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THE KONDO EFFECT, CONFORMAL FIELD THEORY AND FUSION RULES

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In an earlier paper, a relationship was established between the Kondo effect and conformal field theory (CFT) with Kac–Moody (KM) symmetry. Here, we study the case of k degenerate bands, corresponding to a level-k KM algebra, in more detail. At the low-temperature fixed point, the Kac–Moody currents absorb the impurity spin (of size s). We hypothesize that the details of this process are governed by the standard KM fusion rules, the analogue of addition of angular momentum in CFT, and show that this leads to a Fermi liquid and to the expected $\pi/2$ phase shift when $s \geqslant k/2$.

1. Introduction

In an earlier paper [1] a relationship was developed between the Kondo effect [2] and conformal field theory (CFT) with Kac–Moody symmetry [3–5]. This relationship was conjectured independently, using a quite different approach, in ref. [6] and used to obtain critical exponents consistent with those derived exactly from the Bethe ansatz [7]. An apparently different relationship has also been proposed [8]. While all the physical results on the Kondo effect derived here and in ref. [1] (i.e. the spectrum and the Wilson ratio) have been obtained earlier by other methods [7, 9], the novel conceptual framework developed here will be shown to lead to new physical results in a later paper.

We consider here the general case with k degenerate bands of electrons interacting with a spin-s impurity [8, 9]. Working in the s-wave channel, and projecting the incoming and outgoing waves onto the positive and negative x-axis, we arrive at the standard hamiltonian density, involving one-dimensional left-moving fermions:

$$\mathscr{H} = i\psi^{i\alpha\dagger}\frac{\mathrm{d}\psi_{i\alpha}}{\mathrm{d}x} + \lambda\delta(x)\boldsymbol{S}\cdot\psi^{i\alpha\dagger}\frac{\boldsymbol{\sigma}_{\alpha}^{\beta}}{2}\psi_{i\beta}\,. \tag{1}$$

Here, α runs over 1, 2, labelling the electron spin and i runs over 1 to k labelling the k different bands. $\boldsymbol{S}$ is a quantum spin of size s, located at $x=0$. In the antiferromagnetic regime $\lambda > 0$ the model renormalizes to strong coupling at low temperatures.

The free theory can be written in a Sugawara form (see e.g. ref. [5]) involving only the left-moving charge U(1), spin SU(2), and flavor SU(k) currents:

$$J = \psi^{i\alpha\dagger}\psi_{i\alpha}, \qquad \boldsymbol{J} = \psi^{i\alpha\dagger}\frac{\boldsymbol{\sigma}_\alpha^\beta}{2}\psi_{i\beta}, \qquad J^A = \psi^{i\alpha\dagger}(T^A)_i^j\psi_{j\alpha}. \tag{2}$$

[Here the T^A's are generators of SU(k).] The SU(2) and SU(k) currents, $\boldsymbol{J}$ and J^A obey the Kac–Moody algebras at level k and 2 respectively. The free hamiltonian density takes the form

$$\mathscr{H}_0 = \frac{\pi v}{2k}JJ + \frac{2\pi v}{k+2}\boldsymbol{J}\cdot\boldsymbol{J} + \frac{2\pi v}{k+2}J^AJ^A = \mathscr{H}_0^{(Q)} + \mathscr{H}_0^{(s)} + \mathscr{H}_0^{(f)}. \tag{3}$$

States can be described by their charge (Q), spin (s) and flavor (f) quantum numbers. This Sugawara form is useful for studying the Kondo interaction because the latter only involves the SU(2) spin current:

$$\mathscr{H}_{\mathrm{K}} = \lambda\delta(x)\boldsymbol{S}\cdot\boldsymbol{J}. \tag{4}$$

It is now useful to study the universal low-energy spectrum of eq. (1) on a large circle of length $2L$. Retaining only the part of $\mathscr{H}$ involving the SU(2) currents, since the other parts are unaffected by the impurity, we have

$$\mathscr{H}^{(s)} = \mathscr{H}_0^{(s)} + \mathscr{H}_{\mathrm{K}} = \frac{\pi}{L}\sum_{n=-\infty}^{\infty}\left(\frac{v}{2+k}\boldsymbol{J}_n\cdot\boldsymbol{J}_{-n} + \frac{\lambda}{2\pi}\boldsymbol{S}\cdot\boldsymbol{J}_n\right), \tag{5a}$$

where $\boldsymbol{J}_n$ are Fourier modes of the current $\boldsymbol{J}(x,t)$, satisfying the usual KM commutation relations (see e.g. ref. [5])

$$\left[J_n^a, J_m^b\right] = i\epsilon^{abc}J_{n+m}^c + \tfrac{1}{2}kn\delta^{ab}\delta_{n+m,0} \tag{5b}$$

