

# Magnetoresistance in relativistic hydrodynamics without anomalies

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**ABSTRACT:** We present expressions for the magnetoconductivity and the magnetoresistance of a strongly interacting metal in  $3 + 1$  dimensions, derivable from relativistic hydrodynamics. Such an approach is suitable for ultraclean metals with emergent Lorentz invariance. When this relativistic fluid contains chiral anomalies, it is known to exhibit longitudinal negative magnetoresistance. We show that similar effects can arise in non-anomalous relativistic fluids due to the distinctive gradient expansion. In contrast with a Galilean-invariant fluid, the resistivity tensor of a dirty relativistic fluid exhibits similar angular dependence to negative magnetoresistance, even when the constitutive relations and momentum relaxation rate are isotropic. We further account for the effect of magnetic field-dependent corrections to the gradient expansion and the effects of long-wavelength impurities on magnetoresistance. We note that the holographic D3/D7 system exhibits negative magnetoresistance.

**KEYWORDS:** AdS-CFT Correspondence, Holography and condensed matter physics (AdS/CMT), Holography and quark-gluon plasmas

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## 1 Introduction

The behavior of metals in the presence of external electromagnetic fields is of fundamental importance to our understanding of materials and their transport properties. One such property is the magnetoresistance (MR), which characterizes magnetic field dependence of the conductivity tensor. For most metals, the longitudinal conductivity is a monotonically decreasing function of  $B$ , or the resistance is a monotonically increasing function of  $B$  [1]. However, the recent discovery of Weyl semimetals — materials in which conduction bands intersect at distinct points in the Brillouin zone — shows that this is not always the case [2–9].

In Weyl semi-metals, quasi-particles with momentum values near the intersection points (Weyl points) are described by the massless Weyl Hamiltonian

$$H = \pm v \vec{\sigma} \cdot (-i\nabla - e\vec{A}) \quad (1.1)$$

where  $v$  is the velocity,  $\vec{\sigma}$  is the pseudospin operator, and  $\pm$  correspond to the chirality of the quasi-particle. It is a well known result in quantum field theory that in chiral theories such as (1.1), the chiral (axial) current of these fermions is not conserved. Such an anomaly, commonly referred to as the Adler-Bell-Jackiw (ABJ) or chiral anomaly, is due to the breaking of chiral symmetry by the path integral measure [10]. Although this effect is manifestly quantum mechanical, it has important consequences for classical transport.

In particular, chiral fermions can be spontaneously created if parallel electric and magnetic fields are applied to the sample:

$$\partial_\mu j_A^\mu = -\frac{e^2}{8\pi^2} F_{\mu\nu} (\star F)^{\mu\nu} = \frac{e^2}{2\pi^2} \vec{E} \cdot \vec{B} \quad (1.2)$$

The subscript A on this current emphasizes that it is the axial current.

However, it is a theorem [11] that on a lattice, one must have multiple Weyl points, such that the net chirality of the system vanishes. Because these Weyl points are located a finite distance away from each other in the Brillouin zone, there is always some non-vanishing scattering rate for quasiparticles between the two Weyl points. Hence, one must schematically modify (1.2) to

$$\partial_\mu j_A^\mu = \frac{e^2}{2\pi^2} \vec{E} \cdot \vec{B} - \frac{\chi_A \delta\mu_A}{\tau}, \quad (1.3)$$

where  $\tau$  is the scattering time for quasiparticles to scatter from the neighborhood of one Weyl point to another, and  $\chi_A \delta\mu_A = \delta\rho_A$  is the deviation in the axial charge density from equilibrium. Applying an infinitesimal electric field  $E_i$ , and using Ohm's law

$$J_i = \sigma_{ij} E_j \quad (1.4)$$

to extract the electrical conductivity, one finds<sup>1</sup>

$$\sigma_{ij} = \sigma_0 \delta_{ij} + \frac{\tau e^2}{4\pi\chi_A} B_i B_j. \quad (1.5)$$

In the limit  $\tau \rightarrow \infty$ , we find a parametrically large contribution to the conductivity. The specific angular dependence of this enhancement (only in the longitudinal direction oriented along the magnetic field) is a key prediction. It can be found within a kinetic description at weak coupling [12], as well as a hydrodynamic [13, 14] or holographic [15] description at strong coupling. Evidence for such an angular structure was found in the recent experiments [2–9].

Already at weak coupling it has been demonstrated that NMR is possible in non-anomalous systems [16]. In the present paper, we describe the hydrodynamic gradient expansion in background magnetic fields and derive a hydrodynamic equation for  $\sigma_{ij}$ , analogous to (1.5), without any assumption of chirality. We are inspired in part by the holographic D3/D7 system, whose field theory dual is that of  $\mathcal{N} = 2$  super Yang-Mills (SYM) hypermultiplets propagating through an  $\mathcal{N} = 4$  SYM plasma [17]: we will show that this system exhibits NMR. We will also see that it is possible to obtain positive magnetoresistance within hydrodynamics, and will present two different ‘microscopic’ mechanisms for this.

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<sup>1</sup>The anomaly itself drives a charge separation which yields an axial charge growing linearly with time and so without any mechanisms to release axial charge would result in run-away behavior. In the presence of an relaxation mechanism for axial charge with time scale  $\tau$ , one finds a finite build-up of axial charge  $\delta\mu_A$ , which in turn drives an electric current proportional to  $B$  via the chiral magnetic effect.

In fact, an order one contribution to the current of the form  $(\vec{E} \cdot \vec{B})B_i$  is generic. Using symmetries alone, and neglecting the breaking of rotational invariance by the microscopic crystal lattice, we anticipate the following expression for  $\sigma_{ij}$ :

$$\sigma_{ij} = \sigma_0 \delta_{ij} + \sigma_1 \epsilon_{ijk} B_k + \sigma_2 B_i B_j. \quad (1.6)$$

The inverse of  $\sigma_{ij}$ , the resistivity tensor  $\rho_{ij}$ , will also have similar structure:

$$\rho_{ij} \equiv \alpha \delta_{ij} + \beta \epsilon_{ijk} B_k + \gamma \frac{B_i B_j}{B^2}. \quad (1.7)$$

In section 2, we will show how (1.7) generically appears in relativistic hydrodynamics, with all  $\alpha, \beta, \gamma \neq 0$ , and relate these coefficients to the hydrodynamic dissipative coefficients. An important difference between Galilean-invariant fluids and more general fluids (including relativistic fluids) is the fact that the charge current is not proportional to the momentum density in the latter case. Hence, we will find that  $\gamma \neq 0$ , in contrast to the Galilean invariant case. In fact, up to a brief discussion of thermal transport, our discussion of hydrodynamic charge transport is also valid for any non-Galilean invariant system.

Nowhere do we assume the existence of any (emergent) axial anomaly. Unlike in (1.5), there is no reason (a priori) to expect  $\sigma_2$  to be parametrically large. Still, we note that typical anomalous NMR observed in experiment is not an order-of-magnitude enhancement.

