# THE ADLER–BELL–JACKIW ANOMALY AND WEYL FERMIONS IN A CRYSTAL

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The Adler–Bell–Jackiw (ABJ) axial anomaly is derived from the physical point of view as the production of Weyl particles and it is used to show the absence of the net production of particles for lattice regularized chirally invariant theories with locality. An analogy or a simulation is pointed out between the Weyl fermion theory and gapless semiconductors where two energy bands have pointlike degeneracies. For such materials, in the presence of parallel electric and strong magnetic fields, there exists an effect similar to the ABJ anomaly that is the movement of the electrons in the energy–momentum space from the neighborhood of one degeneracy point to another one. The longitudinal magneto-conduction becomes extremely strong.

*1.* It is the purpose of the present article to demonstrate a similarity between the fermion system of lattice gauge theories and the electron system of crystals. We then point out that there exists an effect analogous to the mechanism of the Adler–Bell–Jackiw (ABJ) axial anomaly [1] in solid state physics.

We firstly derive the ABJ anomaly [2] in a physically intuitive way, which was also given independently by Lipatov, Lüscher, Peskin [3] and Susskind [4], in order to apply it to solid state physics. The mechanism of the ABJ anomaly is understood as the production of Weyl fermions in the presence of external electric and magnetic fields. Using this derivation [‡1] we shall demonstrate that in any lattice theory of chirally invariant fermions with locality there is an equal number of production and annihilation of Weyl fermions. Thus there is no *net* production so that the axial charges are conserved. It is based on a theorem [6] proved by the present authors that in any lattice regularized version with locality there appears necessarily an equal number of right-handed (RH) and left-handed (LH) species of Weyl fermions. It should be stressed that this theorem does not hold for the nonlocal theories by Drell et al. [‡2].

The basic similarity between lattice fermions and electrons in crystals is that in both theories there is only the lattice translational invariance. Then the momentum is conserved modulo a multiple of the unit length of the reciprocal lattice so that the momentum space becomes a Brillouin zone which is topologically equivalent to the hypertorus $S^1 \times S^1 \times S^1$. The electrons of the crystal are described by a one-component (nonrelativistic) Schrödinger equation, but the energy eigenvalues form bands. As explained in section 4 using a localized function we can write the electron theory as a multicomponent lattice fermion theory with a matrix hamiltonian.

We will be interested in the situation where two energy bands of the electrons make contact at points in the energy–momentum dispersion law space which

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[‡1] The derivation was applied to the Wilson lattice fermions with a hopping term in order to investigate the anomaly in ref. [5].

[‡2] As for the anomaly on this model see ref. [8].

can be simulated by the relativistic Weyl fermions with $\omega = \pm(P^2)^{1/2}$. We shall refer to this as a generic degeneracy point [6]. Near such a degeneracy point the equation of the two band wave functions of the electrons can be approximated by the Weyl equation. Then according to our theorem such degeneracy points occur in pairs of the RH and LH by Weyl equations.

There is then no net production of electrons like the absence of the net production of Weyl fermions in lattice fermion theories when parallel electric, $E$, and magnetic, $H$, fields are put on. This leads to the conservation of the axial charge. The ABJ anomaly manifests itself by having electrons transferred from the neighborhood of the LH degeneracy point to the RH one in energy–momentum space. This movement, as will be discussed in section 5, gives rise to an electric current which is a special one compared to that of the semiconductors. We compute the magnitude of such a current and suggest that the material – a gapless semiconductor [9–11] may have an exceptionally strong magnetic conductivity. We discuss here our main results [‡3] and leave the detailed argument to a forthcoming paper [13].

*2.* Let us start with a (1 + 1)-dimensional right-handed (RH) Weyl fermion theory coupled to a uniform electric field $\dot{A}^1 = E$ in the temporal gauge. The one component RH Weyl equation for $\psi_R = \frac{1}{2}(1 + \gamma_5)\psi$ reads

$$i\dot{\psi}_R(x) = (-i\partial_x - A^1)\psi_R(x) . \quad (1)$$

The dispersion law is $\omega(P) = P$. Corresponding to the classical equation of a charged particle in the presence of an electric field where $\dot{P} = eE$, the acceleration of the RH particles in quantum theory is given by $\dot{\omega} = \dot{P} = eE$. The creation rate of the RH particles per unit time and unit length is determined by a change of the Fermi surface, which distinguishes the filled and unfilled states as illustrated in fig. 1a. Let the quantization length be $L$; the density of states per length $L$ is $L/2\pi$ and the rate of change of the RH particle number $N_R$ is

$$\dot{N}_R = L^{-1}(L/2\pi)\,\dot{\omega}_{fs} = (e/2\pi)E . \quad (2)$$

‡3 Some results of the present article were reported by one of the authors (ref. [12]).

