Quantum many-body systems (8.513 fa19)
Lecture note 3

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https://stellar.mit.edu/S/course/8/fa19/8.513/index.html

1D field theory to study no $U(1)$ symmetry breaking in 1D

Phase space Lagrangian in "symmetry breaking phase" of 1D XY model: $\phi_i = (\bar\phi + q_i)\mathrm{e}^{\mathrm{i}\theta_i}$, $\bar\phi^2 = \frac{2J-h}{g}$, near the transition $\bar\phi \sim 0$

$$L = \sum_i \mathrm{i}\phi_i^* \dot\phi_i + 2J(\phi_i\phi_{i+1}^* + h.c.) - 2h|\phi_i|^2 - g|\phi_i|^4$$

$$= \sum_i -(\bar\phi + q_i)^2\dot\theta_i + 2J|\bar\phi|^2(\mathrm{e}^{\mathrm{i}(\theta_i-\theta_{i+1})} + h.c.) - 4(2J-h)q_i^2$$

$$= \sum_i -(\bar\phi^2 + 2\bar\phi q_i)\dot\theta_i - 2J|\bar\phi|^2(\theta_i - \theta_{i+1})^2 - 4(2J-h)q_i^2$$

$$= \int \mathrm{d}x\ [-\bar\phi^2 \underbrace{-\frac{2\bar\phi}{a}q(x)}_{K\partial_x\varphi/\pi}]\dot\theta(x) - 2J|\bar\phi|^2 a[\partial_x\theta(x)]^2 - \frac{4(2J-h)}{a}q^2(x)$$

$$= \int \mathrm{d}x\ \frac{K}{\pi}\partial_x\varphi\partial_t\theta - \frac{K}{2\pi}V_1(\partial_x\theta)^2 - \frac{K}{2\pi}V_2(\partial_x\varphi)^2 - \frac{\bar\phi^2}{a}\partial_t\theta$$

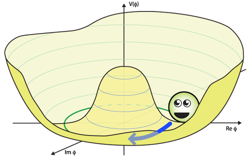
where $V_1 = \frac{4\pi J(2J-h)a}{gK}$, $V_2 = \frac{2gaK}{\pi}$, and $K = \frac{1}{2}$.

- Momentum of uniform $\theta(x)$: $\int \mathrm{d}x \frac{\partial_x\varphi}{2\pi} = \frac{\Delta\varphi}{2\pi} = int. \to \varphi \sim \varphi + 2\pi$

1D field theory – non-linear σ-model

- "Coordinate space" Lagrangian (rotor model): subsitute one of the EOM $\frac{K}{\pi}\partial_t\theta = \frac{K}{\pi}V_2\partial_x\varphi$ into the phase space Lagrangian

$$L = K\int \mathrm{d}x\ \frac{V_2^{-1}}{2\pi}(\partial_t\theta)^2 - \frac{V_1}{2\pi}(\partial_x\theta)^2\ \underbrace{-\frac{\bar\phi^2}{a}\partial_t\theta}_{\text{a topo. term}}$$



$$= K\int \mathrm{d}x\ \frac{V_2^{-1}}{2\pi}(\mathrm{i}\,u^\dagger\partial_t u)^2 - \frac{V_1}{2\pi}(\mathrm{i}\,u^\dagger\partial_x u)^2 + \mathrm{i}\,\frac{\bar\phi^2}{a}u^\dagger\partial_t u$$

- The field is really $u = \mathrm{e}^{\mathrm{i}\theta}$, not θ. So the above is the so call non-linear σ-model, where the field takes value in a manifold G (called target space), *ie* the field is a map from space-time manifold to the target space: $M^{d+1}_{\text{space-time}} \to G$.
- In our case, the target space is a circle $G = S^1$ (the minima of the symmetry breaking potential).

- The above is a **low energy effective theory** for $U(1)$ symmetry breaking phase ($\theta \sim \theta + 2\pi$)
- $V_1 \sim 0$ near the transition, but the low energy effective theory is valid even for arbitary V_1.

1D non-linear σ-model in phase space (another form)

- Introduce $\phi_1 = \theta + \varphi,\ \phi_2 = \theta - \varphi$ (where $\phi_I \sim \phi_I + 2\pi,\ I = 1, 2$)

$$L = K \int \mathrm{d}x\ \frac{1}{\pi}\partial_x\varphi\partial_t\theta - \frac{1}{2\pi}V_1(\partial_x\theta)^2 - \frac{1}{2\pi}V_2(\partial_x\varphi)^2 - \frac{\bar\phi^2}{a}\partial_t\theta$$

$$= K \int \mathrm{d}x\ \frac{1}{\pi}\frac{1}{4}\partial_x(\phi_1 - \phi_2)\partial_t(\phi_1 + \phi_2) - \frac{V_1}{2\pi}(\partial_x\theta)^2 - \frac{V_2}{2\pi}(\partial_x\varphi)^2 - \frac{\bar\phi^2}{a}\partial_t\frac{\phi_1+\phi_2}{2}$$

$$= K \int \mathrm{d}x\ \frac{1}{4\pi}(\partial_x\phi_1\partial_t\phi_1 - \partial_x\phi_2\partial_t\phi_2) - \frac{1}{4\pi}v_{IJ}\partial_x\phi_I\partial_x\phi_J - \frac{\bar\phi^2}{a}\partial_t\frac{\phi_1+\phi_2}{2}$$

where $(v_{IJ}) = \frac{1}{2}\begin{pmatrix} V_1 + V_2 & V_1 - V_2 \\ V_1 - V_2 & V_1 + V_2 \end{pmatrix}$

- More general 1D (chiral) $U(1)$ non-linear σ-model

$$L = \int \mathrm{d}x\ \frac{K_{IJ}}{4\pi}\partial_x\phi_I\partial_t\phi_J - \frac{V_{IJ}}{4\pi}\partial_x\phi_I\partial_x\phi_J, \quad \phi_I \sim \phi_I + 2\pi.$$

V is symm. and positive definite. K^{-1} is a symm. integer matrix.

- Positive eigenvalues of K $\to$ left movers.
  Negative eigenvalues of K $\to$ right movers. (See next page)
- The model is **chiral** if the number of right movers is different from the number of left movers.

1D field theory – chiral boson (rotor) model

Assume $V_1 = V_2$

$$L = \int \mathrm{d}x \; \frac{K}{4\pi}\partial_x\phi_1(\partial_t\phi_1 - v\partial_x\phi_1) - \frac{K}{4\pi}\partial_x\phi_2(\partial_t\phi_2 + v\partial_x\phi_2) - \frac{\bar{\phi}^2}{a}\partial_t\frac{\phi_1+\phi_2}{2}$$

EOM: $\partial_t\phi_1 - v\partial_x\phi_1 = 0$ and $\partial_t\phi_2 + v\partial_x\phi_2 = 0$
$\to \phi_1(x+vt)$: left-mover, $\phi_2(x-vt)$: right-mover.

- Concentrate on the right-mover ($\phi(x) = \sum_n \mathrm{e}^{-\mathrm{i}kx}\phi_n, \; k = \frac{2\pi}{L}n$)

$$L = -\int \mathrm{d}x \; \frac{K}{4\pi}\partial_x\phi(\partial_t\phi + v\partial_x\phi) + \frac{\bar{\phi}^2}{2a}\partial_t\phi, \quad \text{consider only } n \neq 0 \text{ terms}$$

$$= \sum_{n\neq 0} -\frac{KL}{4\pi}(-\mathrm{i}k)\phi_n(\dot{\phi}_{-n} + \mathrm{i}vk\phi_{-n}) = \sum_{n>0} \mathrm{i}nK\phi_n(\dot{\phi}_{-n} + \mathrm{i}vk\phi_{-n})$$

Quantize $[x,p] = \mathrm{i}$: $[\phi_{-n}, \mathrm{i}nK\phi_n] = \mathrm{i}, H = \sum_{n>0} vknK\phi_n\phi_{-n}$
Let $a_n^\dagger = \sqrt{nK}\phi_n \;\to\; a_n = \sqrt{nK}\phi_{-n}$

$$[a_n, a_n^\dagger] = 1, \quad H = \sum_{n>0} vk\frac{a_n^\dagger a_n + a_n a_n^\dagger}{2} = \sum_{n>0} vk(a_n^\dagger a_n + \frac{1}{2}).$$

Time-ordered correlation function

- Equal time correlation $\langle \mathrm{e}^{\mathrm{i}\theta(x)} \mathrm{e}^{-\mathrm{i}\theta(y)} \rangle$ and $\langle \theta(x)\theta(y) \rangle$
- Time dependent operator $O(t) = \mathrm{e}^{\mathrm{i}Ht} O \mathrm{e}^{-\mathrm{i}Ht}$ so that

$$\langle \Phi' | O(t) | \Phi \rangle = \langle \Phi'(t) | O | \Phi(t) \rangle,$$

where $|\Phi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht} |\Phi\rangle,\ |\Phi'(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht} |\Phi'\rangle$. We find

$$a_n^\dagger(t) = \mathrm{e}^{\mathrm{i}vkt} a_n^\dagger, \qquad \phi_n(t) = \mathrm{e}^{\mathrm{i}vkt} \phi_n,$$

$$\phi(x,t) = \sum_n \mathrm{e}^{-\mathrm{i}k(x-vt)} \phi_n, \qquad k = \frac{2\pi}{L} n.$$

- Time-ordered correlation

$$\langle \mathcal{T}[\phi(x,t)\phi(y,0)] \rangle = \begin{cases} \langle \phi(x,t)\phi(y,0) \rangle, & t > 0 \\ \langle \phi(y,0)\phi(x,t) \rangle, & t < 0 \end{cases}$$

For anti-commuting fermion operators

$$\langle \mathcal{T}[\psi(x,t)\tilde{\psi}(y,0)] \rangle = \begin{cases} \langle \psi(x,t)\tilde{\psi}(y,0) \rangle, & t > 0 \\ -\langle \tilde{\psi}(y,0)\psi(x,t) \rangle, & t < 0 \end{cases}$$

Time ordered correlation function

- For $t > 0$

$$\langle \phi(x,t)\phi(0,0)\rangle = \sum_{n_1,n_2} \mathrm{e}^{-\mathrm{i}\, k_1(x-vt)} \langle \phi_{n_1}\phi_{n_2}\rangle = \sum_{n_2>0} \mathrm{e}^{\mathrm{i}\, k_2(x-vt)} \langle \phi_{-n_2}\phi_{n_2}\rangle$$

$$= \sum_{n=1}^{\infty} \mathrm{e}^{\mathrm{i}\, 2\pi \frac{x-vt}{L} n} \frac{1}{nK} = -\frac{1}{K} \log(1 - \mathrm{e}^{\mathrm{i}\, 2\pi \frac{x-vt}{L}})$$

since $\sum_{n=1}^{\infty} \mathrm{e}^{\alpha n} \frac{1}{n} = -\log(1 - \mathrm{e}^{\alpha})$

- For $t < 0$

$$\langle \phi(x,t)\phi(0,0)\rangle = \sum_{n_1,n_2} \mathrm{e}^{-\mathrm{i}\, k_1(x-vt)} \langle \phi_{n_2}\phi_{n_1}\rangle = \sum_{n_1>0} \mathrm{e}^{-\mathrm{i}\, k_1(x-vt)} \langle \phi_{-n_1}\phi_{n_1}\rangle$$

$$= \sum_{n=1}^{\infty} \mathrm{e}^{-\mathrm{i}\, 2\pi \frac{x-vt}{L} n} \frac{1}{nK} = -\frac{1}{K} \log(1 - \mathrm{e}^{-\mathrm{i}\, 2\pi \frac{x-vt}{L}})$$

Correlation function of vertex operator $\mathrm{e}^{\mathrm{i}\phi}$

- Normal ordering ($\mathrm{e}^{A}\mathrm{e}^{B} = \mathrm{e}^{\frac{1}{2}[A,B]}\mathrm{e}^{A+B}$)

$$: \mathrm{e}^{\mathrm{i}\phi(x,t)} := \underbrace{\mathrm{e}^{\mathrm{i}\sum_{n>0}\mathrm{e}^{\mathrm{i}k(x-vt)}\phi_n}}_{\text{creation}}\underbrace{\mathrm{e}^{\mathrm{i}\sum_{n<0}\mathrm{e}^{\mathrm{i}k(x-vt)}\phi_n}}_{\text{annihilation}}$$

$$= \mathrm{e}^{-\frac{1}{2}[\sum_{n>0}\mathrm{e}^{\mathrm{i}k(x-vt)}\phi_n, \sum_{n<0}\mathrm{e}^{\mathrm{i}k(x-vt)}\phi_n]}\mathrm{e}^{\mathrm{i}\phi(x,t)} = \underbrace{\mathrm{e}^{\frac{1}{2K}\sum_n\frac{1}{n}}}_{\sim(\frac{L}{a})^{\frac{1}{2K}}}\mathrm{e}^{\mathrm{i}\phi(x,t)}$$

