Anomalous transport of Weyl fermions in Weyl semimetals

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We present a field theoretical model of anomalous transport in Weyl semimetals. We calculate the chiral magnetic and chiral vortical effect in the electric, axial (valley), and energy current. Our findings coincide with the results of a recent analysis using kinetic theory in the bulk of the material. We point out that the kinetic currents have to be identified with the *covariant* currents in quantum field theory. These currents are anomalous and the CME appears as anomalous charge creation/annihilation at the edges of the Weyl semimetal. We discuss a possible simultaneous experimental test of the chiral magnetic and the chiral vortical effect sensitive to the temperature dependence induced by the gravitational contribution to the axial anomaly.

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I. INTRODUCTION

Weyl semimetals are materials whose electronic structure features quasiparticles that are locally in momentum space well described by massless chiral fermions via the Weyl equation. Realizations of such materials have been suggested in [1–3]. One of the most characteristic properties of the quantum theory of chiral fermions is the presence of chiral anomalies [4–7] (see [8,9] for extensive reviews). Not all symmetries that are present in the classical Hamiltonian or Lagrangian are really present on the quantum level. How the anomaly can be realized effectively in crystal lattices has been explained some time ago in [10]. From a modern point of view the essential feature is that locally around the tip of the Weyl cone in momentum space the electron wave function is subject to a nontrivial Berry phase [11,12] with a Berry curvature of integer flux k around the cone. The anomaly in each Weyl cone takes the form

$$\partial_\mu J^\mu = \frac{k}{4\pi^2}\vec{E}\cdot\vec{B}. \tag{1}$$

A physically insightful derivation of this equation has been presented in [10] in the context of Weyl fermions in a crystal. It can be phrased in the following way: In a magnetic field the spectrum of a Weyl fermion splits into Landau levels with the lowest level behaving as a chiral fermion in one dimension. Its momentum has to be aligned (or antialigned) with the magnetic field. If we also switch on an electric field parallel to the magnetic one the Lorentz force acting on the fermions in the lowest Landau level implies $\dot{p} = \pm E$ depending on chirality, where p is the momentum and E is the electric field. This implies that the Fermi momentum p_F is shifted. The change in the density of states at the Fermi level is given by $dp_F/(2\pi)$ multiplied with the degeneracy of the lowest Landau level, which is $B/(2\pi)$. Therefore, the number of states changes with time as $\dot{n} = \pm EB/(4\pi^2)$, which is nothing but a noncovariant version of the anomaly equation (1). Note that this derivation of the anomaly works separately for each Weyl cone. The Nielsen-Ninomiya no-go theorem [13] furthermore implies that the total sum over the Berry fluxes in the Brillouin zone vanishes. So the simplest realization is a model with $k \in \{+,-\}$, i.e., one right- and one left-handed chiral fermion. It also implies that in metals there is no "real" anomaly but the (axial) anomaly is simulated by electrons moving from one Weyl cone to another one of opposite Berry flux [10]. Equation (1) describes the rate of this effective chirality change in external electric and magnetic fields. The description as Weyl fermions is valid only inside a limited energy range. An electron present in one Weyl cone can scatter outside this energy range and reappear in the other Weyl cone of opposite chirality. For the effective description in terms of Weyl fermions this looks as if one fermion had changed its chirality. Such a chirality changing process is possible even in the absence of external electric and magnetic fields and represents a tree-level breaking of the axial symmetry akin to a mass term for a Dirac fermion.

At finite temperature and density anomalies are intimately related to the existence of nondissipative transport phenomena. More precisely, a magnetic field induces a current via the chiral magnetic effect (CME) and a vortex or rotation of the fluid or gas of chiral fermions also induces a current via the chiral vortical effect (CVE) [14–32] (see also the recent reviews [33–41]) Generalizations to arbitrary dimensions have been discussed in [42]. The hydrodynamic approach has been generalized in [43,44]. Since the effects are nondissipative they can also be obtained from effective action approaches [45,46] or using Ward identities [47]. In the context of high-energy physics these effects are nowadays well established. They have also been found via lattice QCD [48–52] and via holographic methods [21,24,25,38,53–58].

It is however still an open question if they really lead to observable effects in heavy ion collisions, such as charge separation [23], a chiral magnetic wave [59,60], or enhanced production of high spin baryons [61]. A recent review of the experimental situation at heavy ion collisions can be found in [62].

In the context of Weyl semimetals the theoretical situation seems less clear. Statements of existence of the CME in Weyl semimetals are contrasted with some explicit calculations that see no such effect [63–72]. Also different approaches have been put forward to describe anomaly induced transport in Weyl semimetals. One is the point of view of effective field theory of chiral fermions valid at long wavelength and for excitations around the Fermi surface. This can be contrasted with a more down to earth picture in which there are simply electrons filling a particular band structure. The difference between these two points of view is that in the relativistic Weyl fermion picture we need to deal with the intricacies of relativistic field theory, particles and antiparticles, and as a result a complicated vacuum. In contrast, electrons are on a fundamental level simply electrons without any antiparticles even if they partially fill some nonstandard band structures.

The purpose of this article is to develop a field theoretical model. In particular we want to connect the well known anomaly related transport formulas for relativistic Weyl fermions with the results of kinetic theory of electrons at Weyl nodes [12,72–77]. Using a field theoretical model forces us however to also go substantially beyond the level of discussion available in kinetic theory. One of the central issues is what precisely does formula (1) correspond to in field theoretical language? Anomalies come in two formulations, one as *covariant* anomaly and the other as *consistent* anomaly [78]. Accordingly one can define covariant and consistent currents. Whereas the divergence of covariant currents is uniquely defined, the divergence of consistent currents depends on specific regularization schemes. It is this freedom of regularization, more precisely, the freedom to modify effective actions by adding local counterterms, that allows the definition of an exactly conserved electric current. We argue that both formulations of the anomaly, the covariant one and the consistent one, lead to the same physical outcome: Charge separation at the edges perpendicular to the magnetic field even when there is no bulk current according to kinetic theory.

We also extend the results to the CME and CVE in the energy current and prove agreement of our model and kinetic theory. One of the interesting results is that the temperature dependence related to the presence of mixed axial-gravitation anomaly [6,7] enters the vortical effect in the energy current. This is the only vectorlike current, i.e., sum of left-handed and right-handed currents, that is sensitive to it. We suggest an experimental setup that allows us to test the temperature dependence and therefore the gravitational contribution to the axial anomaly.

This paper is organized as follows. In Sec. II we adapt the well-known formulas for the anomalous transport coefficients to the situation present in Weyl semimetals. We then compute the chiral magnetic and chiral vortical conductivities and find that our formulas for the covariant current coincide with [72]. We also compute the chiral conductivities in the energy current leading to the interesting result that the vortical effect in the energy current is sensitive to the T^2 temperature dependence induced by the gravitational anomaly.

