The index of elliptic operators: V

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Introduction

The preceding papers of this series dealt with the index of elliptic pseudo-differential operators and families of such operators. In all this, our operators (and vector bundles) were over the complex numbers. In this paper we want to refine the preceding theory to deal with *real* operators, for example differential operators with real coefficients. For a single real elliptic operator on a compact manifold the index can of course be computed by passing to the complexification. For a family, parametrized by a space Y, the situation is different. For a complex family we defined, in paper IV, an index in $K(Y)$. Similarly for a real family we will get an index in $KR(Y)$, the Grothendieck group of real vector bundles on Y. Complexifying the family leads to complexification of the index, namely the homomorphism

$$KR(Y) \longrightarrow K(Y)$$

induced by $E \mapsto E \otimes_{\mathbf{R}} \mathbf{C}$. If Y is a point this homomorphism is injective, but for general spaces Y it is *not* injective. Thus knowing the index of complex families is not enough to determine the index of real families. This is the justification for the present paper.

In § 1 we show how the main theorem of paper IV has to be modified to take account of reality conditions. The only point that needs special mention here is that the symbol class of a real operator has to be interpreted in the appropriate K-theory, and this is not the K-theory of real vector bundles in the usual sense. Instead we have to use the K-theory developed in [2] for spaces with involution. This was in fact the motivation for [2], as explained in [2, § 5]. Once we have the right K-theory the proof of the main theorem proceeds as before, one just has to watch that the reality conditions are observed throughout.

Perhaps the simplest and most interesting example of a real elliptic family arises from a real skew-adjoint elliptic operator P. As explained in [8] such an operator gives rise to a family $\tilde{P}$ parametrized by the circle S^1 and the index of $\tilde{P}$ in $KR(S^1)$ lies in the reduced group $\widetilde{KR}(S^1) = Z_2$ and coincides with the "mod 2 index" of P:

$$\text{index } \tilde{P} = \dim \text{Ker } P \quad \text{mod } 2 \ .$$

Thus the index theorem for real elliptic families leads in particular to a result on $\mathrm{mod}\,2$ indices of real skew-adjoint elliptic operators. This is explained in § 2.

In § 3 we study some particular examples of skew-adjoint operators defined on manifolds. In particular we show that, for a compact oriented $(4k + 1)$-manifold X the Kervaire semi-characteristic

$$k(X) = \sum_p \dim H^{2p}(X; \mathbf{R}) \qquad \text{mod } 2$$

is such a $\mathrm{mod}\,2$ index. This has some interesting geometrical applications concerned with vector fields [4].

1. Real operators

If E, F are real vector bundles over a compact manifold X then we can consider differential operators

$$P: C^{\infty}(X; E) \longrightarrow C^{\infty}(X; F)$$

with real coefficients. This simply means that, in any local coordinate system, P has real coefficients. Since E, F are *real* vector bundles this statement has invariant meaning.

By complexification P defines an operator

$$P^c: C^{\infty}(X; E^c) \longrightarrow C^{\infty}(X; F^c)$$

where $E^c = E \otimes_{\mathbf{R}} \mathbf{C}$, $F^c = F \otimes_{\mathbf{R}} \mathbf{C}$. This complexified operator satisfies the reality condition

(1.1) $\overline{P^c(u)} = P^c(\bar{u})$

where complex conjugation in the space of sections of E^c, F^c is induced by complex conjugation in the fibres. Conversely any differential operator P^c which satisfies (1.1) is the complexification of a real differential operator. Because the definition of pseudo-differential operators involves the Fourier transform, it is therefore more convenient to work always in the complexification and to impose there the reality condition (1.1). Note however that we continue to work with vector bundles which are complexifications of real bundles.

A real elliptic pseudo-differential operator Q is therefore defined to be an elliptic pseudo-differential operator

$$Q: C^{\infty}(X; E^c) \longrightarrow C^{\infty}(X; F^c)$$

such that $\overline{Q(u)} = Q(\bar{u})$. In euclidean space, if $Q = q(x, D)$ is defined from the function $q(x, \xi)$ by the usual formula

$$Qu = (2\pi)^{-n} \int q(x, \xi)\hat{u}(\xi)e^{i<x, \xi>}d\xi \,,$$

we have $\left(\text{since } D = -i(\partial/\partial x)\right)$

$$\overline{Qu} = \overline{q(x, D)u} = \bar{q}(x, -D)\bar{u} \,.$$

Thus the reality condition becomes

(1.2) $$\bar{q}(x, -\xi) = q(x, \xi) \,.$$

This implies that the symbol $\sigma(x, \xi)$ of a real pseudo-differential operator satisfies

(1.3) $$\bar{\sigma}(x, -\xi) = \sigma(x, \xi) \,.$$

It is clear that a symbol satisfying (1.3) always represents a real operator: if Q is any operator with symbol σ, then $u \mapsto (1/2)\big(Q(u) + \overline{Q(\bar{u})}\big)$ is real and also has symbol σ.

