730$. The first property guarantees that Eq. (11) satisfies the usual sum rule $\int \frac{d\omega}{2\pi} G_{nn}(\omega, k) = \chi T$.

Returning to the lab frame, the fluctuation-dissipation theorem (26) relates the symmetric correlator to the retarded Green's function

$$\operatorname{Im} G^R_{nn}(\omega, k) \simeq \frac{\omega}{2T} G_{nn}(\omega, k) \simeq \frac{\chi\omega}{2\mathcal{D} k^{3/2}} g_{\mathrm{KPZ}}\left(\frac{\omega - ck}{\mathcal{D} k^{3/2}}\right), \tag{36}$$

(in the hydrodynamic regime $\omega \ll T$).

A.2 Retarded response

Analyticity of retarded response $G^R(\omega)$ in the upper-half complex plane implies that its real and imaginary parts are connected by the Kramers-Kronig relation (see e.g. [48]). The dynamic susceptibility can be obtained from Eq. (36) as

$$\chi(z, k) = \int \frac{d\omega}{\pi} \frac{\operatorname{Im} G^R_{nn}(\omega, k)}{\omega - z} \simeq \chi + \chi \frac{z}{\mathcal{D} k^{3/2}} \int \frac{dx}{2\pi} \frac{g_{\mathrm{KPZ}}(x)}{x - \frac{z - ck}{\mathcal{D} k^{3/2}}}, \tag{37}$$

The full retarded Green's function is then the analytic continuation of $\chi(z, k)$ to the upper-half plane

$$G^R_{nn}(\omega, k) = \chi(\omega + i0^+, k) \simeq \chi + \chi \frac{\omega}{\mathcal{D} k^{3/2}} \int \frac{dx}{2\pi} \frac{g_{\mathrm{KPZ}}(x)}{x - \frac{\omega - ck}{\mathcal{D} k^{3/2}} - i0^+}. \tag{38}$$

Non-dissipative response can now be studied from the real part of G^R_{nn}. For example the bulk Hall conductivity is as expected

$$\sigma^{\mathrm{bulk}}_{xy} = \lim_{\omega\to 0}\lim_{k\to 0} G^R_{jn}(\omega, k) = -\lim_{\omega\to 0}\lim_{k\to 0} \frac{\omega}{k} G^R_{nn}(\omega, k) = \chi c \int \frac{dx}{2\pi} g_{\mathrm{KPZ}}(x) = \frac{\nu}{2\pi}, \tag{39}$$

where in the last step we used (35).

B Renormalization of chiral diffusion

B.1 Effective field theory of hydrodynamics

Hydrodynamics has recently been cast into an effective field theory (EFT) framework which is based solely on symmetry principles [44, 45, 49, 50, 51, 52]. Here we briefly review this framework, for a more general discussion we remind the reader to [53]. In the next subsections we will apply the formalism to study the dynamical renormalization group of chiral diffusion. For the "minimal models" described in the main text, the effective action ends up essentially coinciding with that obtained from the stochastic system (16)-(18) using the Martin-Siggia-Rose formalism [46]. The strength of the EFT approach is that it ensures to capture all terms associated to chiral anomaly and fluctuations in a systematic derivative expansion of the action, and it is straightforward to generalize to more complicated systems. A systematic study of renormalization in the context of diffusion was undertaken in [54].

We are interested in studying the low-energy behavior of real-time correlation functions of the charge and current operators $j^0(t,x), j^i(t,x)$ at finite temperature. These correlation functions are conveniently encoded in the Schwinger-Keldysh generating functional. This is obtained by coupling the system to a background gauge field A_μ as $H \to H + \int j^\mu A_\mu$, where H is the Hamiltonian of the underlying microscopic system, and $\mu = 0, i$ is a spacetime index. One then writes the generating functional

$$Z[A_{1\mu}, A_{2\mu}] \equiv \mathrm{Tr}\left(U(A_{1\mu})\rho_0 U^\dagger(A_{2\mu})\right) \ , \tag{40}$$

