12

Generating Functionals and the Effective Potential

12.1 Connected, Disconnected and Irreducible Propagators

In this chapter we return to the structure of perturbation theory in a canonical local field theory that we discussed in Chapter 11. The results that we will derive here for the simpler case of a scalar field theory apply, with some changes, to any local field theory, relativistic or not.

Let us suppose that we want to compute the four-point function in ϕ^4 theory of a scalar field, $G_4(x_1, x_2, x_3, x_4)$. Obviously, there is a set of graphs in which the four-point function is reduced to products of two-point functions

$$\begin{aligned} G_4(x_1, x_2, x_3, x_4) =& \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \\ =& G_2(x_1, x_2)G_2(x_3, x_4) + \text{permutations} + \text{other terms} \end{aligned} \tag{12.1}$$

An example of such diagrams is shown in Fig.12.1. This graph is linked (i.e. it has no vacuum part), but it is disconnected since we can split the graph into two pieces by drawing a line without cutting any propagator line.

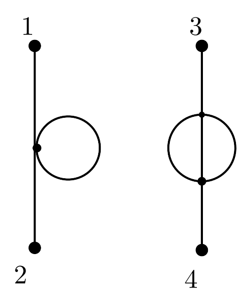


Figure 12.1 A factorized contribution to the four-point function.

On the other hand, as we already saw, the N-point function can be computed from the generating functional $Z[J]$ by functional differentiation with respect to the sources $J(x)$, i.e.

$$G_N(x_1,\dots,x_N) = \frac{1}{Z[J]}\frac{\delta^N Z[J]}{\delta J(x_1)\dots\delta J(x_N)}\bigg|_{J=0} \tag{12.2}$$

Let us now compute instead the following expression

$$G_N^c(x_1,\dots,x_N) = \frac{\delta^N \ln Z[J]}{\delta J(x_1)\dots\delta J(x_N)}\bigg|_{J=0} \tag{12.3}$$

We will now see that $G_N^{(c)}(x_1,\dots,x_N)$ is a an N-point function which contains only *connected* Feynman diagrams.

As an example let us consider first the two-point function $G_2^c(x_1,x_2)$, which is formally given by the expression

$$G_2^c(x_1,x_2) = \frac{\delta^2 \ln Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} \tag{12.4}$$

$$= \frac{\delta}{\delta J(x_1)}\frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x_2)}\bigg|_{J=0} \tag{12.5}$$

$$= \frac{1}{Z[J]}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} - \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x_1)}\bigg|_{J=0}\frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x_2)}\bigg|_{J=0} \tag{12.6}$$

Thus, we find that the connected two-point function can be expressed in terms of the two-point function and the one-point functions,

$$G_2^c(x_1,x_2) = G_2(x_1,x_2) - G_1(x_1)G_1(x_2) \tag{12.7}$$

We can express this result equivalently in the form

$$\langle\phi(x_1)\phi(x_2)\rangle_c = \langle\phi(x_1)\phi(x_2)\rangle - \langle\phi(x_1)\rangle\langle\phi(x_2)\rangle \tag{12.8}$$

and the quantity

$$G_2^c(x_1,x_2) = \langle\phi(x_1)\phi(x_2)\rangle_c = \langle[\phi(x_1)-\langle\phi(x_1)\rangle][\phi(x_2)-\langle\phi(x_2)\rangle]\rangle \tag{12.9}$$

is called the *connected* two-point function. Hence, it is the two point function of the field $\phi(x)-\langle\phi(x)\rangle$ that has been normal-ordered with respect to the *true vacuum*. It is straightforward to show that the same identification holds for the connected N-point functions.

The generating functional of the *connected* N-point functions $F[J]$,

$$F[J] = \ln Z[J] \tag{12.10}$$

is identified with the 'free energy' (or *vacuum energy*) of the system. The connected N-point functions are obtained from the free energy by

$$G_N^c(x_1, \ldots, x_N) = \frac{\delta^N F[J]}{\delta J(x_1) \ldots \delta J(x_N)}\Big|_{J=0} \tag{12.11}$$

Notice in passing that the connected N-point functions play a role analogous to the cumulants (or moments) of a probability distribution.

Let us recall that in the theory of phase transitions the source $J(x)$ plays the role of the symmetry-breaking field $H(x)$ that breaks the global symmetry of the scalar field theory, i.e. the external magnetic field in the Landau theory of magnetism. For an uniform external field $H(x) = H$, one finds

$$\frac{\delta F}{\delta J} = \frac{dF}{dH} = \int d^d x \langle \phi(x) \rangle = V \langle \phi \rangle \tag{12.12}$$

In terms of the free energy density $f = \frac{F}{V}$, we can write

$$\frac{df}{dH} = \langle \phi \rangle = m \tag{12.13}$$

where m is the magnetization density. Similarly, the magnetic susceptibility χ,

$$\chi = \frac{dm}{dH} = \frac{d^2 f}{dH^2} \tag{12.14}$$

is given by an integral of the two-point function,

$$\begin{aligned} \chi =& \frac{1}{V} \frac{d}{dH} \Big\langle \int d^d x_1 \phi(x_1) \Big\rangle \\ =& \frac{1}{V} \Big\langle \int d^d x_1 \int d^d x_2 \phi(x_1) \phi(x_2) \Big\rangle - \frac{1}{V} \Big\langle \int d^d x_1 \phi(x_1) \Big\rangle \Big\langle \int d^d x_2 \phi(x_2) \Big\rangle \end{aligned} \tag{12.15}$$

Using the fact that in a translation invariant system the expectation value of the field is constant, and that the two-point function is only a function of distance, we find

$$\begin{aligned} \chi =& \frac{1}{V} \int d^d x_1 \int d^d x_2 \; G_2(x_1, x_2) - V \langle \phi \rangle^2 \\ =& \int d^d y \; G_2(|\mathbf{y}|) - V \langle \phi \rangle^2 \\ =& \int d^d y \; G_2^c(|\mathbf{y}|) = \lim_{\boldsymbol{k} \to 0} G_2^c(\boldsymbol{k}) \end{aligned} \tag{12.16}$$

Hence, the magnetic susceptibility is the integral of the connected two-point function, which is also known as the correlation function.

12.2 Vertex Functions

So far we have been able to reduce the number of diagrams to be considered by:

1. showing that vacuum parts do not contribute to $G_N(x_1,\ldots,x_N)$,
2. showing that disconnected parts need not be considered by working instead with the connected N point function, $G^c_N(x_1,\ldots,x_N)$,

There is still another set of graphs that can be handled easily. Consider the second order contribution to the connected two-point function $G^c_2(x_1,x_2)$ shown in Fig.12.2. The explicit form of this contribution is, in momentum

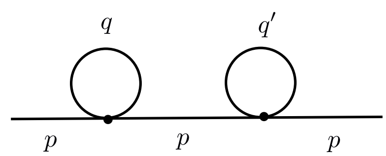


Figure 12.2 A reducible contribution to the two-point function

space, given by the following expression

$$\left(-\frac{\lambda}{4!}\right)^2\frac{1}{2!}(4\times 3)\cdot(4\times 3)\left(G_0(p)\right)^3\int\frac{d^dq}{(2\pi)^d}G_0(q)\int\frac{d^dq'}{(2\pi)^d}G_0(q') \tag{12.17}$$

We should note two features of this contribution. One is that the momentum in the middle propagator line is the same as the momentum p of the external line. This follows from momentum conservation. The other is that this graph can be split in two by a line that cuts either the middle propagator line or any of the two external propagator lines. A graph that can be split into two disjoint parts by cutting single propagator line is said to be one-particle reducible. No matter how complicated is the graph, that line must have the same momentum as the momentum on an incoming leg (again, by momentum conservation).