In Sec. III we emphasize that the currents of the previous section have to be understood as the *covariant* currents in a quantum field theoretical definition. These currents are anomalous and we show that the CME arises as anomalous charge creation and annihilation at the edges of the Weyl semimetal.

In Sec. IV we suggest a simple experiment that uses both the chiral vortical and the chiral magnetic effect. Not only does it test the presence of chiral transport in general but it is also sensitive to the T^2 term related to the gravitational anomaly.

We present some additional discussion of the results and outlook to future work in Sec. V.

II. ANOMALOUS TRANSPORT

The chiral magnetic effect is only the most prominent example of a larger set of anomaly related transport coefficients. The full set of magnetic and vortical effects can be written down as [41,43][1]

$$\sigma^B_{ab} = \frac{1}{4\pi^2} d_{abc}\mu_c, \tag{2}$$

$$\sigma^V_a = \sigma^B_{\varepsilon,a} = \frac{1}{8\pi^2} d_{abc}\mu_b\mu_c + \frac{1}{24} b_a T^2, \tag{3}$$

$$\sigma^V_\varepsilon = \frac{1}{12\pi^2} d_{abc}\mu_a\mu_b\mu_c + \frac{1}{12} b_a\mu_a T^2. \tag{4}$$

The constants d_{abc} and b_a indicate the presence of gauge and gravitational anomalies

$$D_\mu J^\mu_a = \frac{d_{abc}}{32\pi^2}\varepsilon^{\mu\nu\rho\lambda}F^b_{\mu\nu}F^c_{\rho\lambda} + \frac{b_a}{768\pi^2}\varepsilon^{\mu\nu\rho\lambda}R^\alpha{}_{\beta\mu\nu}R^\beta{}_{\alpha\rho\lambda}. \tag{5}$$

These are general expressions valid for a family of symmetries (possibly non-Abelian) labeled by the indices a,b,c. The constants d_{abc} and b_a are

$$d_{abc} = \mathrm{sTr}(T_aT_bT_c)_R - \mathrm{sTr}(T_aT_bT_c)_L, \tag{6}$$

$$b_a = \mathrm{Tr}(T_a)_R - \mathrm{Tr}(T_a)_L, \tag{7}$$

where sTr stands for a symmetrized trace, T_a is the matrix generator of the symmetry labeled by a, and the subscripts R,L denote right- and left-handed fermions, respectively.

We would like to emphasize that the gravitational anomaly is not necessarily related to the presence of space-time curvature. It rather reflects the fact that in the quantum field theory of chiral fermions there is an obstruction to define at the same time conserved chiral currents and a conserved energy momentum tensor. This obstruction is manifest on the level of triangle diagrams with one chiral current and two energy momentum tensors even for quantum field theory in flat Minkowski space.

Note in particular that the temperature dependence in the anomalous conductivities enters via the mixed gauge-gravitational anomaly coefficient b_a. It is worth mentioning that the contribution of the gravitational anomaly to transport at first order in derivatives, such as the responses to magnetic field and vorticity, is surprising because it is of higher (fourth) order in derivatives. How these responses can be obtained in a model independent way using a combination of hydrodynamic and geometric reasoning was recently shown in [46].

The conductivities defined in Eqs. (2)–(4) determine the response of the system in the currents $\vec{J}_a$ and the energy current $J^i_\varepsilon = T^{0i}$ in the presence of a magnetic field $\vec{B}_a$ and rotation measured by the vorticity $\vec{\omega} = \frac{1}{2}\nabla\times\vec{v}$, with $\vec{v}$ as the vector field describing the rotation. The response is summarized in[2]

$$\vec{J}_a = \sigma^B_{ab}\vec{B}_b + 2\sigma^V_a\vec{\omega}, \tag{8}$$

$$\vec{J}_\varepsilon = \sigma^B_{\varepsilon,b}\vec{B}_b + 2\sigma^V_\varepsilon\vec{\omega}. \tag{9}$$

[1]These are one-loop expressions which are known to receive higher loop corrections when dynamical gauge fields are important [79–81]. In the case of Weyl semimetals these corrections can be assumed to be small and will be ignored.

[2]The factor of 2 in the vortical effects arises since we use the transport coefficients normalized to the gravitomagnetic field in which $2\omega_i = \varepsilon_{ijk}\partial_j h_{0k}$ for the metric fluctuation h_{0i} coupling to the energy current.

One way of deriving these anomalous conductivities is via the formalism of Kubo formulas [27,41]

$$\sigma^B = \lim_{k_j \to 0} \sum_{i,k} \varepsilon_{ijk} \frac{i}{2k_j} \langle J^i J^k \rangle|_{\omega=0}, \tag{10}$$

$$\sigma^V = \lim_{k_j \to 0} \sum_{i,k} \varepsilon_{ijk} \frac{i}{2k_j} \langle J^i J_\varepsilon^k \rangle\big|_{\omega=0}, \tag{11}$$

$$\sigma_\varepsilon^V = \lim_{k_j \to 0} \sum_{i,k} \varepsilon_{ijk} \frac{i}{2k_j} \langle J_\varepsilon^i J_\varepsilon^k \rangle\big|_{\omega=0}. \tag{12}$$

There the expressions arise from the purely finite temperature and finite density parts of the amplitudes (i.e., neglecting the vacuum contribution) of current-current and current-energy-momentum tensor two point functions. Without going into detail one finds for one chiral (right-handed) fermion

$$\sigma^B = \frac{1}{4\pi^2} \int_0^\infty dk \left[n_F\left(\frac{k-\mu}{T}\right) - n_F\left(\frac{k+\mu}{T}\right)\right] = \frac{\mu}{4\pi^2}, \tag{13}$$

$$\sigma^V = \sigma_\varepsilon^B = \frac{1}{4\pi^2} \int_0^\infty dk\, k \left[n_F\left(\frac{k-\mu}{T}\right) + n_F\left(\frac{k+\mu}{T}\right)\right] = \frac{\mu^2}{8\pi^2} + \frac{T^2}{24}, \tag{14}$$

$$\sigma^B = \frac{1}{4\pi^2} \int_0^\infty dk\, k^2 \left[n_F\left(\frac{k-\mu}{T}\right) - n_F\left(\frac{k+\mu}{T}\right)\right] = \frac{\mu^3}{12\pi^2} + \frac{\mu T^2}{12}. \tag{15}$$

It is interesting to rewrite these formulas in the following way. Let us assume that $\mu > 0$. Then the argument of the Fermi-Dirac distribution in the first terms in the integrals (13)–(15) are negative in the range $0 < k < \mu$. In this range we use identity $n_F(x) = 1 - n_F(-x)$. The conductivities can then be re-expressed in the form

$$\sigma^B = \frac{1}{4\pi^2} \int_0^\mu dk, \tag{16}$$

$$\sigma^V = \sigma_\varepsilon^B = \frac{1}{4\pi^2} \int_0^\mu k\, dk + \frac{1}{2\pi^2} \int_0^\infty x\, n_F\left(\frac{x}{T}\right) dx, \tag{17}$$

$$\sigma^B = \frac{1}{4\pi^2} \int_0^\mu k^2 dk + \frac{\mu}{\pi^2} \int_0^\infty x\, n_F\left(\frac{x}{T}\right) dx. \tag{18}$$

These expression have the advantage that the contributions from the thermal fluctuations are nicely separated from the finite density contributions.