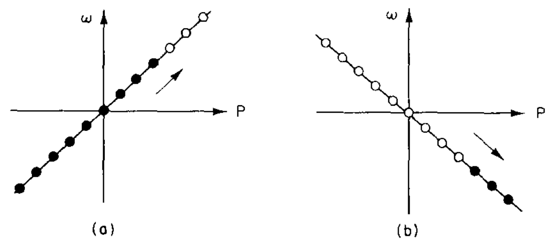


Fig. 1. Dispersion laws for the RH (a) and LH (b) Weyl fermions in 1 + 1 dimensions. The black and white points denote the filled and unfilled levels and the arrows indicate the direction of the movement of the Fermi surface when $E$ in on.

This particle creation is the ABJ anomaly. Consequently the chiral charge $Q_R$ is not conserved and $\dot{Q}_R = \dot{N}_R = (e/2\pi)E$. It follows from an analogous reasoning that the annihilation rate of the LH particles with the dispersion law $\omega = -P$ as shown in fig. 1b is

$$\dot{N}_L = -(e/2\pi)E , \quad (3)$$

thus the creation rate of the LH antiparticles is

$$\dot{\bar{N}}_L = (e/2\pi)E .$$

Therefore the anomaly for the Dirac particles is

$$\dot{N}_R + \dot{\bar{N}}_L = (e/\pi)E ,$$

which gives $\dot{Q}_5 = (e/\pi)E$.

In 3 + 1 dimensions we first calculate the energy levels of the RH Weyl fermion in the presence of the applied uniform magnetic field along the third direction given by

$$A^2 = HX^1 \quad \text{and } A^\mu = 0 \text{ otherwise} .$$

The solution to the equation for a two-component RH field $\psi_R$ of the form

$$[i\partial/\partial t - (\boldsymbol{P} - e\boldsymbol{A})\boldsymbol{\sigma}]\,\psi_R(x) = 0 \quad (4)$$

is expressed in terms of a solution of the auxiliary equation

$$[i\partial/\partial t - (\boldsymbol{P} - e\boldsymbol{A})\boldsymbol{\sigma}]\,[i\partial/\partial t + (\boldsymbol{P} - e\boldsymbol{A})\boldsymbol{\sigma}]\,\Phi = 0 \quad (5)$$

as

$$\psi_R = [i\partial/\partial t + (\boldsymbol{P} - e\boldsymbol{A})\boldsymbol{\sigma}]\,\Phi . \quad (6)$$

From eq. (5) the energy and the $P_2, P_3$ eigenfunction satisfies an equation of the harmonic oscillator type

$$[-(\partial/\partial x^1)^2 + (eH)^2(x^1 + P_2/eH) + (P_3)^2 + eH\sigma_3]\,\Phi$$

$$= \omega^2\Phi ,$$

with $\sigma_3 = \pm 1$. The energy levels are given by the Landau levels,

$$\omega(n, \sigma_3, P_3) = \pm[2eH(n + \tfrac{1}{2}) + (P_3)^2 + eH\sigma_3]^{1/2}$$

$$(n = 0, 1, 2, ...) \qquad (7)$$

except for the $n = 0$ and $\sigma_3 = -1$ mode where

$$\omega(n = 0, \sigma_3 = -1, P_3) = \pm P_3 . \qquad (8)$$

The eigenfunction takes the form

$$\Phi_{n\sigma_3}(x) = Nn\sigma_3 \exp(-iP_2x^2 - iP_3x^3)$$

$$\times \exp[-\tfrac{1}{2}eH(x^1 + P_2/eH)^2]H_n(x^1 + P_2/eH)\chi(\sigma_3) , \qquad (9)$$

