Particle-hole symmetry and electromagnetic response of a half-filled Landau level

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We derive exact physical consequences of particle-hole symmetry of the $\nu = 1/2$ state of electrons in a strong magnetic field. We show that if the symmetry is not spontaneously broken, the Hall conductivity and the susceptibility satisfy an exact relationship, valid at any wave number and frequencies much below the cyclotron frequency. The relationship holds for clean systems and also for systems with statistically particle-hole symmetric disorder. We work out the constraints this relationship imposes on the theory of the Dirac composite fermion. We also argue that that the exact relationship is violated in the Halperin-Lee-Read (HLR) field theory and present an explicit calculation within a Galilean invariant mean-field approximation to the HLR theory to illustrate the breakdown.

I. INTRODUCTION

The $\nu = 1/2$ state of fermions in a strong magnetic field [1] is one of the most important states in quantum Hall physics. It is at and near $\nu = 1/2$ that sharp predictions of the composite fermion (CF) theory were made and successfully compared with experiments [2–4]. The $\nu = 1/2$ state is also the "parent" of the incompressible states in the Jain sequences $\nu = n/(2n \pm 1)$ [5] and of the Moore-Read (MR) state [6].

An important aspect of the $\nu = 1/2$ quantum Hall system is that the Hamiltonian is approximately symmetric under particle-hole (PH) conjugation [7] in the spin-polarized lowest Landau level. In fact, this symmetry is exact for models with no Landau-level mixing, complete spin polarization, and only two-body interactions. As an exact symmetry valid at all length scales, it must be present in any low-energy, long-distance description. At the same time, it is well known that the standard Halperin-Lee-Read (HLR) field theory [1] does not have any explicitly manifest symmetry that can be identified with the particle-hole symmetry. To address this issue, an alternative field theory with explicit particle-hole symmetry has been proposed for the $\nu = 1/2$ state: the Dirac composite fermion theory [8]. In this theory, the composite fermion is a Dirac fermion, characterized by a Berry phase of π around the Fermi line. Numerical simulations have confirmed this Berry phase [9].

The Dirac composite fermion theory solves an old puzzle with the theory of the composite fermion. Particle-hole symmetry implies that, when impurities are particle-hole symmetric (in the statistical sense), the Hall conductivity σ_{xy} is exactly $\frac{1}{2}(e^2/h)$. In the HLR theory, this condition translates into a Hall conductivity $-\frac{1}{2}(e^2/h)$ of the composite fermion [10]. This seems to contradict the fact that the composite fermion feels zero average magnetic field. If one takes the Hall conductivity of the composite fermions to be zero, the electron Hall conductivity is strictly less than $\frac{1}{2}(e^2/h)$, signaling the breakdown of particle-hole symmetry in the HLR theory. Similarly, it was concluded in Ref. [11] that thermoelectric transport in the HLR theory is also inconsistent with particle-hole symmetry. However, a more recent analysis shows that in a certain regime the CF Hall conductivity may actually be $-\frac{1}{2}(e^2/h)$ [12], raising the question of whether the HLR theory is secretly particle-hole symmetric.

In this paper we address the last question by deriving a consequence of particle-hole symmetry for transport at nonzero wave number and frequency. We show that if the $\nu = 1/2$ state coincides with its own PH conjugate, then there exists an exact relationship between two linear response functions, both regarded as functions of wave number q and frequency ω: the Hall conductivity and the susceptibility. The relationship holds in the presence of particle-hole symmetric disorder but remains nontrivial in the absence of disorder. Any low-energy effective theory of the half-filled Landau level must reproduce, within its regime of validity, this exact relationship. We then argue that the relationship can be easily accommodated by the Dirac composite fermion theory but is impossible to satisfy within the HLR theory. This rules out the possibility that the HLR theory has a hidden particle-hole symmetry.[1]

To write down the exact relationship, we first define the two response functions. Consider a small perturbation of the scalar potential A_0. Let $\delta\rho$ be the perturbation of the charge density and j^i be the current (more precisely, the $g = 2$ electromagnetic current that remains finite in the lowest-Landau-level (LLL) limit $m \to 0$ [13]). The linear response of the system to external A_0 is characterized by the susceptibility $\chi(\omega,\mathbf{q})$ and Hall conductivity $\sigma_H(\omega,\mathbf{q})$,

$$\delta\rho = \chi(\omega,\mathbf{q})A_0, \tag{1}$$

$$j^i = \left[\frac{\omega q^i}{q^2}\chi(\omega,\mathbf{q}) + i\sigma_H(\omega,\mathbf{q})\epsilon^{ij}q_j\right]A_0. \tag{2}$$

Alternatively, the Hall conductivity can also be defined through the density response to the perturbation of the magnetic field (the Středa formula) when A_0 is left unperturbed,

$$\delta\rho = \sigma_H \delta B. \tag{3}$$

Our result is that in the LLL limit and assuming that PH symmetry is not spontaneously broken,

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[1]By the HLR theory we have in mind an effective field theory of a Fermi surface of composite fermions coupled to a Chern-Simons gauge field. The composite fermions have nontrivial Landau parameters but have zero Berry phase.

the two response functions satisfy an exact linear relationship,

$$\sigma_H(\omega,q) + \frac{1}{4\pi}\tilde{V}(q)\chi(\omega,q) = \frac{1-e^{-q^2/2}}{2\pi q^2}. \quad (4)$$

In this paper we set $B = 1$, so the magnetic length $\ell_B = 1$, and also the CF Fermi momentum $p_F = 1$. The function $\tilde{V}(q)$ is fully determined by the electron-electron interaction potential $V(r)$ or its Fourier transform $V(q)$:

$$\tilde{V}(q) = \frac{2}{q^2}(1-e^{-q^2/2})V(q)$$
$$- \frac{2}{q^2}\int_0^\infty dp\, p e^{-p^2/2}[1-J_0(pq)]V(p). \quad (5)$$

