Nambu, A Foreteller of Modern Physics III

The Dirac composite fermion of the fractional quantum Hall effect

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We review the recently proposed Dirac composite fermion theory of the half-filled Landau level.
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1. Introduction

The fractional quantum Hall effect (FQHE) was discovered in 1982 [1], only a couple of years following the discovery of the integer quantum Hall effect (IQHE). One of the most nontrivial problems of condensed matter physics, the FQHE has attracted the attention of theorists ever since. (One of the earliest and most influential works is the one by Laughlin [2].) This paper surveys the most recent progress in the understanding of one particular, but very important, aspect of the FQHE: the composite fermion in the half-filled Landau level [3]. In particular, we will review the arguments leading to the Dirac composite fermion theory [4].

The quantum Hall problem is attractive for theorists partly because of its very simple starting point: a Hamiltonian describing particles moving on a two-dimensional plane, in a constant magnetic field, and interacting with each other through a two-body potential,

$$H = \sum_{a=1}^{N} \frac{(\mathbf{p}_a + \mathbf{A}(\mathbf{x}_a))^2}{2m} + \sum_{\langle a,b \rangle} V(|\mathbf{x}_a - \mathbf{x}_b|). \tag{1}$$

Here, $\mathbf{A}$ is the gauge potential corresponding to a constant magnetic field. The two-body potential V is normally taken to be the Coulomb potential $V(r) = e^2/r$, but one believes many results are valid for a large class of repulsive interactions. The quantum Hall states are characterized by many physical properties, including quantized Hall resistivity, vanishing longitudinal resistivity, bulk energy gap, edge modes, etc. For the purpose of this article, we take the existence of an energy gap to be the defining property of the quantum Hall states. A very simplified summary of the experimental situation is as follows: for certain values of the filling factor, defined as

$$\nu = \frac{\rho}{B/2\pi}, \tag{2}$$

where ρ is the two-dimensional electron density, the system is in one of the quantum Hall states with an energy gap. The values of ν for which there is a gap are either integers, in which case we have IQHE, or rational numbers, which correspond to FQHE.

The existence of a gap for integer ν can be understood on the basis of the approximation of noninteracting electrons. In a magnetic field B, the energy eigenvalues of the one-particle Hamiltonian are organized into Landau levels,

$$E_n = \frac{B}{m}\left(n + \frac{1}{2}\right). \tag{3}$$

The degeneracy of each Landau level is $B/2\pi$ per unit area. At integer ν, states with $n \leq \nu$ are filled and those with $n > \nu$ are left empty. The system then has a gap equal to the spacing between Landau levels, which is $\omega_c = B/m$.

In contrast to the IQHE, the fractional quantum Hall effect cannot be understood from the noninteracting limit. For example, when $0 < \nu < 1$, the lowest Landau level (LLL, $n = 0$) is partially filled, so the noninteracting Hamiltonian has an exponentially large (in the number of electrons) ground state degeneracy. The miracle of the FQHE is that for certain rational values of ν, interactions between electrons lead to a gap.

There are two energy scales in the FQH problem. The first scale is the cyclotron energy $\omega_c = B/m$, while the second scale is the interaction energy scale. In the case of the Coulomb interaction, the latter energy scale can be estimated as the potential energy between two neighboring electrons,

$$\Delta = \frac{e^2}{r} \sim e^2\sqrt{B}. \tag{4}$$

The FQH problem is usually considered in the limit $\Delta \ll \omega_c$. This limit is reached experimentally by taking $B \to \infty$ at fixed ν; theoretically, it is also reached by taking $m \to 0$ at fixed B. When $\Delta \ll \omega_c$ one can ignore all Landau levels above the lowest one, and the problem can be reformulated as pertaining to a Hamiltonian which operates only on the LLL,

$$H = \mathcal{P}_{\text{LLL}}\sum_{\langle a,b\rangle} V(|\mathbf{x}_a - \mathbf{x}_b|), \tag{5}$$

where P_{LLL} is the projection to the lowest Landau level. This extremely simple Hamiltonian, believed to underlie all the richness of FQH physics, cannot be solved by traditional methods of perturbation theory due to the lack of a small parameter. In particular, there is only one energy scale—the Coulomb energy scale Δ. The FQH problem is essentially nonperturbative.