Condition (1.3) forces us to introduce an appropriate K-theory as in [2]. We shall recall briefly the definitions given there. We consider in general a compact space X with an involution $x \mapsto \bar{x}$ and we consider complex vector bundles $E \to X$ with an involution $e \mapsto \bar{e}$ which covers the involution on X and is anti-linear on the fibres. The Grothendieck group of such bundles with involution we denote by $KR(X)$. Note that if the involution on X is trivial ($\bar{x} = x$ for all $x \in X$), then the bundle E has a conjugation in each fibre E_x and so is, in a natural way, the complexification of a real bundle E_R (the fixed-points of the involution). Thus $KR(X)$ in this case can be identified with the K-theory of real vector bundles on X (also written $KO(X)$). A suggestive terminology is to call a space X with involution a *Real* space and vector bundles E as above *Real* vector bundles. For a locally compact space X we can then define $KR(X)$ by triples (E, F, α) where E, F are Real vector bundles on X and α is a Real isomorphism outside a compact set (that is, α is compatible with the involutions).

Return now to a real symbol σ. The bundles E, F on which σ acts are complexifications, and so are Real vector bundles over our manifold X (with trivial involution). Now lift to the tangent bundle TX, giving TX the antipodal involution $\xi \mapsto -\xi$. We get Real vector bundles $\pi^*E, \ \pi^*F$ over TX and condition (1.3) just asserts that

$$\sigma: \pi^*E \longrightarrow \pi^*F$$

is a Real homomorphism

$$\overline{\sigma(x, \xi)e} = \bar{\sigma}(x, \xi)\bar{e}$$
$$= \sigma(x, -\xi)\bar{e} \,.$$

Hence an *elliptic* real symbol σ has a symbol class $[\sigma] \in KR(TX)$.

As explained in the Introduction there is no new index theory for a single operator in the real case so we proceed at once to the case of families. A real elliptic family parametrized by a compact space Y is defined just as in [7]. We have a fibre bundle now $Z \to Y$ with fibre the compact smooth manifold X, and for each $y \in Y$ we have a real elliptic operator P_y on the fibre X_y varying continuously with Y. In [7, § 2] we saw how to define the index of a complex family as an element of $K(Y)$. If Ker P_y is of constant dimension we can take

$$(1.4) \qquad\qquad \text{index } P = [\text{Ker } P] - [\text{Coker } P]$$

where Ker P is the vector bundle over Y whose fibre at y is Ker P_y and similarly for Coker P. In the general case we have first to modify P by adding some sections $s_1, \cdots, s_q$ as in [7, (2.2)]. In the real case we can also define index $P \in KR(Y)$ by (1.4) when Ker P_y has constant dimension. In general we proceed as in [7, (2.2)] but using only real sections s_i: we just observe that the construction of the s_i given in [7, (2.2)] also works in the real domain (if all bundles and operators are real).

Of course if we ignore the reality conditions on P then we get a complex family whose index in $K(X)$ is just the image of index $P \in KR(X)$ under the natural homomorphism $KR(X) \to K(X)$. Since this map is not always injective the real index is a more refined object than the complex index, and we propose to refine the index theorem of [7] accordingly.