where $U(A_\mu)$ is the evolution operator from $t = -\infty$ to $t = +\infty$ for the Hamiltonian $H + \int J^\mu A_\mu$, and $\rho_0 = \frac{e^{-H/T}}{\mathrm{Tr}\, e^{-H/T}}$ is the thermal density matrix at temperature T describing the state of the system at the initial time $t = -\infty$. Varying $Z[A_{1\mu}, A_{2\mu}]$ with respect to $A_{1\mu}$ ($A_{2\mu}$) introduces time-ordered (anti time-ordered) insertions of j^μ in the trace, which we denote by j_1^μ (j_2^μ), i.e.:

$$\langle j_1^\mu(t,\vec{x}) j_2^\nu(t',\vec{x}') j_1^\rho(t'',\vec{x}'')\cdots\rangle \equiv \mathrm{Tr}\left(\mathcal{T}(j^\mu(t,\vec{x}) j^\rho(t'',\vec{x}'')\cdots)\rho_0 \tilde{\mathcal{T}}(j^\nu(t',\vec{x}')\cdots)\right) \ , \tag{41}$$

where $\mathcal{T}$ and $\tilde{\mathcal{T}}$ denote time- and anti time-ordering, respectively. This generating functional satisfies the following important constraints:

$$\text{normalization:} \qquad Z[A_\mu, A_\mu] = 1 \tag{42}$$

$$\text{reflectivity:} \qquad Z[A_{1\mu}, A_{2\mu}] = (Z[A_{2\mu}, A_{1\mu}])^* \tag{43}$$

$$\text{KMS invariance:} \qquad Z[A_{1\mu}, A_{2\mu}] = Z[A_{1\mu}(-t,-x,y^A), A_{2\mu}(-t - iT^{-1}, -x, y^A)] \tag{44}$$

$$\text{gauge invariance:} \qquad Z[A_{1\mu}, A_{2\mu}] = Z[A_{1\mu} + \partial_\mu\lambda_1, A_{2\mu} + \partial_\mu\lambda_2] \tag{45}$$

where λ_1, λ_2 are arbitrary functions of t, x. The first three properties are general for any Schwinger-Keldysh generating functional with ρ_0 being thermal at temperature T, where for the third property we additionally assumed the microscopic Hamiltonian H to be invariant under the product of time-reversal and reflection in the x-direction, as is the case for the systems considered in the main text. The fourth property is a consequence of conservation of the currents j_1^μ and j_2^μ.

The existence of long-living diffusive modes makes the generating functional (40) non-local. These modes arise due to the conservation of j_1^μ and j_2^μ. The crucial step is then to integrate back such long-living modes [44], which leads to

$$Z[A_{1\mu}, A_{2\mu}] = \int D\varphi_1 D\varphi_2 e^{iS[B_{1\mu}, B_{2\mu}]} \ , \tag{46}$$

where $S[B_{1\mu}, B_{2\mu}]$ is a *local* functional of $B_{1\mu} = A_{1\mu} - \partial_\mu \varphi_1$ and $B_{2\mu} = A_{2\mu} - \partial_\mu \varphi_2$, and φ_1, φ_2 are the modes associated to the conservation of j_1^μ, j_2^μ, respectively. The dependence of the effective action S on these combinations guarantees that, integrating out φ_1, φ_2, one obtains a gauge-invariant Z, i.e. the equations of motion for φ_1, φ_2 are precisely the conservations of j_1^μ, j_2^μ. Locality of S follows from that, in our case, φ_1, φ_2 are the only long-living modes in the system. The action S should satisfy the same constraints as Z, i.e. eqs. (42)-(45) in terms of $B_{1\mu}, B_{2\mu}$ instead of $A_{1\mu}, A_{2\mu}$. Following the effective field theory approach, one writes down the most general terms for S according to derivative expansion. It will be convenient to introduce a new basis of fields:

$$\varphi_r = \frac{1}{2}(\varphi_1 + \varphi_2), \qquad \varphi_a = \varphi_1 - \varphi_2 \ . \tag{47}$$