2. Flux attachment

One of the most productive ideas in FQH physics has been the idea of the composite fermion (CF). The notion of the CF itself is based on another concept called flux attachment [5], which was applied to the FQHE in a number of groundbreaking works [3,6–8]. I will now review the standard textbook field theory of the composite fermion, although later on I will argue that it needs some nontrivial modification to become the correct low-energy effective theory.

In the FQH case, one "attaches" an even number (in the simplest case, two) of magnetic flux quanta to an electron, transforming it to a new object called the "composite fermion." In field theory

language, one starts from a theory of interacting electrons ψ_e in (2+1) dimensions in a background magnetic field

$$\mathcal{L} = i\psi_e^\dagger(\partial_t - iA_0)\psi_e - \frac{1}{2m}|(\partial_i - iA_i)\psi_e|^2 + \cdots, \tag{6}$$

where $\cdots$ stands for interaction terms, and "derives," following a certain formal procedure, a new Lagrangian for the composite fermion ψ,

$$\mathcal{L} = i\psi^\dagger(\partial_t - iA_0 + ia_0)\psi - \frac{1}{2m}|(\partial_i - iA_i + ia_i)\psi|^2 + \frac{1}{2}\frac{1}{4\pi}\epsilon^{\mu\nu\lambda} + a_\mu\partial_\nu a_\lambda \cdots \tag{7}$$

The Chern–Simons term in Eq. (7) encodes the idea of flux attachment. In fact, the equation of motion obtained by differentiating the action with respect to a_0 reads

$$2\psi^\dagger\psi = \frac{b}{2\pi}, \qquad b = \mathbf{\nabla} \times \mathbf{a}, \tag{8}$$

which means that the magnetic fluxes of the dynamic gauge field a_μ are tied to the location of the composite fermions, with two units of fluxes per particle.

There are two features of the field theory (7)—which will be called the HLR field theory after Halperin, Lee, and Read, who used it to study the half-filled Landau level [3]—which are rather trivial but worth listing here for future reference:

- The number of composite fermion is the same as the number of electrons. It cannot be otherwise if the composite fermion results from attaching magnetic fluxes to an electron.
- The action contains a Chern–Simons term for a_μ. As demonstrated above, this term encodes in mathematical terms the idea of flux attachment.

In the literature, it is often stressed that transformation from (6) to (7) can be done in an exact way (see, e.g., Ref. [8]). "Conservation of difficulty" then implies that the theory (7) cannot be solved exactly. To make any progress at all, one has to start with some approximation scheme, and in every work so far this has been the mean field approximation where one replaces the dynamical gauge field a_μ by its average value determined from Eq. (8). Since in the Lagrangian (7) the gauge fields A and a enter through the difference $A - a$, and the density of the composite fermions is the same as the density of the original electrons, the effective average magnetic field acting on ψ is

$$B_{\text{eff}} = B - \langle b \rangle = B - 4\pi\rho. \tag{9}$$

Translated to the language of the filling factors,

$$\nu = \frac{\rho}{B/2\pi}, \qquad \nu_{\text{CF}} = \frac{\rho}{B_{\text{eff}}/2\pi}, \tag{10}$$

the equation becomes

$$\nu_{\text{CF}}^{-1} = \nu^{-1} - 2. \tag{11}$$

In particular, the values $\nu = \frac{n}{2n+1}$ map to $\nu_{\text{CF}} = n$. In this way we have mapped the FQH problem for the electron to the IQH problem for the composite fermions, which gives an "explanation" for

the emergence of an energy gap. Experimentally, one finds quite robust quantum Hall plateaux at these values of ν, up to $n \approx 10$.

Another sequence of quantum Hall plateaux are found at $\nu = \frac{n+1}{2n+1}$. Now $\nu > \frac{1}{2}$ so the effective average magnetic field B_{eff} is negative, i.e., points in the direction opposite to the direction of the original B. The composite fermion still forms IQH states, with $n + 1$ filled Landau levels ($\nu_{\text{CF}} = -(n + 1)$). Together, the two series of FQH plateaux at $\nu = \frac{n}{2n+1}$ and $\nu = \frac{n+1}{2n+1}$ are called the Jain sequences of plateaux.

