Transport coefficients in strong magnetic fields

J. Phys. C: Solid State Phys., Vol. 10, 1977. Printed in Great Britain.

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Received 9 November 1976

Abstract. Formulae describing the electron transport coefficients as functions of the conductivity tensor are derived on the basis of the linear response theory. It is shown that the Wiedemann–Franz law and the Mott rule are obeyed even in the presence of a quantizing magnetic field $\omega\tau > 1$ if the scattering of electrons is elastic and if $\hbar\omega \gg kT$. As no assumptions about the form of the one-electron potential are made, the conclusions are valid not only in the case of the Shubnikov–de Haas oscillations but also if effects such as magnetic breakdown and the size-effect appear.

1. Introduction

The aim of this paper is to study the relationships between the transport coefficients of a metal in a strong magnetic field and at very low temperatures, and especially to investigate the possibility of expressing them using the conductivity tensor $\boldsymbol{\sigma}$. Examples of such relations are the Wiedemann–Franz law for the heat conductivity $\boldsymbol{K}$,

$$\boldsymbol{K} = LT\boldsymbol{\sigma}, \tag{1}$$

and the Mott rule for the thermopower $\boldsymbol{S}$,

$$\boldsymbol{S} = eLT\boldsymbol{\sigma}^{-1}\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}\mu}. \tag{2}$$

Here T is the temperature, μ denotes the chemical potential and e the electron charge. The Lorentz number $L = \frac{1}{3}(\pi k/e)^2$, where k is the Boltzmann constant, is universal for all metals. The most general derivation of these expressions for the case of zero magnetic field was given by Chester and Thellung (1961), who used the Kubo formulae for the transport coefficients (Kubo *et al* 1957). Chester and Thellung's derivation is based on the following assumptions: (i) the electrons are independent and form a degenerate Fermi–Dirac assembly, (ii) the scattering of electrons is purely elastic, and (iii) $kT \ll \mu$. No assumption is made about the strength of scattering.

In the presence of a magnetic field, equations (1) and (2) were shown to hold when the Boltzmann equation is valid (Lifshits *et al* 1971), i.e. if the scattering is weak enough to satisfy the condition $\hbar/\tau \ll \mu$, where τ is the relaxation time, and if the quantization of the energy levels due to the magntic field is unimportant (the energy distance $\hbar\omega$ between the Landau levels fulfills the conditions $\hbar\omega \ll kT$, $\tau\omega \ll 1$). For strong magnetic fields ($\hbar\omega > kT$, $\tau\omega > 1$) the Boltzmann equation no longer applies and the quantum effects become important and manifest themselves in the oscillatory behaviour of the transport coefficients as functions of magnetic field. Many attempts have been undertaken to prove that (1) and (2) are still valid in this case. Serious difficulties appeared when Kasuya (1959), Nakajima (1958) and Hajdu (1964) found that equations (1) and (2) are violated by the off-diagonal components of the transport coefficients and that the magnetothermopower, for example, goes to infinity with decreasing temperature instead of approaching zero. This problem was resolved for the system of free electrons in a strong magnetic field by Obraztsov (1964), Zyryanov and Silin (1964), Baryakhtar and Peletminskii (1965) and others (for reviews, see Zyryanov and Klinger (1976) and Hajdu (1969)). They pointed out that the averaged values of the current density operators split into macroscopic currents and diamagnetic currents proportional to the curl of the magnetization and that, naturally, (1) and (2) hold only for the macroscopic currents. Later on, these conclusions were generalized by Kuleev (1970), who took into account the scattering on impurities to fourth order of the perturbation theory, and by Okulov (1966), who neglected scattering but considered the anisotropic Fermi surface.

In this paper we shall derive expressions for the transport coefficients using the 'mechanical' approach of Luttinger (1964). The essential point is that the effect of the thermal gradient is replaced by the gravitational potential, which can be easily described by a perturbation Hamiltonian. The resulting Kubo formulae are exact for an arbitrary one-electron potential describing the elastic scattering of electrons. Consequently, all shapes of the Fermi surface, all strengths of scattering and all configurations of scatterers are involved. In the following sections we shall study the relationships between the transport coefficients for this rather general model and shall determine the limits of validity of equations (1) and (2).

