

Supplementary Information Appendix to Hydrodynamic theory of thermoelectric transport and negative magnetoresistance in Weyl semimetals

by

Andrew Lucas^a, Richard A. Davison^a and Subir Sachdev^{a,b}

^a Department of Physics, Harvard University, Cambridge, MA 02138, USA

^b Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada

A Single Chiral Fluid

To first order, the hydrodynamic gradient expansion of a chiral fluid reads [34, 35]

$$J^\mu = nu^\mu - \sigma_Q \mathcal{P}^{\mu\nu} \left(\nabla_\nu \mu - \frac{\mu}{T} \nabla_\nu T - F_{\nu\rho} u^\rho \right) + \mathcal{D}_1 \varepsilon^{\mu\nu\rho\sigma} u_\nu \nabla_\rho u_\sigma + \frac{\mathcal{D}_2}{2} \varepsilon^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma}, \quad (1a)$$

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P \eta^{\mu\nu} - \eta \mathcal{P}^{\mu\rho} \mathcal{P}^{\nu\sigma} (\nabla_\rho u_\sigma + \nabla_\sigma u_\rho) - \left(\zeta - \frac{2\eta}{3} \right) \mathcal{P}^{\mu\nu} \nabla_\rho u^\rho, \quad (1b)$$

where η and ζ are shear and bulk viscosities, σ_Q is a “quantum critical” conductivity,

$$\mathcal{P}^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \quad (2)$$

and

$$\mathcal{D}_1 = \frac{C\mu^2}{2} \left(1 - \frac{2}{3} \frac{n\mu}{\epsilon + P} \right) - \frac{4G\mu n T^2}{\epsilon + P}, \quad (3a)$$

$$\mathcal{D}_2 = C\mu \left(1 - \frac{1}{2} \frac{n\mu}{\epsilon + P} \right) - \frac{GT^2 n}{\epsilon + P}. \quad (3b)$$

There is a further coefficient that is allowed in \mathcal{D}_1 [35, 40], though it does not contribute to transport and so we will neglect it in this paper. The entropy current is given by

$$s^\mu \equiv (\epsilon + P) u^\mu - \frac{\mu}{T} J^\mu + \left(\frac{C\mu^3}{3T} + 2G\mu T \right) \varepsilon^{\mu\nu\rho\sigma} u_\nu \nabla_\rho u_\sigma + \left(\frac{C\mu^2}{2T} + GT \right) \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma}. \quad (4)$$

B Transport in the Weak Disorder Limit

Here we present details of the computation of the thermoelectric conductivity matrix, using the notation for the perturbative transport computation presented in the main text. At the first non-trivial order

in an expansion at small B and $1/\xi$ (assuming that they are of a similar magnitude),¹ the linearized hydrodynamic equations are

$$\partial^i \left[n_a \tilde{v}_{ia} + \tilde{n}_a v_{ia} + \sigma_{\text{Qa}} \left(\tilde{E}_i + \varepsilon_{ijk} B^k \tilde{v}_{ja} - \partial_i \tilde{\mu}_a - \frac{\mu_a}{T} (T \tilde{\zeta}_i - \partial_i \tilde{T}_a) \right) \right] - \partial_i \left\{ \varepsilon^{ijk} \partial_j \mathcal{D}_{1a} \tilde{v}_{ka} - B_i \tilde{\mathcal{D}}_{2a} + \mathcal{D}_{2a} \varepsilon^{ijk} \tilde{v}_{ja} \partial_k \mu_0 \right\} = C_a \tilde{E}_z B - \sum_b \left[\mathcal{R}_{ab} \tilde{\nu}_b + \mathcal{S}_{ab} \tilde{\beta}_b \right], \quad (5a)$$

$$\begin{aligned} \partial_i \left[T s_a \tilde{v}_{ia} + (\tilde{\epsilon}_a + \tilde{P}_a - \mu_0 \tilde{n}_a) v_{ia} - \mu_0 \sigma_{\text{Qa}} \left(\tilde{E}_i + \varepsilon_{ijk} B^k \tilde{v}_{ja} - \partial_i \tilde{\mu}_a - \frac{\mu_a}{T} (T \tilde{\zeta}_i - \partial_i \tilde{T}_a) \right) - v_{ja} \eta_a \partial_j \tilde{v}_{ia} \right. \\ \left. - v_{ia} \left(\zeta_a + \frac{\eta_a}{3} \right) \partial_j \tilde{v}_{ja} \right] - \partial_i \left\{ \mu_0 B^i \tilde{\mathcal{D}}_{2a} - \mu_0 \mathcal{D}_{2a} \varepsilon^{ijk} \tilde{v}_{ja} \partial_k \mu_0 - \mu_0 \mathcal{D}_{1a} \varepsilon^{ijk} \partial_j \tilde{v}_{ka} \right\} \\ = 2G_a T_0^2 B^i \tilde{\zeta}_i + \sum_b \left[(\mu_0 \mathcal{R}_{ab} + \mathcal{U}_{ab}) \tilde{\nu}_b + (\mu_0 \mathcal{S}_{ab} + \mathcal{V}_{ab}) \tilde{\beta}_b \right], \end{aligned} \quad (5b)$$

$$\begin{aligned} n_a (\partial_i \tilde{\mu}_a - \tilde{E}_i) + s_a (\partial_i \tilde{T}_a - T \tilde{\zeta}_i) + \partial^j \left((\epsilon_a + P_a) (v_{ja} \tilde{v}_{ia} + v_{ia} \tilde{v}_{ja}) - \eta_a (\partial_j \tilde{v}_{ia} + \partial_i \tilde{v}_{ja}) - \left(\zeta_a - \frac{2\eta_a}{3} \right) \delta_{ij} \partial_k \tilde{v}_{ka} \right) \\ = \varepsilon_{ijk} \tilde{J}_a^j B^k + v_{ia} \sum_b (\mathcal{U}_{ab} \tilde{\nu}_b + \mathcal{V}_{ab} \tilde{\beta}_b). \end{aligned} \quad (5c)$$

These are respectively the equations of motion for charge, heat and momentum. In order to derive these equations, it is important to use covariant derivatives with respect to the metric. We stress the importance of carefully deriving the $\tilde{\zeta}_i$ -dependent terms in (5). It is crucial that such terms be correctly accounted for in order for the resulting theory of transport to obey Onsager reciprocity at the perturbative level.