A. Choice of vacuum

So far we have described the anomaly related transport for relativistic Weyl fermions. The results in [72] do however not fit into this scheme [Eqs. (2)–(4)]. If we assume a vanishing CME this would automatically imply a vanishing CVE, whereas [72] finds a vanishing CME but a nonvanishing CVE. Indeed if we assume $\mu_L = \mu_R$ the chiral magnetic conductivity, Eq. (2), and the chiral vortical conductivity, Eq. (3), both vanish (the temperature term cancels upon adding left- and right-handed current).

So how can these formulas can be applied to Weyl semimetals? A schematic picture of the electronic structure of a Weyl semimetal is given in Fig. 2. The energy levels are filled up to the Fermi surface denoted $E = \mu$. The locations in the Brillouin zone of the touchings are $(E_R, \vec{k}_R)$ and $(E_L, \vec{k}_L)$. Locally around the conical band touching points the Hamiltonian is linear in the momentum $\vec{p} - \vec{k}_{R,L}$ and takes the form of right- and left-handed Weyl equations

$$H_{R,L} = E_{R,L} \pm \vec{\sigma}(\vec{p} - \vec{k}_{R,L}). \tag{19}$$

Relative to each other the Weyl cones are displaced in energy by $E_R - E_L$ and in momentum by $\vec{k}_R - \vec{k}_L$. An effective action describing this situation is

$$S = \int d^4x\; \bar{\Psi}\gamma^\mu (i\partial_\mu - \gamma_5 b_\mu)\Psi, \tag{20}$$

with $E_L - E_R = 2b_0$ and $\vec{k}_L - \vec{k}_R = 2\vec{b}$. In the terminology of the previous section we can identify the four-vector b_μ with an axial "gauge" potential A_μ^5. But we should not identify b_0 with the chemical potential. In general, the chemical potential is not necessarily a parameter in the Lagrangian, it is only so in a specific gauge, e.g., $A_0 = \mu$. In the context of the chiral magnetic effect it has been shown in [41] that an analogous gauge choice for the axial gauge field has to be avoided. We will therefore define the chemical potential via specific choices of contours in the complex frequency plane. However, before we do so we need to decide what we consider to be the quantum field theoretical vacuum.

In the relativistic theory of a Dirac fermion we have a unique choice of vacuum which is invariant under the discrete symmetries C,P,T. It is the usual normal-ordered vacuum which re-interprets negative energy states as propagating backwards in time via the Feynman boundary conditions giving rise to negatively charged antiparticles. This removes an *a priori* infinite Dirac sea of occupied negative energy states. At finite chemical potential the Feynman integration contour is shifted by μ. The contribution to an amplitude stemming from the occupied on-shell states between $\omega = 0$ and $\omega = \mu$ is given by the difference of the two contours depicted in Fig. 1. Now from Fig. 2 it can already be seen that no such privileged choice of vacuum is available in the case of a Weyl semimetal. In addition, it is clear that the validity of the description of the dynamics of the band electrons by the Weyl Hamiltonian in Eq. (19) is limited to a certain energy range. Following [72] we take E_0 to be the lower cutoff and E_1 the upper cutoff at which the description in terms of Weyl fermions breaks down. Note that from the point of the effective theory of Weyl fermions these are both UV cutoffs. This is obvious for E_1 but even E_0 acts as a UV cutoff since in order to probe that energy we have to poke a hole in the Fermi surface of the order of $\mu - E_0$. From the point of view of the Weyl fermions we have to excite "antiparticles" (=hole states) of energies $E_{R,L} - E_0$ in the presence of an effective chemical potential $\mu_{R,L} = \mu - E_{R,L}$. We will for simplicity

