Quantum Ising Models

Andrew Tan[1]

[1]*Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7.*[*]

(Dated: December 9, 2018)

I. INTRODUCTION

This report is a summary of a few key techniques for solving quantum Ising models and a summary of well-known mappings between classical Ising models and quantum Ising models. The transfer-matrix formalism will be used to outline a mapping between the D-dimensional classical Ising model and a $(D-1)$-dimensional quantum model [3–5]. Furthermore, the quantum Ising model in one-dimension described in the spin-basis will be diagonalized using standard techniques [2, 4] to recover the exact spectrum of its Hamiltonian. Finally, we will investigate the level gap near the critical point of the transverse-field Ising model.

II. THE QUANTUM ISING MODEL IN 1D

In analogy with its classical counterpart, we study a linear chain of two-state systems, such as spin-$\frac{1}{2}$ particles, with a specified external field and local couplings.

One such model is the transverse-field Ising model which can be described with the following Hamiltonian parameterized as in [4]:

$$H = -J\sum^{N}\left(\hat{\sigma}_j^z\hat{\sigma}_{j+1}^z + g\hat{\sigma}_j^x\right) \tag{1}$$

where $\hat{\sigma}_j^x$ and $\hat{\sigma}_j^z$ are the Pauli matrices at each location on the chain, g is a dimensionless nearest-neighbour coupling parameter, and J includes the inverse temperature and sets the energy scale.

More generally, we may consider external fields and nearest neighbour couplings along other axes, spatially varying parameters, and longer-range couplings. The Heisenberg model and XY model are additional examples of commonly studied systems falling in this framework. These models, as well as more complicated ones with Hamiltonians consisting of only two-body interactions, are amenable to similar analysis as presented below for the transverse-field Ising model.

A. Quantum phase transitions

First, we need to make precise the notion of a quantum phase transition. Following the treatment provided in [4], we consider a Hamiltonian with degrees of freedom on a lattice, $H(g)$, parameterized with a dimensionless coupling parameter g. A quantum phase transition is defined to be a point of non-analyticity in the energy of the ground state of $H(g)$. Such non-analyticities are typically the result of near level-crossings in finite systems taken to the infinite system size limit (see Figure 1) [4].

B. Phase transition in the Transverse-Field Ising Model

We can provide heuristic arguments for the existance of a quantum phase transition in the 1D transverse-field Ising model of Equation 1. For $g \ll 1$, the nearest-neighbour coupling term dominates and the ground state is expected to be such that all spins are completely aligned in the up or down direction. For $g \gg 1$, the external field dominates and the ground state is expected to have all spins aligned with the external field.

---

[*]Electronic address: `andr.tan@mail.utoronto.ca`

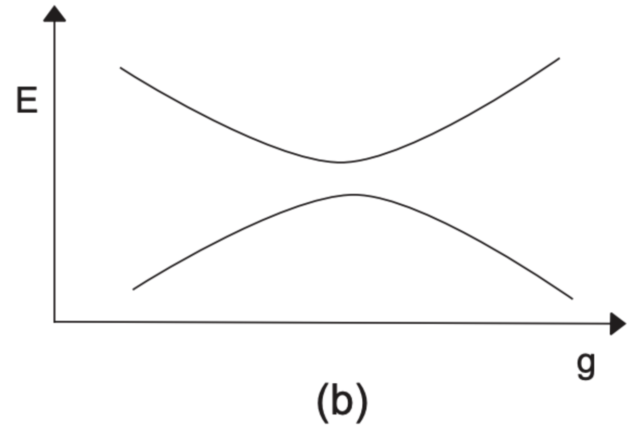

FIG. 1: Shown above is a near level crossing for finite system that can become non-analytic in the infinite system limit. Taken from [4].

In the $g \ll 1$ case, there is a $\mathbb{Z}_2$ symmetry in the ground state corresponding to flipping the spins along the z-axis, $\hat{\sigma}_j^z \mapsto -\hat{\sigma}_j^z$. The spontaneous $\mathbb{Z}_2$ symmetry breaking as $g \to 0$ is indicative of a phase transition at some critical coupling g_c of order one [4]. The transverse-field Ising model, parameterized as in Equation 1 has this quantum phase transition described above at $T = 0 \Rightarrow J \to \infty$, where the ground state level crossing is most important.