One of the most spectacular successes of the composite fermion theory is the prediction of the nature of the $\nu = \frac{1}{2}$ state (the half-filled Landau level) [3]. At this filling fraction, the average effective magnetic field is equal to 0, and the composite fermion should form a gapless Fermi surface. HLR theory thus predicts that the low-energy excitation is the fermionic quasiparticle near the Fermi surface. There is strong experimental evidence that this is indeed the case [9–11]. These experiments give the strongest evidence that the composite fermion is a real physical object—a quasiparticle near half filling—and not just a mathematical construct.

Despite its astounding success, the quantum field theory (7) has been criticized on various grounds. The criticism leveled most often against the theory (7) is the lack of any information about the projection to the lowest Landau level. In particular, the energy gap predicted by the mean-field picture is B_{eff}/m, which for generic ν is of order ω_c, but not Δ. To remedy the issue, one has to assume that the energy gap is determined by an effective mass m_*, postulated to be parametrically B/Δ. In particular, m_* is assumed to remain finite in the limit $m \to 0$.

In my view, there are in reality two energy scale problems. The first problem, which I would call the "grand problem" of energy scale, is to derive, from microscopic calculations, the finite value of m_* in the limit $m \to 0$. The second problem, the more modest "little problem" of energy scale, is to make the low-energy effective field theory with m_* consistent with the fundamental symmetries of the original theory of electrons with a much smaller mass.

The "grand problem" is the one that attracts most attention. We note here a few past attempts to address it [12–14]. However important it is, it will not concern us if our ambition is limited to capturing the low-energy phenomenology, i.e., the physics at energy scales much smaller than Δ. The effective mass m_* would appear simply as an input parameter in a low-energy effective field theory, and we will simply postulate that such an effective mass arises somehow as a result of the renormalization group flow from a UV scale above ω_c to an IR scale below Δ. The "little problem" of energy scale is a fully low-energy question, and it can now be solved, in principle, by using the Newton–Cartan formalism (see, e.g., Refs. [15–18]).

However, the most recent progress in the physics of the half-filled Landau level has arrived from an attempt to address another problem, usually regarded as less important and subordinate to the energy scale problem: the lack of particle–hole (PH) symmetry.

3. The problem of particle–hole symmetry

A system of nonrelativistic particles interacting through a two-body interaction has two discrete symmetries: parity, or spatial reflection ($x \to x, y \to -y$), which we denote as P, and time reversal, which will be called T. In a constant uniform magnetic field both P and T are broken, but PT is preserved. But in the lowest Landau level limit ($\Delta \ll \omega_c$), the projected Hamiltonian (5) has an additional discrete symmetry: the particle–hole symmetry, first considered in Ref. [19].

To define the particle–hole symmetry, one chooses a particular basis of LLL one-particle states $\psi_k(x)$. This basis defines the electron creation and annihilation operators $c_k^\dagger$, c_k. The many-body

LLL Fock space is obtained by acting products of creation operators on the empty Landau level $|\text{empty}\rangle$.

Particle–hole conjugation, Θ, is defined as an antilinear operator, which maps an empty Landau level to a full one:

$$\Theta : |\text{empty}\rangle \rightarrow |\text{full}\rangle = \prod_{k=1}^{M} c_k^\dagger |\text{empty}\rangle, \tag{12}$$

where M is the number of orbitals on the LLL. It also maps a creation operator to an annihilation operator, and vice versa:

$$\Theta : c_k^\dagger \leftrightarrow c_k. \tag{13}$$

One can show that the projected Hamiltonian maps to itself, up to the addition of a chemical potential term,

$$\Theta : H_{\text{LLL}} \rightarrow H_{\text{LLL}} - \mu_0 \sum_k c_k^\dagger c_k, \tag{14}$$

where μ_0 depends on the interaction V. This means that for $\mu = \mu_0/2$, the Hamiltonian $H_{\text{LLL}} - \mu N$ maps to itself: at this chemical potential the Hamiltonian is particle–hole symmetric.

