Viscosity of Quantum Hall Fluids

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The viscosity of quantum fluids with an energy gap at zero temperature is related to the adiabatic curvature on the space parametrizing flat background metrics. For quantum Hall fluids on two-dimensional tori, the quantum viscosity is computed. It turns out to be isotropic, constant, and proportional to the magnetic field strength.

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Classically, the elastic modulus λ and viscosity η are tensors of rank 4, which relate the stress tensor σ to the strain tensor u, and the strain-rate tensor $\dot{u}$. For small deformations [1],

$$\sigma_{\alpha\beta} = \sum_{\gamma,\delta} \lambda_{\alpha\beta\gamma\delta} u_{\gamma\delta} - \sum_{\gamma,\delta} \eta_{\alpha\beta\gamma\delta} \dot{u}_{\gamma\delta}, \quad \alpha,\beta,\gamma,\delta = 1,\dots,d, \tag{1}$$

where d is the dimension of configuration space. The elastic modulus tensor $\lambda_{\alpha\beta\gamma\delta}$ is symmetric in the first and second pairs of indices, and it is also symmetric under the transposition of the first and second pairs of indices.

Since the tensors σ, u, and $\dot{u}$ are symmetric, the components of the viscosity tensor obey the relations $\eta_{\alpha\beta\gamma\delta} = \eta_{\beta\alpha\gamma\delta} = \eta_{\alpha\beta\delta\gamma}$ [1,2]. With respect to the substitution of indices $(\alpha\beta\gamma\delta) \mapsto (\gamma\delta\alpha\beta)$, the viscosity tensor splits into symmetric and antisymmetric parts, $\eta = \eta^S + \eta^A$, where

$$\eta^S_{\alpha\beta\gamma\delta} = \eta^S_{\gamma\delta\alpha\beta}, \qquad \eta^A_{\alpha\beta\gamma\delta} = -\eta^A_{\gamma\delta\alpha\beta}. \tag{2}$$

The symmetric part, being associated with dissipation, is a positive quadratic form in strain rates. For an isotropic fluid, η^S depends on two coefficients of viscosity [3]. This is the normal situation in Newtonian fluid mechanics.

The antisymmetric part of the viscosity tensor η^A describes nondissipative response. It vanishes for systems with time-reversal symmetry (this is a consequence of the Onsager relation [4]). In general, in two dimensions, η^A contains three independent coefficients η^A_{1112}, η^A_{1122}, η^A_{1222}. For an isotropic fluid in Euclidean plane one has

$$\eta^A_{1112} = \eta^A_{1222} = \eta, \qquad \eta^A_{1122} = 0, \tag{3}$$

where η is a single *coefficient of viscosity*. In three dimensions isotropy implies $\eta^A = 0$.

Quantum fluids can have a ground state which is separated by a finite gap from the rest of the spectrum. At zero temperature, such a fluid will have a nondissipative response, with $\eta^S = 0$. The antisymmetric part η^A may or may not vanish at zero temperature. Quantum fluids with energy gap and broken time-reversal symmetry will, in general, have $\eta^A \neq 0$ at zero temperature. A quantum Hall fluid with a full Landau level gives such an example. The viscosity tensor at low temperature could then be dominated by the nondissipative part η^A. The study of the nondissipative viscosity bears analogies with the quantum Hall effect [5], the Magnus force in superconductors [6], and even with gravity [7,8]. Like the Hall conductance and the Magnus force, the (nondissipative) viscosity is related to the adiabatic curvature [9] and to topological invariants [10]. The connection with gravity comes about because the adiabatic curvature relevant for viscosity is a 2-form on the space parametrizing flat background metrics.