In particular, for Coulomb interaction $V(r) = e^2/(\epsilon_0 r)$, $\tilde{V}(q)$ can be computed exactly in closed form,

$$\tilde{V}(q) = \frac{4\pi e^2}{\epsilon_0}\left\{\frac{1-e^{-q^2/2}}{q^3} - \frac{1}{q^2}\sqrt{\frac{\pi}{2}}\left[1-e^{-q^2/4}I_0\left(\frac{q^2}{4}\right)\right]\right\}. \quad (6)$$

Equation (4) is valid for arbitrary wave numbers and for frequencies much smaller than the cyclotron frequency. Equation (4) also holds in the presence of PH symmetric impurities and at finite temperature, provided that the thermal ensemble is PH symmetric. This should be the case at least at sufficiently high temperature, even if the potential $V(r)$ is such that the ground state breaks PH symmetry spontaneously.

Equation (4) provides a nontrivial relationship between two otherwise unrelated response functions. One recalls that in a Fermi liquid the response functions at finite ω have singularities related to particle-hole pairs and other physical excitations. Since the right-hand side of Eq. (4) does not depend on ω at all, the singularities in σ_H and χ cancel exactly on the left-hand side of Eq. (4).

If a state (denoted as A) does not coincide with its particle-hole conjugate (B), one can generalize Eq. (4) to relate the Hall conductivities of the two states, σ_H^A and σ_H^B, with the susceptibility χ (which is the same in the two states),

$$\frac{\sigma_H^A(\omega,q) + \sigma_H^B(\omega,q)}{2} + \frac{1}{4\pi}\tilde{V}(q)\chi(\omega,q) = \frac{1-e^{-q^2/2}}{2\pi q^2}. \quad (7)$$

II. PROOF OF THE EXACT RELATION

A. Outline of the main argument

We start from a microscopic theory describing spin-polarized electrons of mass m, with gyromagnetic factor $g = 2$, in an external magnetic field, interacting through a two-body potential V:

$$H = \int d\mathbf{x}\left[\frac{1}{2m}|(\partial_i - iA_i)\psi|^2 - \left(\frac{B}{2m} + A_0\right)\psi^\dagger\psi\right]$$
$$+ \frac{1}{2}\int d\mathbf{x}\,d\mathbf{y}\,V(|\mathbf{x}-\mathbf{y}|)\psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{y})\psi(\mathbf{y})\psi(\mathbf{x}). \quad (8)$$

The linear electromagnetic response of the system is given by $\delta j^\mu(\omega,\mathbf{q}) = \Pi^{\mu\nu}(\omega,\mathbf{q})A_\nu(\omega,\mathbf{q})$, where

$$\Pi^{00} = \chi, \quad (9)$$

$$\Pi^{0i} = \frac{\omega q^i}{q^2}\chi - i\epsilon^{ij}q_j\sigma_H, \quad (10)$$

$$\Pi^{i0} = \frac{\omega q^i}{q^2}\chi + i\epsilon^{ij}q_j\sigma_H. \quad (11)$$

The same function σ_H governs the current response to the scalar potential and the density response to perturbations of the magnetic field. One can thus find σ_H by calculating the density of the ground state in nonuniform magnetic fields (the Středa formula).

In constant B, PH conjugation flips the sign of A_0 [14,15]. We will show that in a nonuniform magnetic field, the action of PH conjugation is more nontrivial: it flips the sign of A_0 and simultaneously shifts it,

$$\text{PH}: A_0 \to -A_0 + \delta_{\text{PH}}A_0. \quad (12)$$

Here $\delta_{\text{PH}}A_0$ is a functional of B which vanishes when B is uniform. For small perturbations of B, $B = B_0 + \delta B$, $\delta_{\text{PH}}A_0$ is linear in δB. For later convenience, we parametrize the perturbation of the magnetic field through a "Kähler potential" K: $\delta B = \nabla^2 K$ [16]. Then $\delta_{\text{PH}}A_0$ is linear in K, i.e.,

$$\delta_{\text{PH}}A_0(\mathbf{x}) = -2\int d\mathbf{y}\,F(\mathbf{x}-\mathbf{y})K(\mathbf{y}) \equiv -2F*K(\mathbf{x}), \quad (13)$$

with some kernel F.

Another way to write Eq. (12) is to define

$$\tilde{A}_0 = A_0 - \frac{1}{2}\delta_{\text{PH}}A_0[B]. \quad (14)$$

Then particle-hole conjugation simply flips the sign of $\tilde{A}_0$. In particular, if $\tilde{A}_0 = 0$ or

$$A_0 = \frac{1}{2}\delta_{\text{PH}}A_0[B] = -F*K, \quad (15)$$

then the Hamiltonian is particle-hole symmetric.

Accepting the transformation law (12), the argument leading to Eq. (4) goes as follows. First, if the state under consideration is PH symmetric in uniform magnetic field, then it will remain PH symmetric under the small perturbation (15). This means that the particle number density in this state is exactly half of the density of the full Landau level in the (nonuniform) magnetic field. But the density is given by the linear response formula

$$\delta\rho = \Pi^{00}\delta A_0 + \Pi^{0i}\delta A_i = \chi\delta A_0 + \sigma_H\delta B$$
$$= \left(\sigma_H + \frac{F}{q^2}\chi\right)\delta B(q). \quad (16)$$

On the other hand, the density of the full Landau level is computable to linear order in perturbations (see Ref. [17] and below),

$$\delta\rho_{\nu=1}(q) = \frac{1-e^{-q^2/2}}{\pi q^2}\delta B(q). \quad (17)$$

This leads to Eq. (4), with

$$\tilde{V}(q) = \frac{4\pi}{q^2} F(q). \tag{18}$$

Note that the coefficient on the right-hand side of Eq. (17) is simply the Hall conductivity of the full Landau level,

$$\sigma_H^{\nu=1}(q) = \frac{1 - e^{-q^2/2}}{\pi q^2}. \tag{19}$$

If one is dealing with a state which is not its own particle-hole conjugate, repeating the above procedure and remembering that the susceptibility χ is invariant under particle-hole symmetry, one can derive Eq. (7).