We have been able to remove many potential terms in the above equations which end up being proportional to $\varepsilon_{ijk} \partial_j \mu_0 \partial_k \mu_0 = \varepsilon_{ijk} \partial_j \mu_0 \partial_k v_l = 0$. In (5), \tilde{n} and other thermodynamic objects are to be interpreted as $\tilde{n} = (\partial_\mu n) \tilde{\mu} + (\partial_T n) \tilde{T}$, for example. One finds terms $\sim \omega^{-1}$, which vanish identically assuming that the background is a solution to the hydrodynamic equations; higher order terms in ω vanish upon taking $\omega \rightarrow 0$. (5) is written in such a way that the terms on the left hand side are single-valley terms, with non-anomalous contributions to the charge and heat conservation laws written as the divergence of a current in square brackets, and anomalous contributions as the divergence of a current in curly brackets; the terms on the right hand side of the charge and heat conservation laws are spatially homogeneous violations of the conservation laws.

The equations (5) are valid for a disordered chemical potential of any strength. We will now focus on the case where it is perturbatively small, and take the perturbative limit described in the main text.

Let us split all of our perturbations into constants (zero modes of spatial momentum) and spatially fluctuating pieces:

$$\tilde{v}_{ia} = \tilde{w}_{ia} + \frac{\partial \mathbf{v}_i}{\partial \mu} \tilde{\mu}_a + \frac{\partial \mathbf{v}_i}{\partial T} \tilde{T}_a, \quad (6a)$$

$$\tilde{w}_{ia} = \tilde{\mathbb{V}}_{ia} + \tilde{V}_{ia}(\mathbf{x}), \quad (6b)$$

$$\tilde{\mu}_a = \tilde{\mathbb{M}}_a + \tilde{M}_a(\mathbf{x}), \quad (6c)$$

$$\tilde{T}_a = \tilde{\mathbb{T}}_a + \tilde{\theta}_a(\mathbf{x}). \quad (6d)$$

Recall that we defined \mathbf{v} to be the function of μ and T which gives the equilibrium fluid velocity. We will show self-consistently that the functions and constants introduced above scale as

$$\tilde{\mathbb{V}} \sim u^{-2}, \quad \tilde{\mathbb{M}} \sim \tilde{\mathbb{T}} \sim u^{-4}, \quad \tilde{V}, \tilde{M}, \tilde{\theta} \sim u^{-1}, \quad (7)$$

¹It is important to only work to leading order in $1/\xi$ since we only know the background solution to leading order in $1/\xi$.

at leading order in perturbation theory. This will lead to charge and heat currents (and hence a conductivity matrix) which scale as u^{-2} . To correctly capture the leading order response at small u , we do not need every term which has been retained in (5). At leading order, linearized equations of motion reduce to those shown in the main text. Upon replacing \tilde{v} with \tilde{w} , the resulting equations have become much simpler. Note that terms proportional to $\varepsilon_{ijk}\mathbf{v}_j B_k = 0$ (since $\mathbf{v}_i \sim B_i$) can be dropped in the limit $B \rightarrow 0$ and $\xi \rightarrow \infty$, as can viscous terms, which can be shown to contribute extra factors of $1/\xi$ to the answer [32].

Next, we define the charge and heat currents in our hydrodynamic theory. In an individual valley fluid, the leading order contributions to the charge current (1) are

$$\tilde{J}_a^i = n_a \tilde{v}_{ia} + \tilde{n}_a v_{ia} + \tilde{\mathcal{D}}_{2a} B_i = n_a \tilde{\mathbb{V}}_{ia} + C_a \tilde{\mathbb{M}}_a B_i. \quad (8)$$

The last step follows from the definitions of \mathbf{v} and \mathbb{V} . We assume that the total charge current is

$$\tilde{J}^i = \sum_a \tilde{J}_a^i. \quad (9)$$

The canonical definition of the global heat current for all valley fluids is

$$\tilde{Q}^i = \tilde{T}^{ti} - \mu_0 \tilde{J}_i. \quad (10)$$

In order to write \tilde{Q}^i as a sum over valley contributions:

$$\tilde{Q}^i = \sum_a \tilde{Q}_a^i, \quad (11)$$

we define

$$\tilde{Q}_a^i = \tilde{T}_a^{ti} - \mu_0 \tilde{J}_a^i. \quad (12)$$

A simple computation reveals that at leading order in perturbation theory,

$$\tilde{Q}_a^i = T s_a \tilde{\mathbb{V}}_{ia} + 2G_a T_0 \tilde{\mathbb{T}}_a B_i. \quad (13)$$

\tilde{Q}_a^i is not equivalent to the entropy current of an individual valley fluid, even at leading order.

We now proceed to determine the spatially uniform responses $\tilde{\mathbb{V}}_{ia}$, $\tilde{\mathbb{M}}_a$ and $\tilde{\mathbb{T}}_a$ to leading order. We begin by focusing on the inhomogeneous parts of the linearized equations. It is simplest to do so in momentum space. At the leading order $\mathcal{O}(u^{-1})$, the inhomogeneous equations of motion are

$$ik_i \left[n_a(\mathbf{k}) \tilde{\mathbb{V}}_{ia} + n_a \tilde{V}_{ia}(\mathbf{k}) - \sigma_{\text{Q}a} ik_i \left(\tilde{M}_a(\mathbf{k}) - \frac{\mu_0}{T} \tilde{\theta}_a(\mathbf{k}) \right) \right] = 0, \quad (14a)$$

$$ik_i \left[T s_a(\mathbf{k}) \tilde{\mathbb{V}}_{ia} + T s_a \tilde{V}_{ia}(\mathbf{k}) + \mu_0 \sigma_{\text{Q}a} ik_i \left(\tilde{M}_a(\mathbf{k}) - \frac{\mu_0}{T} \tilde{\theta}_a(\mathbf{k}) \right) \right] = 0, \quad (14b)$$

$$n_a \tilde{M}_a(\mathbf{k}) + s_a \tilde{\theta}_a(\mathbf{k}) = 0. \quad (14c)$$

