Supplementary information for "Energy Magnetization and Thermal Hall Effect"

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I. KUBO'S CANONICAL CORRELATION FUNCTION

Kubo's canonical correlation function is defined as [1]:

$$\left\langle \hat{B}; \hat{A} \right\rangle_0 = \frac{1}{\beta_0} \mathrm{Tr} \left[\hat{\rho}_{\mathrm{eq}} \int_0^{\beta_0} d\lambda e^{\lambda \hat{\mathscr{H}}} \hat{B} e^{-\lambda \hat{\mathscr{H}}} \hat{A} \right] , \tag{S1}$$

where $\hat{\mathscr{H}}$ is the Hamiltonian of the system, and $\hat{\rho}_{\mathrm{eq}} = (1/Z) \exp\left(-\beta_0 \hat{\mathscr{H}}\right)$. Some of its properties used in the main text are [1]:

$$\left\langle \Delta\hat{A}; \Delta\hat{B} \right\rangle_0 = \left\langle \Delta\hat{B}; \Delta\hat{A} \right\rangle_0 , \tag{S2}$$

and:

$$\beta_0 \left\langle \Delta\dot{\hat{A}}; \Delta\hat{B} \right\rangle_0 = -\beta_0 \left\langle \Delta\hat{A}; \Delta\dot{\hat{B}} \right\rangle_0 = \frac{1}{\mathrm{i}\hbar} \left\langle \left[\Delta\hat{A}, \Delta\hat{B} \right] \right\rangle_0 , \tag{S3}$$

where $\Delta\hat{A} \equiv \hat{A} - \left\langle \hat{A} \right\rangle_0$ and $\dot{\hat{A}} \equiv (1/\mathrm{i}\hbar)[\hat{A}, \hat{\mathscr{H}}]$.

For a system perturbed by a static external force:

$$\hat{\mathscr{H}}' = \hat{\mathscr{H}} - \hat{A}\delta x . \tag{S4}$$

The change of the expectation value of an operator $\hat{B}$ can be calculated to the linear order [1]:

$$\delta \left\langle \hat{B} \right\rangle = \left\langle \hat{B} \right\rangle_{\delta x} - \left\langle \hat{B} \right\rangle_0 \equiv \chi_{BA} \delta x , \tag{S5}$$

where:

$$\chi_{BA} \equiv \beta_0 \left\langle \Delta\hat{B}; \Delta\hat{A} \right\rangle_0 . \tag{S6}$$

II. DETAILS OF DERIVATION FOR EQS. (7-10), MAGNETIZATION FORMULA

The derivation of Eqs. (7-10) is detailed in the following:

(1) We introduce $\boldsymbol{\chi}_{ij}(\boldsymbol{r}, \boldsymbol{r}') = \beta_0 \left\langle \Delta\hat{n}_j(\boldsymbol{r}') ; \Delta\hat{\boldsymbol{J}}_i(\boldsymbol{r}) \right\rangle_0$ with $i, j = 1, 2$, where $\hat{n}_1(\boldsymbol{r}) \equiv \hat{n}(\boldsymbol{r})$, $\hat{n}_2(\boldsymbol{r}) \equiv \hat{K}(\boldsymbol{r})$, $\hat{\boldsymbol{J}}_1(\boldsymbol{r}) \equiv \hat{\boldsymbol{J}}_N(\boldsymbol{r})$, $\hat{\boldsymbol{J}}_2(\boldsymbol{r}) \equiv \hat{\boldsymbol{J}}_Q(\boldsymbol{r})$, $\Delta\hat{a} \equiv \hat{a} - \langle \hat{a} \rangle_0$ and $\langle \cdots \rangle_0 \equiv \mathrm{Tr}[\hat{\rho}_0 \cdots]$, so we have:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{\chi}_{ij}(\boldsymbol{r}, \boldsymbol{r}') =& \beta_0 \left\langle \Delta\hat{n}_j(\boldsymbol{r}') ; \boldsymbol{\nabla} \cdot \Delta\hat{\boldsymbol{J}}_i(\boldsymbol{r}) \right\rangle_0 , & \text{(S7)} \\ =& -\beta_0 \left\langle \hat{n}_j(\boldsymbol{r}') ; \dot{\hat{n}}_i(\boldsymbol{r}) \right\rangle_0 . & \text{(S8)} \end{aligned}$$

From Eq. (S7) to Eq. (S8), we have used $\boldsymbol{\nabla} \cdot \hat{\boldsymbol{J}}_i(\boldsymbol{r}) = -\dot{\hat{n}}_i$ and $\boldsymbol{\nabla} \cdot \boldsymbol{J}_i^{\mathrm{eq}} = 0$. Using Eq. (S3), we obtain:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\chi}_{ij}(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{\mathrm{i}\hbar} \left\langle \left[\hat{n}_j(\boldsymbol{r}') , \hat{n}_i(\boldsymbol{r}) \right] \right\rangle_0 . \tag{S9}$$

We define $\boldsymbol{\chi}^{\boldsymbol{q}}_{ij}\left(\boldsymbol{r}\right) \equiv \int d\boldsymbol{r}' \boldsymbol{\chi}_{ij}\left(\boldsymbol{r},\boldsymbol{r}'\right) e^{-\mathrm{i}\boldsymbol{q}\cdot\left(\boldsymbol{r}-\boldsymbol{r}'\right)}$, and have:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\chi}^{\boldsymbol{q}}_{ij}\left(\boldsymbol{r}\right)+\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{\chi}^{\boldsymbol{q}}_{ij}\left(\boldsymbol{r}\right)=\frac{1}{\mathrm{i}\hbar}\int d\boldsymbol{r}'\left\langle\left[\hat{n}_j\left(\boldsymbol{r}'\right),\,\hat{n}_i\left(\boldsymbol{r}\right)\right]\right\rangle_0 e^{-\mathrm{i}\boldsymbol{q}\cdot\left(\boldsymbol{r}-\boldsymbol{r}'\right)}\,, \tag{S10}$$

where $i,j=1,2$.

(2) We can obtain the right hand side of Eq. (S10) from the definitions of currents and their scaling laws. Basically, we have:

$$\frac{1}{\mathrm{i}\hbar}[\hat{n}(\boldsymbol{r}),\,\hat{H}^{\phi,\psi}]=-\,\boldsymbol{\nabla}\cdot\hat{\boldsymbol{J}}^{\phi,\psi}_N(\boldsymbol{r})\,, \tag{S11}$$

$$\frac{1}{\mathrm{i}\hbar}[\hat{h}^{\phi,\psi}(\boldsymbol{r}),\,\hat{H}^{\phi,\psi}]=-\,\boldsymbol{\nabla}\cdot\hat{\boldsymbol{J}}^{\phi,\psi}_E(\boldsymbol{r})\,, \tag{S12}$$

and:

$$\hat{\boldsymbol{J}}^{\phi,\psi}_N(\boldsymbol{r})=\left[1+\psi(\boldsymbol{r})\right]\hat{\boldsymbol{J}}_N(\boldsymbol{r})\,, \tag{S13}$$

$$\hat{\boldsymbol{J}}^{\phi,\psi}_E(\boldsymbol{r})=\left[1+\psi(\boldsymbol{r})\right]^2\left[\hat{\boldsymbol{J}}_E(\boldsymbol{r})+\phi\left(\boldsymbol{r}\right)\hat{\boldsymbol{J}}_N(\boldsymbol{r})\right]\,. \tag{S14}$$

If we set $\phi\left(\boldsymbol{r}\right)=0$ and $1+\psi\left(\boldsymbol{r}\right)=e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}$, Eq. (S11) becomes:

$$\frac{1}{\mathrm{i}\hbar}\int d\boldsymbol{r}'\left[e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}'}\hat{h}\left(\boldsymbol{r}'\right),\,\hat{n}\left(\boldsymbol{r}\right)\right]=\boldsymbol{\nabla}\cdot\left(e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\hat{\boldsymbol{J}}_N\left(\boldsymbol{r}\right)\right)\,, \tag{S15}$$

$$=e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\mathrm{i}\boldsymbol{q}\cdot\hat{\boldsymbol{J}}_N\left(\boldsymbol{r}\right)+e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\boldsymbol{\nabla}\cdot\hat{\boldsymbol{J}}_N\left(\boldsymbol{r}\right)\,, \tag{S16}$$

and we obtain:

$$\frac{1}{\mathrm{i}\hbar}\int d\boldsymbol{r}'\left\langle\left[e^{\mathrm{i}\boldsymbol{q}\cdot\left(\boldsymbol{r}'-\boldsymbol{r}\right)}\hat{h}\left(\boldsymbol{r}'\right),\,\hat{n}\left(\boldsymbol{r}\right)\right]\right\rangle_0=\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{\nabla}\times\boldsymbol{M}_N\left(\boldsymbol{r}\right)\,, \tag{S17}$$

where we have used $\boldsymbol{\nabla}\cdot\hat{\boldsymbol{J}}^{\mathrm{eq}}_N\left(\boldsymbol{r}\right)=0$ and $\hat{\boldsymbol{J}}^{\mathrm{eq}}_N\left(\boldsymbol{r}\right)=\boldsymbol{\nabla}\times\boldsymbol{M}_N\left(\boldsymbol{r}\right)$. This is exactly the right hand side of Eq. (S10) for $i=1$, $j=2$.