We also define $A_{r\mu} = \frac{1}{2}(A_{1\mu} + A_{2\mu})$ and $A_{a\mu} = A_{1\mu} - A_{2\mu}$, and similarly we introduce $B_{r\mu}, B_{a\mu}$. As we will see, $\dot{\varphi}_r$ will be identified with μ in (16)-(18), while φ_a will be related to the noise. Since in diffusion μ is the only combination appearing in the equations, we impose the further symmetry on the action:

$$\varphi_r \to \varphi_r + \chi(x) \ , \tag{48}$$

i.e. the action should be invariant under time-independent shifts of φ_r, which allows $\dot{\varphi}_r$ to appear explicitly, but not $\partial_i \varphi_r$.

B.2 Action for chiral diffusion

The general approach is to write down all terms compatible with the symmetries and constraints discussed in the previous subsection, order by order in derivatives. Here, we will simply quote the action with the minimal number of terms that we need for our discussion in the main text. We first write down the non-chiral part of the action, which is separately invariant under spatial inversion in the x- and y^A-directions. The action is

$$S_0 = \int dtd^dx \left(n(B_{r0})B_{a0} - \sigma_x \dot{B}_{rx}B_{ax} - \sigma_\perp \dot{B}_{rA}B_{aA} + i\sigma_x TB_{ax}^2 + i\sigma_\perp TB_{aA}^2\right) \ , \tag{49}$$

where $n = n(B_{r0})$ is an arbitrary function of B_{r0}, and $\sigma_x, \sigma_\perp$ are constants. Note that we imposed rotation invariance in the y^A-directions.

We now add to S_0 the chiral contribution along the x-direction. Besides breaking the inversion symmetry $x \to -x$, the existence of these chiral hydrodynamic modes is tied to the presence of a chiral anomaly in the x-direction, $\partial_\mu j^\mu = CF_{0x}$. For $d = 1$ this was discussed in Section 2. In $d = 2$, for surface chiral metals, one can immediately generalize the $d = 1$ case by considering stacking together quantum Hall layers in the limit of vanishing interlayer coupling, leading to the expected anomaly. Turning on interlayer couplings, as far as the bulk remains gapped, one expects that the anomaly inflow from the three-dimensional bulk to the boundary surface will guarantee that the anomaly is unmodified. For hydrodynamics in $d = 3$ with chiral anomaly $\partial_\mu j^\mu \propto F_{\mu\nu}\tilde{F}^{\mu\nu}$, a background magnetic field parallel to the x-direction $\vec{B} = B_0\hat{x}$ will lead to $\partial_\mu j^\mu \propto F_{\mu\nu}\tilde{F}^{\mu\nu} = CF_{0x}$. To add the effect of the anomaly to the action we proceed in a way similar to [55]. Eq. (45) is modified to

$$Z[A_{1\mu} + \partial_\mu\lambda_1, A_{2\mu} + \partial_\mu\lambda_2] = Z[A_{1\mu}, A_{2\mu}]e^{i\mathcal{A}} \ , \tag{50}$$

where the anomaly is

$$\mathcal{A} = C\int dtd^dx \left[\varepsilon^{\alpha\beta}F^1_{\alpha\beta}\lambda_1 - \varepsilon^{\alpha\beta}F^2_{\alpha\beta}\lambda_2\right] \ , \tag{51}$$

where $\alpha, \beta = t, x$, and $F^1_{\alpha\beta} = \partial_\alpha A_{1\beta} - \partial_\beta A_{1\alpha}$, and similarly for $F^2_{\mu\nu}$, where $\varepsilon^{\alpha\beta}F_{\alpha\beta} = E_x$ for each of the two copies. We add to S_0 a local action S_{ch} which leads to the anomaly (50). Under gauge transformations $\delta A_{1\mu} = \partial_\mu\lambda_1$ and $\delta A_{2\mu} = \partial_\mu\lambda_2$, we must have

$$\delta S_{\text{ch}} = C\int dtd^dx \left[\varepsilon^{\alpha\beta}F^1_{\alpha\beta}\lambda_1 - \varepsilon^{\alpha\beta}F^2_{\alpha\beta}\lambda_2\right] \ . \tag{52}$$