(ϵ^{abc} is the antisymmetric tensor and k is the KM level). At a special, positive value of λ, $\lambda^* = 4\pi v/(k+2)$, $\mathscr{H}^{(s)}$ can be rewritten in terms of new currents,

$$\tilde{\boldsymbol{J}}_n \equiv \boldsymbol{J}_n + \boldsymbol{S}, \tag{6}$$

which obey the same KM algebra eq. (5b),

$$\mathscr{H}^{(s)} = \frac{\pi}{L} \sum_{n=-\infty}^{\infty} \frac{v}{2+k} \tilde{\boldsymbol{J}}_n \cdot \tilde{\boldsymbol{J}}_{-n} . \tag{7}$$

The impurity spin has been "absorbed" by the conduction electrons. Thus we conclude that the hamiltonian $\mathscr{H} = \mathscr{H}_0^{(Q)} + \mathscr{H}^{(s)} + \mathscr{H}_0^{(f)}$ for the strong-coupling fixed point $\lambda = \lambda^*$ is the same as for the zero-coupling fixed point, the "correct" answer to describe the Fermi liquid fixed point à la Nozières [8]. The states can, as in the free case, again be described by charge, spin and flavor quantum numbers.

To confirm this picture it is desirable to study the spectrum of the theory at the strong coupling fixed point, eqs. (6, 7). Only the simplest case, $k = 2s = 1$ (no flavor degrees of freedom) was studied in detail in ref. [1]. There it was shown how the spectrum of the free fermion theory (3) is reproduced by appropriately combining the spectrum of the U(1) charge and SU(2) spin excitations in eq. (3): states in the charge $Q =$ odd sector occur only in combination with states of the spin $j = 0$ KM conformal tower and states in the charge $Q =$ even sector only with states in the spin $j = \frac{1}{2}$ KM tower, or, vice versa, depending on the boundary conditions. The two possibilities turn out to be precisely related by a $\pi/2$ phase shift of the free-fermion wave functions. It was hypothesized in ref. [1] that (for $k = 2s = 1$) the spectrum of the strong-coupling fixed point hamiltonian of eqs. (6) and (7) again consists of combinations of these two conformal towers but that, since $\boldsymbol{J}_n$ has been shifted by a $s = \frac{1}{2}$ operator, the two SU(2) spin conformal towers are interchanged. This leads to the $\pi/2$ phase shift of the free-fermion wave functions, which is believed to characterize the Fermi liquid fixed point [8].

The situation will clearly be more complicated for larger values of k and s, due to the extra flavor degrees of freedom. Now, even in the free theory (3), the relationship between the free-fermion spectrum and the spectra of charge U(1), spin SU(2) and flavor SU(k) excitations is non-trivial. The relationship involves the branching rules for the conformal embedding [10] of $\mathrm{SU(2)}_k \times \mathrm{SU}(k)_2$ in $\mathrm{SU}(2k)_1$ (the latter being related to free fermions by the standard non-abelian bosonization relation [3]). Here the subscript denotes the level [5] of the Kac–Moody representation (which is related to the Virasoro central charge). When the conduction electron spin density, $\boldsymbol{J}$, absorbs the impurity spin $\boldsymbol{S}$ as indicated in eq. (6) it is not obvious what will happen to the $k+1$ conformal towers of $\mathrm{SU(2)}_k$ which will be affected by the absorption process.

In this paper we make a precise hypothesis concerning how the conformal towers are rearranged upon this absorption. We suggest that the process is governed by the same fusion rules [4, 11, 12] that control the operator product expansion for fields in Kac–Moody conformal field theory. They can be thought of as the analogue of angular momentum coupling in CFT. This leads to a natural understanding of the "underscreening" phenomenon [8]: If the impurity spin s is

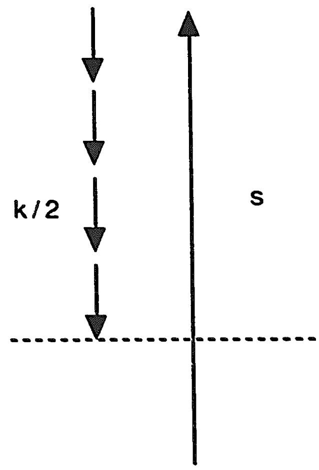

Fig. 1. Physical picture of "underscreening": The k conduction electrons absorb a part of magnitude $k/2$ of the impurity spin $s \geqslant k/2$. An impurity spin of magnitude $(s - k/2)$ is left over.

larger than $k/2$, $s > k/2$, only a part, of size $k/2$, of the impurity spin can be absorbed by the conduction electrons, and a free spin of magnitude $(s - k/2)$ remains at the strong coupling fixed point (fig. 1). By examining the interplay between this rearrangement and the branching rules of the conformal embedding, we show that after absorption of spin $k/2$ the theory consists again solely of free fermions (i.e. is a Fermi liquid), whose wave functions however have precisely suffered a phase shift of $\pi/2$, characteristic of Nozières and Blandin's strong coupling Fermi liquid fixed point [8]. This shows that CFT is the natural framework in which the physical picture of "underscreening" introduced by Nozières and Blandin, takes a precise mathematical form.