These equations are identical to those in [32] (with vanishing viscosity), but in one higher dimension. Note that any term written without an explicit \mathbf{k} dependence denotes the constant $\mathbf{k} = \mathbf{0}$ mode. These equations give the following relations for the spatially dependent parts of the hydrodynamic variables

$$k_i \tilde{V}_{ia}(\mathbf{k}) = -\frac{\mu_0 n_a(\mathbf{k}) + T_0 s_a(\mathbf{k})}{\epsilon_a + P_a} k_i \tilde{\mathbb{V}}_{ia}, \quad (15a)$$

$$\tilde{\theta}_a(\mathbf{k}) = \frac{ik_i \tilde{\mathbb{V}}_{ia} T_0^2 n_a (s_a n_a(\mathbf{k}) - n_a s_a(\mathbf{k}))}{\sigma_{\text{Q}a} k^2 (\epsilon_a + P_a)^2}, \quad (15b)$$

$$\tilde{M}_a(\mathbf{k}) = -\frac{ik_i \tilde{\nabla}_{ia} T_0^2 s_a (s_a n_a(\mathbf{k}) - n_a s_a(\mathbf{k}))}{\sigma_{Qa} k^2 (\epsilon_a + P_a)^2}. \quad (15c)$$

To determine the conductivities, we also require the leading order homogeneous components of the equations of motion. Spatially integrating over the momentum conservation equation, we find the leading order equation (at order $\mathcal{O}(u^0)$)

$$\Gamma_{ija} \tilde{\nabla}_{ja} - n_a \tilde{E}_i - s_a T_0 \tilde{\zeta}_i = \varepsilon_{ijk} n_a \tilde{\nabla}_{ja} B_k, \quad (16)$$

with

$$\Gamma_{ija} \equiv \sum_{\mathbf{k}} \frac{k_i k_j}{k^2} \frac{T_0^2 (s_a (\partial n_a / \partial \mu) - n_a (\partial s_a / \partial \mu))^2}{\sigma_{Qa} (\epsilon_a + P_a)^2} u^2 |\hat{\mu}(\mathbf{k})|^2. \quad (17)$$

Γ_{ija} is proportional to the rate at which momentum relaxes in the fluid due to the effects of the inhomogeneous chemical potential. Henceforth, we will assume isotropy for simplicity: $\Gamma_{ija} \equiv \Gamma_a \delta_{ij}$. It is manifest from the definition that $\Gamma_a > 0$. We can easily solve this equation for $\tilde{\nabla}_{ia}$.

Finally, to see the effects of the anomalies on hydrodynamic transport, we spatially average over the charge and heat conservation equations. At leading order $\mathcal{O}(u^2)$, this gives

$$C_a B E_z = \sum_b \left[\mathcal{R}_{ab} \tilde{\nu}_b + \mathcal{S}_{ab} \tilde{\beta}_b \right], \quad (18a)$$

$$-C_a \mu_0 B E_z - 2G_a T_0^2 B \tilde{\zeta}_z = \sum_b \left[\mathcal{U}_{ab} \tilde{\nu}_b + \mathcal{V}_{ab} \tilde{\beta}_b \right]. \quad (18b)$$

Due to the anomalies, external temperature gradients and electric fields induce changes in the chemical potential and temperature of each fluid, which result in charge and heat flow. In the above equations, and for the rest of this paragraph, the fluctuations $\tilde{\nu}_b$ and $\tilde{\beta}_b$ denote the homogeneous parts of these objects, as it is only these which contribute at leading order to $\tilde{\mathbb{M}}$ and $\tilde{\mathbb{T}}$. Hence, we find that

$$\begin{pmatrix} \tilde{\nu}_a \\ -\tilde{\beta}_a \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{ab} & -\mathcal{S}_{ab} \\ -\mathcal{U}_{ab} & \mathcal{V}_{ab} \end{pmatrix}^{-1} \begin{pmatrix} C_b B E_z \\ C_b \mu_0 B E_z + 2G_b T_0^2 B \tilde{\zeta}_z \end{pmatrix}. \quad (19)$$

We will find useful the relation

$$\begin{pmatrix} \tilde{\mathbb{M}} \\ \tilde{\mathbb{T}} \end{pmatrix} = \begin{pmatrix} T & \mu T \\ 0 & T^2 \end{pmatrix} \begin{pmatrix} \tilde{\nu} \\ -\tilde{\beta} \end{pmatrix}. \quad (20)$$

We are now ready to construct the thermoelectric conductivity matrix. Combining the definition of the thermoelectric conductivity matrix, (9) and (11) with our hydrodynamic results (8), (13), (16), (19) and (20) we obtain the thermoelectric conductivity matrix presented in the main text.

C Simple Example

It is instructive to study the simplest possible system with an anomalous contribution to the conductivity. This is a Weyl semimetal with 2 valley fluids, where the Berry flux

$$k_1 = -k_2 = 1, \quad (21)$$

and $C_{1,2}$ and $G_{1,2}$ are given by the results for a free Weyl fermion [41]. We also assume that the equations of state and disorder for each valley fluid are identical, so that $n_{1,2} = n$, $s_{1,2} = s$ and $\Gamma_{1,2} = \Gamma$. Finally,

we take the simplest possible ansatz for \mathbf{A} consistent with symmetry, positive-definiteness and global conservation laws:

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}_0 & -\mathcal{R}_0 \\ -\mathcal{R}_0 & \mathcal{R}_0 \end{pmatrix}, \quad (22a)$$

$$\mathcal{S} = \mathcal{U} = \begin{pmatrix} \mathcal{S}_0 & -\mathcal{S}_0 \\ -\mathcal{S}_0 & \mathcal{S}_0 \end{pmatrix}, \quad (22b)$$

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_0 & -\mathcal{V}_0 \\ -\mathcal{V}_0 & \mathcal{V}_0 \end{pmatrix}, \quad (22c)$$

with positive-semidefiniteness of \mathbf{A} imposing $\mathcal{R}_0, \mathcal{V}_0 \geq 0$ and

$$\mathcal{R}_0 \mathcal{V}_0 \geq \mathcal{S}_0^2. \quad (23)$$

We find the thermoelectric conductivities

$$\sigma_{xx} = \sigma_{yy} = 2 \frac{n^2 \Gamma}{\Gamma^2 + B^2 n^2}, \quad (24a)$$

$$\sigma_{xy} = 2 \frac{B n^3}{\Gamma^2 + B^2 n^2}, \quad (24b)$$

$$\sigma_{xz} = \sigma_{yz} = 0, \quad (24c)$$

$$\sigma_{zz} = 2 \frac{n^2}{\Gamma} + \frac{T B^2 (\mathcal{R}_0 \mu^2 + 2 \mathcal{S}_0 \mu + \mathcal{V}_0)}{16 \pi^4 (\mathcal{R}_0 \mathcal{V}_0 - \mathcal{S}_0^2)}, \quad (24d)$$

$$\bar{\kappa}_{xx} = \bar{\kappa}_{yy} = 2 \frac{T s^2 \Gamma}{\Gamma^2 + B^2 n^2}, \quad (24e)$$