A minimal action that satisfies this is

$$S_{\text{ch}} = C\int dtd^dx\varepsilon^{\alpha\beta}(\varphi_1 F_{1\alpha\beta} - \varphi_2 F_{2\alpha\beta}) \ . \tag{53}$$

This action breaks (48):[7]

$$\delta_\chi S_{\text{an}} = -2C\int dtd^dx\partial_x\chi A_{a0} \ . \tag{54}$$

[7] We use the convention $\varepsilon^{0x} = -1$.

To restore (48), we need to add another term to $S_{\rm ch}$ which is gauge invariant (so that (52) is still satisfied) and makes $\delta_\chi S_{\rm ch} = 0$. The simplest choice is $S_{\rm ch} \to S_{\rm ch} + S_1$, where

$$S_1 = 2C \int dt d^d x (B_{1x} B_{10} - B_{2x} B_{20}) \ . \tag{55}$$

Written in terms of r/a variables, we then have

$$S_{\rm ch} = C \int dt d^d x \left(\varphi_a \varepsilon^{\alpha\beta} F_{r\alpha\beta} + \varphi_r \varepsilon^{\alpha\beta} F_{a\alpha\beta} + 2B_{ax} B_{r0} + 2B_{rx} B_{a0} \right) \ . \tag{56}$$

One can verify that constraints (42)-(44) applied to $S_{\rm ch}$ are satisfied. The complete minimal action for chiral diffusion is then $S = S_0 + S_{\rm ch}$.

Introduce

$$\tilde{j}_1^\mu = \frac{\delta S}{\delta A_{1\mu}}, \qquad \tilde{j}_2^\mu = \frac{\delta S}{\delta A_{2\mu}} \ . \tag{57}$$

Note that these are the consistent currents, i.e. currents coming from varying the generating functional with respect to the background. Due to (50), these currents are not gauge invariant [36]. The currents can be made gauge invariant by shifting them with the Bardeen-Zumino term

$$j_{1,2}^\mu = \tilde{j}_{1,2}^\mu + 2C\varepsilon^{\mu\nu} A_\nu \ , \tag{58}$$

where j_1^μ and j_2^μ are called covariant currents. In the main text, and in the following discussion, we shall always use covariant currents. Now, defining

$$j_r^\mu = \frac{1}{2}(j_1^\mu + j_2^\mu), \quad j_a^\mu = j_1^\mu - j_2^\mu \ , \tag{59}$$

we find the following explicit expressions

$$j_r^0 = n, \quad j_r^x = -\sigma \partial_0 B_{rx} + 2i\sigma_x T B_{ax} + 4C B_{r0} \tag{60}$$

$$j_r^A = -\sigma \partial_0 B_{rA} + 2i\sigma_\perp T B_{aA}, \quad j_a^0 = \frac{\partial n}{\partial B_{r0}} B_{a0} \tag{61}$$

$$j_a^x = \sigma_x \partial_0 B_{ax} + 4C B_{a0}, \quad j_a^A = \sigma_\perp \partial_0 B_{aA} \ , \tag{62}$$

which are indeed gauge invariant. The a-type fields, as we will see, are related to the noise, while j_r^μ plays the role of j^μ of the main text. The same correspondence holds in the MSR formalism. Setting a-type fields to zero, as well as the background sources $A_{1\mu}, A_{2\mu}$, the current reads

$$j_r^0 = n, \quad j_r^x = 4a\mu - \sigma \partial_x \mu, \quad j_r^A = -\sigma \partial_A \mu \ , \tag{63}$$

where we used the identification $B_{r0} = \partial_0 \varphi_r = \mu$, leading to the correct hydrodynamic constitutive relation of the current (16) upon setting $C = \frac{\nu}{8\pi}$.