with $Nn\sigma_3$ as the normalization constant. Here $\chi(\sigma_3)$ denotes the eigenfunctions of the Pauli spin $\sigma_3$ which can be taken as $\chi(1) = \binom{1}{0}$ and $\chi(-1) = \binom{0}{1}$. The solution of (4) is obtained by inserting (9) into (6). This leads to the relations

$$\psi_{\mathrm{R}}^{(n+1, \sigma_3=-1)} = (N_{n+1, \sigma_3=-1}/N_{n, \sigma_3=1})\psi_{\mathrm{R}}^{(n, \sigma_3=1)} ,$$

for $n = 0, 1, \ldots$ ,

and

$$\psi_{\mathrm{R}}^{(n=0, \sigma_3=-1)} = 0, \quad \text{with } \omega = -P_3 .$$

Thus the energy levels of $\psi_{\mathrm{R}}$ are (4) and

$$\omega(n = 0, \sigma_3 = -1, P_3) = P_3 . \qquad (10)$$

The dispersion laws (7) and (10) are shown in fig. 2.

Next a uniform electric field is turned on along the third direction parallel to $H$. As for the zero mode ($n = 0, \sigma_3 = -1$) the dispersion law is the same as that for 1 + 1 dimensions and the creation rate of the particles is calculated in a similar manner. It should be noticed that when $E$ is on adiabatically there is no particle creation in the $n \neq 0$ modes. The density of the state per length $L$ is $LeH/4\pi^2$ and the creation rate (the ABJ anomaly) is given by

$$\dot{N}_{\mathrm{R}} = L^{-1}(LeH/4\pi^2)\omega_{\mathrm{fs}}(n = 0, \sigma_3 = -1, P_3)$$

$$= (e^2/4\pi^2)EH , \qquad (11)$$

which equals to $\dot{Q}_{\mathrm{R}}$.

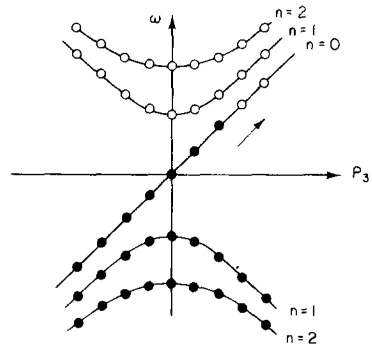


Fig. 2. Dispersion law for the RH Weyl fermion in 3 + 1 dimensions in the presence of a magnetic field in the $x^3$-direction.

For the LH fermions the annihilation rate of the LH particles is

$$\dot{N}_{\mathrm{L}} = -(e^2/4\pi^2)EH , \qquad (12)$$

and the creation rate of the LH antiparticles is

$$\dot{\bar{N}}_{\mathrm{L}} = (e^2/4\pi^2)EH , \qquad (13)$$

which is $\dot{Q}_{\mathrm{L}}$.

We then have for the Dirac field

$$\dot{N}_{\mathrm{R}} + \dot{\bar{N}}_{\mathrm{L}} = (e^2/2\pi^2)EH .$$

that gives $\dot{Q}_5 = (e^2/2\pi^2)EH$.

*3.* Consider next the hamiltonian version of a general lattice regularized two-component chiral fermion theory described by an $N$ component fermion field $\psi_k$. We further assume the locality and hermicity conditions for the hamiltonian. The action is given by

$$S = -\mathrm{i}\int \mathrm{d}t \sum_n \sum_{k=1}^{N} \psi_k^+(na)\psi_k(na)$$

$$- \int \mathrm{d}t \sum_n\sum_m\sum_k\sum_l \psi_k^+(na)H_{kl}((n-m)a)\psi_l(ma) .$$

Here $n$ denotes the set of integers, $a$ the lattice spacing and $H$ a local and harmitian $N \times N$ hamiltonian. A characteristic feature of the lattice fermion theory is that, because $S$ is invariant under a lattice translation with $\psi$ sitting on the lattice sites, the momentum space forms a hypertorus, $\mathrm{T}^3$, which is the Brillouin zone.