Hydrodynamic transport in background fields has been applied successfully to describe strongly correlated materials starting with the work of [18] on 2+1 dimensional physics. More recently, this (relativistic) hydrodynamic approach to transport has also been applied successfully to understand experimental transport data from clean samples of graphene [19, 20]. These studies take as input the hydrodynamic transport coefficients, and then use the structure of the hydrodynamic equations to give expressions for the dependence of the transport properties on external fields and particle number density. Starting from relativistic hydrodynamics instead of Galilean hydrodynamics, one finds a number of distinct predictions in 2+1 dimensions such as  $B$ -dependent lifetimes for cyclotron modes (a violation of Kohn’s theorem) [18].

One important aspect of this procedure is that one has to be careful to work in a consistent expansion scheme. Hydrodynamics itself is a gradient expansion, where  $\partial_\mu$  is treated as a small parameter. In the usual gradient expansion, since  $\vec{B} = \nabla \times \vec{A}$ , the results of [18] are, strictly speaking, only valid to linear orders in  $B$ . In this limit, one must treat multiple small parameters in the theory as “equally small” and only then perform the perturbative expansion. If one wants to, for example, see the motion of hydrodynamic poles (such as the cyclotron resonance) in the conductivity as a function of  $B$ , one needs to include all higher transport coefficients involving arbitrary powers of  $B$ . Since in our work all the interesting physics appears at quadratic order in  $B$ , we will develop an expansion scheme in which  $B$  is considered to be zeroth order in derivatives [21].

The article is organized as followed: in section 2 we present our hydrodynamic calculation, including the full conductivity and resistivity tensors. We then discuss some interesting limits and their physical interpretations. In section 3 we compare our results to that of the D3/D7 system and a simple toy model made of “electron-hole plasma” in 3+1

dimensions. The former is shown to exhibit negative magnetoresistance, while the latter has positive magnetoresistance. We discuss a hydrodynamic model for the momentum relaxation time in section 4, and show how the magnetic field generically leads to anisotropic momentum relaxation. We conclude in section 5.

As this paper was being finalized, [22] appeared, which contains some overlap with section 2.

## 2 Relativistic hydrodynamics in a magnetic field

### 2.1 Weak magnetic fields

Hydrodynamics is the low energy effective description of any interacting quantum field theory, valid for fluctuations whose wavelength is much larger than a ‘thermalization scale’: when quasiparticles are well-defined, this scale is simply the mean free path of the quasiparticles. When we look on length scales long compared to the mean free path, the system appears to be in local thermal equilibrium, thereby allowing us to describe the global dynamics in terms of conserved quantities. In this paper, these conserved quantities will be charge, energy and momentum. The dynamical equations in the presence of external fields are

$$\nabla_\mu J^\mu = 0 \quad (2.1a)$$

$$\nabla_\mu T^{\mu\nu} = F^{\mu\nu} J_\mu + \frac{1}{\tau} (\delta^\mu_\nu + u^\mu u_\nu) T^{\nu\lambda} u_\lambda \quad (2.1b)$$

where the last term allows for the dissipation of momentum due to impurities [18]; it can be derived rigorously when the disorder strength is small from multiple approaches [23]. The form of the  $1/\tau$  term in (2.1) is only sensible to linear order in the spatial components of  $u^\mu$ . Following the conventions of [18], we can write the current and energy-momentum tensor as

$$J^\mu = \rho u^\mu + \nu^\mu + J_{\text{mag}}^\mu \quad (2.2a)$$

$$T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu + p g^{\mu\nu} + \tau^{\mu\nu} + T_{\text{mag}}^{\mu\nu} \quad (2.2b)$$

where  $\nu^\mu$  and  $\tau^{\mu\nu}$  are dissipative contributions which arise at first order in derivatives and a fluid frame can be chosen in which they satisfy the following orthogonality relations

$$u_\mu \nu^\mu = u_\mu \tau^{\mu\nu} = u_\nu \tau^{\mu\nu} = 0. \quad (2.3)$$

These terms can be found to first order in the derivative expansion by requiring positivity of the divergence of the entropy current [24]. Such an analysis gives

$$\begin{aligned} \nu^\mu &= \sigma_Q (g^{\mu\nu} + u^\mu u^\nu) \left[ \left( -\partial_\nu \mu + F_{\nu\lambda} u^\lambda \right) + \mu \frac{\partial_\mu T}{T} \right] \\ \tau^{\mu\nu} &= -(g^{\mu\lambda} + u^\mu u^\lambda) \left[ \eta (\partial_\lambda u^\nu + \partial^\nu u_\lambda) + \left( \zeta - \frac{2}{D} \eta \right) \delta^\nu_\lambda \partial_\alpha u^\alpha \right]. \end{aligned} \quad (2.4)$$

The parameter  $\sigma_Q$  is the “quantum critical conductivity”, the conductivity in the absence of charges and external fields.  $D$  denotes the number of spatial dimensions, which in this paper will generically be 3.

The last term in eqs. (2.2a) and (2.2b) are contributions due to polarization of the material. They are present already in thermal equilibrium. Their contribution to currents in the bulk will be compensated by surface currents, rendering them unmeasurable in any experimental set up [25]. Alternatively, one can derive these terms using variational techniques such as in [21].

Reference [18] presented formulas for the conductivity and resistivity tensors within relativistic hydrodynamics in 2+1 dimensions. Here, we comment on the extension of this work to 3+1 dimensions. Unlike in 2 + 1 dimensions, where  $B$  is a pseudoscalar, in higher dimensions background magnetic fields break rotational invariance. As a consequence, we will see that in 3 + 1 dimensions, anisotropic  $(\vec{E} \cdot \vec{B})B_i$  terms generically arise in  $J_i$ .

## 2.2 Strong magnetic fields

As we mentioned in the introduction, one important caveat in the work of [18] and the work that followed is whether the hydrodynamic expansion has been systematically applied. In the constitutive relation (2.4) we only kept terms to first order in the gradient expansion, treating  $F_{\mu\nu}$  as being first order in the gradient expansion. The final answers hence only are valid up to linear order in  $E$  and  $B$ . Even assuming we may treat  $1/\tau$  as first order in derivatives, all of the novel relativistic phenomenology of [18] requires inclusion of terms proportional to  $\sigma_Q B^2$  in the conductivity! As we emphasize in this paper, there are additional hydrodynamic contributions to the conductivity at order  $B^2$ .

For Abelian background fields, which includes electromagnetism, one can define an alternate expansion scheme in which the magnetic field strength is treated as order 0 in the gradient expansion, and only derivatives of  $\vec{B}$  count as gradients. Namely, we should really keep *all* orders in  $F_{\mu\nu}$  at the start of the calculation. This is especially important for us since we are interested in NMR, which only occurs at order  $B^2$ . Since our goal is to obtain the linear response relation (1.4), we can consider “weak” electric fields and “strong” magnetic fields, i.e.  $E \sim \mathcal{O}(\partial)$ ,  $B \sim \mathcal{O}(1)$ . Thus, we will develop a more general constitutive relation that keeps all orders in  $\vec{B}$ , but is linear in  $\vec{E}$ . This is easiest to do using non-relativistic notation where  $\vec{E}$  and  $\vec{B}$  are treated separately. Note that we require  $B \lesssim T^2$  as a point of principle; if this inequality is violated, then the hydrodynamic description should be replaced by an alternative description. For example, in the limit of extremely large  $B$ -fields, a hydrodynamic framework for low-lying Landau levels becomes appropriate [26].