We shall now rewrite the action in a slightly different form, which will make the physics more transparent. Note that, switching off the background sources $A_{1\mu}, A_{2\mu}$, the action depends on φ_r only through $\partial_0\varphi_r = \mu$, due to (48). We can then make the change of variable to the path integral measure: $\int D\varphi_r \cdots = \int D\mu \cdots$, where we discarded the Jacobian of the transformation, $|\frac{1}{\partial_0}|$, being independent of all fields. We make the further change of variable $\int D\mu \cdots = \int DnJ \cdots$, where $J = \det(\frac{\partial n}{\partial \mu}(t,\vec{x})\delta^{(d)}(\vec{x}-\vec{y})\delta(t-t'))$ is the Jacobian of the change of variable. In what follows, we shall use dimensional regularization, in which case the Jacobian factor in this transformation equals the identity.[8] The path integral then becomes

$$Z = \int DnD\varphi_a e^{iS[n,\varphi_a]} \ , \tag{64}$$

where

$$\begin{aligned} S[n,\varphi_a] &= \int dtd^dx \left(-\varphi_a(\partial_0 n + \partial_x j_r^x + \partial_A j_r^A) + i\sigma_x T(\partial_x\varphi_a)^2 + i\sigma_\perp T(\partial_A\varphi_a)^2\right) \\ &= \int dtd^dx\big(-\varphi_a(\dot{n} + c\partial_x n + c' n\partial_x n - D_x\partial_x^2 n - D_\perp\partial_A^2 n) \\ &\qquad + i\chi D_x T(\partial_x\varphi_a)^2 + i\chi D_\perp T(\partial_A\varphi_a)^2\big) \end{aligned} \tag{65}$$

with $c = \frac{4C}{\chi}$, and $c' = -4C\frac{\chi'}{\chi^2}$, with $\chi' = -\chi^2\partial_n^2\mu$. In the above, we also introduced the constants $D_x = \sigma_x/\chi$ and $D_\perp = \sigma_\perp/\chi$. Using steps similar to those of the MSR formalism, this path integral can be immediately written as a stochastic equation, which is precisely (16)-(18). Following the discussion around Eq. (17), we make the change of coordinates $t' = t$, $x' = x - ct$, $y'^A = y^A$, for which the explicit action becomes (dropping the primes)

$$S = \int dtd^dx \left(-\varphi_a(\dot{n} + c'n\partial_x n - D_x\partial_x^2 n - D_\perp\partial_A^2 n) + i\chi D_x T(\partial_x\varphi_a)^2 + i\chi D_\perp T(\partial_A\varphi_a)^2\right) \ . \tag{66}$$

B.3 Renormalization

We will now evaluate the one-loop correction to the retarded two-point function of the charge, from which we will extract renormalization of transport. This can be written as the two-point function of j_r^0 and j_a^0:[9]

$$G_{nn}^R(t,x,y^A) = i\langle j_r^0(t,x,y^A) j_a^0(0)\rangle \ . \tag{67}$$

In what follows we will neglect contact term contributions. Our computations will be based on the two-point functions of the degrees of freedom n and φ_a which, at the level of the

[8] See [56], Section 18.2.4.