III. CORRESPONDENCE BETWEEN CLASSICAL AND QUANTUM SYSTEMS

There is a correspondence that can be drawn between a $(D+1)$-dimensional classical Ising system and a D-dimensional quantum system [3]. This correspondence makes precise the similarities between the Boltzmann factor $e^{-\beta H}$ of statistical mechanics with the unitary evolution operator, $e^{-iHt/\hbar}$, of quantum mechanics.

A. Transfer-Matrix formalism in 1D

Here we follow the derivation of the Transfer-Matrix formalism of [5] in reducing the 2D classical Ising model to a 1D quantum model.

First, consider the classical 1D Ising model with periodic boundary conditions with partition function as in [5],

$$Z = \sum_{\{\sigma\}} \exp\left(J \sum \sigma_j \sigma_{j+1}\right) \exp\left(h \sum \sigma_n\right) \tag{2}$$

Making use of a slight abuse of notation where we define $|+1\rangle = \begin{pmatrix}1 & 0\end{pmatrix}^T$ and $|-1\rangle = \begin{pmatrix}0 & 1\end{pmatrix}^T$ for $\sigma_i \in \{\pm 1\}$ and

$$\mathbf{V}_1 = \begin{pmatrix} e^J & e^{-J} \\ e^{-J} & e^J \end{pmatrix} \quad \text{and} \quad \mathbf{V}_2 = \begin{pmatrix} e^h & 0 \\ 0 & e^{-h} \end{pmatrix} \tag{3}$$

We can rewrite the 1D Ising Hamiltonian using the matrices above as

$$Z = \sum_{\{\sigma\}} \prod \langle \sigma_i | \mathbf{V}_1 \mathbf{V}_2 | \sigma_{i+1} \rangle = \sum_{\{\sigma\}} \langle \sigma_1 | \mathbf{V}_1 \mathbf{V}_2 | \sigma_2 \rangle \langle \sigma_2 | \ldots | \sigma_{n-1} \rangle \langle \sigma_{n-1} | \mathbf{V}_1 \mathbf{V}_2 | \sigma_1 \rangle = \mathrm{Tr}(\mathbf{V}_1 \mathbf{V}_2)^N \equiv \mathrm{Tr}\, \mathbf{V}^N \tag{4}$$

where $\mathbf{V}$ is known as the transfer matrix and the second last equality above makes use of the fact that $\sum_{\sigma_i} |\sigma_i\rangle \langle \sigma_i| = \mathbb{1}$.

From [5], $\mathbf{V}_1$ and $\mathbf{V}_2$ can be rewritten in terms of the Pauli matrices, $\hat{\sigma}$, as follows,

$$\mathbf{V}_1 = (2 \sinh 2J)^{\frac{1}{2}} \exp(J^* \hat{\sigma}^x) \quad \text{and} \quad \mathbf{V}_2 = \exp(H \hat{\sigma}^z) \tag{5}$$

where $\tanh J^* \equiv e^{-2J}$.

B. Transfer-Matrix formalism in 2D

We now extend the formalism to the 2D Ising model on a lattice of size $N \times M$. From [5], we can write $\mathbf{V}_1$ and $\mathbf{V}_2$ as the sum over all 2^M configurations of the rows

$$\mathbf{V}_1 = (2\sinh 2J)^{\frac{1}{2}} \exp(J^* \sum \hat{\sigma}^x_m) \quad \text{and} \quad \mathbf{V}_2 = \exp\left(H \sum \hat{\sigma}^z_m + J \sum \hat{\sigma}^z_m \hat{\sigma}^z_{m+1}\right) \tag{6}$$

With these definitions as before, we can write the partition function as $Z = \mathrm{Tr}(\mathbf{V}_1\mathbf{V}_2)^N \equiv \mathrm{Tr}\,\mathbf{V}^N$. Here $\hat{\sigma}_m$ represents the Pauli matrix acting on a site in column m.