B. Particle-hole conjugation in a nonuniform magnetic field

To derive (12), we can limit ourselves to perturbations which are translationally invariant in one Cartesian coordinate, chosen to be y. In the Landau gauge $A_x = 0$, $A_y = x + K'(x)$, the normalized LLL orbitals (which are degenerate with zero energy [18]) have the form $\psi_k(x,y) = L_y^{-1/2} \psi_k(x) e^{iky}$, where L_y is the size of the box along the y direction, and to linear order in K

$$\psi_k(x) = \frac{1}{\pi^{1/4}} e^{-(x-k)^2/2} [1 - K(x) + \bar{K}(k)]. \tag{20}$$

Here the function $\bar{K}$ is obtained by smearing K by a Gaussian,

$$\bar{K}(k) = \frac{1}{\sqrt{\pi}} \int dx \, e^{-(x-k)^2} K(x), \tag{21}$$

which implies that the Fourier transforms of K and $\bar{K}$ are related by $\bar{K}(q) = e^{-q^2/4} K(q)$. (We use k, l, etc., for momenta along the y direction and p, q, etc., for momenta along the x direction).

The density of the $\nu = 1$ state in inhomogeneous magnetic field can be computed exactly to linear order in perturbation,

$$\rho(x) = \sum_k |\psi_k(x)|^2$$

$$= \int \frac{dk}{2\pi} \frac{1}{\sqrt{\pi}} e^{-(x-k)^2} [1 - 2K(x) + 2\bar{K}(k)]$$

$$= \frac{1}{2\pi} [1 - 2K(x) + 2\bar{\bar{K}}(x)], \tag{22}$$

where $\bar{\bar{K}}$ is the function K smeared [as in Eq. (21)] twice. In momentum space,

$$\delta\rho(q) = -\frac{K(q)}{\pi} (1 - e^{-q^2/2}), \tag{23}$$

which coincides with Eq. (17).

In the $m \to 0$ limit, the Hamiltonian can be projected to the LLL. Assuming that A_0 is also translationally invariant along the y direction, the projected Hamiltonian is

$$H = \sum_k U_k c_k^\dagger c_k + \frac{1}{2} \sum_{klmn} V_{klmn} c_k^\dagger c_l^\dagger c_m c_n, \tag{24}$$

with

$$U_k = -\int dx \, A_0(x) \psi_k^2(x), \tag{25}$$

$$V_{klmn} = \int dx_1 \, dx_2 \, dy \, V(x_1 - x_2, y) \psi_k(x_1) \psi_l(x_2)$$

$$\times \psi_m(x_2) \psi_n(x_1) \frac{1}{L_y} e^{-i(k-n)y} \delta_{k+l,m+n}. \tag{26}$$

Note that V_{klmn} is real and $V_{klmn} = V_{nmlk}$.

We will assume A_0 is of the same smallness as K; therefore we can replace the wave functions in Eq. (25) by the unperturbed wave functions at $K = 0$. We find

$$U_k = -\tilde{A}_0(k) = -\int \frac{dq}{2\pi} e^{iqk - q^2/4} A_0(q), \tag{27}$$

where $A_0(q)$ is the Fourier transform of $A_0(x)$.

We now perform PH conjugation of the Hamiltonian: $c_k \to c_k^\dagger$.[2] After normal ordering, one finds that the two-body potential remains unchanged, but the one-body potential is modified,

$$U_k \to -U_k + \sum_l (-V_{kllk} + V_{klkl}). \tag{28}$$

We will evaluate explicitly the sum by inserting wave functions (20) into the definition of V_{klmn}. But even without calculating, since we know that the result must be linear in $K(q)$ and respect translational invariance along the y direction, we can write it as

$$\sum_l (-V_{kllk} + V_{klkl}) = 2 \int \frac{dq}{2\pi} e^{iqk - q^2/4} F(q) K(q). \tag{29}$$

From Eqs. (27), (28), and (29), we conclude under PH conjugation

$$A_0(q) \to -A_0(q) - 2F(q)K(q), \tag{30}$$

and so $F(q)$ is the Fourier transform of the function $F(\mathbf{x})$ introduced in Eq. (13).

We now split $F(q) = F^{\mathrm{H}}(q) + F^{\mathrm{ex}}(q)$, where

$$-\sum_l V_{kllk} = 2 \int \frac{dq}{2\pi} e^{iqk - q^2/4} F^{\mathrm{H}}(q) K(q), \tag{31}$$

$$\sum_l V_{klkl} = 2 \int \frac{dq}{2\pi} e^{iqk - q^2/4} F^{\mathrm{ex}}(q) K(q), \tag{32}$$

and compute F^{H} and F^{ex} separately. The calculation is straightforward but somewhat tedious; readers who are not interested in the details can skip to Eqs. (38) and (47). First,

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[2]The PH conjugation used here is a *unitary* transformation which is a product of the antiunitary particle-hole conjugation of Refs. [14,15], spatial reflection (P) $y \to -y$, and time reversal T. The product PT transforms wave functions as $\Psi(x_i, y_i) \to \Psi^*(x_i, -y_i)$ and seems to be a symmetry of all quantum Hall states proposed so far, although, strictly speaking, there seems to be no reason it could not be spontaneously broken. For PT symmetric states, invariance under the unitary PH conjugation implies invariance under the antiunitary PH conjugation, and vice versa.