Just as in [7] the mapping $P \mapsto$ index P defines an analytical index

$$\text{a-ind: } KR(TZ) \longrightarrow KR(Y) \ .$$

To define the topological index we proceed as in [5], [7]. The first important point is that, if N is a tubular neighborhood of X in a euclidean space V, then the identification $TN = \pi^*(N \otimes_{\mathbf{R}} \mathbf{C})$ used in [6, § 3] is compatible with the involutions. This is because we regarded a vector $y + i\eta \in N_x \otimes_{\mathbf{R}} \mathbf{C}$ as representing the tangent vector η at the point $y \in N_x$, so that the antipodal involution on TN corresponds to complex conjugation on $N \otimes_{\mathbf{R}} \mathbf{C}$. The second point to mention is that the Thom isomorphism holds in KR for Real vector bundles. This is proved in [2] and [3]. With these observations made, it is then clear that the real topological index is defined and gives a homomorphism

$$\text{t-ind: } KR(TZ) \longrightarrow KR(Y) \ .$$

Our theorem will of course be

THEOREM (1.5).   *The analytical and topological indices of a real elliptic family coincide.*

The proof proceeds just as in the case of complex families. The only point that calls for special mention is that the fundamental equivariant symbol class $b = i_!(1) \in KR_{SO(n)}(T\mathbf{R}^n)$ (where $i: A \to \mathbf{R}^n$ is the inclusion of the origin) must be shown to have analytical index 1 in $KR_{SO(n)}$ (point) $= RO(SO(n))$ (the real representation ring of $SO(n)$). But for any group G, complexification of representations

$$RO(G) \longrightarrow R(G)$$

is injective. Thus we are reduced to proving a-ind $(b) = 1 \in R(SO(n))$ and this was done in [5].

Remark. So far, although we introduced an involution in TX, we have only considered the trivial involution on X and Y. We can, however, consider involutions on X and Y and Theorem (1.5) continues to hold with essentially the same proof. A particularly interesting case arises when X is a complex algebraic manifold defined over $\mathbf{R}$. The Dolbeault complex of X is then a Real elliptic complex (with respect to the involution $x \mapsto \bar{x}$ on X) and, using a metric, it defines a Real elliptic operator P. If we have a family X_y for which the sheaf cohomology groups $H^q(X_y, \mathcal{O}_y)$ have constant dimension ($\mathcal{O}_y$ denotes the sheaf of holomorphic functions on X_y), then the index of the Real family P_y is just the alternating sum

$$\sum (-1)^q H^q \in KR(Y)$$

where H^q stands for the real vector bundle over Y whose fibre at y is $H^q(X_y, \mathcal{O}_y)$. This case was briefly alluded to in [2, § 5].

2. Skew-adjoint operators

Let P be a real elliptic operator on the compact manifold X and assume that (with respect to given metrics in X and the bundles) P is skew-adjoint: $P^* = -P$. Then Ker $P = $ Ker P^* so that the usual index of P is zero. However, the dimension of Ker P modulo 2 is a new interesting invariant of P. As shown in [8, Prop. (5.1)] this is invariant under continuous deformation of P. As in [8] we denote it by $\mathrm{ind}_1 P$ and refer to it as the mod 2 index of P: the notation is chosen because of generalizations, denoted by $\mathrm{ind}_k P$, which are described in [8, § 5].

Since $\mathrm{ind}_1 P$ is a deformation invariant it should depend only on an appropriate symbol class and it is reasonable to expect to compute $\mathrm{ind}_1 P$ topologically from the symbol. We shall show how to do this by associating to P a family $\tilde{P}$ of real elliptic operators parametrized by the circle and then using Theorem (1.5) to compute $\mathrm{ind}\, \tilde{P} \in KR(S^1)$. As explained in the Introduction the family $\tilde{P}$ will have the property that $\mathrm{ind}\, \tilde{P}$ coincides essentially

(as an element of Z_2) with $\mathrm{ind}_1 P$.

The construction of $\tilde{P}$ will be a minor modification of the construction used in [8]: there we dealt with abstract Fredholm operators in Hilbert space, whereas here we are dealing with pseudo-differential elliptic operators.

It is sufficient to consider the case where P is of order zero, $P \in Q^0(X; E, E)$: the general case where P is of order m follows by putting $R = QPQ^*$ where Q is of order $-m/2$, $\sigma(Q) = 1$ on the unit sphere bundle of X and $\mathrm{Ker}\, Q = \mathrm{Ker}\, Q^* = 0$. Let I denote the unit interval $0 \leqq y \leqq 1$ and define

$$(2.1) \qquad\qquad P_y = 1 \operatorname{Cos} \pi y + P \operatorname{Sin} \pi y$$

where 1 is the identity operator on $C^\infty(X; E)$. Since $P^* = -P$ it follows that, for all y, P is elliptic and, for $y \neq 1/2$, we have

$$\mathrm{Ker}\, P_y = \mathrm{Ker}\, P_y^* = 0 \;.$$

The operators P_y are a family of elliptic operators parametrized by the unit interval I. We want now to construct a family parametrized by the unit circle S^1. To do this we need to identify the points $0, 1 \in I$. Now we have

$$P_0 = 1, \quad P_1 = -1 \;.$$

Hence to construct our family of operators over the circle we must twist one copy of the bundle E with the Hopf bundle[2] H on S^1. The operators P_y define operators[3]

$$\tilde{P}_y\colon C^\infty(X, E) \otimes H_y \longrightarrow C^\infty(X, E) \;,$$

where y is now regarded as a point on S^1.