Let us first recall a general fact from the theory of adiabatic response [11]. Consider a family of Hamiltonians $H(X)$ which depend smoothly on a set of parameters $X = \{X_1,\dots,X_n\}$ (X denotes a point in parameter space while x denotes a point in configuration space). Let $|\psi(X)\rangle$ be a (normalized) nondegenerate state of $H(X)$, with energy $E(X)$. Let $X(t)$ be a path in parameter space which is traversed adiabatically; $\dot{X}$ is the velocity along the path. We assume throughout that the state stays nondegenerate along the path.

By the principle of virtual work, $-\delta H/\delta X$ is the observable corresponding to the generalized force related to δX [12]. Adiabatic response theory says that, in the adiabatic limit,

$$\left\langle \frac{\partial H}{\partial X_j} \right\rangle = \frac{\partial E}{\partial X_j} + \sum_{k=1}^{n} \Omega_{jk} \dot{X}_k, \tag{4}$$

where Ω_{jk} is the (antisymmetric) adiabatic curvature [9]:

$$\Omega_{jk} = \mathrm{Im}\langle \partial_j \psi | \partial_k \psi \rangle, \qquad \Omega_{jk} = -\Omega_{kj}, \quad \partial_j = \frac{\partial}{\partial X_j}. \tag{5}$$

If H is time-reversal invariant, then $\Omega_{ij} = 0$ [9]. The antisymmetry of Ω implies no dissipation: There is no

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change in energy if the system is taken along a closed loop in parameter space.

In the special case where $|\psi\rangle$ is a (normalized) multiparticle state corresponding to N noninteracting fermions in the single particle (normalized) states $|\varphi_\ell\rangle$ with energies $E_\ell(X)$ (all depending smoothly on parameter X), one has

$$\Omega_{jk} = \operatorname{Im} \sum_{\ell=1}^{N} \langle \partial_j \varphi_\ell | \partial_k \varphi_\ell \rangle,$$
$$E(X) = \sum_{\ell=1}^{N} E_\ell(X). \tag{6}$$

To apply all this to viscosity, one needs to identify appropriate parameters so that $-\partial H/\partial X_j$ is related to the stress tensor and $\dot{X}$ to the rate of strain. Consider a fluid confined to a given domain in d-dimensional Euclidean space (planar parallelogram shown in Fig. 1 is an example for $d = 2$). A uniform deformation is represented by the (symmetric) tensor u of constant strain [1]:

$$u_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\beta}{\partial x_\alpha} + \frac{\partial u_\alpha}{\partial x_\beta}\right), \tag{7}$$

where $u_\alpha = x'_\alpha - x_\alpha$ are the coordinates of the *displacement vector,* and x' is a linear transformation of x. In two dimensions, the space of linear deformations (i.e., constant strains) is three dimensional. A two-dimensional subspace is associated with shears (transformations that preserve the volume), and the transverse direction may be associated with scaling.

An equivalent point of view is to keep the domain fixed and to consider deformations of the metric instead. The parameter space of flat metrics is then the same as that of constant strains. Explicit formulas relating strains and metrics, as well as convenient coordinates on the corresponding parameter space, will be given later.

Now let H denote the Hamiltonian in a domain D in d-dimensional space with metric g. By the principle of virtual work [12], $-\delta H/\delta u$ is the observable associated with *total* stress tensor, $\int_D [\delta H/\delta u(x)]\, d\,\mathrm{vol}(x)$.

Adiabatic deformations of constant strain give the quantum version of Eq. (1):

$$\left\langle \frac{\partial H}{\partial u_{\alpha\beta}} \right\rangle = \frac{\partial E}{\partial u_{\alpha\beta}} + \sum_{\gamma,\delta=1}^{d} \Omega_{\alpha\beta\gamma\delta} \dot{u}_{\gamma\delta}. \tag{8}$$

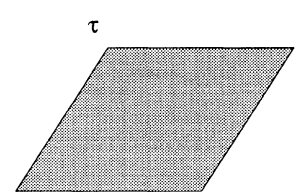

FIG. 1. Parallelogram in the complex plane associated with the lattice with periods $\omega_1 = (V/\tau_2)^{1/2}$, $\omega_2 = (V/\tau_2)^{1/2}\tau$.