2. Phenomenological theory

The macroscopic electric current $\boldsymbol{I}$ and the energy current $\boldsymbol{I}_{\mathrm{E}}$ arising in a non-equilibrium system are proportional to the driving force present. Generally, the phenomenological transport equations are written in the form (Luttinger 1964)

$$\boldsymbol{I} = \boldsymbol{L}_{11}\left[\boldsymbol{E} - \frac{T}{e}\nabla\left(\frac{\mu}{T}\right)\right] + \boldsymbol{L}_{12}\left[T\nabla\left(\frac{1}{T}\right) - \frac{1}{c^2}\nabla\psi\right], \tag{3}$$

$$\boldsymbol{I}_{\mathrm{E}} = \boldsymbol{L}_{21}\left[\boldsymbol{E} - \frac{T}{e}\nabla\left(\frac{\mu}{T}\right)\right] + \boldsymbol{L}_{22}\left[T\nabla\left(\frac{1}{T}\right) - \frac{1}{c^2}\nabla\psi\right]. \tag{4}$$

$\boldsymbol{E} = -\nabla\phi$ and $-\nabla\psi/c^2$ are the dynamical forces, $\phi = -\boldsymbol{E}.\boldsymbol{r}$ and $\psi = \boldsymbol{r}.\nabla\psi$ denote the electrical and gravitational fields, respectively, and c is the velocity of light. The statistical force are $-(T/e)\nabla(\mu/T)$ and $T\nabla(1/T)$. The phenomenological transport coefficients $\boldsymbol{L}_{ij}(i, j = 1, 2)$ are, in general, tensors. The aim of the transport theory is to calculate them from first principles. The experimental transport coefficients, such as the conductivity $\boldsymbol{\sigma}$, the heat conductivity $\boldsymbol{K}$ and the thermoelectric power $\boldsymbol{S}$, are obtained by combining the $\boldsymbol{L}_{ij}$:

$$\boldsymbol{\sigma} = \boldsymbol{L}_{11}, \tag{5}$$

$$\boldsymbol{K} = T^{-1}(\boldsymbol{L}_{22} - \boldsymbol{L}_{21}\boldsymbol{L}_{11}^{-1}\boldsymbol{L}_{12}), \tag{6}$$

$$\boldsymbol{S} = T^{-1}(\boldsymbol{L}_{12}\boldsymbol{L}_{11}^{-1} - \mu/e). \tag{7}$$

In equilibrium, $\boldsymbol{E} = (T/e)\nabla(\mu/T)$, $T\nabla(1/T) = \nabla\psi/c^2$ and $\boldsymbol{I} = \boldsymbol{I}_E = 0$. Therefore, in (3) and (4) the phenomenological transport coefficients at $\boldsymbol{E}$ must be equal to the coefficients at $-(T/e)\nabla(\mu/T)$, and the same is true for the coefficients at $-\nabla\psi/C^2$ and $T\nabla(1/T)$ (Einstein relationships). This means that the dynamical forces $\boldsymbol{E}$ and $-\nabla\psi/c^2$ are fully equivalent to the statistical forces $-(T/e)\nabla(\mu/T)$ and $T\nabla(1/T)$, respectively. When calculating $\boldsymbol{L}_{ij}$ it is enough to consider only the response of the system to the dynamical forces.

3. The model

We shall consider the system of N independent electrons in a magnetic field $\boldsymbol{B}$. The static and, in general, non-uniform field $\boldsymbol{B}$ is made up of two contributions, one of which is the original field present when the specimen was absent and the other is the field due to the magnetization of the sample. The one-electron Hamiltonian describing the system is

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + V(\boldsymbol{r}), \tag{8}$$

where m is the electron mass and $\boldsymbol{A}$ is the vector potential ($\boldsymbol{B} = \operatorname{curl}\boldsymbol{A}$). The electrons move in an arbitrary fixed potential $V(\boldsymbol{r})$ which causes only elastic scattering of electrons. No assumptions are made about the range, the space distribution or the scattering strength of the scatterers, but $V(\boldsymbol{r})$ includes the surface of the sample.