Using the similar approach, we can prove that:

$$\frac{1}{\mathrm{i}\hbar}\int d\boldsymbol{r}'\left\langle\left[\hat{n}_j\left(\boldsymbol{r}'\right),\,\hat{n}_i\left(\boldsymbol{r}\right)\right]\right\rangle_0 e^{-\mathrm{i}\boldsymbol{q}\cdot\left(\boldsymbol{r}-\boldsymbol{r}'\right)}=\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{\nabla}\times\boldsymbol{M}_{ij}\left(\boldsymbol{r}\right)\,, \tag{S18}$$

where $\boldsymbol{M}_{11}(\boldsymbol{r})=0$, $\boldsymbol{M}_{12}(\boldsymbol{r})=\boldsymbol{M}_N(\boldsymbol{r})$, $\boldsymbol{M}_{21}(\boldsymbol{r})=\boldsymbol{M}_N(\boldsymbol{r})$, and $\boldsymbol{M}_{22}(\boldsymbol{r})=2\boldsymbol{M}_Q(\boldsymbol{r})$.

(3) Therefore, $\chi^{\boldsymbol{q}}_{ij}(\boldsymbol{r})$ satisfies the equation:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\chi}^{\boldsymbol{q}}_{ij}\left(\boldsymbol{r}\right)+\mathrm{i}\boldsymbol{q}\cdot\left[\boldsymbol{\chi}^{\boldsymbol{q}}_{ij}\left(\boldsymbol{r}\right)-\boldsymbol{\nabla}\times\boldsymbol{M}_{ij}\left(\boldsymbol{r}\right)\right]=0\,, \tag{S19}$$

and it has the general solution:

$$\boldsymbol{\chi}^{\boldsymbol{q}}_{ij}\left(\boldsymbol{r}\right)=-\mathrm{i}\boldsymbol{q}\times\boldsymbol{M}_{ij}\left(\boldsymbol{r}\right)+e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\boldsymbol{\nabla}\times\boldsymbol{\kappa}^{\boldsymbol{q}}_{ij}\left(\boldsymbol{r}\right)\,, \tag{S20}$$

where $\boldsymbol{\kappa}^{\boldsymbol{q}}_{ij}(\boldsymbol{r})$ is an arbitrary function. This equation can be considered as a decomposition of $\chi^{\boldsymbol{q}}_{ij}(\boldsymbol{r})$. It is important to note that the decomposition is not necessary to be unique, because the magnetization can only be defined up to a gradient. However, the arbitrariness does not affect our result on the total magnetizations, as long as both $\boldsymbol{M}_{ij}(\boldsymbol{r})$ and $\boldsymbol{\kappa}^{\boldsymbol{q}}_{ij}\left(\boldsymbol{r}\right)$ are well behaved functions: i.e., they are bounded for all $\boldsymbol{r}$. The constraint is necessary because, first, magnetizations are properties of materials; second, only when these functions are well behaved, can their contributions presented in Eq. (18-19) be well defined.

(4) We can relate $\boldsymbol{\kappa}^{\boldsymbol{q}=0}_{ij}\left(\boldsymbol{r}\right)$ to the macroscopic thermodynamic quantities. To see this, we use Eq. (17) and see how the equilibrium currents are perturbed by the spatially uniform changes of the chemical potential and the temperature. We have:

$$\delta\boldsymbol{J}^{\mathrm{eq}}_i(\boldsymbol{r})\approx\int \mathrm{d}\boldsymbol{r}'\left[\boldsymbol{\chi}_{i1}(\boldsymbol{r},\boldsymbol{r}')\delta\mu_0-\boldsymbol{\chi}_{i2}(\boldsymbol{r},\boldsymbol{r}')T_0\delta(1/T_0)\right]\,, \tag{S21}$$

$$=\boldsymbol{\chi}^{\boldsymbol{q}=0}_{i1}(\boldsymbol{r})\delta\mu_0-\boldsymbol{\chi}^{\boldsymbol{q}=0}_{i2}(\boldsymbol{r})T_0\delta(1/T_0)\,, \tag{S22}$$

$$=\boldsymbol{\nabla}\times\left[\boldsymbol{\kappa}^{\boldsymbol{q}=0}_{i1}(\boldsymbol{r})\delta\mu_0-\boldsymbol{\kappa}^{\boldsymbol{q}=0}_{i2}(\boldsymbol{r})T_0\delta(1/T_0)\right]\,. \tag{S23}$$

Note that $\delta \boldsymbol{J}_2^{\rm eq}(\boldsymbol{r}) \equiv {\rm Tr}\left[\boldsymbol{J}_E^{\phi,\psi}(\boldsymbol{r})\delta\hat{\rho}_{\rm leq}\right] - \alpha(\boldsymbol{r})\left[\boldsymbol{J}_N^{\phi,\psi}(\boldsymbol{r})\delta\hat{\rho}_{\rm leq}\right] \approx \delta\boldsymbol{J}_E^{\rm eq}(\boldsymbol{r}) - \mu_0\delta\boldsymbol{J}_N^{\rm eq}(\boldsymbol{r})$. On the other hand, $\delta\boldsymbol{J}_i^{\rm eq}$ is, by definition:

$$\delta\boldsymbol{J}_1^{\rm eq} = \boldsymbol{\nabla}\times\left(\delta\boldsymbol{M}_N\right) , \tag{S24}$$
$$\delta\boldsymbol{J}_2^{\rm eq} = \boldsymbol{\nabla}\times\left(\delta\boldsymbol{M}_E\right) - \mu_0\boldsymbol{\nabla}\times\left(\delta\boldsymbol{M}_N\right) . \tag{S25}$$

Comparing the two sides, we obtain:

$$\boldsymbol{\kappa}_{11}^{\boldsymbol{q}=0}\left(\boldsymbol{r}\right) = \left.\frac{\partial\boldsymbol{M}_N\left(\boldsymbol{r}\right)}{\partial\mu_0}\right|_{T_0} , \tag{S26}$$
$$\boldsymbol{\kappa}_{12}^{\boldsymbol{q}=0}\left(\boldsymbol{r}\right) = T_0\left.\frac{\partial\boldsymbol{M}_N\left(\boldsymbol{r}\right)}{\partial T_0}\right|_{\mu_0} , \tag{S27}$$
$$\boldsymbol{\kappa}_{21}^{\boldsymbol{q}=0}\left(\boldsymbol{r}\right) = \left.\frac{\partial\boldsymbol{M}_Q\left(\boldsymbol{r}\right)}{\partial\mu_0}\right|_{T_0} + \boldsymbol{M}_N\left(\boldsymbol{r}\right) , \tag{S28}$$
$$\boldsymbol{\kappa}_{22}^{\boldsymbol{q}=0}\left(\boldsymbol{r}\right) = T_0\left.\frac{\partial\boldsymbol{M}_Q\left(\boldsymbol{r}\right)}{\partial T_0}\right|_{\mu_0} . \tag{S29}$$