Note that the conduction electron spin density and the impurity spin, which are essentially decoupled at high temperatures, due to the asymptotic freedom of the Kondo coupling, "fuse" at low temperatures, in the technical sense of conformal field theory [4, 11, 12].

2. SU(2) fusion rules

The basic fusion rules for KM conformal towers [4, 11] are the analogue of the rules for adding angular momentum in the finite Lie algebra case. For the case of SU(2) which we will be mainly interested in, each tower is characterized by its highest-weight state having spin j. At level k there are $k+1$ KM conformal towers $j = 0, 1/2, \ldots, k/2$ [11]. The operator product expansion of two operators belonging to the conformal towers with highest-weight state of spin j_1 and j_2 contains operators belonging to the spin-j tower with [11]:

$$|j_1 - j_2| \leqslant j \leqslant \min(j_1 + j_2, k - j_1 - j_2). \tag{8}$$

We hypothesize that this rule also governs the mapping of the conformal tower of spin j_1 in eqs. (3,5) upon absorbing an ("external") impurity spin of size s, i.e. $j_2 = s$ in eq. (8). We see immediately that the rule is only defined for $s \leqslant k/2$. This is in accord with Nozières and Blandin's argument [8] that $k/2$ is the maximum spin that can be absorbed by k bands (cf. fig. 1). In the limiting case, relevant for "underscreening", where the absorbed part of the impurity spin has magnitude $k/2$, the fusion rule eq. (8) gives a unique possible spin:

$$(j) \otimes (k/2) = (k/2 - j). \tag{9}$$

In sect. 3 we show that the rearrangement of conformal towers, $j \to (k/2 - j)$, leads to a simple modification of the free-fermion spectrum upon using the branching rules to combine the charge, spin and flavor spectra, corresponding to a $\pi/2$ phase shift.

3. Conformal embedding branching rules and the phase shift

In this section we discuss how the free-fermion spectrum can be regarded as a suitable combination of charge (Q), spin (s) and flavor (f) excitations with certain selection or branching rules, governing how these three spectra are combined. We then study the modification of the combined spectrum due to the absorption of the impurity spin using the hypothesis of sect. 2 and show that the resulting spectrum corresponds again to free fermions which have however suffered a $\pi/2$ phase shift.

To decompose the free-fermion spectrum into charge, spin and flavor excitations, it is convenient to work in two steps. We first use the non-abelian bosonization relation [3] which expresses the free fermions in terms of a U(1) charge boson and a SU($2k$) level-1 KM theory. In the second step we decompose [10] the SU$(2k)_1$ theory into SU$(2)_k \times$ SU$(k)_2$.

For definiteness, consider left-moving free electrons on a circle of length $2L$ with a boundary condition

$$\psi(-L) = -\psi(L). \tag{10}$$

The allowed wave vectors are

$$k = \frac{\pi}{L}(n + 1/2), \quad n = 0, \pm 1, \pm 2, \dots. \tag{11}$$

In the ground state only the levels with $n < 0$ are filled. For a single species of electrons, the energies of the low-lying excited states are (see ref. [1])

$$E = \frac{v\pi}{L}\left[Q^2/2 + \sum_{m=1}^{\infty} m \cdot n_m\right]. \tag{12}$$