- Correlation function ($\mathrm{e}^{A}\mathrm{e}^{B} = \mathrm{e}^{[A,B]}\mathrm{e}^{B}\mathrm{e}^{A}$)

$$\langle : \mathrm{e}^{\mathrm{i}\phi(x,t)} :: \mathrm{e}^{-\mathrm{i}\phi(0,0)} : \rangle = \langle \mathrm{e}^{\mathrm{i}\phi_>(x,t)}\mathrm{e}^{\mathrm{i}\phi_<(x,t)}\mathrm{e}^{-\mathrm{i}\phi_>(0,0)}\mathrm{e}^{-\mathrm{i}\phi_<(0,0)} \rangle$$

$$= \langle \mathrm{e}^{\mathrm{i}\phi_<(x,t)}\mathrm{e}^{-\mathrm{i}\phi_>(0,0)} \rangle = \underbrace{\mathrm{e}^{[\phi_<(x,t),\phi_>(0,0)]}}_{=\mathrm{e}^{\langle\phi(x,t)\phi(0,0)\rangle}}\underbrace{\langle \mathrm{e}^{-\mathrm{i}\phi_>(0,0)}\mathrm{e}^{\mathrm{i}\phi_<(x,t)} \rangle}_{=1}$$

$$= \begin{cases} (1 - \mathrm{e}^{\mathrm{i}2\pi\frac{x-vt+\mathrm{i}0^+}{L}})^{-1/K}, & t > 0 \\ (1 - \mathrm{e}^{-\mathrm{i}2\pi\frac{x-vt-\mathrm{i}0^+}{L}})^{-1/K}, & t < 0 \end{cases}$$

$$\approx \frac{(L/2\pi)^{1/K}}{[-\mathrm{i}(x-vt)\mathrm{sgn}(t) + 0^+]^{1/K}} = \frac{(L/2\pi)^{1/K}\mathrm{e}^{\mathrm{i}\frac{1}{K}\frac{\pi}{2}\mathrm{sgn}((x-vt)t)}}{|x-vt|^{1/K}}$$

Correlation function of $\mathrm{e}^{\mathrm{i}\theta}$ and symmtery breaking

$$
\begin{aligned}
&\langle \mathcal{T}[:\mathrm{e}^{\mathrm{i}\theta(x,t)}::\mathrm{e}^{-\mathrm{i}\theta(0,0)}:]\rangle \\
&= \langle \mathcal{T}[:\mathrm{e}^{\frac{1}{2}\mathrm{i}\phi_1(x,t)}::\mathrm{e}^{-\frac{1}{2}\mathrm{i}\phi_1(0,0)}:]\rangle\langle \mathcal{T}[:\mathrm{e}^{\frac{1}{2}\mathrm{i}\phi_2(x,t)}::\mathrm{e}^{-\frac{1}{2}\mathrm{i}\phi_2(0,0)}:]\rangle \\
&= \begin{cases} (1-\mathrm{e}^{\mathrm{i}2\pi\frac{-x-vt+\mathrm{i}0^+}{L}})^{-1/4K}(1-\mathrm{e}^{\mathrm{i}2\pi\frac{x-vt+\mathrm{i}0^+}{L}})^{-1/4K}, & t>0 \\ (1-\mathrm{e}^{-\mathrm{i}2\pi\frac{-x-vt-\mathrm{i}0^+}{L}})^{-1/4K}(1-\mathrm{e}^{-\mathrm{i}2\pi\frac{x-vt-\mathrm{i}0^+}{L}})^{-1/4K}, & t<0 \end{cases} \\
&= \frac{(L/2\pi)^{1/2K}}{[-\mathrm{i}(x-vt)\mathrm{sgn}(t)+0^+]^{1/4K}[-\mathrm{i}(-x-vt)\mathrm{sgn}(t)+0^+]^{1/4K}} \\
&= \frac{(L/2\pi)^{1/2K}}{(x^2-v^2t^2+\mathrm{i}2vt\mathrm{sgn}(t)0^+ + (0^+)^2)^{\frac{1}{4K}}} = \frac{L/2\pi}{(x^2-v^2t^2+\mathrm{i}0^+)^{\frac{1}{2}}}
\end{aligned}
$$

1D supperfluid (or boson condensation, or $U(1)$ symmetry breaking) only has an **algebraic long range order**, not real **long range order** which requires $\langle :\mathrm{e}^{\mathrm{i}\theta(x,0)}::\mathrm{e}^{-\mathrm{i}\theta(0,0)}:\rangle \to const.$ as $x\to\infty$.

Conitinous symmetry cannot spontaneously broken in 1D, can only "nearly broken"

Correlation function and spectral function of $\mathrm{e}^{\mathrm{i}\theta} \sim \sigma^+$

$$G(x,t) = \mathrm{i}\langle T[:\mathrm{e}^{\mathrm{i}\theta(x,t)}::\mathrm{e}^{-\mathrm{i}\theta(0,0)}:]\rangle$$

$$= \mathrm{i}(1-\mathrm{e}^{\mathrm{i}2\pi\frac{x-vt}{L}\mathrm{sgn}(t)})^{-1/4K}(1-\mathrm{e}^{\mathrm{i}2\pi\frac{-x-vt}{L}\mathrm{sgn}(t)})^{-1/4K}$$

$$= \sum_n C_{m,n}\mathrm{i}\,\mathrm{e}^{\mathrm{i}(m\frac{2\pi}{L}x - n\frac{2\pi v}{L}t)\mathrm{sgn}(t)} = \sum_n C_{m,n}\mathrm{i}\,\mathrm{e}^{\mathrm{i}(K_m x - E_n t)\mathrm{sgn}(t)}$$

$$I(k,\epsilon) = \sum_n C_{m,n}[\delta(k-K_m)\delta(\epsilon-E_n) + \delta(k+K_m)\delta(\epsilon+E_n)]$$

Fourier transformation of $G(x,t)$:

$$\int_0^L \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}t\ \mathrm{e}^{-\mathrm{i}(kx-\epsilon t)}\mathrm{i}\,\mathrm{e}^{\mathrm{i}(K_m x - E_n t)\mathrm{sgn}(t)}$$

$$= \int_0^L \mathrm{d}x \int_0^{\infty} \mathrm{d}t\ \mathrm{e}^{-\mathrm{i}[kx-(\epsilon+\mathrm{i}0^+)t]}\mathrm{i}\,\mathrm{e}^{\mathrm{i}(K_m x - E_n t)} + (t<0)$$

$$= \underbrace{\delta(k-K_m)}_{L\delta_{k,K_m}}\frac{\mathrm{i}}{-\mathrm{i}(\epsilon - E_n + \mathrm{i}0^+)} = \underbrace{\delta(k-K_m)}_{L\delta_{k,K_m}}[\frac{-1}{\epsilon - E_n} + \mathrm{i}\pi\delta(\epsilon - E_n)]$$

$$I(k,\epsilon) = \mathrm{Im}\, G(k,\epsilon)/\pi$$

Correlation function and spectral function of $\mathrm{e}^{\mathrm{i}\theta} \sim \sigma^+$

Correlation function of $\mathrm{e}^{\mathrm{i}\theta} \sim \sigma^+$

$$G(x.t) = \frac{\mathrm{i}(L/2\pi)^{1/2K}}{(x^2 - v^2t^2 + \mathrm{i}0^+)^{1/4K}} = \frac{\mathrm{i}(L/2\pi)^{1/2K}}{(y_1 y_2 + \mathrm{i}0^+)^{1/4K}}$$

where $y_1 = x + vt,\ y_2 = x - vt$. We find

$$\begin{aligned} G(k,\epsilon) &= \int \mathrm{d}x\,\mathrm{d}t\ \mathrm{e}^{-\mathrm{i}(kx-\epsilon t)} \frac{\mathrm{i}(L/2\pi)^{1/2K}}{(x^2 - v^2t^2 + \mathrm{i}0^+)^{1/4K}} \\ &= \int \mathrm{d}x\,\mathrm{d}t\ \mathrm{e}^{-\mathrm{i}\frac{1}{2}[k(y_1+y_2)-v^{-1}\epsilon(y_1-y_2)]} \frac{\mathrm{i}(L/2\pi)^{1/2K}}{(y_1 y_2 + \mathrm{i}0^+)^{1/4K}} \\ &\sim \int \mathrm{d}y_1\,\mathrm{d}y_2\ \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}[(k-\frac{\epsilon}{v})y_1+(k+\frac{\epsilon}{v})y_2]}}{(y_1 y_2 + \mathrm{i}0^+)^{1/4K}} \end{aligned}$$

up to a positive factor.

Correlation function and spectral function of $\mathrm{e}^{\mathrm{i}\theta} \sim \sigma^+$

- $y_1 > 0,\ y_2 > 0$: $\quad \left(\int_0^\infty \mathrm{d}x \frac{\mathrm{e}^{-ax}}{x^\alpha} = \Gamma(1-\alpha)a^{\alpha-1}\right)$

$$G_{++}(k,\epsilon) = \mathrm{i} \int_0^\infty \mathrm{d}y_1 \int_0^\infty \mathrm{d}y_2 \ \frac{\mathrm{e}^{-\mathrm{i}\frac{1}{2}(k-\frac{\epsilon}{v}-\mathrm{i}0^+)y_1}\mathrm{e}^{-\mathrm{i}\frac{1}{2}(k+\frac{\epsilon}{v}-\mathrm{i}0^+)y_2}}{(y_1y_2+\mathrm{i}0^+)^{1/4K}}$$

$$= \mathrm{i}\Big(\frac{\mathrm{i}(k-\frac{\epsilon}{v})+0^+}{2}\Big)^{\frac{1}{4K}-1}\Gamma(1-\frac{1}{4K})\Big(\frac{\mathrm{i}(k+\frac{\epsilon}{v})+0^+}{2}\Big)^{\frac{1}{4K}-1}\Gamma(1-\frac{1}{4K})$$

$$= \mathrm{i}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\frac{1}{4K}-1)[\mathrm{sgn}(vk-\epsilon)+\mathrm{sgn}(vk+\epsilon)]}$$

$$\Big(\frac{|vk-\epsilon|}{2v}\Big)^{\frac{1}{4K}-1}\Big(\frac{|vk+\epsilon|}{2v}\Big)^{\frac{1}{4K}-1}\Gamma^2(1-\frac{1}{4K})$$

Correlation function and spectral function of $\mathrm{e}^{\mathrm{i}\theta} \sim \sigma^+$

- $y_1 > 0,\ y_2 < 0$:

$$G_{+-}(k,\epsilon) = \mathrm{i} \int_0^\infty \mathrm{d}y_1 \int_{-\infty}^0 \mathrm{d}y_2\ \frac{\mathrm{e}^{-\mathrm{i}\frac{1}{2}(k-\frac{\epsilon}{v}-\mathrm{i}0^+)y_1}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}(k+\frac{\epsilon}{v}+\mathrm{i}0^+)y_2}}{(y_1y_2+\mathrm{i}0^+)^{1/4K}}$$

$$= \mathrm{i} \int_0^\infty \mathrm{d}y_1 \int_0^\infty \mathrm{d}y_2\ \frac{\mathrm{e}^{-\mathrm{i}\frac{1}{2}(k-\frac{\epsilon}{v}-\mathrm{i}0^+)y_1}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}(k+\frac{\epsilon}{v}+\mathrm{i}0^+)y_2}}{(-y_1y_2+\mathrm{i}0^+)^{1/4K}}$$

$$= \mathrm{i}\Big(\frac{\mathrm{i}(k-\frac{\epsilon}{v})+0^+}{2}\Big)^{\frac{1}{4K}-1}\Big(\frac{-\mathrm{i}(k+\frac{\epsilon}{v})+0^+}{2}\Big)^{\frac{1}{4K}-1}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4K}}\Gamma^2(1-\frac{1}{4K})$$

$$= \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4K}}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\frac{1}{4K}-1)[\mathrm{sgn}(vk-\epsilon)-\mathrm{sgn}(vk+\epsilon)]}$$

$$\Big(\frac{|vk-\epsilon|}{2v}\Big)^{\frac{1}{4K}-1}\Big(\frac{|vk+\epsilon|}{2v}\Big)^{\frac{1}{4K}-1}\Gamma^2(1-\frac{1}{4K})$$