We consider the thermodynamic quantities in (2.2a) and (2.2b) to be functions of  $\mu$ ,  $T$  and  $B^2$ , and treat  $\mu$ ,  $T$  and  $u^\mu$  as the degrees of freedom that respond to external perturbations to the system. The fluctuations of  $\varepsilon$ ,  $p$  and  $\rho$  will then be determined by the equation of state, as is standard. As in [18], we assume that in equilibrium the fluid velocity is  $u^\mu = (1, \vec{0})$ , and so perturbations from equilibrium allows us to treat the velocity of the fluid  $u^i = v^i$  as  $\mathcal{O}(\partial)$ . Thus, any quadratic terms in  $u^i$  can be ignored in the gradient expansion. We also assume that the applied electromagnetic fields are static

sources; strictly speaking, electromagnetism is a gauge theory and dynamical gauge fields complicate the hydrodynamic description [22, 27], although the physics is only qualitatively different under extreme magnetic fields.

Finally, although it is not rigorous, we will for now assume the “mean field” approximation that disorder modifies the momentum conservation equation simply through the factor  $1/\tau$ , as in (2.1), is correct. These approximations hone in on the changes to the gradient expansion that occur in a background magnetic field. We will relax this assumption in section 4.

Since without gradients and background electric field the hydrodynamic velocity is zero, keeping only terms linear in  $E$  also means keeping only leading orders in  $v$ . Rotation invariance demands that the only tensor structures allowed in  $J^i$  are  $v^i$ ,  $E^i$ ,  $\epsilon_{ijk}B^jv^k$ ,  $\epsilon_{ijk}B^jE^k$  and  $B^iB_jE^j$ . The exact combination of these terms that is allowed to appear is further constrained by boost invariance. Since we are working to linear order in  $E$  and hence  $v$ , we can restrict ourselves to Galilean boosts which act as:

$$t \rightarrow t, \quad x_i \rightarrow x_i + v_i t, \quad J_i \rightarrow J_i + v_i \rho, \quad T^{ti} \rightarrow T^{ti} + (\epsilon + P)v^i, \quad (2.5a)$$

$$\rho \rightarrow \rho, \quad (\epsilon + P) \rightarrow (\epsilon + P), \quad (2.5b)$$

$$B_i \rightarrow B_i, \quad E_i \rightarrow E_i + \epsilon_{ijk}v^j B^k. \quad (2.5c)$$

As we are not considering inhomogeneous flows, we can exploit boost invariance by boosting into the rest frame of the fluid with  $v^i = 0$ . In this frame the most general form of the current reads

$$J^i = c(B^2)E^i + d(B^2)E^j B_j B^i + \tilde{\sigma}(B^2)\epsilon^i_{jk}B^j E^k. \quad (2.6)$$

The full constitutive relation can be recovered by acting on this rest frame current with a Galilean boost:

$$\begin{aligned} J^i = & [\rho(B^2) + \tilde{\sigma}(B^2)B^2]v^i - \tilde{\sigma}(B^2)B^i B_j v^j + c(B^2)(E^i + \epsilon^i_{jk}v^j B^k) \\ & + d(B^2)E^j B_j B^i + \tilde{\sigma}(B^2)\epsilon^i_{jk}B^j E^k. \end{aligned} \quad (2.7)$$

The coefficients can, in general, depend on  $B^2$ . Finally, we use our freedom to redefine the fluid frame to fix<sup>2</sup>

$$T^{ti} \equiv (\epsilon + P)v^i. \quad (2.8)$$

Another way to interpret what we have found is the simple statement that

$$J^i = \rho v^i + \Sigma^{ij} \left( E_j + \epsilon_{jkl}v^k B^l \right), \quad \Sigma^{ij} \equiv c(B^2)\delta^{ij} + \tilde{\sigma}(B^2)\epsilon^{ikj}B_k + d(B^2)B^i B^j. \quad (2.9)$$

The first order correction to the current  $J^i$ , which was before simply proportional to  $\sigma_Q$ , is now proportional to a matrix  $\Sigma^{ij}$  which inherits the rotational symmetry breaking pattern of the external magnetic field. Since entropy production, which occurs at quadratic order

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<sup>2</sup>In principle one could add transport coefficients analogous to  $\tilde{\sigma}$ ,  $c$  and  $d$  that appear in (2.7) also in the momentum current. These can however be absorbed by changing the hydrodynamic frame, that is by redefining the velocity as  $v_i \rightarrow v_i + A_1 E_i + A_2 \epsilon_{ijk} B_j E_k + A_3 E_j B_j B_i$ . The 3 coefficients  $A_{1,2,3}$  contain enough freedom to eliminate the 3 analogues of  $\tilde{\sigma}$ ,  $c$  and  $d$  in the momentum current.

in  $E$ , should be proportional to  $E^i J_i$ , and the first term in  $J^i$  does not contribute to entropy production as it arises at zeroth order in hydrodynamics, we conclude that the matrix  $\Sigma^{ij}$  should be positive definite. This constrains

$$c \geq 0, \quad c + dB^2 \geq 0. \quad (2.10)$$

We also observe that the coefficient  $\tilde{\sigma}$  is dissipationless, and does not appear to be constrained.

The form of the constitutive relation can be further constrained if we impose charge conjugation symmetry. Assuming that the underlying critical theory has charge conjugation symmetry, and latter is only broken by the explicit presence of the charge carries via  $\rho$  demands symmetry under

$$j_i \rightarrow -j_i, \quad T^{ti} \rightarrow T^{ti}, \quad v^i \rightarrow v^i, \quad (E^i, B^i) \rightarrow (-E^i, -B^i), \quad \rho \rightarrow -\rho, \quad \epsilon + P \rightarrow \epsilon + P \quad (2.11)$$

This symmetry requires  $\tilde{\sigma} = 0$ . Generically, it is possible for  $\tilde{\sigma}$  to be an odd function of  $\rho$ . But for simplicity, we will often set  $\tilde{\sigma} = 0$  in what follows to simplify the equations.

We confirm in appendix A that starting with the most general relativistic constitutive relation including terms up to order  $F^3$  indeed yields eq. (2.7) up to  $\mathcal{O}(B^2)$ , when restricting to terms linear in  $E$ . The transport coefficients  $c$ ,  $\tilde{\sigma}$  and  $d$  appear as linear combinations of the various terms appearing in the relativistic analysis.