C. The classical-quantum correspondence

The transfer matrix in 1D, written in terms of the Pauli matrices, can readily be seen as the time-evolution operator of a single spin-$\frac{1}{2}$ particle over an imaginary time axis.

The transfer matrix for the 2D Ising model acts on a 2^M-dimensional Hilbert space and can be seen as an imaginary-time-evolution operator relating adjacent rows. The adjacent rows are $(2-1)$-dimensional quantum Ising chains.

To make this correspondence more precise, consider defining H and Z, the Hamiltonian and partition function of a $(D-1)$-dimensional quantum system as in [3],

$$H \equiv \frac{1}{\Delta\tau} \ln \mathbf{V} \quad \text{and} \quad Z = \mathrm{Tr}(e^{-H\Delta\tau})^{\beta/\Delta\tau} \tag{7}$$

where $\Delta\tau = \beta/N$.

This technique can be generalized for mapping a D-dimensional classical system into a $(D-1)$-dimensional quantum one, where one lattice dimension is chosen as an imaginary time axis on which the transfer matrix acts.

IV. EXACT SPECTRUM OF THE TRANSVERSE-FIELD ISING MODEL

The exact solution for a couple of quantum 1D models was derived in 1961 by Lieb, Schultz, and Mattis [2] through finding a basis in which the Hamiltonian is diagonal. Below we quote the procedure described by Sachdev in [4].

The spin raising and lowering operators, $\hat{\sigma}^{\pm}_i \equiv (\hat{\sigma}^x_i \pm \hat{\sigma}^y_j)/2$, satisfy the following relationships

$$[\hat{\sigma}^{\pm}_i, \hat{\sigma}^{\pm}_j] = 0,\ i \neq j; \quad \left\{\hat{\sigma}^-_i, \hat{\sigma}^+_i\right\} = 1; \quad \left(\hat{\sigma}^+_i\right)^2 = \left(\hat{\sigma}^-_i\right) = 0 \tag{8}$$

The operators satisfy a mixed relationship with the first relationship of Equation 8 resembling Bose operators and the last two resembling Fermi operators. We therefore require an additional step before the Hamiltonian can be diagonalized as these mixed commutation relationships are not preserved by a linear transformation [5].

A. Jordan-Wigner transformation

Following [4], the well-known Jordan-Wigner transformation transforms the spin-$\frac{1}{2}$ operators to spinless fermionic operators as shown below,

$$c_m \equiv \exp\left(i\pi \sum \hat{\sigma}^+_j \hat{\sigma}^-_j\right) \hat{\sigma}^-_m = \left(\prod_{j<i} \hat{\sigma}^z_j\right) \hat{\sigma}^+_j \tag{9a}$$

$$c^\dagger_m \equiv \exp\left(i\pi \sum \hat{\sigma}^+_j \hat{\sigma}^-_j\right) \hat{\sigma}^+_m = \left(\prod_{j<i} \hat{\sigma}^z_j\right) \hat{\sigma}^-_j \tag{9b}$$

The definition of these operators, while seemingly unmotivated at first are designed so that the resulting operators satisfy the following fermionic relationships,

$$\left\{c_i, c^\dagger_j\right\} = \delta_{ij}; \quad \{c_i, c_j\} = \left\{c^\dagger_i, c^\dagger_j\right\} = 0 \tag{10}$$

Following the treatment by [4], we make the following 90° rotation about the y-axis:

$$\hat{\sigma}^z \mapsto \hat{\sigma}^x \quad \text{and} \quad \hat{\sigma}^x \mapsto -\hat{\sigma}^z \tag{11}$$

As in [4], writing the Hamiltonian (Equation 1) in terms of these fermionic operators, we obtain,

$$H = -J \sum \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i - 2g c_i^\dagger c_i - g \right) \tag{12}$$

As noted by [4], the fermion number is not conserved due to the $c^\dagger c^\dagger$ terms, but this too will be remedied soon.