Correlation function and spectral function of $\mathrm{e}^{\mathrm{i}\theta} \sim \sigma^+$

- $y_1 < 0,\ y_2 > 0$:

$$G_{-+}(k,\epsilon) = \mathrm{i} \int_{-\infty}^{0} \mathrm{d}y_1 \int_0^\infty \mathrm{d}y_2 \ \frac{\mathrm{e}^{-\mathrm{i}\frac{1}{2}(k-\frac{\epsilon}{v}+\mathrm{i}0^+)y_1}\, \mathrm{e}^{-\mathrm{i}\frac{1}{2}(k+\frac{\epsilon}{v}-\mathrm{i}0^+)y_2}}{(y_1y_2+\mathrm{i}0^+)^{1/4K}}$$

$$= \mathrm{i} \int_0^\infty \mathrm{d}y_1 \int_0^\infty \mathrm{d}y_2 \ \frac{\mathrm{e}^{\mathrm{i}\frac{1}{2}(k-\frac{\epsilon}{v}+\mathrm{i}0^+)y_1}\, \mathrm{e}^{-\mathrm{i}\frac{1}{2}(k+\frac{\epsilon}{v}-\mathrm{i}0^+)y_2}}{(-y_1y_2+\mathrm{i}0^+)^{1/4K}}$$

$$= \mathrm{i} \Big(\frac{-\mathrm{i}(k-\frac{\epsilon}{v})+0^+}{2}\Big)^{\frac{1}{4K}-1} \Big(\frac{\mathrm{i}(k+\frac{\epsilon}{v})+0^+}{2}\Big)^{\frac{1}{4K}-1} \mathrm{e}^{-\mathrm{i}\frac{\pi}{4K}} \Gamma^2(1-\frac{1}{4K})$$

$$= \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{4K}}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\frac{1}{4K}-1)[-\mathrm{sgn}(vk-\epsilon)+\mathrm{sgn}(vk+\epsilon)]}$$

$$\Big(\frac{|vk-\epsilon|}{2v}\Big)^{\frac{1}{4K}-1} \Big(\frac{|vk+\epsilon|}{2v}\Big)^{\frac{1}{4K}-1} \Gamma^2(1-\frac{1}{4K})$$

Correlation function and spectral function of $\mathrm{e}^{\mathrm{i}\theta} \sim \sigma^+$

- $y_1 < 0,\ y_2 < 0$:

$$G_{--}(k,\epsilon) = \mathrm{i} \int_{-\infty}^{0} \mathrm{d}y_1 \int_{-\infty}^{0} \mathrm{d}y_2 \ \frac{\mathrm{e}^{-\mathrm{i}\frac{1}{2}(k-\frac{\epsilon}{v}+\mathrm{i}0^+)y_1}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}(k+\frac{\epsilon}{v}+\mathrm{i}0^+)y_2}}{(y_1y_2+\mathrm{i}0^+)^{1/4K}}$$

$$= \mathrm{i} \int_{0}^{\infty} \mathrm{d}y_1 \int_{0}^{\infty} \mathrm{d}y_2 \ \frac{\mathrm{e}^{\mathrm{i}\frac{1}{2}(k-\frac{\epsilon}{v}+\mathrm{i}0^+)y_1}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}(k+\frac{\epsilon}{v}+\mathrm{i}0^+)y_2}}{(y_1y_2+\mathrm{i}0^+)^{1/4K}}$$

$$= \mathrm{i}\Big(\frac{-\mathrm{i}(k-\frac{\epsilon}{v})+0^+}{2}\Big)^{\frac{1}{4K}-1}\Big(\frac{-\mathrm{i}(k+\frac{\epsilon}{v})+0^+}{2}\Big)^{\frac{1}{4K}-1}\Gamma^2(1-\frac{1}{4K})$$

$$= \mathrm{i}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\frac{1}{4K}-1)[-\mathrm{sgn}(vk-\epsilon)-\mathrm{sgn}(vk+\epsilon)]}$$

$$\Big(\frac{|vk-\epsilon|}{2v}\Big)^{\frac{1}{4K}-1}\Big(\frac{|vk+\epsilon|}{2v}\Big)^{\frac{1}{4K}-1}\Gamma^2(1-\frac{1}{4K})$$

Correlation function and spectral function of $\mathrm{e}^{\mathrm{i}\theta} \sim \sigma^+$

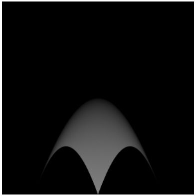
$$G(k,\epsilon) \sim \mathrm{i}\Big(\frac{|vk-\epsilon|}{2v}\Big)^{\frac{1}{4K}-1}\Big(\frac{|vk+\epsilon|}{2v}\Big)^{\frac{1}{4K}-1}\Gamma^2(1-\frac{1}{4K})\times$$
$$\Big(\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\frac{1}{4K}-1)[\mathrm{sgn}(vk-\epsilon)+\mathrm{sgn}(vk+\epsilon)]} + \mathrm{e}^{-\mathrm{i}\frac{\pi}{4K}}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\frac{1}{4K}-1)[\mathrm{sgn}(vk-\epsilon)-\mathrm{sgn}(vk+\epsilon)]}$$
$$+ \mathrm{e}^{-\mathrm{i}\frac{\pi}{4K}}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\frac{1}{4K}-1)[-\mathrm{sgn}(vk-\epsilon)+\mathrm{sgn}(vk+\epsilon)]} + \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\frac{1}{4K}-1)[-\mathrm{sgn}(vk-\epsilon)-\mathrm{sgn}(vk+\epsilon)]}\Big)$$
$$= \mathrm{i}\Big(\frac{|vk-\epsilon|}{2v}\Big)^{\frac{1}{4K}-1}\Big(\frac{|vk+\epsilon|}{2v}\Big)^{\frac{1}{4K}-1}\Gamma^2(1-\frac{1}{4K})\times$$
$$\begin{cases} -\mathrm{e}^{\mathrm{i}\frac{\pi}{4K}} + \mathrm{e}^{-\mathrm{i}\frac{\pi}{4K}} + \mathrm{e}^{-\mathrm{i}\frac{\pi}{4K}} - \mathrm{e}^{-\mathrm{i}\frac{\pi}{4K}} = -2\mathrm{i}\sin(\frac{\pi}{4K}), & vk-\epsilon>0, vk+\epsilon>0 \\ -\mathrm{e}^{-\mathrm{i}\frac{\pi}{4K}} + \mathrm{e}^{-\mathrm{i}\frac{\pi}{4K}} + \mathrm{e}^{-\mathrm{i}\frac{\pi}{4K}} - \mathrm{e}^{\mathrm{i}\frac{\pi}{4K}} = -2\mathrm{i}\sin(\frac{\pi}{4K}), & vk-\epsilon<0, vk+\epsilon<0 \\ 1-1-\mathrm{e}^{-\mathrm{i}\frac{\pi}{2K}}+1 = 1-\mathrm{e}^{-\mathrm{i}\frac{\pi}{2K}}, & vk-\epsilon>0, vk+\epsilon<0 \\ 1-\mathrm{e}^{-\mathrm{i}\frac{\pi}{2K}}-1+1 = 1-\mathrm{e}^{-\mathrm{i}\frac{\pi}{2K}}, & vk-\epsilon<0, vk+\epsilon>0 \end{cases}$$

Spectral function: $I(k,\epsilon) = \Big(\frac{|vk-\epsilon|}{2v}\Big)^{\frac{1}{4K}-1}\Big(\frac{|vk+\epsilon|}{2v}\Big)^{\frac{1}{4K}-1}\Gamma^2(1-\frac{1}{4K})\times$
$$\begin{cases} 0, & (\epsilon-vk)(\epsilon+vk)<0 \\ 1-\cos(\pi/2K), & (\epsilon-vk)(\epsilon+vk)>0 \end{cases}$$

$k = 0$ modes

- Our theory so far can produce exication near $k = 0$, but not near $k = k_B = 2\pi\frac{N}{L}$.
- The correlation $\langle \mathcal{T}[: \mathrm{e}^{\mathrm{i}\theta(x,t)} :: \mathrm{e}^{\mathrm{i}\theta(0,0)} :]\rangle \sim (x^2 - v^2t^2)^{-1/4K}$

  We need to include $k = 0$ modes.



- Concentrate on the right mover

$$\phi(x) = \frac{2\pi W}{L}x + \phi_0 + \sum_{n\neq 0} \mathrm{e}^{-\mathrm{i}\,kx}\phi_n, \quad k = \frac{2\pi}{L}n,\ W = \text{winding number}$$

$$L = \int_0^L \mathrm{d}x\ \frac{K}{4\pi}\partial_x\phi(\partial_t\phi - v\partial_x\phi) - \frac{\bar{\phi}^2}{2a}\partial_t\phi$$
$$= (\frac{K}{2}m - \frac{\bar{\phi}^2 L}{2a})\dot{\phi}_0 - Kv\frac{\pi m^2}{L} + \sum_{n>0} \mathrm{i}\,nK\phi_n(\dot{\phi}_{-n} + \mathrm{i}\,vk\phi_{-n})$$

- $\frac{K}{2}W$ corresponds to angluar momentum of ϕ_0, with a $-\frac{\bar{\phi}^2 L}{2a}$ shift. For integer W, we allow fractional angluar momentum $\frac{K}{2}W = \frac{W}{4}$?

$k = 0$ modes

To understand the above puzzle, from $\phi_1 = \theta + \varphi$ $\phi_2 = \theta - \varphi$, we note that the periodicy $(\theta, \varphi) \sim (\theta + 2\pi, \varphi) \sim (\theta, \varphi + 2\pi)$ implies the periodicy

$$
\begin{aligned}
(\phi_1, \phi_2) &\sim (\phi_1 + 2\pi, \phi_2 + 2\pi) \sim (\phi_1 + 2\pi, \phi_2 - 2\pi) \\
&\sim (\phi_1 + 4\pi, \phi_2) \sim (\phi_1, \phi_2 + 4\pi)
\end{aligned}
$$

Introduce $(\tilde\phi_1, \tilde\phi_2) = \frac{1}{2}(\phi_1, \phi_2)$ so that $\tilde\phi_I$ have independent periodicy $\tilde\phi_I \sim \tilde\phi_I + 2\pi$ ($\tilde K = 4K = 2$):

$$
\begin{aligned}
L &= \int_0^L \mathrm{d}x\ \frac{\tilde K}{4\pi} \partial_x \tilde\phi(\partial_t \tilde\phi - v\partial_x \tilde\phi) - \frac{\bar\phi^2}{a}\partial_t \tilde\phi, \quad \tilde\phi = \frac{2\pi w}{L} x + \tilde\phi_0 + \cdots \\
&= (\frac{\tilde K w}{2} - \frac{\bar\phi^2 L}{a})\dot{\tilde\phi}_0 - \tilde K v \frac{\pi w^2}{L} + \sum_{n>0} \mathrm{i} n \tilde K \tilde\phi_n (\dot{\tilde\phi}_{-n} + \mathrm{i} v k \tilde\phi_{-n})
\end{aligned}
$$

Now winding number $\frac{\tilde K}{2} w = w$ is the angular momentum of $\tilde\phi$ (with $-\frac{\bar\phi^2 L}{a}$ shift), which is quantuized as integers.

$k = 0$ modes

Compare the following two theories

$$L_1 = \int_0^L \mathrm{d}x\ \frac{1}{2\pi}\partial_x\tilde\phi_1\partial_t\tilde\phi_1 - v\frac{1}{2\pi}\partial_x\tilde\phi_1\partial_x\tilde\phi_1 - \frac{\bar\phi^2}{a}\partial_t\tilde\phi_1$$
$$- \frac{1}{2\pi}\partial_x\tilde\phi_2\partial_t\tilde\phi_2 - v\frac{1}{2\pi}\partial_x\tilde\phi_2\partial_x\tilde\phi_2 - \frac{\bar\phi^2}{a}\partial_t\tilde\phi_2$$
$$L_2 = \int \mathrm{d}x\ (\frac{1}{2\pi}\partial_x\varphi - \frac{\bar\phi^2 L}{a})\partial_t\theta - \frac{v}{4\pi}(\partial_x\theta)^2 - \frac{v}{4\pi}(\partial_x\varphi)^2$$

- The $k \neq 0$ modes of the two theories are identical.
- The $k = 0$ modes, the sectors, of the two theories are not the same.