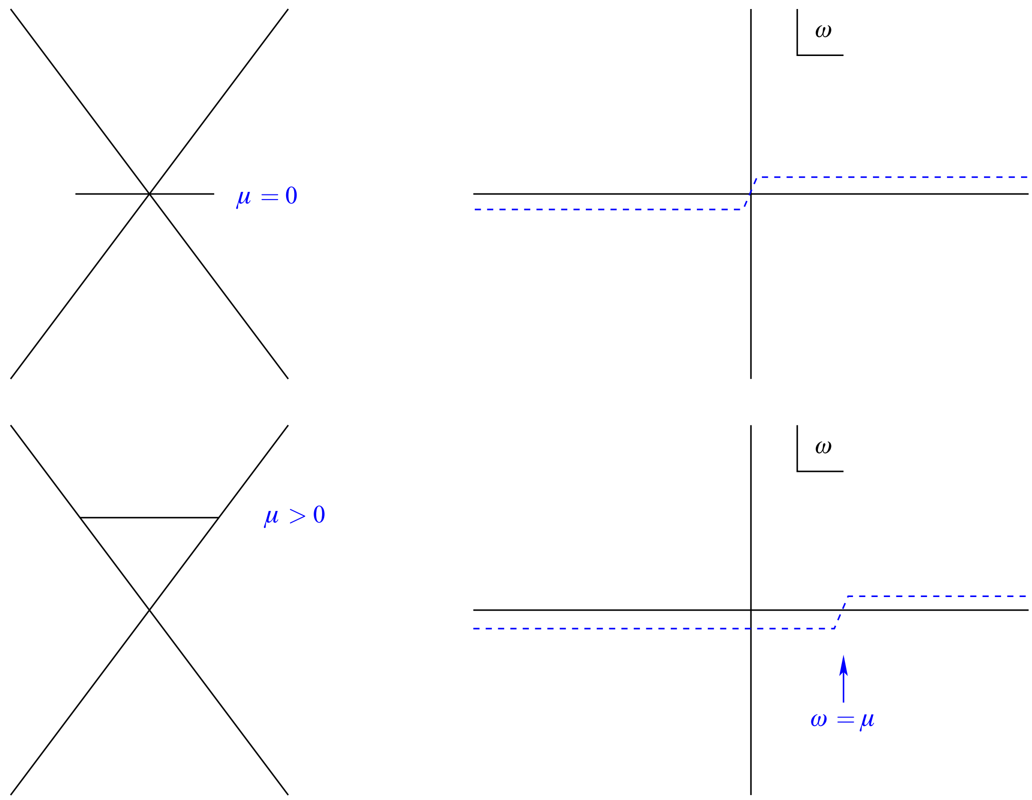

FIG. 1. (Color online) In the relativistic theory of Weyl fermions the choice of vacuum is dictated by symmetry and chosen to lie at the tip of the cone of the dispersion relations. The negative energy states below are reinterpreted as antiparticles via the Feynman boundary conditions. The contour integration in the complex ω plane is shifted slightly below the poles corresponding to the negative energy states and slightly above the positive energy poles. At finite chemical potential the positive energy states between the vacuum $\omega = 0$ and $\omega = \mu$ are occupied. Because of the Pauli principle no further states can be created. But a fermion annihilation operator does create a hole state. This looks like a negative energy state for the Hamiltonian $H - \mu Q$ which justifies the gauge choice $A_0 = \mu$ normally used in quantum field theory. The contour is now shifted below the poles up to $\omega = \mu$ and above the poles for large frequencies. The difference between the two contours is the contribution of the occupied on-shell states at zero temperature.

work with sharp cutoffs $E_{0,1}$. This is most likely a drastic oversimplification. In a real Weyl semimetal the change of the dispersion relation from relativistic to nonrelativistic will be gradual. We think that this situation can however be modeled by effective (model dependent) cutoffs $E_{0,1}$. A rough estimate is given by the energy level at which the dispersion obtains a local minimum as in Fig. 2.[3] Below E_0 we do not even have the relativistic vacuum of the Weyl fermions present. The formulas (2), (3), and (4) however are derived assuming the description in terms of Weyl formulas to be valid down to arbitrarily low energies and by subtracting a suitable chosen vacuum (an infinitely deep Dirac sea). In Weyl semimetals the Dirac sea is however physical and rather shallow, it reaches down from $E_{L,R}$ to E_0 and its contribution has to be taken into account properly. Because of this we suggest to define the "vacuum" at the energy E_0. In that way we include all occupied electron states in the Weyl cones even if they are below the tip of the cone.

[3]A more realistic model would be to include a (small) mass term in (20) and work out the values of the anomalous transport coefficients via Kubo formulas [41].

This is implemented by the following contour deformation for a generic one-loop amplitude at finite chemical potential and finite temperature. Using standard manipulations we can write the sum over Matsubara frequencies for a generic function $f(z)$ as

$$\begin{aligned} T\sum_n f(i\omega_n+\mu) &= \frac{1}{2}\int_{\mathcal{C}} f(z)\tanh\left(\frac{z-\mu}{2T}\right)\frac{dz}{2\pi i} \\ &= \int_{\mu-i\infty}^{\mu+i\infty} f(z)\frac{dz}{2\pi i} + \sum_{\mathrm{Re}(z_i)>\mu} \mathrm{res}(f)(z_i)n_F\left(\frac{z_i-\mu}{T}\right) \\ &\quad - \sum_{\mathrm{Re}(z_i)<\mu} \mathrm{res}(f)(z_i)n_F\left(\frac{\mu-z_i}{T}\right), \end{aligned} \tag{21}$$

where the contours are as in Fig. 3 and z_i are the poles of $f(z)$. We have used $\tanh(x) = 1 - 2n_F(2x)$ for $\mathrm{Re}(z) > \mu$ and $\tanh(x) = -1 + 2n_F(-2x)$ for $\mathrm{Re}(z) < \mu$. This allows us to extract the finite temperature contribution that comes from the fluctuations around the Fermi surface at $E = \mu$ for a general amplitude just as what we did explicitly for the conductivities in (16)–(18). The thermal contribution vanishes at $T = 0$. Furthermore, for $\mathrm{Re}(z_i) \gg \mu$ the thermal

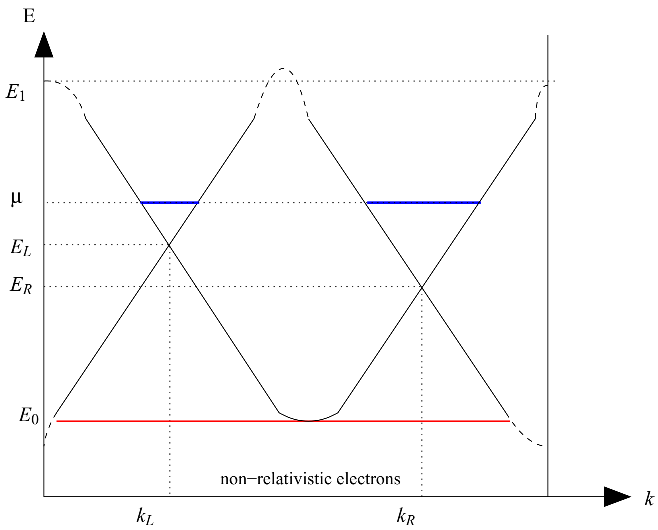

FIG. 2. (Color online) Schematic depiction of the electronic structure of a Weyl semimetal. Two Weyl cones of opposite handedness are located at (E_L,k_L) and (E_R,k_R). The filling level (chemical potential) is given by μ and in equilibrium must be the same for both cones. Below the level denoted by E_0 the description in terms of Weyl fermions is not valid. This provides a natural IR cutoff. From the point of view of the physics of the excitations near the Fermi surface this is however better thought of as an UV cutoff since in order to probe it one needs to create holes (=antiparticles) of energy $\omega \sim \mu - E_0$. In this sense the low energy effective theory near the Fermi surface is the one of two Weyl fermions with chemical potentials $\mu_{R,L} = \mu - E_{R,L}$. The Dirac seas below the tips of the cones is however not of infinite depth but reaches down only to $E = E_0$.

contributions are exponentially suppressed. The temperature independent integral can be Wick rotated and deformed into the vacuum contour at $E = E_0$ (see Fig. 4). We will ignore this vacuum contour for the moment but comment on its significance in the next section.

The remaining contour integration is over a compact cycle and can be decomposed as the sum of $\mathcal{C}_{\mu-E_i}$ and $-\mathcal{C}_{-E_i}$. The latter contours are the same as for a Weyl fermion at chemical potential $\mu_{\rm eff} = \mu - E_i$ and the contour of another Weyl fermion at effective chemical potential $\mu_{\rm eff} = E_0 - E_i$ were it not for the orientation of the contour. Because of this orientation we need to subtract this contribution rather than to add it.