Under particle–hole conjugation the filling factor ν transforms as

$$\nu \rightarrow 1 - \nu. \tag{15}$$

In particular, $\nu = 1/2$ maps to itself under PH conjugation: the half-filled Landau level is at the same time half empty. Moreover, $\nu = \frac{n}{2n+1}$ maps to $\nu = \frac{n+1}{2n+1}$: the two Jain sequences of quantum Hall plateaux form pairs that map to each other under PH conjugation: $\nu = 1/3$ and $\nu = 2/3$, $\nu = 2/5$ and $\nu = 3/5$, etc.

Let us now ask what the discrete symmetries of the HLR field theory (7) are. It is easy to see that there is only one such symmetry, *PT*. The Chern–Simons theory does not have any discrete symmetry that can be associated with particle–hole conjugation. This reflects on the asymmetry in the treatment of quantum Hall plateaux: the $\nu = \frac{n}{2n+1}$ is described by an integer quantum Hall state where the CFs fill n Landau levels, while its PH conjugate $\nu = \frac{n+1}{2n+1}$ by $n + 1$ filled Landau levels.

The Fermi liquid state with $n = 1/2$ presents a particularly baffling problem for particle–hole symmetry. Naively, one expects PH conjugation to map a filled state to an empty state and vice versa. This would mean that the Fermi disk of the CFs, describing the Fermi liquid state, maps to a hollow disk in momentum states: the states with momentum $|\mathbf{k}| > k_F$ are filled, and those with $|\mathbf{k}| < k_F$ are empty. This is obviously silly.

The lack of particle–hole symmetry has been recognized as a problem of the HLR theory from early on. One aspect of this problem was noticed in 1997 by Kivelson, Lee, Krotov, and Gan [20]. When disorders are statistically particle–hole symmetric, particle–hole symmetry implies that at half filling σ_{xy} is exactly $\frac{1}{2}(e^2/h)$, but the HLR theory, in the random phase approximation, implies that $\rho_{xy} = 2(h/e^2)$. These two results disagree with each other when the longitudinal conductivity σ_{xx} (or equivalently, the longitudinal resistivity ρ_{xx}) is nonzero. From time to time, the issue of particle–hole symmetry has been brought up in the literature (for example, it was crucial for the discovery of the

anti-Pfaffian state [21,22]), but no conclusive resolution of the problem of the lack of PH symmetry in the HLR theory has been found.

What makes the PH symmetry problem seem hard is that PH symmetry is not the symmetry of nonrelativistic electrons in a magnetic field [the theory (1)]. It only emerges as the symmetry after taking the lowest Landau level limit [theory (5)]. The particle–hole symmetry of the LLL is not realized as a local operation acting on fields.

It was commonly thought that the PH symmetry problem is part of the energy scale problem: PH symmetry becomes exact in the LLL limit, where the energy scale problem is sharpest. But in fact, the PH symmetry problem is easier than the "grand problem" of energy scale: PH symmetry is a question about the low-energy effective field theory, while the CF effective mass, the object of concern of the energy scale problem, comes mostly from energy scales above Δ.

One can envision three possible scenarios for the problem of particle–hole asymmetry of the HLR theory to resolve itself:

  (i) Despite the lack of an explicit PH symmetry, the HLR theory has a hidden PH symmetry.
 (ii) Particle–hole symmetry is spontaneously broken, and the HLR theory describes only the low-energy excitations around one of the two ground states.
(iii) The effective field theory describing the low-energy excitations is different from HLR. In this theory, particle–hole symmetry is explicitly realized.

Option (i) cannot be ruled out, but a careful diagrammatic analysis by Kivelson et al. [20] does not seem to reveal any mechanism under which particle–hole symmetry may be hidden. How this can be reconciled with the supposed exactness of the flux attachment procedure is not clear, but one should remember that the HLR theory, as applied in practice, makes an additional assumption of the mean field Fermi liquid as the starting point. One thing is clear: if one takes the HLR Lagrangian and declares it (after making some standard modifications like changing the electron mass m to the effective mass m_*, adding Landau's interactions, etc.) to be the Lagrangian of a low-energy effective field theory (with a cutoff much smaller than the Fermi energy), then this effective field theory would show no indication of particle–hole symmetry.