for the Hartree term F^{H},

$$
-\sum_l V_{kllk} = -\frac{1}{L_y}\sum_l \int dx_1\, dx_2\, dy\, V(x_1-x_2,y)
$$

$$
\times \psi_k^2(x_1)\psi_l^2(x_2) \tag{33}
$$

$$
= -\int dx_1\, dx_2\, \frac{dl}{2\pi}\, V_1(x_1-x_2)\psi_k^2(x_1)\psi_l^2(x_2), \tag{34}
$$

where $V_1(x) = \int dy\, V(x,y)$. Corrections of order $O(K)$ appear in both $\psi_k(x_1)$ and $\psi_l(x_2)$, but it is easy to see that the only nontrivial $O(K)$ contribution comes from $\psi_l(x_2)$. We thus have

$$
-\sum_l V_{kllk} = -\int dx_1\, dx_2\, \frac{dl}{2\pi}\, V_1(x_1-x_2)
$$

$$
\times \frac{1}{\pi} e^{-(x_1-k)^2 - (x_2-l)^2} 2[-K(x_2) + \bar{K}(l)]. \tag{35}
$$

Integration over l yields

$$
\frac{1}{\pi}\int dx_1\, dx_2\, V_1(x_1-x_2)\frac{1}{\sqrt{\pi}} e^{-(x_1-k)^2}[K(x_2) - \bar{\bar{K}}(x_2)]. \tag{36}
$$

Rewriting in momentum-space representation, one finds

$$
-\sum_l V_{kllk} = \frac{1}{\pi}\int \frac{dq}{2\pi}\, e^{-q^2/4} e^{iqk} V(q)(1 - e^{-q^2/2})K(q), \tag{37}
$$

which means

$$
F^{\mathrm{H}}(q) = \frac{1 - e^{-q^2/2}}{2\pi} V(q). \tag{38}
$$

Now let us turn to the exchange contribution,

$$
\sum_l V_{klkl} = \int dx_1\, dx_2\, dy\, \frac{dl}{2\pi}\, \psi_k(x_1)\psi_l(x_2)\psi_k(x_2)
$$

$$
\times \psi_l(x_1) e^{-i(k-l)y} V(x_1-x_2,y). \tag{39}
$$

Expanding the wave functions to linear powers in K using Eq. (20), we get

$$
\sum_l V_{klkl} = \frac{2}{\pi}\int dx_1\, dx_2\, dy
$$

$$
\times \frac{dl}{2\pi} e^{-\frac{1}{2}[(x_1-k)^2 + (x_2-l)^2 + (x_2-k)^2 + (x_1-l)^2] - i(k-l)y}
$$

$$
\times V(x_1-x_2,y)[-K(x_1) - K(x_2) + \bar{K}(k) + \bar{K}(l)]. \tag{40}
$$

To evaluate this integral, we represent K, $\bar{K}$, and the potential V in Fourier components,

$$
K(x) = \int \frac{dq}{2\pi} e^{iqx} K(q), \quad \bar{K}(k) = \int \frac{dq}{2\pi} e^{iqk - q^2/4} K(q), \tag{41}
$$

$$
V(x,y) = \int \frac{dp_x\, dp_y}{(2\pi)^2} e^{i(p_x x + p_y y)} V(p),
$$

$$
p \equiv \sqrt{p_x^2 + p_y^2}. \tag{42}
$$

After integrating over y and l, we obtain

$$
\frac{2}{\pi}\int \frac{dq}{2\pi}\frac{dp_x\, dp_y}{(2\pi)^2} V(p)K(q)\int dx_1\, dx_2\, e^{S_0}
$$

$$
\times (-e^{iqx_1} - e^{iqx_2} + e^{iqk - q^2/4} + e^{iq(k-p_y) - q^2/4}), \tag{43}
$$

where

$$
S_0 = -\frac{1}{2}[(x_1-k)^2 + (x_2-k+p_y)^2 + (x_2-k)^2
$$

$$
+ (x_1-k+p_y)^2] + ip_x(x_1-x_2). \tag{44}
$$

The integral over x_1, x_2 is a Gaussian integral which can be evaluated exactly. We get

$$
2\int \frac{dq}{2\pi}\frac{dp_x\, dp_y}{(2\pi)^2} V(p)K(q)e^{-p^2/2}e^{ikq - q^2/4}
$$

$$
\times (-e^{-(p_x + ip_y)q/2} - e^{(p_x - ip_y)q/2} + 1 + e^{-ip_y q}). \tag{45}
$$

Going to polar coordinates in the (p_x, p_y) plane and integrating over the angle, we find

$$
\sum_l V_{klkl} = -2\int \frac{dq}{2\pi}\int_0^\infty \frac{dp}{2\pi}\, pV(p)K(q)e^{-p^2}e^{ikq - q^2/4}
$$

$$
\times [1 - J_0(pq)], \tag{46}
$$

which implies

$$
F^{\mathrm{ex}}(q) = -\int_0^\infty \frac{dp}{2\pi}\, pe^{-p^2/2}[1 - J_0(pq)]V(p). \tag{47}
$$

Summing up F^{H} and F^{ex} and using Eq. (18), we find Eq. (5). The exact result (4) is now proven.

Later, we will need the Taylor expansion of $F^{\mathrm{ex}}(q)$ over q^2,

$$
F^{\mathrm{ex}}(q) = \sum_{n=1}^\infty \frac{(-1)^n}{n!}\left(\frac{q^2}{2}\right)^n V_n, \tag{48a}
$$

$$
V_n = \frac{1}{2^n n!}\int_0^\infty \frac{dp}{2\pi}\, p^{2n+1} e^{-p^2/2} V(p)
$$

$$
= \int_0^\infty dr\, r L_n\left(\frac{r^2}{2}\right) e^{-r^2/2} V(r). \tag{48b}
$$

In contrast to $\tilde{V}^{\mathrm{H}}(q) = 4\pi q^{-2} F^{\mathrm{H}}(q)$, which inherits the singularity at small q of the potential (for example, of the Coulomb potential), $\tilde{V}^{\mathrm{ex}}(q) = 4\pi q^{-2} F^{\mathrm{ex}}(q)$ is regular at $q = 0$ for reasonably behaving potentials.