In general we need to do a sum of diagrams with the structure shown in Fig.12.3, which represents the expression

$$G_0^4(p)(\Sigma(p))^3 \tag{12.18}$$

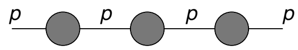


Figure 12.3 Three blobs

and where the 'blobs' of Fig.12.3 represent the self-energy $\Sigma(p)$, i.e. the sum of one-particle irreducible diagrams of the two-point function. In fact, we can do this sum to *all orders* and obtain

$$\begin{aligned} G_2(p) =& G_0(p) + G_0(p)\Sigma(p)G_0(p) + G_0^3(p)(\Sigma(p))^2 + \dots \\ =& G_0(p)\sum_{n=0}^{\infty}(\Sigma(p)G_0(p))^n \\ =& \frac{G_0(p)}{1-\Sigma(p)G_0(p)} \end{aligned} \tag{12.19}$$

We can write this result in the equivalent form

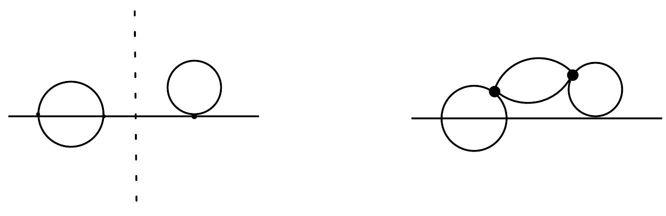


Figure 12.4 One-particle reducible and one-particle irreducible diagrams

$$G_2^{-1}(p) = G_0^{-1}(p) - \Sigma(p) \tag{12.20}$$

Armed with this result, we can express the relation between the bare and the full two-point function as the *Dyson Equation*

$$G_2(p) = G_0(p) + G_0(p)\Sigma(p)G_2(p) \tag{12.21}$$

where, as before, $\Sigma(p)$ represents the set of all possible connected, one-particle irreducible graphs with their external legs amputated.

The one-particle irreducible two-point function $\Sigma(p)$ is known as the mass operator or as the self-energy (or two-point vertex). Why? In the limit $p \to 0$ the inverse bare propagator reduces to the bare mass (squared)

$$G_0^{-1}(0) = m_0^2 \tag{12.22}$$

Similarly, also in the zero momentum limit, the inverse full propagator takes the value of the *effective* (or *renormalized*) mass (squared)

$$G_2^{-1}(0) = m_0^2 - \Sigma(0) = m^2 \tag{12.23}$$

Thus, $\Sigma(0)$ represents a renormalization of the mass.

12.2.1 General Vertex Functions

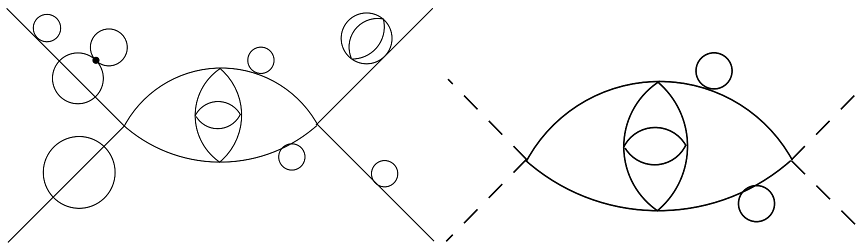


Figure 12.5 A contribution a) to a 1-Particle Reducible vertex function, and b) to a 1-Particle irreducible vertex function

We will now extend the concept of the sum of one-particle irreducible diagrams to a general N-point function, the *vertex functions*. To this end, we will need to find a suitable a generating functional for these correlators.

In previous chapters we considered the "free energy" $F[J]$ and showed that it is the generating functional of the connected N-point functions. $F[J]$ is a function of the external sources J. However, in many cases, this is inconvenient since, in systems that exhibit *spontaneous symmetry breaking*, as $J \to 0$ we may still have $\langle\phi\rangle \neq 0$. Thus, it will be desirable to have a quantity which is a functional of the *expectation values* of the observables instead of the sources. Thus, we will seek instead a functional that is a functional of the expectation values instead of the external sources. We will find this functional by means of a Legendre transformation from the sources J to the expectation values $\langle\phi\rangle$. This procedure is closely analogous to the relation in thermodynamics between the free Helmholtz free energy and the Gibbs free energy.

The local expectation value of the field, $\langle\phi(x)\rangle \equiv \bar{\phi}(x)$, is related to the functional $F[J]$ by

$$\langle\phi(x)\rangle = \frac{\delta F}{\delta J(x)} \tag{12.24}$$

The Legendre transform of $F[J]$, denoted by $\Gamma\left[\bar{\phi}\right]$, is defined by

$$\Gamma[\bar{\phi}] = \int d^d x \, \bar{\phi}(x) J(x) - F[J] \tag{12.25}$$

where, for simplicity, we have omitted all other indices, e.g. components of the scalar field, etc. What we do below can be easily extended to theories with other types of fields and symmetries.

Let us now compute the functional derivative of the generating functional $\Gamma[\bar{\phi}]$ with respect to $\bar{\phi}(x)$. After some simple algebra we find

$$\begin{aligned}\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(x)} =& \int d^d y \, J(y)\delta(y-x) + \int d^d y \, \bar{\phi}(y) \frac{\delta J(y)}{\delta\bar{\phi}(x)} - \int d^d y \, \frac{\delta F}{\delta J(y)} \frac{\delta J(y)}{\delta\bar{\phi}(x)} \\ =& J(x) + \int d^d y \, \bar{\phi}(y) \frac{\delta J(y)}{\delta\bar{\phi}(x)} - \int d^d y \, \bar{\phi}(y) \frac{\delta J(y)}{\delta\bar{\phi}(x)}\end{aligned} \tag{12.26}$$

Since the last two terms of the right hand side cancel each other, we find

$$\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(x)} = J(x) \tag{12.27}$$

However, we can consider a theory in which, even in the limit $J \to 0$, still $\frac{\delta F}{\delta J(x)}|_{J=0} = \bar{\phi}(x)$ may be non-zero. On symmetry grounds one expects that is the source vanishes then the expectation value of the field should also vanish. However, there many situations in which the expectation value of the field does not vanish in the limit of a vanishing source. In this case we say that a symmetry is *spontaneously broken* if $\bar{\phi}(x) \neq 0$ as $J \to 0$. An example is a magnet where ϕ is local magnetization and J is the external magnetic field. Another example is in a theory of Dirac fermions the bilinear $\bar{\phi}\psi$ is the order parameter for chiral symmetry-breaking and the fermion mass is the symmetry-breaking field.

Returning to the general case, since the expectation value $\bar{\phi}(x)$ satisfies the condition $\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(x)} = 0$, this state is an extremum of the potential Γ. Naturally, for the state to be stable, it must also be a minimum, not just an extremum. The value of $\bar{\phi}(x)$ is known as the *classical field*.

The functional $\Gamma[\bar{\phi}]$ can be formally expanded in a Taylor series expansion of the form

$$\Gamma[\bar{\phi}] = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{z_1,\ldots,z_N} \Gamma^{(N)}(z_1,\ldots,z_N)\bar{\phi}(z_1)\ldots\bar{\phi}(z_N) \tag{12.28}$$

The coefficients

$$\Gamma^{(N)}(z_1,\ldots,z_N) = \frac{\delta\Gamma[\phi]}{\delta\bar{\phi}(z_1)\ldots\delta\bar{\phi}(z_N)} \tag{12.29}$$

are the N-point vertex functions.

In order to find relations between the vertex functions and the connected functions we differentiate the classical field $\bar{\phi}(x)$ by $\bar{\phi}(y)$ and find

$$\begin{aligned}\delta(x-y) = & \frac{\delta^2 F}{\delta J(x)\delta\bar{\phi}(y)} = \int_z \frac{\delta^2 F}{\delta J(x)\delta J(z)}\frac{\delta J(z)}{\delta\bar{\phi}(y)} \\ = & \int_z \frac{\delta^2 F}{\delta J(x)\delta J(z)}\frac{\delta^2 \Gamma}{\delta\bar{\phi}(z)\delta\bar{\phi}(y)} \end{aligned} \tag{12.30}$$

Since the connected two-point function $G_2^c(x-z)$ is given by

$$G_2^c(x-z) = \frac{\delta^2 F}{\delta J(x)\delta J(z)}\Big|_{J=0} \tag{12.31}$$

we see that the operator

$$\Gamma^{(2)}(x-y) = \frac{\delta^2 \Gamma}{\delta\bar{\phi}(x)\bar{\phi}(y)}\Big|_{J=0} \tag{12.32}$$

is the inverse of $G_2^c(x-y)$ (as an operator).

We can gain further insight by passing to momentum space where we find

$$\Gamma^{(2)}(p) = \left[G_2^c(p)\right]^{-1} = p^2 + m_0^2 - \Sigma(p) \tag{12.33}$$

Thus, $\Gamma^{(2)}(p)$ is essentially the negative of the self-energy and it is the sum of all the 1PI graphs of the two-point function.