In the presence of electrical and gravitational fields, the Hamiltonian of the system is changed to the form

$$H_T = H + F, \tag{9}$$

where F describes the interaction with the applied fields,

$$F = -e\boldsymbol{E}.\boldsymbol{r} + \tfrac{1}{2}(H\boldsymbol{r} + \boldsymbol{r}H)\frac{\nabla\psi}{c^2}. \tag{10}$$

The operators of the electric current density $\boldsymbol{j}(\boldsymbol{r})$ and the energy current density $\boldsymbol{j}_E(\boldsymbol{r})$ are defined by the equations of continuity:

$$\frac{\partial n(\boldsymbol{r})}{\partial t} = \frac{1}{i\hbar}[n(\boldsymbol{r}), H_T] = -\operatorname{div}\boldsymbol{j}(\boldsymbol{r}), \tag{11}$$

$$\frac{\partial h_T(\boldsymbol{r})}{\partial t} = \frac{1}{i\hbar}[h_T(\boldsymbol{r}), H_T] = -\operatorname{div}\boldsymbol{j}_E(\boldsymbol{r}). \tag{12}$$

Here $n(\boldsymbol{r}) = e\delta(\boldsymbol{r} - \boldsymbol{r}')$ denotes the charge density and $h_T(\boldsymbol{r}) = \frac{1}{2}[H_T\delta(\boldsymbol{r} - \boldsymbol{r}') + \delta(\boldsymbol{r} - \boldsymbol{r}')H_T]$ is the Hamiltonian density operator. From equations (11) and (12) we obtain

$$\boldsymbol{j}(\boldsymbol{r}) = \boldsymbol{j}_0(\boldsymbol{r}) + \boldsymbol{j}_1(\boldsymbol{r}) = \boldsymbol{j}_0(\boldsymbol{r}) + \frac{\psi(\boldsymbol{r})}{c^2}\boldsymbol{j}_0(\boldsymbol{r}) \tag{13}$$

$$\boldsymbol{j}_E(\boldsymbol{r}) = \boldsymbol{j}_{0E}(\boldsymbol{r}) + \boldsymbol{j}^1{}_E(\boldsymbol{r}) = \boldsymbol{j}_{0E}(\boldsymbol{r}) + \phi(\boldsymbol{r})\boldsymbol{j}_0(\boldsymbol{r}) + 2\frac{\psi(\boldsymbol{r})}{c^2}\boldsymbol{j}_{0E}(\boldsymbol{r}), \tag{14}$$

where

$$\boldsymbol{j}_0(\boldsymbol{r}) = \tfrac{1}{2}e[\boldsymbol{v}\delta(\boldsymbol{r} - \boldsymbol{r}') + \delta(\boldsymbol{r} - \boldsymbol{r}')\boldsymbol{v}], \tag{15}$$

$$\boldsymbol{j}_{0E}(\boldsymbol{r}) = \tfrac{1}{2}[H\boldsymbol{j}_0(\boldsymbol{r}) + \boldsymbol{j}_0(\boldsymbol{r})H] \tag{16}$$

and

$$v = \frac{1}{i\hbar}[r', H], \tag{17}$$

r' being the coordinate of the electron. The operators $j_1(r)$ and $j_{1E}(r)$ are linear in the applied fields. Note that equations (10) and (13)–(16) are only valid for Fourier components of long wavelength. Therefore, if $(j_E)_q = \int dr\, j_E(r) \exp(-iq.r)$, for example, then for small q these expressions give errors or relative order q. For further details see Luttinger (1967) and Mori (1958), who obtained equations (13) and (14) for systems without magnetic field.

4. The macroscopic and microscopic currents

The measurable current densities are obtained by the quantum-mechanical and thermodynamical averaging of the current operators. This is done with help of the non-equilibrium density matrix ρ, by expression $\langle j(r)\rangle = \mathrm{Tr}[\rho j(r)]$. In the next section we shall calculate the density matrix to first order in the applied fields via the Kubo formula. We shall express ρ as a sum $\rho = \rho_0 + \rho_1$, where ρ_0 is the equilibrium Fermi–Dirac distribution function

$$\rho_0(H) = \frac{1}{1 + \exp[(H - \mu)/kT]} \tag{18}$$

and ρ_1 describes the deviation from equilibrium due to the presence of the fields. Then the expressions for the current densities, linear in the electrical and gravitational fields, will take the form

$$\langle j(r)\rangle = \mathrm{Tr}[\rho j(r)] = \mathrm{Tr}[\rho_0 j_0(r)] + \mathrm{Tr}[\rho_0 j_1(r)] + \mathrm{Tr}[\rho_1 j_0(r)], \tag{19}$$

$$\langle j_E(r) = \mathrm{Tr}[\rho j_E(r)] = \mathrm{Tr}[\rho_0 j_{0E}(r)] + \mathrm{Tr}[\rho_0 j_{1E}(r)] + \mathrm{Tr}[\rho_1 j_{0E}(r)]. \tag{20}$$

Those terms on the right-hand sides of (19) and (20) containing ρ_0 vanish in zero magnetic field. The expressions $\mathrm{Tr}[\rho_0 j_0(r)]$ and $\mathrm{Tr}[\rho_0 j_{0E}(r)]$ do not depend on the applied fields and describe the equilibrium diamagnetic surface currents. The remaining terms describe partly the change of the diamagnetic currents due to E and $-\nabla\psi/c^2$ and partly the currents arising in these fields.

