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Sufficient conditions for topological order in insulators

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Abstract. – We prove the existence of low-energy excitations in insulating systems at general filling factor under certain conditions, and discuss in which cases these may be identified as topological excitations. In the specific case of half-filling this proof provides a significantly shortened proof of the recent higher-dimensional Lieb-Schultz-Mattis theorem.

The classic 1961 result of Lieb, Schultz, and Mattis (LSM) [1], proving the existence of an excitation within energy $\sim 1/L$ of the ground state for certain one-dimensional spin chains, has had a large effect on the field. While it was then proven by Affleck and Lieb [2] that one-dimensional systems either have gapless localized excitations or a local symmetry breaking, it has long been suspected that in higher dimensions there is a more interesting possibility of topological order [3].

One way to understand topological order is based on flux insertion. We give the physical argument here, and then discuss the difficulties in this argument which give rise to the need for the more careful argument of this paper. We consider a higher-dimensional system which is periodic in one direction. A spin-(1/2) system can be mapped to a hard-core boson system on a lattice, with the presence or absence of a particle denoting spin up or down. If the particle system is superfluid, there is long-range order in the x and y components of the spin in the original system, implying the existence of low-energy excitations. On the other hand, if the particle system is insulating, it should be possible to insert 2π of gauge flux in the hopping of particles across a given line cutting the system, returning the Hamiltonian to the original one, but, for non-integer filling fraction, taking the system to an excited state which is very close in energy to the ground state. Using adiabatic flux insertion, this was suggested as a way to prove a higher-dimensional LSM theorem [4]. The two possibilities would thus seem to be a superfluid system (or other system which resists flux insertion) which has low-energy excitations, or an insulating system which has topological order. In either case, there is a state close in energy to the ground state.

However, there is a serious problem with this argument. The definition of adiabatic flux insertion depends on the existence of a gap (but does not require anything about the magnitude of a gap); however, in a spin system with no disorder, any gap in the spectrum at zero flux must close at some non-zero value of the flux, thus making it impossible even to define an adiabatic flux insertion [5]. In a fractional quantum Hall system with disorder, there is a related problem. In a 1/3 quantum Hall state, the gap between the three approximately degenerate ground states remains open for all values of the flux, even though the gap is exponentially small. However, this means that adiabatic flux insertion leaves the system in the lowest of the three states and does not produce the correct topologically excited state [6].

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Thus flux must be inserted slowly, but not adiabatically. For fractional Hall systems, the experimentally relevant flux insertion is slow enough to avoid exciting states above the three almost degenerate ground states, but fast enough to "shoot through" the level crossing.

A precise definition of this quasi-adiabatic flux insertion was given in [7], and used to prove a version of the LSM theorem for spin systems with total $S^z = 0$ valid in arbitrary dimension. The flux was inserted in such a way that the ground-state wave function was only disturbed near the flux insertion point. Physically, one can imagine that the flux was inserted sufficiently rapidly to prevent influences propagating around the system, but sufficiently slowly to avoid creating localized excitations. Compared to [7], the proof here is shorter and is generalized to systems at general filling fraction ρ instead of just spin systems with total $S^z = 0$. It also explicitly constructs the low-energy states for certain systems. Later, we discuss under what conditions these states can be identified as topological excitations.

Definition of system. – We consider systems defined on a lattice, using letters $i, j, \ldots$ to denote lattice sites (throughout, the term "site" may also be used to refer to a unit cell comprised of several sites), and introducing a metric $d(i, j)$ on the lattice. We assume that there is a conserved charge, $Q = \sum_i Q_i$, where Q_i, the charge on site i, is quantized to be an integer (for example, Q_i may be taken to be the number of particles on a site, or in a spin system of half-integer spins Q_i may be equal to $S^z_i + 1/2$). We assume that the Hamiltonian, $\mathcal{H}$ can be written as $\mathcal{H} = \sum_i \mathcal{H}^i$, where the $\mathcal{H}^i$ obey the finite-range conditions [8,9]: 1) the commutator $[\mathcal{H}^i, O] = 0$ for any operator O which acts only on sites j with $d(i, j) > R$, where R is the range of the Hamiltonian; and 2) the operator norm $||\mathcal{H}^i|| \leq J$ for all i, for some constant J. Finally, the number of sites j with $d(i, j) \leq R$ should be bounded by some number that we denote S. These finite-range conditions include all short-range boson systems with a *finite* number of bosons allowed per site, as well as all short-range spin systems.