To find relations of this type for more general N-point functions we will proceed to differentiate Eq.(12.30) by $J(u)$ to obtain

$$\begin{aligned}\frac{\delta}{\delta J(u)}\delta(x-y) = & 0 \\ = & \int_z \Big[\frac{\delta^3 F}{\delta J(x)\delta J(z)\delta J(u)}\frac{\delta J(y)}{\delta\bar{\phi}(y)} + \frac{\delta^2 F}{\delta J(x)\delta J(z)}\frac{\delta^2 J(z)}{\delta J(u)\delta\bar{\phi}(y)}\Big]\end{aligned} \tag{12.34}$$

But, since

$$\begin{aligned}\frac{\delta^2 J(z)}{\delta J(u)\delta\bar{\phi}(y)} = & \int_w \frac{\delta^3 \Gamma}{\delta\bar{\phi}(w)\delta\bar{\phi}(z)\delta\bar{\phi}(y)}\frac{\delta\bar{\phi}(w)}{\delta J(u)} \\ = & \int_w \frac{\delta^3 \Gamma}{\delta\bar{\phi}(w)\delta\bar{\phi}(z)\delta\bar{\phi}(y)}\frac{\delta^2 F}{\delta J(u)\delta J(w)}\end{aligned} \tag{12.35}$$

So, we get

$$0 = \int_z G_3^c(x,z,y)\Gamma^{(2)}(z-y) + \int_{z,w} G_2^c(x-z)G_2^c(u-w)\Gamma^{(3)}(w,z,y) \tag{12.36}$$

where $\Gamma^{(2)} = [G_2]^{-1}$. Hence, we find the expression for the three-point function at points x_1, x_2 and x_3

$$G_3^c(x_1, x_2, x_3) = -G_2^c(x_1, y_1)G_2^c(x_2, y_2)G_2^c(x_3, y_3)\Gamma^{(3)}(y_1, y_2, y_3) \quad (12.37)$$

where repeated labels are integrated over.

Notice, in passing, that we can write the two-point function in a similar fashion

$$G_2^c(x_1, x_2) = G_2^c(x_1, y_1)G_2^c(x_2, y_2)\Gamma^{(2)}(y_1, y_2) \quad (12.38)$$

(again with repeated labels being integrated over) since $G_2^c = [\Gamma^{(2)}]^{-1}$.

Thus, $\Gamma^{(3)}$ is the 1PI 3-point vertex function. Eq.(12.37) has the pictorial representation shown in Fig.12.6, where the blob is the three-point vertex function and the sticks are connected two-point functions.

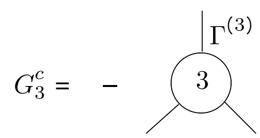


Figure 12.6 The three-point vertex function.

If we now further differentiate Eq.(12.34) with respect to additional fields $\bar{\phi}$ we obtain relations between the four-point function (and the lower point functions) shown pictorially in Fig.12.7. Here the blobs are four and three-point vertex functions and the sticks are, again, connected two-point functions.

This procedure generalizes to the higher point functions. In Fig. 12.8 and Fig.12.9 we present the pictorial representation for the connected five and six-point functions in terms of the corresponding vertex functions and connected two-point functions. In Fig.12.9 the symbol (*) means that the respective diagram is one-particle reducible by a body cut. Also, in each diagram, the summation over all possible equivalent combinations is implied. Clearly, a graph may be reducible either by a cut of only an external line or via a body cut.

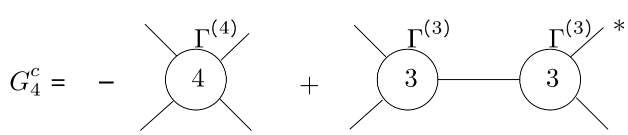


Figure 12.7 The four-point vertex function

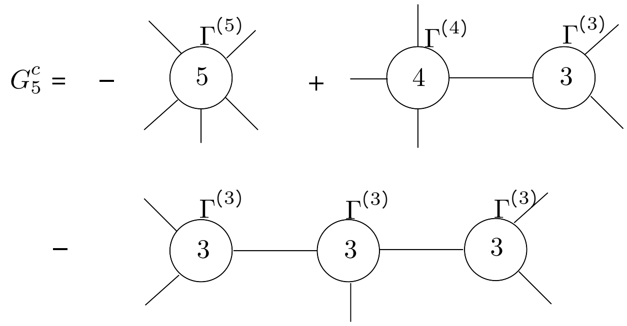


Figure 12.8 The five-point vertex function.

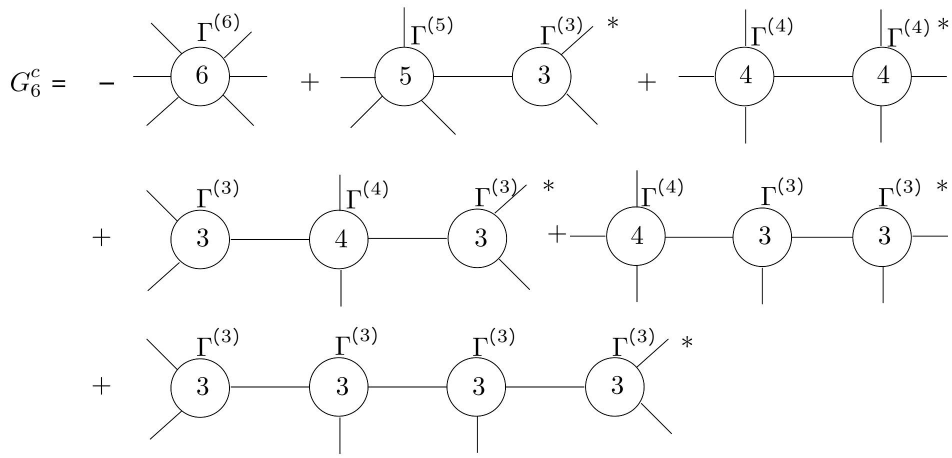


Figure 12.9 The six-point vertex function.

In general, from the definition of the vertex function $\Gamma^{(N)}$

$$\Gamma^{(N)}(1,\dots,N)=\frac{\delta^N\Gamma(\bar{\phi})}{\delta\bar{\phi}(1)\dots\delta\bar{\phi}(N)}\Big|_{J=0} \tag{12.39}$$

we find that the connected N-point function, for $N>2$, is related to the vertex function by (repeated labels are again integrated over)

$$G_N^c(1,\dots,N)=-G_2^c(1,1')\dots G_2^c(N,N')\Gamma^{(N)}(1',\dots,N')+Q^{(N)}(1,\dots,N) \tag{12.40}$$

where the first term is one-particle reducible only via cuts of the external legs and the second by body cuts. Notice that for the r-point function in a ϕ^r theory this second term vanishes.

In momentum space these expressions become simpler. Thus, the connected two-point function obeys

$$G_2^c(k_1, k_2) = (2\pi)^d \delta^d(\mathbf{k}_1 + \mathbf{k}_2) G_2^c(k_1) \tag{12.41}$$

By using Eq.(12.33), we can write the two-point vertex function as

$$\Gamma^{(2)}(k_1, k_2) = (2\pi)^d \delta^d(\mathbf{k}_1 + \mathbf{k}_2) \Gamma^{(2)}(k_1) \tag{12.42}$$

In the general case, $N > 2$, we have

$$G_N^c(k_1, \dots, k_N) = -G_2^c(k_1) \dots G_2^c(k_N) \Gamma^{(N)}(k_1, \dots, k_N) + Q^{(N)}(k_1, \dots, k_N) \tag{12.43}$$

In what follows we will focus our attention on the vertex functions.

12.3 The Effective Potential and Spontaneous Symmetry Breaking

Let $v = \bar{\phi} = \langle \phi \rangle$. Then, with the above definition for the vertex functions $\Gamma^{(N)}$ we may write the generating function $\Gamma[\bar{\phi}]$ as a power series expansion of the form

$$\Gamma[\bar{\phi}] = \\ = \sum_{N=1}^{\infty} \frac{1}{N!} \int d^d x_1 \dots \int d^d x_N \Gamma^{(N)}(x_1, \dots, x_N | v) [\bar{\phi}(x_1) - v] \dots [\bar{\phi}(x_N) - v] \tag{12.44}$$

If $J \to 0$, then the sum starts at $N = 2$. Here $v = \lim_{J \to 0} \bar{\phi}$ which is a local minimum of $\Gamma[\bar{\phi}]$ since

$$\left. \frac{\delta \Gamma}{\delta \bar{\phi}} \right|_{\bar{\phi}=v} = J \mapsto 0, \quad \text{and} \qquad \left. \Gamma^{(2)} \right|_{\bar{\phi}=v} \geq 0 \tag{12.45}$$

In the symmetric phase of the theory the generating function $\Gamma[\bar{\phi}]$ has the form

$$\Gamma[\bar{\phi}] = \sum_{N=1}^{\infty} \frac{1}{N!} \int d^d x_1 \dots \int d^d x_N \Gamma^{(N)}(x_1, \dots, x_N) \bar{\phi}(x_1) \dots \bar{\phi}(x_N) \tag{12.46}$$