These considerations lead to the general formula for the anomalous transport in Weyl semimetals

$$\sigma_{\rm wsm} = \sigma(\mu - E_i,T) - \sigma(E_0 - E_i,0). \tag{22}$$

B. Anomalous conductivities

Let us now apply this formula to the chiral magnetic effect. For each isolated Weyl cone we find

$$\begin{aligned}\vec{J}_{R,L} &= \pm\left(\frac{\mu - E_{R,L}}{4\pi^2} - \frac{E_0 - E_{R,L}}{4\pi^2}\right)\vec{B}\\ &= \pm\frac{\mu - E_0}{4\pi^2}\vec{B}.\end{aligned} \tag{23}$$

Therefore the (covariant vector) current $J_R + J_L$ vanishes and there is no CME in a Weyl semimetal in equilibrium! While we find that there is no chiral magnetic current we will see however in the next section that the anomaly will still give rise to a charge separation effect once the effects from the edges are taken into account. On the other hand, the chiral separation effect (CSE), i.e., the generation of an axial current,[4] is fully realized:

$$\vec{J} = 0, \tag{24}$$

$$\vec{J}_5 = \frac{\mu - E_0}{2\pi^2}\vec{B}. \tag{25}$$

The driving force for the CME is an imbalance in the Fermi surfaces of left- and right-handed fermions [23]. It has been suggested that this can be achieved by placing the system in parallel electric and magnetic fields [12,82] which will pump such an imbalance into the Weyl semimetal via the anomaly equation (1). We will discuss another way of at least locally achieving such an imbalance in Sec. V.

Now we turn to the chiral vortical effect. According to our formula (22) it is given by

$$\begin{aligned}\vec{J}_{R,L} &= \pm\left(\frac{(\mu - E_{R,L})^2}{4\pi^2} + \frac{T^2}{12} - \frac{(E_0 - E_{R,L})^2}{4\pi^2}\right)\vec{\omega}\\ &= \pm\left(\frac{(\mu - E_0)(\mu + E_0 - 2E_{R,L})}{4\pi^2} + \frac{T^2}{12}\right)\vec{\omega}.\end{aligned} \tag{26}$$

The temperature independent part coincides exactly with the result of [72]. The vortical effect in the electric current is given by

$$\vec{J} = \frac{(E_L - E_R)(\mu - E_0)}{2\pi^2}\vec{\omega}. \tag{27}$$

Let us now turn to the energy current. The general formulas (2)–(4) suggest also anomalous transport in the energy current. In particular, our model predicts an energy current in a magnetic field of

$$\vec{J}_{\varepsilon,R,L} = \pm\left(\frac{(\mu - E_0)(\mu + E_0 - 2E_{R,L})}{8\pi^2} + \frac{T^2}{24}\right)\vec{B}. \tag{28}$$

The equality of the chiral magnetic conductivity for the energy current and the chiral vortical conductivity for the charge current follows from the Kubo formulas [41].

However, this is not yet the total energy flow generated. The tips of the cones are located at $E = E_{R,L}$ and they can be understood as background values for left-handed and right-handed temporal components of gauge fields. They contribute terms to the action of the form

$$\int A_\mu^{R,L} J_{R,L}^\mu. \tag{29}$$

The energy-momentum tensor can be understood as the reaction of the system to a variation in the metric. In particular, if we switch on a mixed time-spatial component h_{0i} in the

[4] In the condensed matter literature this is usually referred to as "valley" current.

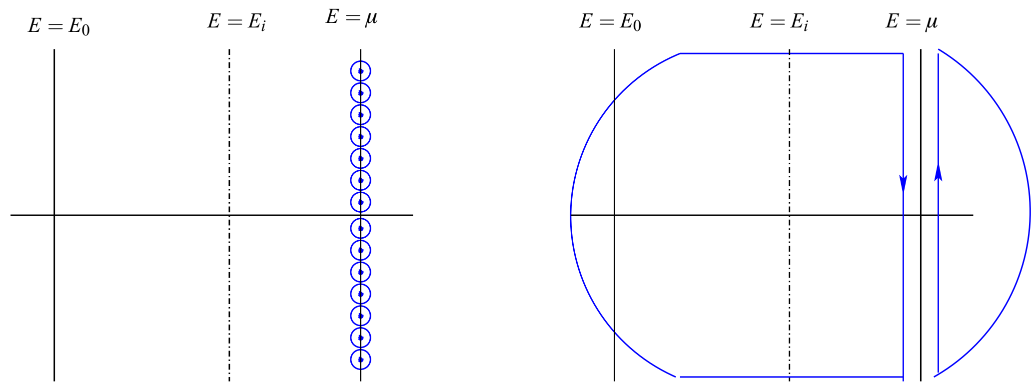

FIG. 3. (Color online) Contour integration for the summation over Matsubara frequencies for a generic one-loop amplitude at finite chemical potential μ. The finite temperature contribution is captured by the sum over the residues weighted by the Fermi-Dirac distribution inside the two D shaped contours in the right figure.

metric we induce a term[5]

$$\int A^0 j^i h_{0i} = \int E_{R,L} j^i_{R,L} h_{0i}, \tag{30}$$

and therefore there is the additional "drift" in the energy flow of $E_{R,L}\vec{j}_{R,L}$. For each Weyl cone we thus find

$$\begin{aligned}\vec{J}_{\varepsilon,R,L} &= \pm\bigg(\frac{(\mu - E_{R,L})(\mu + E_0 - 2E_{R,L})}{8\pi^2} \\ &\quad + \frac{T^2}{24} + \frac{E_{R,L}(\mu - E_0)}{4\pi^2}\bigg)\vec{B} \\ &= \pm\bigg(\frac{\mu^2 - E_0^2}{8\pi^2} + \frac{T^2}{24}\bigg)\vec{B}.\end{aligned} \tag{31}$$

[5]There is also the term $k^i_{R,L} J^0_{R,L} h_{0i}$ which might give additional contributions if J^0 is not vanishing.

