Nuclear Physics B243 (1984) 205–211


# NON PERTURBATIVE ANOMALIES IN HIGHER DIMENSIONS

S. ELITZUR[1] and V.P. NAIR

*The Institute for Advanced Study, Princeton, New Jersey 08540, USA*

Received 16 March 1984

We study how nonperturbative anomalies can occur in dimensions higher than four and their implications on the consistency of the theory.

The Adler–Bell–Jackiw type perturbative anomalies have recently been calculated for any even dimensional gauge theory. Their relation to the higher homotopy groups of the gauge group has also been clarified [1, 2]. Nearly two years ago Witten showed that an SU(2) gauge theory in four dimensions with an odd number of Weyl fermions was inconsistent due to a nonperturbative anomaly [3]. A natural question would thus be: are there simple higher dimensional generalizations of the nonperturbative anomaly? We analyze this question in this paper. The search for general nonperturbative anomalies, we believe, clarifies the nature of the original SU(2) anomaly. This may also be of some relevance to certain Kaluza–Klein type theories, although this aspect will not be developed here.

We start with a rephrasing of the SU(2) anomaly in terms of the SU(3) perturbative anomaly (see ref. [4]). This will set the pattern for the general case where we study the nonperturbative anomalies of a gauge group H (which is perturbatively anomaly free) by embedding H in a gauge group G and studying the perturbative anomaly of G. The starting point is the transformation property of the fermion measure $\mathrm{d}\mu(\psi)$ (in the functional integral) under finite gauge transformations [5, 6]. If $g$ denotes an element of the gauge group G,

$$\mathrm{d}\mu(g\psi) = \mathrm{d}\mu(\psi) \exp\left[i\Gamma(g, A, F)\right], \tag{1}$$

where $\Gamma(g, A, F)$ is the Wess–Zumino action [2, 4, 7]. For $g = e^{\varepsilon}$ $\delta\Gamma/\delta\varepsilon$ is the perturbative anomaly of the theory associated with the direction of $\varepsilon$ in the Lie algebra $\underset{\sim}{G}$ of G. $\Gamma(g, A, F)$ can be obtained by integration of the anomaly or more directly as follows. For a gauge theory in $2n$ dimensions, we start with the form $\mathrm{Tr}\,(F^{n+1})$ in $2n+2$ dimensions. Here $F = \mathrm{d}A + AA$. $F$ is the differential two-form corresponding to the field strength tensor. $A$ and $F$ are also Lie algebra matrices in the representation corresponding to the fermions, which is in general reducible. (In

[1] Permanent address: Racah Institute of Physics, Hebrew University, Jerusalem, Israel.

writing $F^{n+1}$, standard wedge products are understood.) We then have [2]*

$$\mathrm{Tr}\,(F^{n+1}) = \mathrm{d}\omega_{2n+1}(A, F)\,, \tag{2}$$

Now define

$$\gamma(g, A, F) = \frac{i^{n+1}}{(2\pi)^n (n+1)!}[\omega_{2n+1}(A^g, F^g) - \omega_{2n+1}(A, F)]\,, \tag{3}$$

where $A^g$, $F^g$ denote the gauge transforms of $A$ and $F$ by $g$. $\Gamma(g, A, F)$ is now defined in terms of $\gamma$ as

$$\Gamma(g, A, F) = \int_{\mathrm{D}} \gamma(g, A, F)\,. \tag{4}$$

$\gamma(g, A, F)$ is a $(2n+1)$-form. Eqs. (3) and (4) thus involve an extension of $g$, $A$, $F$ from the $2n$-dimensional spacetime (which we take to be compactified to $\mathrm{S}^{2n}$) to a $(2n+1)$-dimensional manifold. This $(2n+1)$-dimensional manifold is taken to be a disc D whose boundary is spacetime $\mathrm{S}^{2n}$. This extension is not unique. (There are several discs in $(2n+1)$ dimensions whose boundary is $\mathrm{S}^{2n}$.) If D and D′ are two discs with the spacetime $\mathrm{S}^{2n}$ as boundary, then we have between D and D′

$$\Gamma_{\mathrm{D}}(g, A, F) - \Gamma_{\mathrm{D}'}(g, A, F) = \int_{\mathrm{S}^{2n+1}} \gamma(g, A, F)\,. \tag{5}$$