The classical field $\bar{\phi} = v$ is defined by the condition $\frac{\delta \Gamma}{\delta \bar{\phi}} = 0$. If $\bar{\phi} \neq 0$ then the global symmetry $\phi \leftrightarrow -\phi$ is spontaneously broken. Moreover, for $\langle \phi \rangle = \bar{\phi} = \text{const.}$, the generating functional Γ becomes

$$\Gamma[\bar{\phi}] = \sum_{N=2}^{\infty} \frac{1}{N!} \left[\int d^d x_1 \dots \int d^d x_N \Gamma^{(N)}(x_1, \dots, x_N) \right] \bar{\phi}^N \tag{12.47}$$

The Fourier transform of $\Gamma^{(N)}(x_1, \ldots, x_N)$ is

$$\Gamma^{(N)}(x_1, \ldots, x_N) = \int \frac{d^d k_1}{(2\pi)^d} \cdots \int \frac{d^d k_N}{(2\pi)^d} \Gamma^{(N)}(k_1, \ldots, k_N) e^{-i\mathbf{k}_j \cdot \mathbf{x}_j} \quad (12.48)$$

Momentum conservation requires that $\Gamma^{(N)}(k_1, \ldots, k_N)$ should take the form

$$\Gamma^{(N)}(k_1, \ldots, k_N) = (2\pi)^d \delta^d(\sum_j \mathbf{k}_j) \tilde{\Gamma}^{(N)}(k_1, \ldots, k_N) \quad (12.49)$$

So we find that $\Gamma(\bar{\phi})$ is given by the expression

$$\Gamma(\bar{\phi}) = V \sum_{N=2}^{\infty} \frac{1}{N!} \tilde{\Gamma}^{(N)}(0, \ldots, 0)\, \bar{\phi}^N \quad (12.50)$$

where V is the volume of Euclidean space-time. Clearly we can also write $\Gamma(\bar{\phi}) = VU(\bar{\phi})$ where

$$U(\bar{\phi}) = \sum_{N=2}^{\infty} \frac{1}{N!} \tilde{\Gamma}^{(N)}(0, \ldots, 0)\, \bar{\phi}^N \quad (12.51)$$

is the *effective potential*.

Note that the $\tilde{\Gamma}^{(N)}(0, \ldots, 0)$'s are computed in the *symmetric theory*. In this framework, if U has a minimum at $\bar{\phi} \neq 0$ for $J = 0$, we will conclude that the vacuum state (i.e. the ground state) is not invariant under the global symmetry of the theory: we have a *spontaneously broken global symmetry* (or, spontaneous symmetry breaking). If we identify $J(x) \equiv H$ with the external physical field, then it follows from Eq.(12.27) that $\frac{dU}{d\bar{\phi}} = H$. From this relation the equation of state follows:

$$H = \sum_{N=1}^{\infty} \frac{1}{N!} \tilde{\Gamma}^{(N+1)}(0, \ldots, 0)\, \bar{\phi}^N \quad (12.52)$$

These results provides the following strategy. We one computes the effective potential and from it the vacuum (ground state). Next one computes the full vertex functions, either in the symmetric or broken symmetry state, by identifying in $\Gamma[\bar{\phi}]$ the coefficients of the products $\prod_i (\bar{\phi}(x_i) - v)$, where v is the classical field that minimizes the effective potential $U(\bar{\phi})$, i.e.

$$\Gamma^{(N)}(1, \ldots, N | v) = \frac{\delta^N \Gamma[\bar{\phi}]}{\delta\bar{\phi}(1) \ldots \delta\bar{\phi}(N)}\Big|_{\bar{\phi}=v} \quad (12.53)$$

12.4 Ward Identities

Let us now discuss the consequences of the existence of a continuous global symmetry G. We will begin with a discussion of the simpler case in which the symmetry group is $G = O(2) \simeq U(1)$. Let us consider the case of a two-component real scalar field $\boldsymbol{\phi}(x) = (\phi_\pi(x), \phi_\sigma(x))$ whose Euclidean Lagrangian is

$$\mathcal{L}(\boldsymbol{\phi}) = \frac{1}{2}\left[(\partial\boldsymbol{\phi})^2 + m_0^2\boldsymbol{\phi}^2\right] + \frac{\lambda}{4!}(\boldsymbol{\phi}^2)^2 + \boldsymbol{J}(x)\cdot\boldsymbol{\phi}(x) \tag{12.54}$$

where $\boldsymbol{J}(x)$ are a set of sources. In the absence of such sources, $\boldsymbol{J} = 0$, Lagrangian $\mathcal{L}$ is invariant under global $O(2)$ transformations

$$\boldsymbol{\phi}' = \exp(i\theta\sigma_2)\boldsymbol{\phi} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\boldsymbol{\phi} \equiv T\boldsymbol{\phi} \tag{12.55}$$

For an infinitesimal angle θ we can approximate

$$T = I + \epsilon\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \dots \tag{12.56}$$

The partition function $Z[\boldsymbol{J}]$

$$Z[\boldsymbol{J}] = \int \mathcal{D}\boldsymbol{\phi}\exp(-\int d^dx\mathcal{L}[\boldsymbol{\phi}, \boldsymbol{J}]) \tag{12.57}$$

is invariant under the global symmetry $\boldsymbol{\phi}' = T\boldsymbol{\phi}$ if the sources transform accordingly, $\boldsymbol{J}' = T\boldsymbol{J}$. Then, $\boldsymbol{J}(x)\cdot\boldsymbol{\phi}(x)$ is also invariant. Provided the integration measure of the path-integral is invariant, $\mathcal{D}\boldsymbol{\phi} = \mathcal{D}\boldsymbol{\phi}'$, it follows that

$$Z[\boldsymbol{J}'] = Z[\boldsymbol{J}] \tag{12.58}$$

Thus, the partition function $Z[J]$, the generating function of the connected correlators $F[\boldsymbol{J}]$, and the generating functional of the vertex (one-particle irreducible) functions $\Gamma[\boldsymbol{\phi}]$, are invariant under the action of the global symmetry.

In our discussion of classical field theory in section 3.1, we proved Noether's theorem which states that a system with a global continuous symmetry has a locally conserved current and a globally conserved charge. However this result only held at the classical level since the derivation required the use of the classical equations of motion. We will now show that in the full quantum theory the correlators of the fields obey a set of identities, known as *Ward identities*, that follow from the existence of a global continuous symmetry. Moreover, these identities will also allow us to find consequences which hold if the global continuous symmetry is spontaneously broken.

To derive the Ward identities will consider the action of infinitesimal global transformations on the generating functionals. For an infinitesimal transformation T the sources transform as

$$\boldsymbol{J}' = \boldsymbol{J} + \epsilon \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{J} \tag{12.59}$$

In terms of the components of the source, $\boldsymbol{J} = (J_\sigma, J_\pi)$, the infinitesimal transformation is

$$J'_\sigma = J_\sigma + \epsilon J_\pi \tag{12.60}$$

$$J'_\pi = J_\pi - \epsilon J_\sigma \tag{12.61}$$

Or, equivalently, $\delta J_\sigma = \epsilon J_\pi$ and $\delta J_\pi = -\epsilon J_\sigma$. Since the generating functional $F[\boldsymbol{J}]$ is invariant under the global symmetry, we find

$$\begin{aligned} \delta F &= \int d^d x \left[\frac{\delta F[\boldsymbol{J}]}{\delta J_\sigma(x)} \delta J_\sigma(x) + \frac{\delta F[\boldsymbol{J}]}{\delta J_\pi(x)} \delta J_\pi(x) \right] = 0 \\ &= \int d^d x \; \epsilon \left[\frac{\delta F[\boldsymbol{J}]}{\delta J_\sigma(x)} J_\pi(x) - \frac{\delta F[\boldsymbol{J}]}{\delta J_\pi(x)} J_\sigma(x) \right] = 0 \end{aligned} \tag{12.62}$$

which implies that

$$\int d^d x \left[\bar{\phi}_\sigma(x) J_\pi(x) - \bar{\phi}_\pi(x) J_\sigma(x) \right] = 0 \tag{12.63}$$

Therefore, the generating functional $\Gamma[\bar{\boldsymbol{\phi}}]$ satisfies the identity

$$\int d^d x \left[\bar{\phi}_\sigma(x) \frac{\delta \Gamma[\bar{\boldsymbol{\phi}}]}{\delta \bar{\phi}_\pi(x)} - \bar{\phi}_\pi(x) \frac{\delta \Gamma[\bar{\boldsymbol{\phi}}]}{\delta \bar{\phi}_\sigma(x)} \right] = 0 \tag{12.64}$$

The above equation, Eq.(12.64) is called *Ward Identity* for the generating functional $\Gamma[\boldsymbol{\phi}]$. It says that $\Gamma[\boldsymbol{\phi}]$ is invariant under the global transformation $\boldsymbol{\phi} \to T\boldsymbol{\phi}$. This identity is always valid (i.e. to all orders in perturbation theory).