It follows from equations (19) and (20) that $\langle j(r)\rangle$ and $\langle j_E(r)\rangle$ are the microscopic currents rather than the macroscopic ones (defined as the charge or the energy crossing a unit area of the total cross section of the sample per unit time) which appear in the transport equations (3) and (4). It is well known from the theory of electromagnetism that the microscopic current $\langle j(r)\rangle$ is composed of two parts:

$$\langle j(r)\rangle = I(r) + c\,\mathrm{curl}\, M(r), \tag{21}$$

where $I(r)$ is the macroscopic current and $M(r)$ is the magnetization of the sample. As pointed out by Zyryanov and Klinger (1976), the space average of $\langle j(r)\rangle$ is simply equal to $I(r)$. Indeed, using (21), we can write

$$\int \langle j(r)\rangle\, ds = \int I(r)\, ds + c \int \mathrm{curl}\, M(r)\, ds \tag{22}$$

and

$$\int \operatorname{curl} \boldsymbol{M}(\boldsymbol{r})\, \mathrm{d}s = \oint \boldsymbol{M}(\boldsymbol{r})\, \mathrm{d}l = 0, \tag{23}$$

if the perimeter l of the cross section S of the sample lies just above the sample surface where $\boldsymbol{M}(\boldsymbol{r}) = 0$. Then

$$\boldsymbol{I} = \frac{1}{L} \int_V \langle \boldsymbol{j}(\boldsymbol{r}) \rangle\, \mathrm{d}\boldsymbol{r}. \tag{24}$$

In the following we shall assume that the volume V and the length L of the sample are equal to one. A similar argument also holds for $\langle \boldsymbol{j}_{\mathrm{E}}(\boldsymbol{r}) \rangle$. Thus, in order to obtain the macroscopic currents $\boldsymbol{I}$ and $\boldsymbol{I}_{\mathrm{E}}$ it is necessary to perform not only the quantum-mechanical and thermodynamical averaging, but also the average over the volume of the sample.

5. Transport coefficients

The equation of motion of the density matrix

$$\frac{\partial \rho}{\partial t} + \frac{1}{\mathrm{i}\hbar} [\rho, H + F \exp(-\epsilon t)] = 0 \tag{25}$$

yields the solution in the form

$$\rho = \rho_0 + \rho_1 = \rho_0 + \frac{\mathrm{i}}{\hbar} \lim_{\epsilon \to 0^+} \int_0^{+\infty} \exp(-\epsilon t) \exp\left(-\frac{\mathrm{i}}{\hbar} Ht\right) [\rho_0, F] \exp\left(\frac{\mathrm{i}}{\hbar} Ht\right) \mathrm{d}t. \tag{26}$$

Introducing (26) into (19) and (20) we arrive, after space-averaging and integration over the time variable, at the following expression for the phenomenological transport coefficients:

$$L_{ij}^{\alpha\beta} = M_{ij}^{\alpha\beta} + S_{ij}^{\alpha\beta}. \tag{27}$$

Here $i, j = 1, 2$ and α, β denote the space coordinates x, y, z.

First of all it is obvious that the space average of the terms $\mathrm{Tr}[\rho_0 \boldsymbol{j}_0(\boldsymbol{r})]$ and $\mathrm{Tr}[\rho_0 \boldsymbol{j}_{0\mathrm{E}}(\boldsymbol{r})]$ yields zero, as the equilibrium diamagnetic currents do not contribute to the total currents through the cross section of the sample, i.e. to the macroscopic currents $\boldsymbol{I}$ and $\boldsymbol{I}_{\mathrm{E}}$.