We shall now compute $\mathrm{ind}\, \tilde{P} \in KR(S^1)$. According to the prescription given in [7, (2.2)] (applied to the real case) we must first choose sections $s_1, \cdots, s_q$ of the (trivial) bundle $C^\infty(X, E) \times S^1$ so that, for all $y \in S^1$ the map

$$Q_y\colon C^\infty(X, E) \otimes H_y \oplus \mathbf{R}^q \longrightarrow C^\infty(X, E)$$

given by

$$Q_y(u; \lambda_1, \cdots, \lambda_q) = \tilde{P}_y(u) + \sum_{i=1}^q \lambda_i s_i(y)$$

is surjective. In our case, since P_y is already surjective for all $y \neq 1/2$, it is enough to take $s_1, \cdots, s_q$ to be the constant sections given by a basis of $\mathrm{Ker}\, P$. The kernel of Q_y is then naturally isomorphic to the kernel of the map

$$C^\infty(X, E) \otimes H_y \longrightarrow (\mathrm{Ker}\, P)^\perp$$

given by composing[4] $\tilde{P}_y$ with orthogonal projection on $(\mathrm{Ker}\, P)^\perp$. But this

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[2] H is the line bundle over S^1 obtained from $I \times \mathbf{R}^1$ by the identification $(0, u) \leftrightarrow (1, -u)$.

[3] From now on we think of E as a real vector bundle, not the complexification of one.

[4] Note that, from the definition (2.1), P_y commutes with projection on $(\mathrm{Ker}\, P_y)^\perp$.

kernel is clearly Ker $P \otimes H_y$, and so

$$\text{ind } \tilde{P} = [\text{Ker } P \otimes H] - [\text{Ker } P] \in KR(S^1)$$
$$= \dim \text{Ker } P([H] - [1]) \,.$$

Now $\widetilde{KR}(S^1) \cong Z_2$ with generator $[H] - [1]$. Hence, identifying $\widetilde{KR}(S^1)$ with Z_2, we see that

(2.2) $\text{ind } \tilde{P} = \text{ind}_1 P \,.$

The symbol $\tilde{\sigma}$ of $\tilde{P}$ defines a symbol class $[\tilde{\sigma}] \in KR(S^1 \times TX)$ which is trivial on KR (point $\times TX$) and so can be regarded as an element of $KR^{-1}(TX)$. In analogy with the notation $\text{ind}_1 P$, we shall denote this element of $KR^{-1}(TX)$ by $[\sigma(P)]_1$, and call it the skew-symbol class.

The topological index (for families over S^1)

$$\text{t-ind: } KR(S^1 \times TX) \longrightarrow KR(S^1)$$

induces (by restriction) a homomorphism

$$\text{t-ind}_1 \colon KR^{-1}(TX) \longrightarrow KR^{-1}(\text{point}) \,.$$

This is just the natural suspension of the topological index $KR(TX) \to KR$ (point). Thus from Theorem (1.5) we deduce

THEOREM (2.3). *Let P be a real skew-adjoint elliptic operator on a compact manifold X. Let $[\sigma(P)]_1 \in KR^{-1}(TX)$ be the skew-symbol class of P, and let*

$$\text{t-ind}_1 \colon KR^{-1}(TX) \longrightarrow KR^{-1}(\text{point}) = Z_2$$

be the topological index. Then

$$\dim \text{Ker } P = \text{t-ind}_1 [\sigma(P)]_1 \qquad \text{mod } 2 \,.$$

When X is a spin-manifold we can simplify things somewhat by using the Thom isomorphism [3, Th. (6.2)]

$$KR^{-1}(TX) \cong KR^{n-1}(X) \,.$$

The homomorphism t-ind_1 now becomes the direct image (or Gysin) homomorphism for spin-manifolds [9]:

