

# Thermal Hall conductance and a relative topological invariant of gapped two-dimensional systems

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## Abstract

We derive a Kubo-like formula for the thermal Hall conductance of a 2d lattice systems which is free from ambiguities associated with the definition of energy magnetization. We use it to define a relative topological invariant of gapped 2d lattice systems at zero temperature. Up to a numerical factor, it can be identified with the difference of chiral central charges for the corresponding edge modes. This establishes the bulk-boundary correspondence for the chiral central charge. We also show that for any Local Commuting Projector Hamiltonian the relative chiral central charge vanishes, while for free fermionic systems it is related to the zero-temperature electric Hall conductance via the Wiedemann-Franz law.

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## I. INTRODUCTION AND SUMMARY

There has been much theoretical as well as experimental interest in the thermal Hall effect. Just to give a couple of recent examples: (1) thermal Hall effect has been used to probe the non-Abelian nature of the  $\nu = 5/2$  FQHE state [1]; (2) an unusual behavior of thermal Hall conductivity at low temperatures was observed in cuprate superconductors in the pseudogap region [2].

Despite many theoretical works on the thermal Hall effect (see e.g. [3–7]), there are still unresolved issues with the very definition of thermal Hall conductivity. In fact, all known approaches to defining thermal Hall conductivity as a bulk property are plagued with ambiguities. To see what the issue is in the simplest possible setting, consider a macroscopic system where the only locally conserved quantity is energy, such as an insulating crystal at positive temperature. One could expect that thermal Hall conductivity appears as a transport coefficient in the hydrodynamics of phonons, but this is not the case: the hydrodynamics of phonons does not allow for a time-reversal-odd transport coefficient at leading order in the derivative expansion. The conservation law for the energy density  $\epsilon$  is

$$\frac{\partial \epsilon}{\partial t} = -\nabla \cdot \mathbf{j}^E. \quad (1)$$

In the hydrodynamic limit one can expand the energy current  $\mathbf{j}^E$  to first order in derivatives of  $\epsilon$ , or equivalently in derivatives of the temperature  $T$ :

$$\mathbf{j}_m^E = -\kappa_{m\ell}(T)\partial_\ell T. \quad (2)$$

Hence the conservation law becomes

$$c(T)\frac{\partial T}{\partial t} = \kappa_{m\ell}\partial_m\partial_\ell T + \kappa'_{m\ell}\partial_m T\partial_\ell T, \quad (3)$$

where  $c(T)$  is the heat capacity and the prime denotes derivative with respect to  $T$ . The r.h.s. of this equation depends only on the symmetric part  $\kappa_{m\ell}^S$  of the tensor  $\kappa_{m\ell}$ , which by Onsager reciprocity is the same as its time-reversal-even part. The anti-symmetric part  $\kappa_{m\ell}^A$  has no observable effect. While the energy current through a surface (or, in the 2d context, through a line) depends on the whole tensor  $\kappa_{m\ell}$ , the contribution of  $\kappa_{m\ell}^A$  can be thought of as a boundary effect. Indeed, if we define

$$\beta_{m\ell}^A(T) = \int^T \kappa_{m\ell}^A(u)du, \quad (4)$$

then in 3d the Stokes' theorem gives

$$-\int_{\Sigma} d\Sigma_m \kappa_{m\ell}^A \partial_{\ell} T = -\frac{1}{2} \int_{\partial\Sigma} d\mathbf{l}_k \epsilon_{kpm} \beta_{pm}^A(T). \quad (5)$$

Similarly, in 2d the contribution of  $\kappa_{m\ell}^A$  to the energy current through a line can be written as a boundary term. The conclusion seems to be that thermal Hall conductivity has no meaning as a bulk transport property, either in 3d or 2d. One manifestation of this is that Kubo-type formulas for thermal Hall conductivity involve “energy magnetization”, which is defined only up to an arbitrary function of temperature and other parameters [3, 4, 7].

In the 2d case the tensor  $\kappa_{m\ell}^A$  reduces to a single quantity <sup>1</sup>, the thermal Hall conductivity  $\kappa^A = \frac{1}{2} \epsilon_{m\ell} \kappa_{m\ell}^A$ , and there is an alternative line of reasoning which suggests that in certain circumstances  $\kappa^A$  can be defined in bulk terms. At low temperatures and for systems with a bulk energy gap, it seems natural to relate  $\kappa^A(T)$  to the chiral central charge of the edge CFT:

$$\kappa^A(T) \simeq \frac{\pi T}{6} (c_R - c_L). \quad (6)$$

This follows from the fact that a chiral 1+1d CFT at temperature  $T$  carries an equilibrium energy current  $I^E = \frac{\pi T^2}{12} (c_R - c_L)$  [11, 12]. Thus in a strip of a 2d material whose boundaries are kept at temperatures  $T$  and  $T + \Delta T$  which are close to each other and much smaller than the bulk energy gap there is a net energy current

$$I^E \simeq \frac{\pi T}{6} (c_R - c_L) \Delta T. \quad (7)$$

If we define  $\kappa^A = I^E / \Delta T$ , we get (6). It has been shown in [13] that the chiral central charge of the edge modes (and more generally, the equilibrium energy current carried by the edge modes) is independent of the particular edge. Hence the low-temperature thermal Hall conductivity of a gapped 2d material defined via (6) is a well-defined bulk property. <sup>2</sup> This argument does not say anything about thermal Hall conductivity at general temperatures or about thermal Hall conductivity of materials which are not gapped. Note also that the

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<sup>1</sup> We use the notation  $\kappa^A$  instead of the more standard  $\kappa_{xy}$  to avoid confusion with the off-diagonal component of  $\kappa^S$  which may be nonzero if rotational invariance is broken.

<sup>2</sup> For gapped 2d systems at low temperatures, one can try to define thermal Hall conductivity as the coefficient of the gravitational Chern-Simons term in the low-energy effective action [8–10]. As explained in [5], the energy current corresponding to the gravitational Chern-Simons term is of higher order in derivatives, in agreement with the above discussion. However, there is no natural way to couple a typical condensed matter system to gravity, therefore this prescription is ambiguous.

argument makes no assumption about the way the temperature varies within the strip. Thus the low-temperature thermal Hall *conductance* of a strip of a gapped material coincides with its thermal Hall conductivity and is a well-defined bulk property as well. One does not expect this to hold at arbitrary temperatures, or for gapless materials.

In this paper we derive Kubo-like formulas for thermal Hall conductance which are valid for at arbitrary temperatures and are free of ambiguities. We also show that for gapped materials at low temperatures thermal Hall conductance becomes linear in  $T$  and independent of the parameters of the Hamiltonian as well as of the precise profile of the temperature gradient. Thus  $\lim_{T \rightarrow 0} \kappa^A/T$  is a topological invariant of a gapped 2d material. In the case when the edge is described by a CFT, the above argument shows that this topological invariant can be identified with  $\pi/6$  times the chiral central charge.

To explain in what way our formula for thermal Hall conductance differ from the existing ones, it is instructive to view thermal Hall conductance as a relative quantity. That is, properly speaking, one should talk about the relative thermal Hall conductance  $\kappa_{\mathcal{M}\mathcal{M}'}^A$  of materials  $\mathcal{M}$  and  $\mathcal{M}'$ . To measure it, one needs to study energy flow along the surface of a cylinder made out of  $\mathcal{M}$  and  $\mathcal{M}'$  (see Fig. 1) in the situation when the two junctions between  $\mathcal{M}$  and  $\mathcal{M}'$  are maintained at different temperatures. For a 2d crystal with a square symmetry or an isotropic disordered material where the off-diagonal component of  $\kappa^S$  is negligible,  $\kappa_{\mathcal{M}\mathcal{M}'}^A$  is the energy flow perpendicular to the temperature gradient divided by the temperature difference. More generally, one would need to repeat the measurement for the materials  $\bar{\mathcal{M}}$  and  $\bar{\mathcal{M}}'$  where time-reversal invariance is broken in the opposite way in order to be able to separate  $\kappa^A$  from the off-diagonal part of  $\kappa^S$ . While the result of such measurement will depend on the way  $\mathcal{M}$  and  $\mathcal{M}'$  are glued together, we show that this dependence becomes vanishingly small when the diameter of the cylinder is much larger than the correlation length. Further, for three materials  $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ , we have a natural property  $\kappa_{\mathcal{M}\mathcal{M}'}^A + \kappa_{\mathcal{M}'\mathcal{M}''}^A = \kappa_{\mathcal{M}\mathcal{M}''}^A$ .

In practice one of the materials is usually the vacuum (or trivial insulator), and what one usually calls the thermal Hall conductance of  $\mathcal{M}$  is the thermal conductance of  $\mathcal{M}$  relative to vacuum. The importance of the relative viewpoint is that it suggests that no physical significance can be attached to  $\kappa^A$  unless a reference point in the space of materials (such as the vacuum) has been chosen. In contrast, one can formulate the procedure for measuring ordinary (electric) Hall conductance of a material  $\mathcal{M}$  in the torus geometry, and no reference

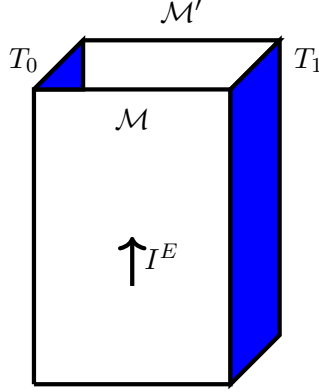


FIG. 1: Protocol for measuring the relative thermal Hall conductance of materials  $\mathcal{M}$  and  $\mathcal{M}'$ . If  $\kappa_{xy}^S$  is non-zero, one needs to repeat the measurement with the materials  $\bar{\mathcal{M}}$  and  $\bar{\mathcal{M}'}$  where time-reversal symmetry is broken in the opposite way.

point needs to be chosen.

From the mathematical viewpoint, the distinction between a relative and absolute material quantity is analogous to the distinction between an exact 1-form and a function. Given an exact 1-form  $\psi = \psi_i dx^i$  on a connected space  $X$  and a reference point  $x_0$ , one can define a function  $f(x) = \int_\gamma \psi$ , where  $\gamma$  is any path from  $x_0$  to  $x$ . The choice of  $\gamma$  is immaterial, but the choice of  $x_0$  matters: changing it shifts  $f(x)$  by a constant. On the other hand, the integral  $f(x, x') = \int_x^{x'} \psi$  is a well-defined function of a pair of points which does not depend on any reference point and satisfies  $f(x, x') + f(x', x'') = f(x, x'')$ .

To make this analogy precise, we introduce the parameter space  $M$  whose points label the choice of a lattice Hamiltonian on an infinite lattice as well as temperature  $T$ . It is understood that  $M$  includes only systems with short-range correlations, that is, loci where phase transitions occur are excluded.<sup>3</sup> We construct an exact 1-form  $\Psi$  on  $M$  such that its integral along a path connecting points  $\mathcal{M}, \mathcal{M}' \in M$  is equal to  $\kappa_{\mathcal{M}\mathcal{M}'}^A/T$ . We give an explicit formula for  $\Psi$  in terms of microscopic correlators.

At positive temperatures, the 1-form  $\Psi$  as well as relative thermal Hall conductance  $\kappa_{\mathcal{M}\mathcal{M}'}^A$  depend on the precise temperature profile as well as the contour along which the thermal

<sup>3</sup> It seems plausible that for  $T > 0$  the space of 2d materials with a finite correlation length is connected, in the sense that one can deform any  $\mathcal{M}$  to a trivial insulator without going through a phase transition. Phase transitions at nonzero temperatures are usually associated with spontaneous symmetry breaking and typically can be turned into cross-overs by adding suitable symmetry-breaking perturbations.

Hall current is measured. To eliminate this dependence and get thermal Hall *conductivity*, one needs to perform suitable averagings. On the other hand, we will argue that if both  $\mathcal{M}$  and  $\mathcal{M}'$  are gapped at  $T = 0$ , the dependence on such details is exponentially suppressed at low  $T$ , and thermal Hall conductance becomes identical to thermal Hall conductivity. Moreover, properties of the 1-form  $\Psi$  are such that the quantity

$$I(\mathcal{M}, \mathcal{M}') = \lim_{T \rightarrow 0} \frac{\kappa_{\mathcal{M}\mathcal{M}'}^A}{T} = \int_{\mathcal{M}}^{\mathcal{M}'} \Psi \quad (8)$$

does not change under deformations of zero-temperature systems  $\mathcal{M}$  and  $\mathcal{M}'$  provided the gap does not close. Thus  $I(\mathcal{M}, \mathcal{M}')$  is a relative topological invariant of gapped 2d phases at  $T = 0$ .

For technical reasons, in this paper we confine ourselves to lattice systems, but the discussion can be extended to systems of particles interacting via a short-range potential interaction. Ambiguities also affect Kubo formulas for certain thermoelectric coefficients. They can be dealt with in the same manner as in the case of thermal Hall conductance. These issues will be discussed elsewhere.

Since the definition of thermal Hall conductance is rather subtle, we begin with a discussion of electric Hall conductance. Some of the subtleties arise already in this context. Then we move on to the case of thermal Hall effect, define the 1-form  $\Psi$ , discuss its properties and define a relative topological invariant of gapped 2d systems. We will see that for systems described by Local Commuting Projector Hamiltonians  $\Psi$  vanishes identically, and thus the thermal Hall conductance vanishes too. Therefore such systems cannot have edge modes with a nonzero chiral central charge. This is an energy counterpart of the recently proved result that in such systems the zero-temperature electric Hall conductance vanishes [14]. In one of the Appendices, we show by a direct computation that for free fermionic systems the relative thermal Hall conductance of the  $T = 0$  and  $T = \infty$  states is related to the zero-temperature electric Hall conductance through a version of the Wiedemann-Franz law.