Thus between D and D′, the transformation of $\mathrm{d}\mu(\psi)$ differs by a phase factor $\exp[i\int_{\mathrm{S}^{2n+1}} \gamma(g, A, F)]$. The fermion measure can be single-valued and eq. (1) meaningful only if we have the condition [4, 6],

$$\exp\left[i\int_{\mathrm{S}^{2n+1}} \gamma(g, A, F)\right] = 1\,. \tag{6}$$

This is a consistency requirement for any gauge group G. Now $\gamma(g, A, F)$ can be written as $\mathrm{d}\alpha_{2n}(g, A, F) + \tilde{\gamma}(g)$. $\tilde{\gamma}(g)$ is a (closed) $(2n+1)$-form proportional to $\mathrm{Tr}\,[(g^{-1}\,\mathrm{d}g)^{2n+1}]$. $\alpha_{2n}(g, A, F)$ is irrelevant for eq. (6).

We now turn to nonperturbative anomalies for a gauge group H. Of course, H has to be free of perturbative anomalies for this question to be meaningful. We embed H in a gauge group G. The H gauge theory is obtained on spacetime $\mathrm{S}^{2n}$ by restricting $A$ and $F$ to be in the Lie algebra $\underline{\mathrm{H}}$. Further the fermion content of G is so chosen that upon reduction to H we get the required H representations and a number of H singlets. Consider now $\gamma(g, A, F)$. We allow $g$ to be an arbitrary element of G with the condition that $g \in \mathrm{H}$ on the boundary $\partial\mathrm{D} = \mathrm{S}^{2n}$. (In the interior of the disc, $g$ can move outside H. $A$ and $F$ can be kept in H throughout the interior of D.) The motivation for this construction comes about in the following way. Nonperturbative anomalies are associated with gauge transformations which cannot

* We use the notation of ref. [2] in the first part of our discussion.

be continuously connected to the identity. Thus for example for SU(2) we have two types of gauge transformations which approach 1 at infinity, since $\pi_4[\mathrm{SU}(2) = Z_2$. A transformation $\tilde{h}$ which belongs to the nontrivial type cannot be connected to the identity by SU(2) transformations. Since $\pi_4[\mathrm{SU}(3)] = 0$, if we embed SU(2) in SU(3), $\tilde{h}$ can be connected to the identity by an SU(3) transformation. Any nonperturbative anomaly for SU(2) would then result from the perturbative anomalies of SU(3) transformations, which are given by $\gamma(g, A, F)$. Now, since H is an anomaly free subgroup of G, $\gamma$ is unchanged under H gauge transformations, namely $\gamma(gh, A^{h^{-1}}, F^{h^{-1}}) = \gamma(g, A, F)$ for $h \in \mathrm{H}$. This can be easily checked using definition (3) noting that by (2), $\omega_{2n+1}(A, F) = 0$ for any $A$ and $F$ in the anomaly free subgroup H. Thus $\gamma$ really depends on G/H. Since $g \in \mathrm{H}$ on $\partial\mathrm{D}$ and $\gamma$ depends only on G/H,

$$\int_{\mathrm{D}} \gamma(g, A, F) = \int_{\mathrm{S}^{2n+1}} \gamma(g, A, F) . \tag{7}$$

The transformation of the measure is now

$$\mathrm{d}\mu(g\psi) = \mathrm{d}\mu(\psi) \exp\left[ i \int_{\mathrm{S}^{2n+1}} \gamma(g, A, F) \right] \tag{8}$$

for those $g$ which go to an element of H on $\partial\mathrm{D}$. (Notice that in (7) we are integrating over a disc in G which becomes a sphere in G/H. Thus the integral is not the same as in (6) where we have a sphere in G.) Since $\mathrm{d}\gamma = 0$, $\int_{\mathrm{S}^{2n+1}} \gamma$ is a homotopic invariant. $\int_{\mathrm{S}^{2n+1}} \gamma(g, A, F)$ thus gives a mapping from $\pi_{2n+1}(\mathrm{G/H})$ into real numbers, i.e. for any $[g]$, the equivalence class of $g$'s which are homotopic to one another, we get a real number. The composition laws in $\pi_{2n+1}(\mathrm{G/H})$ are also preserved, i.e. the mapping is a homomorphism. The normalization of this integral and its value for the nontrivial H transformations can be obtained by considering exact homotopy sequences.