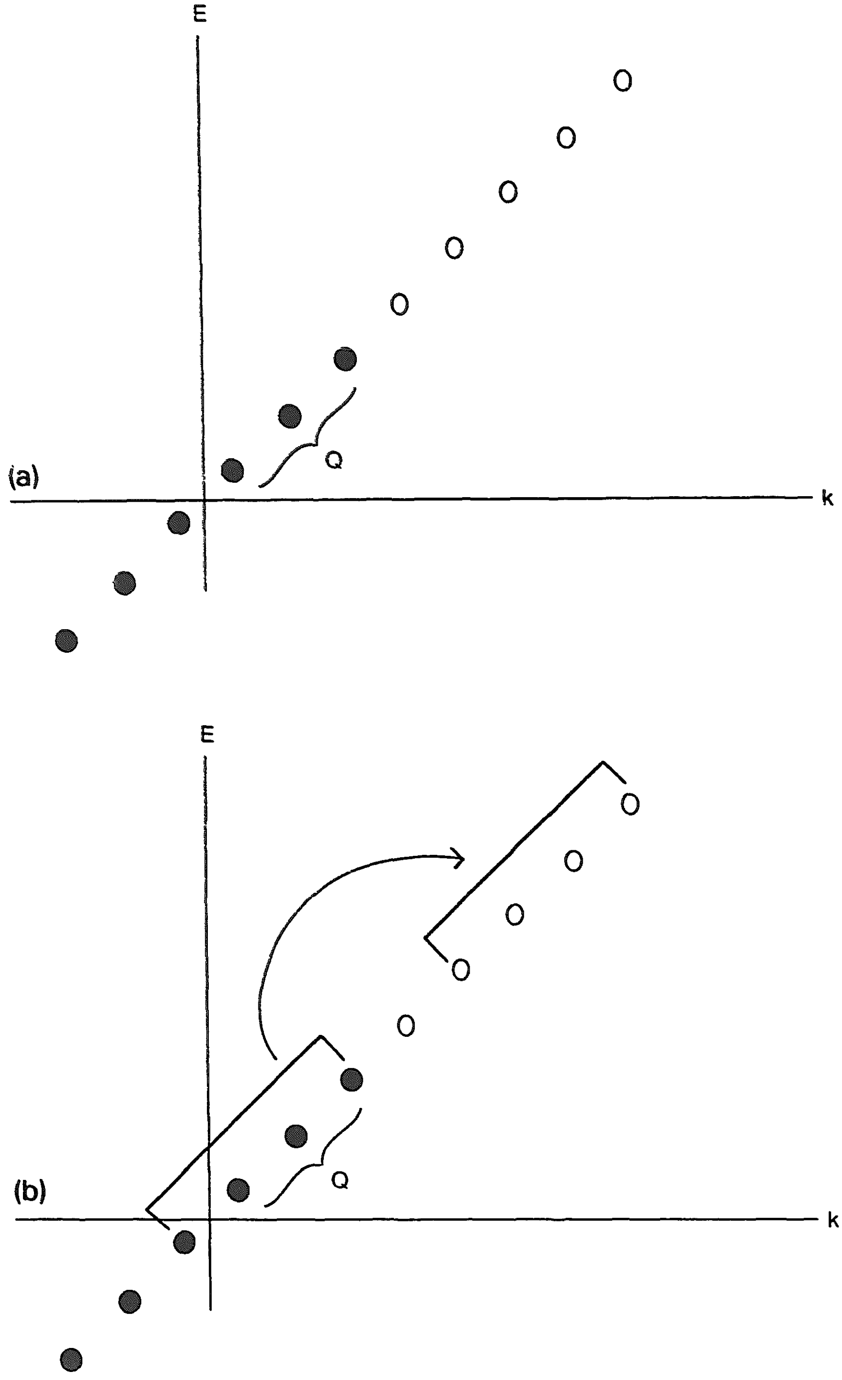

Fig. 2. (a) Adding Q electrons to the Fermi sea creates a state of U(1) charge Q (cf. eq. (12)). (b) Additional bosonic particle–hole excitations are created by then moving n_m electrons m steps up.

Here Q is the U(1) charge measuring the total number of fermions relative to that of the ground state and n_m is the number of electrons which have been raised by m levels relative to the lowest energy configuration of charge Q (describing bosonic particle–hole excitations, see fig. 2). We may introduce boson annihilation operators, a_m for states of momentum $(\pi/L)m$, and regard n_m as the occupation number of the bosonic level. Q can be interpreted as the soliton number of the boson field.

3.1. SU($2k$) NON-ABELIAN BOSONIZATION AND THE $\pi/2$ PHASE SHIFT

Free fermions. We are interested in the case where there are $N \equiv 2k$ different flavors of free electrons. In this case we obtain eq. (12) for each species and have N different fermion charges Q_i. This theory can be bosonized [3] in terms of a level-1 SU($N = 2k$) KM theory together with a free boson representing flavor and charge excitations respectively: Suppress $\mathscr{H}_0^{(s)}$ and let $k \to 2k$ in eq. (3).

We may introduce charge boson operators,

$$a_m^{(c)} \equiv \sum_{i=1}^{N} a_{im}/\sqrt{N}\,, \tag{13}$$

associated occupation numbers, $n_m^{(c)} \equiv a_m^{(c)\dagger} a_m^{(c)}$, and the total charge quantum number

$$Q \equiv \sum_{i=1}^{N} Q_i\,. \tag{14}$$

Introducing other linearly independent combinations of the a_i and Q_i, the part of the energy of a general excited state depending on Q and the $n_m^{(c)}$ is

$$E^{(c)} = \frac{v\pi}{L}\left[Q^2/2N + \sum_{m=1}^{\infty} m \cdot n_m^{(c)}\right] \qquad (\text{“charge excitations”})\,. \tag{15}$$

The additional terms in the formula for the energy of a general excited state $E_{\text{tot}} = \Sigma_{i=1}^{N} E_i = E^{(c)} + E^{(f)}$ (each E_i as in eq. (12)) can be expressed in terms of the spectrum of flavor SU(N)$_1$ KM representations. Allowed at level 1 are only [11] the N conformal towers with highest-weight state in the antisymmetric tensor representations of SU(N) with p indices (i.e. Young tableaux with p boxes and a single column, fig. 3a) with $p = 0, 1, 2, \ldots, N-1$. The energies of states in the pth conformal tower are [4]

$$E_p^{(f)} = \frac{v\pi}{L}\left[\frac{C_p}{2(N+1)} + n_p^{(f)}\right] \qquad (\text{“SU}(N)\text{ flavor excitations”})\,, \tag{16}$$