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## II. ELECTRIC HALL CONDUCTANCE

### A. Electric currents on a lattice

A 2d lattice system has a Hilbert space  $V = \otimes_{p \in \Lambda} V_p$ , where  $\Lambda$  (“the lattice”) is a uniformly discrete subset of  $\mathbb{R}^2$ , and all  $V_p$  are finite-dimensional. An observable is localized at a point  $p \in \Lambda$  if it has the form  $A \otimes 1_{\Lambda \setminus p}$  for some  $A : V_p \rightarrow V_p$ . An observable is localized on a subset  $\Lambda' \subset \Lambda$  if it commutes with all observables localized at any  $p \notin \Lambda'$ . A local observable  $A$  is an observable localized on a compact set  $\Lambda' \subset \Lambda$ , which will be called the support of  $A$ .

Hamiltonian of a lattice system has the form

$$H = \sum_{p \in \Lambda} H_p, \quad (9)$$

where the operators  $H_p : V \rightarrow V$  are Hermitian and local. We will assume that the Hamiltonian has a finite range  $\delta$ , which means that each  $H_p$  is a local observable supported in a ball of radius  $\delta$  centered at  $x$ . This implies that  $[H_p, H_q] = 0$  whenever  $|p - q| > 2\delta$ . We will also assume that the operators  $H_p$  are uniformly bounded, i.e. there exists  $C > 0$  such that  $\|H_p\| < C$  for all  $p \in \Lambda$ .

To define electric currents, we assume that the system has an on-site  $U(1)$  symmetry. Thus we are given a  $U(1)$  action on each  $V_p$ , with the generator  $Q_p$  (a Hermitian operator on  $V_p$  with integral eigenvalues). The total  $U(1)$  charge is  $Q_{tot} = \sum_{p \in \Lambda} Q_p$ . Further, we assume that  $[Q_{tot}, H_p] = 0$  for any  $p \in \Lambda$ . Since the time derivative of  $Q_q$  is

$$\frac{dQ_q}{dt} = i \sum_{p \in \Lambda} [H_p, Q_q], \quad (10)$$

it appears natural to define the  $U(1)$  current from  $q$  to  $p$  by  $J_{pq} = -i[H_p, Q_q]$ . However, this does not satisfy a physically desirable property  $J_{qp} = -J_{pq}$ . Instead we define

$$J_{pq} = i[H_q, Q_p] - i[H_p, Q_q]. \quad (11)$$

The lattice current thus defined satisfies  $J_{qp} = -J_{pq}$  as well as

$$\frac{dQ_q}{dt} = - \sum_{p \in \Lambda} J_{pq}. \quad (12)$$

Suppose  $\Lambda$  is decomposed into a disjoint union of two sets,  $\Lambda = A \cup B$ ,  $A \cap B = \emptyset$ . The current from  $B$  to  $A$  is defined as

$$J(A, B) = \sum_{p \in A} \sum_{q \in B} J_{pq}. \quad (13)$$

More generally, given a function  $\eta : \Lambda \times \Lambda \rightarrow \mathbb{R}$ , one can define

$$J(\eta) = \frac{1}{2} \sum_{p,q} \eta(p,q) J_{pq}. \quad (14)$$

In the case  $\eta(p,q) = \chi_B(q) - \chi_B(p)$ , where  $\chi_B(p) = 1$  for  $p \in B$  and  $\chi_B(p) = 0$  otherwise,  $J(\eta)$  reduces to  $J(A,B)$ .

## B. Chains and cochains

So far, we have encountered local operators  $H_p$  and  $Q_p$  which depend on a point and operators  $J_{pq}$  which depend on a pair of points. Later we will also encounter objects which depend on three points. It is useful to introduce a suitable terminology for such objects. A quantity  $A(p_0, \dots, p_n)$  which depends on  $n+1$  points of  $\Lambda$ , is skew-symmetric under the exchange of points, and decays rapidly (say, faster than any power of the distance) when the distance between any two points becomes large, will be called a chain of degree  $n$ , or an  $n$ -chain. (One can visualize  $n+1$  ordered points of  $\Lambda$  as the vertices of an abstract oriented  $n$ -dimensional simplex, and a formal linear combination of simplices is usually called a chain.) For example, the operators  $J_{pq}$  form an operator-valued 1-chain. The decay condition is satisfied here because  $[H_p, Q_q] = 0$  for  $|p - q| > \delta$ .

There is an operation  $\partial$  on chains which lowers the degree by 1:

$$(\partial A)(p_1, \dots, p_n) = \sum_{q \in \Lambda} A(q, p_1, \dots, p_n). \quad (15)$$

Although the sum is infinite, the operation is well-defined for  $n > 0$  since we assumed rapid decay when  $q$  is far away from any of the points  $p_1, \dots, p_n$ . This operation satisfies  $\partial \circ \partial = 0$ . The chain  $\partial A$  is called the boundary of the chain  $A$ . Using this notation, the conservation equation (12) can be written as

$$\frac{dQ}{dt} = -\partial J. \quad (16)$$

Applying  $\partial$  to both sides, we get the conservation of the total electric charge  $Q_{tot} = \partial Q$ . Some care is needed here, since  $Q_{tot}$  is an infinite sum and thus is an unbounded operator.

For a given  $\Lambda$ , the space of chains of all degrees (real or operator-valued) equipped with the boundary operator  $\partial$  forms a simplicial complex. A closely related complex (the Vietoris-Rips complex) appears in mathematics and data science, see e.g. [15]. Chains



in the Vietoris-Rips complex satisfy a stronger decay condition: there exists a  $\delta > 0$  such that  $A(p_0, \dots, p_n)$  vanishes if  $|p_i - p_j| > \delta$  for some  $i, j$ . The current 1-chain  $J_{pq}$  is in fact an operator-valued Vietoris-Rips chain, and we will make occasional use of this. Some properties of the Vietoris-Rips complex are discussed in Appendix A.

Dually, an  $n$ -cochain is a function of  $n + 1$  points of  $\Lambda$  which is skew-symmetric, but need not decay when one of the points is far from the rest. We will only consider real-valued cochains. A natural operation on cochains is:

$$(\delta\alpha)(p_0, \dots, p_{n+1}) = \sum_{j=0}^{n+1} (-1)^j \alpha(p_0, \dots, p_{j-1}, p_{j+1}, \dots, p_{n+1}). \quad (17)$$

It increases the degree by 1 and satisfies  $\delta \circ \delta = 0$ . The cochain  $\delta\alpha$  is called the coboundary of the cochain  $\alpha$ . The evaluation of an  $n$ -chain  $A$  on an  $n$ -cochain  $\alpha$  is

$$A(\alpha) = \frac{1}{(n+1)!} \sum_{p_0, \dots, p_n} A(p_0, \dots, p_n) \alpha(p_0, \dots, p_n). \quad (18)$$

An example of a 1-cochain is a function  $\eta(p, q)$  which appears in (14), then the operator  $J(\eta)$  is simply the evaluation of the operator-valued 1-chain  $J$  on a 1-cochain  $\eta$ . This operator is, in general, unbounded. Note that the evaluation of a real-valued or complex-valued chain on a cochain does not necessarily make sense, because of convergence issues. To make it well-defined, one can require the chain to decay rapidly or vanish when any of the points is far from a particular point of  $\Lambda$ . This, however, does not hold for most chains we naturally encounter here. Alternatively, we can impose an additional condition on cochains: if we truncate the cochain by declaring that it vanishes whenever the distance between any two points is larger than some  $\delta$ , then we may require the truncated cochain to vanish when any of the points is far from some fixed point. We may call such cochains "compactly supported". For example, if we regard  $\chi_B$  as a 0-cochain, then  $\eta = \delta\chi_B$  is compactly-supported if either  $A$  or  $B$  are compact. One can evaluate an arbitrary complex-valued chain on this cochain and get a well-defined number. Or, when one evaluates an operator-valued chain like  $J$  on this cochain, one gets an operator  $J(A, B) = J(\eta)$  which is bounded.

With this being said, we can state a kind of "Stokes' theorem"

$$A(\delta\beta) = \partial A(\beta) \quad (19)$$

which applies to any  $n$ -chain  $A$  and any  $(n-1)$ -cochain  $\beta$  for which both sides of the equation are well-defined. In the special case  $A = J$  and  $\beta = \chi_B$  for some finite set  $B$ , combining

(19) and the conservation equation (16) we get that the current through the boundary of  $B$  (represented by the 1-cocycle  $\delta\chi_B$ ) is equal to minus the rate of change of the total charge in  $B$ .

Given an  $n$ -cochain  $\alpha$  and an  $m$ -cochain  $\gamma$  one can define an  $(n+m)$ -cochain  $\alpha \cup \gamma$  by

$$(\alpha \cup \gamma)(p_0, \dots, p_{n+m}) = \frac{1}{(n+m+1)!} \sum_{\sigma \in \mathcal{S}_{n+m+1}} (-1)^\sigma \alpha(p_{\sigma(0)}, \dots, p_{\sigma(n)}) \gamma(p_{\sigma(n)}, \dots, p_{\sigma(n+m)}). \quad (20)$$

where  $\mathcal{S}_{n+m+1}$  is the permutation group on  $n+m+1$  objects. This operation satisfies

$$\alpha \cup \gamma = (-1)^{nm} \gamma \cup \alpha, \quad \delta(\alpha \cup \gamma) = \delta\alpha \cup \gamma + (-1)^n \alpha \cup \delta\gamma. \quad (21)$$

The operations  $\delta$  and  $\cup$  on cochains are analogous to operations  $d$  and  $\wedge$  on differential forms.

Finally, we note that if an  $n$ -chain  $A(p_0, \dots, p_n)$  is nonzero only if  $|p_i - p_j| \leq \delta$  for all  $i, j$ , then its contraction with an  $n$ -cochain  $\alpha$  is well-defined even if  $\alpha(p_0, \dots, p_n)$  is only defined for  $|p_i - p_j| \leq \delta$ . We will make use of such partially-defined cochains below.

As a simple application of these constructions, we consider the definition of magnetization. The equilibrium expectation value of the electric current satisfies

$$\partial \langle J \rangle = 0. \quad (22)$$

One can show (see Appendix A) that there exists a 2-chain  $M(p, q, r)$  such that

$$\langle J \rangle = \partial M. \quad (23)$$

This is a lattice analog of the continuum equation

$$\langle J_k(p) \rangle = -\epsilon_{kj} \partial_j M(p) \quad (24)$$

which defines magnetization  $M(p)$ . Thus one can regard the 2-chain  $M(p, q, r)$  as a lattice analog of magnetization. Unfortunately,  $M$  is not unique: one can always redefine

$$M \mapsto M + \partial P, \quad (25)$$

where  $P$  is an arbitrary 3-chain. There is no natural way to fix this ambiguity. On the other hand there is a natural expression for the derivatives of  $M$  with respect to the parameters

of the Hamiltonian. That is, there is a canonically-defined 2-chain  $\mu(p, q, r)$  with values in 1-forms on the parameter space such that

$$d\langle J \rangle = \partial\mu. \quad (26)$$

The 2-chain  $\mu$  is given by

$$\mu(p, q, r) = -\beta\langle\langle dH_p; J_{qr} \rangle\rangle - \beta\langle\langle dH_r; J_{pq} \rangle\rangle - \beta\langle\langle dH_q; J_{rp} \rangle\rangle, \quad (27)$$

where  $\langle\langle A; B \rangle\rangle$  denotes the Kubo canonical pairing [16]. Using the properties of the Kubo pairing (see Appendix B), one can easily verify the identity (26).

### C. Kubo formula for the electric Hall conductance

Usually, Kubo formula is written down for conductivity rather than conductance. That is, it is assumed that the electric field is uniform across all relevant scales. For our purposes, it will be important to have a formula for the electric Hall current which does not assume that the electric field is uniform.

Consider a time-dependent perturbation of the Hamiltonian of the form

$$\Delta H(t) = \epsilon e^{st} \sum_{p \in \Lambda} g(p) Q_p, \quad (28)$$

where the real parameter  $\epsilon$  is small and  $g : \Lambda \rightarrow \mathbb{R}$  is arbitrary for now. This perturbation corresponds to adiabatically switching on an electric potential  $\epsilon g$ . Assuming that at  $t = -\infty$  the system is in an equilibrium state at temperature  $T$ , at  $t = 0$  the system will be in a non-equilibrium steady state. The change in the expectation value of an observable  $A$  at  $t = 0$  relative to the expectation value at  $t = -\infty$  is given by the general Kubo formula

$$\Delta\langle A \rangle = \epsilon \lim_{s \rightarrow 0+} \beta \int_0^\infty e^{-st} \left\langle \left\langle A(t); \sum_p \frac{1}{i} [H, g(p) Q_p] \right\rangle \right\rangle dt. \quad (29)$$

Here Heisenberg-picture operators are defined as usual,  $A(t) = e^{iHt} A e^{-iHt}$ , and double brackets  $\langle\langle \dots \rangle\rangle$  denote Kubo's canonical pairing, see Appendix B. We also assumed that  $A$  doesn't have an explicit dependence on  $\epsilon$ .