$$\bar{\kappa}_{xy} = 2 \frac{B T n s^2}{\Gamma^2 + B^2 n^2}, \quad (24f)$$

$$\bar{\kappa}_{xz} = \bar{\kappa}_{yz} = 0, \quad (24g)$$

$$\bar{\kappa}_{zz} = 2 \frac{T s^2}{\Gamma} + \frac{T^4 B^2 \mathcal{R}_0}{144 (\mathcal{R}_0 \mathcal{V}_0 - \mathcal{S}_0^2)}, \quad (24h)$$

$$\alpha_{xx} = \alpha_{yy} = 2 \frac{n s \Gamma}{\Gamma^2 + B^2 n^2}, \quad (24i)$$

$$\alpha_{xy} = -\alpha_{yx} = 2 \frac{B n^2 s}{\Gamma^2 + B^2 n^2}, \quad (24j)$$

$$\alpha_{xz} = \alpha_{yz} = 0, \quad (24k)$$

$$\alpha_{zz} = 2 \frac{n s}{\Gamma} + \frac{T^2 B^2 (\mathcal{R}_0 \mu + \mathcal{S}_0)}{48 \pi^2 (\mathcal{R}_0 \mathcal{V}_0 - \mathcal{S}_0^2)}. \quad (24l)$$

As expected due to the matrix inverse in the expressions for \mathfrak{s} , \mathfrak{a} and \mathfrak{h} , we see that the anomalous contributions to the conductivities depend on the intervalley scattering rates for charge and energy in a rather complicated way.

D Imposing External Sources Through Boundary Conditions

The derivation of the thermoelectric conductivity matrix presented above applied \tilde{E}_i and $\tilde{\zeta}_i$ by particular deformations to background fields. As in [46], one might also wish to impose electric fields and temperature

gradients in a space with boundaries, as is done in a real experiment. In this case, we do not need to deform the metric from Minkowski space, nor the external gauge field, as we did in the main text.

For example, let us keep the x and y directions periodic, but consider a Weyl fluid in the domain $0 \leq z \leq L$, subject to the boundary conditions

$$\mu(z=0) = \mu_0, \quad \mu(z=L) = \mu_0 - \tilde{E}_z L, \quad (25a)$$

$$T(z=0) = T_0, \quad T(z=L) = T_0 - \tilde{\zeta}_z T_0 L. \quad (25b)$$

The hydrodynamic variables become

$$\mu_a = \mu_0 + \tilde{\mu}_a - \tilde{E}_z z, \quad (26a)$$

$$T_a = T_0 + \tilde{T}_a - T_0 \tilde{\zeta}_z z, \quad (26b)$$

$$u_a^\mu = (1, v_{ia} + \tilde{v}_{ia}), \quad (26c)$$

and, in linear response, we can also arrive at (5). The simplest way to see this as follows. In equilibrium the hydrodynamic equations are satisfied (at leading order in B and ξ). After taking spatial derivatives in $\partial_\mu J_a^\mu$ (for example), it is possible to obtain terms of the form $-\tilde{E}_z z \times \partial_i \mu_0$ which are linear in z . However, all such terms must identically cancel, because the background solution is independent of a global spatial shift in μ and T . We have, in fact, already seen this explicitly – the coefficients of \mathbb{M}_a and \mathbb{T}_a in the charge and heat currents (8) and (13) are all independent of \mathbf{x} .

Hence, upon plugging in (26) into the equations of motion, the only terms which do not vanish at leading order in B and $1/\xi$ are

$$\partial_z J_a^z = \partial_z (-C_a B E_z z + \dots) = - \sum_b \left[\mathcal{R}_{ab} \tilde{\nu}_b + \mathcal{S}_{ab} \tilde{\beta}_b \right] \quad (27a)$$

$$\partial_z (T_a^{tz} - \mu_0 J_a^z) = \partial_z \left(-2G_a B T_0^2 \tilde{\zeta}_z z + \dots \right) = \sum_b \left[(\mu_0 \mathcal{R}_{ab} + \mathcal{U}_{ab}) \tilde{\nu}_b + (\mu_0 \mathcal{S}_{ab} + \mathcal{V}_{ab}) \tilde{\beta}_b \right], \quad (27b)$$

$$\partial_z T_a^{zz} = -n_a \tilde{E}_z - T_0 s_a \tilde{\zeta}_z + \dots \quad (27c)$$

The \dots terms above are linear in $\tilde{\mu}_a$, \tilde{T}_a or \tilde{v}_{ia} , and are the same as found in (5). Upon comparing with (5), we see that the source (\tilde{E}_z and $\tilde{\zeta}_z$) terms are identical.

Hence our equations of motion (5) are unchanged, and our perturbative theory of transport can be recovered regardless of the choice of boundary conditions. This is important as experiments will always impose temperature gradients through the boundary conditions on a finite domain.

E Coulomb Screening

Coulomb screening alters the electric field seen by the charges. In our equilibrium solution, it leads to an effective change in $\mu_0(\mathbf{x})$, the disorder profile seen by the fluid. We may account for it by replacing

$$\mu_0 \rightarrow \mu_0 - \varphi \equiv \mu_0 - \int d^3\mathbf{y} K(\mathbf{x}; \mathbf{y}) \sum_a n_a(\mathbf{y}), \quad (28)$$

with $K \sim 1/r$ the Coulomb kernel (its precise form is not important, and we could include thermal screening effects if we wish). However, as pointed out in [32, 33], by simply redefining μ_0 to be the equilibrium electrochemical potential, one can neglect this effect.