Here $-\langle \partial H/\partial u_{\alpha\beta}\rangle$ is the total stress tensor which is related to the (ordinary) stress tensor σ by $\sigma_{\alpha\beta} = -(1/V)\langle \partial H/\partial u_{\alpha\beta}\rangle$. The first term on the right-hand side of Eq. (8) means that the elastic modulus tensor is $\lambda_{\alpha\beta\gamma\delta} = \frac{1}{2}\,\partial^2 E/\partial u_{\alpha\beta}\,\partial u_{\gamma\delta}$. It is symmetric in the pairs of indices $(\alpha\beta)$ and $(\gamma\delta)$ and under the transposition of $(\alpha\beta)$ and $(\gamma\delta)$.

The adiabatic curvature $\Omega_{\alpha\beta\gamma\delta}$ is antisymmetric with respect to the transposition of the first and second pairs of indices, and plays the role of nondissipative viscosity. The adiabatic curvature Ω and the viscosity η_A are two related notions of viscosity, in rough analogy to conductance and conductivity. Here Ω is a dimensionless measure of quantum viscosity and has units of Planck constant $\hbar$. It relates the total stress (whose dimension coincides with energy) to the strain rate. The conventional viscosity η_A has a dimension of $\hbar/V$, where V is the volume and relates the stress (whose dimension coincides with pressure) to the strain rate. The two notions of viscosity are related by

$$\eta^A = \frac{\Omega}{V}. \tag{9}$$

Whether one chooses to focus on Ω or on η^A is to an extent a matter of taste. In the example we shall consider, Ω is quantized, whereas η_A is a local characteristic of the fluid.

Let us illustrate these ideas with a concrete example. Consider two-dimensional quantum Hall fluid on a torus represented by the unit square in the configuration space $(x,y) \in Q = [0,1] \times [0,1]$ with opposite sides identified. As coordinates on the space of flat metrics on Q, we use the area $V = (\det g)^{1/2}$ and complex parameter $\tau = \tau_1 + i\tau_2$ describing shears, i.e., deformations that preserve the volume. In this parametrization, flat metrics are

$$g(V,\tau) = \frac{V}{\tau_2}(dx^2 + 2\tau_1\, dx\, dy + |\tau|^2 dy^2),$$
$$x, y, \in Q. \tag{10}$$

An equivalent point of view is to associate this configuration torus to the lattice in the complex plane with periods

$$\omega_1 = \left(\frac{V}{\tau_2}\right)^{1/2}, \qquad \omega_2 = \left(\frac{V}{\tau_2}\right)^{1/2}\tau, \tag{11}$$

as shown in Fig. 1. The complex coordinate z on the parallelogram in Fig. 1 is related to coordinates $(x,y) \in Q$ by $z = (V/\tau_2)^{1/2}\,(x + \tau y)$. In terms of coordinate z, one has $g = |dz|^2$; i.e., the metric is Euclidean.

The Landau Hamiltonian describes the kinetic energy of a charged (spinless) particle in a constant magnetic field and Aharonov-Bohm gauge fields. It is given by the 5-parameter family

$$H(V,\tau,\phi) = \frac{1}{V\tau_2}[|\tau|^2 D_x^2 - \tau_1(D_x D_y + D_y D_x) + D_y^2], \tag{12}$$

where $D_x = -i\partial_x + 2\pi(By + \phi_1 + B/2)$, $D_y = -i\partial_y + 2\pi(\phi_2 + B/2)$; ϕ_1 and ϕ_2 are associated with two Aharonov-Bohm fluxes, and the integer B is the number of magnetic flux quanta through torus. (Note that in our units, 1 is the unit of quantum flux hc/e.) We impose the usual magnetic translation boundary conditions [13]:

$$\psi(x+1,y) = \psi(x,y),$$
$$\psi(x,y+1) = e^{-2\pi i B x}\psi(x,y). \tag{13}$$