We suppose that there are a total of V sites on the d-dimensional lattice. We assume that the Hamiltonian is translationally invariant in one of the directions of the lattice, with period L: this is the length of the system. We define the filling fraction of the system to be $\rho = Q/V$. For each site i, we introduce coordinates (x_i, y_i) to specify sites, where x_i labels the distance along the length direction and y_i labels the transverse directions. The coordinate x is periodic with period L and should be compatible with the metric $d(i, j)$: any two sites i, j should have $d(i, j)$ greater than or equal to the minimum over integers n of $|x_i - x_j - nL|$.

We also define a twisted Hamiltonian and a rotation operator. Let

$$R(\theta) = \prod_{j, 0 < x_j \leq L/2} \exp[i\theta(Q_j - \rho)], \tag{1}$$

where the product ranges over all sites j such that $0 < x_j \leq L/2$. We introduce two boundary condition twists as follows. Let $\mathcal{H}_{\theta_1,\theta_2} = \sum_i \mathcal{H}^i_{\theta_1,\theta_2}$, where $\mathcal{H}^i_{\theta_1,\theta_2} = R(\theta_1)\mathcal{H}^i R(-\theta_1)$ if $-R \leq x_i \leq R$ and $\mathcal{H}^i_{\theta_1,\theta_2} = R(-\theta_2)\mathcal{H}^i R(\theta_2)$ if $L/2 - R \leq x_i \leq L/2 + R$ and otherwise $\mathcal{H}^i_{\theta_1,\theta_2} = \mathcal{H}^i$. Thus, angle θ_1 defines a twist in boundary conditions between $x = 0$ and $x = 1$, while angle θ_2 defines a twist between $x = L/2$ and $x = L/2 + 1$. These twists correspond to inserting gauge flux between $x = 0$ and $x = 1$ or between $x = L/2$ and $x = L/2 + 1$. Let there be a total of N_c sites i with $-R \leq x_i \leq R$, and let $||\partial_\theta \mathcal{H}^i_{\theta_1,\theta_2}|| \leq K$ for all i for some constant K. For a d-dimensional system with linear size L and aspect ratio close to unity, $N_c \sim L^{d-1}$.

Let $\Psi_0(\theta_1, \theta_2)$ be the ground state of $\mathcal{H}_{\theta_1,\theta_2}$, and let Ψ_0 be the ground state of $\mathcal{H}_{0,0} = \mathcal{H}$. Note that $\mathcal{H}_{\theta_1,\theta_2} = R(-\theta_2)\mathcal{H}_{\theta_1+\theta_2,0}R(\theta_2)$ and in particular $\mathcal{H}_{\theta,-\theta} = R(\theta)\mathcal{H}R(-\theta)$ and $\Psi_0(\theta, -\theta) = R(\theta)\Psi_0$. The second boundary condition angle thus appears to be redundant, since it is only the sum of the two angles, $\theta_1 + \theta_2$, that affects the eigenspectrum of $\mathcal{H}_{\theta_1,\theta_2}$.

That is, the physical properties depend only on the total gauge flux $\theta_1 + \theta_2$. However, the introduction of the second boundary angle will be extremely useful below.

Results. – The main result is that for a short-range Hamiltonian as defined above we are able to construct topologically excited states under certain assumptions on the spectrum of the Hamiltonian. The topologically excited states have a low energy but momentum that differs from the ground-state momentum as given in eqs. (4), (5), (6). We begin by specifying the conditions on the spectrum.

Suppose that, for $\theta_1 = \theta_2 = 0$, there is a gap ΔE in energy between the ground state and a set of eigenstates, that we refer to as local excitations; we define P_{high} to be the projection operator onto this space of states. There may be other eigenstates with energy below this gap; we define $P_{low} = 1 - P_{high}$ to project onto this space. We assume that for certain local operators O, namely $O = \partial_\theta \mathcal{H}_{\theta,-\theta}$ and $O = \partial_\theta R(\theta)$, the matrix elements of these operators, between Ψ_0 and any other normalized state Ψ which is a linear combination of eigenstates below the gap and which is orthogonal to the ground state, are bounded by $\epsilon||O||$ for some small constant ϵ. Physically, we are interested in this case because in many situations the matrix elements of local operators between the ground state and topologically excited states are very small, often exponentially small in the system size. Since the Hamiltonian is translationally invariant, the ground state, Ψ_0, may be taken translationally invariant, with eigenvalue z_0: $T\Psi_0 = z_0\Psi_0$, where T is the translation operator which increases the x-coordinate by 1.