For an infinite system, the existence of the limit  $s \rightarrow 0+$  in eq. (29) is far from obvious. When both the perturbation  $\Delta H$  and the observable  $A$  are supported on a compact set

$K \subset \Lambda$ , the existence of the limit has been proved in [17]. When  $g$  is nonzero only on a compact set  $K \subset \Lambda$ , but  $A$  is supported on a non-compact set, we still expect the limit to exist, at least away from phase transitions. Indeed, if the correlation length is finite, the state of the system far from  $K$  is unaffected by the perturbation, and we can effectively truncate the support  $A$  to be compact, thereby reducing to the case when both  $g$  and  $A$  are compactly supported. More generally, when the intersection of the supports of  $g$  and  $A$  is compact, the same argument suggests that  $\Delta\langle A \rangle$  is well-defined.

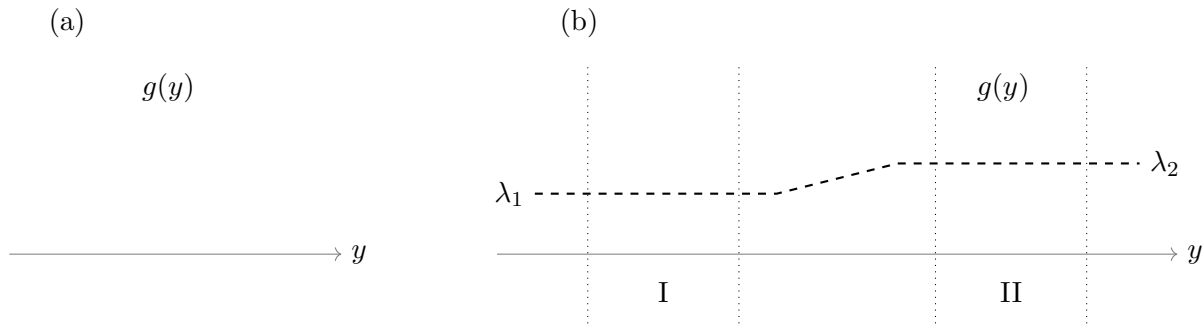


FIG. 2

To compute quantum Hall conductance we would like  $A$  to be the electric current across a vertical line  $x = a$ , and to let  $g$  be a function which depends only on  $y$ , vanishes at  $y = +\infty$  and approaches 1 at  $y = -\infty$ , see Fig. 2a. Such a function  $g$  corresponds to the net electric potential change  $-\epsilon$  from  $y = -\infty$  to  $y = +\infty$ . However, these  $A$  and  $g$  do not satisfy the condition on supports explained above. Another way to explain a potential problem is to note that while the electric field corresponding to such a function  $g$  is vanishingly small for  $y \ll 0$  and all  $t$ , the state of the system at  $t = 0$  and  $y \ll 0$  is different from that at  $t = -\infty$  and  $y \ll 0$  because the chemical potential changes by  $-\epsilon$ . Since the expectation value of the current density is nonzero even in equilibrium and may depend on the chemical potential, the change in the current density between  $t = 0$  and  $t = -\infty$  need not vanish at large negative  $y$ , and then the change in the net current across the line  $x = a$  will be ill-defined.

One way to avoid this difficulty is to make the  $y$  direction periodic and to perturb the system by a constant vector potential rather than a scalar potential. However, this approach does not have an analog in the case of thermal transport, which is our primary interest. Alternatively, one can take  $g$  to vanish both for  $y \ll 0$  and  $y \gg 0$ . For example, one can

take  $g$  to look as in Fig. 2b. Then the electric field is smooth in the regions  $I$  and  $II$  and has opposite magnitudes there. Elsewhere it is zero. If the system is homogeneous, the net electric Hall current in the  $x$  direction will be zero. However, if the system is inhomogeneous, then the electric Hall conductance of the two regions may be different, and the net electric Hall current will be given by

$$\Delta\langle A \rangle = -\epsilon \int_{-\infty}^{+\infty} \sigma_{xy} \partial_y g dy = \epsilon \int_{-\infty}^{+\infty} g \partial_y \sigma_{xy} dx = \epsilon \int_{\lambda_1}^{\lambda_2} \frac{\partial \sigma_{xy}}{\partial \lambda} d\lambda. \quad (30)$$

Here we assumed that the system is homogeneous in regions  $I$  and  $II$ , while in the intermediate region some parameter of the Hamiltonian  $\lambda$  varies from  $\lambda_1$  to  $\lambda_2$  as  $y$  is increased. This approach allows one to compute the derivatives of the Hall conductance with respect to parameters. Integrating these derivatives along a path in the space of parameters, one can compute the relative electric Hall conductance of two systems, provided the path avoids phase transitions. This is good enough, since in practice one usually measures the relative electric Hall conductance of a particular material and vacuum.

As discussed in the previous section, the net current through a vertical line  $x = a$  is defined as

$$J_a = \sum_{x(p) > a, x(q) < a} J_{pq} = J(\delta f), \quad (31)$$

where  $f(p) = \theta(a - x(p))$  is a step-function. More generally, one can take

$$A = J(\delta f), \quad (32)$$

where  $f : \Lambda \rightarrow \mathbb{R}$  is equal to 1 if  $x(p) \ll 0$  and equal to 0 if  $x(p) \gg 0$ . The corresponding operator is a smeared current from the left half-plane to the right half-plane. Then (29) takes the form

$$\Delta\langle J(\delta f) \rangle = \epsilon\beta \lim_{s \rightarrow 0+} \int_0^\infty e^{-st} \langle J(\delta f, t); J(\delta g) \rangle dt. \quad (33)$$

Here we used the conservation law (10) and the Stokes theorem.

Recall now that we consider a Hamiltonian depending on a parameter  $\lambda$  which varies with  $y$  such that in region  $I$   $\lambda = \lambda_1$ , in region  $II$   $\lambda = \lambda_2$ , and in the intermediate region  $\lambda$  interpolates between these two values without crossing a phase transition. We assume that  $\lambda_2 - \lambda_1$  is small. We also choose  $g$  as in Fig. 2b. Then  $J(\delta g)$  is a sum of operators supported in regions  $I$  and  $II$ . We can make this explicit by writing  $\delta g = \delta g_I + \delta g_{II}$ , where  $g_{II}$  interpolates between 0 and 1 as one moves from  $y = +\infty$  to the intermediate region, and

$g_I$  interpolates from 1 to 0 as one moves from the intermediate region to  $y = -\infty$ . If the electric field in region  $I$  is minus the translate of the electric field in region  $II$ , then  $J(\delta g_I)$  is minus the translate of  $J(\delta g_{II})$ , and to linear order in  $\Delta\lambda$  we get

$$\Delta\langle J(\delta f) \rangle = \epsilon(\lambda_2 - \lambda_1) \frac{\partial}{\partial \lambda} \left[ \beta \lim_{s \rightarrow 0+} \int_0^\infty e^{-st} \langle\langle J(\delta f, t); J(\delta g_{II}) \rangle\rangle dt \right] + O((\lambda_2 - \lambda_1)^2). \quad (34)$$

Here we implicitly assumed that the correlator

$$\lim_{s \rightarrow 0+} \int_0^\infty dt e^{-st} \langle\langle J(\delta f, t); J(\alpha) \rangle\rangle, \quad (35)$$

where  $\alpha$  is a 1-cochain, depends only on the Hamiltonian in some neighborhood of the intersection of supports of  $J(\delta f)$  and  $J(\alpha)$ , and thus when evaluating it one may assume that either  $\lambda = \lambda_1$  everywhere (when  $\alpha = \delta g_I$ ) or  $\lambda = \lambda_2$  everywhere (when  $\alpha = \delta g_{II}$ ).

Comparing eq. (34) with eq. (30), we get

$$\frac{\partial \sigma_{xy}(f, g)}{\partial \lambda} = \frac{\partial}{\partial \lambda} \left[ \beta \lim_{s \rightarrow 0+} \int_0^\infty e^{-st} \langle\langle J(\delta f, t); J(\delta g) \rangle\rangle dt \right], \quad (36)$$

where  $g$  is now a function depending only on  $y$  which interpolates between 1 and 0 as  $y$  varies from  $-\infty$  to  $+\infty$ , and  $f(p) = \theta(a - x(p))$ . This formula determines the electric Hall conductance up to an arbitrary constant. If we define the electric Hall conductance of vacuum to be zero, then we get a Kubo formula for the electric Hall conductance itself:

$$\sigma_{xy}(f, g) = \beta \lim_{s \rightarrow 0+} \int_0^\infty e^{-st} \langle\langle J(\delta f, t); J(\delta g) \rangle\rangle dt. \quad (37)$$

Note that it still depends on the exact profile of the electric potential  $g$  as well as on the choice of  $a$ . To get the electric Hall conductivity one needs to take the limit where  $g$  is linear over a very large region in  $y$  and also to average over  $a$ . We will soon see that at  $T = 0$  the precise choice of  $f$  and  $g$  becomes immaterial.

#### D. Zero-temperature electric Hall conductance as a topological invariant

In this section we argue that for a gapped system at  $T = 0$  the electric Hall conductance  $\sigma_{xy}(f, g)$  becomes independent of the precise choice of functions  $f$  and  $g$  and unchanged under variations of the Hamiltonian which do not close the gap. This is an adaptation of the arguments in Ref. [19]. We will also make use of the recent rigorous results on the decay of correlation functions in gapped systems [18]. Since [18] assumes that the system is finite,

strictly speaking we need to work on  $T^2$  instead of  $\mathbb{R}^2$ . That is, we need to assume that the system is periodic with some large period  $L$  (both in  $x$  and  $y$  directions) and consider only quantities which are  $L$ -periodic. If  $L$  is much larger than all other scales of the problem, electric Hall conductance should be unaffected.

Constructing the  $T^2$  analog of eq. (37) is slightly non-trivial, since the (smeared) step-functions like  $f$  and  $g$  are not  $L$ -periodic and thus are not defined on  $T^2$ . But to define the  $T^2$  analog of the expression (37) we only need an analog of the closed 1-cochains  $\delta f$  and  $\delta g$ . Since  $J_{pq}$  vanishes when  $|p - q| > \delta$ , it can be contracted with a 1-cochain  $\alpha(p, q)$  which is defined only for  $|p - q| \leq \delta$ . Such partially-defined cochains are not only allowed but are required if we want the cohomology of the cochain complex to be nontrivial on  $T^2$  and to agree with its de Rham cohomology.

Specifically, we replace  $\delta f$  (resp.  $\delta g$ ) with the 1-cochain  $\alpha(p, q)$  (resp.  $\gamma(p, q)$ ) which is equal to  $f(q) - f(p)$  (resp.  $g(q) - g(p)$ ) when  $p, q$  are no farther than  $\delta$  and undefined otherwise. Then the  $T^2$  analog of eq. (37) is

$$\sigma_{xy}(\alpha, \gamma; L) = \beta \lim_{s \rightarrow 0^+} \int_0^\infty e^{-st} \langle J(\alpha, t); J(\gamma) \rangle dt. \quad (38)$$

Assuming that such correlators are determined by the state of the system in some neighborhood of the intersection of the supports of  $J(\alpha)$  and  $J(\gamma)$ , the  $L \rightarrow \infty$  volume of this expression will coincide with  $\sigma_{xy}(f, g)$ . Note that since the operators  $J(\delta f)$  and  $J(\delta g)$  are supported on a vertical and a horizontal strip in  $\mathbb{R}^2$ , the operators  $J(\alpha)$  and  $J(\gamma)$  are supported on a vertical and a horizontal strip on  $T^2$ . We will denote these strips  $\mathcal{V}_a$  and  $\mathcal{H}_b$ , where  $a, b \in \mathbb{R}$ , defined modulo  $L$ , are  $x$  and  $y$  coordinates of their centers, respectively.

Next we specialize to  $T = 0$ . On torus the spectrum of the Hamiltonian is discrete, and by inserting a complete set of states it is easy to write  $\sigma_{xy}(\alpha, \gamma; L)$  in terms of the many-body Green's function  $G = (z - H)^{-1}$ :

$$\sigma_{xy}(\alpha, \gamma; L) = -i \oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} (G J(\alpha) G^2 J(\gamma)). \quad (39)$$

Here  $E_0$  is the energy of the ground state, the contour of integration encloses the point  $z = E_0$  counter-clockwise and trace is taken over Hilbert space.

Let us say that a quantity depending on  $L$  is negligible if it is of order  $L^{-\infty}$  for large  $L$ . We will argue that shifting the strips  $\mathcal{V}_a$  and  $\mathcal{H}_b$  horizontally and vertically, respectively (i.e. changing  $a$  and  $b$ ) affects  $\sigma_{xy}(\alpha, \gamma; L)$  by a negligible amount. Changing  $a$  can be

accomplished by replacing  $\alpha$  with  $\alpha + \delta f_0$ , where  $f_0$  is a function on  $T^2$  supported on a vertical strip in  $T^2$  and independent of  $y$ . Then the change in  $\sigma_{xy}(\alpha, \gamma; L)$  is

$$-i \oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} (GJ(\delta f_0)G^2 J(\gamma)) = - \oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} (G[H, Q(f_0)]G^2 J(\gamma)) . \quad (40)$$

Here we used the Stokes' theorem. Using the identity  $[H, A] = -[G^{-1}, A]$ , this expression can be written as

$$\oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} (Q(f_0)G^2 J(\gamma)) - \oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} (GQ(f_0)GJ(\gamma)) . \quad (41)$$

The first term can be written as

$$\oint_{z=E_0} \frac{dz}{2\pi i} \frac{\partial}{\partial z} \text{Tr} (Q(f_0)GJ(\gamma)) \quad (42)$$

and thus vanishes.