### 2.3 Linear response

We are now in a position to determine  $\sigma_{ij}$  in linear response. The energy and charge conservation equations can be ignored so long as the fluid is homogeneous [18]; the momentum conservation equation reads:

$$(\epsilon + P) \left( -i\omega + \frac{1}{\tau} \right) v^i \equiv \Gamma v^i = \rho E^i + \varepsilon^i_{jk} J^j B^k. \quad (2.12)$$

Plugging (2.12) into the constitutive relation (2.7) gives the following matrix expression

$$\begin{aligned} & \left( \delta^i_j + \frac{c(B^2)}{\Gamma} \{B^2 \delta^i_j - B^i B_j\} - \frac{\rho}{\Gamma} \varepsilon^i_{jk} B^k \right) J^j \\ &= \left( \left\{ \frac{\rho^2}{\Gamma} + c(B^2) \right\} \delta^i_j + \left\{ c(B^2) \frac{\rho}{\Gamma} \right\} \varepsilon^i_{jk} B^k + dB^i B_j \right) E^j. \end{aligned} \quad (2.13)$$

Without loss of generality, we let  $\vec{B} = B\hat{z}$ . With this, we obtain the following expressions for the conductivity:

$$\sigma_{zz} = \frac{\rho^2}{\Gamma} + c(B^2) + dB^2 \quad (2.14a)$$

$$\sigma_{xx} = \sigma_{yy} = \Gamma \frac{\rho^2 + \Gamma c(B^2) + B^2 c(B^2)^2}{(\Gamma + c(B^2)B^2)^2 + \rho^2 B^2} \quad (2.14b)$$

$$\sigma_{xy} = -\sigma_{yx} = B\rho \frac{\rho^2 + 2\Gamma c(B^2) + B^2 c(B^2)^2}{(\Gamma + c(B^2)B^2)^2 + \rho^2 B^2}. \quad (2.14c)$$



with the corresponding resistivity given by the inverse matrix:

$$\rho_{zz} = \frac{\Gamma}{d\Gamma B^2 + \rho^2 + \Gamma c(B^2)} \quad (2.15a)$$

$$\rho_{xx} = \rho_{yy} = \frac{\Gamma \rho^2 + c(B^2) (B^2 \Gamma c(B^2) + \Gamma^2)}{\rho^4 + (\Gamma^2 + B^2 \rho^2) c(B^2)^2 + 2\rho^2 \Gamma c(B^2)^2} \quad (2.15b)$$

$$\rho_{xy} = -\rho_{yx} = -B \frac{B^2 \rho c(B^2)^2 + 2\Gamma \rho c(B^2) + \rho^3}{\rho^4 + (\Gamma^2 + B^2 \rho^2) c(B^2)^2 + 2\rho^2 \Gamma c(B^2)^2}. \quad (2.15c)$$

The constants in  $c(B^2)$  and  $d$  will depend on the microscopic details of the theory, and their sign will determine if NMR is present. Indeed, (2.14a) is reminiscent of (1.5), even though this fluid is not chiral: letting  $c \approx c_0 + c_1 B^2$ , and similarly for  $d$ , we see that so long as  $c_1 + d_0 > 0$   $\sigma_{zz}$  is an increasing function of  $B^2$ , and  $\rho_{zz}$  is a decreasing function of  $B^2$ , as is (1.5). However, the main experimental test for anomaly-induced NMR is the dramatic angular dependence of the resistivity. Using the definitions in (1.7), it is straightforward to show that

$$\alpha = \frac{\Gamma \rho^2 + c(B^2) (B^2 \Gamma c(B^2) + \Gamma^2)}{\rho^4 + (\Gamma^2 + B^2 \rho^2) c(B^2)^2 + 2\rho^2 \Gamma c(B^2)^2}, \quad (2.16a)$$

$$\beta = -B \frac{B^2 \rho c(B^2)^2 + 2\Gamma \rho c(B^2) + \rho^3}{\rho^4 + (\Gamma^2 + B^2 \rho^2) c(B^2)^2 + 2\rho^2 \Gamma c(B^2)^2}, \quad (2.16b)$$

$$\gamma = \alpha \left[ -1 + \frac{1 + \frac{c^2 B^2}{\rho^2 + c\Gamma} \frac{\rho^2}{\rho^2 + c\Gamma}}{\left(1 + \frac{c^2 B^2}{\rho^2 + c\Gamma}\right) \left(1 + \frac{d\Gamma B^2}{\rho^2 + c\Gamma}\right)} \right]. \quad (2.16c)$$

At  $B = 0$ , one can easily check that  $\beta = \gamma = 0$ , as must happen, since the theory becomes isotropic. Allowing  $B \neq 0$  but assuming  $d = 0$ , we find that

$$\frac{\gamma}{\alpha} = - \frac{\frac{c^2 B^2}{\rho^2 + c\Gamma} \frac{c\Gamma}{\rho^2 + c\Gamma}}{1 + \frac{c^2 B^2}{\rho^2 + c\Gamma}} < 0. \quad (2.17)$$

We hence conclude that the hydrodynamics of [18] can exhibit similar angular dependence in the resistivity to (1.5), despite the fact that the hydrodynamic equations are manifestly isotropic. This effect is dependent on the breaking of Galilean invariance, which allows for the coefficient  $c \neq 0$ . Adding the  $d$ -dependence back in, we conclude that only for  $d$  sufficiently negative is it possible for  $\gamma > 0$ . At small  $B$ , one can show that

$$-d > \frac{c_0^3}{c_0 \Gamma + \rho^2} \quad (2.18)$$

is necessary in order for the angular dependence to appear as “positive” magnetoresistance.

If we include non-vanishing  $\tilde{\sigma}$ , then the conductivity matrix generalizes to:

$$\sigma_{zz} = \frac{\rho^2}{\Gamma} + c(B^2) + d(B^2)B^2 \quad (2.19a)$$

$$\sigma_{xx} = \sigma_{yy} = \Gamma \frac{\rho\tilde{\rho} + B^2\tilde{\rho}\tilde{\sigma}(B^2) + \Gamma c(B^2) + B^2c(B^2)^2}{(\Gamma + c(B^2)B^2)^2 + \rho^2B^2} \quad (2.19b)$$

$$\sigma_{xy} = -\sigma_{yx} = B \frac{\tilde{\rho}^2\rho + \Gamma\tilde{\rho}c(B^2) + \Gamma\rho c(B^2) - \Gamma^2\tilde{\sigma}(B^2) + B^2c(B^2)^2\rho - B^2c(B^2)\tilde{\sigma}(B^2)}{(\Gamma + c(B^2)B^2)^2 + \rho^2B^2} \quad (2.19c)$$

where  $\tilde{\rho} = \rho - \tilde{\sigma}B^2$ .

## 2.4 Thermal transport

Let us now briefly discuss thermal transport. In this case, we wish to compute the charge current and the heat current in response to applied electric fields and thermal gradients:

$$\begin{pmatrix} J^i \\ Q^i \end{pmatrix} = \begin{pmatrix} \sigma^{ij} & T\alpha^{ij} \\ T\bar{\alpha}^{ij} & T\bar{\kappa}^{ij} \end{pmatrix} \begin{pmatrix} E_j \\ -T^{-1}\partial_j T \end{pmatrix} \quad (2.20)$$

The heat current is defined as [18]

$$Q^i \equiv T^{ti} - \mu J^i \quad (2.21)$$

with  $\mu$  the chemical potential of the fluid.

A priori, such a computation can be quite subtle, since it appears as though we need to account for more terms in the derivative expansion to fix the constitutive relations for  $\partial_j T$ . However, consider the following arguments. Firstly, we may use the standard Landau frame in which (2.8) is exact. Using the thermodynamic relation  $\epsilon + P - \mu\rho = Ts$ , with  $s$  the entropy density, we conclude that in an electric field  $E^i$  (but keeping  $\partial_i T = 0$ ):

$$Q^i = Tsv^i - \mu\Sigma^{ij}(E_j + \epsilon_{jkl}v^k B^l). \quad (2.22)$$

We have already solved for the velocity field  $v^i$  in an applied electric field in our computation of  $\sigma^{ij}$ , so hence we obtain straightforwardly the matrix

$$\begin{aligned} \bar{\alpha}_{xx} &= \frac{\Gamma\rho sT - c\Gamma\mu(B^2c + \Gamma)}{T\left((B^2c + \Gamma)^2 + B^2\rho^2\right)} \\ \bar{\alpha}_{xy} &= \frac{B(csT(B^2c + \Gamma) - c\Gamma\mu\rho + \rho^2sT)}{T\left((B^2c + \Gamma)^2 + B^2\rho^2\right)} \\ \bar{\alpha}_{zz} &= \frac{\rho s}{\Gamma} - \frac{\mu(B^2d + c)}{T}. \end{aligned} \quad (2.23)$$

Secondly, we use Onsager reciprocity which states that  $\alpha_{ij}(B) = \bar{\alpha}_{ji}(-B)$ ; since all off-diagonal elements of the transport matrices are antisymmetric, we conclude that  $\alpha_{ij} = \bar{\alpha}_{ij}$ .