Then, we introduce the twist operator

$$W_1(\phi) = \Theta \exp\left\{ -\int_0^\phi \mathrm{d}\theta \int_0^\infty \mathrm{d}\tau \exp\left[-(\tau\Delta E)^2/(2q)\right] \left[\tilde{w}^+_{1,\theta}(i\tau) - \text{H.c.}\right]\right\}, \tag{2}$$

where the symbol Θ denotes that the exponential is θ-ordered, in analogy to the usual time-ordered or path-ordered exponentials. We define $w_{1,\theta} = \partial_\theta \mathcal{H}_{\theta,0}$ and define $\tilde{w}^+_{1,\theta}(i\tau)$ following [9]: for any operator A

$$\tilde{A}(t) \equiv A(t) \exp\left[-(t\Delta E)^2/(2q)\right]; \qquad \tilde{A}^\pm(\pm i\tau) = \frac{1}{2\pi} \int \mathrm{d}t\, \tilde{A}(t) \frac{1}{\pm it + \tau}\,. \tag{3}$$

The time evolution of operators is defined by $A(t) = \exp[i\mathcal{H}_{1;\theta}t] A \exp[-i\mathcal{H}_{1;\theta}t]$, where we define the Hamiltonian $\mathcal{H}_{1;\theta} = \sum_i \mathcal{H}^i_{\theta,0}$, with the sum ranging over $-L/4+R < x_i < L/4-R$. The Hermitian conjugate in eq. (2) of $\tilde{w}^+_{1,\theta}(i\tau)$ is $\tilde{w}^-_{1,\theta}(-i\tau)$, and $W_1(\theta)$ is a unitary operator. We define $W_1 = W_1(2\pi)$.

To understand the definition (3), for any operator A define $A^+(i\tau)$ to be the positive energy part of A at imaginary time $i\tau$. That is, in a basis of eigenstates of the Hamiltonian with energies E_i, we have $A^+_{ij}(i\tau) = A_{ij} \exp[-\tau(E_i - E_j)]\Theta(E_i - E_j)$, with the step function $\Theta(x) = 1$ for $x > 0$, $\Theta(0) = 1/2$, and $\Theta(x) = 0$ for $x < 0$. The operator $\exp[-(t\Delta E)^2/(2q)]\tilde{A}^+(i\tau)$ is a good approximation to $A^+(i\tau)$ in the following sense: the difference $|P_{high} \exp[-(\tau\Delta E)^2/(2q)]\tilde{A}^+(i\tau)\Psi_0 - P_{high}A^+(i\tau)\Psi_0|$ is bounded by $\exp[-(\tau\Delta E)^2/(2q)]\times \exp[-q/2]||A||$ for $\tau < q/\Delta E$ and is bounded by $\exp[-\tau\Delta E]||A||$ for $\tau > q/\Delta E$. This result follows from elementary integrations [9] and is a kind of uncertainty relation: $\tilde{A}(t)$ is cutoff by the Gaussian at times of order $1/\Delta E$ and so can be used to approximate states with energies of order ΔE. In the limit $q \to \infty$, when the approximation $\tilde{A}^+(i\tau)$ becomes exact, eq. (2) adiabatically twists θ_1 by ϕ; we instead keep q finite as chosen later.

The use of $\mathcal{H}_{1;\theta}$ to define the time evolution, rather than $\mathcal{H}_{\theta,0}$ as in [7] is a very useful technical trick which simplifies the proof. Using $\mathcal{H}_{1;\theta}$, we have $[W_1, O] = 0$ for any operator O which acts only on a sites j with $L/4 \leq x_j \leq 3L/4$. That is, W_1 acts only on sites j with $-L/4 < x_j < L/4$.