Among $N$ discrete energy eigenvalues determined by

$$\sum_{l=1}^{N} H_{kl}(p)\psi_l^{(i)}(P) = \omega_i \psi_k(P) \quad (i = 1, ..., N) \tag{14}$$

suppose that the $i$th level $\psi_i(P)$ and the $(i + 1)$th $\omega_i(P)$ are degenerate at several different points in the dispersion space $(\omega(P), P)$. Then expand $H(P)$ around one of the degeneracy point $(\omega_d(P_d), P_d)$ in powers of $(P - P_d)$. It can be shown that the shifts of the energy, $\omega(P) - \omega_d(P_d)$ to the first order in $(P - P_d)$ is determined by the 2 X 2 submatrix $H^{(2)}(P)$ formed by the $i$th and $(i + 1)$th entries of the $N \times N$ matrix for $H$. In the expansion of $H^{(2)}(P)$

$$H^{(2)}(P) = H^{(2)}(P_d) + (P - P_d)\partial H^{(2)}(P)/\partial P|_{P=P_d}$$

$$+ O((P - P_d)^2) ,$$

the derivative term is expressed by the Pauli matrices $(1, \sigma_\alpha)$ as

$$\partial H^{(2)}/\partial P_k|_{P=P_d} = a_k(P_d)1 + V_k^\alpha(P_d)\sigma_\alpha .$$

Here $a$, $\mathbf{V}$ are constants depending on $P_d$. Thus near $P = P_d$, $H^{(2)}(P)$ takes the form

$$H^{(2)}(P) = \omega_d 1 + (P - P_d)a1 + (P - P_d)_k V_\alpha{}^k \sigma^\alpha . \tag{15}$$

The eigenvalue equation for the $i$th and $(i + 1)$th energy eigenvalues near $P = P_d$, $H^{(2)}(P)u = \omega u$ can be rewritten by using a new set of variables

$$\hat{P} = P - P_d, \quad P^0 = \omega - \omega_d - \hat{p}a , \tag{16}$$

as

$$\hat{p}\mathbf{V}\boldsymbol{\sigma}u = P^0 u . \tag{17}$$

We introduce $k^0 = P^0$ and $k = \pm\hat{p}\mathbf{V}$, where $\pm$ corresponds to the sign of det $V$. The (17) becomes

$$\boldsymbol{k\sigma}u = \pm k^0 u . \tag{18}$$

In this way the RH or LH Weyl equation describes the two energy levels near each degeneracy point. Thus each degeneracy point in $(\omega(P), P)$ space correspond to a species of Weyl fermions contained in the theory.

The theorem that the RH and LH degeneracy points appear necessarily as a pair because of the Brillouin zone structure was proved by us based only on topological arguments together with locality. Detailed argument of this is referred to in ref. [6]. Fig. 3 illustrates the doubling of the Weyl fermions.

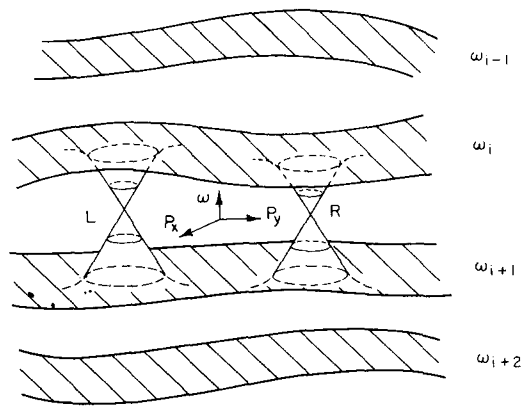


Fig. 3. Generic dispersion law for the two degeneracy points denoted by R and L between the energy levels $\omega_i$ and $\omega_{i+1}$. The shaded surfaces in (3 + 1)-dimensional $\omega - P$ space denote the layers of the energy bands. One momentum axis, $P_z$, is suppressed. Near the degeneracy points R and L, the dispersion laws are cones. The equation near R is the RH Weyl equation and the one near L is LH one.

When the parallel electric and magnetic fields are turned on, the production or annihilation of the RH or LH Weyl particles takes place around a RH or LH degeneracy point respectively, according to the argument in section 2 for eqs. (11) and (12). As was stated above there is an equal number of species of RH and LH particles within each Brillouin zone, there is no net production. Therefore from eqs. (11) and (13) the axial charge satisfies

$$\dot{Q}_5 = (\dot{N}_R - \dot{N}_L) \times \tfrac{1}{2}(\#\,\text{deg.pt.}) = 0 .$$

where (# deg.pt.) denotes the number of degeneracy points. The absence of net production however means that the particles move from the neighborhood of the LH degeneracy point to that of RH one by the amount given by eq. (11) [or equivalently (12)] because all the cones are connected smoothly by the dispersion law in fig. 3. It is this movement of the particles in the energy–momentum space that we call the analogy of the ABJ anomaly on the lattice. This is indicated by the solid line with arrows in fig. 6.