Next, we can imagine turning off the electric field and only applying an external temperature gradient. The momentum conservation equation then reads

$$\Gamma v^i = \epsilon^{ijk} J_j B_k - T s \frac{\partial^i T}{T}. \quad (2.24)$$

Using the fact that

$$J^i = -\bar{\alpha}_{ij} \partial^j T, \quad (2.25)$$

we may combine (2.21), (2.24) and (2.25) to obtain  $\bar{\kappa}_{ij}$ :

$$\bar{\kappa}_{xx} = \frac{B^2 c (c\Gamma\mu^2 + (\mu\rho + sT)^2) + \Gamma (c\Gamma\mu^2 + s^2 T^2)}{T \left( (B^2 c + \Gamma)^2 + B^2 \rho^2 \right)} \quad (2.26a)$$

$$\bar{\kappa}_{xy} = \frac{BsT(\rho sT - 2c\Gamma\mu) - B^3 c^2 \mu(\mu\rho + 2sT)}{T \left( (B^2 c + \Gamma)^2 + B^2 \rho^2 \right)} \quad (2.26b)$$

$$\bar{\kappa}_{zz} = \frac{\mu^2 (B^2 d + c)}{T} + \frac{s^2 T}{\Gamma}. \quad (2.26c)$$

From these results we can conclude that the constitutive relation for the current must include a linear temperature gradient as

$$J^i = \rho v^i + \Sigma^{ij} \left( E^j + \epsilon^{jkl} v_k B_l - \frac{\mu}{T} \partial^j T \right). \quad (2.27)$$

As in an ordinary relativistic fluid, we conclude that there are no new dissipative coefficients associated with thermo-magnetic response.

We note that all thermoelectric coefficients in the  $xy$  plane are identical to those presented in [18]. This is to be expected, since only the longitudinal components of the transport matrix are sensitive to the anisotropies introduced by the  $E^j B_j B^i$  term in (2.7).

### 3 “Microscopic” examples

#### 3.1 $\mathcal{N} = 4$ SYM plasma

We now wish to compare our formalism with the conductivity of  $N_f$  massive  $\mathcal{N} = 2$  supersymmetric hypermultiplets flowing through an  $\mathcal{N} = 4$  SYM plasma with gauge group  $SU(N_c)$  at temperature  $T$ . This model was studied extensively starting with [28] and the conductivities in the background of constant electromagnetic field with generic orientations was worked out in [17]. We take the limits  $N_c \rightarrow \infty$  with large but finite 't Hooft coupling  $\lambda = g_{YM}^2 N_c$ , allowing the use of holographic techniques. The flavor hypermultiplets are dual to  $N_f$  D7 branes [29] embedded in a fixed AdS-Schwarzschild background. Furthermore, we will work in the probe limit  $N_f \ll N_c$  so that we may neglect the back reaction of the probe branes on the supergravity fields. This allows us to treat the plasma as stationary, and focus on the dynamics of the flavor fields alone. Specifically, this limit allows for an apparent dissipation of momentum. The flavor fields lose energy to the plasma at a rate of order  $N_c$  so only at times of order  $N_c$  will the back reaction on the  $N_c^2$  plasma degrees of freedom be non-negligible.

This momentum relaxation is rather ‘peculiar’, and so the theory of transport in probe brane models differs in important ways from other models of transport [23]. In particular, it is unclear whether or not a ‘weak disorder’ limit exists. As such a limit was required in order to rigorously include  $\Gamma$  in our hydrodynamic model of transport, there is a priori no reason to expect exact quantitative agreement between our hydrodynamic model and this holographic model. It is known that generic holographic models disagree with the hydrodynamics of [30–32] at next-to-leading order in  $\Gamma$ : this can crudely be thought of as arising due to  $\Gamma$ -dependent corrections to the hydrodynamic constitutive relations.

The conductivity of the propagating hypermultiplets in generic constant background fields was found in [17]. They considered an  $E$  field that is fixed along the  $x$ -axis, while the  $B$  field lies in the  $x - z$  plane. This can be mapped onto our formalism by rewriting their results in terms of the basic constitutive relation (1.7). For small electric fields, they found

$$\sigma_{xx} = \tilde{\rho} \frac{1 + b_x^2}{1 + b^2} \sqrt{1 + \frac{N_f^2 N_c^2 T^2}{\tilde{\rho}^2 16 \pi^2} \cos^6 \theta^* (1 + b^2)}, \quad \sigma_{xy} = \frac{\tilde{\rho} b_z}{1 + b^2}, \quad \sigma_{xz} = \frac{b_x b_z}{1 + b_x^2} \sigma_{xz} \quad (3.1)$$

where  $b_i = \frac{B_i}{qT^2}$  and  $\tilde{\rho} = \frac{\rho}{qT^2}$ . This can be brought into the form (1.7) with coefficients

$$\alpha = \frac{q\rho T^2 \sqrt{\frac{2\sigma_0 T^2 (B^2 + q^2 T^4)}{q\rho^2} + 1}}{\rho^2 + 2q\sigma_0 T^6} \quad (3.2a)$$

$$\beta = -\frac{\rho}{\rho^2 + 2q\sigma_0 T^6} \quad (3.2b)$$

$$\gamma = \frac{qT^2}{\rho \sqrt{\frac{2\sigma_0 T^2 (B^2 + q^2 T^4)}{q\rho^2} + 1}} - \alpha \quad (3.2c)$$

where  $\sigma_0 = \frac{N_f^2 N_c^2 \sqrt{\lambda}}{64\pi} \cos^6 \theta^*$  and  $q = \frac{\pi}{2} \sqrt{\lambda}$ .

This results look quite different from the hydrodynamic form. One could ask whether the D3/D7 answer can be fit into the hydrodynamic framework by a particular choice of transport coefficients. At large  $\rho$ , the coefficients  $c$ ,  $d$  and  $\tilde{\sigma}$  are generically  $\rho$ -dependent. In the limits  $\rho \gg T^3$  and  $B \ll qT^2$  limits, [17] have shown that probe brane models appear consistent with

$$\Gamma = qT^2 \rho$$

at leading order in  $\rho$ . At this order, one trivially finds a Drude-like conductivity:  $\sigma_{ij} \approx \rho^2 \delta_{ij} / \Gamma$ . Since  $c$ ,  $d$  and  $\tilde{\sigma}$  arise at next-to-leading order in this limit, their unique determination requires specifying  $\Gamma$  at order  $\rho^0$ . It is unclear whether this question is even ‘well-posed’, in light of the subtleties that arise in transport beyond the weak disorder limit [23, 30–32].