We define $\Psi_n = W_1^n \Psi_0$ and define $E_n = \langle \Psi_n | \mathcal{H} | \Psi_n \rangle - \langle \mathcal{H} \rangle$, where the first and second pairs of angle brackets denote the inner product and ground-state expectation value, respectively. We prove below that

$$E_n \leq VJe_n/2, \tag{4}$$

where e_n is given by eq. (14) and

$$|\langle \Psi_n | T | \Psi_n \rangle - \langle T \rangle \exp[i2\pi n \rho(V/L)]| \leq e_n \,. \tag{5}$$

For large L, we will find that e_n is of order

$$e_n \sim (K/\Delta E)\big[N_c^2 \exp[-c_1 L \Delta E/8] + N_c^2 \sqrt{c_1 L \Delta E} \exp[-L/4\xi_C] + N_c \epsilon \sqrt{c_1 L \Delta E}\big], \tag{6}$$

where the constants c_1, ξ_C are defined below. Thus, for a d-dimensional system with $N_c \propto L^{d-1}$ and with fixed ΔE and vanishingly small ϵ, we have e_n, E_n going to zero exponentially in L.

One consequence of these results is a higher-dimensional Lieb-Schultz-Mattis theorem for systems of with $\rho(V/L)$ non-integer, a case which includes spin systems with half-integer spin per unit cell, total $S^z = 0$, and odd width V/L [7]. If we set ΔE to be equal to the gap between the ground state and the first excited state, then $\epsilon = 0$ as the ground state is the only state with energy less than ΔE. Then, if $\Delta E = k \log(L)/c_1 L$, the exponential $\exp[-c_1 L \Delta E/8]$ decays as $L^{-k/8}$, and for large enough k the state Ψ_1 will be less than ΔE in energy, while we can use the expectation value of T for state Ψ_1 to bound the overlap of Ψ_1 with the ground state, leading to a contradiction and thus bounding ΔE.

To show these results, we will need a few additional definitions defined in the next two sections. For use later, we introduce the operator

$$W_2(\phi) = \Theta \exp\left[-\int_0^{-\phi} \mathrm{d}\theta \int_0^{\infty} \mathrm{d}\tau \exp\big[-(\tau \Delta E)^2/(2q)\big] \big[\tilde{w}_{2,\theta}^{+}(i\tau) - \mathrm{H.c.}\big] \right], \tag{7}$$

where $w_{2,\theta} = \partial_\theta \mathcal{H}_{0,\theta}$. We write $W_2 = W_2(2\pi)$. Here, the time evolution of operators is defined using the Hamiltonian $\mathcal{H}_{2;\theta} \equiv \sum_i \mathcal{H}_{0,\theta}^i$, with the sum ranging over $L/4 + R < x_i < 3L/4 - R$. Finally, we define $W(\phi) = \Theta \exp[-\int_0^\phi \mathrm{d}\theta \int_0^\infty \mathrm{d}\tau \exp[-(\tau\Delta E)^2/(2q)][\tilde{w}_\theta^{+}(i\tau) - \mathrm{H.c.}]]$, where $w_\theta = \partial_\theta \mathcal{H}_{\theta,-\theta}$ and where the time evolution is defined using the Hamiltonian $\mathcal{H}_{\theta,-\theta}$.

Twist in boundary condition. – This and the next section show two basic facts about W, W_1, W_2. First, the difference $|W(\phi)\Psi_0 - R(\phi)\Psi_0|$ is exponentially small in q. Thus, W "twists" the boundary conditions, twisting θ_1 by ϕ and θ_2 by $-\phi$. Second, the difference $||W_1(\phi)W_2(\phi) - W(\phi)||$ is exponentially small in $(c_1 L \Delta E)^2/q$, and thus one can approximately factor the operator W into the product of two commuting operators W_1, W_2. Physically, the operator $W_1(\phi)$ twists θ_1 by ϕ while $W_2(\phi)$ twists θ_2 by $-\phi$. We will then be able to pick a q of order L such that both differences are exponentially small in L. Given these two results, eqs. (10), (12), it will be easy to derive the main results (4), (5) above. The reader who would prefer to skip the derivation of these two results may skip to the section *Bound on energy.*