[9] See [53], Section II B.

quadratic part of the action (66), are[10]

$$\langle n\varphi_a\rangle_0(p) = \frac{i}{i\omega - D_x k_x^2 - D_\perp k_\perp^2} \tag{69}$$

$$\langle nn\rangle_0(p) = \frac{\chi T(D_x k_x^2 + D_\perp k_\perp^2)}{|i\omega - D_x k_x^2 - D_\perp k_\perp^2|^2} \ , \tag{70}$$

where $p^\mu = (\omega, \vec{k}) = (\omega, k^x, k_\perp^A)$, and $\langle n\varphi_a\rangle_0(p)$ is the tree-level part of $\langle n(p)\varphi_a(-p)\rangle$, and similarly for $\langle nn\rangle_0(p)$. Recall that we are in the coordinate frame following the chiral front. In the lab frame, the frequency is shifted to $\omega \to \omega + ck_x$. Notice that, from (60), the retarded two-point function of the charge density can be related to that of n and φ_a:

$$i\langle J_r^0(p) J_a^0(-p)\rangle = -\chi\omega\langle n(p)\varphi_a(-p)\rangle \ , \tag{71}$$

where we neglected the non-linear term in $J_a^0 = \chi\partial_0\varphi_a + \chi' n\partial_0\varphi_a$ as this is subleading at low energy. This can be inferred from the scaling analysis below Eq. (17).

We are now ready to compute loop corrections to (69). To obtain this we expand the path integral (46) in the interaction coupling c' and perform Wick contractions

$$\langle n\varphi_a\rangle = \langle n\varphi_a\rangle_0 + i\langle n\varphi_a S_{\text{ing}}\rangle_0 - \frac{1}{2}\langle n\varphi_a S_{\text{ing}}^2\rangle_0 \ , \tag{72}$$

where $S_{\text{int}} = c' \int dt d^d x \, n\partial_x n$. For this simple action there is only one diagram contributing, shown in Fig. 2.

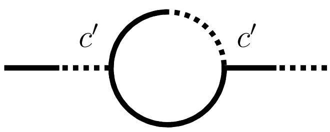

Figure 2: One-loop contribution to $\langle n\varphi_a\rangle$. Solid lines represent n and dashed lines represent φ_a.

We represent this contribution as

$$i\langle n\varphi_a\rangle_0(p)\Sigma(p)\langle n\varphi_a\rangle_0(p) \ . \tag{73}$$

In going to higher loops the structure of the diagram in Fig. 2 naturally leads to a geometric series, thus resumming into a self-energy contribution:

$$\langle n(p)\varphi_a(-p)\rangle = \frac{i}{i\omega - D_x k_x^2 - D_\perp k_\perp^2 + \Sigma(\omega, k_x, k_A)} \ , \tag{74}$$

[10]We use

$$\langle \mathcal{O}_1(p)\mathcal{O}_2(-p)\rangle = \int dt d^d x e^{i\omega t - i\vec{k}\cdot\vec{x}} \langle \mathcal{O}_1(t,\vec{x})\mathcal{O}_2(0)\rangle \ . \tag{68}$$

where the self-energy is given by

$$
\begin{aligned}
-i\Sigma(\omega, k_x, k_A) &= c'^2 k_x \int \frac{d^d k'}{(2\pi)^d} \int \frac{d\omega'}{2\pi} k'_x \langle nn\rangle(p-p')\langle n\varphi_a\rangle(p') \\
&= -i\chi T c'^2 \frac{k_x^2}{2} \frac{1}{\sqrt{2^d D_x D_\perp^{d-1}}} \int \frac{d^d q}{(2\pi)^d} \frac{1}{\left(-i\omega + \frac{1}{2}(D_x k_x^2 + D_\perp k_A^2)\right) + q_x^2 + q_A^2} \ ,
\end{aligned}
\tag{75}
$$

where in the last step we performed the frequency integral, and changed to variables $(k'_x, k'_A) = \left(\frac{q_x}{\sqrt{2D_x}}, \frac{q_A}{\sqrt{2D_\perp}}\right) + \frac{1}{2}(k_x, k_A)$. In dimensional regularization,