(5) Equation (S20) can be rewritten as:

$$\boldsymbol{\chi}_{ij}^{\boldsymbol{q}}\left(\boldsymbol{r}\right) = -{\rm i}\boldsymbol{q}\times\left[\boldsymbol{M}_{ij}\left(\boldsymbol{r}\right) - e^{-{\rm i}\boldsymbol{q}\cdot\boldsymbol{r}}\boldsymbol{\kappa}_{ij}^{\boldsymbol{q}}\left(\boldsymbol{r}\right)\right] + \boldsymbol{\nabla}\times\left[e^{-{\rm i}\boldsymbol{q}\cdot\boldsymbol{r}}\boldsymbol{\kappa}_{ij}^{\boldsymbol{q}}\left(\boldsymbol{r}\right)\right] , \tag{S30}$$

Applying $\boldsymbol{\nabla}_{\boldsymbol{q}}\times$ to the both sides of Eq. (S30) and setting $\boldsymbol{q}\to 0$, we obtain:

$$\frac{\rm i}{2}\left.\boldsymbol{\nabla}_{\boldsymbol{q}}\times\boldsymbol{\chi}_{ij}^{\boldsymbol{q}}\left(\boldsymbol{r}\right)\right|_{\boldsymbol{q}\to 0} = -\boldsymbol{M}_{ij}\left(\boldsymbol{r}\right) + \boldsymbol{\kappa}_{ij}^{\boldsymbol{q}=0}\left(\boldsymbol{r}\right) - \boldsymbol{\nabla}\times\boldsymbol{U}_{ij}\left(\boldsymbol{r}\right) , \tag{S31}$$

where $\boldsymbol{U}_{ij}\left(\boldsymbol{r}\right) = \frac{\rm i}{2}\boldsymbol{\nabla}_{\boldsymbol{q}}\times\left.\left(e^{-{\rm i}\boldsymbol{q}\cdot\boldsymbol{r}}\boldsymbol{\kappa}_{ij}^{\boldsymbol{q}}\left(\boldsymbol{r}\right)\right)\right|_{\boldsymbol{q}\to 0}$. After substituting different components of $\boldsymbol{M}_{ij}\left(\boldsymbol{r}\right)$ and $\boldsymbol{\kappa}_{ij}^{\boldsymbol{q}=0}\left(\boldsymbol{r}\right)$ and integrating over $\boldsymbol{r}$ we come to the formulae for the total magnetizations:

$$-\frac{\partial\boldsymbol{M}_N}{\partial\mu_0} = \frac{\beta_0}{2{\rm i}}\left.\boldsymbol{\nabla}_{\boldsymbol{q}}\times\left\langle\hat{n}_{-\boldsymbol{q}};\hat{\boldsymbol{J}}_{N,\boldsymbol{q}}\right\rangle_0\right|_{\boldsymbol{q}\to 0} , \tag{S32}$$
$$\boldsymbol{M}_N - T_0\frac{\partial\boldsymbol{M}_N}{\partial T_0} = \frac{\beta_0}{2{\rm i}}\left.\boldsymbol{\nabla}_{\boldsymbol{q}}\times\left\langle\hat{K}_{-\boldsymbol{q}};\hat{\boldsymbol{J}}_{N,\boldsymbol{q}}\right\rangle_0\right|_{\boldsymbol{q}\to 0} , \tag{S33}$$
$$-\frac{\partial\boldsymbol{M}_Q}{\partial\mu_0} = \frac{\beta_0}{2{\rm i}}\left.\boldsymbol{\nabla}_{\boldsymbol{q}}\times\left\langle\hat{n}_{-\boldsymbol{q}};\hat{\boldsymbol{J}}_{Q,\boldsymbol{q}}\right\rangle_0\right|_{\boldsymbol{q}\to 0} , \tag{S34}$$
$$2\boldsymbol{M}_Q - T_0\frac{\partial\boldsymbol{M}_Q}{\partial T_0} = \frac{\beta_0}{2{\rm i}}\left.\boldsymbol{\nabla}_{\boldsymbol{q}}\times\left\langle\hat{K}_{-\boldsymbol{q}};\hat{\boldsymbol{J}}_{Q,\boldsymbol{q}}\right\rangle_0\right|_{\boldsymbol{q}\to 0} . \tag{S35}$$

In the derivation, we assume that $\int {\rm d}\boldsymbol{r}\boldsymbol{\nabla}\times\boldsymbol{U}_{ij}(\boldsymbol{r}) = 0$. This is guaranteed because $\boldsymbol{\kappa}_{ij}^{\boldsymbol{q}}$ is a well behaved function.

III. DETAILS OF DERIVATION FOR EQS. (18-19), LOCAL EQUILIBRIUM CURRENTS

Inserting Eq. (13) into Eq. (17), we obtain:

$$\boldsymbol{J}_i^{\rm leq}(\boldsymbol{r}) \approx \boldsymbol{J}_i^{\rm eq}(\boldsymbol{r}) + \sum_{j=1}^{2}\left(\boldsymbol{M}_{ij}\left(\boldsymbol{r}\right)\times\boldsymbol{\nabla}x_j\left(\boldsymbol{r}\right) + \int\frac{d\boldsymbol{q}}{\left(2\pi\right)^3}\boldsymbol{\nabla}\times\boldsymbol{\kappa}_{ij}^{\boldsymbol{q}}\left(\boldsymbol{r}\right)x_{j\boldsymbol{q}}\right) . \tag{S36}$$

Because $x_{j\boldsymbol{q}} = \int d\boldsymbol{r}' x_j\left(\boldsymbol{r}'\right)e^{-{\rm i}\boldsymbol{q}\cdot\boldsymbol{r}'}$, we have:

$$\boldsymbol{J}_i^{\rm leq}(\boldsymbol{r}) \approx \boldsymbol{J}_i^{\rm eq}(\boldsymbol{r}) + \sum_{j=1}^{2}\left(\boldsymbol{M}_{ij}\left(\boldsymbol{r}\right)\times\boldsymbol{\nabla}x_j\left(\boldsymbol{r}\right) + \boldsymbol{\nabla}\times\int d\boldsymbol{r}'\boldsymbol{\kappa}_{ij}\left(\boldsymbol{r},\boldsymbol{r}'\right)x_j\left(\boldsymbol{r}'\right)\right) , \tag{S37}$$

where $\boldsymbol{\kappa}_{ij}\left(\boldsymbol{r},\boldsymbol{r}'\right) = \int d\boldsymbol{q}/\left(2\pi\right)^3\boldsymbol{\kappa}_{ij}^{\boldsymbol{q}}\left(\boldsymbol{r}\right)e^{-{\rm i}\boldsymbol{q}\cdot\boldsymbol{r}'}$.

We can obtain $\boldsymbol{J}_i^{\mathrm{eq}}(\boldsymbol{r})$ through the scaling law. Without ψ and ϕ, we have:

$$\boldsymbol{J}_{N(E)}^{\mathrm{eq}}(\boldsymbol{r}) = \boldsymbol{\nabla} \times \boldsymbol{M}_{N(E)}(\boldsymbol{r}) \,. \tag{S38}$$

When $\psi(\boldsymbol{r})$ and $\phi(\boldsymbol{r})$ are present, according to the scaling law in Eq. (S13) and (S14) we have:

$$\boldsymbol{J}_1^{\mathrm{eq}}(\boldsymbol{r}) = [1+\psi(\boldsymbol{r})]\, \boldsymbol{\nabla} \times \boldsymbol{M}_N(\boldsymbol{r}) \,, \tag{S39}$$

$$\boldsymbol{J}_2^{\mathrm{eq}}(\boldsymbol{r}) = [1+\psi(\boldsymbol{r})]^2 \left[\boldsymbol{\nabla} \times \boldsymbol{M}_E(\boldsymbol{r}) - \mu(\boldsymbol{r}) \boldsymbol{\nabla} \times \boldsymbol{M}_N(\boldsymbol{r})\right] . \tag{S40}$$