We start by using linear perturbation theory to compute $\partial_\theta \Psi_0(\theta, -\theta) = \partial_\theta R(\theta)\Psi_0$. We have $\mathcal{H}_{\theta,-\theta} R(\theta)\Psi_0 = E_0 R(\theta)\Psi_0$. Taking derivatives of both sides with respect to θ at $\theta = 0$, and working in a basis of eigenstates Ψ_i of $\mathcal{H}$ with energies E_i, we can compute the matrix element $(\partial_\theta R(\theta))_{i0}$ if $E_i \neq E_0$: $(\partial_\theta R(\theta))_{i0} = -(E_i - E_0)^{-1}(\partial_\theta \mathcal{H}_{\theta,-\theta})_{i0}$. Thus,

$$\partial_\theta \Psi_0(\theta,-\theta) = -\sum_{i, E_i \neq E_0} \frac{(\partial_\theta \mathcal{H}_{\theta,-\theta})_{i0}}{E_i - E_0} \Psi_i + Z_{i0}\Psi_i = -\int_0^\infty \mathrm{d}\tau [(\partial_\theta \mathcal{H}_{\theta,-\theta})^{+}(i\tau) - \mathrm{H.c.}]\Psi_0 + Z\Psi_0 \,, \tag{8}$$

where we define the operator Z by $Z_{ij} = (\partial_\theta R(\theta))_{ij}$ if $E_i = E_j$ and $Z_{ij} = 0$ otherwise. The second equality in (8) may be verified by computing the integral over τ. In the limit $q \to \infty$, the operator $-\int_0^\infty \mathrm{d}\tau [(\partial_{\theta'} \mathcal{H}_{\theta',-\theta'})^{+}(i\tau) - \mathrm{H.c.}]$ equals the integrand in the exponent of W and so in that limit, for a non-degenerate ground state, $W(\phi)\Psi_0 = R(\phi)\Psi_0$.

We now consider the case of finite q and bound the difference $|W(\phi)\Psi_0 - R(\phi)\Psi_0|$. We use $\partial_\theta \mathcal{H}_{\theta,-\theta} = R(\theta)\partial_{\theta'}\mathcal{H}_{\theta',-\theta'}R(-\theta)$, taking the derivatives at $\theta' = 0$, to rewrite $W(\phi) = \Theta \exp[-\int_0^\phi d\theta \int_0^\infty d\tau \exp[-(\tau\Delta E)^2/(2q)]R(\theta)[\tilde{w}_0^+(i\tau) - \mathrm{H.c.}]R(-\theta)]$, where the time evolution of w_0 is defined using the Hamiltonian $\mathcal{H}_{0,0} = \mathcal{H}$. Thus, $W(\phi) = R(\phi)\exp[-\phi\int_0^\infty d\tau \times \exp[-(\tau\Delta E)^2/(2q)][\tilde{w}_0^+(i\tau) - \mathrm{H.c.}] - \phi\partial_\theta R(\theta)]$, taking the derivatives at $\theta = 0$.

Then, $|W(\phi)\Psi_0 - R(\Phi)\Psi_0| = |R(-\phi)W(\phi)\Psi_0 - \Psi_0|$ and

$$R(-\phi)W(\phi)\Psi_0 = \exp\left[-\phi\int_0^\infty d\tau\{\exp\left[-(\tau\Delta E)^2/(2q)\right]\tilde{w}_0^+(i\tau) - w_0^+(i\tau) - \mathrm{H.c.}\} - \phi Z\right]\Psi_0 . \quad (9)$$

We now bound the norm of the state $\Psi \equiv (-\int_0^\infty d\tau\{\exp[-(\tau\Delta E)^2/(2q)]\tilde{w}_0^+(i\tau) - w_0^+(i\tau) - \mathrm{H.c.}\} - Z)\Psi_0$. The norm $|\Psi| \le |P_{high}\Psi| + |P_{low}\Psi|$. We have $P_{high}Z\Psi_0 = 0$, while using the bounds below eq. (3) $|P_{high}(-\int_0^\infty d\tau\{\exp[-(\tau\Delta E)^2/(2q)]\tilde{w}_0^+(i\tau) - w_0^+(i\tau) - \mathrm{H.c.}\})\Psi_0| \le N_c(K/\Delta E)(2\exp[-q] + \exp[-q/2]\sqrt{2\pi q})$.

On the other hand, $P_{low}\Psi = P_{low}[-\int_0^\infty d\tau \exp[-(\tau\Delta E)^2/(2q)]\tilde{w}_0^+(i\tau) - \mathrm{H.c.} - \partial_\theta R(\theta)]\Psi_0$. The expectation values $\langle \partial_\theta R(\theta)\rangle$ and $\langle \partial_\theta \mathcal{H}_{\theta,-\theta}\rangle$ both vanish, due to the translation symmetry of the ground state, so we may assume that $P_{low}\Psi$ is orthogonal to Ψ_0. Then, we can use the bound on the matrix elements between Ψ_0 and other states below the gap to show that $|P_{low}\Psi| \le \epsilon\sqrt{2\pi q}N_c(K/\Delta E) + \epsilon||\partial_{\theta'}R(\theta')||$. Therefore, $|\Psi| \le N_c(K/\Delta E)[2\exp[-q] + \exp[-q/2]\sqrt{2\pi q}] + \epsilon\sqrt{2\pi q}N_c(K/\Delta E) + \epsilon||\partial_{\theta'}R(\theta')||$.