The terms $M_{ij}^{\alpha\beta}$ are obtained from the expressions for $\mathrm{Tr}[\rho_0 \boldsymbol{j}_1(\boldsymbol{r})]$ and $\mathrm{Tr}[\rho_0 \boldsymbol{j}_{1\mathrm{E}}(\boldsymbol{r})]$ in equations (19) and (20) and can be written as

$$M_{11}^{\alpha\beta} = 0, \tag{28}$$

$$M_{12}^{\alpha\beta} = M_{21}^{\alpha\beta} = -\tfrac{1}{2} e \int \rho_0(\eta)\, \mathrm{Tr}[\delta(\eta - H)(v^\alpha r^\beta + r^\beta v^\alpha)]\, \mathrm{d}\eta, \tag{29}$$

$$M_{22}^{\alpha\beta} = -\int \eta \rho_0(\eta)\, \mathrm{Tr}[\delta(\eta - H)(v^\alpha r^\beta + r^\beta v^\alpha)]\, \mathrm{d}\eta. \tag{30}$$

The terms $S_{ij}^{\alpha\beta}$ originate in the expressions $\mathrm{Tr}[\rho_1 \boldsymbol{j}_0(\boldsymbol{r})]$ and $\mathrm{Tr}[\rho_1 \boldsymbol{j}_{0\mathrm{E}}(\boldsymbol{r})]$ and can be written

$$S_{ij}^{\alpha\beta} = \mathrm{i}\hbar \int \rho_0(\eta)\, \mathrm{Tr}\left(j_i^\alpha \frac{\mathrm{d}G^+}{\mathrm{d}\eta} j_j^\beta \delta(\eta - H) - j_i^\alpha \delta(\eta - H) j_j^\beta \frac{\mathrm{d}G^-}{\mathrm{d}\eta}\right) \mathrm{d}\eta, \tag{31}$$

where

$$j_1^\alpha = ev^\alpha \tag{32}$$

and

$$j_2^\alpha = \tfrac{1}{2}(v^\alpha H + Hv^\alpha) \tag{33}$$

are the space-averaged operators of the electric current density $j_0(\boldsymbol{r})$ and the energy current density $j_{0\mathrm{E}}(\boldsymbol{r})$, respectively. The coefficient $S_{11}^{\alpha\beta}$, i.e. the conductivity tensor $\sigma^{\alpha\beta}$, was given in a form identical to (31) by Bastin *et al* (1971) and by Středa and Smrčka (1975). The Green function (resolvent) G, defined by

$$G(z) = (z - H)^{-1}, \tag{34}$$

is introduced into (31) via the expression

$$\delta(\eta - H) = -\frac{1}{2\pi\mathrm{i}}(G^+ - G^-). \tag{35}$$

Here $G^\pm$ denotes $G(\eta \pm \mathrm{i}\epsilon)$.

It is obvious by inspection of equations (27)–(31) that $L_{ij}^{\alpha\beta}$ are real numbers,

$$L_{ij}^{\alpha\beta *}(\boldsymbol{B}) = L_{ij}^{\alpha\beta}(\boldsymbol{B}), \tag{36}$$

and that they satisfy the Onsager relations

$$L_{ij}^{\alpha\beta}(\boldsymbol{B}) = L_{ji}^{\beta\alpha}(-\boldsymbol{B}). \tag{37}$$

Integrating the formulae for $L_{ij}^{\alpha\beta}$ by parts we can express the coefficients at finite temperature via the coefficients at zero temperature:

$$L_{ij}^{\alpha\beta}(T, \mu) = -\int \frac{\mathrm{d}\rho_0(\eta)}{\mathrm{d}\eta} L_{ij}^{\alpha\beta}(0, \eta)\, \mathrm{d}\eta. \tag{38}$$

For this reason, only the zero-temperature coefficients will be considered in the following.

The expression for $L_{ij}^{\alpha\beta}$ can be given a more convenient form from which the relationships between different coefficients will be obvious. First we shall write for $M_{ij}^{\alpha\beta}$, using the identity $\mathrm{Tr}(\rho_0 v^\alpha r^\beta) = -\mathrm{Tr}(\rho_0 r^\alpha v^\beta)$,

$$M_{11}^{\alpha\beta} = 0, \tag{39}$$

$$M_{12}^{\alpha\beta} = M_{21}^{\alpha\beta} = \tfrac{1}{2} e \int_{-\infty}^{\mu} \mathrm{Tr}[\delta(\eta - H)(r^\alpha v^\beta - r^\beta v^\alpha)]\, \mathrm{d}\eta, \tag{40}$$

$$M_{22}^{\alpha\beta} = \int_{-\infty}^{\mu} \eta\, \mathrm{Tr}\,[\delta(\eta - H)(r^\alpha v^\beta - r^\beta v^\alpha)]\, \mathrm{d}\eta. \tag{41}$$