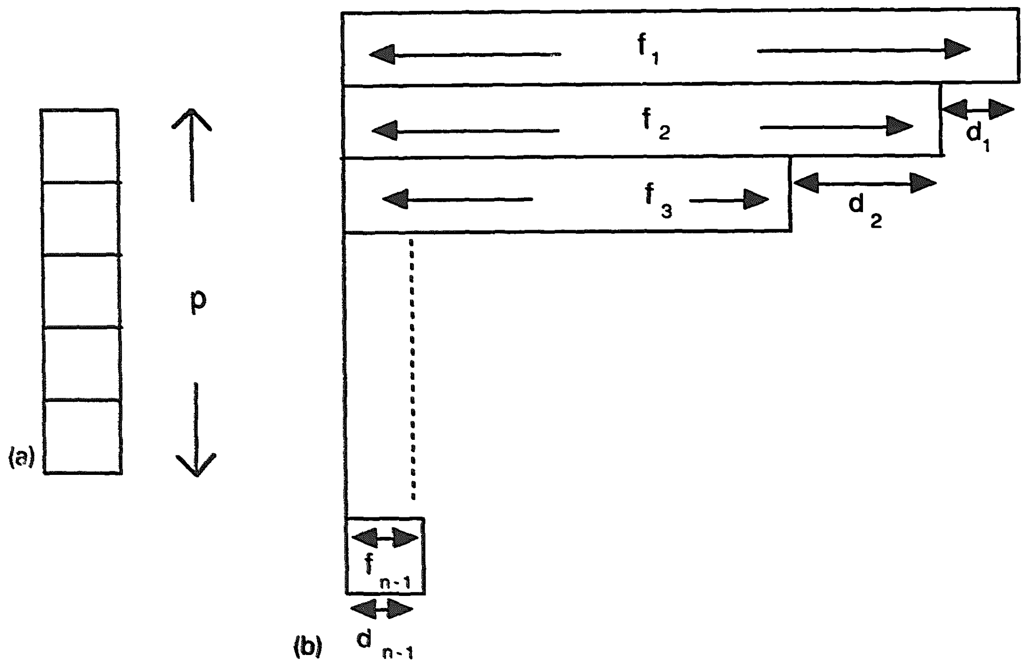

Fig. 3. (a) Young tableau describing the completely antisymmetric tensor representation of SU(N) (cf. eq. (16)). (b) Young tableau and Dynkin indices for general SU(N) representations (cf. eq. (34)).

where the integer $n_p^{(f)}$ is the grading (level of KM descendant) and C_p is the quadratic Casimir of the pth representation,

$$C_p = p(N-p)(N+1)/N, \tag{17}$$

$$E_p^{(f)} = \frac{v\pi}{L}\left[\frac{p(N-p)}{2N} + n_p^{(f)}\right]. \tag{18}$$

Naively, we might expect that states in the tower of arbitrary U(1) charge Q can occur together with states in the tower of arbitrary SU(N) symmetry p. However this cannot be the case as can be seen by examining the total energy:

$$E_{\text{tot}} = \frac{v\pi}{L}\left[\frac{Q^2}{2N} + \frac{p(N-p)}{2N} + n^{(c)} + n_p^{(f)}\right]. \tag{19}$$

Here we have introduced the integers

$$n^{(c)} = \sum_{m=1}^{\infty} m \cdot n_m^{(c)}, \tag{20}$$

representing the total energy of charge excitations coming from occupation of the finite momentum bosonic levels (U(1) KM descendants). Note that by eq. (12), valid for each E_i, the free-fermion energies always give an integer value of $2LE_{\mathrm{tot}}/v\pi$; but this is only true in eq. (19) if

$$Q = p(\mathrm{mod}\, N). \tag{21}$$

The necessity of this condition can also be seen by considering the U(1) charge and SU(N) flavor quantum numbers of an arbitrary multi-electron state. If we considered adding Q electrons (i.e. charge Q) in their lowest unoccupied fermion momentum states, corresponding to $n_p^{(f)} = n^{(c)} = 0$, then this state necessarily transforms in the antisymmetric p-tensor representation of the flavor group, where p satisfies eq. (21), due to Fermi statistics. All representations of SU(N) have definite N-ality [i.e. transformation property under the action of the center, Z_N, of SU(N)]. All states in the pth conformal tower [eq. (16)] are in representations of SU(N) of N-ality p. In the free-fermion basis, any state of Q fermions transforms in a representation of N-ality $p = Q(\mathrm{mod}\, N)$.

Eq. (21) appears to be the only restriction on the spectrum; otherwise arbitrary combinations of charge and flavor excitations can occur.