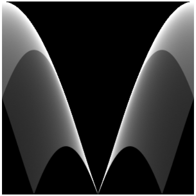
- The sectors for L_1 are labeled by w_1, w_2: $E = \frac{2\pi}{L}v(w_1^2 + w_2^2)$
- The sectors for L_2 are labeled by w_θ, w_φ (Only $q = \partial\varphi$ is physical): $\theta = w_\theta\frac{2\pi}{L}x + \theta_0, \quad \varphi = w_\varphi\frac{2\pi}{L}x.$
  $L_2 = (w_\varphi - \frac{\bar\phi^2 L}{a})\dot\theta_0 - \frac{1}{2}\frac{2\pi}{L}v(w_\theta^2 + w_\varphi^2) \ \to E = \frac{1}{2}\frac{2\pi}{L}v(w_\theta^2 + w_\varphi^2)$

- The spectra match when $w_\theta = w_1 + w_2$, $w_\varphi = w_1 - w_2$, and we allow $w_1, w_2 = int.$ and $w_1, w_2 = \frac{1}{2} + int.$.

$k = 0$ modes

- **What is the meaning of w_φ (angular momentum of θ_0)?**
  We note that $-2\bar\phi a^{-1} q = K\partial_x\varphi/\pi = \partial_x\varphi/2\pi = w_\varphi/L$.
  So $w_\varphi = \int \mathrm{d}x\ (-2\bar\phi a^{-1} q) = \sum_i (-2\bar\phi q_i)$

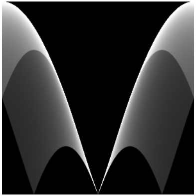
  Spectral of n_i

  But what is $\sum_i (-2\bar\phi q_i)$?
  Remember that $|\phi_i\rangle = \frac{|\uparrow\rangle + \phi_i|\downarrow\rangle}{\sqrt{1+|\phi_i|^2}} = \frac{|0\rangle + \phi_i|1\rangle}{\sqrt{1+|\phi_i|^2}}$.
  So $\langle n_i\rangle = \frac{|\phi_i|^2}{1+|\phi_i|^2} \approx |\phi_i|^2$
  Thus the canonical momentum of θ_0,
  $-\frac{\bar\phi^2 L}{a} + w_\varphi = \sum_i (-\bar\phi^2 - 2\bar\phi q_i) = -\sum_i n_i = -N$, is the total number of the bosons (with a minus sign).
  Under the $U(1)$ symmetry transformation, $\theta_0 \to \theta_0 + \Delta\theta$. The angular momentum of θ_0 is the total number of the bosons.
- **What is the meaning of w_θ?**
  A non-zero w_θ gives rise $\phi_i = \bar\phi \mathrm{e}^{\mathrm{i}\, w_\theta x \frac{2\pi}{L}}$. Each boson carries momentum $w_\theta \frac{2\pi}{L}$. The total momentum is $w_\theta \frac{2\pi N_0}{L} = w_\theta k_B$.

Winding-number changing operators

$$L=\int \mathrm{d}x\ (\frac{1}{2\pi}\partial_x\varphi-\frac{\bar\phi^2 L}{a})\partial_t\theta-\frac{v}{4\pi}(\partial_x\theta)^2-\frac{v}{4\pi}(\partial_x\varphi)^2$$



- The **local operator** $\mathrm{e}^{\mathrm{i}\theta}=\mathrm{e}^{\frac{\mathrm{i}}{2}(\phi_1+\phi_2)}$ changes the particle number N by -1, *ie* change the winding number of φ, w_φ, by 1.
- To see this explicitly $[\theta(x),\frac{1}{2\pi}\partial_y\varphi(y)]=\mathrm{i}\delta(x-y)$
  We find $[\theta(x),\Delta\varphi]=\mathrm{i}2\pi$ where $\Delta\varphi=\varphi(+\infty)-\varphi(-\infty)$.
  Thus $\theta(x)=\mathrm{i}2\pi\frac{\mathrm{d}}{\mathrm{d}\Delta\varphi}$, and $\mathrm{e}^{\mathrm{i}\theta(x)}=\mathrm{e}^{-2\pi\frac{\mathrm{d}}{\mathrm{d}\Delta\varphi}}$ is an operator that changes $\Delta\varphi$ by -2π, or w_φ by -1, or particle number by 1
- Similarly, we have $[\theta(x),\varphi(y)]=-\mathrm{i}2\pi\Theta(x-y)$
  $\to[\partial_x\theta(x),\varphi(y)]=-\mathrm{i}2\pi\delta(x-y)$
  We find $[\Delta\theta,\varphi(y)]=-\mathrm{i}2\pi$ where $\Delta\theta=\theta(+\infty)-\theta(-\infty)$.
  Thus $\varphi(y)=\mathrm{i}2\pi\frac{\mathrm{d}}{\mathrm{d}\Delta\theta}$, and $\mathrm{e}^{\mathrm{i}\varphi(x)}=\mathrm{e}^{-2\pi\frac{\mathrm{d}}{\mathrm{d}\Delta\theta}}$ is an operator that changes $\Delta\theta$ by -2π, or change w_θ by -1, or change total momentum by $-k_B$.

Local operators in 1D XY-model (superfluid)

- Lattice operators

$$\sigma_i^z = (\#\partial_x\theta + \#\partial_x\varphi) + \#\mathrm{e}^{-\mathrm{i}k_B x}\mathrm{e}^{\mathrm{i}\varphi(x)} + \cdots$$

$$\sigma_i^+ = (\# + \#\partial_x\theta + \#\partial_x\varphi)\mathrm{e}^{-\mathrm{i}\theta(x)} + \#\mathrm{e}^{-\mathrm{i}k_B x}\mathrm{e}^{-\mathrm{i}\theta(x)}\mathrm{e}^{\mathrm{i}\varphi(x)} + \cdots$$

- Set of local operators: $\partial_x\theta,\ \partial_x\varphi,\ \underbrace{\mathrm{e}^{\mathrm{i}(m_\theta\theta+m_\varphi\varphi)}}_{\text{change sectors}}$

or $\theta = \phi_1 + \phi_2$, $\varphi = \phi_1 - \phi_2$

$$\partial_x\phi_1,\ \partial_x\phi_2,\ \underbrace{\mathrm{e}^{\mathrm{i}(m_1\phi_1+m_2\phi_2)}}_{\text{change sectors}}$$

where $m_1 + m_2 = m_\theta,\quad m_1 - m_2 = m_\varphi,\qquad m_\theta, m_\varphi \in \mathbb{Z}.$
We see that $m_1,\ m_2$ are both integers or both half-integers.

- The sectors are labeled by w_θ, w_φ (or $w_1 = \frac{w_\theta+w_\varphi}{2}, w_2 = \frac{w_\theta-w_\varphi}{2}$).
The sectors are also labeled by m_θ, m_φ (or m_1, m_2):

$$m_\theta = -w_\varphi, \qquad m_\varphi = -w_\theta.$$

$$m_1 = -w_1, \qquad m_2 = w_2.$$

Fractionalization in XY-model (superfluid)

- A boson creation operator $\sigma^+ \sim \mathrm{e}^{\mathrm{i}\theta}$ (spin flip operator $\Delta S^z = 1$)

$$\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}\frac{1}{2}(\phi_1+\phi_2)}, \quad \phi_1 \text{ left-mover}, \quad \phi_2 \text{ right-mover}$$

$\mathrm{e}^{\mathrm{i}\frac{1}{2}\phi_2}$ creats half boson (spin-1/2) in right-moving sector
$\mathrm{e}^{\mathrm{i}\frac{1}{2}\phi_1}$ creats half boson (spin-1/2) in left-moving sector
for a model

$$L = \int \mathrm{d}x \; \frac{K}{\pi}\partial_x\varphi\partial_t\theta - \frac{K}{2\pi}V_1(\partial_x\theta)^2 - \frac{K}{2\pi}V_2(\partial_x\varphi)^2 - \frac{\bar{\phi}^2}{a}\partial_t\theta$$

with $V_1 = V_2,\ K = \frac{1}{2}$.

- In general $V_1 \neq V_2$, $\mathrm{e}^{\mathrm{i}\theta}$ creats q bosons in right-moving sector and $1-q$ bosons in left-moving sector.

The partition function of 1D XY-model (superfluid)

- The partition function:

$$Z(\beta, L) = \mathrm{Tr}\, \mathrm{e}^{-\beta \hat{H}} = \sum_n D_n \mathrm{e}^{-\beta E_n}$$

where D_n is the number of states with energy E_n

- The generalize partition function:

$$Z(\beta, , L, X) = \mathrm{Tr}\, \mathrm{e}^{-\beta \hat{H} + \mathrm{i} X \hat{K}} = \sum_{n,m} D_{n,m} \mathrm{e}^{-\beta E_n + \mathrm{i} X K_m}$$

where $D_{n,m}$ is the number of states with E_n, K_m

- The generalize partition function for the $K \sim 0$ sector

$$\hat{H} = \sum_{n_R > 0} v \frac{2\pi n_R}{L} (a^\dagger_{n_R} a_{n_R} + \frac{1}{2}) + \sum_{n_L > 0} v \frac{2\pi n_L}{L} (a^\dagger_{n_L} a_{n_L} + \frac{1}{2}) + \rho_\epsilon L.$$

$$\hat{K} = \sum_{n_R > 0} \underbrace{\frac{2\pi n_R}{L}}_{k_R} a^\dagger_{n_R} a_{n_R} + \sum_{n_L > 0} \underbrace{-\frac{2\pi n_L}{L}}_{k_L} a^\dagger_{n_L} a_{n_L},$$

The partition function of 1D XY-model (superfluid)

$$Z(\beta, L, X) = \mathrm{e}^{-\beta\rho_\epsilon L} \prod_{m_R>0, m_L>0} Z_{m_R}(\beta, X) Z_{m_L}(\beta, X)$$

$$Z_{m_R}(\beta, L, X) = \sum_{n=0}^{\infty} \mathrm{e}^{(-\beta v \frac{2\pi m_R}{L} + \mathrm{i} X \frac{2\pi m_R}{L})(n+\frac{1}{2})} = \sum_{n=0}^{\infty} q^{m_R(n+\frac{1}{2})} = \frac{q^{\frac{m_R}{2}}}{1 - q^{N_R}}$$

$$Z_{m_L}(\beta, L, X) = \sum_{n=0}^{\infty} \mathrm{e}^{(-\beta v \frac{2\pi m_L}{L} - \mathrm{i} X \frac{2\pi m_L}{L})(n+\frac{1}{2})} = \sum_{n=0}^{\infty} \bar{q}^{m_R(n+\frac{1}{2})} = \frac{\bar{q}^{\frac{m_L}{2}}}{1 - \bar{q}^{N_L}}$$

where $q = \mathrm{e}^{(-\beta v + \mathrm{i} X)\frac{2\pi}{L}}$

$$Z(\beta, L, X) = \mathrm{e}^{-\beta\rho_\epsilon L} \prod_{m_R=1}^{\infty} \frac{\bar{q}^{\frac{m_R}{2}}}{1 - \bar{q}^{N_R}} \prod_{m_L=1}^{\infty} \frac{\bar{q}^{\frac{m_L}{2}}}{1 - \bar{q}^{N_L}}$$

$$= \mathrm{e}^{-\beta\tilde{\rho}_\epsilon L} \frac{q^{-\frac{1}{24}}}{\prod_{n=1}^{\infty}(1 - q^n)} \frac{\bar{q}^{-\frac{1}{24}}}{\prod_{n=1}^{\infty}(1 - \bar{q}^n)}$$

where we have used

$$\sum_{n=1}^{\infty} n = \#(\frac{L}{a})^2 - \frac{1}{12}$$

Heat kernal regularization

$$\sum_{n=1}^{\infty} n e^{-\alpha n} = -\frac{d}{d\alpha}\sum_{n=1}^{\infty} e^{-\alpha n} = -\frac{d}{d\alpha}\frac{e^{-\alpha}}{1-e^{-\alpha}} = -\frac{d}{d\alpha}\frac{1}{e^{\alpha}-1}$$
$$= \frac{e^{\alpha}}{(e^{\alpha}-1)^2} = \frac{1}{(e^{\alpha/2}-e^{-\alpha/2})^2} = \frac{1}{(2\frac{\alpha}{2}+2\frac{1}{3!}(\frac{\alpha}{2})^3)^2} = \frac{1}{\alpha^2(1+\frac{1}{24}\alpha^2)^2}$$
$$= \frac{1}{\alpha^2} - \frac{1}{12} \quad \to \quad \sum_{n=1}^{\infty} n \text{ “}=\text{”} -\frac{1}{12}$$