Option (ii) is self-consistent and was investigated by Barkeshli et al. [23]. If that is the case, there are two states at $\nu = 1/2$: one corresponds to a Fermi surface of "composite particles" and the other to that of "composite holes." However, there is no numerical or experimental evidence for this kind of spontaneous particle–hole symmetry breaking. In fact, the experimental result of Ref. [24] seems to indicate, at least naively, that the $\nu = 1/2$ Fermi liquid is equally well interpreted as being made out of "composite particles" or "composite holes." There is now strong numerical evidence that the $\nu = 1/2$ state is particle–hole symmetric [25].

We will now try to make sense of option (iii).

4. Dirac composite fermion

There exists an alternative theory that satisfies particle–hole symmetry but also preserves all successful phenomenological predictions of the HLR theory. This theory is the Dirac composite fermion theory, first proposed in Ref. [4] as the low-energy effective field theory of the half-filled Landau level. The essence of the theory is that the composite fermion does not transform into a "composite hole" under particle–hole symmetry, but remains a composite particle. Only the momentum of the

composite fermion flips sign under particle–hole conjugation,

$$\Theta : \mathbf{k} \rightarrow -\mathbf{k}. \tag{16}$$

Implicitly, we assume that the Fermi disk of the composite fermion transforms into itself (a filled disk, not a hollow disk).

Equation (16) is how time reversal usually works. In the theory of the Dirac composite fermion, the CF is described by a two-component spinor field ψ, which transforms under PH conjugation following the formula usually associated with time reversal,

$$\psi \rightarrow i\sigma_2\psi. \tag{17}$$

There are several arguments one can put forward to argue that the composite fermion has to be a massless Dirac particle. One argument, or rather a hint, comes from the CF interpretation of the Jain-sequence states. Recall that one problem with the standard CF picture is that $\nu = \frac{n}{2n+1}$ corresponds to the composite fermion filling factor $\nu_{CF} = n$, while $\nu = \frac{n+1}{2n+1}$ maps to $\nu_{CF} = n + 1$ (ignoring the sign). On the other hand, these two states are PH-conjugate pairs and should be described by the same filling factor of the composite fermion in any PH-symmetric theory. The most naive way to reconcile these different pictures is to replace the filling factors $\nu_{CF} = n$ and $\nu_{CF} = n + 1$ with the average value $\nu_{CF} = n + \frac{1}{2}$. But now we have a problem: we want to map the FQHE in the Jain sequences to the IQHE of the composite fermions, but is it possible to have an IQH state with half-integer filling factor? Indeed it is, if the composite fermion is a massless Dirac fermion. Half-integer quantization of the Hall conductivity is a characteristic feature of the Dirac fermion, confirmed in experiments with graphene [26,27].

The second argument in favor of the Dirac nature of the CF relies on a property of the square of the particle–hole conjugation operator Θ^2 [25].[1] It is intuitively clear that applying particle–hole conjugation twice maps a given state to itself, but there is a nontrivial factor of ± 1 that one gains by doing so.

Consider a generic state on the LLL with N_e electrons,

$$|\psi\rangle = \prod_{i=1}^{N_e} c_{k_i}^\dagger |\text{empty}\rangle. \tag{18}$$

Then, under PH conjugation,

$$\Theta : |\psi\rangle \rightarrow \prod_{i=1}^{N_e} c_{k_i} |\text{full}\rangle = \prod_{i=1}^{N_e} c_{k_i} \prod_{j=1}^{M} c_j^\dagger |\text{empty}\rangle. \tag{19}$$

Applying Θ again, one finds

$$\Theta^2 : |\psi\rangle \rightarrow \prod_{i=1}^{N_e} c_{k_i}^\dagger \prod_{j=1}^{M} c_j |\text{full}\rangle = \prod_{i=1}^{N_e} c_{k_i}^\dagger \prod_{j=1}^{M} c_j \prod_{k=1}^{M} c_k^\dagger |\text{empty}\rangle = (-1)^{M(M-1)/2} |\psi\rangle. \tag{20}$$

This relationship is quite easy to interpret when M is an even number: $M = 2N_{CF}$. Then,

$$\Theta^2 : |\psi\rangle \rightarrow (-1)^{N_{CF}} |\psi\rangle. \tag{21}$$

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[1] Also, M. Levin and D. T. Son, unpublished (2015).