As an example consider the RH Weyl theory in 1 + 1 dimensions given by

$$i\dot{\psi}_R(na) = -(i/2a)[\psi_R((n + 1)a) - \psi_R((n - 1)a)]$$

that is a naive lattice version of eq. (1) with $A^1 = 0$.

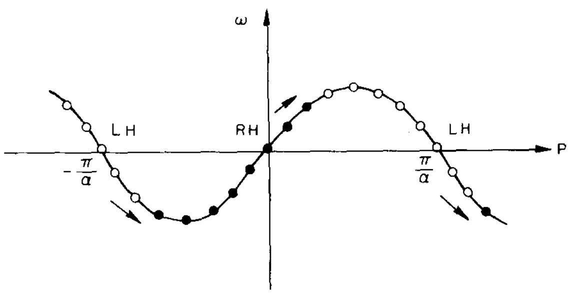


Fig. 4. Dispersion law for the RH Weyl equation on the lattice in 1 + 1 dimensions. The arrow indicates the direction of the shift of the particles when $E$ is on. Note that two points, $P = -\pi/a$, and $P = \pi/a$, are identified.

In the dispersion law $\omega = (1/a) \sin Pa$ there is one RH species with $\omega = P$ and one LH species with $\omega = -(P - \pi/a)$ near $P = 0$ and $P = \pi/a$ respectively as illustrated in fig. 4. When $E$ is on the production of RH particles takes place near $P = 0$ by the amount given by (2) while at the same time near $P = \pi/a$ LH particles are annihilated by the same amount according to (3). This means that particles are shifted from the neighborhood of $P = 0$ to that of $P = \pi/a$ along the dispersion law as shown in fig. 4.

*4.* We make here the analogy between the lattice fermion theory described in section 3 and the electrons in a crystal [14]. Let us consider the electron system with a generic hamiltonian obeying the Schrödinger equation

$$\mathrm{i}\dot{\Psi}(x) = H\Psi(x) = \omega\Psi(x). \tag{19}$$

In order to obtain a close analogy with lattice fermion theory, we expand in terms of an orthonormal localized functions $f_l(x - na)$ ($l = 1, ..., N$), which goes to zero exponentially as $|x - na| \to \infty$. Inserting into (19) the expansion

$$\Psi(x) = \sum_l \sum_n \psi_l(na) f_l(x - na) ,$$

and using the orthogonality of $f_l$, we obtain the equation

$$\sum_l \sum_n H_{kl}(ma - na)\, \psi_l(ma) = \omega \psi_k(ma) .$$

Here the matrix $H_{kl}$ is defined by

$$H_{kl}(ma - na) = \int \mathrm{d}^3 x\, f_k^*(x - ma) H f_l(x - na) .$$

By making a Fourier transform, we obtain

$$\sum_l H_{kl}(P)\, \psi_l(P) = \omega \psi_k(P) . \tag{20}$$

This is precisely the same form as that of lattice fermion theory (14).

As was discussed in ref. [6] the generic behavior of the dispersion laws $\omega_i(P)$ for the general complex hamiltonian $H(P)$ corresponds to the case where the two energy levels $\omega_i(P)$ and $\omega_{i+1}(P)$ have a finite number of discrete degeneracy points $(\omega_\mathrm{d}(P), P_\mathrm{d})$ like the case of the Weyl fermion theory discussed in section 2. However for $H(P)$ to be complex, either time reversal symmetry or parity (space inversion) symmetry must be broken. We consider the case where the latter symmetry is broken. For such parity noninvariant zero-gap materials, the two energy bands near the degeneracy point are described by either RH or LH Weyl equations (17) by the same argument as in section 3. The only difference from the lattice fermion theory is that the vierbein $\mathbf{V}$, which describes the slope of the dispersion law, i.e., the velocity of particles near the degeneracy point, is much smaller than the velocity of light $c = 1$. If the $PT$ symmetry is strongly broken (i.e., of order of 1), we may expect the elements of $V$ to be of the order of magnitude of the typical velocity $v$ of an electron in a crystal. For simplicity we may take as an example $V_{k\alpha} = v\delta_{k\alpha}$ ($k, \alpha = 1, 2, 3$) and then the Weyl equation (17) reads