- Zero-point energy of a chiral boson with velocity v

$$E_0(L) = \sum_{n=1}^{\infty}\frac{1}{2}vk = \sum_{n=1}^{\infty}\frac{1}{2}v\frac{2\pi}{L}n = \frac{1}{2}v\frac{2\pi}{L}\Big(\#(\frac{L}{a})^2 - \frac{1}{12}\Big)$$
$$= \#L\frac{v}{a} - \frac{1}{24}v\frac{2\pi}{L}$$

\- For 1D superfluid (XY-model) with both left and right movers

$$E_{\text{grnd}}(L) = \#L\frac{v}{a} - \frac{c_L+c_R}{24}v\frac{2\pi}{L} = \Big|_{c_L=c_R=1} \#L\frac{v}{a} - \frac{1}{12}v\frac{2\pi}{L}$$

A story about central charge c (conformal field theory)

- It is a property of 1D gapless system with a finite and unique velocity. $c = c_L + c_R = 0$ for gapped systems.
- It has an additive property: $A \boxtimes B = C \to c_A + c_B = c_C$
- It measures how many low energy excitation are there. Specific heat (heat capacity per unit length) $C = c\frac{\pi}{6}\frac{T}{v}$

A story about central charge c (conformal field theory)

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\- Why? $E = \rho_\epsilon L - \frac{c}{24}\frac{2\pi}{L}$ (**assume** $v = 1$) $\to$

Partition function $Z(\beta, L) = \mathrm{Tr}(\mathrm{e}^{-\beta H}) = \mathrm{e}^{-\beta L \rho_\epsilon - \frac{2\pi\beta}{L}\frac{c}{24}}|_{\beta\to\infty}$

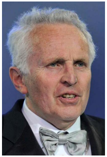
- A magic: emergence of $O(2)$ symmetry in space-(imaginary-)time

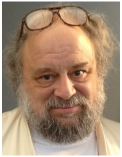
$$Z(\beta, L) = Z(L, \beta), \qquad \text{have used } v = 1.$$

This allows us to find $Z(\beta, L) = \mathrm{e}^{-\beta L \rho_\epsilon - \frac{2\pi L}{\beta}\frac{c}{24}}|_{L\to\infty}$

Free energy density $f = \rho_\epsilon - \frac{2\pi}{(\beta)^2}\frac{c}{24}$
$= \rho_\epsilon - 2\pi T^2 \frac{c}{24}$

Specific heat $C = -T\frac{\partial^2 F}{\partial T^2} = T\frac{\pi}{6}c$





Belavin-Polyakov-Zamolodchikov NPB 241,333(84); Ginsparg hep-th/9108028

Examples of conformal field theories (CFT)

- **Non-chiral CFT**

- For the 1+1D superfluid

$$c = c_R + c_L = 2 \text{ and } c_R = c_L = 1.$$

- For the 1+1D free fermion metal

$$c = c_R + c_L = 2 \text{ and } c_R = c_L = 1.$$

- For the 1+1D Z_2 symmetry breaking transition (Ising model)

$$c = c_R + c_L = 1 \text{ and } c_R = c_L = 1/2.$$

- For the 1+1D Z_3 symm. breaking transition (3-state Potts model)

$$c = c_R + c_L = \frac{8}{5} \text{ and } c_R = c_L = \frac{4}{5}.$$

- **Chiral CFT**

- Boundary of $\nu = 1$ IQH state (Chern insulator)

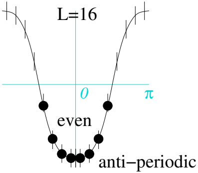
$$c = c_R + c_L = 1 \text{ and } c_R = 1,\ c_L = 0.$$

- Boundary of 2+1D $p + \mathrm{i}\, p$ topological superconductor

$$c = c_R + c_L = \frac{1}{2} \text{ and } c_R = \frac{1}{2},\ c_L = 0.$$

Ground state energy of non-interacting fermions on a ring

$$H = -\sum_{i=1}^{L} t \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right), \qquad \sum_{n=n_1}^{n_2} x^{n+\alpha} = \frac{x^{n_1+\alpha} - x^{n_2+\alpha} x}{1-x}$$



anti-periodic boundary condition $c_{L+1} = -c_1$

The Hamiltonian in k-space is given by

$$H = \sum_k -2t\cos(k)\psi_k^\dagger \psi_k, \qquad k = \frac{2\pi}{L}(n + \frac{1}{2})$$

The ground state is obtained by filling all negative energy levels

$$E(L) = \sum_{n=-\frac{L}{4}}^{\frac{L}{4}-1} -2t \underbrace{\cos[\frac{2\pi}{L}(n+\frac{1}{2})]}_{e^{i\#n}+e^{-i\#n}}, \qquad \text{assume } L = 0 \bmod 4$$

$$= -t\left(\frac{e^{i\frac{2\pi}{L}(-\frac{L}{4}+\frac{1}{2})} - e^{i\frac{2\pi}{L}(\frac{L}{4}-\frac{1}{2})} e^{i\frac{2\pi}{L}}}{1 - e^{i\frac{2\pi}{L}}} + h.c. \right) = -2t\frac{\sin(\frac{2\pi}{L}\frac{L}{4})}{\sin(\frac{\pi}{L})}$$

$$= -2t\frac{1}{\sin(\frac{\pi}{L})} = -2t(\frac{L}{\pi} + \frac{1}{6}\frac{\pi}{L}) = -2t\frac{L}{\pi} - \frac{1}{12}v\frac{2\pi}{L}$$

Partition function on space-time with different shape

We have calculated XY partition function ($q = e^{(-\beta + i X)\frac{2\pi}{L}}$):

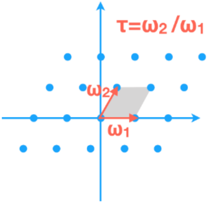
$$Z(\beta, L, X) = e^{-\beta \rho_\epsilon L} \frac{q^{-\frac{1}{24}}}{\prod_{n=1}^{\infty}(1-q^n)} \frac{\bar{q}^{-\frac{1}{24}}}{\prod_{n=1}^{\infty}(1-\bar{q}^n)}$$

Does it really satisfies $Z(\beta, L, 0) = Z(L, \beta, 0)$?

- Simplify partition function:

- Set $\rho_\epsilon = 0$ to remove size dependence. (This is not setting the ground state energy to zero, but set the linear L-term to zero.)

- Let $\tau = \frac{X + i\beta}{L}$ describing the shape of space-time. $q = e^{2\pi\tau}$

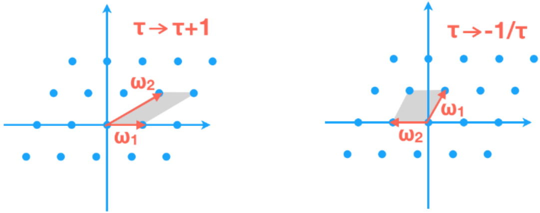


$$Z(\tau) = \frac{1}{\eta(q)\bar{\eta}(\bar{q})}, \qquad \eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n).$$

$$= q^{-\frac{1}{24}} \bar{q}^{-\frac{1}{24}} \sum_{m,n} D_{m,n} q^m \bar{q}^n$$

$m + n$ energy in unit of $\frac{2\pi v}{L}$ (ground state energy shift $\frac{1}{12}\frac{2\pi v}{L}$)
$m - n$ momentum in unit of $\frac{2\pi}{L}$
$D_{m,n} \in \mathbb{N}$ degeneracies.

Modular invariance of CFT partition function



Partition function only dependends on the shape of space-time "manifold", not on the underlying lattice.

The essence of quantum field theory - long-distance effective theory

- Since τ, $\tau+1$, $-1/\tau$ all described the same shape

$$Z(\tau,\bar\tau) = Z(\tau+1,\bar\tau+1) = Z(-1/\tau,-1/\bar\tau).$$

- Modular transformation of η-function

$$\eta(\tau+1) = \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{24}}\eta(\tau), \qquad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$$

https://math.stackexchange.com/questions/1815212/modular-transformations-of-eta-tau

$$Z(-1/\tau,-1/\bar\tau) = |\tau|^{-1} Z(\tau,\bar\tau), \qquad \text{not modular invaraint!}$$

Other sectors and state-operator correspondence

But $Z(\tau,\bar\tau) = 1/\eta(\tau)\bar\eta(\bar\tau)$ is only the $K \sim 0$ sector, there are other sectors. May be the full partition function is modular invariant.

- Consider the right-movers ϕ_2:

$$L = \int_0^L \mathrm{d}x \; -\frac{1}{8\pi}\partial_x\phi_2\partial_t\phi_2 - v\frac{1}{8\pi}\partial_x\phi_2\partial_x\phi_2 - \frac{\bar\phi^2}{a}\partial_t\phi_2$$

\- The sector w_2: $\phi_2 = w_2\frac{4\pi}{L}x + \cdots$.
The ground state in the sector has $E = v\frac{2\pi}{L}w_2^2$, $K = \frac{2\pi}{L}w_2^2 \underbrace{-k_B w_2}_{\text{dropped}}$

- The partition function of the sector w_2:

$$\frac{q^{w_2^2}}{\eta(\tau)} \quad \to \quad \text{lowest energy } E = (-\frac{1}{24} + w_2^2)v\frac{2\pi}{L}$$

- The sector is created by $\mathrm{e}^{\mathrm{i}\, m_2\phi_2}$, $m_2 = w_2$ from the sector-0

$$\langle \mathrm{e}^{\mathrm{i}\, m_2\phi_2(x)}\mathrm{e}^{-\mathrm{i}\, m_2\phi_2(y)}\rangle \sim \mathrm{e}^{m_2^2\langle\phi_2(x)\phi_2(y)\rangle} \sim \mathrm{e}^{m_2^2[-2\log(x-y)]} \sim \frac{1}{(x-y)^{2m_2^2}}$$

- $\mathrm{e}^{\mathrm{i}\, m_2\phi_2}$ has **scaling dimension** $h = m_2^2$. The corresponding state has energy $E = (-\frac{c}{24} + h)v\frac{2\pi}{L}$. This is true for any operator and state: **state-operator correspondence**.

Chiral boson partition functions and CFT characters

The total partition function for "integer sectors" $m_2 \in \mathbb{Z}$

$$Z_0(\tau) = \frac{1}{\eta(\tau)} \sum_{m_2 \in \mathbb{Z}} q^{m_2^2} = \chi_0^{u1_2}(\tau),$$

The total partition function for "half-integer sectors" $m_2 \in \mathbb{Z} + \frac{1}{2}$

$$Z_1(\tau) = \frac{1}{\eta(\tau)} \sum_{m_2 \in \mathbb{Z}} q^{(m_2+\frac{1}{2})^2} = \chi_1^{u1_2}(\tau)$$

where $u1$-CFT characters are given by

$$\chi_m^{u1_M}(\tau) = \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(nR+\frac{m}{R})^2} = \chi_{m+M}^{u1_M}(\tau),$$

$$R = \sqrt{M}, \qquad m = 0, \cdots, M-1.$$

Modular invariance of CFT partition function

The CFT characters transforms simply under the modular transformation

$$\chi_i^{u1_M}(-\frac{1}{\tau}) = S_{ij}\chi_j^{u1_M}(\tau), \qquad S_{ij} = \frac{\mathrm{e}^{-\mathrm{i}\,2\pi\,\frac{ij}{M}}}{\sqrt{M}},$$

$$\chi_i^{u1_M}(\tau+1) = T_{ij}\chi_j^{u1_M}(\tau), \qquad T_{ij} = \mathrm{e}^{-\mathrm{i}\,\frac{2\pi}{24}}\,\mathrm{e}^{\mathrm{i}\,2\pi\,\frac{i^2}{2M}}\delta_{ij}.$$

- The total partition function of both right- and left-movers

$$Z(\tau,\bar\tau) = \underbrace{Z_0(\tau)\bar Z_0(\bar\tau)}_{m_1,m_2\in\mathbb{Z}} + \underbrace{Z_1(\tau)\bar Z_1(\bar\tau)}_{m_1,m_2\in\mathbb{Z}+\frac{1}{2}}$$

$$= \sum_{m=0,1} \chi_m^{u1_2}(\tau)\bar\chi_m^{u1_2}(\bar\tau)$$

Such a combination is modular in variant.