The coefficient $M_{12}^{\alpha\beta}$ is just c times the magnetization of the sample $(e/2c)\langle \boldsymbol{r} \wedge \boldsymbol{v}\rangle$, the average of the magnetic momentum, which is proportional to the vector product of $\boldsymbol{r}$ and $\boldsymbol{v}$. $M_{22}^{\alpha\beta}$ differs by a factor of η in the integrand. It is clear from (39) to (41) that their diagonal elements vanish:

$$M_{ij}^{\alpha\alpha} = 0. \tag{42}$$

The diagonal elements of the coefficients $S_{ij}^{\alpha\beta}$ can be written as

$$S_{ij}^{\alpha\alpha} = \pi\hbar\,\mathrm{Tr}[j_i^\alpha\,\delta(\mu - H)\,j_j^\alpha\,\delta(\mu - H)]. \tag{43}$$

Introducing now the expressions (32) and (33) for the j_i^α and bearing in mind equations (27) and (42) we get simple relations between the diagonal elements of the transport coefficients $L_{ij}^{\alpha\beta}$. If we write

$$L_{11}^{\alpha\alpha} \equiv \sigma^{\alpha\alpha} = \pi\hbar e^2\,\mathrm{Tr}[v^\alpha\delta(\mu - H)\,v^\alpha\delta(\mu - H)], \tag{44}$$

then immediately

$$L_{12}^{\alpha\alpha} = L_{21}^{\alpha\alpha} = \frac{\mu}{e}\sigma^{\alpha\alpha} \tag{45}$$

$$L_{22}^{\alpha\alpha} = \left(\frac{\mu}{e}\right)^2 \sigma^{\alpha\alpha}. \tag{46}$$

These expressions for the diagonal elements can also be found in the paper by Baryakhtar and Peletminskii (1965), for example.

It is more difficult to obtain similar expressions for the off-diagonal elements, as the $M_{ij}^{\alpha\beta}$ are, in general, non-zero. In the following we shall show that these are compensated by the corresponding terms in $S_{ij}^{\alpha\beta}$. Assuming $\alpha \neq \beta$ and making use of equations (32) and (33) we acquire

$$S_{11}^{\alpha\beta} = e^2 \int_{-\infty}^{\mu} A(\eta)\,\mathrm{d}\eta, \tag{47}$$

$$S_{12}^{\alpha\beta} = S_{21}^{\alpha\beta} = e\int_{-\infty}^{\mu} \eta A(\eta)\,\mathrm{d}\eta + \frac{e}{2}\int_{-\infty}^{\mu} B(\eta)\,\mathrm{d}\eta, \tag{48}$$

$$S_{22}^{\alpha\beta} = \int_{-\infty}^{\mu} \eta^2 A(\eta)\,\mathrm{d}\eta + \int_{-\infty}^{\mu} \eta B(\eta)\,\mathrm{d}\eta, \tag{49}$$

where

$$A(\eta) = \mathrm{i}\hbar\,\mathrm{Tr}\left(v^\alpha \frac{\mathrm{d}G^+}{\mathrm{d}\eta}\,v^\beta\,\delta(\eta - H) - v^\alpha\,\delta(\eta - H)\,v^\beta \frac{\mathrm{d}G^-}{\mathrm{d}\eta}\right) \tag{50}$$

and

$$B(\eta) = \mathrm{i}\hbar\,\mathrm{Tr}[v^\alpha G^+ v^\beta \delta(\eta - H) - v^\alpha\delta(\eta - H)\,v^\beta G^-]. \tag{51}$$

Equations (48) and (49) may be written, after integration by parts, as

$$S_{12}^{\alpha\beta} = \frac{\mu}{e}\sigma^{\alpha\beta} + e\int_{-\infty}^{\mu} (\eta - \mu)\left(A(\eta) - \tfrac{1}{2}\frac{\mathrm{d}B(\eta)}{\mathrm{d}\eta}\right)\mathrm{d}\eta \tag{52}$$

and

$$S_{22}^{\alpha\beta} = \left(\frac{\mu}{e}\right)^2 \sigma^{\alpha\beta} + \int_{-\infty}^{\mu} (\eta - \mu)^2 \left(A(\eta) - \tfrac{1}{2}\frac{\mathrm{d}B(\eta)}{\mathrm{d}\eta}\right)\mathrm{d}\eta. \tag{53}$$