We must still account for the effects of Coulomb screening on the linear response around the equilibrium state. In our perturbative formalism, the leading order conductivities are governed by the equations of

motion (5), and the simplification in the main text. We may account for Coulomb screening in these equations by modifying the external electric field to

$$\tilde{E}_i \rightarrow \tilde{E}_i - \partial_i \tilde{\varphi}, \quad (29)$$

where $\tilde{\varphi}$ is the convolution of the Coulomb kernel with $\sum_a \tilde{n}_a$. The equations of motion then become

$$\partial_i \left[n_a \tilde{w}_{ia} + \sigma_{qa} \left(\tilde{E}_i - \partial_i \tilde{\Phi}_a - \frac{\mu_0}{T} (T \tilde{\zeta}_i - \partial_i \tilde{T}_a) \right) \right] = C_a \tilde{E}_i B_i - \sum_b \left[\mathcal{R}_{ab} \tilde{\nu}_b + \mathcal{S}_{ab} \tilde{\beta}_b \right], \quad (30a)$$

$$\begin{aligned} \partial_i \left[T s_a \tilde{w}_{ia} - \mu_0 \sigma_{qa} \left(\tilde{E}_i - \partial_i \tilde{\Phi}_a - \frac{\mu_0}{T} (T \tilde{\zeta}_i - \partial_i \tilde{T}_a) \right) \right] &= 2 G_a T_0^2 \tilde{\zeta}_i B_i \\ &+ \sum_b \left[(\mathcal{R}_{ab} \mu_0 + \mathcal{U}_{ab}) \tilde{\nu}_b + (\mathcal{S}_{ab} \mu_0 + \mathcal{V}_{ab}) \tilde{\beta}_b \right], \end{aligned} \quad (30b)$$

$$n_a (\partial_i \tilde{\Phi}_a - \tilde{E}_i) + s_a (\partial_i \tilde{T}_a - T \tilde{\zeta}_i) = \varepsilon_{ijk} \tilde{w}_{ja} n_a B_k, \quad (30c)$$

where we have defined

$$\tilde{\Phi}_a \equiv \tilde{\varphi} + \tilde{\mu}_a. \quad (31)$$

We have neglected the contribution of the Coulomb kernel to the anomalous creation of charge in a single valley in (30). This is because, in our perturbative limit, only the homogeneous part of this term is important. Since

$$\tilde{\nu}_a = \frac{\tilde{\mu}_a}{T_0} + \tilde{\beta}_a \mu_0 = \frac{\tilde{\Phi}_a - \tilde{\varphi}}{T_0} + \tilde{\beta}_a \mu_0, \quad (32)$$

it follows from the fact that $\sum_b \mathcal{R}_{ab} = \sum_b \mathcal{S}_{ab} = 0$ that the $\tilde{\varphi}$ -dependent corrections to the inter-valley terms exactly cancel. Mathematically, we now see that (30) are the same as the linearized equations of motion in the main text, up to a relabeling of the variables. The long-range Coulomb interactions, introduced in hydrodynamics through $F_{\mu\nu}$, do not alter our definitions of the charge current (8) or the heat current (13) at leading order in perturbation theory. Hence, our expressions for the conductivities are not affected by long-range Coulomb interactions, confirming our claim in the main text. The interactions may alter the specific values of the parameters in the hydrodynamic equations (1).²

Finite frequency transport is generally sensitive to long-range Coulomb interactions, although in (disordered) charge-neutral systems the effect is likely much more suppressed (see [33] for a recent discussion in two spatial dimensions).

F Violation of the Wiedemann-Franz Law in the Hydrodynamic Regime

Since σ_{ij} , α_{ij} and $\bar{\kappa}_{ij}$ are block diagonal in our perturbative hydrodynamic formalism, κ_{ij} will be as well. We begin by focusing on the longitudinal (zz) conductivities. A simple computation gives

$$\kappa_{zz} = T \sum_a \frac{s_a^2}{\Gamma_a} + \mathfrak{h} B^2 - T \left(\mathfrak{a} B^2 + \sum_a \frac{s_a n_a}{\Gamma_a} \right)^2 \left(\mathfrak{s} B^2 + \sum_a \frac{n_a^2}{\Gamma_a} \right)^{-1}. \quad (33)$$

Firstly, consider the case $B = 0$. In this case, there are two possibilities of interest. If³

$$s_a = s \quad \text{and} \quad n_a = n, \quad (34)$$

²More carefully, if we place our equations on a periodic space, where the transport problem is still well-posed, then the boundary conditions on \tilde{w}_{ia} , $\tilde{\mu}_a$ and \tilde{T}_a are all periodic boundary conditions. Hence, \tilde{w}_{ia} , $\tilde{\Phi}_a$ and \tilde{T}_a all have periodic boundary conditions and so the change of variables between the linearized equations presented in the main text and (30) does not affect the transport problem even via non-trivial boundary conditions.

³This is a stronger statement than necessary for this equation to hold for κ_{zz} . It is sufficient for the ratio s_a/n_a to be identical for all valley fluids for (35) to hold. However, this stricter requirement is necessary for (39) to hold.

for all valley fluids, then

$$\kappa_{zz}(B=0) \sim \mathcal{O}(u^0), \quad (35)$$

is subleading in perturbation theory. Hence, assuming $n \neq 0$ (i.e., the system is at finite charge density), we find that $\mathcal{L}_{zz} \ll \mathcal{L}_{WF}$ in the perturbative limit $u \rightarrow 0$. That a charged fluid has a highly suppressed κ is by now a well-appreciated effect in normal relativistic fluids with a single valley [31, 54]. If the valley fluids are indistinguishable as in (34), then they behave as a “single valley” at $B = 0$ and so the considerations of [31, 54] apply here. The reason that (35) is small relative to $\bar{\kappa}_{zz}$ is that the boundary condition $\tilde{J} = 0$ forces us to set (at leading order in u) the velocity $\tilde{\mathbf{V}} = 0$, which means that both the leading order charge and heat currents vanish.

However, at a non-zero value of B , the leading order contribution does not vanish: $\kappa_{zz}(B) \sim u^{-2}$. In particular, as $B \rightarrow 0$

$$\kappa_{zz}(B \rightarrow 0) \approx \left(\mathfrak{h} + Ts \frac{s\mathfrak{s} - 2n\mathfrak{a}}{n^2} \right) B^2, \quad (36)$$

while at larger B (such that n^2/\mathfrak{s} , sn/\mathfrak{a} , $Ts^2/\mathfrak{h} \ll \Gamma B^2$, while keeping $B \ll T^2$)

$$\kappa_{zz} \approx \left(\mathfrak{h} - \frac{T\mathfrak{a}^2}{\mathfrak{s}} \right) B^2. \quad (37)$$

Hence, as $B \rightarrow 0$, $\mathcal{L}_{zz} \sim B^2$ is parametrically small, but at larger B it approaches a finite value

$$\mathcal{L}_{zz} \rightarrow \frac{\mathfrak{h}}{T\mathfrak{s}} - \frac{\mathfrak{a}^2}{\mathfrak{s}^2}. \quad (38)$$

If (34) does not hold, then we instead find that $\kappa_{zz}(B)$ is finite and $\mathcal{O}(u^{-2})$, just as $\sigma_{zz}(B)$. Hence, we find that the Lorenz ratio \mathcal{L}_{zz} is generally $\mathcal{O}(1)$ and there are no parametric violations. However, the Wiedemann-Franz law will not hold in any quantitative sense, and \mathcal{L}_{zz} can easily be B -dependent.