- **An 1D CFT always gives rise to a modular invariant partition function $Z(\tau,\bar\tau)$.**
- **A modular invariant partition function $Z(\tau,\bar\tau)$ "always" gives rise to a CFT – an 1D gapless state.**

Spectrum of exponents and state-operator correspondence

- Spectrum of R,L-exponents $\{h_R, h_L\}$

$$\mathrm{Tr}[\mathrm{e}^{-(\beta-\tau)H} O_{h_R,h_L}(x)\mathrm{e}^{-\tau H} O_{h_R,h_L}(0)] \equiv \langle O_{h_R,h_L}(x,\tau) O_{h_R,h_L}(0)\rangle$$

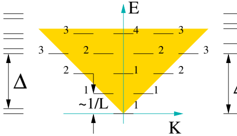
$$\sim \frac{1}{z^{2h_R}\bar{z}^{2h_L}}, \qquad z = x - vt = x + \mathrm{i}v\tau, \quad \bar{z} = x + vt = x - \mathrm{i}v\tau.$$

We view $O_{h_R,h_L} = O_{h_R} O_{h_L}$

$$\langle O_{h_R}(x,\tau) O_{h_R}(0)\rangle \sim \frac{1}{z^{2h_R}}, \qquad \langle \bar{O}_{h_L}(x,\tau) \bar{O}_{h_L}(0)\rangle \sim \frac{1}{\bar{z}^{2h_L}}.$$

- Lowest energy created by O_{h_R} corresponds to $h_R = \frac{E-E_{\text{grnd}}}{2\pi v/L}$,
  Lowest momentum created by O_{h_R} corresponds to $K = \frac{2\pi}{L} h_R$.
- Lowest energy created by $\bar{O}_{h_L}$ corresponds to $h_L = \frac{E-E_{\text{grnd}}}{2\pi v/L}$,
  Lowest momentum created by $\bar{O}_{h_L}$ corresponds to $K = -\frac{2\pi}{L} h_L$.

- **State-operator correspondence**:
  Spectrum of $(h_R + h_L, h_L - h_R)$ = spectrum of (total energy $E - E_{\text{grnd}}$, total momentum K) of the excitations, in units $\Delta E = \frac{2\pi v}{L}$, $\Delta K = \frac{2\pi}{L}$.



An example of state-operator correspondence

Consider an 1D free fermion system $H_f = -\sum_i (\frac{\mathrm{i}}{2} c_i^\dagger c_{i+1} + h.c.)$
Fermion dispertion relation $\epsilon_k = -\sin(k)$. Fermi velocity $v_F = 1$

- **CFT**: Left-movers near $k = 0$. Right-movers near $k = \pi$.

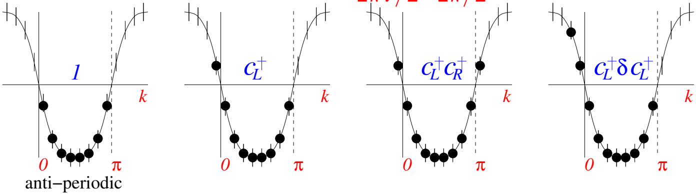
- The correlation $\langle c_L(x) c_L(y)\rangle \sim \frac{1}{x-y}$ has scaling dimension 1 (ie 1/length) $\to$ c_L has scaling dimension $h^{c_L} = \frac{1}{2}$

- $\langle c_R(x) c_R(y)\rangle \sim \frac{\mathrm{e}^{\mathrm{i}\,\pi(x-y)}}{x-y}$ $\to$ c_R has scaling dimension $h^{c_R} = \frac{1}{2}$

- $h^{c_R c_L} = h_L^{c_R c_L} + h_R^{c_R c_L} = 1$, $\quad h^{c_L \partial_x c_L} = h_L^{c_L \partial_x c_L} + \underbrace{h_R^{c_L \partial_x c_L}}_{=0} = 2$,

$h^{c_R \partial_x c_R} = h_L^{c_R \partial_x c_R} + h_R^{c_R \partial_x c_R} = 2$, ...

- State-operator correspondence $(\frac{E}{2\pi v/L}, \frac{K}{2\pi/L}) = (h^R + h^L, h^R - h^L)$



$(h_L, h_R) = (0, 0)$, $\quad (\frac{1}{2}, 0)$, $\quad (\frac{1}{2}, \frac{1}{2})$, $\quad (\frac{1}{2} + \frac{3}{2} = 2, 0)$

$U(1)$ quantum number of the spectrum

Partition function gives rise energy E, momentum K

$$Z(\tau,\bar\tau) = \frac{\sum_{m_1,m_2} q^{m_2^2}\bar q^{m_1^2}}{\eta(\tau)\bar\eta(\bar\tau)} = \frac{\sum_{m_\theta,m_\varphi\in\mathbb{Z}} q^{\frac12(m_\theta-m_\varphi)^2}\bar q^{\frac12(m_\theta+m_\varphi)^2}}{\eta(\tau)\bar\eta(\bar\tau)}$$

Duality in transverse Ising model and $h_c =?$

Let $A_\alpha = \sigma^x_i \sigma^x_{i+1}$, $B_i = \sigma^z_i$, where α labels the **dual lattice** sites

Operator algebra of local operators A_α and B_i:

$[A_\alpha, B_i] = 0, \quad \alpha, i$ are not neighbors $\quad \{A_\alpha, B_i\} = 0, \quad \alpha, i$ are neighbors.

$A^2_\alpha = 1, \qquad B^2_i = 1.$

We may also view $A_\alpha = \tau^z_\alpha$ and $B_i = \tau^x_\alpha \tau^x_{\alpha+1}$ $\to$ the same algebra

- Transverse Ising model

$$H = -h^{-1} \sum_\alpha A_\alpha - h \sum_i B_i$$

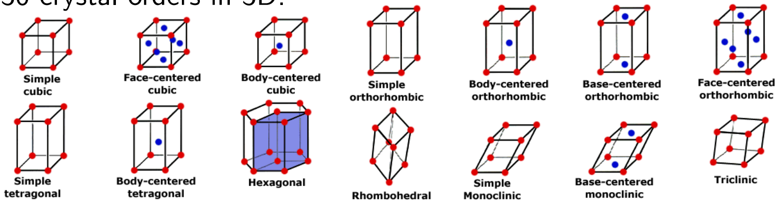
- Duality transformation: $A_\alpha \leftrightarrow B_i$, $h \leftrightarrow h^{-1}$; or $\sigma_i \leftrightarrow \tau_\alpha$
- Only one transition $\to$ transition at $h = 1$ – the self dual point.

Exercise: Find an exact finite self-dual lattice model.

For more details, see: *Confusion about duality transformation in 1+1d Ising model in a transverse field*
http://physics.stackexchange.com/questions/135098

Symmetry breaking transition between gapped systems

- A symmetry breaking phase transition happens at $h = h_c$, where
  - Ground state energy density $\epsilon_h = \min[\epsilon_h(\phi)]$ has a singularity
  - Energy gaps for excitations Δ, Δ_p also have singularities, and vanish at the transition (more gapless excitations at transition)
  - Every physical quantities have singularities at the transition
- **The math foundation is group theory:** classified by (G_H, G_Ψ)
  From 230 ways of translation symmetry breaking, we obtain the 230 crystal orders in 3D.



Phase transition in 1+1D XY model with $U(1)$ symmetry: beyond Landau symmetry break theory

- Superfluid to Mott insulator transition and XY transition:

$$H = -\sum_i (2J\sigma_i^+ \sigma_{i+1}^- + h.c.) - h\sigma_i^z = -\sum_i J(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) - h\sigma_i^z$$

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$$H = -\sum_i (2J\sigma_i^+\sigma_{i+1}^- + h.c.) - h\sigma_i^z = -\sum_i J(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) - h\sigma_i^z$$

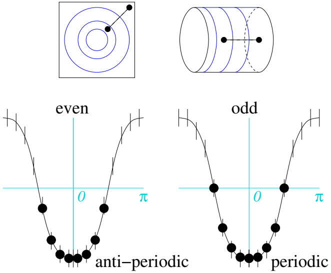
- Jordan-Wigner: $H \to H_f = \sum_i(-2Jc_i^\dagger c_{i+1} + h.c.) + 2h(n_i - \frac{1}{2})$
- Superfluid phase does not break the $U(1)$ symmetry
  $\langle\sigma^+(x)\sigma^-(y)\rangle \sim |x-y|^{-\alpha}$ algebraic long-range correlation.

\- How to calculate the exponent α

$$\langle\sigma^+(x)\sigma^-(y)\rangle \sim \langle c_x[\prod_{x<i<y}(-)^{n_i}]c_y^\dagger\rangle,$$

$$E_{\text{anti-perio}}^{\text{even}} = \epsilon L + \frac{2\pi v}{L}(-\frac{c}{24}),$$

$$E_{\text{perio}}^{\text{odd}} - E_{\text{anti-perio}}^{\text{even}} = \frac{2\pi v}{L}\frac{\alpha}{2}.$$



- Central charge $c = 1$

Computing α

In Prob. Set 7, we have computed: $\epsilon_k = -4J\cos(k) + 2h$
for aPBC, n_f = even ($k_F = \frac{n_f}{2}\frac{2\pi}{L}$):

$$E^{\text{even}}_{\text{aPBC}}(h, L) = -L\Big(h - 2h\frac{k_F}{\pi} + \frac{4\sin(k_F)}{\pi}\Big) - \frac{1}{12}\frac{2\pi}{L}v$$

for PBC, n_f = odd ($k_F = \frac{n_f-1}{2}\frac{2\pi}{L}$):

$$E^{\text{odd}}_{\text{PBC}}(h, L) = -L\Big(h - 2h\frac{k_F}{\pi} + \frac{4\sin(k_F)}{\pi}\Big) \overbrace{-4\cos(k_F) + 2h}^{=0 \text{ as } L\to\infty} + \frac{1}{6}\frac{2\pi}{L}v$$

$$= -L\Big(h - 2h\frac{k_F}{\pi} + \frac{4\sin(k_F)}{\pi}\Big) + \Big(-\frac{1}{12} + \frac{1}{4}\Big)\frac{2\pi}{L}v$$

We find that $\alpha = 1/2$: $\langle\sigma^+(x)\sigma^-(y)\rangle \sim \frac{1}{|x-y|^{1/2}}$. Another way

$$\langle\sigma^+(x)\sigma^+(x+1)\sigma^-(y)\sigma^-(y+1)\rangle \sim \langle c(x)c(x+1)c^\dagger(y)c^\dagger(y+1)\rangle$$

$$\sim \langle c_R(x)c_L(x)c_R^\dagger(y)c_L^\dagger(y)\rangle \sim \frac{1}{|x-y|^2},$$

$$\langle\sigma^+(x)\sigma^-(y)\rangle \sim \langle\sqrt{c_R(x)c_L(x)}\sqrt{c_R^\dagger(y)c_L^\dagger(y)}\rangle \sim \frac{1}{|x-y|^{\frac{2}{4}}} \sim \frac{1}{|x-y|^{\frac{1}{2}}}.$$

Condensation picture of the phase transition in Ising model

$$H = -J \sum \sigma_i^x \sigma_{i+1}^x - h \sum \sigma_i^z$$

- $h > 0$: **from symmetric phase to symmetry breaking phase**
  Two cases: $J > 0$ and $J < 0$.

- Condensing particle carry $k = \pi$ crystal momentum for $J < 0$
  $\to$ condensed state break the translation symmetry:

$$|0\rangle + |1\rangle + |2\rangle + \cdots \to |0\rangle - |1\rangle - |2\rangle + \cdots$$

- $J > 0$: **from symmetry breaking phase to symmetric phase**
  Two cases: $h > 0$ and $h < 0$.

- Condensing domain-wall carry $k = \pi$ crystal momentum $h < 0 \to$
  condensed state break the translation symmetry ?

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- $J > 0$: **from symmetry breaking phase to symmetric phase**
  Two cases: $h > 0$ and $h < 0$.