For the Coulomb potential $V(r) = e^2/(\epsilon_0 r)$, using the formulas above, the function $F(q)$ can be evaluated exactly to be the one given by Eqs. (6) and (18).

C. Discussion

Our formula (4) does not determine σ_H and χ separately and only fixes a linear combination of the two. However, even this limited statement can give very interesting results, which, in principle, should be verifiable by numerical simulation of the half-filled Landau level.

First, we notice that for a purely repulsive potential [$V(r) > 0$ for all r], $F(q)$ vanishes at some value of q. In fact, at $q \to 0$, one can show that

$$
F(0) = \frac{q^2}{2}\int dr\, r V(r)\left[1 - e^{-r^2/2}\left(1 - \frac{r^2}{2}\right)\right] \tag{49}
$$

and hence is positive, while for $q \to \infty$ the exchange part F^{ex} dominates, and according to Eq. (47), it approaches a negative constant, which can be shown to be $F(\infty) = -\int_0^\infty dr\, e^{-r^2/2} V(r)$. At q where $F(q) = 0$, the Hall conductivity is exactly determined by Eq. (4). For example, for the Coulomb interaction $F(q)$ vanishes at $q = q_0 \approx 1.4197\ell_B^{-1}$. The Hall conductivity σ_H at this wave number can be predicted to be $0.3150\, e^2/h$ and is independent of frequency.

Another interesting value of q is $q = 2$, corresponding to the $2k_F$ singularity in the response functions. Our result implies that although each function χ and σ_H may show Friedel-type singular behavior at this wave number, the linear combination

$$\sigma_H + \frac{1}{4} F(2)\chi, \qquad F(2) \approx -0.2372 \frac{e^2}{\epsilon_0} \qquad (50)$$

should be free of all singularities for all ω.

Finally, in the limit $q \to 0$, $\omega \to 0$, $q/\omega \to 0$, the density-density correlation function is expected to behave as $\chi(q) \sim q^4$ (the q^2 term in χ is fixed by Kohn's theorem and vanishes in the LLL limit $m \to 0$). For any potential which is less singular than $1/q^2$ at small q, the q^2 correction to the ac Hall conductivity is then completely fixed,

$$\sigma_H(\omega,q) = \frac{1}{4\pi}\left(1 - \frac{q^2}{4}\right) + o(q^2), \qquad v_F q \ll \omega. \quad (51)$$

To order q^2, this is exactly one half of σ_H for a filled Landau level.

III. CONSTRAINTS ON THE DIRAC COMPOSITE FERMION THEORY

We now show that the exact relationship can be accommodated by the Dirac composite fermion theory. To illustrate how such a theory can be constructed, we start with the simplest model Lagrangian and then improve it.

Let us start from the action providing the dual description of Dirac fermions [19–23]

$$S = i\bar\psi\gamma^\mu(\partial_\mu - ia_\mu)\psi - \frac{1}{4\pi} A da, \quad (52)$$

where $A da \equiv \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda$ and, for convenience, here and below integration over space-time is implied in the action. As we are interested only in local response, we ignore issues related to the parity anomaly and the fractional coefficients of the CS terms. This action has a CP symmetry, under which $x \to x$, $y \to -y$, and

$$A_0 \to -A_0, \quad A_x \to -A_x, \quad A_y \to A_y, \quad a_0 \to a_0,$$
$$a_x \to a_x, \quad a_y \to -a_y. \quad (53)$$

Under CP the Hall conductivity flips sign; therefore any CP symmetric state must have zero Hall conductivity.

Consider now the half-filled Landau level with a long-range electron-electron interaction. At minimum, one has to add two more terms to the action

$$S = S_{\text{CF}}(\psi,a) - \frac{1}{4\pi} A da + \frac{1}{8\pi} A dA - \frac{1}{2} V\delta\rho^2, \quad (54)$$

with

$$\delta\rho = \frac{1}{4\pi}(\delta B - b) \quad (55)$$

being the density perturbation, and we use the shorthand notation $V\delta\rho^2 \equiv \int d\mathbf{x}\, d\mathbf{y}\, V(\mathbf{x} - \mathbf{y})\delta\rho(\mathbf{x})\delta\rho(\mathbf{y})$. The two terms added are the AdA term, which shifts the particle-hole symmetric value of σ_H from 0 to $\frac{1}{2}(e^2/h)$, and the term containing V, which describes the long-range density-density interaction. The additional terms violate the CP symmetry (53). However, if one separates out the Chern-Simons term AdA, which depends exclusively on the background field,

$$S = S_{\text{ph}}(\psi,\tilde A,a) + \frac{1}{8\pi} A dA, \quad (56)$$

the remaining part S_{ph} can be put into the form

$$S_{\text{ph}} = S(\psi,a) - \frac{1}{4\pi}\tilde A da - \frac{1}{2}\frac{1}{(4\pi)^2}(V\delta B^2 + Vb^2), \quad (57)$$

where

$$\tilde A_0 = A_0 - \frac{V * \delta B}{4\pi}, \qquad \tilde A_i = A_i. \quad (58)$$

Now one can see that S_{ph} has a modified CP symmetry, under which

$$\tilde A_0 \to -\tilde A_0, \quad A_x \to -A_x, \quad A_y \to A_y,$$
$$a_0 \to a_0, \quad a_x \to a_x, \quad a_y \to -a_y. \quad (59)$$