$$v\boldsymbol{k\sigma}u = k^0 u , \tag{21}$$

with the dispersion law $(k^0)^2 = v^2 \boldsymbol{k}^2$. The spin indicated by the Pauli matrices $\sigma_i$ in eq. (21) is not the intrinsic electron spin. This formal artificial spin is introduced in order to describe the wave functions of the two energy bands $\omega_i$ and $\omega_{i+1}$ in terms of a two-component field $u$. Since the momentum space forms the Brillouin zone, our theorem holds and there is an equal number of RH and LH degeneracy points in the Brillouin zone.

*5.* We assume that we have found a parity non-invariant zero-gap semiconductors which can be simulated by a Weyl fermion theory with a dispersion law $\epsilon^2 = v^2 P^2$. The effect analogous to the ABJ anomaly gives rise to a peculiar behavior of the conductivity of the electric current in the presence of the magnetic field. It is enough to consider one conduction band $\omega_i$ in order to compute the electric current induced by the external electric field. The valence band $\omega_{i+1}$ (negative energy state) is assumed to be completely filled. In the absence of the external field, the single electron states of the conduction band, the upper cone e.g., around the RH degeneracy point of fig. 3 are filled up to the Fermi level energy $\mu$. Then the electron distribution function in the thermodynamical equilibrium is of the form

$$f_0(P) = \{1 + \exp[(\epsilon(P) - \mu)/kT]\}^{-1} . \qquad (22)$$

In the presence of $E$ and $H = 0$ there occurs a small deviation from $f_0$ so that $f = f_0 + \delta f$. The $E$ field accelerates the electrons in the same direction and then

$$(\partial f/\partial t)_{\text{drift}} = eE\,\partial f/\partial P_z .$$

At the same time the accelerated electrons get scattered back into some state in the same cone as shown in fig. 5. We assume that $f$ falls back into $f_0$ exponentially with a relaxation time $\tau_0$ so that $\delta f \propto \exp(-t/\tau_0)$. Then

$$(\partial f/\partial t)_{\text{coll}} = -\tau_0^{-1}(f - f_0) .$$

Therefore the steady state condition is $(\partial f/\partial t)_{\text{drift}} = -(\partial f/\partial t)_{\text{coll}}$ (Boltzmann equation). The solution of this is in the lowest order in $E$

$$f(P) = f_0(\epsilon) + eE\tau_0\,\partial f_0(\epsilon)/\partial P_z .$$

Then the longitudinal current density [15] is given by

$$J_0 = \frac{1}{L^3}\sum_P (-e)v_z f(P)\ (\#\,\text{deg.pt.})$$

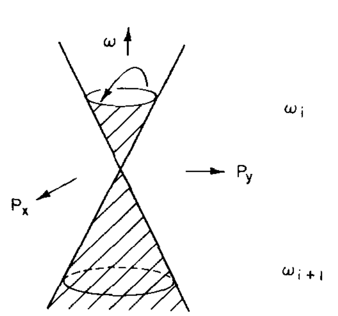


Fig. 5. The accelerated electrons near the Fermi surface in one of the cones in the levels $\omega_i$ are scattered back into the Fermi surface in the same cone as indicated by the arrow. The shaded region is filled by electrons.

where $v_z = \partial\epsilon/\partial P_z$ and (# deg.pt.) denotes the number of degeneracy points. In the low temperature approximation $f_0(\epsilon) = \theta(\mu - \epsilon)$ so that

$$J_0 = (1/6\pi^2)e^2 E(\mu^2/v)\tau_0\ (\#\,\text{deg.pt.}) .$$

The relation time is given in terms of the transition probability of the electron from the state with $\boldsymbol{P}$ into one with $\boldsymbol{P}'$, $W(\boldsymbol{P} \to \boldsymbol{P}')$, by

$$\frac{1}{\tau_0} = \frac{1}{L^3}\sum_{P'}\frac{P_z - P_z'}{P_z}\,W(\boldsymbol{P} \to \boldsymbol{P}') . \qquad (23)$$