For $i=1$, inserting Eq. (S39) into Eq. (S37),

$$\boldsymbol{J}_1^{\mathrm{leq}}(\boldsymbol{r}) \approx [1+\psi(\boldsymbol{r})]\, \boldsymbol{\nabla} \times \boldsymbol{M}_N(\boldsymbol{r}) - \boldsymbol{M}_N(\boldsymbol{r}) \times T_0 \boldsymbol{\nabla} \frac{1}{T} + \boldsymbol{\nabla} \times \int d\boldsymbol{r}' \sum_{j=1}^{2} \boldsymbol{\kappa}_{1j}(\boldsymbol{r},\boldsymbol{r}')\, x_j(\boldsymbol{r}') \,, \tag{S41}$$

$$= \boldsymbol{\nabla} \times \left([1+\psi(\boldsymbol{r})]\, \boldsymbol{M}_N(\boldsymbol{r})\right) - \frac{1}{\beta} \boldsymbol{M}_N(\boldsymbol{r}) \times \boldsymbol{X}_2 + \boldsymbol{\nabla} \times \int d\boldsymbol{r}' \sum_{j=1}^{2} \boldsymbol{\kappa}_{1j}(\boldsymbol{r},\boldsymbol{r}')\, x_j(\boldsymbol{r}') \,, \tag{S42}$$

so we can write:

$$\boldsymbol{J}_1^{\mathrm{leq}}(\boldsymbol{r}) \approx \boldsymbol{\nabla} \times \boldsymbol{M}_N^{\phi,\psi}(\boldsymbol{r}) - \frac{1}{\beta} \boldsymbol{M}_N(\boldsymbol{r}) \times \boldsymbol{X}_2 \,, \tag{S43}$$

where $\boldsymbol{M}_N^{\phi,\psi}(\boldsymbol{r}) \equiv [1+\psi(\boldsymbol{r})]\, \boldsymbol{M}_N(\boldsymbol{r},T_0,\mu_0) + \delta\boldsymbol{M}_N(\boldsymbol{r})$ and $\delta\boldsymbol{M}_N(\boldsymbol{r}) \equiv \sum_{j=1}^{2} \int d\boldsymbol{r}' \boldsymbol{\kappa}_{1j}(\boldsymbol{r},\boldsymbol{r}')\, x_j(\boldsymbol{r}')$.

Similarly, for $i=2$, inserting Eq. (S40) into Eq. (S37),

$$\boldsymbol{J}_2^{\mathrm{leq}}(\boldsymbol{r}) \approx [1+\psi(\boldsymbol{r})]^2 \left[\boldsymbol{\nabla} \times \boldsymbol{M}_E(\boldsymbol{r}) - \mu(\boldsymbol{r}) \boldsymbol{\nabla} \times \boldsymbol{M}_N(\boldsymbol{r})\right] + \boldsymbol{M}_N(\boldsymbol{r}) \times \boldsymbol{\nabla}\mu - 2\boldsymbol{M}_Q \times T_0 \boldsymbol{\nabla} \frac{1}{T} \tag{S44}$$

$$+ \boldsymbol{\nabla} \times \int d\boldsymbol{r}' \sum_{j=1}^{2} \boldsymbol{\kappa}_{2j}(\boldsymbol{r},\boldsymbol{r}')\, x_j(\boldsymbol{r}') \,, \tag{S45}$$

$$= \boldsymbol{\nabla} \times \left[(1+\psi(\boldsymbol{r}))^2 \left(\boldsymbol{M}_E(\boldsymbol{r}) + \phi(\boldsymbol{r})\, \boldsymbol{M}_N(\boldsymbol{r})\right)\right] - \alpha(\boldsymbol{r}) \boldsymbol{\nabla} \times \left((1+\psi(\boldsymbol{r}))\, \boldsymbol{M}_N(\boldsymbol{r})\right) \tag{S46}$$

$$- \frac{1}{\beta} \boldsymbol{M}_N(\boldsymbol{r}) \times \boldsymbol{X}_1 - \frac{2}{\beta} \boldsymbol{M}_Q(\boldsymbol{r}) \times \boldsymbol{X}_2 + \boldsymbol{\nabla} \times \sum_{j=1}^{2} \int d\boldsymbol{r}' \boldsymbol{\kappa}_{2j}(\boldsymbol{r},\boldsymbol{r}')\, x_j(\boldsymbol{r}') \,. \tag{S47}$$

Further, by substituting $\boldsymbol{M}_N^{\phi,\psi}(\boldsymbol{r})$ into Eq. (S46) we can write $\boldsymbol{J}_2^{\mathrm{leq}}(\boldsymbol{r})$ as:

$$\boldsymbol{J}_2^{\mathrm{leq}}(\boldsymbol{r}) \approx \boldsymbol{\nabla} \times \boldsymbol{M}_E^{\phi,\psi}(\boldsymbol{r}) - \alpha(\boldsymbol{r}) \boldsymbol{\nabla} \times \boldsymbol{M}_N^{\phi,\psi}(\boldsymbol{r}) - \frac{1}{\beta} \boldsymbol{M}_N(\boldsymbol{r}) \times \boldsymbol{X}_1 - \frac{2}{\beta} \boldsymbol{M}_Q(\boldsymbol{r}) \times \boldsymbol{X}_2 \,, \tag{S48}$$

where $\boldsymbol{M}_E^{\phi,\psi}(\boldsymbol{r}) \equiv (1+\psi(\boldsymbol{r}))^2 \left(\boldsymbol{M}_E(\boldsymbol{r},T_0,\mu_0) + \phi(\boldsymbol{r})\, \boldsymbol{M}_N(\boldsymbol{r},T_0,\mu_0)\right) + \delta\boldsymbol{M}_E(\boldsymbol{r})$, $\delta\boldsymbol{M}_E(\boldsymbol{r}) \equiv \sum_{j=1}^{2} \int d\boldsymbol{r}' \boldsymbol{\kappa}'_{2j}(\boldsymbol{r},\boldsymbol{r}')\, x_j(\boldsymbol{r}')$, and $\boldsymbol{\kappa}'_{2j} \equiv \boldsymbol{\kappa}_{2j} + \mu_0 \boldsymbol{\kappa}_{1j}$.

IV. DETAILS OF DERIVATION FOR EQ. (22), DEFINITION OF ENERGY CURRENT

The energy density can be written as:

$$\hat{h}^{\phi,\psi}(\boldsymbol{r}) = [1+\psi(\boldsymbol{r})] \left\{ \frac{m}{2} \left[\hat{\boldsymbol{v}}\hat{\varphi}(\boldsymbol{r})\right]^\dagger \cdot \left[\hat{\boldsymbol{v}}\hat{\varphi}(\boldsymbol{r})\right] + \hat{\varphi}^\dagger(\boldsymbol{r}) \left[V(\boldsymbol{r}) + \phi(\boldsymbol{r})\right] \hat{\varphi}(\boldsymbol{r}) \right\} . \tag{S49}$$

The Schrödinger equation for the system is $\mathrm{i}\hbar \frac{\partial \hat{\varphi}}{\partial t} = \hat{\mathcal{H}}^{\phi,\psi} \hat{\varphi}$, where $\hat{\mathcal{H}}^{\phi,\psi} \equiv \frac{m}{2} \hat{\boldsymbol{v}} \cdot [1+\psi(\boldsymbol{r})]\, \hat{\boldsymbol{v}} + [1+\psi(\boldsymbol{r})] \left[V(\boldsymbol{r}) + \phi(\boldsymbol{r})\right]$. Therefore, we have:

$$\frac{\partial \hat{h}^{\phi,\psi}(\boldsymbol{r})}{\partial t} = \frac{1}{\mathrm{i}\hbar} [1+\psi(\boldsymbol{r})] \left\{ \frac{m}{2} \left[\hat{\boldsymbol{v}}\hat{\varphi}(\boldsymbol{r})\right]^\dagger \cdot \left[\hat{\boldsymbol{v}}\hat{\mathcal{H}}^{\phi,\psi}\hat{\varphi}(\boldsymbol{r})\right] - \frac{m}{2} \left[\hat{\boldsymbol{v}}\hat{\mathcal{H}}^{\phi,\psi}\hat{\varphi}(\boldsymbol{r})\right]^\dagger \cdot \left[\hat{\boldsymbol{v}}\hat{\varphi}(\boldsymbol{r})\right] \right. \tag{S50}$$

$$\left. + \hat{\varphi}^\dagger(\boldsymbol{r}) \left[V(\boldsymbol{r}) + \phi(\boldsymbol{r})\right] \left[\hat{\mathcal{H}}^{\phi,\psi}\hat{\varphi}(\boldsymbol{r})\right] - \left[\hat{\mathcal{H}}^{\phi,\psi}\hat{\varphi}(\boldsymbol{r})\right]^\dagger \left[V(\boldsymbol{r}) + \phi(\boldsymbol{r})\right] \hat{\varphi}(\boldsymbol{r}) \right\} , \tag{S51}$$