Let us consider the SU(2) gauge theory in four dimensions in detail. Here H = SU(2), G = SU(3). We have the exact sequence [8],

$$\begin{array}{ccccccc}
\pi_5[\mathrm{SU}(3)] & \to & \pi_5\left[\dfrac{\mathrm{SU}(3)}{\mathrm{SU}(2)}\right] & \to & \pi_4[\mathrm{SU}(2)] & \to & \pi_4[\mathrm{SU}(3)] \,, \\
Z & \to & Z & \to & Z_2 & \to & 0 \,, \\
 & & m = 2k+1 & \to & -1 & \to & 0 \,, \\
k & \to & m = 2k & \to & 1 & \to & 0 \,.
\end{array} \tag{9}$$

Exactness dictates the mappings indicated. Since $\pi_4[\mathrm{SU}(3)] = 0$, all of the $\pi_4[\mathrm{SU}(2)] = Z_2$ should be the image of $\pi_5[\mathrm{SU}(3)/\mathrm{SU}(2)] = Z$. Elements of $\pi_5[\mathrm{SU}(3)/\mathrm{SU}(2)]$ corresponding to odd integers map onto the nontrivial SU(2) transformations, i.e. corresponding to $-1$ of $Z_2$. All of $\pi_5[\mathrm{SU}(3)/\mathrm{SU}(2)]$ cannot be the image of $\pi_5[\mathrm{SU}(3)]$. $\pi_5[\mathrm{SU}(3)]$ has to map onto the even elements of

$\pi_5$[SU(3)/SU(2)]. For one Weyl triplet under SU(3) (doublet under SU(2)) of fermions, we have [1, 2]

$$\int_{S^5} \gamma(g_1, A, F) = 2\pi\,, \tag{10}$$

where $g_1$ is the mapping $S^5 \to SU(3)$ which corresponds to the generating element of $\pi_5$[SU(3)], i.e. represented by $1 \in Z = \pi_5[SU(3)]$.

Consider $\tilde{h}(x)$ corresponding to the nontrivial gauge transformation in SU(2). We extend this to $\tilde{g}(x)$ on D such that $\tilde{g}(x) = \tilde{h}(x)$ on $\partial D = S^4$. $\tilde{g}(x)$ is not a mapping from $S^5 \to SU(3)$, but it defines a mapping $S^5 \to SU(3)/SU(2)$. Choose a $\tilde{g}(x)$ which corresponds to the generating element of $\pi_5$[SU(3)/SU(2)]. Let

$$\int_D \gamma(\tilde{g}, A, F) = \int_{S^5} \gamma(\tilde{g}, A, F) = Q[\tilde{g}]\,. \tag{11}$$

If we take $\tilde{g}$ twice, we get

$$\int \gamma(\tilde{g}^2) = 2Q[\tilde{g}] \tag{12}$$

by linearity of $\int \gamma$. $\tilde{g}$ taken twice goes to an element of SU(2) on $\partial$D which can be deformed to the identity continuously. So $\tilde{g}$ taken twice corresponds to a mapping $S^5 \to SU(3)$. Since $\tilde{g}$ was chosen to generate $\pi_5$[SU(3)/SU(2)], by the exactness of (9), $\tilde{g}$ taken twice generates $\pi_5$[SU(3)], i.e. $\tilde{g}^2$ is homotopic to $g_1$ of eq. (10) and we have $\int \gamma(\tilde{g}^2) = 2\pi$. Eq. (12) then implies

$$Q[\tilde{g}] = \int_D \gamma(\tilde{g}, A, F) = \int_{S^5} \gamma(\tilde{g}, A, F) = \pi\,. \tag{13}$$

Thus for nontrivial SU(2) gauge transformations, corresponding to odd elements of $\pi_5$[SU(3)/SU(2)], $d\mu(\psi) \to -d\mu(\psi)$. The theory has a discrete anomaly and is inconsistent.