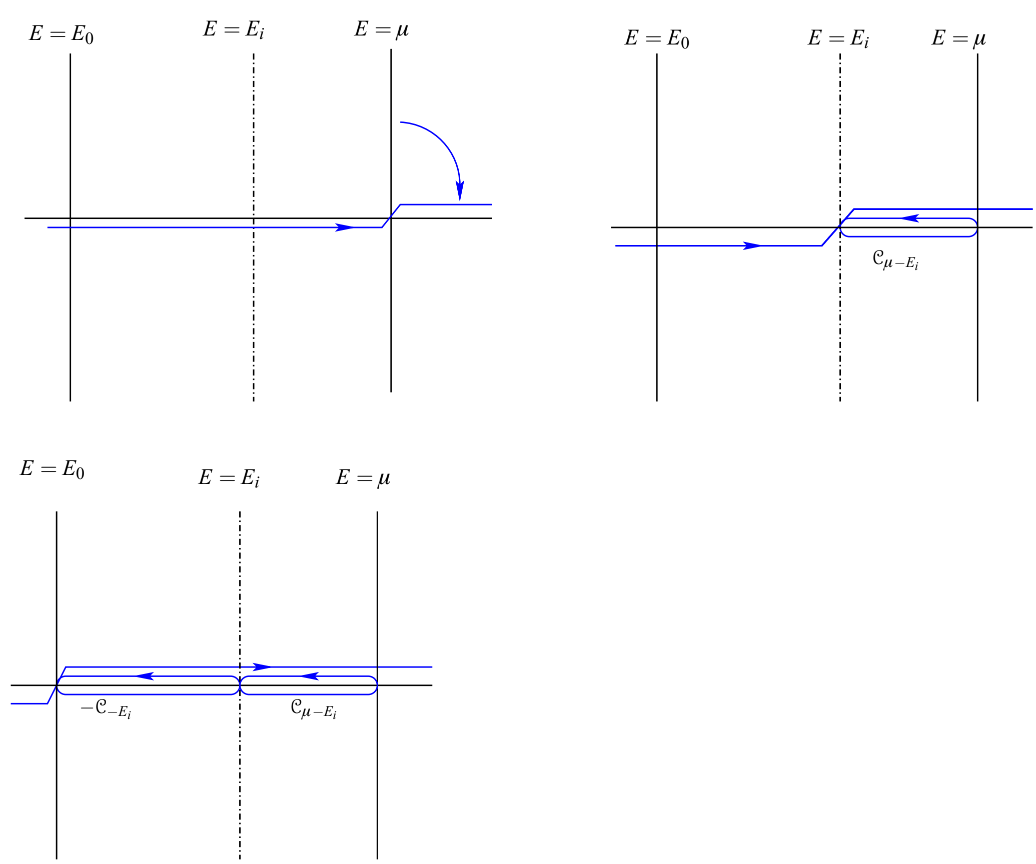

FIG. 4. (Color online) Wick rotation and contour deformation for the noncompact vacuum contour. The contour at $E = \mu$ describes particles for $z > \mu$ and the holes as antiparticles for $z < \mu$. In the second figure we have the vacuum contour of the usual relativistic Weyl fermion with antiparticles below $z < \mu - E_{R,L}$ and particles above and the occupied particle states between $E = E_{R,L}$ and $E = \mu$. In the third part of the figure we have the vacuum down at $E = E_0$ and the states occupied between $E = E_0$ and $E = E_{R,L}$.

The sum of the contributions of the two Weyl cones vanishes.

Let us compare this now with kinetic theory. The energy flow in one Weyl cone we compute as

$$\vec{J}_\varepsilon = \int \frac{d^3p}{(2\pi)^3}\sqrt{G}n_F(E)\,E\dot{\vec{x}}, \tag{32}$$

where $G = 1 + \vec{\Omega}\cdot\vec{B}$ is the phase space measure in presence of the Berry curvature and magnetic field [12,74]. And using the steps outlined in [72] this evaluates to

$$\begin{aligned}\vec{J}_\varepsilon &= \pm\frac{1}{4\pi^2}\vec{B}\int_{E_0}^\infty E n_F(E)dE\\ &= \pm\frac{\mu^2 - E_0^2}{4\pi^2}\vec{B} + O(T).\end{aligned} \tag{33}$$

Where in the last line we have taken the zero temperature limit. Again this agrees with our Weyl fermion model.

Finally, let us also compute the energy flow due to rotation. According to what we outlined before the response in T^{0i} is the one that does not take into account the energy drift due to Eq. (30). Our Weyl fermion model predicts (ignoring for a moment the drift term)

$$\begin{aligned}\vec{J}_{\varepsilon,R,L} = \pm\bigg(&\frac{(\mu - E_{R,L})^3}{6\pi^2} + \frac{(\mu - E_{R,L})T^2}{6}\\ &- \frac{(E_{R,L} - E_0)^3}{6\pi^2}\bigg)\vec{\omega}.\end{aligned} \tag{34}$$

We can also subtract the drift term in the kinetic model. It corresponds to inserting $E - E_{R,L}$ instead of E in the phase space integrals. We use also the heuristic substitution $\vec{B} = 2(E - E_{R,L})\vec{\omega}$ suggested in [72,74] to find the kinetic theory based expression

$$\begin{aligned}\vec{J}_{\varepsilon,R,L} &= \pm\vec{\omega}\frac{1}{2\pi^2}\int_{E_0}^\infty dE(E - E_{R,L})^2 n_F(E)\\ &= \pm\bigg(\frac{(\mu - E_{R,L})^3}{6\pi^2} - \frac{(E_0 - E_{R,L})^3}{6\pi^2} + O(T)\bigg)\vec{\omega}.\end{aligned} \tag{35}$$

Again we find agreement with the Weyl fermion model. For completeness we also state the result for the total energy flow due to rotation including the drift terms

$$\vec{J}_\varepsilon = \bigg(\frac{(\mu^2 - E_0^2)(E_L - E_R)}{4\pi^2} + \frac{(E_L - E_R)T^2}{12}\bigg)\vec{\omega}. \tag{36}$$

This result is of some significance since it is the only vectorlike (i.e., sum as opposed to difference of right- and left-handed contributions) current sensitive to the T^2 temperature dependence and therefore to the mixed gauge-gravitational anomaly.

III. ANOMALY AND CME AT THE EDGE

In this section we will try to connect formula (1) with standard field theoretical language.

In the quantum field theory of Weyl fermions there are at least two conventions on how to phrase the anomaly. The first one, called the *consistent* anomaly, defines the current as the functional derivative of the effective action with respect to to the gauge field. The anomaly comes from a triangle diagram with three currents at the vertices. For the moment we think of having only one (right-handed) Weyl fermion, all three currents are the same and that imposes a Bose symmetry of the vertices. In this case the anomaly is given by [8,9]

$$\partial_\mu J^\mu_{\text{cons}} = \frac{1}{96\pi^2}\varepsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda} = \frac{1}{12\pi^2}\vec{E}\cdot\vec{B}. \tag{37}$$

If we compare this to (1) we see that this gives an anomaly coefficient that is smaller by a factor of 1/3. The deeper reason is that because of the Bose symmetry at the vertices the anomaly is distributed equally amongst them.

There is however another definition of the current that does not follow from functional differentiation of the effective action. Instead, we could define the current as a gauge-invariant operator, e.g., by gauge-covariant point splitting. The covariant current is gauge invariant even if the gauge transformations are effected by an anomaly. The quantum operator J^μ_{cov} defined in this way obeys the *covariant* anomaly equation [8,9,78]

$$\partial_\mu J^\mu_{\text{cov}} = \frac{1}{32\pi^2}\varepsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda} = \frac{1}{4\pi^2}\vec{E}\cdot\vec{B}. \tag{38}$$

The two definitions of currents differ by a Chern-Simons current

$$J^\mu_{\text{cov}} = J^\mu_{\text{cons}} + \frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\lambda}A_\nu F_{\rho\lambda}. \tag{39}$$

Now the important point is that the kinetic definition of the current obeys the covariant anomaly equation. Therefore, we should identify the kinetic current with the covariant current of quantum field theory.