We assume that the interaction between the electrons and the ionized impurities is given by the screened Coulomb potential of the form

$$V(x) = \left(\frac{4\pi e^2}{k}\right)\exp(-|\boldsymbol{x}|/r_0)/|\boldsymbol{x}| ,$$

with the screening length $r_0$ and $k$ the dielectric constant. Computing $\tau_0$ in the first order perturbation we obtain the current as

$$J_0 = (4e^2E/3\pi n_I)(k/4\pi e^2)^2(\mu^4/v^2)$$
$$\times\ [\ln(1 + \beta) - \beta/(1 + \beta)]^{-1}(\#\,\text{deg.pt.}) . \qquad (24)$$

with $\beta = 2\pi kv/e^2$ (# deg.pt.) and $n_I$ the density of the impurity.

Next we compute the magneto-conductivity when $H$, parallel to $E$, is so strong that only the lowest states $n = 0$, $\sigma_3 = -1$ with the dispersion law $\epsilon = vP_z$ or $\epsilon = -vP_z$ near the RH and LH degeneracy point are filled. The ABJ anomaly effect will cause to move the electrons in the momentum space from the lowest Landau level ($n = 0$, $\sigma_3 = -1$) at one degeneracy point in the LH cone to the corresponding one ($n = 0$, $\sigma_3 = -1$) at the other one in the RH cone. Thus there will appear a deviation from the thermodynamical equilibrium that can be expressed by different chemical potentials for the electrons at the RH degeneracy point, $\mu_R$, and at the LH one, $\mu_L$. If one had calculated the relaxation time in the approximation where only one degeneracy point at a time was relevant – such as we did above in the $H = 0$ case – we would have found $1/\tau = 0$. This comes out of such a calculation due to the energy conservation factor $\delta(\epsilon - \epsilon') = (1/v)\delta(P_z - P_z')$ contained in $W(P_z \to P_z')$ which makes (23) give $1/\tau = 0$. However of course we cannot neglect scattering processes involging two degeneracy points.

The mechanism for the electric current with both

$E$ and $H$ on is a peculiar one different from one with a negligibly weak $H$. In the presence of strong $H$ the lattice analogy of the ABJ anomaly takes place and the transfer of the particles from the LH degeneracy point to the RH one acts as a drift term, $\dot{N}|_{\text{drift}}$, in the Boltzmann equation. On the other hand for negligible $H$ each degeneracy points acts independently. By the ABJ anomaly, the Fermi energy level $\mu_R$ in the RH cone goes up compared to that of the $H = 0$ case, $\mu$, and $\mu_L$ in the LH cone is lowered. See fig. 2: and fig. 6. In order that the system is in the steady state the excess electrons by the ABJ anomaly in the RH cone must be scattered back to another state. But they can not be scattered back into the states in the same cone because, as was explained above, $\tau = \infty$, when we neglect scattering from one degeneracy point to another. Therefore they must transit into the states in another cone, from the RH cone into the LH cone. We call this the intercone scattering and denote the corresponding relaxation time by $\tau_I$. If the intercone transition probability $W(P_z \to P_z')$ from the RH cone into the LH cone is calculated then the collision term is given by

$$\dot{N}_R|_{\text{coll}} = \frac{2}{L} \sum_{P_z} [f(P_z) - f_0(P_z)] \frac{1}{L} \sum_{P_z'} W(P_z \to P_z')$$

$$\equiv -\tau_I^{-1}(N_R - N_R^0) .$$

Here $N_R$ and $N_R^0$ denote the total electron numbers in the RH cone above the degeneracy energy in the $H \neq 0$ and $H = 0$ cases, respectively. Thus

$$\frac{1}{\tau_I} = \frac{2eH}{(2\pi)^2} \frac{1}{L} \sum_{P_z'} W(P_z \to P_z') . \tag{25}$$

The generation of a current associated with the ABJ anomaly can be shown by the following energy conservation argument. The ABJ anomaly indicates that the electrons are transferred from the LH cone into the RH cone by the rate of $e^2EH/(2\pi)^2$ per unit time and unit volume from eqs. (11) and (12). Notice that the dispersion law is continuous and the RH and LH cones are connected smoothly by it as shown in fig. 6. Since the Fermi level energies are $\mu_R > \mu_L$, the transfer costs the energy $[e^2EH/(2\pi)^2](\mu_R - \mu_L)$. This energy must be taken from the $E$ field by the presence of a current $J_A$ determined by the energy balance