Free fermions with $\pi/2$ phase shift. Now let us consider the effect of the $\pi/2$ phase shift of the free-fermion wave functions, i.e. we shift the allowed fermion wave vectors by half a spacing:

$$k = \frac{\pi}{L}\left(n + \frac{1}{2}\right) \to \frac{\pi}{L} n. \tag{22}$$

This shifts the energies of the excited states for a single species as follows (cf. ref. [1]):

$$E_i = \frac{v\pi}{L}\left[\frac{Q_i^2}{2} + n_i\right] \to \frac{v\pi}{L}\left[\frac{Q_i^2}{2} - \frac{Q_i}{2} + n_i\right], \tag{23}$$

where

$$n_i = \sum_{m=1}^{\infty} m \cdot n_{im}. \tag{24}$$

Note that the state with $Q_i = 1$ is now degenerate with the ground state since the zero-momentum fermion level may be filled or empty. For N flavors the spectrum is shifted to (from eqs. (14)–(16))

$$\begin{aligned} E_{\mathrm{tot}} &= \frac{v\pi}{L}\left[\frac{Q^2}{2N} - \frac{Q}{2} + \frac{p(N-p)}{2N} + n^{(c)} + n_p^{(f)}\right] \\ &= \frac{v\pi}{L}\left[\frac{(Q-N/2)^2}{2N} + \frac{p(N-p)}{2N} + n^{(c)} + n_p^{(f)}\right] + \text{constant}. \end{aligned} \tag{25}$$

Since $N/2$ is itself the charge of one of the $2N$ degenerate ground states we may redefine the charge of the ground state by $N/2$ (recall that $N=2k$ is even), so that the formula for E_{tot} returns to its original form, before the phase shift. However, this redefinition of Q now affects the rule for assembling charge U(1) and flavor SU(N) excitations:

$$Q=p(\operatorname{mod} N)(\text{"before phase shift"}) \rightarrow Q=p+\frac{N}{2}(\operatorname{mod} N)(\text{"after phase shift"}) . \tag{26}$$

Thus we see that a $\pi/2$ phase shift in the free-fermion basis corresponds to a permutation of the SU(N) KM conformal towers $p \rightarrow p+N/2$ (mod N). Such a permutation corresponds to the action of an element of the center Z_N of SU(N) on the representations due to the cyclic symmetry of the Dynkin diagram [5, 10, 13].

3.2. CONFORMAL EMBEDDING BRANCHING RULES

We now consider the decomposition of the $\mathrm{SU}(2k)_1$ spectrum:

$$\mathrm{SU}(k)_2 \times \mathrm{SU}(2)_k \subset \mathrm{SU}(2k)_1 . \tag{27}$$

This has been analyzed extensively in a recent paper by Altschüler, Bauer and Itzykson [10] (ABI) and we merely quote some relevant results. That expression (27) is a conformal embedding follows from the fact that the Sugawara formula [5] for $2k$ free complex fermions [cf. eq. (3)] can be written in terms of the U(1) current and in terms of either the full SU($2k$) currents or the SU(2) $\times$ SU(k) currents only. This implies that the conformal anomaly parameters add up correctly,

$$c=\left[(2k)^2-1\right]/(2k+1)=2(k^2-1)/(2+k)+k(2^2-1)/(2+k) . \tag{28}$$

We label the $\mathrm{SU}(2k)_1$ conformal towers [4, 5] $V_p^{(2k)}$ by the SU($2k$) representation of the highest-weight state, corresponding to a p-index antisymmetric tensor (as in eqs. (16) and (17)) defined cyclically, i.e.

$$V_{p+2k}^{(2k)}=V_p^{(2k)} \tag{29}$$

and label the $\mathrm{SU}(2)_k$ conformal towers $V_j^{(2)}$ by the spin j of the highest-weight state. $\mathrm{SU}(k)_2$ conformal towers $V_f^{(k)}$ are labelled by the SU(k) representation of the highest-weight state, characterized by a vector f whose elements give the lengths of the rows of the Young tableau [13] (see eq. (34) below and fig. 3b).