Let us now add a left-handed and a right-handed fermion. The covariant vector and axial currents are simply defined as sum and difference of right-handed and left-handed currents. We also introduce vectorlike gauge fields A_μ that couple with the same sign to left- and right-handed fermions and axial gauge fields that couple with opposite signs to left- and right-handed fermions. The anomalies in these *covariant* currents can be written as

$$\partial_\mu J^\mu_{V,\text{cov}} = \frac{1}{2\pi^2}(\vec{E}_5\cdot\vec{B} + \vec{E}\cdot\vec{B}_5), \tag{40}$$

$$\partial_\mu J^\mu_{5,\text{cov}} = \frac{1}{2\pi^2}(\vec{E}\cdot\vec{B} + \vec{E}_5\cdot\vec{B}_5). \tag{41}$$

The important lesson is that the vector current, i.e., the sum of left-handed and right-handed fermion number currents, is *not* conserved!

Let us see now what consequence this nonconservation has if we have a finite region in space in which the temporal component of the axial gauge field is switched on. In the case of the Weyl semimetal we see from the effective action (20) that the energy difference of left- and right-handed cones $1/2(E_R - E_L)$ plays the role of A^5_0! We will model therefore the Weyl semimetal as a region of space in which $A^5_0 \neq 0$ and constant. So for a cubic body of extensions $\{L_1,L_2,L_3\}$,

$$A^5_0 = \frac{E_R - E_L}{2}\prod_{i=1}^3\left[\Theta\left(x_i + \frac{L_i}{2}\right) - \Theta\left(x_i - \frac{L_i}{2}\right)\right]. \tag{42}$$

If we apply a magnetic field in the 3 direction it follows from (38) and the fact that there is no bulk current that at the edges

$$\partial_t \rho|_{x_3=\pm L_3/2} = \pm \frac{E_R - E_L}{4\pi^2} B. \tag{43}$$

Although in this description based on the covariant current we found that there is no bulk chiral magnetic current, there is effective accumulation of electric charges on the edges of the Weyl semimetal because of the anomaly in the covariant current! We note that no net charge is created and this is guaranteed as long as at spatial infinity we are dealing with the trivial vacuum, i.e., the one in which the axial gauge fields are zero! Indeed, since the anomaly is a total derivative

$$\begin{aligned}\int_\Omega \partial_t \rho &= \frac{1}{4\pi^2}\int_\Omega \varepsilon^{ijk}\left(\partial_i A_0^5 F_{jk}\right)\\ &= \frac{1}{4\pi^2}\int_{\partial\Omega} dS_i \varepsilon^{ijk}\left(A_0^5 F_{jk}\right),\end{aligned} \tag{44}$$

and we can take the surface bounding the volume Ω to lie outside the region where A_0^5 is nonzero.

We also note that this is precisely the charge that would accumulate at the edges if there was a bulk current:

$$\vec{J} = \frac{E_L - E_R}{4\pi^2}\vec{B}. \tag{45}$$

This looks like the chiral magnetic effect but it its origin is rather different. It is present even if there is no genuine axial chemical potential, i.e., if there is no imbalance in the number of right-handed and left-handed occupied fermion states. Chiral kinetic theory, which is concerned only with the occupied on-shell states within the local Weyl cones, does not have this type of bulk current. However, the kinetic (covariant) current is anomalous and therefore it is the anomaly that induces the edge charges. We interpret this phenomenon as follows: Due to the anomaly the Fermi level gets diminished at one edge, whereas it expands at the other edge. In the bulk in between that necessitates a reshuffling of the electronic states below our cutoff E_0 and this is the origin of the current (45). In our simple model with hard cutoff at $E = E_0$ we have no direct access to this reshuffling of states. In quantum field theory with a hard cutoff this current appears if one imposes a regularization scheme that preserves invariance under the gauge transformations associated with the gauge field A_μ as has been noted in [55]. In fact, if one does not calculate the covariant current but rather the consistent current in a scheme that preserves vectorlike gauge invariance one finds the *consistent* current [83,84]

$$J^\mu_{V,\text{cons}} = J^\mu_{V,\text{cov}} - \frac{1}{4\pi^2}\varepsilon^{\mu\nu\rho\lambda}A_\nu^5 F_{\rho\lambda}. \tag{46}$$

As we have argued, here it is however not necessary to introduce the consistent and exactly conserved current (46). We can work with the covariant current more natural to kinetic theory but have to take into account the anomaly at the edges.[6]

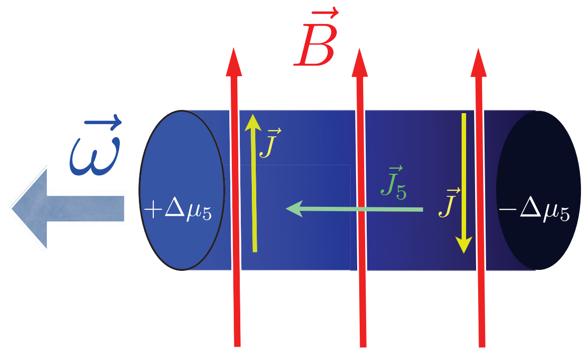

FIG. 5. (Color online) It should be possible to test the CVE and CME at once with a cylinder shaped Weyl semimetal. The rotation will induce an axial current that will lead to a stationary state with effective axial chemical potentials of equal magnitude but opposite sign at the ends of the cylinder. A magnetic field will induce an electric current that depending on the sign of the local axial chemical potential will either flow along the magnetic field or contrary to it.

Since vorticity does not source the anomaly (38) (a constant vorticity does not excite the higher derivative gravitational anomaly) there is no such edge contribution present in the chiral vortical effect!

IV. EXPERIMENTAL TEST OF CME AND CVE

Now we are going to suggest an experiment that can test some of the main points discussed in the previous sections. We ask how can we induce the chiral magnetic effect and at the same time test for the T^2 term related to the gravitational contribution to the axial anomaly.