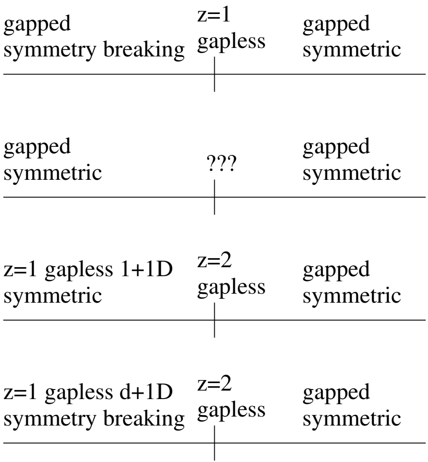
\- Condensing domain-wall carry $k = \pi$ crystal momentum $h < 0 \to$ condensed state break the translation symmetry ?

\- But a single domain-wall cannot be created alone, and a pair always carry $k = 0$, and no translation symmetry breaking.

\- In fact, we can make condensing domain-wall to carry $k = 0$ crystal momentum, reference to $|\downarrow\rangle$ or $-|\downarrow\rangle$.
$\to$ Two different condensed states $\to$ Two different symmetric phases (**Homework**).

Phase transitions induced by condensation

- Condensing particles
  local excitation with
  non-trivial symmetry
  quantum number
  Induced phase transition
  symmetry breaking
  phase transition
  $\rightarrow$ symmetry breaking phase
- Condensing particles
  topological excitation
  (domain-wall) with
  non-trivial symmetry
  quantum number
  Induced phase transition
  symmetry-restoring
  phase transition
  $\rightarrow$ SPT phase

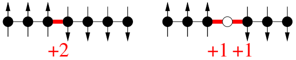


Example: A $Z_2^x \times Z_2^z$ spin-1 chain, & its symmetric phases

$$|\uparrow_z\rangle = \frac{|x\rangle + \mathrm{i}|y\rangle}{\sqrt{2}} \quad ,|0_z\rangle = |z\rangle, \quad |\downarrow_z\rangle = \frac{|x\rangle - \mathrm{i}|y\rangle}{\sqrt{2}}$$

$$S^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad S^y = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad S^z = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$Z_2^x \times Z_2^z$ symmetry: $U^x : (|x\rangle, |y\rangle, |z\rangle) \to (-|x\rangle, |y\rangle, |z\rangle)$
$U^z : (|x\rangle, |y\rangle, |z\rangle) \to (|x\rangle, |y\rangle, -|z\rangle)$.

$H^0 = \sum_i -J_z S_i^z S_{i+1}^z \;\to\; Z_2^x$ breaking 

- Two kinds of domain walls with the same energy, but different Z_2^z-charges and different hopping operators:

$$H_1^{\text{hop}} = \sum_i -K[(S_i^+)^2 + h.c.], \qquad H_2^{\text{hop}} = \sum_i -J_{xy}(S_i^+ S_{i+1}^+ + h.c.).$$

- $H^0 + H_1^{\text{hop}}$ & $H^0 + H_2^{\text{hop}}$ $\to$ different symm. ground states

Topological invariant for the symmetric states

How to show the two symm. ground states are in different phases?

- U^x-symmetry transformation:
  $$(U^x)^2 = 1,\ U^x S^x U^x = S^x,\ \ U^x S^{y,z} U^x = -S^{y,z}.$$
- U^x-symmetry twist:

$$S_i^z S_{i+1}^z \to U_i^x S_i^z U_i^x S_{i+1}^z = -S_i^z S_{i+1}^z$$
$$S_i^+ S_{i+1}^+ \to U_i^x S_i^+ U_i^x S_{i+1}^+ = S_i^- S_{i+1}^+$$

- The two models with U^x symmetry twist:

$$H_1 = \sum_{i=1}^{L-1}[-J_z S_i^z S_{i+1}^z - \sum_{i=1}^{L} K[(S_i^+)^2 + h.c.] + J_z S_L^z S_1^z$$
$$H_2 = \sum_{i=1}^{L-1}[-J_z S_i^z S_{i+1}^z - J_{xy}(S_i^+ S_{i+1}^+ + h.c.)] + J_z S_L^z S_1^z - J_{xy}(S_L^- S_1^+ + h.c.).$$

The twisted ground state of H_1 has trivial (even) Z_2^z-charge
The twisted ground state of H_2 has odd Z_2^z-charge (**Homework)**

Topological invariant for the symmetric states

- Put the untwisted H_1 and H_2 on a ring $\to$ non-degenerate ground state.

- Put the untwisted H_1 and H_2 on an open line $\to$
non-degenerate ground state for H_1
four-fold nearly degenerate ground states for H_2, two for each end.
(**Homework**)

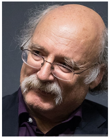
The symmetric ground state of H_2 is a non-trivial SPT state

Haldane phase of spin-1 chain

$$H_2 = \sum[-J_z S_i^z S_{i+1}^z - J_{xy}(S_i^+ S_{i+1}^+ + h.c.)] \qquad S^\pm = (S^x \pm \mathrm{i} S^y)/\sqrt{2}$$

$$= \sum[-J_z S_i^z S_{i+1}^z - J_{xy}(S_i^x S_{i+1}^x - S_i^y S_{i+1}^y)]$$

$$\to \sum[J_z S_i^z S_{i+1}^z + J_{xy}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y)]$$

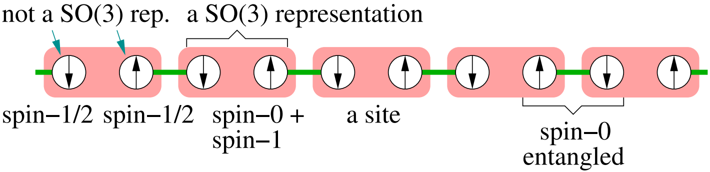


after 180° S^y rotation on odd sites.

Haldane, PRL 50, 1153 (1983)

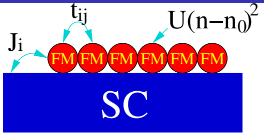
- When $J_z = J_{xy} \to SO(3)$ spin rotation symmetry. Gapped ground state that does not break $SO(3)$ symmetry. Four-fold nearly degenerate ground states for H_2 on an open line, spin-$1/2$ for each end $\to SO(3)$ **symmetry fractionalization** on a defect.

Last example has $Z_2^x \times Z_2^z$ **symmetry fractionalization** (?)



Z_2^f symmetry breaking for fermions and topo. degeneracy

Consider an 1+1D system of ferromagnetic particles/molecules on a superconductor:
Klassen-Wen, arXiv:1412.5985



- Symmetry: no $SO(3)$ spin rotation, no $U(1)$ electron number conservation symmetry, but electron number mod 2 is conserved.
  $\to$ Z_2^f **fermion-number-parity symmetry**, which is an unbreakable symmetry from fermion systems.

- $Z_2^f \subset SU(2)$ spin rotation symmetry
- $Z_2^f \subset U(1)$ electron number conservation symmetry
- Z_2^f is generated by $(-1)^{\hat{N}_f} = \hat{U}_\pi$ (the $U(1)$ by $\hat{U}_\theta = \mathrm{e}^{\mathrm{i}\theta \hat{N}_f}$)

- Is there an 1D state that spontaneously break the Z_2^f symmetry?
  $\to$ **Topological 2-fold degeneracy** ($\Delta \sim \mathrm{e}^{-L/\xi}$) that is robust against any perturbations that can break any symm. (except Z_2^f).

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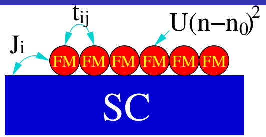

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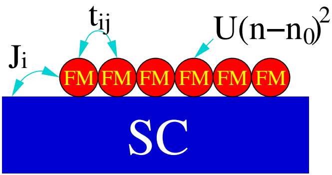
  $\to$ **Topological 2-fold degeneracy** ($\Delta \sim \mathrm{e}^{-L/\xi}$) that is robust against any perturbations that can break any symm. (except Z_2^f).
  Yes. We can map the 1D fermion system with Z_2^f symmetry on a line to an 1D spin system with Z_2 symmetry on a line and find the Z_2 spontaneous-symmetry-breaking state for the spin system.

Model Hamiltonian for fermionic chain on superconductor

$$H = \sum_i [t \hat{c}^\dagger_{i+1} \hat{c}_i + J \hat{c}_i \hat{c}_i + h.c.]$$

$$+ \sum_i \Big[U(\hat{n}_i - n_0)^2 + \Delta \frac{(-)^{\hat{n}_i} - 1}{2} \Big],$$



where $\hat{n}_i$ is the fermion number operator and $\hat{c}_i$ is the effective (spinless) fermion operator acting on the Hilbert space $\mathcal{V}_i$ on site-i. $\mathcal{V}_i$ is formed by states of n-fermions, $n = 0, \pm 1, \pm 2,$ etc and $\hat{n}_i$ and $\hat{c}_i$ satisfy

$$\{\hat{c}_i, \hat{c}_j\} = \{\hat{c}_i, \hat{c}^\dagger_j\} = [\hat{c}_i, \hat{n}_j] = 0, \quad i \neq j,$$
$$\hat{c}_i |n\rangle = |n-1\rangle, \qquad \hat{n}_i |n\rangle = n |n\rangle.$$

Note that the eigenvalue of $\hat{n}_i$ can be any integer n, and $\hat{c}_i$ is not the standard fermionic operator.

Map to spin/boson system

Jordan-Wigner transformation

$$\hat{c}_i^\dagger = \hat{n}_i^+ \prod_{j<i} (-1)^{\hat{n}_j} \qquad \hat{c}_i = \hat{n}_i^- \prod_{j<i} (-1)^{\hat{n}_j},$$

where the action of these operators are as follows

$$\hat{n}_i |n\rangle = n|n\rangle, \qquad \hat{n}_i^+ |n\rangle = |n+1\rangle, \qquad \hat{n}_i^- |n\rangle = |n-1\rangle.$$

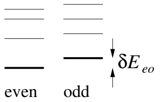
Our bosonic effective Hamiltonian then takes the form

$$H = \sum_i \Big[U(\hat{n}_i - n_0)^2 + \Delta \frac{(-1)^{\hat{n}_i} - 1}{2} + (J \hat{n}_i^+ \hat{n}_i^+ + h.c.) + (t \hat{n}_i^+ (-1)^{\hat{n}_i} \hat{n}_{i+1}^- + h.c.) \Big]$$

The Z_2^f transformation is generated by $(-)^{\sum_i \hat{n}_i}$, which is a symmetry of the above effective Hamiltonian.

The Z_2^f symmetry breaking state

- For small t, we first solve the one-site Hamiltonian
  $H_i = U(\hat{n}_i - n_0)^2 + \Delta\frac{(-1)^{\hat{n}_i}-1}{2} + (J\hat{n}_i^+\hat{n}_i^+ + h.c.)$



- Project into the even-fermion state $|\uparrow\rangle$, the odd-fermion state $|\downarrow\rangle$.
- In the subspace $(-)^{\hat{n}_i} = \sigma_i^z$ and $\hat{n}_i^+$ has a form
  $\hat{n}_i^+ = \mathrm{e}^{\mathrm{i}\phi}(h_x\sigma_i^x + \mathrm{i}\, h_y\sigma_i^y)$, where $h_{x,y} \sim O(1)$ are real and positive.

$$H = \sum_i \Big[-\frac{\delta E_{eo}}{2}\sigma_i^z + 2\mathrm{Re}(t)h_x h_y(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) + 2\mathrm{Im}(t)(h_x^2\sigma_i^y\sigma_{i+1}^x - h_y^2\sigma_i^x\sigma_{i+1}^y)\Big]$$

- Have Z_2^f breaking if

- the electron hopping t_{ij} between dots is larger than the energy splitting δE_{eo} between states of even and odd electrons on a dot,
- the Josephson coupling J_i between the superconducting substrate and the dot satisfy $|J_i| \gtrsim \delta E_{eo}$,
- the electron hopping amplitude t_{ij} is complex, or more precisely, the phase of the gauge invariant combination $J_i t_{ij}^2 J_j^*$ is not zero.