(2.4) $KR^{n-1}(X) \longrightarrow KR^{-1}(\text{point}) \,.$

Even in this form it must be admitted that this topological index is very hard to compute in practice. One might ask whether it is possible to compute this using cohomology as is done with the usual index of elliptic operators in [6]. Unfortunately this does not seem to be possible, for rather fundamental reasons. The point is that the mod 2 index comes ultimately from the Z_2 homotopy group $\pi_{8k+1}(O)$ of the stable orthogonal group, and it is known that

this cannot be detected directly by mod 2 cohomology. The usual integer index on the other hand comes from $\pi_{2n-1}(U)$ and this is detectable by rational cohomology. In the next section we shall show, by an example, that the mod 2 index behaves in a considerably more suitable way than the usual index.

3. Examples

Let X be a compact manifold of dimension $4q + 1$ and define the real Kervaire semi-characteristic $k(X)$ by

$$k(X) = \sum_p \dim_{\mathbf{R}} H^{2p}(X; \mathbf{R}) \qquad \text{mod } 2 .$$

We shall show that this is the mod 2 index of a certain skew-adjoint elliptic operator.

Choose a riemannian metric and let $*$ be the usual duality operator on forms. Since $\dim X$ is odd we have $*^2 = 1$ and $d*\varphi = (-1)^p *d*\varphi$ for $\varphi \in \Omega^p$. Hence, on even forms, $d*$ is self-adjoint and $*d$ is skew-adjoint. Moreover, since $\dim X \equiv 1 \bmod 4$, $*d$ (on even forms) preserves the degree mod 4 while $d*$ reverses it. Thus the operator D on even forms defined by[5]

$$D\varphi = (-1)^p d*\varphi + *d\varphi \qquad\qquad \varphi \in \Omega^{2p}$$

is skew-adjoint. Since $d^2 = 0$, we have

$$D^*D = -D^2 = dd^* + d^*d = \Delta$$

and so D is elliptic and $\operatorname{Ker} \varphi = \sum H^{2p}$ where H^{2p} is the space of harmonic forms of degree $2p$ (solutions of $\Delta u = 0$, $u \in \Omega^{2p}$). By the Hodge theory $H^{2p} \cong H^{2p}(X; \mathbf{R})$ and so

$$\dim \operatorname{Ker} D = k(X) \qquad \text{mod } 2$$

as required.

Remark. When $\dim X \equiv -1 \bmod 4$ we get a self-adjoint operator (if we choose the sign as in [4]) and $k(X)$ does not appear as a mod 2 index. This difference between the two cases $\dim X \equiv \pm 1 \bmod 4$ reflects significant topological differences: for example the existence of two linearly independent vector fields on X implies that $k(X) = 0$ when $\dim X \equiv -1 \bmod 4$ (as shown by taking X to be the 3-sphere).

Theorem (2.3) gives therefore a K-theoretical evaluation of the Kervaire semi-characteristic $k(X)$ for $\dim X \equiv 1 \bmod 4$. This has an interesting connection with vector fields (see [4; Th. (5.1)]) which will be treated in detail elsewhere.

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[5] From the point of view of Clifford algebras it is more natural to define D by $D\varphi = (-1)^p d*\varphi + (-1)^{p+1} *d\varphi$ as in [4]. This would do just as well for our purposes.

As a second example we shall study the Dirac operator on a spin-manifold X of dimension $8q + 1$.

Let $M = M^0 \oplus M^1$ be an irreducible graded module for the Clifford algebra C_{8q+1} and let $E = E^0 \oplus E^1$ be the associated graded vector bundle over X associated to the representation E of Spin $(8q + 1) \subset C_{8q+1}$. As in [6, § 5] the Dirac operator D acts on the sections of E (interchanging E^0 and E^1) and is given in terms of an orthonormal base e_i of T_x by

$$Ds = \sum e_i(\partial_i s)$$

where $\partial_i s$ is the covariant derivative of s in the direction e_i and $e_i(\ )$ denotes Clifford multiplication. D is self-adjoint and elliptic. Now M^0 and M^1 are isomorphic representations of the even part of C_{8q+1} and so of Spin $(8q + 1)$: the isomorphism is given by Clifford multiplication by the usual element $\omega = e_1 e_2 \cdots e_{8q+1}$. Hence on the manifold X Clifford multiplication by the volume form ω maps E^0 isomorphically onto E^1 and commutes with D. Since $\omega^2 = 1$ and is orthogonal we have $\omega^* = \omega$ and therefore $P = \omega D$ is skew-adjoint. P preserves E^0 and E^1 and so we have a skew-adjoint operator P^0 acting on E^0. We shall call P^0 the skew-Dirac operator of the spin-manifold X. Note that Ker $P^0 =$ Ker $D \mid E^0$ is the space of "harmonic spinors."