To derive the consequences of this symmetry for transport, imagine that we integrate out the dynamical ψ and a. We now have an effective action for A_μ which is constrained by the symmetry (59). To the quadratic order

$$S_{\text{eff}}[A] = \frac{\chi}{2}\tilde A_0^2 - \frac{\chi_M}{\delta}B^2 + \frac{1}{8\pi} A dA, \quad (60)$$

where χ is the susceptibility and χ_M is a coefficient related to the magnetic susceptibility. From this effective action it is straightforward to derive

$$\sigma_H + \frac{V}{4\pi}\chi = \frac{1}{4\pi}. \quad (61)$$

This is similar to but not yet the exact relationship (4): instead of $\tilde V(q)$ we have only the leading part $V(q)$, and on the right-hand side is just a constant instead of the full function of q. However, it is easy to modify the action to reproduce correctly the exact relationship. Let $S_{\nu=1}[A]$ be the action describing the full lowest Landau level. This is a complicated functional of A, but to the quadratic level it is completely determined by the Hall conductivity (19),

$$S_{\nu=1}[A] = \frac{1}{4\pi}\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\nu - \frac{1}{8\pi}(\nabla \cdot \mathbf{E})\frac{1 - e^{\nabla^2/2} + \nabla^2/2}{\nabla^4/8}B. \quad (62)$$

With a bit of hindsight, consider the following action:

$$S = S_0(\psi,a,A) + \frac{1}{2}S_{\nu=1}[A] - \frac{V}{2}\left(\frac{\delta S_0}{\delta A_0} + \frac{1}{2}\frac{\delta S_{\nu=1}}{\delta A_0} - \rho_0\right)^2$$
$$+ \sum_{n=0}^\infty C_n \frac{\delta S_0}{\delta A_0}\nabla^{2n}\delta B, \quad (63)$$

where $S_0[\psi,a,A]$ is a local Lagrangian involving the Dirac composite fermion field ψ, the emergent gauge field a, and the external gauge field A (the Chern-Simons term $-\frac{1}{4\pi}Ada$ is included in S_{CF}). We assume for simplicity that S_0 is linear in A_0. Since the electron density is

$$\rho = \frac{\delta S_0}{\delta A_0} + \frac{1}{2}\frac{\delta S_{\nu=1}}{\delta A_0}, \qquad (64)$$

the third term on the right-hand side of Eq. (63) is the interaction energy.

In Eq. (63) we have included an infinite number of local interaction terms between the composite fermion charge density $\delta S_0/\delta A_0$ and the magnetic field. Since these terms are all local, nothing prevents them from arising in the low-energy effective theory. We now rewrite the action to the form

$$S = S_0(\psi,a,\tilde{A}) - \frac{V}{2}\left(\frac{\delta S_0}{\delta A_0}\right)^2 - \frac{V}{8}\left(\frac{\delta S_{\nu=1}}{\delta A_0} - \rho_0\right)^2$$
$$+ \frac{1}{2}S_{\nu=1}[A], \qquad (65)$$

where ρ_0 is the background density and

$$\tilde{A}_0 = A_0 - \frac{V}{2}\left(\frac{\delta S_{\nu=1}}{\delta A_0} - \rho_0\right) + \sum_{n=0}^{\infty}C_n\nabla^{2n}\delta B. \qquad (66)$$

To linear order,

$$\tilde{A}_0(q) = A_0(q) - \left[\frac{1}{4\pi}\frac{2}{q^2}(1 - e^{-q^2/2})V(q)\right.$$
$$\left. - \sum_{n=0}^{\infty}(-1)^nC_nq^{2n}\right]\delta B. \qquad (67)$$

Now if C_n are related to the coefficients V_n in Eq. (48b) by

$$C_n = \frac{V_{n+1}}{2^{n+1}(n+1)!}, \qquad (68)$$

then one recognizes the expression in the square brackets to be $\tilde{V}(q)$, so

$$\tilde{A}_0 = A_0 - \frac{\tilde{V}*\delta B}{4\pi}. \qquad (69)$$

Again, the action can be broken into two parts,

$$S = S_{\mathrm{ph}}[\psi,a,\tilde{A}] + \frac{1}{2}S_{\nu=1}[A], \qquad (70)$$

with the S_{ph} part having the symmetry (59). From this we find the exact relation (4). We conclude that if the action can be brought to the form (63), where S_0 depends linearly on A_0 and has the symmetry (53), then the exact relationship between σ_H and χ is guaranteed.

At this moment we do not know how to write the action (63) in a more compact and more natural form. We expect additional terms to appear if one goes to higher orders in δB. One should be able to write these terms down by repeating the calculations of Sec. II B beyond linear order in δB.

IV. CONCLUSION AND COMMENTS ON THE HLR THEORY

We have shown that particle-hole symmetry of the lowest Landau level leads to an exact relationship between the Hall conductivity and the susceptibility. The relationship is valid at all wave numbers and for all frequencies (much smaller than the cyclotron frequency, but the frequency can be in any relationship with the interaction energy) in the absence of disorder or in the presence of statistically particle-hole symmetric disorder. This relationship should be verifiable in numerical simulations. For the latter, the Hall conductivity can be found, e.g., by measuring the density in inhomogeneous magnetic field.

We have shown that the exact relationship requires that the action of the Dirac composite fermion theory contains an infinite number of local terms, with coefficients fixed by the two-body electron-electron potential. This seems to be related to a discrete symmetry and thus should not be viewed as fine-tuning.

The HLR field theory does not have an explicit particle-hole symmetry, and hence *a priori* it is not clear if it can be modified so that the exact relationship (4) holds for all ω and q. While a full analysis is still to be made, one problem can already be identified in the limit of small ω and q, $q/\omega \to 0$, where Eq. (51) predicts the value for the coefficient of the q^2 correction to the Hall conductivity. One might think that this coefficient can be tuned to any value by adding higher-derivative terms to the HLR Lagrangian. However, this is not true. It is known that Galilean invariance relates this coefficient to the (frequency-dependent) Hall viscosity η_H at zero q [24,25]. For the $g = 2$ current considered in this paper, the relationship reads

$$\sigma_H(\omega,q) = \frac{\nu}{2\pi} + \left(\eta_H(\omega) - \frac{\rho}{2}\right)q^2 + o(q^2), \qquad (71)$$

where ρ is the particle number density. The Hall viscosity is related to the average orbital spin per particle s by [26]

$$\eta_H = \frac{\rho s}{2}. \qquad (72)$$

Equation (51) then translates to an average orbital spin $s = \frac{1}{2}$ per particle in the Fermi-liquid state, exactly the same value as in the filled Landau level. On the other hand, in the HLR theory the composite fermion acquires orbital spin from flux attachment: each unit of flux increases the orbital spin of the composite particle by $\frac{1}{2}$ [27], making the composite fermion of the half-filled Landau level have orbital spin 1, which differs from the value we have just deduced from particle-hole symmetry and which would lead to a vanishing q^2 correction to σ_H.