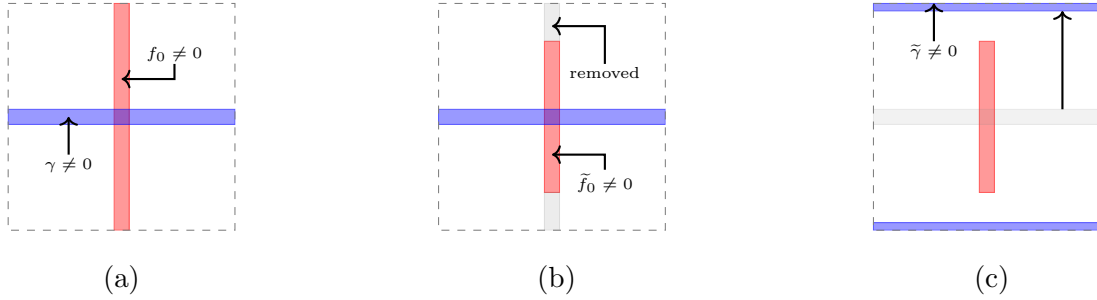


FIG. 3: (a) The red vertical line represents the support of  $Q(f_0)$ , the blue horizontal line represents the support of  $J(\gamma)$ . Sides of the square are identified. (b) Grey parts are far away from the blue line, give a negligible contribution and can be dropped. (c) One can use the conservation law to move the blue line so that the blue and red lines are separated by a large distance.

The second term in (41) is negligible. Indeed, it is shown in [18] that correlators of the form

$$\oint_{z=E_0} \text{Tr} (GAGB) \quad (43)$$

are negligible when the supports of  $A$  and  $B$  are separated by a distance of order  $L$ . Thus we make a negligible error when we replace the function  $f_0$  with a function  $\tilde{f}_0$  which is equal to  $f_0$  for  $|y - b| < L/3$  but vanishes for  $|y - b| \geq L/3$ . Let  $\tilde{\gamma}$  denote the translate of  $\gamma$  in the  $y$  direction by  $L/2$ , see Fig 3. Clearly,  $\gamma - \tilde{\gamma}$  differ by  $\delta g_0$ , where  $g_0$  is a function on  $T^2$ .



Therefore

$$\oint \frac{dz}{2\pi i} \text{Tr} (GQ(f_0)GJ(\gamma)) = \oint \frac{dz}{2\pi i} \text{Tr} (GQ(f_0)GJ(\tilde{\gamma})) - i \oint \frac{dz}{2\pi i} \text{Tr} \left( GQ(\tilde{f}_0)G[H, Q(g_0)] \right) + O(L^{-\infty}). \quad (44)$$

The first term here is negligible since the supports of  $Q(\tilde{f}_0)$  and  $J(\tilde{\gamma})$  are separated by a distance of order  $L$ . The second term is zero, since

$$\begin{aligned} \oint \frac{dz}{2\pi i} \text{Tr} \left( GQ(\tilde{f}_0)G[H, Q(g_0)] \right) &= - \oint \frac{dz}{2\pi i} \text{Tr} \left( GQ(\tilde{f}_0)G[G^{-1}, Q(g_0)] \right) = \\ &= - \oint \frac{dz}{2\pi i} \text{Tr} \left( G[Q(\tilde{f}_0), Q(g_0)] \right) = 0, \end{aligned} \quad (45)$$

due to ultra-locality of the charge. This concludes the proof that  $\sigma_{xy}(\alpha, \gamma; L)$  is independent of  $a$  up to negligible quantities. Independence of  $b$  is proved similarly.

Note that the status of  $f$  and  $g$ , as well as of  $\alpha$  and  $\gamma$ , was somewhat different until now. The function  $g$  described the profile of the electric potential and thus was a smeared step-function of nonzero width. The physically preferred value for  $f$  was an unsmeared step-function of the  $x$  coordinate. However, the difference between a smeared and unsmeared step-function is a function  $f_0$  supported on an interval. The above argument shows that for  $T = 0$  adding to  $\alpha$  a 1-cochain of the form  $\delta f_0$  changes  $\sigma_{xy}$  by a negligible amount. Thus at  $T = 0$  we can take both  $f$  and  $g$  to be unsmeared step-functions centered at  $x = a$  and  $y = b$ , respectively. Exchanging  $x$  and  $y$  is then the same as exchanging  $a$  and  $b$ . It is easy to see that  $\sigma_{xy}$  is anti-symmetric under such an exchange, hence at  $T = 0$  it coincides with  $\sigma^A$ . This is to be expected, since at  $T = 0$  the dissipative part of the conductance tensor vanishes.

Next we show that deformations of the Hamiltonian which do not close the energy gap affect  $\sigma_{xy}(\alpha, \gamma; L)$  by a negligible amount. It is sufficient to show this for families of Hamiltonians of the form  $H(\lambda) = H + \lambda V$ , where  $V$  is a local operator supported on a region of a fixed diameter  $D \ll L$ . The general case is an immediate consequence, since we can write an arbitrary deformation as a sum of such deformations. The number of terms in this sum scales like  $L^2$ , hence we still get a negligible derivative.

As explained above, we can choose  $\alpha$  and  $\gamma$  so that  $J(\alpha)$  and  $J(\gamma)$  are supported on strips of width  $\delta$  centered at  $x = a$  and  $y = b$ , respectively. Accordingly, we will denote the

corresponding operators  $J_a$  and  $J_b$  and write

$$\sigma^A(a, b; L) = -i \oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} (G J_a G^2 J_b). \quad (46)$$

Since changing  $a$  and  $b$  affects  $\sigma^A(a, b; L)$  only by a negligible amount, we can choose them so that the distance between the support of the perturbation  $V$  and the lines  $x = a$  and  $y = b$  is of order  $L$ . The variation of (46) is proportional to

$$\begin{aligned} \frac{\partial}{\partial \lambda} \oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} (G J_a G^2 J_b) &= \oint_{z=E_0} \frac{dz}{2\pi i} \{ \text{Tr} (G V G J_a G G J_b) \\ &+ \text{Tr} (G J_a G V G G J_b) + \text{Tr} (G J_a G G V G J_b) \}, \end{aligned} \quad (47)$$

where we have used the fact that variations of  $J_a, J_b$  are zero since the supports of  $J_a$  and  $J_b$  are more than a distance  $2\delta$  away from the support of  $V$ . We also used

$$\frac{\partial G}{\partial \lambda} = G \frac{\partial H}{\partial \lambda} G = G V G. \quad (48)$$

Subtracting a total derivative

$$\begin{aligned} 0 &= \oint \frac{dz}{2\pi i} \text{Tr} \frac{\partial}{\partial z} (G J_a G V G J_b) = \oint \frac{dz}{2\pi i} \{ \text{Tr} (G G J_a G V G J_b) \\ &+ \text{Tr} (G J_a G G V G J_b) + \text{Tr} (G J_a G V G G J_b) \}, \end{aligned} \quad (49)$$

from the above expression, we get

$$\frac{\partial}{\partial \lambda} \oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} (G J_a G^2 J_b) = - \oint \frac{dz}{2\pi i} \text{Tr} ([V, G J_a G] G J_b G). \quad (50)$$

In Appendix C we show that correlators of the form

$$\oint \frac{dz}{2\pi i} \text{Tr} ([A, G B G] G C G), \quad (51)$$

where  $A, B, C$  are local operators and the support of  $A$  is away from the support of  $B$ , are exponentially suppressed for gapped systems. Therefore we have

$$\frac{\partial \sigma^A(a, b; L)}{\partial \lambda} = -i \oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} (G J_a(x_0) G^2 J_b(y_0)) = O(L^{-\infty}), \quad (52)$$

as claimed.

### III. THERMAL HALL CONDUCTANCE

#### A. Energy currents and energy magnetization on a lattice

The energy current from site  $q$  to site  $p$  is defined to be [20, 21]

$$J_{pq}^E = -i[H_p, H_q], \quad (53)$$

so that

$$\frac{dH_q}{dt} = - \sum_{p \in \Lambda} J_{pq}^E. \quad (54)$$

Since  $[H_p, H_q] = 0$  whenever  $|p - q| > 2\delta$ ,  $J_{pq}^E$  is an operator-valued 1-chain. The energy flux from  $B$  to  $A = \Lambda \setminus B$  is defined to be

$$J^E(A, B) = \sum_{p \in A} \sum_{q \in B} J_{pq}^E = J^E(\delta\chi_B), \quad (55)$$

where  $\chi_B$  is the same as before.

In an equilibrium state we have

$$\partial\langle J^E \rangle = 0. \quad (56)$$

This suggests that there might exist a real-valued 2-chain  $M$  such that

$$\langle J^E \rangle = \partial M^E. \quad (57)$$

In fact, such a chain necessarily exists because the homology of  $\partial$  in degree 1 is trivial (see Appendix A).  $M^E$  is not unique: it is defined up to

$$M^E \mapsto M^E + \partial P, \quad (58)$$

where  $P$  is an arbitrary real-valued 3-chain. The equation (57) is a lattice analog of the continuum equation

$$\langle J_k^E(p) \rangle = -\epsilon_{kj} \partial_j M^E(p), \quad (59)$$

which defines “energy magnetization”  $M^E$  [3, 4]. Note that in the continuum energy magnetization is a function of spatial coordinates, while on the lattice it is a 2-chain. Unfortunately, there is no canonical choice of  $M^E$ , either in the continuum or on the lattice. Thus one cannot regard it as a local property of the system. However, as noted in [20], if we let the Hamiltonian depend on some parameters  $\lambda_\ell$ , then there is a canonical expression for the

2-chain  $\mu^E = dM^E$  valued in the space of 1-forms on the parameter space which solves the equation

$$d\langle J^E \rangle = \partial\mu^E. \quad (60)$$

Namely:

$$\mu^E(p, q, r) = -\beta\langle\langle dH_p; J_{qr}^E \rangle\rangle - \beta\langle\langle dH_r; J_{pq}^E \rangle\rangle - \beta\langle\langle dH_q; J_{rp}^E \rangle\rangle, \quad (61)$$

where  $d = \sum_l d\lambda_l \frac{\partial}{\partial\lambda_l}$  is the differential on space of local Hamiltonians. The identity (60) is easily verified using properties of the Kubo pairing (see Appendix B) and the definition of the energy current (53). Note that if the correlation length is finite,  $\mu_E(p, q, r)$  is exponentially small when any two of the points  $p, q, r$  are far from each other. Thus  $\mu^E$  is a well-defined 2-chain with values in 1-forms on the parameter space.

## B. Kubo formula for the derivatives of the thermal Hall conductance

To derive a Kubo-like formula we follow the same strategy as in the case of electric Hall conductance. Following [22] we perturb the Hamiltonian by a term

$$\Delta H(t) = \epsilon e^{st} \sum_{p \in \Lambda} g(p) H_p, \quad t \in (-\infty, 0]. \quad (62)$$

It is shown in [22] that this is equivalent to a time-dependent and space-dependent infinitesimal temperature deformation

$$\delta T(t, \mathbf{r}) = \epsilon T e^{st} g(\mathbf{r}).$$

As in the electric case, we cannot take  $g$  to be a smeared step-function of  $y$ , since then the change in the expectation value of the net energy current across a line  $x = a$  will be ill-defined. Instead we take  $g$  to be a function as in Fig. 2b, and consider an inhomogeneous system whose Hamiltonian depends on a parameter  $\lambda$  which varies with  $y$  as in Fig. 2b. This allows one to compute the derivative of the thermal Hall conductance with respect to  $\lambda$ .

One difference compared to the electric case is that the energy current operator  $A = J^E(\delta f)$  now has an explicit dependence on  $\epsilon$  (the magnitude of the perturbation). This happens because  $[H_p, H_q] \neq 0$ , in general. The change in  $A$  due to this explicit dependence is

$$\Delta A = -\epsilon \frac{1}{2} \sum_{p, q} i[H_p, H_q] (f(q) - f(p)) (g(p) + g(q)) = 2\epsilon J^E(g \cup \delta f), \quad (63)$$

The corresponding change in the expectation value of  $A$  is

$$2\epsilon\langle J^E(g \cup \delta f) \rangle = 2\epsilon M^E(\delta g \cup \delta f). \quad (64)$$

This expression receives contributions only from the regions  $I$  and  $II$  where the temperature gradient is nonzero. We make this explicit by writing  $\delta g = \delta g_I + \delta g_{II}$ , where  $g_{II}(p)$  depends only on  $y(p)$  and interpolates between 0 and 1 as one moves from  $y = +\infty$  to the intermediate region, and  $g_I(p)$  depends only on  $y(p)$  and interpolates between 1 and 0 as one moves from the intermediate region to  $y = -\infty$ . If the temperature gradients in regions  $I$  and  $II$  are equal and opposite,  $\delta g_I$  is minus the translate of  $\delta g_{II}$ . In these two regions the parameter  $\lambda$  takes constant values  $\lambda_1$  and  $\lambda_2$ , respectively. Therefore the expression (64) can be written as

$$2\epsilon(\lambda_2 - \lambda_1) \mu^E(\delta g_{II} \cup \delta f) + O((\lambda_2 - \lambda_1)^2), \quad (65)$$

where  $\mu^E = \partial M^E / \partial \lambda$ . Combining this with the change in  $\langle A \rangle$  arising from the change in the state of the system, we get

$$\Delta \langle A \rangle \approx \epsilon(\lambda_2 - \lambda_1) \frac{\partial}{\partial \lambda} \left[ \beta \lim_{s \rightarrow 0^+} \int_0^\infty e^{-st} \langle \langle J^E(\delta f, t); J^E(\delta g) \rangle \rangle dt \right] + 2\epsilon(\lambda_2 - \lambda_1) \mu^E(\delta g \cup \delta f). \quad (66)$$

Here to simplify notation we denoted by  $\delta g$  what previously was denoted  $\delta g_{II}$ . That is,  $g(p)$  now denotes a function of  $y(p)$  which interpolates from 1 at  $y \ll 0$  to 0 at  $y \gg 0$ . On the other hand, the expected net energy current from the left half-plane to the right half-plane is

$$-\epsilon T \int_{-\infty}^{+\infty} \kappa_{xy} \partial_y g dy = \epsilon T \int_{\lambda_1}^{\lambda_2} \frac{\partial \kappa_{xy}}{\partial \lambda} d\lambda.$$

Comparing these two expressions we get a formula for the  $\lambda$ -derivative of the thermal Hall conductance:

$$d\kappa_{xy}(f, g) = d \left[ \beta^2 \lim_{s \rightarrow 0^+} \int_0^\infty e^{-st} \langle \langle J^E(\delta f, t); J^E(\delta g) \rangle \rangle dt \right] - 2\beta \mu^E(\delta f \cup \delta g). \quad (67)$$

Here  $\mu^E$  is the 2-chain with values in 1-forms on the parameter space defined by eq. (61).