$$= -\boldsymbol{\nabla} \cdot \left\{ \frac{1}{2} [1+\psi(\boldsymbol{r})] \left(\left[\hat{\boldsymbol{v}}\hat{\varphi}(\boldsymbol{r})\right]^\dagger \left[\hat{\mathcal{H}}^{\phi,\psi}\hat{\varphi}(\boldsymbol{r})\right] + \left[\hat{\mathcal{H}}^{\phi,\psi}\hat{\varphi}(\boldsymbol{r})\right]^\dagger \left[\hat{\boldsymbol{v}}\hat{\varphi}(\boldsymbol{r})\right] \right) \right\} , \tag{S52}$$

so we can identify $\hat{\boldsymbol{J}}_E^{\phi,\psi}(\boldsymbol{r})$ as:

$$\hat{\boldsymbol{J}}_E^{\phi,\psi}(\boldsymbol{r}) = \frac{1}{2}\left[1+\psi(\boldsymbol{r})\right]\left\{\left[\hat{\boldsymbol{v}}\hat{\varphi}(\boldsymbol{r})\right]^{\dagger}\left[\hat{\mathcal{H}}^{\phi,\psi}\hat{\varphi}(\boldsymbol{r})\right] + \left[\hat{\mathcal{H}}^{\phi,\psi}\hat{\varphi}(\boldsymbol{r})\right]^{\dagger}\left[\hat{\boldsymbol{v}}\hat{\varphi}(\boldsymbol{r})\right]\right\}. \tag{S53}$$

Because:

$$\hat{\mathcal{H}}^{\phi,\psi} = \left[1+\psi(\boldsymbol{r})\right]\left[\hat{\mathcal{H}}_0 + \phi(\boldsymbol{r})\right] - \frac{\mathrm{i}\hbar}{2}\left[\boldsymbol{\nabla}\psi(\boldsymbol{r})\right]\cdot\hat{\boldsymbol{v}}\,, \tag{S54}$$

where $\hat{\mathcal{H}}_0 \equiv \hat{\mathcal{H}}^{\phi=0,\psi=0}$, we obtain the following scaling equation:

$$\hat{\boldsymbol{J}}_E^{\phi,\psi}(\boldsymbol{r}) = \left[1+\psi(\boldsymbol{r})\right]^2\left[\hat{\boldsymbol{J}}_E(\boldsymbol{r}) + \phi(\boldsymbol{r})\,\hat{\boldsymbol{J}}_N(\boldsymbol{r})\right] + \boldsymbol{\nabla}\left(1+\psi(\boldsymbol{r})\right)^2 \times \hat{\boldsymbol{\Lambda}}(\boldsymbol{r})\,, \tag{S55}$$

where $\hat{\boldsymbol{\Lambda}}(\boldsymbol{r}) = \frac{\hbar}{8\mathrm{i}}\left(\hat{\boldsymbol{v}}\hat{\varphi}\right)^{\dagger}\times\left(\hat{\boldsymbol{v}}\hat{\varphi}\right)$.

In order to satisfy the scaling law Eq. (5), we redefine the energy current operator as:

$$\hat{\boldsymbol{J}}_E^{\phi,\psi}(\boldsymbol{r}) \;\to\; \hat{\boldsymbol{J}}_E^{\phi,\psi}(\boldsymbol{r}) - \boldsymbol{\nabla}\times\left(\left(1+\psi(\boldsymbol{r})\right)^2\hat{\boldsymbol{\Lambda}}(\boldsymbol{r})\right), \tag{S56}$$

$$\hat{\boldsymbol{J}}_E(\boldsymbol{r}) \;\to\; \hat{\boldsymbol{J}}_E(\boldsymbol{r}) - \boldsymbol{\nabla}\times\hat{\boldsymbol{\Lambda}}(\boldsymbol{r})\,. \tag{S57}$$

This is exactly the energy current definition Eq. (22). It is straightforward to show that modified energy current operator satisfies the scaling law Eq. (5).

The particle current operator is defined as usual. It automatically satisfies the corresponding scaling law Eq. (4).

V. DETAILS OF DERIVATION FOR EQ. (23), KUBO CONTRIBUTION

The thermal current operator $\hat{J}_{Qx}(\boldsymbol{r})$ is:

$$\hat{J}_{Qx}(\boldsymbol{r}) = \frac{\left(\hat{K}\hat{\varphi}(\boldsymbol{r})\right)^{\dagger}\hat{v}_x\hat{\varphi}(\boldsymbol{r}) + \left(\hat{v}_x\hat{\varphi}(\boldsymbol{r})\right)^{\dagger}\hat{K}\hat{\varphi}(\boldsymbol{r})}{2} - \frac{\hbar}{8\mathrm{i}}\sum_{\gamma}\nabla_{\gamma}\left(\left(\hat{v}_x\hat{\varphi}(\boldsymbol{r})\right)^{\dagger}\hat{v}_{\gamma}\hat{\varphi}(\boldsymbol{r}) - \left(\hat{v}_{\gamma}\hat{\varphi}(\boldsymbol{r})\right)^{\dagger}\hat{v}_x\hat{\varphi}(\boldsymbol{r})\right), \tag{S58}$$

where $\gamma = x, y, z$ and we have set $\phi(\boldsymbol{r}) = 0$ and $\psi(\boldsymbol{r}) = 0$. According to our definition, we have [2] $\kappa_{xy}^{\mathrm{Kubo}} \equiv \frac{L_{xy}^{(22)}}{k_B T_0^2}$ and:

$$L_{xy}^{(22)} = \frac{1}{V}\int_0^{\infty} dt e^{-st}\left\langle \hat{J}_{Qy};\, \hat{J}_{Qx}(t)\right\rangle_0\,, \tag{S59}$$

$$= -\frac{\hbar}{\beta_0 V}\sum_{n\boldsymbol{k},n'\boldsymbol{k}'}\frac{f_{n\boldsymbol{k}} - f_{n'\boldsymbol{k}'}}{\mathrm{i}\left(\epsilon_{n\boldsymbol{k}} - \epsilon_{n'\boldsymbol{k}'}\right)^2}\left\langle\psi_{n\boldsymbol{k}}\right|\hat{J}_{Qy}\left|\psi_{n'\boldsymbol{k}'}\right\rangle\left\langle\psi_{n'\boldsymbol{k}'}\right|\hat{J}_{Qx}\left|\psi_{n\boldsymbol{k}}\right\rangle\,, \tag{S60}$$

where $\psi_{n\boldsymbol{k}}$ is the Bloch wave function for band n and quasi-momentum $\boldsymbol{k}$, $f_{n\boldsymbol{k}} \equiv f(\epsilon_{n\boldsymbol{k}})$ is the Fermi distribution function, and $\epsilon_{n\boldsymbol{k}}$ is the electron dispersion. According to our definition Eq. (S58) for $\hat{J}_{Qx}$, we have:

$$\left\langle\psi_{n'\boldsymbol{k}'}\right|\hat{J}_{Qx}\left|\psi_{n\boldsymbol{k}}\right\rangle = \frac{\left\langle\hat{K}\psi_{n'\boldsymbol{k}'}\right|\hat{v}_x\psi_{n\boldsymbol{k}}\rangle + \left\langle\hat{v}_x\psi_{n'\boldsymbol{k}'}\right|\hat{K}\psi_{n\boldsymbol{k}}\rangle}{2} - \frac{\hbar}{8\mathrm{i}}\sum_{\gamma}\left[\left\langle\nabla_{\gamma}\hat{v}_x\psi_{n'\boldsymbol{k}'}\right|\hat{v}_{\gamma}\psi_{n\boldsymbol{k}}\rangle + \left\langle\hat{v}_x\psi_{n'\boldsymbol{k}'}\right|\nabla_{\gamma}\hat{v}_{\gamma}\psi_{n\boldsymbol{k}}\rangle\right. \tag{S61}$$

$$\left. - \left\langle\nabla_{\gamma}\hat{v}_{\gamma}\psi_{n'\boldsymbol{k}'}\right|\hat{v}_x\psi_{n\boldsymbol{k}}\rangle - \left\langle\hat{v}_{\gamma}\psi_{n'\boldsymbol{k}'}\right|\nabla_{\gamma}\hat{v}_x\psi_{n\boldsymbol{k}}\rangle\right]. \tag{S62}$$