One may wonder if it is possible to modify the HLR theory so that the composite fermion would have orbital spin $\frac{1}{2}$ instead of 1. However, such a modification would break the orbital spin of the gapped states derived from the HLR Fermi-liquid state: the Jain-sequence states $\nu = \frac{n}{2n+1}$ and $\nu = \frac{n+1}{2n+1}$ and the Pfaffian state. The $\nu = \frac{n}{2n+1}$ state is obtained from the HLR state by placing the composite fermions into n filled Landau levels, increasing the orbital spin per fermion by $\frac{n}{2}$ to a total of $1 + \frac{n}{2} = \frac{n+2}{2}$, consistent with the shift [28] $n + 2$ for this state. In the PH conjugate state $\nu = \frac{n+1}{2n+1}$ the CFs fill $n + 1$ LLs in a magnetic field of opposite sign, making the orbital

spin $1 - \frac{n+1}{2} = \frac{1-n}{2}$, again matching the shift $1 - n$. In the Moore-Read state the composite fermions form spin-1 Cooper pairs in which each fermion receives an additional orbital spin $\frac{1}{2}$ for a total of $1 + \frac{1}{2} = \frac{3}{2}$, matching the shift $\mathcal{S} = 3$. Any modification of the HLR theory that changes the orbital spin of the CF would destroy these agreements.

To further illustrate the difficulties of the HLR theory with particle-hole symmetry, in the Appendix we consider a concrete realization of this theory: an approximation scheme developed in Refs. [13,29] under the name "magnetized modified random-phase approximation" (MMRPA). We show that this approximation accurately reproduces the q^2 term in the dc Hall conductivity of the Jain-sequence states but fails to get the PH symmetric coefficient of this term in the ac Hall conductivity of the Fermi-liquid state.

APPENDIX: AN HLR MEAN-FIELD CALCULATION OF THE q^2 CORRECTION TO HALL CONDUCTIVITY

We start from the HLR effective theory,

$$S = S[\psi, A - a] + \frac{1}{8\pi} a da + \frac{V}{2(4\pi)^2} (\nabla \times a)^2, \quad (A1)$$

where $S[\psi, A - a]$ is the action describing the coupling of the nonrelativistic CF with the gauge field $(A - a)_\mu$. Integrating over ψ and keeping only the quadratic term, this action is replaced by $\frac{1}{2}(A - a)\Pi_{CF}(A - a)$, where $\Pi_{CF}^{\mu\nu}$ is the electromagnetic response function of the CF. Working in the temporal gauge, this matrix becomes a 2×2 matrix of the spatial components. In this gauge it is easy to integrate out a to get the response matrix of the electron Π,

$$\Pi = -Q(\Pi_{CF} + Q)^{-1} Q + Q, \quad (A2)$$

where the matrix Q is defined as

$$Q_{ij} = \frac{i\omega}{4\pi} \epsilon_{ij} - \frac{V(q)}{(4\pi)^2}(q^2 \delta_{ij} - q_i q_j). \quad (A3)$$

In the most naive random-phase approximation (RPA), Π_{CF} is assumed to be the response function of a free composite Fermi gas Π_{CF}^{free} computed assuming the CF has effective mass m_*. This approximation breaks Galilean invariance when m_* differs from the bare-electron mass m and hence cannot be used if one wants to take the LLL $m \to 0$. To correct the problem, a modification of the RPA was proposed in Ref. [29] which was later amended in Ref. [13] into a scheme called MMRPA. We reinterpret this modification as replacing $S[\psi, A - a]$ not by an action of a free fermion coupled to a_μ but by

$$S[\psi, A - a, v^i] = i\psi^\dagger D_t \psi - \frac{|D_i \psi|^2}{2m(1 + F_1)}$$
$$+ \frac{F_1}{1 + F_1} \frac{i}{2} v^i \psi^\dagger \overset{\leftrightarrow}{D}_i \psi + \frac{F_1}{1 + F_1} \frac{mv^2}{2} \psi^\dagger \psi, \quad (A4)$$

where v^i is a field to be integrated over and F_1 is a parameter. This action describes a theory of a fermion with an effective mass $m_* = m(1 + F_1)$ but is consistent with the Galilean invariance of the original electrons with mass m. One quick way to see that is to follow Ref. [30] to check, with the help of the equation of motion $\delta S/\delta v^i = 0$, that the momentum density T^{0i} is the particle number j^i times the bare-electron mass m:

$$\frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} D_i \psi + D_i \psi^\dagger \frac{\partial \mathcal{L}}{\partial(\partial_t \psi^\dagger)} = m \frac{\partial \mathcal{L}}{\partial A_i}. \quad (A5)$$

If one integrates out v^i first, one generates a contact interaction for the fermion. Hence one can interpret F_1 as the p-wave Landau parameter. The LLL limit corresponds to taking $F_1 \to \infty$, with $m_* = m(1 + F_1)$ fixed. In this limit the last term on the right-hand side of Eq. (A4) disappears.