Therefore, $|W(\phi)\Psi_0 - R(\phi)\Psi_0| \le c_2(\phi)$, where we define

$$\begin{aligned} c_2(\phi) \;=\; & \phi\big\{N_c(K/\Delta E)\big[2\exp[-q] + \\ & + \exp[-q/2]\sqrt{2\pi q}\big] + \epsilon\sqrt{2\pi q}N_c(K/\Delta E) + \epsilon||\partial_{\theta'}R(\theta')||\big\}. \end{aligned} \quad (10)$$

Locality bounds. – The next result we need is a bound on $||W_1(\phi)W_2(\phi) - W(\phi)||$. Note that $W_1(\phi)W_2(\phi) = \Theta\exp[-\int_0^\phi d\theta\int_0^\infty d\tau\exp[-(\tau\Delta E)^2/(2q)][\tilde{w}_\theta^+(i\tau) - \mathrm{H.c.}]]$, where $w_\theta = \partial_\theta\mathcal{H}_{\theta,-\theta}$ and where the time evolution is defined using the Hamiltonian $\mathcal{H}_{1;\theta,} + \mathcal{H}_{2;-\theta}$. This differs from the definition of W in only one respect: the use of $\mathcal{H}_{1;\theta,} + \mathcal{H}_{2;-\theta}$ to define the time evolution rather than $\mathcal{H}_{\theta,-\theta}$.

We next recall the finite group velocity result, proven in [7, 8], that there exists a function $g(t,l)$, which depends on J, R, and the lattice structure, such that $||[A(t), B(0)]|| \le ||A||||B||\sum_j g(t,l_j)$, where the sum ranges over sites j which appear in operator B, and $l_j = d(j,i)$ is the distance from j to the closest site i in the operator A. It was shown that there exists some constant c_1 such that for $|t| \le c_1 l$, $g(c_1 l, l)$ is exponentially decaying in l for large l with correlation length ξ_C.

Now, we pick $B = \mathcal{H}_{\theta,-\theta} - \mathcal{H}_{1;\theta} - \mathcal{H}_{2;-\theta}$, with $||B|| \le 2N_cJ$. The operator B can be written as a sum of $\mathcal{H}^i$, with each such $\mathcal{H}^i$ having at most S sites. Then, we can bound the difference between the two different definitions of $\tilde{w}_\theta^+(i\tau)$ using the two different definitions of time evolution. We have

$$\begin{aligned} (2\pi)^{-1}\Big|\int dt(it+\tau)^{-1}\exp\big[-(t\Delta E)^2/(2q)\big]\{&\exp[i\mathcal{H}_{\theta,-\theta}t]w_\theta\exp[-i\mathcal{H}_{\theta,-\theta}t] - \\ &- \exp[i(\mathcal{H}_{\theta,-\theta} - B)t]w_\theta\exp[-i(\mathcal{H}_{\theta,-\theta} - B)t]\}\Big| \le \\ \le (2\pi)^{-1}N_c||w_\theta||\int dt(it+\tau)^{-1}\exp\big[-(t\Delta E)^2/(2q)\big]&\int_0^t dt' 2JSg(t', L/4 - 2R). \end{aligned} \quad (11)$$

The integral $\int_0^t dt' 2JSg(t', L/4-2R)$ is bounded by $g(t, L/4)$ [7]. Thus, we find that the difference using the two different definitions of time evolution is bounded by $(2\pi)^{-1}N_c||w_\theta||\int dt(it+$