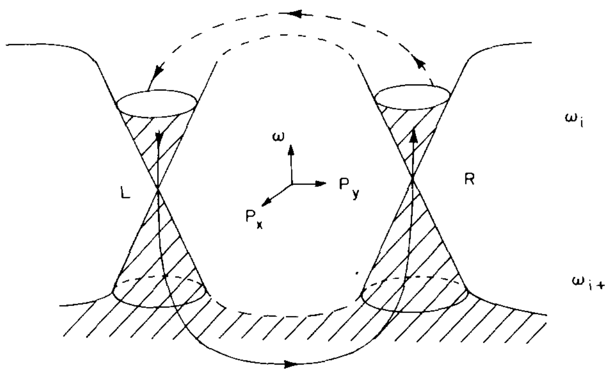


Fig. 6. The electrons in the cone near L in the level $\omega_i$ are transferred to R as shown by the solid line arrow. The exceeding electrons in R are scattered back to L as indicated by the dashed line arrow.

$$EJ_A = [e^2EH/(2\pi)^2](\mu_R - \mu_L) .$$

At the zero temperature, in the RH cone $f_0(\epsilon) = \theta(\mu_R - \epsilon)$ and then

$$N_R = \frac{1}{L^3} \sum_{P_y, P_z} f_0(\epsilon) = [eH/(2\pi)^2] \mu_R / v$$

$$\approx N_R^0 + (\mu_R - \mu)\partial N_R/\partial\mu .$$

Inserting this into the Boltzmann equation

$$\dot{N}_R|_{\text{drift}} = -\dot{N}_R|_{\text{coll}} ,$$

we obtain

$$\mu_R - \mu_L = evE\tau_I .$$

Thus

$$J_A = [eve^2EH/(2\pi)^2]\, \tau_I\, (\#\,\text{deg.pt.}) . \tag{26}$$

Here the subscript A stands for the anomalous current, the one associated with the analogue of the ABJ anomaly. In the definition of $\tau_I$ (25) we may approximate $W(P_z \to P_z') \simeq W(P \to P')$, then

$$W(P_z \to P_z') \simeq (4\pi e^2/k)^2 n_I [(P - P')^2 + 1/r_H^2]^{-2}$$

$$\times\ 2\pi\delta(\epsilon - \epsilon') , \tag{27}$$

with $1/r_H^2 = (EH/kv)$ (# deg.pt.). According to eq. (16) $P - P' = P_d - P_d' + \hat{P} - \hat{P}'$, where $\hat{P}$ and $\hat{P}'$ are oscillating around $P_d$ and $P_d'$ with the order of $(eH)^{1/2}$. We may ignore the oscillatory part $(\hat{P} - \hat{P}')$ and the $1/r_H^2$ term in the denominator of (27) when compared to the distance of the RH and LH degeneracy points $P_d - P'_d$. In this approximation we obtain

$$J_A = (e^2 v^2 E/2\pi n_1)(k/4\pi e^2)^2 (P_d - P'_d)^4 (\#\,\mathrm{deg.pt.}) . \tag{28}$$

From eqs. (28) and (24) we obtain the ratio of the conductivity that is defined by $J = \sigma E$, as

$$\sigma_A/\sigma_0 = \tfrac{3}{16}(v/\mu)^4 [\ln(1+\beta) - \beta/(1+\beta)] \times (P_d - P'_d)^4 . \tag{29}$$

With (27) substituted in (25) for the intercone relaxation time $\tau_I$ the electrons must travel a long "distance", $(P_d - P'_d)^4$, in momentum space and thus $\tau_I$ is expected to be a large value compared to $\tau_0$ for $H = 0$. Therefore $\sigma_A/\sigma_0$ given by (29) is large.

To end the present paper we suggest possible candidates for the zero-gap semiconductors for our purpose of simulating the relativistic Weyl equation. The type of compounds with a zinc-blend structure [9, 16] which do not possess parity symmetry such as $Hg_{1-x}Cd_xTe$ or $Pb_{1-x}Sn_xTe$ with an appropriate range of composition $x$ may have the generic type of Weyl degeneracy points.

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