The pattern and generalization are now clear. In $2n$ dimensions, we consider a gauge theory with gauge group H and fermion representations $r$ such that (i) H is free of perturbative anomalies, (ii) $\pi_{2n}(H) \neq 0$. We then embed H in a group G for which $\pi_{2n}(G) = 0$. Fermions are put in a representation $R$ of G such that $R = r + 1$'s upon reduction to H. With $A$ and $F$ in H, $\int_D \gamma(g, A, F)$ gives a mapping $\pi_{2n+1}(G/H) \to$ real numbers. The normalization is fixed by an equation similar to (10) for the lowest nontrivial element in $\pi_{2n+1}(G)$, i.e. for every generator of $\pi_{2n+1}(G)$. The homotopy sequence

$$\pi_{2n+1}(G) \to \pi_{2n+1}(G/H) \to \pi_{2n}(H) \to \pi_{2n}(G) = 0 \tag{14}$$

then gives the value of $\int \gamma$ for nontrivial H gauge transformations.

The simplest cases to analyze would be for $\pi_{2n+1}(\mathrm{G})=\mathrm{Z}$, $\pi_{2n+1}(\mathrm{G}/\mathrm{H})=\mathrm{Z}$. SU($n$) theories in $2n$ dimensions satisfy these conditions [9]*.

$$\pi_{2n}[\mathrm{SU}(n)]=\mathrm{Z}_{n!}\,,\qquad \pi_{2n+1}\left[\frac{\mathrm{SU}(n+1)}{\mathrm{SU}(n)}\right]=\mathrm{Z}\,,\qquad \pi_{2n+1}[\mathrm{SU}(n+1)]=\mathrm{Z}\,.$$

We thus have $n!$ distinct gauge transformations in $2n$ dimensions. But first we have to ensure that SU($n$) perturbative anomalies are cancelled. Unlike SU(2) in four dimensions this is not zero for the fundamental representation (denoted $f$.) We thus choose a representation $r$ for which the anomalies are zero. Thus in $\gamma(g, A, F)$, $g$, $A$, $F$ are matrices in the representation $R$ of SU($n+1$). The homotopy sequence is

$$\begin{array}{ccccccc} \pi_{2n+1}[\mathrm{SU}(n+1)] & \to & \pi_{2n+1}\left[\dfrac{\mathrm{SU}(n+1)}{\mathrm{SU}(n)}\right] & \to & \pi_{2n}[\mathrm{SU}(n)] & \to & \pi_{2n}[\mathrm{SU}(n+1)]\,, \\ \mathrm{Z} & \to & \mathrm{Z} & \to & \mathrm{Z}_{n!} & \to & 0\,. \end{array} \tag{15}$$

Call the basic nontrivial transformation (generator of $\pi_{2n}[\mathrm{SU}(n)]$) as $\tilde{h}$, extended to $\tilde{g}$. By our earlier arguments, $\tilde{g}$ taken $n!$ times corresponds to a mapping $g_1:\mathrm{S}^{2n+1}\to \mathrm{SU}(n+1)$. Call

$$\int_{\mathrm{D}} \gamma(\tilde{g}, A, F)_R=\int_{\mathrm{S}^{2n+1}} \gamma(\tilde{g}, A, F)_R = Q[\tilde{g}]\,. \tag{16}$$

Then

$$\int_{\mathrm{S}^{2n+1}} \gamma(g_1, A, F)_R = n!\int_{\mathrm{D}} \gamma(\tilde{g}, A, F)_R = n!Q[\tilde{g}]\,. \tag{17}$$