However, given the exact magnetic field dependence of the resistivity, we can non-perturbatively compute the magnetoresistance in both  $B$  and  $\rho$ . Firstly, one can explicitly compute

$$\rho_{zz} = \frac{qT^2}{\sqrt{\rho^2 + \frac{2}{q} T^2 \sigma_0 (B^2 + q^2 T^4)}}, \quad (3.3)$$

which is clearly a decreasing function of  $B^2$ . Secondly, the ratio of the longitudinal to transverse resistivities is

$$\frac{\rho_{xx}}{\rho_{zz}} = 1 + \frac{2\sigma_0 T^2}{q(\rho^2 + 2q\sigma_0 T^6)} B^2 > 1 \quad (3.4)$$

implying the resistivity along the direction of the magnetic field is suppressed relative to the transverse directions. As expected, in the  $B \rightarrow 0$  limit, the ratio goes to one since the theory becomes isotropic.

### 3.2 Cartoon of electron-hole plasma

In this section we present a simple classical cartoon of a fluid where we can compute the coefficients  $c$  and  $d \neq 0$ . More precisely, let us consider a toy model of two charged fluids, one of charge density  $\hat{\rho}$  (the  $+$  fluid) and the other of charge density  $-\hat{\rho}$  (the  $-$  fluid), analogous to electron and hole fluids in graphene. Unlike in graphene [33], we will suppose that the momentum of these two charged fluids is also an almost conserved quantity. For simplicity, we assume that all other properties, such as enthalpy  $\hat{\mathcal{M}}$ , of these two fluids are identical, and we also only consider the hydrodynamic gradient expansion to first order in derivatives.

The net current is given by the sum of currents in the  $\pm$  fluids:  $J^\mu = J_+^\mu + J_-^\mu$ . The spatial components of these currents are given by

$$J_+^i = \hat{\rho} v_+^i + \hat{\sigma}_Q \left( E^i + \varepsilon^{ijk} v_+^j B^k \right), \quad (3.5a)$$

$$J_-^i = -\hat{\rho} v_-^i + \hat{\sigma}_Q \left( E^i + \varepsilon^{ijk} v_-^j B^k \right). \quad (3.5b)$$

$\hat{\sigma}_Q$  is the ‘quantum critical conductivity’ for each microscopic fluid, and will be important to include. The presence of  $\hat{\sigma}_Q$  has broken Galilean invariance microscopically, and will result in a non-Galilean magnetoresistance. The momentum quasi-conservation equations of the two fluids are

$$-i\omega \hat{\mathcal{M}} v_+^i = \hat{\rho} E^i + \varepsilon^{ijk} J_+^j B^k - \alpha (v_+^i - v_-^i), \quad (3.6a)$$

$$-i\omega \hat{\mathcal{M}} v_-^i = -\hat{\rho} E^i + \varepsilon^{ijk} J_-^j B^k - \alpha (v_-^i - v_+^i). \quad (3.6b)$$

$\alpha/\hat{\mathcal{M}}$  governs the rate at which the electron/hole fluids exchange momentum. (3.5) and (3.6) form a set of equations which can be solved straightforwardly: upon doing so, we find that

$$\begin{aligned} \sigma_{xx}(\omega) &= \frac{-2i\omega \hat{\mathcal{M}} \left( \hat{\rho}^2 + \hat{\sigma}_Q \left( 2\alpha + B^2 \hat{\sigma}_Q - i\omega \hat{\mathcal{M}} \right) \right)}{B^4 \hat{\sigma}_Q^2 - 2i\alpha \hat{\mathcal{M}} \omega - (\omega \hat{\mathcal{M}})^2 + B^2 \left( \hat{\rho}^2 + 2\hat{\sigma}_Q \left( \alpha - i\omega \hat{\mathcal{M}} \right) \right)} \\ &\approx \frac{-2i\omega \hat{\mathcal{M}} \left( \hat{\rho}^2 + \hat{\sigma}_Q \left( 2\alpha + B^2 \hat{\sigma}_Q \right) \right)}{B^4 \hat{\sigma}_Q^2 - 2i\alpha \hat{\mathcal{M}} \omega + B^2 \left( \hat{\rho}^2 + 2\hat{\sigma}_Q \alpha \right)}, \end{aligned} \quad (3.7a)$$

$$\sigma_{zz}(\omega) = 2 \frac{\hat{\rho}^2 + (2\alpha - i\hat{\mathcal{M}}\omega) \hat{\sigma}_Q}{2\alpha - i\omega \hat{\mathcal{M}}} \approx 2\hat{\sigma}_Q + \frac{\hat{\rho}^2}{\alpha}. \quad (3.7b)$$

The parameter  $\alpha$  governs the rate of relaxation between the two fluids. Assuming that  $\alpha$  is small enough that it can be treated within the gradient expansion of hydrodynamics, one can show using that the second law of thermodynamics implies

$$\alpha > 0. \quad (3.8)$$

Perhaps more intuitively, (3.8) can also be understood from the requirement that thermal equilibrium  $v_{\pm}^i = 0$  is stable.

The approximations we have made in the last step of (3.7) are valid in the limit  $\alpha \gg \omega \hat{\mathcal{M}}$ . In this limit, we expect single fluid hydrodynamics with net charge density zero. From (2.14) (with  $\rho = 0$  and  $\Gamma = -i\omega \mathcal{M}$ ) we should find

$$\sigma_{xx}(\omega) = \frac{-i\omega \mathcal{M} \sigma_{\perp}}{B^2 \sigma_{\perp} - i\omega \mathcal{M}} + \mathcal{O}(\omega^3), \quad (3.9a)$$

$$\sigma_{zz}(\omega) = \sigma_{\parallel} + \mathcal{O}(\omega) \quad (3.9b)$$

with  $\sigma_{\perp} = c$  and  $\sigma_{\parallel} = c + dB^2$ . Upon making the identifications

$$\mathcal{M} = 2\hat{\mathcal{M}}, \quad (3.10a)$$

$$\sigma_{\parallel} = 2\hat{\sigma}_{\text{Q}} + \frac{\hat{\rho}^2}{\alpha} \quad (3.10b)$$

$$\sigma_{\perp} = \sigma_{\parallel} + \frac{B^2 \hat{\sigma}_{\text{Q}}}{\alpha}, \quad (3.10c)$$

we see that (2.14) and (3.7) agree. Furthermore, we find an expression for

$$d = -\frac{\hat{\sigma}_{\text{Q}}}{\alpha}. \quad (3.11)$$

From (3.8), we conclude that  $d < 0$ . Using (2.18), we see that this model will exhibit positive magnetoresistance if momentum relaxation is strong enough.