$$
L_d(A) \equiv \int \frac{d^d u}{(2\pi)^d} \frac{1}{A + u^2} = \frac{(A/\pi)^{\frac{d}{2}-1}}{2^d \Gamma\left(\frac{d}{2}\right) \sin\left(\frac{d\pi}{2}\right)} \ , \tag{76}
$$

which diverges in even spatial dimensions (in dimensional regularization, power divergences vanish). From the scaling argument below Eq. (17) we already expect that the critical dimension is $d = 2$, so we shall take $d = 2 - \epsilon$ with ϵ small, for which

$$
L_{2-\epsilon}(A) = \frac{1}{2\pi\epsilon} - \frac{1}{4\pi}\left(\gamma + \log\left(\frac{A}{4\pi}\right)\right) + O(\epsilon) \ , \tag{77}
$$

where γ is the Euler-Mascheroni constant, and the momentum cutoff Λ is determined by $\frac{1}{\epsilon} = \log\Lambda$. We then find, in $d = 2$,

$$
\Sigma(\omega, k_x, k_A) = -\frac{\chi T c'^2 k_x}{16\pi\sqrt{D_x D_\perp}} \log\left(-i\omega + \frac{D_x k_x^2}{2} + \frac{D_\perp k_A^2}{2}\right) \ , \tag{78}
$$

where we dropped analytic terms that can be absorbed in the diffusion constants. The diffusion constant D_x receives a UV divergent contribution

$$
D_x^{(\Lambda)} = D_x + \frac{\chi T c'^2}{8\pi\sqrt{D_x D_\perp}} \log\Lambda \ , \tag{79}
$$

while $D_\perp$ does not renormalize at this order.

Let us now derive the β functions of various couplings in dimension $d = 2 - \varepsilon$. Integrating out modes with $\mu < |k| < \Lambda$, Eq. (79) leads to

$$
\beta_{D_x} = \frac{\partial D_x}{\partial \log\mu} = \varepsilon D_x - \frac{\chi T c'^2}{8\pi\sqrt{D_x D_\perp}} \ , \tag{80}
$$

where the first term comes from tree-level scaling (see below Eq. (17).

From the form of (74) we see that there is no wavefunction renormalization, i.e. the first term of (66) remains equal to -1. Also, going back to the lab frame $\omega \to \omega + ck_x$ it is obvious that c in (65) does not renormalize. Finally, we note that the action (66) (or (65)) has an emergent symmetry:

$$
n \to n + \epsilon, \qquad x \to x + 2\epsilon c' t \ , \tag{81}
$$

which implies that c' does not renormalize, thanks to absence of wavefunction renormalization. This can also be verified by directly compute renormalization of c'. In conclusion, at one loop, only D_x undergoes renormalization.

Solving the RG flow equation (80) gives, at leading order as $\mu \to 0$,

$$D_x(\mu) = \left(-\frac{3\chi T c'^2}{16\pi\sqrt{D_\perp}} \log \mu \right)^{2/3} . \tag{82}$$

Now, at leading order the dispersion relation is $\omega \sim k^2 \sim \mu^2$, and thus we find, for the conductivity of the system,

$$\sigma_{xx}(\omega) = \chi D_x(\omega) \to \chi \left(-\frac{3\chi T c'^2}{32\pi\sqrt{D_\perp}} \log \omega \right)^{2/3} . \tag{83}$$

For nonvanishing ω, k_x, k_y, one can fix μ by comparing to (78), which gives $\mu^2 \propto -i\omega + \frac{1}{2}D_x k_x^2 + \frac{1}{2}D_\perp k_A^2$. Then the full Green's function should be

$$G^R_{nn}(\omega) = \frac{\chi\omega}{\omega + iD_\perp k_\perp^2 + ik_x^2 \left\{ -\frac{3\chi T c'^2}{32\pi} \log \frac{-i\omega + \frac{1}{2}D_x k_x^2 + \frac{1}{2}D_\perp k_y^2}{\Lambda^2} \right\}^{2/3}} . \tag{84}$$