According to (26) rotation gives rise to an axial current. That is true even when $E_R = E_L$. This fact could be used to perform an experiment that at once tests both the CME and the CVE (Fig. 5). If we have a cylinder of Weyl semimetal and we rotate it along its axes the vortical CSE will lead to the production of local imbalances between the right-handed and left-handed Fermi surfaces. More precisely, the rotation will accumulate axial charge on one side of the cylinder and deplete it on the other side. We want to take into account that in a real situation axial charge will not be conserved. We take τ to be the lifetime of a chiral quasiparticle. The continuity equation for axial charge is then modified to

$$\partial_t \rho_5 = -\frac{1}{\tau}\rho_5 - \vec{\nabla}\vec{J}_5. \tag{47}$$

Combining the diffusion law with the chiral vortical conductivity we find the axial current to be

$$\vec{J}_5 = -D_5\vec{\nabla}\rho_5 + \sigma_V\vec{\omega}, \tag{48}$$

where D_5 is the diffusion constant for axial charge. We assume now that ρ_5 depends only on the coordinate along the cylinder which we denote by x. Since the divergence of the vorticity

[6]The covariant current can of course not be coupled directly to a gauge field obeying Maxwell's equations since $J^\mu = \partial_\nu F^{\mu\nu}$ entails that the current must be conserved. Therefore, it is better to think of the covariant current to be the source of a Chern-Simons modification of Maxwell's equations $J^\mu_{\text{cov}} = \partial_\mu F^{\mu\nu} + \frac{1}{4\pi^2}\varepsilon^{\mu\nu\rho\lambda}A^5_\nu F_{\rho\lambda}$.

vanishes we can combine these two equations to

$$\partial_t \rho_5 = -\frac{1}{\tau}\rho_5 + D_5 \partial_x^2 \rho_5. \tag{49}$$

To find a stationary state we look for time independent solutions obeying the boundary conditions that the current (48) vanishes at the edges of the cylinder. We take the longitudinal extension to be L. The solution is then given by

$$\rho_5(x) = \sigma_V \omega \frac{\sqrt{\tau}}{\sqrt{D_5}} \sinh\left(\frac{x}{\sqrt{D_5\tau}}\right) \mathrm{sech}\left(\frac{L}{2\sqrt{D_5\tau}}\right) \tag{50}$$

in coordinates in which the cylinder ends are at $x = \pm\frac{L}{2}$. Applying now a perpendicular magnetic field produces an electric current via the chiral magnetic effect in the bulk (as opposed to the edge effect described in Sec. III). The profile of the electric current along the cylinder should be directly proportional to the profile of the axial charge since this current is

$$J_z(x) = \frac{\rho_5(x)}{(\chi_5 2\pi^2)} B_z, \tag{51}$$

with χ_5 the axial charge susceptibility. In Fig. 6 we plot the axial charge profile for three cases, the diffusion dominated case in which the lifetime of the chiral quasiparticles is long, an intermediate regime, and a regime that is dominated by axial charge decay. In the latter the axial charge will be confined to a narrow region at the edges of order $\delta x \sim \sqrt{D_5\tau}$. Note that the decay will always dominate for long cylinders.

This effect depends of course on the material constants τ, D_5, and χ_5 whose determination goes beyond the purpose of this paper. Regardless of the their precise values the effect is directly proportional to the chiral vortical conductivity in the axial (valley) current. We find this effect particularly interesting because this conductivity does depend on temperature $\sigma_V = \text{const.} + \frac{T^2}{6}$. As outlined previously the temperature dependence of the anomalous transport phenomena can be understood to be a direct consequence of the gravitational contribution to the axial anomaly. From the point of view of high-energy particle physics the gravitational anomaly describes the decay of a neutral pion into two gravitons, which due to the weakness of the gravitational interactions at the energy scales available at colliders such as LHC, is impossible to observe. Therefore, the anomalous transport properties of Weyl semimetals offer a unique and exciting opportunity to connect such a subtle quantum effect as the gravitational anomaly to phenomena observable in the laboratory.

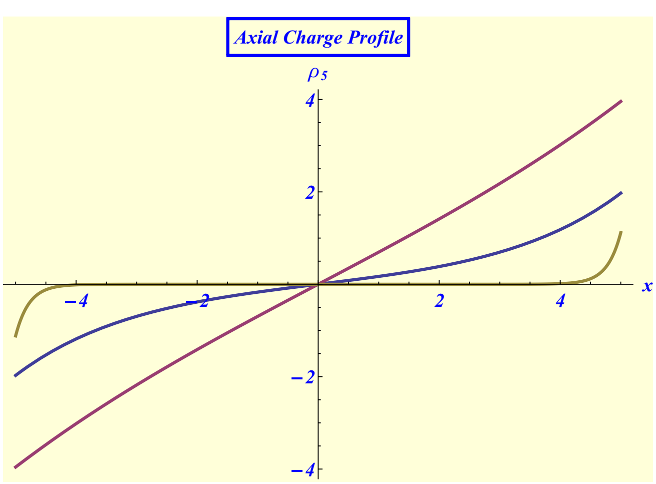

FIG. 6. (Color online) The figure shows the axial charge profile along the rotating cylinder induced via the chiral vortical effect in the axial (valley) current. For long lived chiral quasiparticles diffusion dominates and the profile is simply a straight line. For short lifetimes or long cylinders decay in the bulk dominates and we get axial charge accumulation only at the edges. The blue curve shows an intermediate case.

V. DISCUSSION

We have presented a simple field theoretical model for anomaly related transport in Weyl semimetals. The basic idea was to model the Weyl semimetal as far as possible by the known physics of Weyl fermions. Deviations arise because the choice of vacuum is not dictated by symmetries. This forces us to consider the contributions of the finite Dirac sea, following in this point [72]. The resulting conductivities for the covariant currents follow the law (22). Where they overlap our results agree with the recent results derived using kinetic theory [72]. In particular, the chiral magnetic current vanishes, whereas the chiral vortical current does not. The energy current follows this pattern. The chiral separation effect on the other hand is fully present.

The temperature dependencies are given by a simple T^2 scaling law. This assumes of course that the thermal fluctuations follow the spectrum present inside the Weyl cones. If the temperature gets too high, i.e., either of the order of $T \approx E_1 - \mu$ or $T \approx \mu - E_0$, i.e., if the temperature is high enough to probe the cutoffs, we expect deviations from the simple T^2 behavior.

Even though we found a vanishing chiral magnetic current in the bulk of the Weyl semimetal, we argued that there is still an anomaly induced charge accumulation at the edges, positive on one side and negative on the other. This arises because we based our discussion on the *covariant* current. We also argued that it is this quantum definition of current that should be identified with the current of kinetic theory. We also showed that the total charge is conserved and that the local charge production rate can be rewritten as an inflow of charge via a Chern-Simons current not contained in the kinetic description.

We have also discussed how the combination of the CME and the CVE in the axial current can be used to test for the presence of the T^2 term related to the gravitational anomaly. This constitutes an independent experimental setup to that already proposed in [85].

Finally, we also note that in a complete treatment the anomalous Hall effect should be included and the distinction between covariant and consistent current might also be of relevance there. We leave this problem for future investigation.

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