#### 4 Momentum relaxation rate

So far, we have used a ‘mean field’ description of momentum relaxation. It is also possible that upon adding a magnetic field, momentum can relax more or less efficiently in the direction of the magnetic field. In this section, we will perturbatively compute the rate of momentum relaxation in a fluid, disordered by very long wavelength inhomogeneity in an externally imposed chemical potential [20, 34]. In such a limit, the transport coefficients may be computed by solving the hydrodynamic equations in an inhomogeneous medium, which can be shown to be:

$$\partial_i J^i = \partial_i \left( \rho(\mu(x)) \delta v^i - \Sigma^{ij} \left( \partial_j \delta \mu - \frac{\mu(x)}{T} \partial_j \delta T(x) - \epsilon_{jkl} v^k B^l \right) \right) = 0, \quad (4.1a)$$

$$\partial_i \left( T s(\mu(x)) \delta v^i - \mu(x) \Sigma^{ij} \left( \partial_j \delta \mu - \frac{\mu(x)}{T} \partial_j \delta T(x) - \epsilon_{jkl} v^k B^l \right) \right) = 0, \quad (4.1b)$$

$$\rho(\mu(x)) \partial_i \delta \mu + s(\mu(x)) \partial_i \delta T - \partial_j \left( \eta \left( \partial_i \delta v_j + \partial_j \delta v_i - \frac{2}{3} \delta_{ij} \partial_k \delta v_k \right) \right) = \epsilon_{ijk} J^j B^k. \quad (4.1c)$$

Suppose that  $\mu(x) = \bar{\mu} + u\hat{\mu}(x)$ , with  $u$  perturbatively small. One can compute  $\Gamma$  to leading order in  $u$  by either solving (4.1) in an inhomogeneous background [20, 34, 35], or by using the memory function formalism [23, 36–38]. For our purposes, it will be easier to do the latter. What one finds is that  $\Gamma v_i$  in (2.12) should be replaced by  $\Gamma_{ij}v_j$ , with

$$\Gamma_{ij} \equiv \int \frac{d^3k}{(2\pi)^3} k_i k_j |\mu(k)|^2 \times \lim_{\omega \rightarrow 0} \frac{\text{Im}(G_{\rho\rho}^R(\omega, k))}{\omega}, \quad (4.2)$$

with the retarded Green’s function evaluated in the translation invariant theory, and  $\mu(k)$  the Fourier transform of  $\mu(x)$ . The only non-zero contributions to  $\Gamma_{ij}$  will be at least  $\mathcal{O}(u^2)$ . For simplicity in what follows, we will neglect the anisotropic corrections to the ‘quantum critical’ conductivity  $\sigma_Q$  which can arise in a magnetic field.

The hydrodynamic Green’s functions may be found by the following prescription [39]. Suppose the hydrodynamic equations of motion take the schematic form

$$\partial_t \varphi_A + M_{AB} \varphi_B = 0. \quad (4.3)$$

Let  $\chi_{AB}$  be the susceptibility matrix:  $\chi_{AB} = \partial \varphi_A / \partial \lambda_B$ , with  $\lambda_B$  the thermodynamic conjugate variables to  $\varphi_A$ . For us,  $\varphi_A = (\epsilon, \rho, T^{ti})$  and  $\delta \lambda_A = (\delta T/T, \delta \mu - \mu \delta T/T, \delta v_i)$ , and [39]

$$\chi_{AB} = \begin{pmatrix} T(\partial_T \epsilon)_{\mu/T} & (\partial_\mu \epsilon)_T & 0 \\ (\partial_\mu \epsilon)_T & (\partial_\mu \rho)_T & 0 \\ 0 & 0 & (\epsilon + P)\delta_{ij} \end{pmatrix}. \quad (4.4)$$

The hydrodynamic retarded Green’s function is

$$G_{AB}^R(k, \omega) = [M(k)((M(k) - i\omega)^{-1} \chi)]_{AB}. \quad (4.5)$$

From the equations of motion in a magnetic field, we find (neglecting viscous effects, for simplicity, as these are subleading in the limit where  $\hat{\mu}(x)$  is extremely slowly varying [20, 34, 35])

$$M_{AB} = \begin{pmatrix} 0 & 0 & ik_i \\ \sigma_1 k^2 & \sigma_2 k^2 & ik_j \frac{\rho \delta_{ij} + c \epsilon_{ijk} B_k}{\epsilon + P} \\ ik_i b_1 + i\sigma_1 \epsilon_{ijk} k_j B_k & ik_i b_2 + i\sigma_2 \epsilon_{ijk} k_j B_k & \epsilon_{imk} B_k \left( \frac{\rho}{\epsilon + P} \delta_{jm} + \frac{c}{\epsilon + P} \epsilon^{mjn} B_n \right) \end{pmatrix}, \quad (4.6)$$

with the parameters

$$\frac{\sigma_1}{c} = \left( \frac{\partial \mu}{\partial \epsilon} \right)_\rho - \frac{\mu}{T} \left( \frac{\partial T}{\partial \epsilon} \right)_\rho, \quad \frac{\sigma_2}{c} = \left( \frac{\partial \mu}{\partial \rho} \right)_\epsilon - \frac{\mu}{T} \left( \frac{\partial T}{\partial \rho} \right)_\epsilon, \quad (4.7a)$$

$$b_1 = \left( \frac{\partial P}{\partial \epsilon} \right)_\rho, \quad b_2 = \left( \frac{\partial P}{\partial \rho} \right)_\epsilon. \quad (4.7b)$$

Recall the definition of  $c$  in (2.7).

Using these equations, we now compute the  $(k, \omega) \rightarrow 0$  limit of  $G^R(k, \omega)$ . What we find is that the spectral weight then diverges as  $k^{-2}$ :

$$\lim_{\omega \rightarrow 0} \frac{\text{Im}(G_{\rho\rho}^R(\omega, k))}{\omega} = \mathcal{A} \frac{\rho^2 + c^2 B^2}{c [k^2 (\rho^2 + 2c^2 B^2) - c^2 (k_i B_i)^2]} + \dots, \quad (4.8)$$

where the thermodynamic prefactor

$$\mathcal{A} = \frac{(\partial_\epsilon P)_\rho (\rho(\partial_\mu \epsilon)_T - (\epsilon + P)(\partial_\mu \rho)_T)}{T(\epsilon + P)((\partial_\rho P)_\epsilon (\partial_\epsilon \frac{\mu}{T})_\rho - (\partial_\epsilon P)_\rho (\partial_\rho \frac{\mu}{T})_\epsilon)}. \quad (4.9)$$

It is straightforward to see that spectral weight is enhanced by a magnetic field parallel to the wave vector. Hence, assuming  $\mu(k)$  is random, upon performing the angular integral in (4.2), we find that  $\Gamma_{zz} > \Gamma_{xx} = \Gamma_{yy}$ . Since  $\Gamma_{zz}/\Gamma_{xx} > 1$ , this type of inhomogeneity tends towards ‘positive magnetoresistance’ by causing  $\rho_{zz}/\rho_{xx}$  to increase, relative to the scenario with  $\Gamma$  homogeneous.