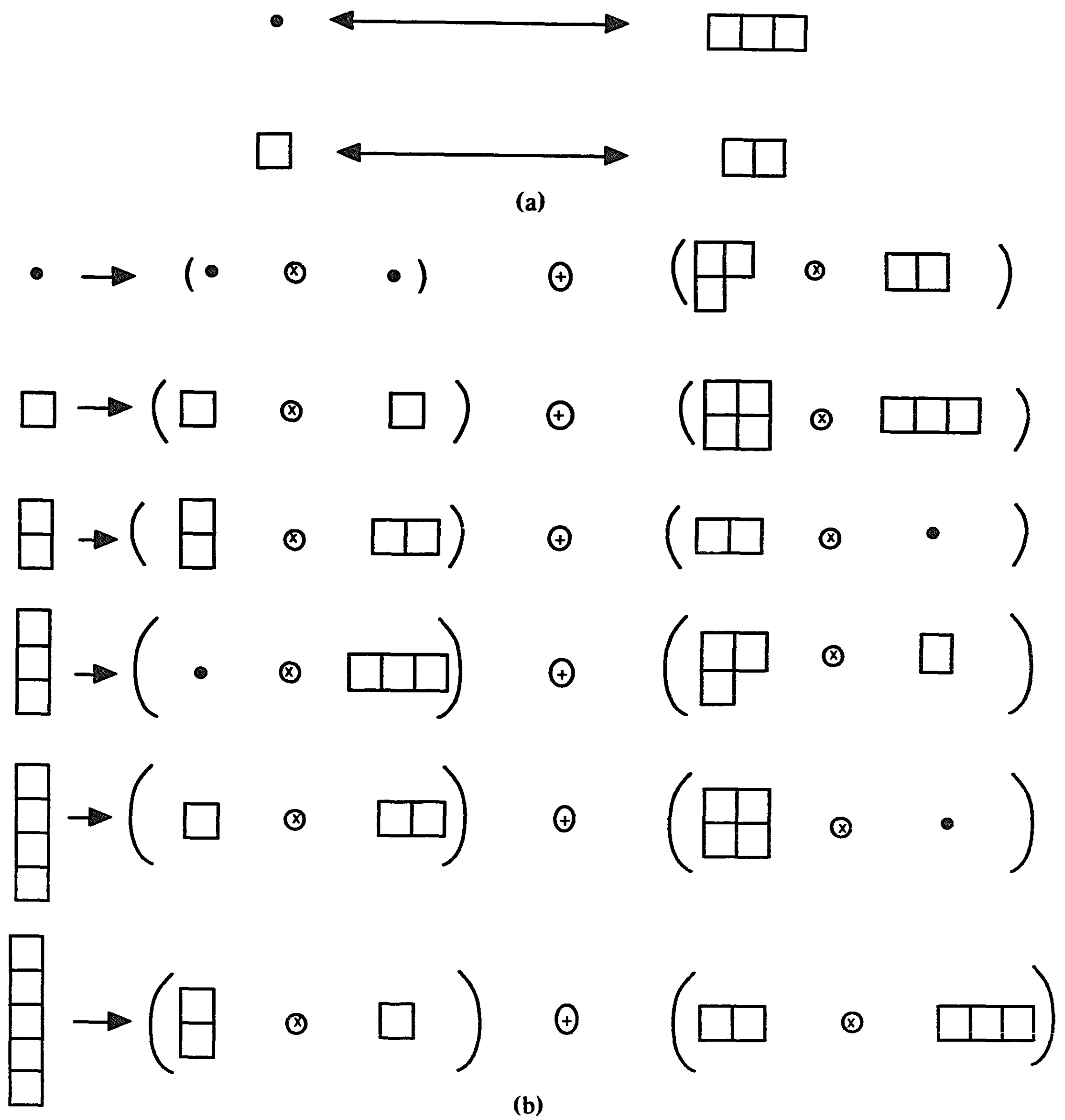

Fig. 4. (a) Interchange of $SU(2)_k$ conformal towers corresponding to the fusion process for the case $k=3$. (b) Explicit form of the conformal embedding $SU(k)_2 \times SU(2)_k \subset SU(2k)_1$, eq. (30), for the case $k=3$.

The essence of the conformal embedding is that each state in the conformal tower of $SU(2k)_1$ can be written as a sum of products of states in a $SU(k)_2$ and $SU(2)_k$ conformal tower respectively [10] (generalizing the standard decomposition of representations known for ordinary groups)

$$V_p^{(2k)} = \sum n_{f,j}^p \cdot V_f^{(k)} \otimes V_j^{(2)}. \tag{30}$$

The integer multiplicities $n_{f,j}^p$ have been calculated explicitly by ABI. In particular, it was shown that they take on only the values 0 and 1. Fig. 4b illustrates eq. (30) for the case $k=3$.

3.3. ABSORPTION OF THE IMPURITY SPIN AND THE $\pi/2$ PHASE SHIFT

We now have all the machinery in place to demonstrate that the absorption of the impurity spin by the conduction electron spin density leads to a $\pi/2$ phase shift. We hypothesized in sect. 2 that the absorption of an impurity spin of magnitude $k/2$ leads to a rearrangement of the $\mathrm{SU}(2)_k$ conformal towers governed by the standard fusion rules of KM conformal field theory (see fig. 4a), i.e.

$$\mathrm{V}_j^{(2)} \to \mathrm{V}_{k/2-j}^{(2)} . \tag{31}$$

We now wish to consider the effect on the free-fermion spectrum of this fusion process (see also fig. 4). To this end we first consider the modification of the $\mathrm{SU}(2k)_1$ conformal towers induced by the fusion in the embedded $\mathrm{SU}(2)_k$ loop group [eq. (27)]. The pth conformal tower is modified by

$$\mathrm{V}_p^{(2k)} \to \sum_{f,j} n_{f,j}^p \cdot \mathrm{V}_f^{(k)} \otimes \mathrm{V}_{k/2-j}^{(2)} \tag{32}$$

as compared to eq. (30).