The (single particle) ground state is B-fold degenerate with energy $E = 2\pi B/V$ independent of τ and ϕ. The ground state of a full Landau level has energy $E = 2\pi B^2/V$ and is separated by a gap from the rest of the spectrum. (For a nonrelativistic spinning electron, described by the Pauli equation, one has $E = 0$.) It follows from Eq. (8) that the only nonzero component of the elastic modulus tensor is $\lambda_{VV} = 2\pi B^2/V^3$. All other components of λ vanish, as they should, for a fluid; the two shear modes are soft [14].

An orthonormal, smooth, family of single particle states that span the lowest Landau level is given by theta functions [15,16]:

$$\varphi_\ell(x,y) = \frac{(2\tau_2 B)^{1/4}}{V^{1/2}} \times \sum_{n=-\infty}^{\infty} e^{i\pi\tau B(\tilde{y}+n)^2} \times e^{-2i\pi(\phi_2+B/2)(\tilde{y}+n)} e^{2\pi i(nB+\ell)x}, \tag{14}$$

where $\tilde{y} = y + \ell/B + \phi_1/B + 1/2$ and $\ell = 0,\ldots,B-1$.

A computation shows that the component of the adiabatic curvature in Eq. (6), which corresponds to the ℓth state, is

$$\sum_{i<j} \Omega^\ell_{ij}\, dX_i \wedge dX_j = \frac{d\tau_1 \wedge d\tau_2}{4\tau_2^2} - \frac{2\pi}{B}\, d\phi_1 \wedge d\phi_2 \tag{15}$$

(cf. Ref. [17]). The basic equations of transport follow from Eqs. (6), (8), and (15):

$$\left\langle \frac{\partial H}{\partial V} \right\rangle = -2\pi B^2/V^2, \qquad \left\langle \frac{\partial H}{\partial \tau_1} \right\rangle = \frac{B}{4\tau_2^2}\,\dot{\tau}_2,$$
$$\left\langle \frac{\partial H}{\partial \tau_2} \right\rangle = -\frac{B}{4\tau_2^2}\,\dot{\tau}_1. \tag{16}$$

To translate these into the language of viscosity tensor, let us note that from the definition of strain, Eq. (7), we get the following formulas for the strain rates:

$$du_{11} = \frac{1}{2}\left(\frac{dV}{V} - \frac{d\tau_2}{\tau_2}\right), \qquad du_{12} = \frac{1}{2}\frac{d\tau_1}{\tau_2},$$
$$du_{22} = \frac{1}{2}\left(\frac{dV}{V} + \frac{d\tau_2}{\tau_2}\right). \tag{17}$$

Combining them with Eqs. (9) and (16), we obtain for the viscosity part of the adiabatic curvature

$$\Omega = \frac{B}{2}\, du_{12} \wedge (du_{11} - du_{22}). \tag{18}$$

It follows that the viscosity tensor has components

$$\eta^A_{1112} = \eta^A_{1222} = \frac{B}{8V}, \qquad \eta^A_{1122} = 0. \tag{19}$$

Comparison with Eq. (3) shows that the viscosity is isotropic, and the coefficient of viscosity $\eta = B/8V$ is proportional to the magnetic field strength B/V. (In our notation, B is the total flux.) It is remarkable that the viscosity is independent of the geometry of the torus (expressed by τ). If we choose to characterize the viscosity by the adiabatic curvature, then the corresponding viscosity constant $B/8$ is quantized (since B is an integer), though not in general an integer.

The physical interpretation of the nondissipative quantum viscosity is as follows. Consider two-dimensional Hall fluid on a surface of a cylinder. Compressing it in the radial direction (or in the axial direction) results in a twist rate of one boundary circle relative to the other. Inversely, a shear of the two boundary circles results in a compression rate of the radial direction and stretching rate of the axial direction.