Unlike in the electric case, there is no canonical formula for  $\kappa_{xy}(f, g)$ . We can still define the difference of thermal Hall conductances of two materials  $\mathcal{M}$  and  $\mathcal{M}'$  by integrating the 1-form  $d\kappa_{xy}$  along a path in the parameter space connecting  $\mathcal{M}$  and  $\mathcal{M}'$ . This path must avoid phase transitions, otherwise objects like  $\mu^E(\delta f \cup \delta g)$  might diverge.

### C. Path-independence of the thermal Hall conductance

We have defined a 1-form  $d\kappa_{xy}$  on the space of parameters of a lattice system whose integral along a curve  $\Gamma$  can be identified with the difference of thermal Hall conductances of the initial and final points of  $\Gamma$ . The definition of the 1-form depended on the rapid spatial decay of the Kubo pairings of local operators. Thus when choosing a curve connecting two points  $\mathcal{M}$  and  $\mathcal{M}'$  in the parameter space, one needs to avoid loci where phase transitions occur. Since we are allowed to enlarge the parameter space by adding arbitrary local terms to the Hamiltonian, it is very plausible that such a curve exists for any two points  $\mathcal{M}$  and  $\mathcal{M}'$ .

An important consistency requirement is that the difference of the thermal Hall conductances thus computed does not depend on the choice of  $\Gamma$ . To show this, consider an arbitrary closed loop  $\Gamma$  in the parameter space. By assumption, the “Kubo” part of the thermal Hall conductance

$$\kappa_{xy}^{\text{Kubo}}(f, g) = \beta^2 \lim_{s \rightarrow 0+} \int_0^\infty e^{-st} \langle \langle J^E(\delta f, t); J^E(\delta g) \rangle \rangle dt, \quad (68)$$

is well-defined for each point of  $\Gamma$ . Therefore  $d\kappa_{xy}^{\text{Kubo}}(f, g)$  is an exact 1-form and its integral over any closed curve vanishes.

Let us now show the energy magnetization contribution  $\mu^E(\delta f \cup \delta g)$  is also exact. This is a 1-form on the parameter space which depends on  $f$  and  $g$ . Its physical meaning is the differential of the energy magnetization in the region where both  $f$  and  $g$  vary substantially. We are going to show that the integral of this 1-form along loop  $\Gamma$  avoiding phase transitions is zero. Intuitively, this must be true in order to avoid contradiction with the theorem about the absence of net energy currents in equilibrium quasi-1d systems [13]. Imagine slowly varying the parameters of the system as a function of  $y \in [0, L]$  while following a loop  $\Gamma$ . Then we can compactify the  $y$  direction with period  $L$ , and regard this as a quasi-1d system. If  $L$  is large compared to the correlation length, this should not affect local properties, including the differential of the energy magnetization. The energy current in the  $x$  direction can be computed using the continuum equation (59). Since the net energy current should vanish, we get

$$0 = \int \langle J_x^E \rangle dy = \int_0^L \partial_y M^E dy \simeq \int \partial_\lambda M^E d\lambda = \int_\Gamma \mu^E. \quad (69)$$

The error in this computation should become arbitrarily small for  $L \rightarrow \infty$ , so we get the desired result.

A more precise argument goes as follows. As above, we identify the  $y$  direction with period  $L$  thereby replacing  $\mathbb{R}^2$  with a cylinder  $\mathbb{R}_x \times S_y^1$ . For large  $L$  this will change local quantities such as the 2-cochain  $\mu^E(p, q, r)$  by negligible amount. In place of  $\delta g$ , one has to use a partially-defined 1-cochain  $\gamma$  which is closed but not exact. One technical problem is that the evaluation of  $\mu^E$  on  $\delta f \cup \gamma$  is not well-defined, since  $\delta f \cup \gamma$  is only partially defined. One can deal with this by truncating  $\mu^E(p, q, r)$  to zero whenever any of the two points are farther apart than some distance of order  $L$ . This introduces an error of order  $L^{-\infty}$ . Let us denote that corresponding 2-cochain on  $\mathbb{R}^x \times S_y^1$  by  $\tilde{\mu}^E$ . Because of truncation, we now have an approximate equality  $\partial \tilde{\mu}^E = d\langle J^E \rangle + O(L^{-\infty})$ . Naively, one can deduce the desired result using the Stokes' theorem:

$$\int_{\Gamma} \tilde{\mu}^E(\delta f \cup \gamma) = \int_{\Gamma} d\langle J^E \rangle(f \cup \gamma) + O(L^{-\infty}) = O(L^{-\infty}). \quad (70)$$

This argument is not correct because the 1-cochain  $f \cup \gamma$  has non-compact support (because the function  $f : \mathbb{R}_x \rightarrow \mathbb{R}$  is a smeared step-function  $\theta(a - x)$ ), and the evaluation of  $d\langle J^E \rangle$  on such a cochain is not well-defined. To fix this, we first modify the Hamiltonian for  $x < a - L$  by scaling it to zero. Because of absence of phase transitions in 1d systems, the correlation length remains finite, and therefore the effect of such a modification on all local observables near  $x = a$  will be negligible. Then the operator-valued chain  $J^E$  also becomes zero for  $x \ll a$ , and the application of the Stokes' theorem becomes legitimate. This concludes the argument.

Since by definition  $\mu^E(\delta f \cup \delta g)$  is the differential of energy magnetization in the neighborhood of the intersection of supports of  $\delta f$  and  $\delta g$ , this result means that energy magnetization exists as a globally-defined function on the parameter space. This function is defined up to an additive constant.

#### D. A relative invariant of gapped 2d lattice systems

In this section we use the 1-form  $d\kappa_{xy}$  to define a relative topological invariant of gapped 2d lattice systems at zero temperature. We anticipate that in the case when both lattice systems admit a conformally-invariant edge, the invariant will be equal to  $\pi/6$  times the

difference of the chiral central charges for the two systems. We cannot necessarily connect two such systems by a curve  $\Gamma$  in the space of Hamiltonians without encountering a bulk phase transition. If we could, this would mean that they are in the same zero-temperature phase, and then by the result of [13] they would have to have the same chiral central charge for the edge modes, and therefore the relative invariant would vanish. Rather, the idea is to treat the temperature  $T$  as yet another parameter, and connect the two systems in the enlarged parameter space  $(T, \lambda)$ . At positive temperatures quantum phase transitions are smoothed out into cross-overs, and the two systems can now be deformed into each other while maintaining a finite correlation length.

Formally, the temperature can be regarded as a parameter because re-scaling the temperature by a positive factor is equivalent to re-scaling the Hamiltonian by the inverse factor. Therefore one can extend the form  $\kappa_{xy}(f, g)$  to the open subset of the enlarged parameter space given by  $T > 0$ . In detail, this is done as follows. Given a Hamiltonian  $H$ , we define a one-parameter family of Hamiltonians by  $H(\lambda) = \lambda H$ . Then the above mentioned scaling symmetry implies

$$T \frac{d}{dT} \frac{\kappa_{xy}^{\text{Kubo}}(f, g)}{T} = - \lambda \frac{d}{d\lambda} \bigg|_{\lambda=1} \frac{\kappa_{xy}^{\text{Kubo}}(f, g; \lambda)}{T}, \quad (71)$$

where  $\kappa_{xy}^{\text{Kubo}}(f, g; \lambda)$  denotes the Kubo part of  $\kappa_{xy}$  computed with the Hamiltonian  $H(\lambda)$ . We have to divide  $\kappa_{xy}$  by  $T$  in order to get an observable which is invariant under the rescaling  $H \mapsto \lambda H, T \mapsto \lambda T$ . The variation of magnetization under temperature change is given by 2-chain  $\tau^E$  defined as

$$\tau^E(p, q, r) = \beta \langle \langle H_p; J_{qr}^E \rangle \rangle + \beta \langle \langle H_r; J_{pq}^E \rangle \rangle + \beta \langle \langle H_q; J_{rp}^E \rangle \rangle. \quad (72)$$

This is just  $-\mu^E$  with  $dH_p$  replaced with  $H_p$ .

We can now define a 1-form on the subset  $T > 0$  of the enlarged parameter space:

$$\Psi(f, g) = \frac{d\kappa_{xy}^{\text{Kubo}}(f, g)}{T} - \frac{2}{T^2} \mu^E(\delta f \cup \delta g) + \frac{d}{dT} \left( \frac{\kappa_{xy}^{\text{Kubo}}(f, g)}{T} \right) dT - 2\tau^E(\delta f \cup \delta g) \frac{dT}{T^3}. \quad (73)$$

Its integral around any closed curve in the  $(T, \lambda)$  space is zero by the same argument as before, therefore  $\Psi$  is exact.

Given any two gapped zero-temperature lattice systems  $\mathcal{M}$  and  $\mathcal{M}'$ , we would like to define a relative topological invariant by integrating  $\Psi$  along a curve in the enlarged parameter space which connects  $\mathcal{M}$  and  $\mathcal{M}'$ . We need to check three things: that the integral



converges, that it does not change as one deforms  $\mathcal{M}$  and  $\mathcal{M}'$  while keeping  $T = 0$  and finite correlation length, and that result of integration does not change as we shift cochains  $f, g$ . Neither of these is obvious. The  $T$ -component of the 1-form  $\Psi$  is

$$\begin{aligned}\Psi_n(f, g) &= \frac{d}{dT} \left( \frac{\kappa_{xy}^{\text{Kubo}}(\delta f, \delta g)}{T} \right) - \frac{2}{T^3} \tau^E(\delta f \cup \delta g) = \\ &= -\frac{1}{T^3} \left[ \frac{d}{d\lambda} \Big|_{\lambda=1} \int_0^\infty \beta \langle \langle J_\lambda^E(\delta f, t); J_\lambda^E(\delta g) \rangle \rangle_\lambda dt + 2\tau^E(\delta f \cup \delta g) \right].\end{aligned}\quad (74)$$

Here  $\langle \langle \dots \rangle \rangle_\lambda$  denotes the Kubo pairing at temperature  $T$  with respect to the Hamiltonian  $H(\lambda) = \lambda H$ , and  $J_\lambda^E$  is the energy current for the Hamiltonian  $H(\lambda)$ . We denoted the  $T$ -component  $\Psi_n$  to emphasize that it is the normal component to the boundary  $T = 0$  of the enlarged parameter space. The convergence of the integral of  $\Psi$  requires the expression in parentheses to vanish faster than  $T^2$  as  $T \rightarrow 0$ . Similarly, the independence of the integral of  $\Psi$  on the deformation of the endpoints requires the tangential component of  $\Psi$ ,

$$\Psi_t(f, g) = \frac{1}{T^2} \left( d \int_0^\infty \beta \langle \langle J^E(\delta f, t); J^E(\delta g) \rangle \rangle dt - 2\mu^E(\delta f \cup \delta g) \right). \quad (75)$$

to vanish at  $T = 0$ . Thus the expression in parentheses should vanish faster than  $T^2$  as  $T \rightarrow 0$ .

In Appendix D we argue (not completely rigorously) that both expressions vanish exponentially fast as  $T \rightarrow 0$ . To see why this is plausible, consider eq. (75) for definiteness and denote the expression in parentheses as  $\Omega(T)$ . It is a 1-form on the space of parameters of the Hamiltonian. The first term in  $\Omega(T)$  is the exterior derivative of the same kind of current correlator which defines the electric Hall conductance, except that the electric current  $J$  is replaced with the energy current  $J^E$ . The same argument as in Section IID shows that at  $T = 0$  this exterior derivative is of order  $L^{-\infty}$  when evaluated on any perturbation of the Hamiltonian which is a distance  $L$  away from the support of  $\delta f \cup \delta g$ . The same is true for the second term, because of the assumed decay of Kubo pairings. Since the sum of the two terms does not change as one varies  $f$  and  $g$ ,  $L$  can be arbitrarily large, and we conclude that  $\Omega(0) = 0$  when evaluated on any deformation of the Hamiltonian supported on a quadrant in  $\mathbb{R}^2$ . Therefore  $\Omega(0) = 0$  identically. Further, in the presence of the energy gap one expects the low-temperature expansion to have a finite radius of convergence, therefore  $\Omega(T) - \Omega(0)$  is exponentially suppressed for low  $T$ . Combining these statements, we show that integral converges and independent of deformations of  $\mathcal{M}$  and  $\mathcal{M}'$  which don't cross

phase transitions. In order to show that the value of integral is independent of the shift of cochains  $f, g$  we can use the fact that one form  $\Psi(f, g)$  is exact and integral is given by difference of antiderivatives of  $\Psi$  at endpoints. The latter can be shown to vanish if either  $f$  or  $g$  is hat-shaped as in Fig. 2b.