Here we use the notation $\sigma^{\alpha\beta} = L_{11}^{\alpha\beta} = S_{11}^{\alpha\beta}$. Utilizing the possibility of the cyclic interchange of operators in $\mathrm{Tr}(\ldots)$ and with the help of the expression

$$v^\alpha = -\frac{1}{\mathrm{i}\hbar}[r^\alpha, G^{-1}] \tag{54}$$

we get, after some algebra,

$$A(\eta) - \tfrac{1}{2}\frac{\mathrm{d}B(\eta)}{\mathrm{d}\eta} = \tfrac{1}{2}\,\mathrm{Tr}\left(\frac{\mathrm{d}\delta(\eta - H)}{\mathrm{d}\eta}(r^{\alpha}v^{\beta} - r^{\beta}v^{\alpha})\right). \tag{55}$$

Inserting this equation into (52) and (53) and integrating by parts again, the second terms on the right-hand sides of (52) and (53) become identical to $-M_{12}^{\alpha\beta}$ and $-M_{22}^{\alpha\beta}$, respectively. Therefore these terms cancel in expressions for $L_{ij}^{\alpha\beta}$ and we arrive at the expected results

$$L_{12}^{\alpha\beta} = L_{21}^{\alpha\beta} = \frac{\mu}{e}\sigma^{\alpha\beta}, \tag{56}$$

$$L_{22}^{\alpha\beta} = \left(\frac{\mu}{e}\right)^2 \sigma^{\alpha\beta}. \tag{57}$$

Thus, at zero temperature and for the case of elastic scattering, equations (45) and (46) hold quite generally.

6. Conclusions

In the preceding sections we have investigated the relationships between the phenomenological transport coefficients. It turns out that these can be expressed via the conductivity tensor $L_{11}^{\alpha\beta} = \sigma^{\alpha\beta}$. Using equations (38), (45) and (46) we get

$$L_{12}^{\alpha\beta}(T,\mu) = L_{21}^{\alpha\beta}(T,\mu) = -\int \frac{\mathrm{d}\rho_0}{\mathrm{d}\eta}\frac{\eta}{e}\sigma^{\alpha\beta}(0,\eta)\,\mathrm{d}\eta, \tag{58}$$

$$L_{22}^{\alpha\beta}(T,\mu) = -\int \frac{\mathrm{d}\rho_0}{\mathrm{d}\eta}\left(\frac{\eta}{e}\right)^2 \sigma^{\alpha\beta}(0,\eta)\,\mathrm{d}\eta. \tag{59}$$

These expressions were derived under the assumptions that (i) the electrons can be considered as independent particles, (ii) the scattering of electrons is elastic (does not change their energy), and (iii) the response of the system to the applied fields is linear. These assumptions are rather general and cover many possible physical situations. We can consider, for example, an anisotropic medium with an arbitrary shape of Fermi surface and arbitrary angles between the magnetic field, the applied fields and the main crystallographic axes. Also, the strength of scattering is not limited and neither is the geometrical arrangement of scatterers, so (58) and (59) hold for alloys, crystals containing dislocations and impurities, and even for thin samples where the scattering on the surface becomes important. Thus, not only the Shubnikov–de Haas oscillations but also magnetic breakdown and the size-effect are taken into account. On the other hand, (58) and (59) fail when the energy current is not due to electrons only or when inelastic scattering of electrons by phonons, for example, takes place.

If the electrons form the degenerate Fermi–Dirac assembly, we can make use of the low-temperature expansion

$$-\frac{\mathrm{d}\rho_0(\eta)}{\mathrm{d}\eta} = \delta(\eta - \mu) + \frac{\pi^2}{6}(kT)^2\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\delta(\eta - \mu) + \ldots. \tag{60}$$

In the case considered here, the zero-temperature coefficients are proportional to the density of states, which is an oscillatory function of magnetic field with period $\hbar\omega$.

Therefore, the criterion of validity of (60) will be $kT \ll \hbar\omega$. An exception is the case when $\tau\omega \ll 1$ and when the oscillations are completely smeared out because of the strong scattering of electrons. The condition of validity is then changed to the usual form $kT \ll \mu$. If we introduce expansion (60) into (38) and use these expressions in (5), (6) and (7), we immediately prove the Wiedemann–Franz law and the Mott rule, equations (1) and (2)

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