This formula suggests the following interpretation: N_{CF} is the number of composite fermions of the state $|\psi\rangle$, and each composite fermion is associated with a factor of -1 under Θ^2. This -1 factor is natural for the Dirac fermion.

In order to have a correct Θ^2, we have to identify the number of composite fermions with half the number of orbitals on the LLL: $N_{CF} = M/2$, which is *independent* of the number of electrons N_e. This contradicts the intuitive picture of flux attachment, in which the composite fermion is obtained by attaching two units of flux quanta to an electron. On the other hand, that is expected: in a theory that treats particles and holes in a symmetric way, the number of composite fermions has to be in general different from the number of electrons, otherwise it would have to be equal to the number of holes as well.

The tentative theory of the composite fermion can be written as follows:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu + 2ia_\mu)\psi + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda + \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \tag{22}$$

(with a speed of light which is determined by microscopic physics). There are two differences between (22) and (7). One is the Dirac nature of the composite fermion ψ. The other is the absence of the Chern–Simons term *ada* in the Lagrangian: such a term (as also the mass term for ψ), if present, would disallow any discrete symmetry that could be identified with particle–hole symmetry. Interestingly, each such modification to the HLR theory would shift the filling factors of the Jain-sequence plateaux, but together the shifts cancel each other and the Jain sequences remain unchanged, as shown below.

How should one visualize the composite fermion? In Ref. [4] it was suggested that the CF is better interpreted as a type of fermionic vortex, arising from a fermionic particle–vortex duality. Particle–vortex duality is well known for bosons [28,29], but we are dealing here with a new duality for fermions. The salient feature of particle–vortex duality is that it switches the roles of particle number and magnetic field. Differentiating (22) with respect to A_0, one obtains the electron density

$$\rho = \frac{\delta S}{\delta A_0} = \frac{b}{2\pi} + \frac{B}{4\pi}. \tag{23}$$

On the other hand, the equation of motion obtained by differentiating the action with respect to a_0 is

$$\bar{\psi}\gamma^0\psi = \frac{B}{4\pi}, \tag{24}$$

i.e., the CF density is set by the external magnetic field.

If one defines the filling factors of the electron and the composite fermion as

$$\nu = \frac{2\pi\rho}{B}, \qquad \nu_{CF} = \frac{2\pi\rho_{CF}}{b}, \tag{25}$$

then from Eqs. (23) and (24) we find that they are related by

$$\nu_{CF} = -\frac{1}{4(\nu - \frac{1}{2})}. \tag{26}$$

In particular, $\nu = \frac{n}{2n+1}$ maps to $\nu_{CF} = n + \frac{1}{2}$, which is the filling factor of an integer quantum Hall state of the Dirac fermion.

It should be emphasized that the Dirac nature of the CF does not mean that there is a Dirac cone for the CF. The tip of the cone is at $\mathbf{k} = 0$ while the CF, as a low-energy mode, exists only near the Fermi surface. The Dirac nature of the CF, strictly speaking, only means that the fermionic quasiparticle has a Berry phase of π around the Fermi surface. It is easy to show that such a Berry phase follows from Eqs. (16) and (21). The quasiparticle Berry phase has been identified as an important ingredient of Fermi liquids [30], but the possibility of such a phase for the composite fermion in FQHE has been overlooked in the literature until very recently.

5. Consequences of Dirac composite fermion

The Dirac composite fermion theory has distinct consequences, in principle verifiable in experiments and numerical simulations.

It is numerical simulations [25] that provide the currently most nontrivial test of the Dirac nature of the composite fermion. The numerical finding is the disappearance, attributable to particle–hole symmetry, of the leading $2k_F$ singularity in certain correlation functions. It is well known that for the (2+1)D massless Dirac fermion, two-point correlation functions of time-reversal-invariant operators are free from the leading $2k_F$ singularity in a generic two-point correlator, a fact that originates from the quasiparticle Berry phase π around the Fermi surface. In the half-filled Landau level, the role of time reversal is played by particle–hole symmetry, therefore to test the Berry phase one should look for the absence of the leading $2k_F$ singularity in correlation functions of PH symmetric operator. The electron density operator $\rho = \psi_e^\dagger \psi_e$ is not PH symmetric (the deviation of the density from the mean density, $\delta\rho = \rho - \rho_0$, flips sign under PH conjugation) but one can easily write down more complicated operators that are PH symmetric, for example $\delta\rho \nabla^2 \rho$. In Ref. [25] the leading $2k_F$ singularity in the correlation function of such an operator was shown to disappear when PH symmetry is made exact (and to reappear when PH symmetry is violated), confirming the Dirac nature of the composite fermion.