A free fermion system with the Z_2^f symmetry breaking

Consider an 1+1D p-wave superconductor: Kitaev cond-mat/0010440

$$H = \sum_i \mu c_i^\dagger c_i - t(c_i^\dagger c_{i+1} - c_i c_{i+1} + h.c.]$$

- Introduce Majorana fermion operators:

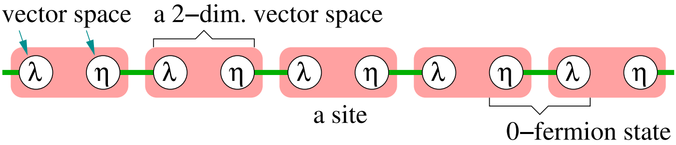
$$c_i = \frac{1}{2}(\lambda_i + \mathrm{i}\eta_i), \quad \lambda_i^\dagger = \lambda_i, \quad \eta_i^\dagger = \eta_i, \quad \mathrm{i}\eta_i\lambda_i = (-)^{c_i^\dagger c_i},$$

$$\lambda_i^2 = \eta_i^2 = 1, \quad \{\lambda_i, \lambda_j\} = \{\eta_i, \eta_j\} = \{\lambda_i, \eta_j\} = 0.$$

- Rewrite the above in terms of Majorana fermion operators:

$$H = \sum_i [\frac{1}{2}\mu \mathrm{i}\lambda_i\eta_i - t\,\mathrm{i}\lambda_{i+1}\eta_i]$$

Topo. degeneracy $\leftrightarrow$ **Vector-space fractionalization** on defect.



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- Rewrite the above in terms of Majorana fermion or spin operators:

$$H = \sum_i [\frac{1}{2}\mu \mathrm{i}\lambda_i\eta_i - t\,\mathrm{i}\lambda_{i+1}\eta_i] = \sum_i [\frac{1}{2}\mu\sigma_i^z + t\sigma_i^x\sigma_{i+1}^x]$$

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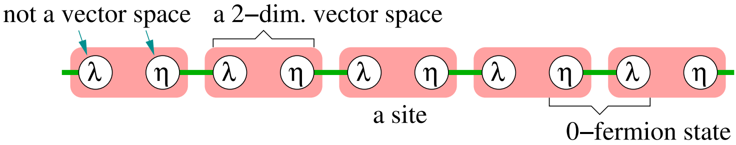

Properties of the Z_2^f-symm. breaking (topological) phase

- The fermion-number-parity (FNP)

$$FNP = \prod_i (-)^{c_i^\dagger c_i} = \prod_i \mathrm{i}\eta_i\lambda_i = \cdots \eta_{i+1}\ \mathrm{i}\lambda_{i+1}\eta_i\ \mathrm{i}\lambda_i \cdots$$
$$= \eta_L \mathrm{i}\lambda_1 \prod_i \mathrm{i}\lambda_{i+1}\eta_i \sim \mathrm{i}\eta_L\lambda_1\Big|_{t>0,\mu=0}$$

  The two degenerate states has opposite FNP.
- An effective zero-energy level of a fermion C_{eff} but with $\mathrm{Re}C_{\text{eff}} = \lambda_1$ at one end and $\mathrm{Im}C_{\text{eff}} = \eta_L$ at the other end.

Fractional statistics and hopping algebra

- If they can move, defects with fractional vector space will have non-abelian statistics.
- What is Fermi statistics? What is fractional statistics? What is non-Abelian statistics?

- Fermi statistics $\leftrightarrow$ Pauli exclusion principle.

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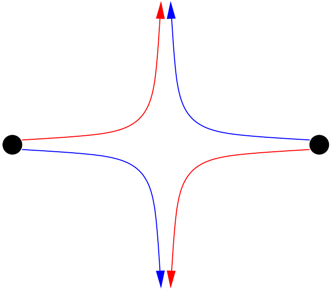
- All physical concepts should be defined via experiments. To define a physical concept is to design an experiment.



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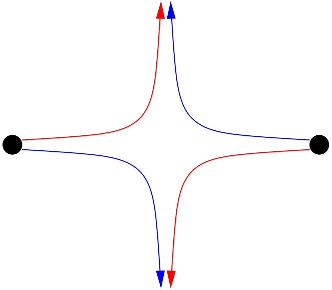
- Hopping operators
  $|\vec{i}, \vec{k}, \cdots\rangle = \hat{t}_{\vec{i},\vec{j}} |\vec{j}, \vec{k}, \cdots\rangle$
  Alegebra of $\hat{t}_{\vec{i},\vec{j}}$ determines statistics
  $[\hat{t}_{\vec{i},\vec{j}}, \hat{t}_{\vec{k},\vec{l}}] = 0, \quad \vec{i}, \vec{j} \neq \vec{k}, \vec{l}$, ...

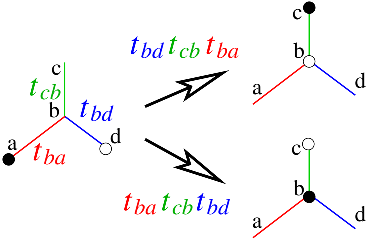


Statistics from the algebra of the hopping operators

- The statistics is determined by the algebra of the particle hopping operators Levin-Wen cond-mat/0302460:

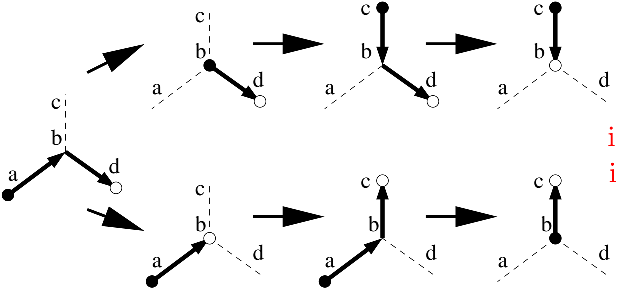
$$\hat{t}_{bd}\hat{t}_{cb}\hat{t}_{ba} = e^{i\theta_{\text{sta}}}\hat{t}_{ba}\hat{t}_{cb}\hat{t}_{bd}, \quad \hat{t}_{bd}\hat{t}_{cb}\hat{t}_{ba} = U_{\text{sta}}\hat{t}_{ba}\hat{t}_{cb}\hat{t}_{bd}$$

- Works for Abelian statistics and non-Abelian statistics



- Consistant with the Berry's phase picture: $\langle \vec{x}, \vec{y}|\vec{x}+\delta\vec{x}, \vec{y}\rangle = e^{i\delta\phi}$
The sum of $\delta\phi$'s for the above two paths may differ by θ_{sta}.

Robust non-Abelian geometric phase (pseudo non-Abelian statistics) for point-defect with fractionalized vector space



$|ad\rangle = |\text{even}\rangle$

$\mathrm{i}\eta_d\lambda_a|ad\rangle = |ad\rangle$

$\mathrm{i}\gamma_d\gamma_a|ad\rangle = |ad\rangle$

$$c^\dagger_{ad} = \frac{\lambda_a - \mathrm{i}\eta_d}{2} = \frac{\gamma_a - \mathrm{i}\gamma_d}{2}$$

- Compare "phases" $\langle\vec{x},\vec{y};\alpha|\vec{x}+\delta\vec{x},\vec{y};\beta\rangle = U_{\alpha\beta}$
- Parallel transport $|\vec{x},\vec{y};\alpha\rangle = |\vec{x}+\delta\vec{x},\vec{y};\alpha\rangle^0 = U^*_{\alpha\beta}|\vec{x}+\delta\vec{x},\vec{y};\beta\rangle$
- Path 1: $(|ad\rangle, c^\dagger_{ad}|ad\rangle) \to \mathrm{e}^{-\mathrm{i}\theta^{bd}_{ab}}(|bd\rangle, c^\dagger_{bd}|bd\rangle)$

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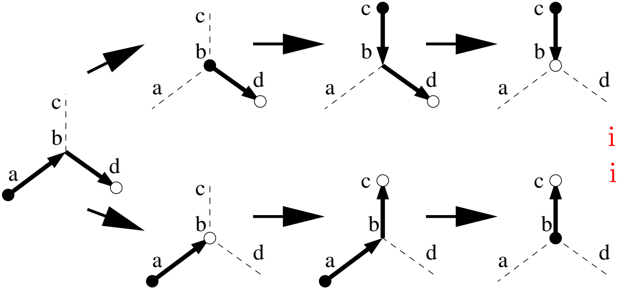

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  $\to \mathrm{e}^{\mathrm{i}(-\theta^{bd}_{ab}+\theta^{bd}_{bc})}(|cd\rangle, c^\dagger_{cd}|cd\rangle) \to \mathrm{e}^{\mathrm{i}(-\theta^{bd}_{ab}+\theta^{bd}_{bc}-\theta^{bc}_{bd})}(|cb\rangle, c^\dagger_{cb}|cb\rangle)$

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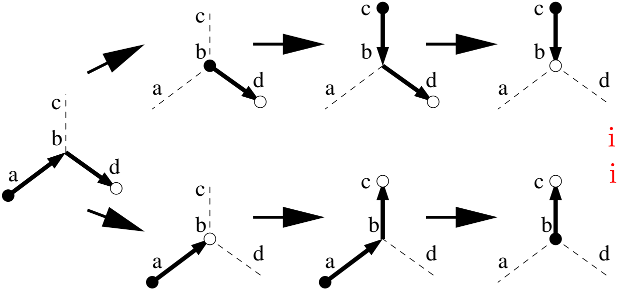

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- Path 2: $(|ad\rangle, c^\dagger_{ad}|ad\rangle) \to \mathrm{e}^{-\mathrm{i}\theta^{ab}_{bd}}(|ab\rangle, c^\dagger_{ab}|ab\rangle)$
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Robust non-Abelian geometric phase (pseudo non-Abelian statistics) for point-defect with fractionalized vector space

- The two paths differ by $\mathrm{e}^{\mathrm{i}(-\theta_{ab}^{bd}+\theta_{bc}^{bd}-\theta_{bd}^{bc})-(-\theta_{ab}^{cb}+\theta_{bc}^{ab}-\theta_{bd}^{ab})}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

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  But the above result is **wrong**. This is because the exchange also change the $+p$-wave SC to a $-p$-wave SC. We need continuously deform the $+p$-wave SC to the $-p$-wave SC to complete the loop.

- $-p$-wave SC to the $+p$-wave SC through the $U(1)$ rotation: $c_- \to \mathrm{i}\,c_- = c_+, \quad \to \quad \mathrm{i}(\lambda^- + \mathrm{i}\eta^-) = \lambda^+ + \mathrm{i}\eta^+$ or $\lambda^- = \eta^+,\ -\eta^- = \lambda^+$.

- **compare** $|bc\rangle_-$ and $|cb\rangle_+$:
  $\mathrm{i}\eta^-_c\lambda^-_b|bc\rangle_- = |bc\rangle_-,\ \mathrm{i}\eta^+_b\lambda^+_c|bc\rangle_- = |cb\rangle_+,\ \mathrm{i}\eta^-_c\lambda^-_b = \mathrm{i}\eta^+_b\lambda^+_c.$
  $\to |bc\rangle_- = \mathrm{e}^{\mathrm{i}\theta}|cb\rangle_+$

- **compare** $c^\dagger_{bc}|bc\rangle_-$ and $c^\dagger_{cb}|cb\rangle_+$:
  $c^\dagger_{bc} = \frac{\lambda^-_b - \mathrm{i}\eta^-_c}{2} = \frac{\eta^+_b + \mathrm{i}\lambda^+_c}{2} = \mathrm{i}\,\frac{\lambda^+_c - \mathrm{i}\eta^+_b}{2} = \mathrm{i}\,c^\dagger_{cb}$
  $\to c^\dagger_{bc}|bc\rangle_- = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}c^\dagger_{cb}|cb\rangle_+$

Alicea, Oreg, Refael, von Oppen, Fisher; arXiv:1006.4395

Pseudo non-Abelian statistics

- The two paths differ by $\mathrm{e}^{\mathrm{i}(-\theta_{ab}^{bd}+\theta_{bc}^{bd}-\theta_{bd}^{bc})-(-\theta_{ab}^{cb}+\theta_{bc}^{ab}-\theta_{bd}^{ab})}\begin{pmatrix}1 & 0\\ 0 & \mathrm{i}\end{pmatrix}$

- The difference is not a pure $U(1)$ phase $\to$ **non-Abelian**
- The $U(1)$ phase is not universal $\to$ **pseudo** non-Abelian statistics
  (The string is observable. String-string interaction at the corner.)

- Another representation of the non-Abelian geometric phase

$$U = \mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{\frac{\pi}{4}\gamma_b\gamma_c}$$