The scheme developed in Refs. [13,29] is essentially the RPA in the theory (A1) and (A4). One integrates out ψ first and keep only terms quadratic in a and v,

$$\frac{1}{2}[A - a - (m_* - m)v]\Pi_{CF}^{free}[A - a - (m_* - m)v]$$
$$+ \frac{\rho}{2}(m_* - m)v^2, \quad (A6)$$

where ρ is the particle number density. Then one performs the Gaussian integration over v^i in the Gaussian approximation to obtain $\frac{1}{2}(A - a)\Pi_{CF}(A - a)$, where

$$\Pi_{CF} = \left(1 + \frac{m_* - m}{\rho}\Pi_{CF}^{free}\right)^{-1} \Pi_{CF}^{free}. \quad (A7)$$

This is exactly the prescription of the modified RPA [29]. The later improvement, magnetized modified RPA [13], simply declares Π obtained in Eq. (A2) to correspond to the $g = 2$ electromagnetic response.

From now on we choose **q** to point along the x axis, $\mathbf{q} = (q, 0)$. For the $\nu = \frac{n}{2n+1}$ state, the composite fermions live in a reduced magnetic field $b = \frac{B}{2n+1}$. In the small-ω, small-q limit, $\omega/q \to 0$, the response function of a free gas is [31]

$$\left(\Pi_{CF}^{free}\right)_{11} = \frac{n}{2\pi} \frac{m_*}{b} \omega^2 + O(\omega^2 q^2), \quad (A8)$$

$$\left(\Pi_{CF}^{free}\right)_{12} = i\omega \frac{n}{2\pi}\left(1 - \frac{3n}{4}(q\ell_b)^2\right) + O(\omega q^4), \quad (A9)$$

$$\left(\Pi_{CF}^{free}\right)_{21} = -\left(\Pi_{CF}^{free}\right)_{12}, \quad (A10)$$

$$\left(\Pi_{CF}^{free}\right)_{22} = -\frac{n^2}{2\pi}\frac{q^2}{m_*} + O(q^4), \quad (A11)$$

where $\ell_b^2 = 1/b = (2n + 1)\ell_B^2$. Inserting these formulas into Eqs. (A2) and (A7), one finds the q dependence of the dc Hall conductivity $\sigma_H = -i \lim_{\omega \to 0} \omega^{-1} \Pi_{12}$ to be

$$\sigma_H(q) = \frac{1}{2\pi}\frac{n}{2n+1}\left(1 + \frac{n}{4}q^2 \ell_B^2\right). \quad (A12)$$

This agrees with the general formula

$$\sigma_H(q) = \frac{\nu}{2\pi}\left(1 + \frac{\mathcal{S} - 2}{4}q^2\ell_B^2\right) \tag{A13}$$

when one substitutes in the latter the value of the shift of the Jain state, $\mathcal{S} = n + 2$.

Analogously, for the $\nu = \frac{n+1}{2n+1}$ state,

$$\left(\Pi_{\mathrm{CF}}^{\mathrm{free}}\right)_{11} = \frac{n+1}{2\pi}\frac{m_*}{|b|}\omega^2 + O(\omega^2 q^2), \tag{A14}$$

$$\left(\Pi_{\mathrm{CF}}^{\mathrm{free}}\right)_{12} = -i\omega\frac{n+1}{2\pi}\left(1 - \frac{3(n+1)}{4}(q\ell_b)^2\right) + O(\omega q^4), \tag{A15}$$

$$\left(\Pi_{\mathrm{CF}}^{\mathrm{free}}\right)_{21} = -\left(\Pi_{\mathrm{CF}}^{\mathrm{free}}\right)_{12}, \tag{A16}$$

$$\left(\Pi_{\mathrm{CF}}^{\mathrm{free}}\right)_{22} = -\frac{(n+1)^2}{2\pi}\frac{q^2}{m_*} + O(q^4), \tag{A17}$$

and after some calculation one obtains

$$\sigma_H = \frac{1}{2\pi}\frac{n+1}{2n+1}\left(1 - \frac{n+1}{4}q^2\ell_B^2\right), \tag{A18}$$

which matches the value of the shift $\mathcal{S} = -n + 1$.

Let us now turn to the Fermi-liquid state. For small ω and q but $\omega/v_F q \gg 1$, the response function of a Fermi gas is

$$\left(\Pi_{\mathrm{CF}}^{\mathrm{free}}\right)_{11} = -\frac{\rho}{m_*}\left(1 + \frac{3}{4}\frac{v_F^2 q^2}{\omega^2}\right), \tag{A19}$$

$$\left(\Pi_{\mathrm{CF}}^{\mathrm{free}}\right)_{12} = \left(\Pi_{\mathrm{CF}}^{\mathrm{free}}\right)_{21} = 0, \tag{A20}$$

$$\left(\Pi_{\mathrm{CF}}^{\mathrm{free}}\right)_{22} = -\frac{\rho}{m_*}\left(1 + \frac{1}{4}\frac{v_F^2 q^2}{\omega^2}\right). \tag{A21}$$

Applying formulas (A2) and (A7), we find

$$\sigma_H = \frac{1}{4\pi} + O(q^3), \tag{A22}$$

with vanishing coefficient in front of the q^2 term, which does not match the requirement of particle-hole symmetry (51).

One can also put the action (A1) and (A4) in curved space by using the metric tensor g^{ij} to sum over spatial indices and replacing the covariant derivative by $D_\mu\psi = (\partial_\mu - iA_\mu + ia_\mu - is\omega_\mu)$, with s being the orbital spin of the composite fermion and the spin connection ω_μ defined as in Ref. [24] so that it vanishes in flat space and any electromagnetic field. It happens that for $s = 1$ one does not need to introduce any higher-order term to make the theory consistent with the nonrelativistic general coordinate invariance of the original electron theory with $g = 2$ [32]. This explains why the MMRPA reproduces correctly the $O(q^2)$ correction to the Hall conductivity for the Jain sequence at $\nu = \frac{n}{2n\pm1}$. It also implies that for Jain states around other even-denominator filling fractions, e.g., $\nu = \frac{1}{4}$, the unmodified MMRPA will not give the correct q^2 correction to σ_H.

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