At higher orders in  $k$ , the spectral weight of the charge density picks up additional contributions. Some of these contributions are proportional to viscosity, which has been neglected so far in our discussion. In the presence of a finite magnetic field, the viscosity becomes a fourth-rank tensor with multiple non-vanishing coefficients: see e.g. [22, 27]. Other contributions are proportional to third-order corrections (in spatial gradients) to the charge current  $J^\mu$ ; these can be thought of as  $k^2$  corrections to the coefficient  $c$  in (4.8). Nevertheless, it is instructive to compare (4.8) with a calculation of the spectral weight only accounting for viscous dissipation. Hence, let us set  $\Sigma_{ij} = 0$ , and assume that the only dissipative coefficient is the isotropic shear viscosity  $\eta$ . In this case, one finds that

$$\lim_{\omega \rightarrow 0} \frac{\text{Im}(G_{\rho\rho}^R(\omega, k))}{\omega} = \mathcal{A}' \frac{B^2 \rho^2 (3k^2 + k_z^2) + k^6 \eta^2}{B^2 \rho^2 k_z^2 + k^6 \eta^2} \quad (4.10)$$

with

$$\mathcal{A}' = \frac{\rho(\epsilon + P)((\partial_\epsilon P)_\rho (\partial_\mu \epsilon)_T + (\partial_\rho P)_\epsilon (\partial_\mu \rho)_T)}{3(\rho(\partial_\rho P)_\epsilon + (\epsilon + P)(\partial_\epsilon P)_\rho)^2}. \quad (4.11)$$

Due to the divergence in the denominator of (4.10) as  $k_z \rightarrow 0$ , it is clear from (4.2) that  $\Gamma_{zz}$  will be smaller than  $\Gamma_{xx}$ ; the factor of  $k_z^2$  in the formula for  $\Gamma_{zz}$  will cancel this small factor in the denominator, while no such cancellation will arise for  $\Gamma_{xx}$ . If viscous effects are large, inhomogeneous chemical potentials lead to negative magnetoresistance. The angular dependence is thus qualitatively distinct from the case where  $\Sigma_{ij} \neq 0$ .

## 5 Conclusion

In this article we have presented a relativistic hydrodynamic theory of magnetotransport in  $3 + 1$  dimensions. Depending on the ‘microscopic’ models of interest, it is possible to obtain both positive and negative magnetoresistance within our framework. NMR is, in some ways, a more generic effect for a relativistic or non-Galilean invariant fluid: arising not only from the spatial anisotropies caused by the presence of background magnetic fields, but also from the particular structure of relativistic hydrodynamics. In particular, we found that the holographic D3/D7 system exhibited negative magnetoresistance. A two-fluid cartoon of electron-hole plasma which exhibits positive magnetoresistance can also be found. Smooth disorder potentials also imply a slight positive magnetoresistance.

In Galilean invariant fluids, it has been shown [40–42] that magnetoresistance is sensitive only to viscosity. However, we have shown in section 4 that for other gradient



expansions, this no longer remains the case. It is not clear whether magnetoresistance is a good viscometer for electron fluids arising from general band structures. This may not be the case, even if the temperature is low enough that thermal effects are negligible. One difficulty, as we have noted, is that the viscosity itself is a fourth-rank tensor which can become anisotropic in the presence of a magnetic field. Further work to resolve this question is warranted.

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## A Constitutive relations to third order in $F_{\mu\nu}$

Here we show that the non-relativistic constitutive relations can be derived from the most general covariant expression up to third order in the field strength. The expression is

$$\begin{aligned} J^\mu = & \rho u^\mu + \sigma_{Q1} F_\lambda^\mu u^\lambda + \sigma_{Q2} F^{\mu\rho} F_{\rho\sigma} u^\sigma + \sigma_{Q3} F^2 F_\rho^\mu u^\rho + \sigma_{Q4} F^{\mu\rho} F_{\rho\sigma} F_\lambda^\sigma u^\lambda \\ & + \sigma_{Q5} F_\rho^\mu F_\lambda^\sigma F_{\sigma\eta} u^\rho u^\lambda u^\eta + \sigma_{Q6} F_\rho^\mu F_\sigma^\rho (\star F)_\nu^\sigma u^\nu + \sigma_{Q7} \varepsilon^{\mu\nu\rho\sigma} F_{\nu\tilde{\nu}} F_{\rho\sigma} u^{\tilde{\nu}} \\ & + \sigma_{Q8} F_\rho^\mu (\star F)_\sigma^\rho (\star F)_\nu^\sigma u^\nu + \sigma_{Q9} (\star F)_{\rho\sigma} F^{\rho\sigma} F_\nu^\mu u^\nu + \sigma_{Q10} (\star F)_{\rho\sigma} F^{\rho\sigma} (\star F)_\nu^\mu u^\nu. \end{aligned} \quad (\text{A.1})$$

We will only keep track of terms to “first order” in  $E$ , recalling that  $v_i$  is first order in  $E$  while  $B$  is zeroth order. The first term simply gives

$$\sigma_{Q1} F_\lambda^\mu u^\lambda \rightarrow \sigma_{Q1} (E^i + \varepsilon^{ijk} v_j B_k) \quad (\text{A.2})$$

which is expected by Galilean invariance. The second term gives

$$\begin{aligned} \sigma_{Q2} F^{\mu\rho} F_{\rho\sigma} u^\sigma & \rightarrow \sigma_{Q2} \left( \varepsilon^{ijn} \varepsilon_{jkm} B_n B_m v^k - \varepsilon^{ijk} E_j B_k \right) \\ & = \sigma_{Q2} \left\{ (\delta_k^n \delta_m^i - \delta_m^n \delta_k^i) B_n B_m v^k - \varepsilon^{ijk} E_j B_k \right\} \\ & = \sigma_{Q2} \left( (v \cdot B) B^i - B^2 v^i - \varepsilon^{ijk} E_j B_k \right). \end{aligned} \quad (\text{A.3})$$

The third term gives a term identical to the first but with a  $B^2$ :

$$\sigma_{Q3} F^2 F_\rho^\mu u^\rho \rightarrow 2\sigma_{Q3} B^2 \left( E^i + \varepsilon^{ijk} v_j B_k \right). \quad (\text{A.4})$$

The fourth term gives

$$\sigma_{Q4} F^{\mu\rho} F_{\rho\sigma} F_\lambda^\sigma u^\lambda \rightarrow -\sigma_{Q4} \left( B^2 \delta_{ij} - B_i B_j \right) \left( E^i + \varepsilon^{ijk} v_j B_k \right). \quad (\text{A.5})$$

The fifth, sixth and seventh terms contribute at  $\mathcal{O}(E^2)$ . The eighth term gives

$$\sigma_{Q8} F_\rho^\mu (\star F)_\sigma^\rho (\star F)_\nu^\sigma u^\nu \rightarrow \sigma_{Q8} (-B^2 E^i + B^2 E_i - (E \cdot B) B^i). \quad (\text{A.6})$$

The term  $(\star F)_{\rho\sigma} F^{\rho\sigma}$  in the ninth and tenth terms gives  $-4E \cdot B$ , and so

$$\sigma_{Q9}(\star F)_{\rho\sigma} F^{\rho\sigma} F_{\nu}^{\mu} u^{\nu} \rightarrow -4\sigma_{Q9}(E \cdot B)(E^i + \varepsilon^{ijk} v_j B_k) \quad (\text{A.7a})$$

$$\sigma_{Q10}(\star F)_{\rho\sigma} F^{\rho\sigma} (\star F)_{\nu}^{\mu} u^{\nu} \rightarrow 4\sigma_{Q10}(E \cdot B)B^i. \quad (\text{A.7b})$$

Only the latter term contributes at linear order in  $E$  and  $v$ . Collecting these equations, we find that

$$c = \sigma_{Q1} + (2\sigma_{Q3} - \sigma_{Q4})B^2 + \dots, \quad (\text{A.8a})$$

$$d = \sigma_{Q4} + 4\sigma_{Q10} - \sigma_{Q8} + \dots. \quad (\text{A.8b})$$

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