Note:

$$\left\langle\hat{K}\psi_{n'\boldsymbol{k}'}\right|\hat{v}_x\psi_{n\boldsymbol{k}}\rangle = \left(\epsilon_{n'\boldsymbol{k}} - \mu_0\right)\left\langle u_{n'\boldsymbol{k}}\right|\hat{v}_{\boldsymbol{k}x}\left|u_{n\boldsymbol{k}}\right\rangle\delta_{\boldsymbol{k}\boldsymbol{k}'}\,, \tag{S63}$$

with $\hat{v}_{\boldsymbol{k}x} = \partial\hat{\mathcal{H}}_{\boldsymbol{k}}/\partial\left(\hbar k_x\right)$, and:

$$\left\langle\nabla_{\gamma}\hat{v}_x\psi_{n'\boldsymbol{k}'}\right|\hat{v}_{\gamma}\psi_{n\boldsymbol{k}}\rangle + \left\langle\hat{v}_x\psi_{n'\boldsymbol{k}'}\right|\nabla_{\gamma}\hat{v}_{\gamma}\psi_{n\boldsymbol{k}}\rangle = -\left\langle\psi_{n'\boldsymbol{k}'}\right|\hat{v}_x\nabla_{\gamma}\hat{v}_{\gamma}\left|\psi_{n\boldsymbol{k}}\right\rangle + \left\langle\psi_{n'\boldsymbol{k}'}\right|\hat{v}_x\nabla_{\gamma}\hat{v}_{\gamma}\left|\psi_{n\boldsymbol{k}}\right\rangle\,, \tag{S64}$$

$$= 0\,, \tag{S65}$$

and similarly, $\langle \nabla_\gamma \hat{v}_\gamma \psi_{n'\boldsymbol{k}'} | \hat{v}_x \psi_{n\boldsymbol{k}} \rangle + \langle \hat{v}_\gamma \psi_{n'\boldsymbol{k}'} | \nabla_\gamma \hat{v}_x \psi_{n\boldsymbol{k}} \rangle = 0$, so we come to:

$$L_{xy}^{(22)} = -\frac{\hbar}{\beta_0 V} \sum_{n\neq n'\boldsymbol{k}} \frac{\left(f_{n\boldsymbol{k}} - f_{n'\boldsymbol{k}}\right)\left(\epsilon_{n\boldsymbol{k}} + \epsilon_{n'\boldsymbol{k}} - 2\mu_0\right)^2}{4\mathrm{i}\left(\epsilon_{n\boldsymbol{k}} - \epsilon_{n'\boldsymbol{k}}\right)^2} \langle u_{n\boldsymbol{k}} | \hat{v}_{\boldsymbol{k}y} | u_{n'\boldsymbol{k}} \rangle \langle \psi_{n'\boldsymbol{k}} | \hat{v}_{\boldsymbol{k}x} | \psi_{n\boldsymbol{k}} \rangle \ , \tag{S66}$$

$$= -\frac{\hbar}{2\beta_0 V} \sum_{n\neq n'\boldsymbol{k}} \frac{f_{n\boldsymbol{k}}\left(\epsilon_{n\boldsymbol{k}} + \epsilon_{n'\boldsymbol{k}} - 2\mu_0\right)^2}{\left(\epsilon_{n\boldsymbol{k}} - \epsilon_{n'\boldsymbol{k}}\right)^2} \mathrm{Im}\left[\langle u_{n\boldsymbol{k}} | \hat{v}_{\boldsymbol{k}y} | u_{n'\boldsymbol{k}} \rangle \langle u_{n'\boldsymbol{k}} | \hat{v}_{\boldsymbol{k}x} | u_{n\boldsymbol{k}} \rangle\right] . \tag{S67}$$

Using the identity:

$$\langle u_{n'\boldsymbol{k}} | \hat{v}_{\boldsymbol{k}x} | u_{n\boldsymbol{k}} \rangle = \frac{1}{\hbar}\left(\epsilon_{n\boldsymbol{k}} - \epsilon_{n'\boldsymbol{k}}\right) \left\langle u_{n'\boldsymbol{k}} \middle| \frac{\partial u_{n\boldsymbol{k}}}{\partial k_x} \right\rangle \ , \tag{S68}$$

we have:

$$L_{xy}^{(22)} = \frac{1}{2\beta_0 \hbar V} \sum_{n\boldsymbol{k}} \mathrm{Im}\left[\left\langle \frac{\partial u_{n\boldsymbol{k}}}{\partial k_x} \middle| \left(\hat{\mathcal{H}}_{\boldsymbol{k}} + \epsilon_{n\boldsymbol{k}} - 2\mu_0\right)^2 \middle| \frac{\partial u_{n\boldsymbol{k}}}{\partial k_y} \right\rangle\right] f_{n\boldsymbol{k}} . \tag{S69}$$

The formula can be rewritten as the alternative form. We introduce the new notations:

$$m_2(\epsilon) \equiv \frac{1}{\hbar} \mathrm{Im} \sum_{n\boldsymbol{k}} \left\langle \frac{\partial u_{n\boldsymbol{k}}}{\partial k_x} \middle| \left(\hat{\mathcal{H}}_{\boldsymbol{k}} - \epsilon\right)^2 \middle| \frac{\partial u_{n\boldsymbol{k}}}{\partial k_y} \right\rangle \delta\left(\epsilon - \epsilon_{n\boldsymbol{k}}\right) \ , \tag{S70}$$

$$m_1(\epsilon) \equiv \frac{1}{\hbar} \mathrm{Im} \sum_{n\boldsymbol{k}} \left\langle \frac{\partial u_{n\boldsymbol{k}}}{\partial k_x} \middle| \left(\hat{\mathcal{H}}_{\boldsymbol{k}} - \epsilon\right) \middle| \frac{\partial u_{n\boldsymbol{k}}}{\partial k_y} \right\rangle \delta\left(\epsilon - \epsilon_{n\boldsymbol{k}}\right) \ , \tag{S71}$$

$$\Omega_z(\epsilon) \equiv -\frac{2}{\hbar} \mathrm{Im} \sum_{n\boldsymbol{k}} \left\langle \frac{\partial u_{n\boldsymbol{k}}}{\partial k_x} \middle| \frac{\partial u_{n\boldsymbol{k}}}{\partial k_y} \right\rangle \delta\left(\epsilon - \epsilon_{n\boldsymbol{k}}\right) \ . \tag{S72}$$

Therefore, we can express $\kappa_{xy}^{\mathrm{Kubo}}$ as:

$$\kappa_{xy}^{\mathrm{Kubo}} = \frac{1}{2T_0 V} \int d\epsilon \left[m_2(\epsilon) + 4\left(\epsilon - \mu_0\right) m_1(\epsilon) - 2\left(\epsilon - \mu_0\right)^2 \Omega_z(\epsilon)\right] f(\epsilon) \ . \tag{S73}$$

It is easy to see $\kappa_{xy}^{\mathrm{Kubo}}$ is divergent in the low temperature limit.