We will now find several (many!) Ward identities which follow by differentiation of the Ward identity of Eq.(12.64). By differentiating Eq.(12.64) with respect to $\bar{\phi}_\pi(y)$ we find

$$0 = \int d^d x \left\{ \frac{\delta^2 \Gamma}{\delta \bar{\phi}_\pi(y) \delta \bar{\phi}_\pi(x)} \bar{\phi}_\sigma(x) - \frac{\delta^2 \Gamma}{\delta \bar{\phi}_\sigma(y) \delta \bar{\phi}_\pi(x)} \bar{\phi}_\pi(x) - \frac{\delta \Gamma}{\delta \bar{\phi}_\sigma(x)} \delta^d(x-y) \right\} \tag{12.65}$$

From this equation it follows that

$$\frac{\delta \Gamma}{\delta \bar{\phi}_\sigma(y)} = \int d^d x \left[\frac{\delta^2 \Gamma}{\delta \bar{\phi}_\pi(x) \delta \bar{\phi}_\pi(y)} \bar{\phi}_\sigma(x) - \frac{\delta^2 \Gamma}{\delta \bar{\phi}_\sigma(x) \delta \bar{\phi}_\pi(y)} \bar{\phi}_\pi(x) \right] \tag{12.66}$$

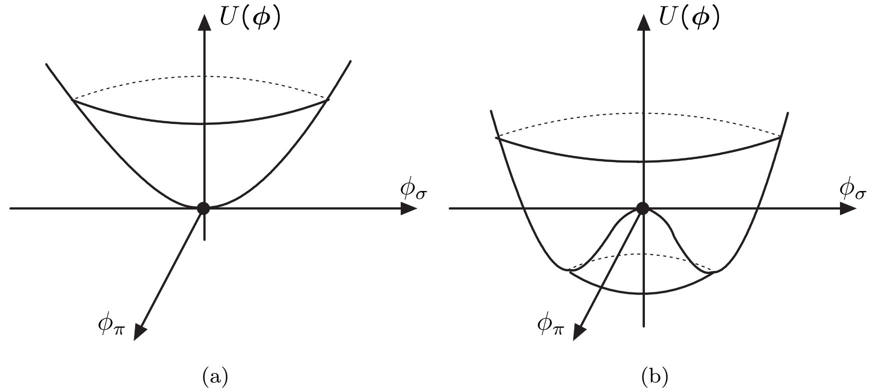


Figure 12.10 The effective potential with $O(2)$ symmetry a) in the symmetric phase and b) in the broken symmetry phase.

If the $O(2)$ symmetry is spontaneously broken, and the minimum of the effective potential $U(\boldsymbol{\phi})$ is on the circle shown in Fig. 12.10(b), say $\bar{\boldsymbol{\phi}} = (v, 0)$, then Eq.(12.66) becomes

$$v \int d^d x \frac{\delta^2 \Gamma}{\delta \bar{\phi}_\pi(x) \delta \bar{\phi}_\pi(y)} = H \tag{12.67}$$

where we denoted by H the uniform component of $\boldsymbol{J}$ along the direction of symmetry breaking, $J_\sigma = H$. Eq. (12.67) can be recast as

$$v \int d^d x \Gamma^{(2)}_{\pi\pi}(x - y) = H \tag{12.68}$$

or, equivalently,

$$\lim_{\boldsymbol{p}\to 0} v \tilde{\Gamma}^{(2)}_{\pi\pi}(\boldsymbol{p}) = H \tag{12.69}$$

So, if the $O(2)$ symmetry is broken spontaneously, then the vacuum expectation value is non zero $v \neq 0$ as the symmetry-breaking field is removed, $H \to 0$. Then, the 1-PI 2-point function of the *transverse* components vanishes at long wavelengths, $\lim_{\boldsymbol{p}\to 0} \tilde{\Gamma}^{(2)}_{\pi\pi}(\boldsymbol{p}) \to 0$, as $H \to 0$. Notice the important order of limits: first $\boldsymbol{p} \to 0$ and then $H \to 0$. Therefore, the connected transverse 2-point function $\widetilde{G}^c_{2,\pi\pi}(\boldsymbol{p})$ has a pole at zero momentum with zero energy in the *spontaneously* broken phase. In other terms, in the broken symmetry state, the transverse components of the field, ϕ_π, describe a massless excitation known as the *Goldstone boson*.

Conversely, in the symmetric phase, in which $v \to 0$ as $H \to 0$, we find instead that

$$\lim_{H\to 0}\lim_{\boldsymbol{p}\to 0}\tilde{\Gamma}^{(2)}_{\pi\pi}(\boldsymbol{p}) = \lim_{H\to 0}\frac{H}{v} \neq 0 \tag{12.70}$$

and the "transverse" modes are also massive. We will see shortly that in the symmetric phase all masses are equal (as they should be!). In fact, the limiting value of $\frac{H}{v}$ in the symmetric phase is just equal to the inverse susceptibility $\chi^{-1}_{\pi\pi}$.

Thus we conclude that there is an alternative: either (a) the theory is in the symmetric phase , i.e. $v = 0$, or (b) the symmetry is spontaneously broken, $v \neq 0$ with $J \to 0$, and there are massless excitations (*Goldstone bosons*). This is result, known as the Goldstone theorem, is actually generally valid in a broken symmetry state of a system with a global continuous symmetry.

Let us now consider the more general case of a global symmetry with group $O(N)$. The analysis for other Lie groups, e.g. $U(N)$, is similar. In the $O(N)$ case the group has $N(N-1)/2$ generators

$$(L_{ij})_{kl} = -i\left[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right] \tag{12.71}$$

where $i.j,k,l = 1,\dots,N$. Let us assume that the $O(N)$ symmetry is spontaneously broken along the direction $\boldsymbol{\phi} = (v,\boldsymbol{0})$, where $\boldsymbol{0}$ has $N-1$ components. More generally, we can write the field as $\bar{\boldsymbol{\phi}} = (\bar{\phi}_\sigma, \bar{\boldsymbol{\phi}}_\pi)$. In the broken symmetry state only the $\bar{\phi}_\sigma$ component has a non-vanishing expectation value. In thsi state there is a residual, unbroken, $O(N-1)$ symmetry of rotating the transverse components among each other. Thus, the symmetry which is actually broken, rather than $O(N)$, is the in the coset $O(N)/O(N-1)$ which is isomorphic to the N-dimensional sphere S_N.

Under the cation of $O(N)$, the field $\bar{\boldsymbol{\phi}}$ transforms as follows

$$\bar{\phi}'_a = \left(e^{i\vec{\lambda}\cdot\vec{L}}\right)_{ab}\bar{\phi}_b \tag{12.72}$$

where $\lambda_{ij} = -\lambda_{ji}$ (with $i,j = 1,\dots,N$) are the Euler angles of the S_N sphere. The broken generators (the generators that mix with the direction of the broken symmetry) are L_{i1}, and L_{ij} (with $i,j \neq 1$) are the generators of the unbroken $O(N-1)$ symmetry.

