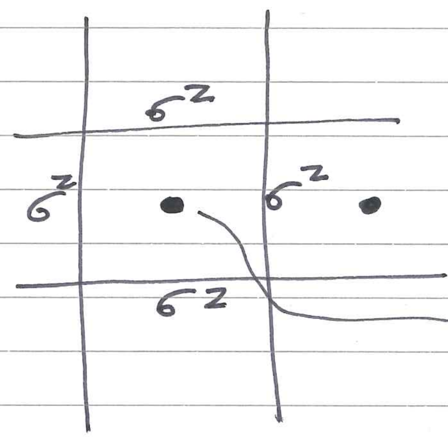
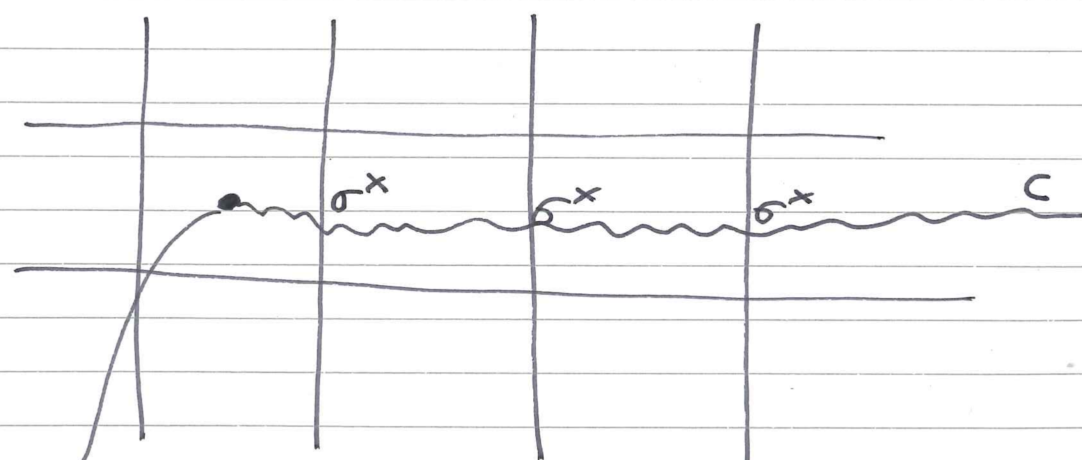
Duality of Z_2 gauge theory and quantum Ising model

Ising gauge theory
$$H = -K \sum_{\square} \prod_{\ell \in \square} \sigma^z_\ell - g \sum_\ell \sigma^x_\ell$$