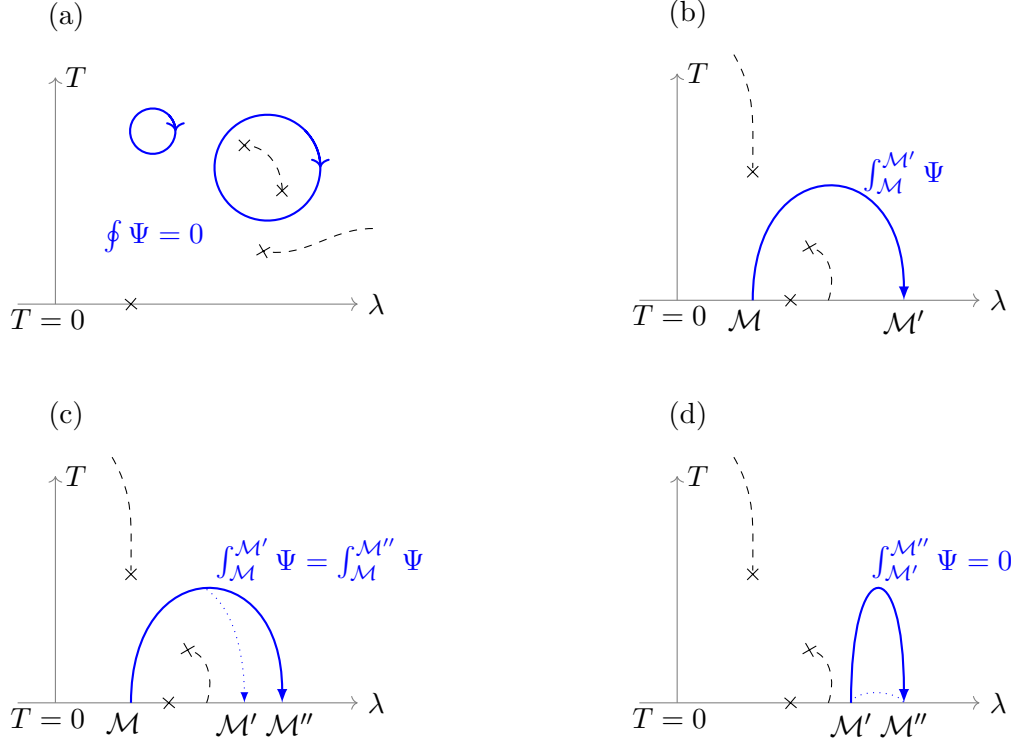


FIG. 4: Phase diagrams. The horizontal axis represents a parameter of the Hamiltonian, the vertical axis is temperature. Dashed lines and crosses represent phase transitions. Blue lines are integration contours.

(a) The integral of  $\Psi$  along a loop is zero regardless of whether there are phase transitions in the interior.

(b) The invariant  $I(\mathcal{M}, \mathcal{M}')$  for zero-temperature phases  $\mathcal{M}$  and  $\mathcal{M}'$  can be computed by integrating  $\Psi$

along the blue line. (c) The points  $\mathcal{M}'$  and  $\mathcal{M}''$  are in the same phase, therefore one expects the integrals

along the solid and dotted blue lines to be the same. (d) The difference of the two paths can be deformed

to a near-zero-temperature path from  $\mathcal{M}''$  to  $\mathcal{M}'$ .  $\Psi$  is exponentially small on this path.

There is another limit where one can evaluate  $\Psi$ , namely  $T \rightarrow \infty$ . In this limit the expectation value  $\langle A \rangle$  of a local operator  $A$  becomes the normalized trace over the local Hilbert space, while the Kubo pairing becomes

$$\lim_{T \rightarrow \infty} \langle \langle A; B \rangle \rangle = \langle AB \rangle - \langle A \rangle \langle B \rangle. \quad (76)$$

Thus all components of  $\Psi$  are of order  $1/T^3$  for large  $T$ , and therefore the relative thermal Hall conductance of any two high-temperature states is of order  $1/T^2$ . Hence another natural choice of a reference state (apart from the trivial insulator at  $T = 0$ ) is the  $T = \infty$  state. That is, one can define an absolute topological invariant of a gapped zero-temperature system  $\mathcal{M}$  by integrating the 1-form  $\Psi$  along any path connecting  $\mathcal{M}$  to the  $T = \infty$  state.

The case of a Locally Commuting Projector Hamiltonian is particularly simple. In this case, since  $J_{pq}^E = -i[H_p, H_q] = 0$  for all  $p, q$ , the  $T$ -component of the 1-form  $\Psi$  vanishes identically. Integrating  $\Psi$  along a path  $\Gamma$  along which only  $T$  changes, we find that  $\kappa^A(T) - \kappa^A(\infty) = 0$ . Thus the thermal Hall conductance relative to the  $T = \infty$  state is zero for all temperatures.<sup>4</sup> This implies that the chiral central charge of the edge modes must vanish for such a Hamiltonian. One can also show that the zero-temperature electric Hall conductance vanishes for such systems, but the proof is very different [14].

The case of gapped systems of free fermions is also fairly simple, since there are no phase transitions at any  $T > 0$ , and one can again integrate  $\Psi$  along a path with only  $T$  varying. Then one only needs to know the  $T$ -component of  $\Psi$ , which can be evaluated in complete generality. This computation is performed in Appendix E where it is shown that

$$\int_{T=0}^{T=\infty} \Psi = \left. \frac{\kappa^A}{T} \right|_{T=\infty} - \left. \frac{\kappa^A}{T} \right|_{T=0} = -\frac{\pi^2}{3} \sigma^A, \quad (77)$$

where  $\sigma^A$  is the electric Hall conductance at  $T = 0$ . If one defines  $\kappa^A/T$  to vanish at  $T = \infty$ , then this can be regarded as a form of the Wiedemann-Franz law. Note however that it cannot be interpreted too naively. For example, since  $\Psi$  is exponentially small for low  $T$ , most of the contribution to the integral (77) comes from  $T$  of order of the energy gap. Although one can define the absolute thermal Hall conductance at temperature  $T$  as

$$\kappa^A(T) = T \int_{\infty}^T \Psi, \quad (78)$$

and it will obey the Wiedemann-Franz law  $\kappa^A \simeq \frac{\pi^2}{3} T \sigma^A$  at low  $T$ ,  $\kappa^A(T)$  is not determined by correlators measured at temperature  $T$  and a fixed Hamiltonian.

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<sup>4</sup> Strictly speaking, to avoid potential phase transitions at  $T > 0$ , one needs to work with a finite-volume version of  $\Psi$  defined on torus. Its  $T$ -component still vanishes for a system described by a Local Commuting Projector Hamiltonian, so the integral from any  $T$  to  $T = \infty$  is still zero. Taking the infinite-volume limit we conclude that the relative thermal Hall conductance is identically zero.

## Appendix A: Vietoris-Rips complex

Let  $\Lambda$  be a countable subset of  $\mathbb{R}^d$  such that the distance between distinct points of  $\Lambda$  is bounded from below by some  $\epsilon > 0$ . Fix  $\delta > 0$ . The Vietoris-Rips complex  $VR(\Lambda; \delta)$  has  $\Lambda$  as its vertex set. An  $(n + 1)$ -tuple of distinct points  $p_0, \dots, p_n \in \Lambda$  spans an  $n$ -simplex of  $VR(\Lambda; \delta)$  if and only if  $|p_i - p_j| \leq \delta$  for all  $i, j$ . In data science  $\Lambda$  represents data points, and the variation of the topology of  $VR(\Lambda; \delta)$  with  $\delta$  is used to characterize the data.

Another important simplicial complex associated to  $\Lambda$  is the Čech complex which is defined as follows. Fix  $\delta > 0$ . The Čech complex  $C(\Lambda; \delta)$  has  $\Lambda$  as its vertex set. Let  $B(p; \delta)$  be a closed ball of radius  $\delta$  centered at  $p \in \Lambda$ . An  $(n + 1)$ -tuple of distinct points  $p_0, \dots, p_n \in \Lambda$  spans an  $n$ -simplex in the Čech complex if and only if the intersection of the balls  $B(p_i; \delta/2)$ ,  $i = 0, \dots, n$ , is non-empty. Clearly, every simplex of  $C(\Lambda; \delta)$  is a simplex of  $VR(\Lambda; \delta)$ , thus  $C(\Lambda; \delta)$  is a sub-complex of  $VR(\Lambda; \delta)$ . It is also easy to see that  $VR(\Lambda; \delta)$  is a sub-complex of  $C(\Lambda; 2\delta)$ .

Both the Vietoris-Rips and the Čech complexes may contain  $n$ -simplices with  $n$  much larger than the dimension of the embedding space  $\mathbb{R}^d$ . The homology of the Čech complex is nevertheless very constrained thanks to the “Nerve Theorem” [23]. It states that the Čech complex is homotopy equivalent to the union  $\bigcup_{p \in \Lambda} B(p; \delta/2)$ . Therefore the homology groups of  $C(\Lambda; \delta)$  in degree  $n > d$  are all trivial, regardless of  $\delta$ .

The Vietoris-Rips complex does not have such a simple behavior, in general. Suppose however that the set  $\Lambda$  covers  $\mathbb{R}^d$  sufficiently uniformly, in the sense there exists a number  $\epsilon' > 0$  such that every point  $p \in \mathbb{R}^d$  is at most  $\epsilon'$  away from some point of  $\Lambda$ . Then for  $\delta > \epsilon'$  every  $n$ -cycle of  $VR(\Lambda; \delta)$  with  $n > 0$  becomes a boundary when mapped to  $VR(\Lambda; 2\delta)$ . Indeed, recall that we have inclusions:

$$VR(\Lambda; \delta) \hookrightarrow C(\Lambda; 2\delta) \hookrightarrow VR(\Lambda; 2\delta). \quad (\text{A1})$$

The union  $\bigcup_{p \in \Lambda} B(p; \delta)$  is the whole  $\mathbb{R}^d$ , hence by the Nerve Theorem the homology of  $C(\Lambda; 2\delta)$  is trivial for  $n > 0$ . Hence the inclusion  $VR(\Lambda; \delta) \hookrightarrow VR(\Lambda; 2\delta)$  maps every homology class of degree  $n > 0$  to zero.

Chains in the Vietoris-Rips complex are finite linear combinations of simplices. If  $\Lambda$  is infinite, one can also consider infinite linear combinations of simplices. It is easy to see that the boundary operator is still well-defined and satisfies  $\partial^2 = 0$ . The corresponding

simplicial complex may be called the locally-finite Vietoris-Rips complex and will be denoted  $VR^{lf}(\Lambda; \delta)$ . It is the more relevant complex for applications to lattice models, since objects like  $J_{pq}$  and  $\langle J_{pq} \rangle$  are nonzero for an infinite number of pairs of points  $p, q$ .

Obviously, if  $\delta' > \delta$ , then  $VR^{lf}(\Lambda; \delta)$  is a sub-complex of  $VR^{lf}(\Lambda; \delta')$ . The homology of  $VR^{lf}(\Lambda; \delta)$  is hard to compute, in general. Nevertheless, if  $\Lambda$  covers  $\mathbb{R}^d$  uniformly, one can show that every cycle in  $VR^{lf}(\Lambda; \delta)$  with  $n \neq d$  becomes a boundary when mapped to  $VR^{lf}(\Lambda; \delta')$  with a sufficiently large  $\delta'$ . To show this, one uses the locally-finite version of the Čech complex  $C^{lf}(\Lambda; \delta)$ . On the one hand, we still have a chain of inclusions:

$$VR^{lf}(\Lambda; \delta) \hookrightarrow C^{lf}(\Lambda; 2\delta) \hookrightarrow VR^{lf}(\Lambda; 2\delta). \quad (\text{A2})$$

On the other hand, it is shown in [24, 25] that the direct limit of the homology groups  $H_n(C^{lf}(\Lambda; \delta))$  with respect to  $\delta$  is isomorphic to  $H_n^{lf}(\mathbb{R}^d)$ , the locally-finite (or Borel-Moore) homology of  $\mathbb{R}^d$ . It is well-known that the latter is trivial for  $n \neq d$ . Therefore for  $n \neq d$  every  $n$ -cycle of  $C^{lf}(\Lambda; \delta)$  becomes a boundary when mapped to  $C^{lf}(\Lambda; \delta')$  with a sufficiently large  $\delta'$ . Combining these two observations, we get the desired result.

These results can be used to prove the existence of magnetization 2-chains  $M(p, q, r)$  and  $M^E(p, q, r)$ . These are solutions of the equations  $\langle J \rangle = \partial M$  and  $\langle J^E \rangle = \partial M^E$ , respectively. We know that the 1-chains  $\langle J \rangle$  and  $\langle J^E \rangle$  are cycles in  $VR^{lf}(\Lambda; \delta)$ , where  $\delta$  is the interaction range. Therefore there exists  $\delta' > \delta$  and  $M, M^E \in VR_2^{lf}(\Lambda; \delta')$  solving these equations. These 2-chains vanish whenever pairwise distances between  $p, q, r$  exceed  $\delta'$  and thus satisfy our decay assumptions.