Let us perform an infinitesimal transformation away from the direction of symmetry breaking with the L_{i1} (with $i = 2,\dots,N$) generators

$$\delta\bar{\phi}_a = i\lambda_{i1}\,(L_{i1})_{ab}\,\bar{\phi}_b = \lambda_{i1}[\delta_{ia}\delta_{1b} - \delta_{ib}\delta_{1a}]\bar{\phi}_b = \lambda_{a1}\bar{\phi}_1 - \lambda_{b1}\delta_{1a}\bar{\phi}_b \tag{12.73}$$

or, what is the same,

$$\delta\bar{\phi}_\sigma = \lambda_{1b}\bar{\phi}_{\pi,b}, \qquad \delta\bar{\phi}_{\pi,a} = -\lambda_{1a}\bar{\phi}_\sigma \tag{12.74}$$

Likewise,

$$\delta J_\sigma = \lambda_{1b} J_{\pi_b}, \qquad \delta J_{\pi_a} = -\lambda_{1a} J_\sigma \tag{12.75}$$

Thus,

$$\begin{aligned}\delta F &= \int d^d x \left[\frac{\delta F}{\delta J_\sigma(x)} \delta J_\sigma(x) + \frac{\delta F}{\delta J_{\pi,a}(x)} \delta J_{\pi,a}(x) \right] \\ &= \int d^d x \lambda_{1a} \left[-\frac{\delta \Gamma}{\delta \bar{\phi}_\sigma(x)} \bar{\phi}_{\pi,a}(x) + \frac{\delta \Gamma}{\delta \bar{\phi}_{\pi,a}(x)} \bar{\phi}_\sigma(x) \right] = 0 \end{aligned} \tag{12.76}$$

Since the infinitesimal Euler angles λ_{1a} are arbitrary, we have a Ward identity for each component ($a = 2, \ldots, N$):

$$\int d^d x \left[\frac{\delta \Gamma}{\delta \bar{\phi}_\sigma(x)} \bar{\phi}_{\pi,a}(x) - \frac{\delta \Gamma}{\delta \bar{\phi}_{\pi,a}(x)} \bar{\phi}_\sigma(x) \right] = 0 \tag{12.77}$$

We will proceed as in the $O(2)$ theory and differentiate Eq.(12.76) with respect to a transverse field component, $\bar{\phi}_{\pi,b}$, to obtain

$$\int d^d x \left[\frac{\delta^2 \Gamma}{\delta \bar{\phi}_\sigma(x) \delta \bar{\phi}_{\pi,b}(y)} \bar{\phi}_{\pi,a}(x) + \frac{\delta \Gamma}{\delta \bar{\phi}_\sigma(x)} \delta_{ab} \delta(x-y) - \frac{\delta^2 \Gamma}{\delta \bar{\phi}_{\pi,a}(x) \delta \bar{\phi}_{\pi,b}(y)} \bar{\phi}_\sigma(x) \right] = 0 \tag{12.78}$$

Let us now assume that the $O(N)$ symmetry is spontaneously broken and that the field has the expectation value $\bar{\boldsymbol{\phi}} = (v, \mathbf{0})$. Then, Eq.(12.78) becomes

$$\delta_{ab} J_\sigma(y) = v \int d^d x \Gamma^{(2)}_{\pi_a, \pi_b}(x-y) = v \lim_{p \to 0} \Gamma^{(2)}_{\pi_a \pi_b}(p) \tag{12.79}$$

which requires that J_σ must be uniform. From this equation, the following conclusions can be made:

1. $\Gamma^{(2)}_{\pi_a \pi_b}(0)$ must be diagonal: $\Gamma^{(2)}_{\pi_a \pi_b}(0) \equiv \delta_{ab} \Gamma_{\pi\pi}(0)$. Hence the masses of the transverse components, $\bar{\phi}_a$, are equal, $m^2_{\pi_a} = m^2_{\pi_b}$, and we have a degenerate multiplet.
2. In the limit in which the source along the symmetry breaking direction is removed, $J_\sigma \to 0$, it must hold that $v \lim_{p \to} \Gamma^{(2)}_{\pi_a \pi_a}(p) = 0$. Thus, we find that there are two possible cases

2a. if $v \neq 0$, all transverse $\bar{\phi}_{\pi,a}$ excitations are massless, and there are $N-1$ massless excitations (Goldstone bosons).

2b. if $v = 0$, the theory is in the symmetric phase, all excitations are massive, and $m^2_\sigma = \Gamma^{(2)}_{\sigma\sigma}(0) \neq 0$.

These results are exact identities and, hence, are valid order by order in perturbation theory.

Following the same procedure, we can get, in fact, an infinite set of identities. For example, in the $O(2)$ theory (for simplicity), by differentiating Eq.(12.65) with respect to the field ϕ_σ we find the relation

$$\Gamma^{(2)}_{\sigma\sigma}(p) - \Gamma^{(2)}_{\pi\pi}(p) = v\, \Gamma^{(3)}_{\sigma\pi\pi}(0, p, -p) \tag{12.80}$$

In the symmetric phase, $v = 0$, Eq.(12.80) implies that the irreducible two-point functions for the ϕ_σ and the ϕ_π components must be equal: $\Gamma^{(2)}_{\sigma\sigma}(p) = \Gamma^{(2)}_{\pi\pi}(p)$, and that in particular the masses must be equal, $m_\sigma^2 = m_\pi^2$. On the other hand, in the broken symmetry phase, where $v \neq 0$, this equation and the requirement that the ϕ_π field must be a Goldstone boson (and hence massless) implies that the mass of the ϕ_σ field must be related to to the three-point function (with all momenta equal zero): $\Gamma^{(2)}_{\sigma\sigma}(0) = v\Gamma^{(3)}_{\sigma\pi\pi}(0,0,0)$. This result implies that in the broken symmetry phase the ϕ_σ field is massive. In the Minkowski space interpretation this result also implies that the ϕ_σ particle has an amplitude to decay into two Goldstone modes (at zero momentum) and hence that, as a state, it must have a finite width (or lifetime) determined by it mass and by the vacuum expectation value v.

Also, by further differentiating the identity of Eq.(12.65) with respect to the ϕ_σ field two more times, and using the identity Eq.(12.80), we can derive one more identity relating the four-point functions of the ϕ_σ and the ϕ_π fields:

$$\Gamma^{(4)}_{\pi\pi\sigma\sigma}(z,y,t,w) + \Gamma^{(4)}_{\pi\pi\sigma\sigma}(w,y,z,t) + \Gamma^{(4)}_{\pi\pi\sigma\sigma}(t,y,z,w) = \Gamma^{(4)}_{\sigma\sigma\sigma\sigma}(y,z,t,w) \tag{12.81}$$

The Fourier transforms of the above identity at a symmetric point of the four incoming momenta (e.g. for $p \to 0$) satisfy the relation $3\Gamma^{(4)}_{\pi\pi\sigma\sigma}|_{S.P.} = \Gamma^{(4)}_{\sigma\sigma\sigma\sigma}|_{S.P.}$, which insures that $O(2)$ invariance holds.

12.5 The Low Energy Effective Action and the Non-Linear Sigma Model

In the preceding section we showed that if a continuous global symmetry is spontaneously broken then the theory has exactly massless excitations known as Goldstone bosons. These states constitute the low-energy manifold of states. It make sense to see what is the possible form that the effective action for these low energy states. We will now see that the form of their effective action is completely determined by the global symmetry.

To find the effective low-energy action it will be more instructive to, rather than the “Cartesian” decomposition into longitudinal and transverse fields that we used above, $\boldsymbol{\phi}(x) = (\phi_\pi(x), \phi_\sigma(x))$, to use instead a non-linear a non-linear representation. In the case of the $O(N)$ theory we will write the field $\boldsymbol{\phi}(x)$ in terms of an amplitude field $\rho(x)$ and an N-component unit-vector field $\boldsymbol{n}(x)$,

$$\boldsymbol{\phi}(x) = \rho(x)\, \boldsymbol{n}(x), \qquad \text{provided } \boldsymbol{n}^2(x) = 1 \tag{12.82}$$

where we imposed the unit-length condition as a constraint. This constraint insures that we have not changed the number of degrees of freedom in the factorization of Eq.(12.82). We will see that the constrained field $\boldsymbol{n}(x)$ describes the manifold of Goldstone states. Clearly the target space of the field $\boldsymbol{n}$ is the sphere S_{N-1}.

Formally, the partition function now becomes

$$Z[\boldsymbol{J}] = \int \mathcal{D}\rho\mathcal{D}\boldsymbol{n} \prod_x \delta(\boldsymbol{n}^2(x)-1)\; \exp\Big(-\int d^dx\mathcal{L}[\rho,\boldsymbol{n}]+\int d^dx\rho(x)\boldsymbol{n}(x)\cdot\boldsymbol{J}(x)\Big) \tag{12.83}$$

where the Lagrangian $\mathcal{L}[\rho,\boldsymbol{n}]$ is

$$\mathcal{L}[\rho,\boldsymbol{n}] = \frac{1}{2}\left(\partial_\mu\rho\right)^2 + \frac{1}{2}\rho^2\left(\partial_\mu\boldsymbol{n}\right)^2 + \frac{m_0^2}{2}\rho^2 + \frac{\lambda}{4!}\rho^4 \tag{12.84}$$

Notice that, due to the constraint on the $\boldsymbol{n}$ field, the integration measure of the path-integral has changed:

$$\mathcal{D}\boldsymbol{\phi} \to \mathcal{D}\rho\mathcal{D}\boldsymbol{n}\prod_x \delta(\boldsymbol{n}^2(x) - 1) \tag{12.85}$$

in order to preserve the number of degrees of freedom.