Now

$$\int_{\mathrm{S}^{2n+1}} \gamma(g_1, A, F)_R = A_R\int_{\mathrm{S}^{2n+1}} \gamma(g_1, A, F)_f = 2\pi A_R\,. \tag{18}$$

Thus

$$\int_{\mathrm{D}} \gamma(\tilde{g}, A, F)_R = 2\pi\frac{A_R}{n!}\,. \tag{19}$$

$A_R$ is a certain index relating representation $R$ to the fundamental representation. We define this below. For SU($n+1$) theories, it is $\gamma$ in the fundamental representation which is normalized to $2\pi$ for $g_1$, generator of $\pi_{2n+1}[\mathrm{SU}(n+1)]$. This is why we need a factor $A_R$ in eq. (18). $\gamma(g, A, F)_R$ is derived from $\mathrm{Tr}\,(F^{n+1})_R$ and has the same Lie algebra structure. Now

$$\mathrm{Tr}\,(F^{n+1})_R = A_R\,\mathrm{Tr}\,(F^{n+1})_f + \sum_{k>0}^{n} C_{k,n}\,\mathrm{Tr}\,(F^k)_f\,\mathrm{Tr}\,(F^{n+1-k})_f+\cdots\,. \tag{20}$$

* For large enough groups, the homotopy groups follow the Bott periodicity pattern. Notice that we are still below this limit.

This defines $A_R$. The lower traces do not contribute in $\int_{S^{2n+1}} \gamma$. Consider for instance a term like $\mathrm{Tr}\, F^2\, \mathrm{Tr}\, F^3$ relevant in eight dimensions. We have

$$\mathrm{Tr}\, F^2\, \mathrm{Tr}\, F^3 = \mathrm{d}\omega_3\, \mathrm{d}\omega_5 = \mathrm{d}(\omega_3\, \mathrm{d}\omega_5)\,. \tag{21}$$

The contribution of this term to $\gamma$ is

$$[\omega_3(A^g, F^g) - \omega_3(A, F)]\, \mathrm{d}\omega_5(A, F)\,, \tag{22}$$

since $\mathrm{d}\omega_5$ is gauge invariant. This is equal to

$$\gamma_3\, \mathrm{d}\omega_5 = -\mathrm{d}(\gamma_3 \omega_5)\,, \tag{23}$$

since $\mathrm{d}\gamma_3 = 0$. Thus the lower trace $\mathrm{Tr}\, F^2\, \mathrm{Tr}\, F^3$ adds only an exact term to $\gamma(g, A, F)$ and gives zero upon integration.

Going back to eq. (19) we see that under a nontrivial SU($n$) transformation

$$\mathrm{d}\mu(\psi) \to \mathrm{d}\mu(\psi) \exp\left[2\pi i \frac{A_R}{n!}\right]. \tag{24}$$

If $A_R$ is not a multiple of $n!$, the theory has a nonperturbative anomaly and is inconsistent.

As a first example consider SU(4) theory in eight dimensions. The homotopy pattern is

$$\begin{array}{ccccccc} \pi_9[\mathrm{SU}(5)] & \to & \pi_9\left[\dfrac{\mathrm{SU}(5)}{\mathrm{SU}(4)}\right] & \to & \pi_8[\mathrm{SU}(4)] & \to & \pi_8[\mathrm{SU}(5)]\,, \\ \mathrm{Z} & \to & \mathrm{Z} & \to & \mathrm{Z}_{24} & \to & 0\,. \end{array}$$

$\gamma(g, A, F)$ is derived from $\mathrm{Tr}\, F^5$. For SU(4)

$$\mathrm{Tr}\,(F^5)_r = a_r\, \mathrm{Tr}\,(F^5)_4\,, \tag{25}$$

while for SU(5)

$$\mathrm{Tr}\,(F^5)_R = A_R\, \mathrm{Tr}\,(F^5)_5 + (\text{lower traces})\,.$$

From values of $a_r$ we see that an SU(4) theory with the fermion content $4+6+\bar{4}$ is free of perturbative anomalies. This can be embedded in $10+\bar{5}$ of SU(5). $A_{10}+A_{\bar{5}} = -12$. Thus the phase factor in (24) is $\exp\,[\frac{2}{24}\pi i(-12)] \in \mathrm{Z}_2$. We have a $\mathrm{Z}_2$ nonperturbative anomaly. We give some more examples in table 1. All the examples we have explored in six dimensions are free of discrete anomalies. Spacetimes with $4n$ dimensions yield several examples.