B. Bogoliubov transformation

The Hamiltonian maintains its quadratic form as expressed in terms of these new Jordan-Wigner fermions suggesting that it can be diagonalized in the Fourier basis. Under the Fourier transformation $c_k \equiv \frac{1}{\sqrt{M}} \sum c_j e^{ikja}$ where M is the number of sites and a is the lattice spacing, we obtain as in [4],

$$H = J \sum \left[2\left(g - \cos(ka)\right) c_k^\dagger c_k - i \sin(ka) \left(c_{-k}^\dagger c_k^\dagger + c_{-k} c_k \right) - g \right] \tag{13}$$

Now that the Hamiltonian has been diagonalized, the only remaining problem is that the Jordan-Wigner fermions are not conserved due to the $c_{-k}^\dagger c_k^\dagger$ terms. We seek to find new fermionic operators, linear combinations of c_k and $c_{-k}^\dagger$, that are conserved; to accomplish this we affect a Bogoliubov transformation mapping to new operators, γ_k as follows:

$$\gamma_k = u_k c_k - i v_k c_{-k}^\dagger, \quad u_k, v_k \in \mathbb{R} \quad \text{satisfying} \quad u_k^2 + v_k^2 = 1 \quad \text{and} \quad u_k = u_{-k} \tag{14}$$

We further require a choice of u_k and v_k results in the vanishing of terms like $\gamma^\dagger \gamma^\dagger$. To this end, we choose $u_k \equiv \cos(\theta_k/2)$, $v_k \equiv \sin(\theta_k/2)$ with $\tan\theta_k = \sin(ka)/(\cos(ka) - g)$. The diagonalized Hamiltonian becomes,

$$H = \sum \varepsilon_j \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right) \tag{15a}$$

$$\varepsilon_j \equiv 2J(1 + g^2 - 2g\cos k)^{\frac{1}{2}} \tag{15b}$$

Through this procedure, we have mapped the spin system to a system of free spinless fermions providing significant analytic insight into the behaviour of the ground state as well as excitations. For example, the ground state of the fermionic system is such that $\gamma_k |0\rangle = 0, \forall k$ and excitations are described using creation operators on the ground state. In the standard way, we can define Majorana fermions, χ, from the γ fermions as $\chi_k = (\gamma_k + \gamma_k^\dagger)/2$ and $\chi_{-k} = (\gamma_k - \gamma_k^\dagger)/(2i)$.

C. Critical coupling parameter

Looking at energy ε_k from Equation 15b, it is noted that the gap at $k = 0$ vanishes for $g = 1$. This is the level crossing that marks the boundary between the ordered and paramagnetic phases [4].

As we approach the critical $g \to 1$ and $T \to 0$, there are arbitrarily many states for which $\varepsilon_k \to 0$. That is to say, near the critical point we approach a continuum of states that can be described using a continuum quantum field theory [4].

V. SUMMARY

A well-known mapping between 2D classical Ising models to a 1D quantum model, as a special case of a D-dimensional to $D-1$-dimensional classical-to-quantum correspondence has been outlined. This provides additional motivation for studying quantum Ising models as well as demonstrating that the 1D quantum Ising model's phase transition belongs to the 2D classical Ising class [4, 5]. Additionally, a few important techniques for the treatment of quantum Ising models and similar have been derived in this report [2, 4, 5]. Notably, near the critical point as $g \to 1$

and $T \to 0$, the free fermion picture of the transverse-field Ising model shows that we approach a continuum of states that can be described using a continuum quantum field theory [1, 3, 4].

[1] Paul Ginsparg. Applied conformal field theory. Aug 1988.

[2] E. H. Lieb, T. D. Schultz, and D. C. Mattis. Two soluble models of an antiferromagnetic chain. *Annals of Physics*, 16:407–466, Dec 1961.

[3] Molignini. Analyzing the two dimensional ising model with conformal field theory. Mar 2013.

[4] Subir Sachdev. *Quantum Phase Transitions.* Cambridge University Press, 2011.

[5] T. D. Schultz, D. C. Mattis, and E. H. Lieb. Two-dimensional ising model as a soluble problem of many fermions. *Rev. Mod. Phys.*, 36:856–871, Jul 1964.