It is clear that in this non-linear representation the global $O(N)$ transformations leave the amplitude field $\rho(x)$ invariant, and only act on the non-linear field $\boldsymbol{n}(x)$, representing the Goldstone manifold, which only enters in the second term of the Lagrangian of Eq.(12.84). In fact the field $\rho(x)$ represents the fluctuations of the amplitude about the minimum of the potential of Fig.12.10(b). Instead, the field $\boldsymbol{n}$ represents the Goldstone bosons: the field fluctuations along the “flat” directions of the potential at the bottom of the “Mexican” hat (or a wine bottle). The other important observation is that only the derivatives of the field $\boldsymbol{n}$ enter in the Lagrangian and that, in particular, there is no mass term for this field. This is a consequence of the symmetry and of the Ward identities.

Let us look closer at what happens in the broken symmetry state, where $m_0^2 < 0$. In this phase the field ρ has a vacuum expectation value equal to $\bar{\rho} = v = \sqrt{|m_0^2|/(6\lambda)}$. Let us represent the amplitude fluctuations in terms of the field $\eta = \rho - v$. The Lagrangian for the fields η and $\boldsymbol{n}$ reads

$$\mathcal{L}[\eta, \boldsymbol{n}] = \frac{1}{2}\left(\partial_\mu \eta\right)^2 + \frac{m_{\text{eff}}^2}{2}\eta^2 + \frac{\lambda}{6}v\eta^3 + \frac{\lambda}{4!}\eta^4 + \frac{v^2}{2}\left(\partial_\mu \boldsymbol{n}\right)^2 + v\eta\left(\partial_\mu \boldsymbol{n}\right)^2 + \frac{1}{2}\eta^2\left(\partial_\mu \boldsymbol{n}\right)^2 \tag{12.86}$$

where $m_{\text{eff}}^2 = \lambda v^2/3 = 2|m_0^2|$ is the effective mass (squared) of the amplitude field ρ. (Here we ignored the downward shift of the classical vacuum energy, $-\lambda v^4/24$, in the broken symmetry phase.)

Several comments are now useful to make. One is that the amplitude field η is massive and that its mass grows parametrically larger deeper in the broken symmetry state. This means that its effects should become weak at low energies (and long distances). The other comment is that in the broken symmetry state there is a trilinear coupling (the term next to last in Eq.(12.86)) which is linear in the amplitude field η and quadratic in the Goldstone field $\boldsymbol{n}$, and whose coupling constant is the expectation value v. This is precisely what follows from the Ward identities, c.f. Eq.(12.80). For this reason v^2 plays the role of the decay rate of the (massive) amplitude mode (into massless Goldstone modes).

We can now ask for the effective Lagrangian of the massless Goldstone field $\boldsymbol{n}$. We can deduce whet it is by integrating out the amplitude fluctuations, the massive field η, in perturbation theory. By carrying this elementary calculation one finds that, to lowest order, the effective Lagrangian for the Goldstone field $\boldsymbol{n}$ has the form (recall that we have the constraint $\boldsymbol{n}^2 = 1$)

$$\mathcal{L}_{\text{eff}}[\boldsymbol{n}] = \frac{1}{2g^2}\left(\partial_\mu \boldsymbol{n}\right)^2 + \frac{1}{2g^2 m_{\text{eff}}^2}\left(\partial_\mu \boldsymbol{n}\right)^4 + \dots \tag{12.87}$$

where $g^2 = 1/v^2$ plays the role of the coupling constant. The first term of this effective Lagrangian is known, in this case, as the $O(N)$ non-linear sigma model. This theory seems free since it is quadratic in $\boldsymbol{n}$, but it is actually non-linear due to the constraint, $\boldsymbol{n}^2 = 1$. The correction term is clearly small in the low-energy regime, $\left(\partial_\mu \boldsymbol{n}\right)^2 \ll m_{\text{eff}}^2$. So this is actually a gradient expansion which is accurate in the asymptotic low energy regime.

The non-linear sigma model is of interest in wide areas of physics. It was originally introduced as a model for pion physics and chiral symmetry breaking in high-energy physics. In this context, the coupling constant g^2 is identified with the inverse of the pion decay constant. It is also of wide interest in classical statistical mechanics where it is a model of the long-wavelength behavior of the free energy of the classical Heisenberg model of a ferromagnet, and in quantum magnets where it is the effective Lagrangian for a quantum antiferromagnet. We will return to detailed discussion the behavior of the non-linear sigma model in Section 16.4 where we discuss the perturbative renormalization group and in Section 17.1 where we discuss the large-N limit.

12.6 Ward Identities, Schwinger-Dyson Equations, and Gauge Invariance

We will now take a somewhat different look at Ward identities. We will focus on the implications of the global symmetry on the correlators of a theory. Here we will derive a set of equations obeyed by the correlators as a consequence of the global symmetry. These equations take the form of Schwinger-Dyson equations. We will discuss two cases. The first will be a scalar field theory which, for simplicity we will take to be that of a single complex scalar field and the global symmetry is $U(1)$. In the other case we will discuss the global $U(1)$ symmetry of quantum electrodynamics (QED). Both cases can be easily generalized to other symmetry groups.

12.6.1 Schwinger-Dyson equation for the complex scalar field

Let us consider a complex scalar field with Lagrangian

$$\mathcal{L}(\phi,\phi^*) = |\partial_\mu \phi|^2 - V(|\phi|^2) \tag{12.88}$$

which has the global $U(1)$ symmetry

$$\phi(x) \mapsto e^{i\theta}\phi(x), \qquad \phi^*(x) \mapsto e^{-i\theta}\phi^*(x) \tag{12.89}$$

Or, in infinitesimal form,

$$\delta\phi(x) = i\theta\phi(x), \qquad \delta\phi^*(x) = -i\theta\phi^*(x) \tag{12.90}$$

The (Euclidean) two-point function

$$G^{(2)}(x-y) = \langle \phi(x)\phi^*(y)\rangle = \frac{1}{Z}\int \mathcal{D}\phi\mathcal{D}\phi^*\ \phi(x)\phi^*(y)\ e^{-\int d^Dx\mathcal{L}(\phi,\phi^*)} \tag{12.91}$$

and the partition function Z

$$Z = \int \mathcal{D}\phi\mathcal{D}\phi^*\ e^{-\int d^Dx\mathcal{L}(\phi,\phi^*)} \tag{12.92}$$

are invariant under the global $U(1)$ symmetry transformation.

Next we make the (infinitesimal) change of variables,

$$\phi(x) \to \phi'(x) = \phi(x)+i\theta(x)\phi(x), \qquad \phi(x)^* \to \phi'(x)^* = \phi(x)^*-i\theta(x)\phi(x)^* \tag{12.93}$$

where $\theta(x)$ is an arbitrary (and small) function of the coordinate x, that vanishes at infinity. From the fact that the path integral of Eq.(12.91) does not change under a change of the integration variable (the field), since the integration measure is invariant, we readily derive the identity

$$\partial_\mu^z \langle j^\mu(z)\phi(x)\phi^*(y)\rangle = \delta(z-x)\langle\phi(x)\phi^*(y)\rangle - \delta(z-y)\langle\phi(x)\phi^*(y)\rangle \quad (12.94)$$

where $j_\mu(z) = i(\phi(z)\partial_\mu\phi^*(z) - \phi^*(z)\partial_\mu\phi(z))$ is the Noether current for this global symmetry. An expression of this type is known as a Schwinger-Dyson equation. It has an obvious generalization to a general N-point function, and for any global symmetry group. Notice that, at the classical level, we would have predicted that the left hand side of Eq.(12.94) should be identically zero. The so-called *contact terms* that appear on the right hand side of this equation have a quantum origin.

12.6.2 Ward identity and gauge invariance

The result that we just presented can be easily generalized to a the case of a gauge theory such as QED. The QED Lagrangian

$$\mathcal{L}_{\text{QED}}[\phi, \bar\psi, A_\mu] = \bar\psi i\gamma^\mu\partial_\mu\psi - eA_\mu\bar\psi\gamma^\mu\psi - \frac{1}{4}F_{\mu\nu}^2 \quad (12.95)$$

is invariant under the *local* $U(1)$ gauge transformations, $\psi(x) \to e^{ie\theta(x)}\psi(x)$, $\bar\psi(x) \to e^{-ie\theta(x)}\bar\psi(x)$, and $A_\mu(x) \to A_\mu(x) + \partial_\mu\theta(x)$.