VI. DETAILS OF DERIVATION FOR EQ. (24), ENERGY MAGNETIZATION

To calculate $M_{Q,z}$, we use $2\boldsymbol{M}_Q - T_0 \frac{\partial \boldsymbol{M}_Q}{\partial T_0} = \frac{\beta_0}{2\mathrm{i}} \boldsymbol{\nabla}_{\boldsymbol{q}} \times \left\langle \hat{K}_{-\boldsymbol{q}}; \hat{\boldsymbol{J}}_{Q,\boldsymbol{q}} \right\rangle_0 \Big|_{\boldsymbol{q}\to 0}$. We can show:

$$\tilde{M}_{Q,z} \equiv \frac{\beta_0}{2\mathrm{i}} \boldsymbol{\nabla}_{\boldsymbol{q}} \times \left\langle \hat{K}_{-\boldsymbol{q}}; \hat{\boldsymbol{J}}_{Q,\boldsymbol{q}} \right\rangle_0 \Big|_{z,\boldsymbol{q}\to 0} = -\beta_0 \frac{\partial}{\mathrm{i}\partial q_y} \left\langle \hat{K}_{-\boldsymbol{q}}; \hat{J}_{Qx,\boldsymbol{q}} \right\rangle_0 \Big|_{\boldsymbol{q}\to 0} \ . \tag{S74}$$

So we have:

$$\tilde{M}_{Q,z} = \frac{\partial}{\mathrm{i}\partial q_y} \sum_{n\boldsymbol{k},n'\boldsymbol{k}'} \frac{f_{n\boldsymbol{k}} - f_{n'\boldsymbol{k}'}}{\epsilon_{n\boldsymbol{k}} - \epsilon_{n'\boldsymbol{k}'}} \langle \psi_{n\boldsymbol{k}} | \frac{\hat{K} e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} + e^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \hat{K}}{2} | \psi_{n'\boldsymbol{k}'} \rangle \langle \psi_{n'\boldsymbol{k}'} | \hat{J}_{Qx} e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} | \psi_{n\boldsymbol{k}} \rangle \ . \tag{S75}$$

We have a careful calculation of $\langle \psi_{n'\boldsymbol{k}'} | \hat{J}_{Qx} e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} | \psi_{n\boldsymbol{k}} \rangle$,

$$\langle \psi_{n'\boldsymbol{k}'} | \hat{J}_{Qx} e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} | \psi_{n\boldsymbol{k}} \rangle = \frac{\left\langle \hat{K}\psi_{n'\boldsymbol{k}'} \middle| e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} | \hat{v}_x \psi_{n\boldsymbol{k}} \right\rangle + \left\langle \hat{v}_x \psi_{n'\boldsymbol{k}'} \right| e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \left| \hat{K}\psi_{n\boldsymbol{k}} \right\rangle}{2} \tag{S76}$$

$$- \frac{\hbar}{8\mathrm{i}} \sum_\gamma \left[\langle \nabla_\gamma \hat{v}_x \psi_{n'\boldsymbol{k}'} | e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} | \hat{v}_\gamma \psi_{n\boldsymbol{k}} \rangle + \langle \hat{v}_x \psi_{n'\boldsymbol{k}'} | e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} | \nabla_\gamma \hat{v}_\gamma \psi_{n\boldsymbol{k}} \rangle\right. \tag{S77}$$

$$\left. - \langle \nabla_\gamma \hat{v}_\gamma \psi_{n'\boldsymbol{k}'} | e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} | \hat{v}_x \psi_{n\boldsymbol{k}} \rangle - \langle \hat{v}_\gamma \psi_{n'\boldsymbol{k}'} | e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} | \nabla_\gamma \hat{v}_x \psi_{n\boldsymbol{k}} \rangle\right] . \tag{S78}$$

In Eq. (S76), we can show:

$$\left\langle \hat{K}\psi_{n'\boldsymbol{k}'}\right| e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \left|\hat{v}_x\psi_{n\boldsymbol{k}}\right\rangle = \left\langle u_{n'\boldsymbol{k}'}\right| e^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}}\hat{K}e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\hat{v}_x e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \left|u_{n\boldsymbol{k}}\right\rangle \,, \tag{S79}$$

$$= \left\langle u_{n'\boldsymbol{k}-\boldsymbol{q}}\right| \hat{K}_{\boldsymbol{k}-\boldsymbol{q}}\hat{v}_{\boldsymbol{k}x} \left|u_{n\boldsymbol{k}}\right\rangle \delta_{\boldsymbol{k}',\boldsymbol{k}-\boldsymbol{q}} \,. \tag{S80}$$

In Eq. (S77), similarly:

$$\left\langle \nabla_\gamma \hat{v}_x\psi_{n'\boldsymbol{k}'}\right| e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \left|\hat{v}_\gamma\psi_{n\boldsymbol{k}}\right\rangle = -\left\langle u_{n'\boldsymbol{k}'}\right| e^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r}}\hat{v}_x\nabla_\gamma e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\hat{v}_\gamma e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \left|u_{n\boldsymbol{k}}\right\rangle \,, \tag{S81}$$

$$= -\left\langle u_{n'\boldsymbol{k}-\boldsymbol{q}}\right| \hat{v}_{\boldsymbol{k}-\boldsymbol{q}x}\left(\nabla_\gamma + \mathrm{i}k_\gamma - \mathrm{i}q_\gamma\right)\hat{v}_{\boldsymbol{k}\gamma} \left|u_{n\boldsymbol{k}}\right\rangle \delta_{\boldsymbol{k}',\boldsymbol{k}-\boldsymbol{q}} \,. \tag{S82}$$

and:

$$\left\langle \hat{v}_x\psi_{n'\boldsymbol{k}'}\right| e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \left|\nabla_\gamma\hat{v}_\gamma\psi_{n\boldsymbol{k}}\right\rangle = \left\langle u_{n'\boldsymbol{k}-\boldsymbol{q}}\right| \hat{v}_{\boldsymbol{k}-\boldsymbol{q}x}\left(\nabla_\gamma + \mathrm{i}k_\gamma\right)\hat{v}_{\boldsymbol{k}\gamma} \left|u_{n\boldsymbol{k}}\right\rangle \delta_{\boldsymbol{k}',\boldsymbol{k}-\boldsymbol{q}} \,, \tag{S83}$$

$$\left\langle \nabla_\gamma\hat{v}_\gamma\psi_{n'\boldsymbol{k}'}\right| e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \left|\hat{v}_x\psi_{n\boldsymbol{k}}\right\rangle = -\left\langle u_{n'\boldsymbol{k}-\boldsymbol{q}}\right| \hat{v}_{\boldsymbol{k}-\boldsymbol{q}\gamma}\left(\nabla_\gamma + \mathrm{i}k_\gamma - \mathrm{i}q_\gamma\right)\hat{v}_{\boldsymbol{k}x} \left|u_{n\boldsymbol{k}}\right\rangle \delta_{\boldsymbol{k}',\boldsymbol{k}-\boldsymbol{q}} \,, \tag{S84}$$

$$\left\langle \hat{v}_\gamma\psi_{n'\boldsymbol{k}'}\right| e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \left|\nabla_\gamma\hat{v}_x\psi_{n\boldsymbol{k}}\right\rangle = \left\langle u_{n'\boldsymbol{k}-\boldsymbol{q}}\right| \hat{v}_{\boldsymbol{k}-\boldsymbol{q}\gamma}\left(\nabla_\gamma + \mathrm{i}k_\gamma\right)\hat{v}_{\boldsymbol{k}x} \left|u_{n\boldsymbol{k}}\right\rangle \delta_{\boldsymbol{k}',\boldsymbol{k}-\boldsymbol{q}} \,. \tag{S85}$$

Therefore, $\left\langle \psi_{n'\boldsymbol{k}'}\right| \hat{J}_{Qx}e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \left|\psi_{n\boldsymbol{k}}\right\rangle$ is,

$$\left\langle \psi_{n'\boldsymbol{k}'}\right| \hat{J}_{Qx}e^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \left|\psi_{n\boldsymbol{k}}\right\rangle = \left\langle u_{n'\boldsymbol{k}-\boldsymbol{q}}\right| \frac{\hat{K}_{\boldsymbol{k}-\boldsymbol{q}}\hat{v}_{\boldsymbol{k}x} + \hat{v}_{\boldsymbol{k}-\boldsymbol{q}x}\hat{K}_{\boldsymbol{k}}}{2} - \frac{\sum_\gamma \hbar q_\gamma\left(\hat{v}_{\boldsymbol{k}-\boldsymbol{q}x}\hat{v}_{\boldsymbol{k}\gamma} - \hat{v}_{\boldsymbol{k}-\boldsymbol{q}\gamma}\hat{v}_{\boldsymbol{k}x}\right)}{8} \left|u_{n\boldsymbol{k}}\right\rangle \delta_{\boldsymbol{k}',\boldsymbol{k}-\boldsymbol{q}} \,. \tag{S86}$$