$$\mu^x = \prod_{\square} \sigma^z$$

Define operators on the dual lattice sites μ^x and μ^z



$$\mu^z_i = \prod_{\ell \in C} \sigma^x_\ell$$

From this defintion it is easy to see that

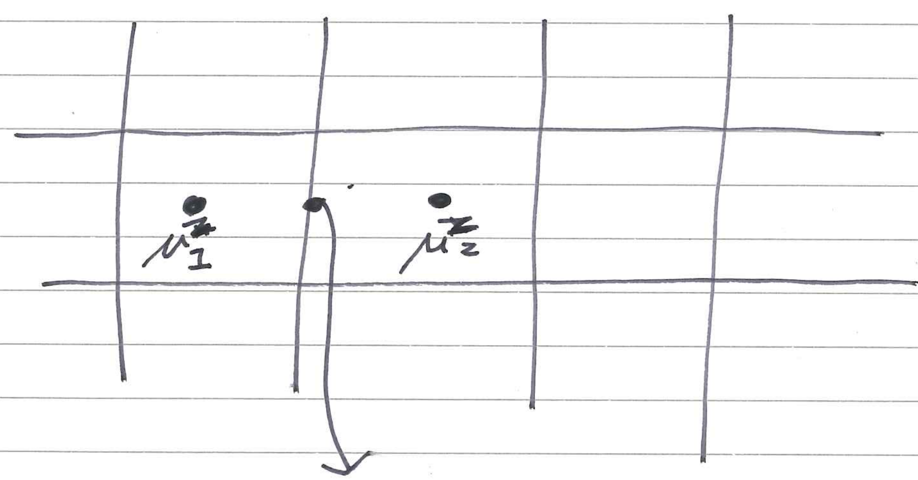
$$\mu_l^z \mu_l^x = - \mu_l^x \mu_l^z$$

and $\mu_l^z \mu_{l'}^x = \mu_{l'}^x \mu_l^z$ for $l \neq l'$

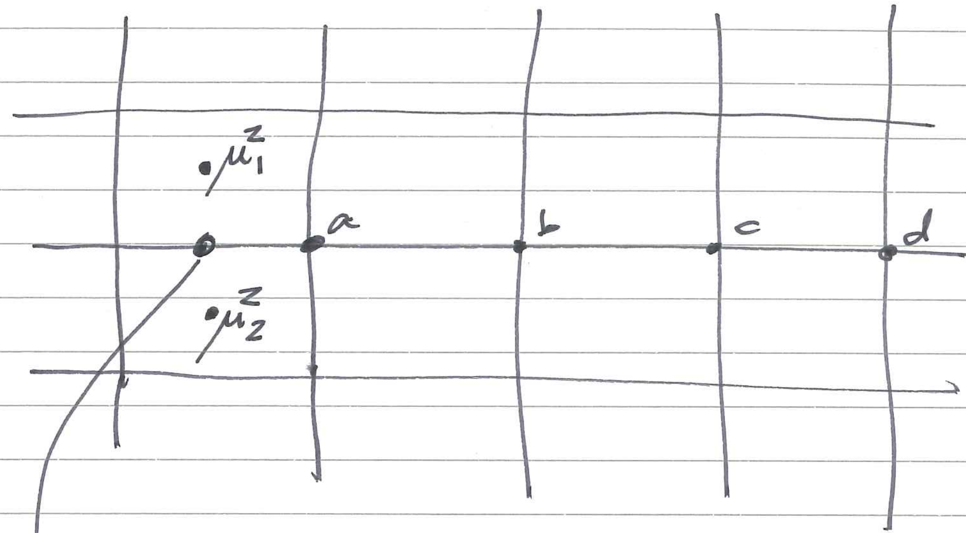
and $\mu_l^{x\,2} = \mu^{z\,2} = 1$.

So μ_l^x and μ_l^z are a dual set of qubits.

Now note that



$$\sigma^x = \mu_1^z \mu_2^z$$



$$\sigma^x = \mu^z_1 \mu^z_2 \underbrace{G_a \, G_b \, G_c \, G_d \cdots}_{1.}$$

So

$$H = -g \sum_{\langle ij \rangle} \mu^z_i \mu^z_j - k \sum_i \mu^x_i$$

This is the square lattice Ising model in a transverse field.

Mapping between quantum Ising model in a transverse field ~~and~~ in d dim. and classical Ising model in d+1 dim.:

First consider $d=0$ i.e. a single quantum spin

$$H = -g\sigma^x$$

$$Z = \mathrm{Tr}\, e^{-\beta H} = 2\cosh(\beta g)$$

$$= \mathrm{Tr}\, \underbrace{e^{-\Delta\tau H} e^{-\Delta\tau H} e^{-\Delta\tau H} \cdots\cdots}_{N \text{ times}}$$

$$N\Delta\tau = \beta.$$

Insert complete set of states between each ~~σ_i~~ exponential.

Use eigenstates of $\sigma^z |s\rangle = s|s\rangle$

$s = \pm 1.$

Now $\langle s_1 | e^{+\Delta\tau g \sigma^x} | s_2 \rangle$

$$= \langle s_1 | \cosh(\Delta\tau g) + \sigma^x \sinh(\Delta\tau g) | s_2 \rangle$$

$$\stackrel{?}{=} A \exp(B s_1 s_2).$$

$s_1 = s_2 = 1$

$$\cosh(\Delta\tau g) = A e^{B}$$

$s_1 = -s_2 = 1$ $$+\sinh(\Delta\tau g) = A e^{-B}$$

$$\Rightarrow \quad A^2 = \cosh(\Delta\tau g) \sinh(\Delta\tau g)$$

$$e^{-2B} = \tanh(\Delta\tau g).$$

So $$Z = A^N \sum_{s_i = \pm 1} \exp\left(B s_i s_{i+1} \right)$$

$$s_1 = s_{N+1}$$

$\Rightarrow$ Classical Ising chain in 1 dimension with periodic boundary conditions!