## Appendix B: Kubo canonical pairing

Kubo canonical pairing of two operators  $A, B$  is defined as follows [16]:

$$\langle\langle A; B \rangle\rangle = \frac{1}{\beta} \int_0^\beta \langle A(-i\tau)B \rangle d\tau - \langle A \rangle \langle B \rangle. \quad (\text{B1})$$

Here  $\langle \dots \rangle$  denotes average over a Gibbs state at temperature  $T = 1/\beta$  (or more generally, over a state satisfying the Kubo-Martin-Schwinger condition), and  $A(-i\tau) = e^{H\tau} A e^{-H\tau}$ . Kubo pairing determines static linear response: if the Hamiltonian is perturbed by  $\lambda B$ , where  $\lambda$  is infinitesimal, then the change in the equilibrium expectation value of  $A$  is

$$\Delta \langle A \rangle = \langle \Delta A \rangle - \beta \lambda \langle\langle A; B \rangle\rangle + O(\lambda^2). \quad (\text{B2})$$

Here the first term is due to the possible explicit dependence of  $A$  on the Hamiltonian, while the second term is the change in the expectation value of  $A$  due to the change in the equilibrium state.

Kubo pairing is symmetric,  $\langle\langle A; B \rangle\rangle = \langle\langle B; A \rangle\rangle$ , and satisfies

$$\beta \langle\langle i[H, A]; B \rangle\rangle = \langle i[B, A] \rangle. \quad (\text{B3})$$

In finite volume, one can write it in terms of the energy eigenstates as follows:

$$\langle\langle A; B \rangle\rangle = Z^{-1} \sum_{n,m} \langle n | \bar{A} | m \rangle \langle m | \bar{B} | n \rangle \frac{e^{-\beta E_m} - e^{-\beta E_n}}{\beta(E_n - E_m)}, \quad (\text{B4})$$

where  $\bar{A} = A - \langle A \rangle$ , and  $\bar{B} = B - \langle B \rangle$ .

### Appendix C: Exponential decay of certain correlators in a gapped phase

Let  $A$ ,  $B$ , and  $C$  be local operators such that the supports of  $A$  and  $B$  are separated by at least  $L$ . Let  $G = (z - H)^{-1}$  be the Green's function of a gapped Hamiltonian, and let  $E_0$  be the energy of the ground state. For the time being we assume that the ground state is unique and comment on the more general case later. We are going to prove that the correlator

$$\oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} ([A, GBG] GCG), \quad (\text{C1})$$

is exponentially suppressed for large  $L$ . Note that the support of the operator  $C$  is not required to be separated from the supports of  $A$  and  $B$ . By performing the  $z$  integration we get

$$\begin{aligned} \oint \frac{dz}{2\pi i} \text{Tr} ([A, GBG] GCG) &= \langle AG_0 BG_0^2 C \rangle + \langle BG_0^2 CG_0 A \rangle - \langle CG_0^2 AG_0 B \rangle \\ &\quad - \langle CG_0 AG_0^2 B \rangle + \langle BG_0^2 AG_0 C \rangle + \langle BG_0 AG_0^2 C \rangle - \langle AG_0 CG_0^2 B \rangle \\ &\quad - \langle CG_0^2 BG_0 A \rangle + 2 (\langle CG_0^3 B \rangle - \langle BG_0^3 C \rangle) \langle A \rangle \\ &\quad + (\langle AG_0^3 B \rangle - \langle BG_0^3 A \rangle) \langle C \rangle + (\langle CG_0^3 A \rangle - \langle AG_0^3 C \rangle) \langle B \rangle, \end{aligned} \quad (\text{C2})$$

where  $\langle \dots \rangle$  denotes the average over the ground state and we have introduced the notation

$$G_0 = \sum_{n \neq 0} \frac{|n\rangle \langle n|}{E_0 - E_n}. \quad (\text{C3})$$

Now we use the following facts from [18] and other similar identities:

$$\begin{aligned}
\langle O_1 G_0^m O_2 G_0^m O_3 \rangle &= \langle O_1 G_0^{m+m} O_3 \rangle \langle O_2 \rangle + O(e^{-L/\xi}), \\
\langle O_2 G_0^m O_1 G_0^m O_3 \rangle &= O(e^{-L/\xi}), \\
\langle O_1 G_0^m O_2 \rangle &= O(e^{-L/\xi}),
\end{aligned} \tag{C4}$$

if  $n, m > 0$  and the support of operator  $O_2$  is at least  $L$  distance away from the supports of  $O_1$  and  $O_3$ . Here  $\xi > 0$  is a scale parameter which is finite for gapped systems. See [18] for the derivation of these identities.

Using these we can simplify the first term in (C2). Separating  $C$  (which is by assumption a sum of local operators) into two parts  $C = C_A + C_B$  where the support of  $C_A$  is far away from  $B$  and the support of  $C_B$  is far away from  $A$ , we get

$$\begin{aligned}
\langle AG_0 B G_0^2 C \rangle &= \langle AG_0 B G_0^2 C_A \rangle + \langle AG_0 B G_0^2 C_B \rangle \\
&= \langle AG_0^3 C_A \rangle \langle B \rangle + O(e^{-L/\xi}) = \langle AG_0^3 C \rangle \langle B \rangle + O(e^{-L/\xi}).
\end{aligned} \tag{C5}$$

Similarly, we have

$$\langle B G_0^2 C G_0 A \rangle = \langle B G_0^3 A \rangle \langle C \rangle + O(e^{-L/\xi}), \tag{C6}$$

$$-\langle C G_0^2 A G_0 B \rangle = -\langle C G_0^3 B \rangle \langle A \rangle + O(e^{-L/\xi}), \tag{C7}$$

$$-\langle C G_0 A G_0^2 B \rangle = -\langle C G_0^3 B \rangle \langle A \rangle + O(e^{-L/\xi}), \tag{C8}$$

$$\langle B G_0^2 A G_0 C \rangle = \langle B G_0^3 C \rangle \langle A \rangle + O(e^{-L/\xi}), \tag{C9}$$

$$\langle B G_0 A G_0^2 C \rangle = \langle B G_0^3 C \rangle \langle A \rangle + O(e^{-L/\xi}), \tag{C10}$$

$$-\langle A G_0 C G_0^2 B \rangle = -\langle A G_0^3 B \rangle \langle C \rangle + O(e^{-L/\xi}), \tag{C11}$$

$$-\langle C G_0^2 B G_0 A \rangle = -\langle C G_0^3 A \rangle \langle B \rangle + O(e^{-L/\xi}). \tag{C12}$$

These eight terms exactly cancel the remaining six terms in (C2). Putting everything together, we get

$$\oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} ([A, G B G] G C G) = O(e^{-L/\xi}). \tag{C13}$$

We have assumed a single ground state in the above derivation. However, as noted in [18], exactly the same arguments work for a  $q$ -fold degenerate ground state assuming that they are indistinguishable by local operators, i.e. if

$$\langle p | O | q \rangle = \delta_{pq} \langle p | O | p \rangle + O(L^{-\infty}) \tag{C14}$$

where  $|p\rangle, |q\rangle$  are ground states,  $O$  is a local operator, and  $L$  is the size of the system.

## Appendix D: The low-temperature behavior of the 1-form $\Psi$ in a gapped system

In this appendix we analyze the properties of the 1-form  $\Psi(f, g)$  whose integral defines the relative invariant of gapped 2d systems. We will have to use estimates on the behavior of certain correlation functions at low but non-zero temperature. More precisely, we will assume that if the  $T \rightarrow 0$  limit of a correlator is well-defined, then at sufficiently low temperature deviations from the  $T = 0$  value are of order  $O(e^{-T^*/T})$  for some  $T^* > 0$ . Physically, this is what one expects for a Hamiltonian with a gap for localized excitations.

One could try to prove it by putting the system on a torus of finite size  $L$ . Then for a correlation function  $C(T)$  one can construct a finite-size analog  $C(T, L)$  such that  $C(T) = \lim_{L \rightarrow \infty} C(T, L)$ . The correlation function  $C(T, L)$  can be rewritten in terms of many-body Green's function  $G(z) = (z - H)^{-1}$ . For example, one can write

$$\int_0^\beta \langle A(-i\tau)B \rangle_L d\tau = Z^{-1} \oint e^{-\beta z} \frac{dz}{2\pi i} \text{Tr}(G(z)AG(z)B), \quad (\text{D1})$$

where  $Z$  is the partition function, and the contour surrounds all the eigenvalues of  $H$ . Now if we deform the contour into a pair of contours, one surrounding  $z = E_0$  and the other surrounding all other eigenvalues, we see that for low  $T$  the contribution of the first contour is exponentially close to its  $T \rightarrow 0$  limit, while the contribution of the second one is exponentially small at low  $T$ . Thus  $C(T, L) - C(0, L)$  is exponentially small at low  $T$ . If we assume that the order of limits  $T \rightarrow 0$  and  $L \rightarrow \infty$  can be interchanged, we can conclude that  $C(T)$  is exponentially small at low  $T$ . These arguments are at best heuristic, since it is far from clear when the interchange of the order of limits is legitimate.

For simplicity of presentation we will work on  $\mathbb{R}^2$  and assume that correlation functions in gapped phase at non-zero temperature are exponentially closed to their zero-temperature expectation value. More rigorously, one should carefully relate cochains and operator-valued chains on the infinite plane with their torus analogs as was done in Section IID, but we will drop these technical details. Also, we will consider the system at a fixed non-zero temperature  $T$  and will vary only the Hamiltonian. As was explained in Section IIID, rescaling the temperature is equivalent to rescaling the Hamiltonian.

Consider the integral of  $\Psi(f, g)$  along a path connecting two zero-temperature phases  $\mathcal{M}$  and  $\mathcal{M}'$ :

$$I(\mathcal{M}, \mathcal{M}') = \int_{\mathcal{M}}^{\mathcal{M}'} \Psi(f, g). \quad (\text{D2})$$



We will argue that it converges, is independent of the choice of end points as long as one stays within the same phase, and does not change under suitable deformations of  $f, g$ .

Let us start with the last property. We will consider adding to  $f$  a function of  $x$  which has compact support. We need to show that

$$\int_{\mathcal{M}}^{\mathcal{M}'} \Psi(f_0, g) = 0, \quad (\text{D3})$$

where  $f_0$  is as in Fig 2b. Since the path in the parameter space is away from phase transitions, the correlation length is finite everywhere along the path. Replacing  $f_0$  with zero far away from the support of  $\delta g$  will introduce negligible error. Denote the truncated cochain  $\tilde{f}_0$ . It has compact support, therefore we can rewrite the magnetization term as

$$\mu^E(\delta \tilde{f}_0 \cup \delta g) = \partial \mu^E(\tilde{f}_0 \cup \delta g) = d\langle J^E(\tilde{f}_0 \cup \delta g) \rangle = -\frac{1}{2} d\langle i[H(\tilde{f}_0), H(g)] \rangle, \quad (\text{D4})$$

where in the last step we have used the definition of  $J^E$  and cup product. The Kubo term, on the other hand, can be rewritten as

$$\begin{aligned} \kappa_{xy}^{\text{Kubo}}(\tilde{f}_0, g) &= -\beta^2 \lim_{s \rightarrow 0+} \int_0^\infty dt e^{-st} \langle \langle \frac{dH(\tilde{f}_0, t)}{dt}; J^E(\delta g) \rangle \rangle \\ &= \beta^2 \langle \langle H(\tilde{f}_0); J^E(\delta g) \rangle \rangle + \beta^2 \lim_{s \rightarrow 0+} s \int_0^\infty dt e^{-st} \langle \langle H(\tilde{f}_0, t); J^E(\delta g) \rangle \rangle. \end{aligned} \quad (\text{D5})$$

The last term is in general non-zero since  $\langle \langle H(\tilde{f}_0, t); J^E(\delta g) \rangle \rangle$  does not have to converge to zero as  $t \rightarrow \infty$ . However, at zero temperature and for a gapped Hamiltonian one can explicitly check that this term is zero. Indeed, expanding the expression in the energy eigenbasis we get

$$\begin{aligned} \lim_{s \rightarrow 0+} s \int_0^\infty dt e^{-st} \langle \langle H(\tilde{f}_0, t); J^E(\delta g) \rangle \rangle \\ = -i \lim_{s \rightarrow 0+} s \sum_{n>0} \frac{\langle 0|H(\tilde{f}_0)|n \rangle \langle n|J^E(\delta g)|0 \rangle - \langle 0|J^E(\delta g)|n \rangle \langle n|H(\tilde{f}_0)|0 \rangle}{(E_0 - E_n)^2} = 0. \end{aligned}$$

Therefore at small but non-zero temperature we expect the second term in (D5) to be exponentially suppressed. The remaining term can be rewritten as

$$\beta^2 d\langle \langle H(\tilde{f}_0); J^E(\delta g) \rangle \rangle = \beta^2 d\langle \langle H(\tilde{f}_0); -i[H, H(g)] \rangle \rangle = -\beta d\langle i[H(\tilde{f}_0), H(g)] \rangle. \quad (\text{D6})$$

This term cancels the energy magnetization contribution (D4). Therefore for a hat-shaped function  $f_0$ ,  $\Psi(f_0, g)$  is a differential of a function which is exponentially small for  $T \rightarrow 0$ .

Hence the integral of  $\Psi(f_0, g)$  along a path connecting two gapped zero-temperature systems is zero. Similarly, one can prove that  $I(\mathcal{M}, \mathcal{M}')$  does not change if we add to  $g$  a compactly supported function of  $y$ .