$\tilde{M}_{Q,z}$ can be simplified as,

$$\tilde{M}_{Q,z} = \frac{\partial}{\mathrm{i}\partial q_y}\sum_{n\boldsymbol{k},n'}\frac{f_{n\boldsymbol{k}} - f_{n'\boldsymbol{k}-\boldsymbol{q}}}{\epsilon_{n\boldsymbol{k}} - \epsilon_{n'\boldsymbol{k}-\boldsymbol{q}}}\left\langle u_{n\boldsymbol{k}}\right| \frac{\hat{K}_{\boldsymbol{k}} + \hat{K}_{\boldsymbol{k}-\boldsymbol{q}}}{2} \left|u_{n'\boldsymbol{k}-\boldsymbol{q}}\right\rangle \left\langle u_{n'\boldsymbol{k}-\boldsymbol{q}}\right| \frac{\hat{K}_{\boldsymbol{k}-\boldsymbol{q}}\hat{v}_{\boldsymbol{k}x} + \hat{v}_{\boldsymbol{k}-\boldsymbol{q}x}\hat{K}_{\boldsymbol{k}}}{2} \tag{S87}$$

$$- \frac{\sum_\gamma \hbar q_\gamma\left(\hat{v}_{\boldsymbol{k}-\boldsymbol{q}x}\hat{v}_{\boldsymbol{k}\gamma} - \hat{v}_{\boldsymbol{k}-\boldsymbol{q}\gamma}\hat{v}_{\boldsymbol{k}x}\right)}{8}\left|u_{n\boldsymbol{k}}\right\rangle \,. \tag{S88}$$

First, we calculate $\tilde{M}_{Q,z}^{\mathrm{inter}}$ for $n \neq n'$. When $\boldsymbol{q} \to 0$, we have:

$$\tilde{M}_{Q,z}^{\mathrm{inter}} = -\frac{1}{4}\sum_{n\neq n'\boldsymbol{k}}\left(\epsilon_{n\boldsymbol{k}} + \epsilon_{n'\boldsymbol{k}} - 2\mu_0\right)^2 \mathrm{Im}\left[\left\langle u_{n\boldsymbol{k}} \middle| \frac{\partial u_{n'\boldsymbol{k}}}{\partial k_y}\right\rangle \left\langle u_{n'\boldsymbol{k}}\right| \hat{v}_{\boldsymbol{k}x} \left|u_{n\boldsymbol{k}}\right\rangle\right] \frac{f_{n\boldsymbol{k}} - f_{n'\boldsymbol{k}}}{\epsilon_{n\boldsymbol{k}} - \epsilon_{n'\boldsymbol{k}}} \,. \tag{S89}$$

Using the identity Eq. (S68), we finally come to:

$$\tilde{M}_{Q,z}^{\mathrm{inter}} = -\frac{1}{2\hbar}\sum_{n\boldsymbol{k}}\mathrm{Im}\left[\left\langle \frac{\partial u_{n\boldsymbol{k}}}{\partial k_x}\middle| \left(\hat{\mathcal{H}}_{\boldsymbol{k}} + \epsilon_{n\boldsymbol{k}} - 2\mu_0\right)^2 \middle| \frac{\partial u_{n\boldsymbol{k}}}{\partial k_y}\right\rangle\right] f_{n\boldsymbol{k}} \,. \tag{S90}$$

Next, we calculate $\tilde{M}_{Q,z}^{\mathrm{intra}}$ for $n = n'$. When $\boldsymbol{q} \to 0$, we have:

$$\tilde{M}_{Q,z}^{\mathrm{intra}} = -\frac{1}{4}\sum_{n\boldsymbol{k}} 4\left(\epsilon_{n\boldsymbol{k}} - \mu_0\right)^2 \mathrm{Im}\left[\left\langle u_{n\boldsymbol{k}} \middle| \frac{\partial u_{n\boldsymbol{k}}}{\partial k_y}\right\rangle \left\langle u_{n\boldsymbol{k}}\right| \hat{v}_{\boldsymbol{k}x} \left|u_{n\boldsymbol{k}}\right\rangle\right] f'_{n\boldsymbol{k}} \tag{S91}$$

$$-\frac{1}{4}\sum_{n\boldsymbol{k}} 2\left(\epsilon_{n\boldsymbol{k}} - \mu_0\right) \mathrm{Im}\left[\left\langle \frac{\partial u_{n\boldsymbol{k}}}{\partial k_y}\middle| \hat{K}_{\boldsymbol{k}}\hat{v}_{\boldsymbol{k}x} + \hat{v}_{\boldsymbol{k}x}\hat{K}_{\boldsymbol{k}} \left|u_{n\boldsymbol{k}}\right\rangle\right] f'_{n\boldsymbol{k}} \tag{S92}$$

$$-\frac{\hbar}{4}\sum_{n\boldsymbol{k}}\left(\epsilon_{n\boldsymbol{k}} - \mu_0\right) \mathrm{Im}\left[\left\langle u_{n\boldsymbol{k}}\right| \hat{v}_{\boldsymbol{k}y}\hat{v}_{\boldsymbol{k}x} \left|u_{n\boldsymbol{k}}\right\rangle\right] f'_{n\boldsymbol{k}} \,. \tag{S93}$$

Using the identity Eq. (S68) and after some simple algebra, we obtain:

$$\tilde{M}_{Q,z}^{\mathrm{intra}} = -\frac{1}{4\hbar}\sum_{n\boldsymbol{k}}\mathrm{Im}\left[\left\langle \frac{\partial u_{n\boldsymbol{k}}}{\partial k_x}\middle| \left(\epsilon_{n\boldsymbol{k}} - \hat{\mathcal{H}}_{\boldsymbol{k}}\right)^2 - 4\left(\epsilon_{n\boldsymbol{k}} - \mu_0\right)\left(\epsilon_{n\boldsymbol{k}} - \hat{\mathcal{H}}_{\boldsymbol{k}}\right) \middle| \frac{\partial u_{n\boldsymbol{k}}}{\partial k_y}\right\rangle\right]\left(\epsilon_{n\boldsymbol{k}} - \mu_0\right) f'_{n\boldsymbol{k}} \,. \tag{S94}$$

Therefore, we have:

$$\tilde{M}_{Q,z} = -\frac{1}{2\hbar}\sum_{n\boldsymbol{k}} \mathrm{Im}\left[\left\langle \frac{\partial u_{n\boldsymbol{k}}}{\partial k_x}\right| \left(\hat{\mathcal{H}}_{\boldsymbol{k}} + \epsilon_{n\boldsymbol{k}} - 2\mu_0\right)^2 \left|\frac{\partial u_{n\boldsymbol{k}}}{\partial k_y}\right\rangle\right] f_{n\boldsymbol{k}} \tag{S95}$$

$$-\frac{1}{4\hbar}\sum_{n\boldsymbol{k}} \mathrm{Im}\left[\left\langle \frac{\partial u_{n\boldsymbol{k}}}{\partial k_x}\right| \left(\epsilon_{n\boldsymbol{k}} - \hat{\mathcal{H}}_{\boldsymbol{k}}\right)^2 - 4\left(\epsilon_{n\boldsymbol{k}} - \mu_0\right)\left(\epsilon_{n\boldsymbol{k}} - \hat{\mathcal{H}}_{\boldsymbol{k}}\right) \left|\frac{\partial u_{n\boldsymbol{k}}}{\partial k_y}\right\rangle\right] \left(\epsilon_{n\boldsymbol{k}} - \mu_0\right) f'_{n\boldsymbol{k}}\,. \tag{S96}$$

We use $2M_{Q,z} - T_0(\partial M_{Q,z}/\partial T_0) = \tilde{M}_{Q,z}$ to obtain $M_{Q,z}$. Using the notations of Eqs. (S70)–(S72), we obtain:

$$M_{Q,z} = -\frac{1}{2}\int d\epsilon \left[\frac{1}{2} m_2\left(\epsilon\right) f\left(\epsilon\right) + 2\left(\epsilon - \mu_0\right) m_1\left(\epsilon\right) f\left(\epsilon\right) - 2\Omega_z\left(\epsilon\right)\int_0^{\epsilon-\mu_0} dx x f\left(x\right)\right]\,. \tag{S97}$$

[1] R. Kubo, M. Toda and N. Hashitsume, *Statistical Physics II: Nonequilibrium statistical mechanics*, (Springer-Verlag, 1983).
[2] G.D. Mahan, *Many-Particle Physics, Third Edition*, (Kluwer Academic, 2000).