There are also predictions about transport that are, strictly speaking, consequences of particle–hole symmetry. If one introduces the conductivities σ_{xx}, σ_{xy}, and the thermoelectric coefficients α_{xx} and α_{xy},

$$\mathbf{j} = \sigma_{xx}\mathbf{E} + \sigma_{xy}\mathbf{E} \times \hat{\mathbf{z}} + \alpha_{xx}\nabla T + \alpha_{xy}\nabla T \times \hat{\mathbf{z}}; \tag{27}$$

then, at exact half filling, particle–hole symmetry implies [4,31]

$$\sigma_{xy} = \frac{1}{2}\frac{e^2}{h}, \qquad \alpha_{xx} = 0. \tag{28}$$

A manifestly particle–hole symmetric theory like the Dirac composite fermion theory reproduces these results automatically. On the other hand, the HLR theory, supplemented by the usual approximations to make it suitable for computation (e.g., the random phase approximation) would, in general, break both relationships [20,31].

6. Conclusion

We have presented arguments in favor of the Dirac nature of the composite fermion. The Dirac composite fermion provides a very simple solution to a number of puzzles that have been plaguing the quantum field theory of the composite fermion for a long time.

A simple demonstration that the Dirac composite fermion emerges from the dynamics of interacting electrons on the lowest Landau level is still lacking. (For a recent attempt to address this question see Ref. [32].) One may wonder how the flux attachment procedure, supposed to be exact, can lead us to something so different from Eq. (7). The situation becomes less puzzling if one remembers that the Lagrangian (22) is a low-energy effective Lagrangian, while the action of the type (7) obtained from the exact flux attachment procedure contains information about all energy scales. One may also be bothered by the emergence of a Dirac fermion out of the initial nonrelativistic fermion. Here again, the situation is not as strange as it sounds: what is important is not really the nonrelativistic Hamiltonian (1), but the LLL projected Hamiltonian (5), which applies equally well if the original fermion is a Dirac fermion (e.g., the gapless mode on the surface of a topological insulator). In this case the duality is one between two theories, both involving Dirac fermions.

Going beyond quantum Hall physics, a very interesting possibility is that the duality between the free Dirac fermion (the electron theory) and Dirac fermion interacting with a gauge field is valid even at zero magnetic field. Such a duality would have consequences for interacting surfaces of topological insulators: for example, the so-called T-Pfaffian state [33–36], otherwise difficult to derive, could be understood simply from the dual picture (the quantum Hall analog of this state is the state called PH-Pfaffian in Ref. [4] and involves BCS pairing of Dirac composite fermions in the *s*-wave channel). Much effort has been made to derive such duality [37–42]. In one approach, one discretizes the system in one spatial dimension and utilizes (1+1)D bosonization [40]. In another approach, the duality between the two fermion theories appears as one particular case of a whole web of (2+1)D dualities that can be derived from an elemental duality between a bosonic field theory and a fermionc field theory [41,42], establishing a connection with an extensive literature on duality between (2+1)D Chern–Simons theories (see, e.g., [43]). The latter approach, in particular, clarifies issues related to the parity anomaly matching. It is unclear, however, if a single two-component fermion coupled to a dynamical gauge field is stable with respect to spontaneous symmetry breaking. Numerical efforts are required to settle this question. There is a claim that QED$_3$ does not spontaneously generate a gap for two flavors of two-component fermion [44], in contrast to the general belief. The situation with one flavor is not clear.

According to P. Freund [45], Nambu was fascinated with the philosophy of science of Mitsuo Take-tani, according to which scientific development passes through three stages: Phenomenon, Substance, and Essence. In the story that we have just surveyed, I guess Nambu would pick the FQH plateaux as the Phenomenon and the composite fermion as the Substance. Are we catching, in the fermionic particle–vortex duality and other field-theoretic dualities in (2+1)D, a glimpse of the Essence?

Acknowledgements

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