The original proof of the SU(2) anomaly relied on a mod 2 index theorem for the Dirac operator in five dimensions [3]. The above approach (based on [4]) avoids the index theorem. This has virtues and vices. The index theorem approach requires some knowledge of the spectrum of the Dirac operator. When we have cancellation of perturbative anomalies among different chiral representations this is not particularly simple. Our approach is easier in this case. However the proof applies only

TABLE 1
Some examples in eight dimensions

| SU(4) representation | SU(5) representation | $A_R$ for SU(5) | Discrete anomaly |
|---|---|---|---|
| $4+6+\bar{4}$ | $10+\bar{5}$ | $-12$ | $Z_2$ |
| $4+10+33\ \bar{4}$'s | $15+33\ \bar{5}$'s | $-12$ | $Z_2$ |
| $\bar{4}+6+\overline{10}+20$ | $40$ | | |
| $2\ 4+10$'s | $2\ 15$'s | 96 | none |
| $2\ \bar{4}$'s | $2\ \bar{5}$'s | | |
| $\bar{4}+6+\overline{10}+20$ | $40$ | | |
| $2\ 4+10$'s | $2\ 15$'s | 108 | $Z_2$ |
| $\bar{4}+\bar{6}$ | $\overline{10}$ | | |
| $\bar{4}$ | $\bar{5}$ | | |
| $4+\bar{4}+10+15+36$ | $70$ | | |
| $2\,(\bar{4}+6+\overline{10}+20)$'s | $2\ 40$'s | 168 | none |
| $\bar{4}+\overline{10}$ | $\overline{15}$ | | |
| $12\ \bar{4}$'s | $12\ \bar{5}$'s | | |
| $4+\bar{4}+10+15+36$ | $70$ | | |
| $\bar{4}+\overline{10}+20''$ | $35$ | 276 | $Z_2$ |
| $2\,(4+\bar{6}+10+\overline{20})$'s | $2\ \overline{40}$'s | | |
| $7\,(4+6)$'s | $7\ 10$'s | | |

for cases where the homotopy groups fall into a certain pattern. We may miss some cases of discrete anomalies.

The simplest way to make a theory with $Z_N$ anomaly consistent is to replicate the fermion content $N$ times, constructing $N$ generations. In general this still gives a chiral theory. Although our examples give only $Z_2$ anomalies, $Z_N$ anomalies for $N>2$ do not seem to be ruled out.

We thank A.P. Balachandran and E. Witten for discussions. V.P.N. was supported by the US Department of Energy under contract number DE AC02 76 ER 02220.

## References

[1] P.H. Frampton and T.W. Kephart, Phys. Rev. Lett. 50 (1983) 1343, 1347;
P.K. Townsend and G. Sierra, Nucl. Phys. B222 (1983) 493
B. Zumino, W. Yong-Shi and A. Zee, Nucl. Phys. B239 (1984) 477
Y. Matsuki and A. Hill, OSU preprints (1983)
[2] B. Zumino, Lectures at Les Houches, August 1983 (LBL preprint 16747, October 1983)
[3] E. Witten, Phys. Lett. 117B (1982) 324
[4] E. Witten, Nucl. Phys. B223 (1983) 422
[5] S.N. Vergeles, quoted in A.A. Migdal, Phys. Lett. 81B (1979) 37;
K. Fujikawa, Phys. Rev. Lett. 44 (1980) 1733
[6] A.P. Balachandran, V.P. Nair and C.G. Trahern, Phys. Rev. D27 (1983) 1369
[7] J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95
[8] N. Steenrod, *The topology of fibre bundles* (Princeton University Press, 1951)
[9] A. Borel, Bull. Amer. Math. Soc. 61 (1955) 397