It may be instructive to compare the transport equations associated with the quantum viscosity with the transport equations associated with the Hall conductance for Landau levels on the torus. These, too, follow from Eqs. (4) and (15), namely,

$$\left\langle \frac{\partial H}{\partial \phi_1} \right\rangle = -2\pi\dot{\phi}_2, \qquad \left\langle \frac{\partial H}{\partial \phi_2} \right\rangle = 2\pi\dot{\phi}_1. \tag{20}$$

The generalized force in this case is the current operator $-\partial H/\partial\phi$, and the generalized velocity $\dot{\phi}$ is the electromotive force. In Eq. (20) the conductance of a full Landau level is 2π in our units and e^2/h in ordinary units. It is isotropic and quantized.

To appreciate the geometric significance of Eq. (16), we will look at the space of parameters (V,τ,ϕ) more thoroughly. The modular group SL(2, Z) acts naturally on this parameter space as a symmetry group of the family of Landau Hamiltonians $H(V,\tau,\phi)$, Eq. (12). To describe this action explicitly, let us note first that an element

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,Z)$$

provides an isometry between metrics $g(V,\tau)$ and $g(V,\tau')$ with $\tau' = (a\tau + b)/(c\tau + d)$ by the formula

$$x \mapsto ax - by, \qquad y \mapsto -cx + dy.$$

We can therefore consider M as a unitary operator in the Hilbert space. The operator $MH(V,\tau,\phi)M^{-1}$ coincides, up to a gauge transformation, with $H(V,\tau',\phi')$, where

$$\phi_1' = d\phi_1 + b\phi_2, \quad \phi_2' = c\phi_1 + a\phi_2.$$

The last formula shows, in particular, that the family of Landau Hamiltonians $H(V,\tau,0)$ is SL(2, Z) invariant and the corresponding parameter space is nothing but the moduli space of elliptic curves [15]. It is a 2-sphere with two conical points and one puncture. It can be conveniently represented by the fundamental domain of SL(2, Z) action on the upper half plane of complex variable τ, Fig. 2. It coincides with the parameter space of flat metrics on a torus of fixed area. This is an analog of the Aharonov-Bohm flux torus in the theory of Hall conductance.

The geometric significance of the viscosity in Eq. (16) is now apparent: The first term in Eq. (15) is the invariant area form on the upper half plane of complex variable τ. Similar to the Hall conductance, which is constant in the flux space, the viscosity is proportional to the area form on the moduli space of elliptic curves.

In the theory of quantum Hall effect, the conductance is associated with a topological invariant Chern number, which is an integer. This integer comes from integrating the adiabatic curvature over the parameter space flux torus. In the case of Eq. (20) this integer is -1. For the viscosity, the situation is almost the same. Integration of the adiabatic curvature over the moduli space [or fundamental domain F of the group SL(2, Z)] gives

$$\frac{1}{2\pi}\int_F \frac{B}{4}\frac{d\tau_1 \wedge d\tau_2}{\tau_2^2} = \frac{B}{24}, \tag{21}$$

where we have used the fact that the area of the fundamental domain is $\pi/3$. Though not an integer in general, this is still a topological invariant Chern number (in the orbifold sense) of the ground state bundle on the parameter space. (It is not an integer, because the parameter space is not a smooth compact manifold in this case.)

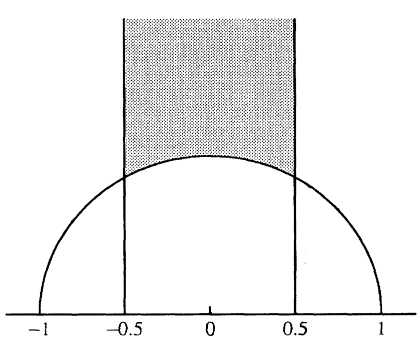

FIG. 2. Fundamental domain of SL(2, Z).

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