In the xy -plane, the Wiedemann-Franz law has a somewhat similar fate. If (34) holds, then we find that the leading order contributions to the thermal conductivities all vanish at leading order, so that

$$\kappa_{xx}(B) \sim \kappa_{yy}(B) \sim \kappa_{xy}(B) \sim \kappa_{yx}(B) \sim \mathcal{O}(u^0), \quad (39)$$

at all values of B , and so the corresponding components of κ_{ij} will be parametrically small. If (34) does not hold, then $\kappa_{ij}(B) \sim u^{-2}$ is never parametrically small, and so the Wiedemann-Franz law will not be violated parametrically, but will be violated by an $\mathcal{O}(1)$ B -dependent function.

G Weak Intervalley Scattering in a Weakly Interacting Weyl Gas

As noted in the main text, it is possible to employ our hydrodynamic formalism even when the fluids at each node are weakly interacting. In fact, the only requirement to use our formalism for computing the anomalous thermoelectric conductivities is that the time scales set by \mathbf{A} are the longest time scales in the problem (in particular: slower than any thermalization time scale within a given valley fluid). Now, we consider a weakly interacting Weyl semimetal with long lived quasiparticles, but where the intervalley scattering is weak enough that our formalism nevertheless is valid. We will employ semiclassical kinetic theory to find relations between \mathcal{R}_0 , \mathcal{S}_0 and \mathcal{V}_0 under reasonable assumptions.

For simplicity, as in our simple example above, we will consider a pair of nodes with opposite Berry flux, but otherwise identical equations of state. We suppose that the two Weyl nodes (located at the same Fermi energy) are at points $\mathbf{K}_{1,2}$ in the Brillouin zone, with $|\mathbf{K}_1 - \mathbf{K}_2| \gg \mu, T$.

Denote with $f(\mathbf{k})$ the number density of quasiparticles at momentum \mathbf{k} . Under very basic assumptions about the nature of weak scattering off of impurities, assuming all scattering off of impurities is elastic, one finds the kinetic theory result [50]

$$\frac{df(\mathbf{k})}{dt} = \int \frac{d^3\mathbf{k}'}{(2\pi)^3} W(\mathbf{k}, \mathbf{k}') [f(\mathbf{k}') - f(\mathbf{k})] \quad (40)$$

where for simplicity we assume spatial homogeneity. $W(\mathbf{k}, \mathbf{k}')$ denotes the scattering rate of a quasiparticle from momentum \mathbf{k} to \mathbf{k}' – under the assumptions listed above, this is a symmetric function which may be perturbatively computed using Fermi's golden rule [50]. Since scattering is elastic, we have $W(\mathbf{k}, \mathbf{q}) \sim \delta(k - q)$. Using that⁴

$$\frac{dn_1}{dt} = -\frac{dn_2}{dt} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big|_{\text{node 1}} \frac{d^3\mathbf{q}}{(2\pi)^3} \Big|_{\text{node 2}} W(\mathbf{k}, \mathbf{q}) [f(\mathbf{q}) - f(\mathbf{k})] \quad (41a)$$

$$\frac{d\epsilon_1}{dt} = -\frac{d\epsilon_2}{dt} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big|_{\text{node 1}} \frac{d^3\mathbf{q}}{(2\pi)^3} \Big|_{\text{node 2}} W(\mathbf{k}, \mathbf{q}) [f(\mathbf{q}) - f(\mathbf{k})] |\mathbf{k}| \quad (41b)$$

In the above integrals, the subscript node 1 implies that the momentum integral is shifted so that $\mathbf{k} = \mathbf{0}$ at the point \mathbf{K}_1 ; a similar statement holds for node 2. All low energy quasiparticles are readily identified as either belonging to node 1 or 2. Since we have set $v_F = 1$, the energy of a quasiparticle of momentum \mathbf{k} (near node 1) is simply $|\mathbf{k}|$; a similar statement holds for quasiparticles near node 2.

The simplest possible assumption is that

$$W(\mathbf{k}, \mathbf{q}) = W_0(k) \delta(|\mathbf{k}| - |\mathbf{q}|), \quad (42)$$

and we may further take W_0 to be a constant if we desire. For our hydrodynamic description to be valid, nodes 1 and 2 are in thermal equilibrium, up to a relative infinitesimal shift in temperature and chemical potential. For simplicity suppose that node 1 is at a different β and ν . Then the infinitesimal change in the rate of charge and energy transfer is

$$\frac{dn_1}{dt} = - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big|_{\text{node 1}} \frac{d^3\mathbf{q}}{(2\pi)^3} \Big|_{\text{node 2}} W(\mathbf{k}, \mathbf{q}) n_F(\beta k - \nu) \quad (43a)$$

$$\frac{d\epsilon_1}{dt} = - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big|_{\text{node 1}} \frac{d^3\mathbf{q}}{(2\pi)^3} \Big|_{\text{node 2}} W(\mathbf{k}, \mathbf{q}) n_F(\beta k - \nu) k \quad (43b)$$

We can now read off

$$\mathcal{R}_0 = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big|_{\text{node 1}} \frac{d^3\mathbf{q}}{(2\pi)^3} \Big|_{\text{node 2}} W(\mathbf{k}, \mathbf{q}) (-n'_F(\beta k - \nu)), \quad (44a)$$

$$\mathcal{S}_0 = \mathcal{U}_0 = - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big|_{\text{node 1}} \frac{d^3\mathbf{q}}{(2\pi)^3} \Big|_{\text{node 2}} W(\mathbf{k}, \mathbf{q}) (-n'_F(\beta k - \nu)) k, \quad (44b)$$

$$\mathcal{V}_0 = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big|_{\text{node 1}} \frac{d^3\mathbf{q}}{(2\pi)^3} \Big|_{\text{node 2}} W(\mathbf{k}, \mathbf{q}) (-n'_F(\beta k - \nu)) k^2. \quad (44c)$$

Our kinetic theory computation gives $\mathcal{S}_0 = \mathcal{U}_0$, as required by general quantum mechanical principles, and serves as a consistency check on our kinetic theory approximations.

⁴In these equations we have noted that the integrand is odd under exchanging \mathbf{k} and \mathbf{q} (and thus vanishes upon integration over \mathbf{k} and \mathbf{q}) if \mathbf{k} and \mathbf{q} belong to the same node.