Since X is a spin-manifold of dimension $8q + 1$ we have the Thom isomorphism

$$\varphi \colon KR(X) \longrightarrow KR^{-1}(TX)$$

and it is a routine matter to check that the symbol class $[\sigma(P^0)]_1 \in KR^{-1}(TX)$ is just $\varphi(1)$: both elements are constructed explicitly from Clifford multiplication and we just have to check the involutions. Thus as a special case of Theorem (2.3) (and using (2.4)) we have

THEOREM (3.1). *Let* X *be a spin-manifold of dimension* $8q + 1$, *and let* H *denote the space of harmonic spinors on* X. *Then* dim H mod 2 *is equal to* $f_!(1)$ *where*

$$f_! \colon KR(X) \longrightarrow KR^{-1}(\text{point}) = Z_2$$

is the direct image homomorphism for spin-manifolds.

Remark. $f_!(1)$ is an example of what is called a *KO*-characteristic number. Such invariants of spin-manifolds have proved important in spin-cobordism (see [1]).

For a spin $(8q + 1)$-manifold the operator giving the Kervaire semi-characteristic is closely related to the skew-Dirac operator P^0. On the symbolic level it is the product of P^0 and the spin bundle of X — we omit the details. Hence we obtain

THEOREM (3.2). *Let X be a spin-manifold of dimension $8q + 1$, then the Kervaire semi-characteristic $k(X)$ is equal to $f_!(\Delta(X))$ where $f_!: KR(X) \to KR^{-1}(\text{point}) = Z_2$ is the direct image homomorphism and $\Delta(X)$ is the spin bundle of X.*

Thus $k(X)$ is also a KO-characteristic number.

For spin-manifolds of dimension $8q + 2$, the spin bundle $\Delta(X)$ is complex, the complex structure being given by the element ω. The Dirac operator now anti-commutes with ω and so the space H of harmonic spinors is naturally a complex vector space. On the same lines as (3.1) we have

THEOREM (3.3). *Let X be a spin-manifold of dimension $8q + 2$, and let H denote the (complex vector space) of harmonic spinors on X. Then $\dim_C H$ mod 2 is equal to $f_!(1)$ where*

$$f_!: KR(X) \longrightarrow KR^{-2}(\text{point}) = Z_2 \,.$$

Theorem (3.3) can be turned into a theorem for families over S^2 using the ideas in [8], and we then apply Theorem (2.3). The details are quite similar to the proof of (3.1) and we shall omit them.

Returning to the Kervaire semi-characteristic we want now to point out a rather interesting result which shows that mod 2 indices differ significantly from the usual integer index.

PROPOSITION (3.4). *Let X be a compact oriented $(4q + 1)$-manifold and let $\widetilde{X}$ be a double covering given by an element $\alpha \in H^1(X; Z_2)$. Then the Kervaire semi-characteristic of $\widetilde{X}$ is given by the formula*

$$k(\widetilde{X}) = \alpha \cdot \omega_{4q}(X)[X] \,.$$

There is an interesting proof of this using symbols which will be given elsewhere and a direct geometrical argument due to G. Lusztig. We shall not give the proof here but we will discuss the implications of this proposition. Note first that there are examples with $k(\widetilde{X}) \neq 0$: we take $X = P_{4q+1}(\mathbf{R})$ and $\widetilde{X} = S^{4q+1}$. Now the usual index of an elliptic operator always behaves multiplicatively for finite coverings, and this is connected with the fact that there are integral expressions for index P involving only local data. The example just mentioned shows that $k(X)$ is *not* multiplicative for double coverings (otherwise we would have $k(\widetilde{X}) = 2k(X) = 0$). It is thus not possible to find a canonical expression for $k(X)$ as an integral involving only local data. Of course $k(X)$ is an integer mod 2 and to hope for an integral expression is perhaps unnatural in any case, but the argument with double coverings is more conclusive.

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