Using the results of ABI we will now show that this implies

$$\mathrm{V}_p^{(2k)} \to \mathrm{V}_{p+k}^{(2k)} , \tag{33}$$

i.e. that the r.h.s. of eq. (32) equals $\mathrm{V}_{p+k}^{(2k)}$. This is illustrated in fig. 4 for the special case $k=3$.

This result essentially follows from the covariance of the extended Dynkin diagram under the center of the ordinary Lie group [5, 10]. An irreducible representation of $\mathrm{SU}(n)$ is labelled by $f=(f_1, f_2, \dots, f_n)$, where f_i is the length of the ith row in the corresponding Young tableau [13]. Alternatively (fig. 3b), representations are labelled by the Dynkin indices $\boldsymbol{d}=(d_1, d_2, \dots, d_{n-1})$

$$d_i = f_i - f_{i+1} \geqslant 0 . \tag{34}$$

For the $\mathrm{SU}(n)_k$ KM algebra (as opposed to the ordinary group $\mathrm{SU}(n)$) there is one more Dynkin index,

$$d_n = k - \sum_{i=1}^{n-1} d_i . \tag{35}$$

The set of highest-weight representations of $\mathrm{SU}(n)_k$ is determined (see e.g. ref. [5]) by the set of all non-negative integer Dynkin indices $(d_1, \dots, d_n)$ such that

$$\sum_{i=1}^{n} d_i = k . \tag{36}$$

The elements of the center Z_n of SU(n) act on the KM representations by cyclically permuting the Dynkin indices:

$$(d_1, d_2, \dots, d_n) \to (d_m, d_{m+1}, \dots, d_n, d_1, d_2, \dots, d_{m-1}), \quad m = 1, 2, \dots, n. \tag{37}$$

In the case of SU$(2k)_1$ only one of the d_i is non-zero and the representation $V_p^{(2k)}$ corresponds to the Dynkin indices

$$V_p^{(2k)} \Leftrightarrow d_p = 1, \quad d_i = 0, i \neq p, \quad 1 \leqslant i \leqslant n. \tag{38}$$

The SU$(2)_k$ conformal towers have Dynkin indices

$$V_j^{(2)} \Leftrightarrow (d_1, d_2) = (2j, k - 2j). \tag{39}$$

Note that the fusion process: $j \to (k/2 - j)$, permutes these two Dynkin indices and hence corresponds to the action of Z_2, the center of SU(2), as defined in eq. (9).

Under the corresponding embedding of the ordinary Lie groups the tensor product of the SU(2) matrix g and the SU(k) matrix h is considered as a SU($2k$) matrix G:

$$\left\{G^{(\alpha,i)}_{(\beta,j)}\right\} = \left\{g^{\alpha}_{\beta}\right\} \otimes \left\{h^{i}_{j}\right\}. \tag{40}$$

The SU(2) group center operation $g \to -g$, corresponds then to an element of the center of SU($2k$) as well: $G \to -G$. If this correspondence holds also at the loop-group (KM) level then we should expect that the Z_2 transformation [eq. (39)] on the SU$(2)_k$ representations (coming from the absorption of the impurity spin) induces the Z_2 element of Z_{2k} acting on the SU$(2k)_1$ representations, i.e.

$$V_p^{(2k)} \to V_{(p+k)}^{(2k)}. \tag{41}$$

ABI have shown that this is in fact the case. This follows from Theorem 1 of ABI in ref. [10]. They show, as already pointed out, that the multiplicities, $n^p_{f,j}$, are all either zero or one. In particular they show, that if a multiplicity is one and we act with the Z_2 center operation (σ in the notation of ABI) on the SU$(2)_k$ representation then this multiplicity will continue to be one if and only if we shift p (δ in the notation of ABI) by k [cf. eq. (2.27) of ref. [10]],

$$n^p_{f,j+k/2} = n^{p+k}_{f,j}.$$

Therefore eq. (33) follows from eq. (32) upon using eq. (30). Fig. 4 illustrates this process for $k = 3$. We have thus shown that the fusion with the impurity spin induces the shift of eq. (41) in the SU$(2k)_1$ representations. The final step is to use the non-abelian bosonization rules discussed in subsect. 3.1. As explained there, permuting the $SU(2k)_1$ representations in this way corresponds to the $\pi/2$ phase shift of the fermion wave functions, predicted by Nozières and Blandin [8].

To summarize: We have introduced the precise hypothesis that the conformal fusion rule for Kac–Moody conformal towers governs also the effect on the conformal towers upon absorbing an external impurity spin. We have shown, using properties of the conformal embedding, that this leads to a concise mathematical description in terms of conformal field theory of the physical picture for the "underscreening" phenomenon in the multiband Kondo effect, introduced by Nozières and Blandin.

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