Now consider $d > 1$

$$H = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - g \sum_i \sigma_i^x$$

$$Z = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\Big(\underbrace{e^{-\Delta\tau H} e^{-\Delta\tau H} \cdots\cdots e^{-\Delta\tau H}}_{N \text{ times}}\Big).$$

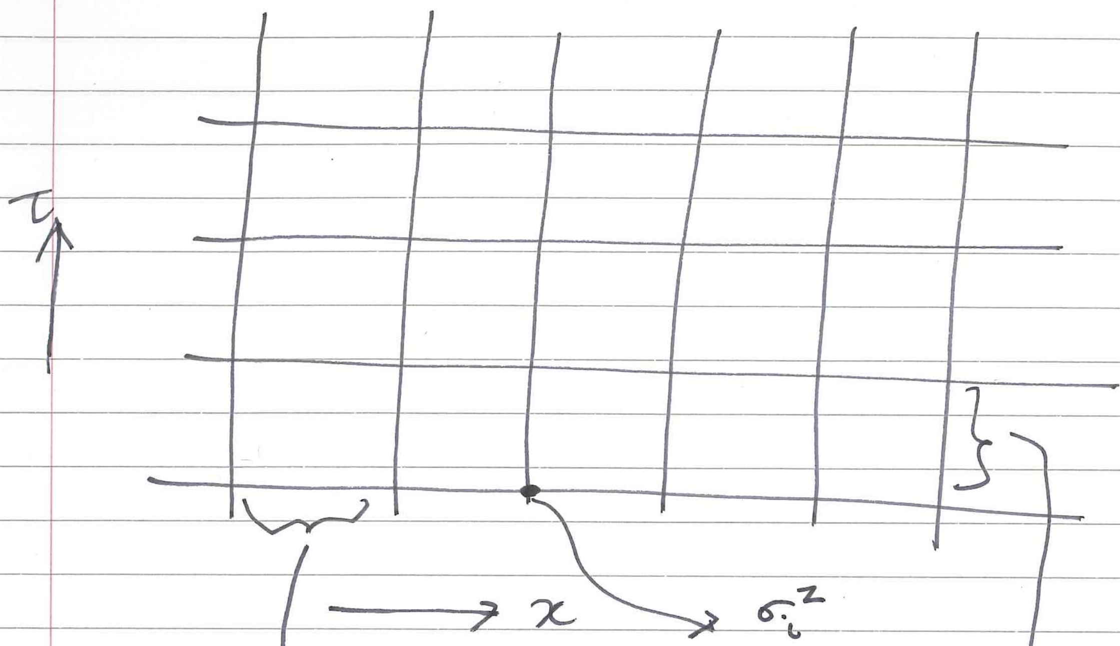
Now use $\Delta\tau \to 0$, $N \to \infty$, $\Delta\tau N = \beta$ fixed.

$$e^{\Delta\tau J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z + \Delta\tau g \sum_i \sigma^x}$$

$$\approx \exp\Big(\Delta\tau J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z\Big) \exp\Big(\Delta\tau g \sum_i \sigma_i^x\Big) + \mathcal{O}\big((\Delta\tau)^2\big)$$

Inserting complete set of states as in $d=0$ we ~~...~~ obtain a $(d+1)$ dim. Ising model

$H =$



$$-\Delta\tau J \sigma_i^z \sigma_{i+x}^z$$

$$-B\sigma_i^z \sigma_{i+\tau}^z.$$

$\mathbb{Z}_2$ gauge theory in a classical form.

$$Z = \mathrm{Tr}\, e^{-H}$$

$$H = \sum_{\square} \prod_{\square} \sigma^z \sigma^z \sigma^z \sigma^z$$

where $\square$ includes <u>spacetime plaquettes</u>.