It is tempting to use the same argument with  $f_0$  replaced with  $f$  to show that  $I(\mathcal{M}, \mathcal{M}')$  is zero. But the argument cannot be carried through because it is impossible to truncate  $f$  and make its support compact in such a way that the support of  $\delta f \cup \delta g$  coincides with the support of  $\delta \tilde{f} \cup \delta g$ . There will necessarily be additional intersections.

In order to show that the integral (D2) defining  $I(\mathcal{M}, \mathcal{M}')$  converges and is independent of the precise choice of endpoints, consider a variation of the Hamiltonian supported in a quadrant of  $\mathbb{R}^2$ . A general perturbation can be decomposed into a sum of four such perturbations. As discussed in the body of the paper, in order to show that  $I(\mathcal{M}, \mathcal{M}')$  is independent of endpoints and converges it is sufficient to show that all components of the 1-form  $\Psi(f, g)$  are exponentially small as  $T \rightarrow 0$ . Following the same logic as before, we can shift  $f, g$  in  $\Psi(f, g)$  away from the support of the variation introducing an error which is exponentially small in temperature. Recall that the 1-form  $\Psi$  is defined as

$$\Psi(f, g) = \beta^2 \left[ d \int_0^\infty \beta e^{-st} \langle \langle J^E(\delta\alpha, t); J^E(\delta\gamma) \rangle \rangle dt - 2\mu^E(\delta\alpha \cup \delta\gamma) \right]. \quad (\text{D7})$$

Using the same arguments as in Section II D, one can show that expression in square brackets is zero at  $T = 0$ . Therefore, it is exponentially small at zero temperature, and the same applies to  $\Psi(f, g)$ .

## Appendix E: Free fermion systems

Consider a free fermionic system on a lattice with a Hamiltonian

$$H = \sum_{p,q} a^\dagger(p) h(p, q) a(q). \quad (\text{E1})$$

The infinite matrix  $h(p, q)$  is assumed Hermitian,  $h(p, q)^* = h(q, p)$ . The energy on site  $p$  is taken to be

$$H_p = \frac{1}{2} \sum_m (a^\dagger(p) h(p, m) a(m) + a^\dagger(m) h(m, p) a(p)). \quad (\text{E2})$$

Defining the charge operator as a 0-chain

$$Q_p = a^\dagger(p) a(p), \quad (\text{E3})$$

we find the electric current 1-chain

$$J_{pq} = i(a^\dagger(q)h(q,p)a(p) - a^\dagger(p)h(p,q)a(q)). \quad (\text{E4})$$

Contracting it with a 1-cochain  $\alpha(q) - \alpha(p)$  for some function  $\alpha : \Lambda \rightarrow \mathbb{R}$ , we get

$$J(\delta\alpha) = -ia^\dagger[h, \alpha]a, \quad (\text{E5})$$

where we now regard  $\alpha$  as an operator in the one-particle Hilbert space.

Similarly, the energy current operator is a 1-chain

$$\begin{aligned} J_{pq}^E = \frac{-i}{4} \sum_m & (a^\dagger(p)h(p,q)h(q,m)a(m) - a^\dagger(q)h(q,p)h(p,m)a(m) \\ & - a^\dagger(m)h(m,q)h(q,p)a(q) + a^\dagger(m)h(m,p)h(p,q)a(q) \\ & + a^\dagger(p)h(p,m)h(m,q)a(q) - a^\dagger(q)h(q,m)h(m,p)a(p)) . \end{aligned} \quad (\text{E6})$$

Contracting it with a 1-cochain  $\alpha(q) - \alpha(p)$ , we get

$$J^E(\delta\alpha) = -\frac{i}{2}a^\dagger[h^2, \alpha]a. \quad (\text{E7})$$

The Gibbs state at temperature  $T = 1/\beta$  is defined via

$$\langle a(p, t)a^\dagger(q, 0) \rangle = \left\langle p \left| \frac{e^{-iht}}{1 + e^{-\beta h}} \right| q \right\rangle, \quad (\text{E8})$$

$$\langle a(p, t)^\dagger a(q, 0) \rangle = \left\langle q \left| \frac{e^{iht}}{1 + e^{\beta h}} \right| p \right\rangle, \quad (\text{E9})$$

and Wick's theorem. Then

$$\langle J(\delta f, t)J(\delta g) \rangle = -\text{Tr} \left( [h, f] \frac{e^{-iht}}{1 + e^{-\beta h}} [h, g] \frac{e^{iht}}{1 + e^{\beta h}} \right), \quad (\text{E10})$$

where the trace on the r.h.s. is taken over the 1-particle Hilbert space  $L^2(\Lambda)$ , and the functions  $f : \Lambda \rightarrow \mathbb{R}$  and  $g : \Lambda \rightarrow \mathbb{R}$  are regarded as Hermitian operators on this Hilbert space. The operators  $[h, f]$  and  $[h, g]$  are supported on a vertical and a horizontal strips, respectively.

Going to the energy basis, replacing  $t \rightarrow t - i\tau$  and integrating over  $\tau$  from 0 to  $\beta$  we get

$$\langle\langle J(\delta f, t); J(\delta g) \rangle\rangle = \frac{-1}{\beta} \sum_{n,m} \langle n|[h, f]|m \rangle \langle m|[h, g]|n \rangle e^{i(\varepsilon_n - \varepsilon_m)t} \frac{e^{\beta\varepsilon_n} - e^{\beta\varepsilon_m}}{(1 + e^{\beta\varepsilon_n})(1 + e^{\beta\varepsilon_m})(\varepsilon_n - \varepsilon_m)}, \quad (\text{E11})$$

where  $\varepsilon_n$  are 1-particle energy levels. Note that in the limit  $T \rightarrow 0$ , the fraction in this equation is equal to  $\frac{\theta(\varepsilon_n) - \theta(\varepsilon_m)}{\varepsilon_n - \varepsilon_m}$  plus exponentially small terms. Thus at zero temperature  $\varepsilon_m$  and  $\varepsilon_n$  must have opposite signs. More generally, we can re-write the fraction as

$$\frac{f(\varepsilon_m) - f(\varepsilon_n)}{\varepsilon_n - \varepsilon_m} \quad (\text{E12})$$

where  $f(\varepsilon) = \frac{1}{1+e^{\beta\varepsilon}}$  is the Fermi-Dirac distribution.

Integrating over  $t$ , we get

$$\sigma(f, g) = i \sum_{n, m} \frac{\langle n | [h, f] | m \rangle \langle m | [h, g] | n \rangle}{\varepsilon_n - \varepsilon_m + is} \frac{f(\varepsilon_n) - f(\varepsilon_m)}{\varepsilon_n - \varepsilon_m} \quad (\text{E13})$$

It is convenient to rewrite this expression using the 1-particle Green's functions  $G_{\pm}(z) = 1/(z - h \pm i0)$ . The following formulas are useful:

$$\langle a^\dagger A a \rangle = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz f(z) \text{Tr} \left( [G_+ - G_-] A \right), \quad (\text{E14})$$

$$\begin{aligned} -\beta \langle \langle a^\dagger A a; a^\dagger B a \rangle \rangle &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz f(z) \text{Tr} \left( [G_+ - G_-] A G_+ B + G_- A [G_+ - G_-] B \right) \\ &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz f(z) \text{Tr} \left( G_+ A G_+ B - G_- A G_- B \right), \end{aligned} \quad (\text{E15})$$

where we have suppressed the argument  $z$  for  $G_{\pm}(z)$ . Here  $A$  and  $B$  are operators on the 1-particle Hilbert space and we have assumed  $\langle a^\dagger A a \rangle = \langle a^\dagger B a \rangle = 0$  in the last formula. Also note that

$$h G_{\pm} = G_{\pm} h = z G_{\pm} - 1, \quad [G_{\pm}, A] = G_{\pm} [h, A] G_{\pm}. \quad (\text{E16})$$

Using the Green's functions, the electric conductance can be rewritten as

$$\sigma(f, g) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} dz f(z) \text{Tr} \{ [h, f] G_+^2 [h, g] (G_+ - G_-) - [h, f] (G_+ - G_-) [h, g] G_-^2 \}, \quad (\text{E17})$$

and the Kubo part of the thermal conductance as

$$\kappa_{xy}^{\text{Kubo}}(f, g) = -\frac{\beta}{8\pi} \int_{-\infty}^{\infty} dz f(z) \text{Tr} \{ [h^2, f] G_+^2 [h^2, g] (G_+ - G_-) - [h^2, f] (G_+ - G_-) [h^2, g] G_-^2 \}. \quad (\text{E18})$$

The value of energy magnetization  $\mu^E$  on a 2-cochain  $\delta f \cup \delta g$  can be found to be

$$\begin{aligned} \mu^E(\delta f \cup \delta g) &= \frac{1}{16\pi} \int_{-\infty}^{\infty} dz f(z) \text{Tr} \left( G_+ d h G_+ \left\{ [[h, f], [h, g]] + [h^2, f] G_+ [h, g] + [h, f] G_+ [h^2, g] \right. \right. \\ &\quad \left. \left. - [h^2, g] G_+ [h, f] - [h, g] G_+ [h^2, f] \right\} \right) - (G_+ \rightarrow G_-), \end{aligned} \quad (\text{E19})$$

where  $dh$  is the variation of the 1-particle Hamiltonian. In the translationally invariant case, one can replace  $f$  and  $g$  with momentum derivatives.

Using the above formulas, it is straightforward to compute the 1-form  $\Psi$  for any free system. Let us demonstrate this by computing the  $T$ -component of the 1-form  $\Psi$ .

For a global re-scaling of the Hamiltonian we have  $dh = h$ , and eq. (E19) can be simplified

$$\begin{aligned} \tau^E(\delta f \cup \delta g) = & -\frac{1}{16\pi} \int_{-\infty}^{\infty} dz \operatorname{Tr} \left\{ 2\mathfrak{f}(z)G_-^2[h^2, f](G_+ - G_-)[h^2, g] \right. \\ & - 2\mathfrak{f}(z)(G_+ - G_-)[h^2, f]G_+^2[h^2, g] + 4\mathfrak{f}'(z)h^2(G_+ - G_-)[h, f]G_+[h, g] \\ & \left. - 4\mathfrak{f}'(z)G_-[h, f]h^2(G_+ - G_-)[h, g] - \mathfrak{f}'(z)h(G_+ - G_-)[[h, f], [h, \beta]] \right\}. \quad (\text{E20}) \end{aligned}$$

Variation of  $\kappa_{xy}^{\text{Kubo}}(f, g)$  contains two pieces:

$$\begin{aligned} & -\frac{\beta}{8\pi} d \left( \int_{-\infty}^{\infty} dz \mathfrak{f}(z) \operatorname{Tr} \left\{ [h^2, f]G_+^2[h^2, g](G_+ - G_-) \right\} \right) \\ & = \frac{\beta}{8\pi} \int_{-\infty}^{\infty} dz \operatorname{Tr} \left\{ -2\mathfrak{f}(z)[h^2, f]G_+^2[h^2, g](G_+ - G_-) - 4\mathfrak{f}'(z)[h, f]G_+^2[h, g]h^3(G_+ - G_-) \right. \\ & \quad \left. + 4\mathfrak{f}'(z)[h, f]G_+[h, g]h^2(G_+ - G_-) - \mathfrak{f}'(z)[h, f][h, g]h(G_+ - G_-) \right\} \quad (\text{E21}) \end{aligned}$$

and

$$\begin{aligned} & \frac{\beta}{8\pi} d \left( \int_{-\infty}^{\infty} dz \mathfrak{f}(z) \operatorname{Tr} \left\{ [h^2, g]G_-^2[h^2, f](G_+ - G_-) \right\} \right) \\ & = \frac{\beta}{8\pi} \int_{-\infty}^{\infty} dz \operatorname{Tr} \left\{ 2\mathfrak{f}(z)[h^2, g]G_-^2[h^2, f](G_+ - G_-) + 4\mathfrak{f}'(z)[h, g]G_-^2[h, f]h^3(G_+ - G_-) \right. \\ & \quad \left. - 4\mathfrak{f}'(z)[h, g]G_-[h, f]h^2(G_+ - G_-) + \mathfrak{f}'(z)[h, g][h, f]h(G_+ - G_-) \right\}. \quad (\text{E22}) \end{aligned}$$

Inserting these three contributions into eq. (74) we arrive at

$$\begin{aligned} \frac{d}{dT} \left( \frac{\kappa_{xy}(f, g)}{T} \right) = & \frac{1}{2\pi T^3} \int_{-\infty}^{\infty} dz \operatorname{Tr} \left\{ \mathfrak{f}'(z)[h, f]G_+^2[h, g]z^3(G_+ - G_-) \right. \\ & \left. - \mathfrak{f}'(z)[h, g]G_-^2[h, f]z^3(G_+ - G_-) \right\}. \quad (\text{E23}) \end{aligned}$$

The right-hand side looks very similar to the electric conductance (E17). Indeed, integrating it over temperature from 0 to  $\infty$  and using the formula

$$\int_0^\infty \frac{dT}{T^3} \mathfrak{f}'(z) = -\frac{\pi^2}{6|z|^3} = -\frac{\pi^2}{3z^3} (\mathfrak{f}(0) - \mathfrak{f}(\infty)) \quad (\text{E24})$$

gives

$$\frac{\kappa^A}{T} \Big|_{T=\infty} - \frac{\kappa^A}{T} \Big|_{T=0} = \frac{\pi^2}{3} \left( \sigma^A \Big|_{T=\infty} - \sigma^A \Big|_{T=0} \right). \quad (\text{E25})$$

Since at infinite temperature the electric Hall conductance vanishes, while the thermal Hall conductance can be defined to vanish, we arrive at the Wiedemann-Franz law.

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