As is reasonable for many materials, we first approximate that $\mu \gg T$. Using the Sommerfeld expansion of the Fermi function, we find that the leading and next-to-leading order terms as $T/\mu \rightarrow 0$ are:

$$\mathcal{R}_0 \approx TA(\mu) + \frac{A''(\mu)}{2} \frac{\pi^2}{3} T^3, \quad (45a)$$

$$\mathcal{S}_0 \approx -T\mu A(\mu) - \mu \frac{A''(\mu)}{2} \frac{\pi^2}{3} T^3 - A'(\mu) \frac{\pi^2}{3} T^3 \quad (45b)$$

$$\mathcal{V}_0 \approx T\mu^2 A(\mu) + \frac{\pi^2}{3} T^3 \left(A(\mu) + 2\mu A'(\mu) + \frac{A''(\mu)}{2} \mu^2 \right) \quad (45c)$$

where we have defined

$$A(\mu) = \frac{\mu^4}{4\pi^4} W_0(\mu). \quad (46)$$

The leading order anomalous conductivities are:

$$\sigma_{zz} = \frac{B^2}{16\pi^4} \frac{1}{A(\mu)}, \quad (47a)$$

$$\alpha_{zz} = \frac{\pi^2 T}{3} \frac{\partial \sigma_{zz}(\mu, T=0)}{\partial \mu}, \quad (47b)$$

$$\bar{\kappa}_{zz} = \frac{\pi^2 T}{3} \sigma_{zz}. \quad (47c)$$

Remarkably, we recover the Wiedemann-Franz law exactly in the limit $\mu \gg T$. This is rather surprising, as the assumptions that have gone into our derivation are subtly different than the standard assumptions about metallic transport. In particular, the ordinary derivation of the Wiedemann-Franz law assumes that elastic scattering off of all disorder is much faster than any thermalization time scale (hence, the conductivity can be written as the sum of conductivities of quasiparticles at each energy scale [50]). In our derivation, we have assumed that intravalley thermalization is much faster than intervalley scattering, which may be the case when the dominant source of disorder is long wavelength [46, 32]. At a technical level, the integral in the numerator of $\sigma_{zz}^{\text{anom}}$ looks much like the standard integral for $\bar{\kappa}_{zz}$ in a metal, and vice versa. The Wiedemann-Franz law is restored by a factor of $(\pi^2/3)^2$ coming from the ratio $(2G/C)^2$. That the Wiedemann-Franz law can arise in a subtle way is emphasized by our interesting violation of the standard Mott relation, which states that $\alpha_{zz} = -(\pi^2 T/3)(\partial \sigma_{zz}(T=0)/\partial \mu)$. This Mott relation differs by a minus sign from the result derived above. The origin of this minus sign is that in our theory, the rate of intervalley scattering is the sum of rates at each quasiparticle energy, as opposed to the net conductivity.

Let us also mention what happens in the regime $\mu \sim T$. In an ordinary semiconductor [51], or a Dirac semimetal such as graphene [55], kinetic theory predicts an $\mathcal{O}(1)$ violation of the Wiedemann-Franz law whereby $\mathcal{L}_{zz} > \mathcal{L}_0$. This is called bipolar diffusion, and is due to the fact that multiple bands with opposite charge carriers are thermally populated, and the thermal conductivity is enhanced by the combined flow of these carriers. Figure 1 shows the fate of the anomalous Wiedemann-Franz law in a Weyl semimetal where intervalley scattering is the slowest timescale in the problem. Here we see the opposite effect – the Wiedemann-Franz law is reduced. The physical explanation of this effect immediately follows from the previous paragraph – bipolar diffusion applies to the scattering rates and not to the conductivities, and hence κ_{zz} is reduced below σ_{zz} as $\mu/T \rightarrow 0$. This discussion should be taken with a grain of salt – it is worth keeping in mind that the regime $\mu/T \rightarrow 0$ is associated with stronger interactions, and so (as in graphene) the quasiparticle description of transport may completely breakdown [29, 32].

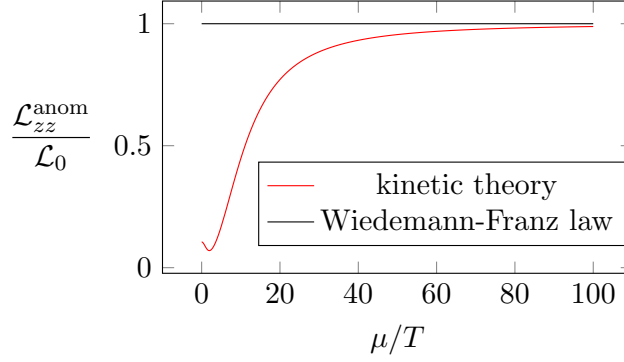


Figure 1: Breakdown of the anomalous Wiedemann-Franz law in the regime where intervalley exchange of charge and energy occurs via quasiparticle scattering and may be treated with kinetic theory. The violation of the Wiedemann-Franz law is opposite to what would be expected in semiconductors or Dirac semimetals such as graphene. We have assumed that W_0 is a constant, though this plot looks qualitatively similar for other choices.

H Memory Matrix Formalism

So far, our theory of transport has relied entirely on a classical theory of anomalous hydrodynamics. Nonetheless, we expect that our results can be computed perturbatively using a more general, inherently quantum mechanical formalism called the memory matrix formalism [45, 39]. The memory matrix formalism is an old many-body approach to transport which does not rely on the existence of long-lived quasiparticles. It is particularly useful in a “hydrodynamic” regime in which only a small number of quantities are long-lived. In such a regime, memory matrix results can be understood for many purposes entirely from classical hydrodynamics [46]. Nevertheless, the memory matrix formalism has some distinct advantages. In particular, it gives microscopic relations for the unknown parameters of the hydrodynamic theory.