However, the QED Lagrangian is also invariant under the *global* symmetry $\psi(x) \to e^{i\theta}\psi(x)$, $\bar\psi(x) \to e^{-i\theta}\bar\psi(x)$, and $A_\mu(x) \to A_\mu(x)$. This is a global symmetry of the Dirac sector of the theory. We can now derive a Ward identity (or Schwinger-Dyson equation) for the global $U(1)$ symmetry of QED. Upon repeating the same arguments used in the derivation of the Ward identity of Eq.(12.94), we now obtain the similar looking relation for the two-point function of the Dirac field

$$\partial_\mu^z\langle j^\mu(z)\psi(x)\bar\psi(y)\rangle = -e\delta(z-x)\langle\psi(x)\bar\psi(y)\rangle + e\delta(z-y)\langle\psi(x)\bar\psi(y)\rangle \quad (12.96)$$

where $j_\mu = e\bar\psi\gamma_\mu\psi$ is the (gauge-invariant) Dirac current. Equivalently, we can write

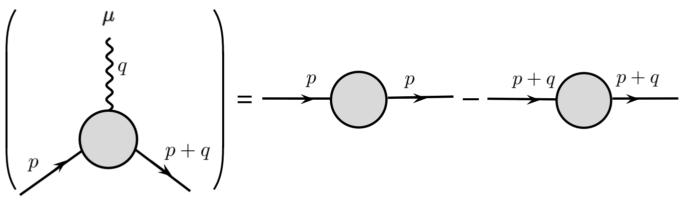
$$-i\partial_\mu^z\langle j^\mu(z)\psi(x)\bar\psi(y)\rangle = -e\delta(z-x)S_F(x-y) + e\delta(z-y)S_F(x-y) \quad (12.97)$$

where $S_F(x-y) = -i\langle\psi(x)\bar\psi(y)\rangle$ is the (Feynman) Dirac propagator. This Ward identity relates the change of the Dirac propagator to the vertex function resulting from the insertion of a current, i.e. the coupling to the gauge field. In momentum space, this Ward identity can be brought to the form

$$-iq_\mu \Gamma^\mu(p+q,p,q) = S_F^{-1}(p+q) - S_F^{-1}(p) \tag{12.98}$$

where $\Gamma^\mu(p+q,p)$ is the vertex function and $S_F(p)$ is the Dirac Feynman propagator. Pictorially, we can represent the Ward identity of Eq.(12.98) in terms of Feynman diagrams by the equation

$q_\mu \cdot$



(12.99)

which relates the insertion of a photon of momentum q to the change of the fermion propagator with the momentum of the photon.

We will now discuss a different derivation of the Ward identities following an approach similar to what we did in ϕ^4 theory with $O(N)$ symmetry. Let us consider the QED Lagrangian in the Feynman-'t Hooft gauges parametrized by α, coupled to a conserved external current $J_\mu(x)$ and to Dirac sources $\eta(x)$ and $\bar\eta(x)$. The full gauge-fixed Lagrangian now is

$$\mathcal{L} = \bar\psi i \not{D} \psi - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + J_\mu(x)A^\mu(x) + \bar\eta(x)\psi(x) + \bar\psi(x)\eta(x) \tag{12.100}$$

where, as usual, $\not{D} = \gamma^\mu(\partial_\mu + ieA_\mu(x))$. The partition function now is

$$Z[J_\mu, \eta, \bar\eta] = \int \mathcal{D}\bar\psi \mathcal{D}\psi \mathcal{D}A_\mu e^{i\int d^4x \mathcal{L}} \tag{12.101}$$

where $\mathcal{L}$ is the Lagrangian of QED with sources, Eq.(12.100). Here we ignored the Faddeev-Popov determinant factor which, as we showed before, is unimportant in QED.

As in other theories we will focus now on $F = i \ln Z[J_\mu, \eta, \bar\eta]$, the generating function of connected N-point functions of Dirac fermions and gauge fields, which satisfies

$$\frac{\delta F}{\delta J_\mu(x)} = A_\mu(x), \quad \frac{\delta F}{\delta \eta(x)} = \bar\psi(x), \quad \frac{\delta F}{\delta \bar\eta(x)} = \psi(x) \tag{12.102}$$

where we omitted the expectation values to simplify the notation. Let us now define $\Gamma[A_\mu, \psi, \bar\psi]$ the generating function of the one-particle irreducible vertex functions, obtained from F by the Legendre transform

$$\Gamma[A_\mu, \psi, \bar\psi] = \int d^4x (J_\mu(x)A^\mu(x) + \bar\eta(x)\psi(x) + \bar\psi(x)\eta(x)) - F \tag{12.103}$$

such that

$$\frac{\delta\Gamma}{\delta A_\mu(x)} = J_\mu(x), \quad \frac{\delta\Gamma}{\delta\psi(x)} = \bar{\eta}(x), \quad \frac{\delta\Gamma}{\delta\bar{\psi}(x)} = \eta(x) \tag{12.104}$$

We next make a change of variables in the partition function $A_\mu \to A_\mu + \delta A_\mu$, $\psi \to \psi + \delta\psi$ and $\bar{\psi} \to \bar{\psi} + \delta\bar{\psi}$ that corresponds to an infinitesimal gauge transformation

$$\delta A_\mu(x) = \partial\Lambda(x), \quad \delta\psi(x) = -ie\Lambda(x)\psi(x), \quad \delta\bar{\psi}(x) = ie\Lambda(x)\bar{\psi}(x) \tag{12.105}$$

where $\Lambda(x)$ is infinitesimal and local. From here one readily finds that the generating function $\Gamma[A_\mu, \psi, \bar{\psi}]$ satisfies the generation function of all Ward identities

$$\partial_\mu \frac{\delta\Gamma}{\delta A_\mu(x)} - ie\left(\psi(x)\frac{\delta\Gamma}{\delta\psi(x)} - \bar{\psi}(x)\frac{\delta\Gamma}{\delta\bar{\psi}(x)}\right) = \frac{1}{\alpha}\partial^2\partial_\mu A^\mu(x) \tag{12.106}$$

where the right hand side arises from the gauge fixing condition.

Now, upon functional differentiation of Eq.(12.106) with respect to $\psi(x)$ and $\bar{\psi}(x)$ we obtain the identity

$$\partial_z^\mu \frac{\delta^3\Gamma}{\delta\bar{\psi}(x)\delta\psi(y)\delta A_\mu(z)} + ie\left(\frac{\delta^2\Gamma}{\delta\bar{\psi}(x)\delta\psi(y)}\delta(x-z) - \frac{\delta^2\Gamma}{\delta\bar{\psi}(x)\delta\psi(y)}\delta(y-z)\right) = 0 \tag{12.107}$$

where we readily identify the first term with the one-particle irreducible three-point vertex function $\langle\psi(x)\bar{\psi}(y)A_\mu(z)\rangle$ and the second and third terms with the one-particle irreducible propagators of the Dirac fields. The Fourier transform of this identity is just Eq.(12.98).

By functional differentiation of Eq.(12.106) with respect to $A_\nu(y)$ we obtain the Ward identity

$$\partial_\mu^x \frac{\delta^2\Gamma}{\delta A_\mu(x)\delta A_\nu(y)} = \partial_x^\mu \Gamma^{(2)}_{\mu\nu}(x-y) = \frac{1}{\alpha}\partial_x^2\partial_x^\mu\delta(x-y) \tag{12.108}$$

On the other hand te 1PI photon two-point function $\Gamma^{(2)}_{\mu\nu}(x-y)$ is given by

$$\Gamma^{(2)}_{\mu\nu}(x-y) = G^{-1}_{\mu\nu}(x-y) - \Pi_{\mu\nu}(x-y) \tag{12.109}$$

where $G_{\mu\nu}(x-y)$ is the bare photon propagator (i.e. in free Maxwell theory) in the 't Hooft-Feynman gauges. Since the Ward identity Eq.(12.108) is also obeyed by the free field propagator $G^{(2)}_{\mu\nu}(x-y)$, we find that the QED photon self-energy, the polarization function $\Pi_{\mu\nu}(x-y)$, must satisfy the exact identity

$$\partial_x^\mu \Pi_{\mu\nu}(x-y) = 0 \tag{12.110}$$

We should note that, although we derived it in the context of QED, this Ward identity all that requires is the global symmetry. In fact, it applies to non-relativistic systems such as the electron gas and Fermi liquids, see Section 10.11. Naturally, the propagators that enter in the expressions above will be different in these theories.