Let us give a simple example of how the memory matrix formalism works, leaving technical details to [45, 39]. Suppose we have a system in which the momentum operator P_i is almost exactly conserved. Assuming isotropy, and that there are no other long-lived vector operators, it can be formally shown that the expectation value of P_i will evolve according to

$$\frac{d\langle P_i \rangle}{dt} = -\frac{M_{PP}}{\chi_{PP}} \langle P_i \rangle, \quad (48)$$

where $\chi_{PP} = \text{Re}(G_{P_x P_x}^R(\mathbf{k} = \mathbf{0}, \omega = 0))$ is the momentum-momentum susceptibility, and M_{PP} is a component of the memory matrix, which is schematically given by

$$M_{PP} \approx \lim_{\omega \rightarrow 0} \frac{\text{Im} \left(G_{\dot{P}_x \dot{P}_x}^R(\mathbf{k} = \mathbf{0}, \omega) \right)}{\omega}. \quad (49)$$

More formal expressions may be found in [39, 56]. Note the presence of operator time derivatives (i.e. $\dot{P} = i[H, P]$, with H the global Hamiltonian) in the expression for M_{PP} . From the hydrodynamic equation (48), it is clear that the momentum relaxation rate is determined by M_{PP} . For a given microscopic Hamiltonian H , we can therefore simply evaluate this element of the memory matrix to obtain the value of the momentum relaxation rate in the hydrodynamic theory.

The memory matrix formalism is very naturally suited to the computation of our hydrodynamic parameters \mathcal{R}_{ab} , \mathcal{S}_{ab} , \mathcal{U}_{ab} and \mathcal{V}_{ab} . As these only affect the conductivities at $\mathcal{O}(B^2)$, it is sufficient to evaluate these in the $B = 0$ state. We assume that we may cleanly divide up the low energy effective theory for our Weyl semimetal into “node fluids” labeled by indices a , just as in the main text. To each node fluid, we assign a charge current operator J_a^μ and a stress tensor $T_a^{\mu\nu}$, which need not be exactly conserved in the presence of intervalley scattering and anomalies. We then define the valley charge and energy operators as

$$n_a \equiv \frac{1}{V_3} \int d^3\mathbf{x} J_a^t, \quad (50a)$$

$$\epsilon_a \equiv \frac{1}{V_3} \int d^3\mathbf{x} T_a^{tt}, \quad (50b)$$

respectively. We have assumed that the fluid is at rest when deriving the above. For later reference, we also define \mathcal{J}_a^i as the zero mode of the operator J_a^i , and P_a^i as the zero mode of the operator T_a^{ti} , analogous to (50).

Now suppose that we take our Weyl semimetal, and “populate” valley fluids at various chemical potentials and temperatures. Let us define the vector of operators

$$x_I = \begin{pmatrix} n_a - n_a^0 \\ \epsilon_a - \epsilon_a^0 \end{pmatrix}, \quad (51)$$

where $n_a^0 = \langle n_a \rangle$ and $\epsilon_a^0 = \langle \epsilon_a \rangle$, with averages over quantum and thermal fluctuations taken in equilibrium. I indices run over the operators n_a and ϵ_a . Assuming that there are no other long-lived modes operators in the system which overlap with the charge and energy of each valley fluid, we can use memory matrix techniques to show that the expectations values of these objects will evolve according to the hydrodynamic equations

$$\frac{d\langle x_I \rangle}{dt} = -M_{IJ} \chi_{JK}^{-1} \langle x_K \rangle, \quad (52)$$

where the matrices M and χ have entries

$$M_{IJ} \approx \lim_{\omega \rightarrow 0} \frac{\text{Im} (G_{x_I x_J}^R(\mathbf{k} = \mathbf{0}, \omega))}{\omega}, \quad (53a)$$

$$\chi_{IJ} = \text{Re} (G_{x_I x_J}^R(\mathbf{k} = \mathbf{0}, \omega = 0)). \quad (53b)$$

These formulae should be valid to leading order in a perturbative expansion in the small intervalley coupling strength.

The easiest way to compute χ_{IJ} is to identify the thermodynamic conjugate variable to x_I (let us call it y_I), and then employ the linear response formula

$$\frac{\partial \langle x_I \rangle}{\partial y_J} = \chi_{IJ}. \quad (54)$$

If the valley fluids interact weakly then we may approximate χ_{JK} as a block diagonal matrix to leading order, with $T\nu_a$ the canonically conjugate variable to μ_a , and $-T\beta_a$ the canonically conjugate variable to ϵ_a . Thus

$$\begin{pmatrix} \langle n_a \rangle - n_a^0 \\ \langle \epsilon_a \rangle - \epsilon_a^0 \end{pmatrix} = \chi_{IJ} \begin{pmatrix} T(\nu_a - \nu_a^0) \\ -T(\beta_a - \beta_a^0) \end{pmatrix}. \quad (55)$$

Comparing (52), (55) and our hydrodynamic definition of \mathbf{A} , we conclude that A_{IJ} , the elements of the intervalley scattering matrix, are related to microscopic Green’s functions by

$$A_{IJ} = TM_{IJ}. \quad (56)$$

Using the symmetry properties of Green's functions, we see that $M_{IJ} = M_{JI}$, thus proving that \mathbf{A} is a symmetric matrix, as we claimed previously. From (53), it is clear that global charge and energy conservation among all valleys enforces $\sum_b \mathcal{R}_{ab} = \sum_b \mathcal{S}_{ab} = \sum_b \mathcal{V}_{ab} = 0$ in the memory matrix formalism. For completeness, we note that the susceptibility matrix is given by

$$(\chi_{IJ})_{a \text{ indices}} = \begin{pmatrix} (\partial_\mu n)_a & 3n_a \\ 3n_a & 12P_a \end{pmatrix}, \quad (57)$$

assuming that the free energy of each fluid depends only on μ_a and T_a .

We finish by reviewing the well-known microscopic expressions for the other parameters in our hydrodynamic theory (see [39] for more details). Using the fact that velocity is conjugate to momentum, and combining (1) and (54), we obtain

$$n_a \delta^{ij} \equiv \chi_{\mathcal{J}_a^i P_a^j}, \quad (58a)$$

$$(\epsilon_a + P_a) \delta^{ij} \equiv \chi_{P_a^i P_a^j}. \quad (58b)$$

The Gibbs-Duhem relation implies that (to good approximation if valley fluids nearly decouple)

$$Ts_a = \epsilon_a + P_a - \mu n_a. \quad (59)$$

Together with the memory matrix result for the momentum relaxation time

$$M_{P_a^i P_a^j} = \delta_{ij} \Gamma_a, \quad (60)$$

we have a microscopic expression for all of the hydrodynamic parameters in our formulas for the conductivities, written in the main text, via the memory matrix formalism. The expression (17) for Γ_a that we derived from hydrodynamics agrees with that obtained by explicitly evaluating M_{PP} [57].

It is possible that the presence of anomalies complicates the memory matrix formalism beyond what is anticipated above. However, as the anomalous contributions to the hydrodynamic equations vanish in the absence of external electromagnetic fields, we do not expect any difficulties when the memory matrices are computed in the absence of background magnetic fields.

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