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$\tau)^{-1}\exp[-(t\Delta E)^2/(2q)]g(t,L/4)$. To bound this integral, for $t < c_1L/4$, we can use the locality bound, while for $t > c_1L/4$, we can use the Gaussian in t. The result gives a bound $(2\pi)^{-1}2N_c||w_\theta||\{g(c_1L/4,L/4)+2[\sqrt{2\pi q}/(\Delta E c_1 L/4)]\exp[-(c_1L\Delta E/4)^2/(2q)]\}$. Integrating over τ, and using $||w_\theta|| \leq N_cK$, we find $||W_1(\phi)W_2(\phi)-W(\phi)|| \leq c_3(\phi)$, where we define

$$\begin{aligned} c_3(\phi) &= \sqrt{2\pi q}(K/\Delta E)(\phi/2\pi)2N_c^2\times \\ &\times\Big\{g(c_1L/4,L/4)+2\frac{\sqrt{2\pi q}}{\Delta E c_1 L/4}\exp\big[-(c_1L\Delta E/4)^2/(2q)\big]\Big\}. \end{aligned} \tag{12}$$

Bound on energy. – We compute $\sum_i\langle W_1^{-n}\mathcal{H}^iW_1^n-\mathcal{H}^i\rangle$. Note that $W_1(-\theta)=W_1^\dagger(\theta)$. First we note that for sites i with $L/4\leq x_i\leq 3L/4$, $\langle W_1^{-n}\mathcal{H}^iW_1^n-\mathcal{H}^i\rangle=0$, since W_1 commutes with H^i for such sites.

We now consider the change in expectation value of the energy for sites i with $-L/4\leq x_i<L/4$. For such sites, $|\langle W_1^{-n}\mathcal{H}^iW_1^n\rangle=\langle W_2^{-n}W_1^{-n}\mathcal{H}^iW_1^nW_2^n\rangle|$. Thus, we can bound $|\langle W_1^{-n}\mathcal{H}^iW_1^n\rangle-\langle\mathcal{H}^i\rangle|\leq 2J|W_1^nW_2^n\Psi_0-\Psi_0|$. Then, we can combine eqs. (10), (12) with the fact that $R(2\pi n)\Psi_0=\Psi_0$ to bound $|W_1^nW_2^n\Psi_0-\Psi_0|\leq c_2+c_3$. Summing over all $V/2$ such i, we find

$$\sum_i|\langle W_1^{-n}\mathcal{H}^iW_1^n\rangle-\langle\mathcal{H}^i\rangle|=E_n\leq VJe_n/2, \tag{13}$$

where

$$e_n=2[c_2(2\pi n)+c_3(2\pi n)]. \tag{14}$$

Finally, we pick q to be equal to $c_1(L/2)\Delta E$ to obtain the lowest possible E_n, giving the exponential decay of e_n with L claimed in the results.

We have thus bounded the difference in energies. The same argument would also work to bound the difference $\langle W_1(-\phi)\mathcal{H}_{\phi,0}W_1(\phi)\rangle-\langle\mathcal{H}\rangle$ for general ϕ.

Expectation of translation operator. – We next consider $\langle\Psi_n|T|\Psi_n\rangle$. We define a twisted translation operator, $T_{\theta,\theta'}=R_1(\theta_1)R_2(\theta_2)T$, where $R_1(\theta_1)=\prod_{j,x_j=1}\exp[i\theta_1(Q_j-\rho)]$ and $R_2(\theta_2)=\prod_{j,x_j=L/2+1}\exp[i\theta_2(Q_j-\rho)]$. Then, T_{θ_1,θ_2} is a symmetry of $\mathcal{H}_{\theta_1,\theta_2}$ and $T_{\theta,-\theta}=R(\theta)TR(-\theta)$, so that $T_{\theta,-\theta}\Psi_0(\theta,-\theta)=z_0\Psi_0(\theta,-\theta)$. Since Q_j is quantized, we have $T_{2\pi n,0}=\exp[-i2\pi n\rho(V/L)]T$.

In this section, we will prove a bound on the difference $|\langle W_1(-\phi)T_{\phi,0}W_1(\phi)\rangle z_0^{-1}-1|$. Taking $\phi=2\pi n$, this will give a bound on the difference $|\langle\Psi_n|T|\Psi_n\rangle-z_0\exp[i2\pi n\rho(V/L)]|$ as desired. We have $\langle W_1(-\phi)T_{\phi,0}W_1(\phi)\rangle z_0^{-1}=\langle W_1(-\phi)R_1(\phi)[TW_1(\phi)T^{-1}]\rangle$. Physically, this expectation value is close to unity for the following reason: the operator W_1 inserts a twist between $x=0$ and $x=1$, while $TW_1(\phi)T^{-1}$ inserts the twist between $x=1$ and $x=2$. The difference is the rotation of the sites with $x=1$. Thus, $R_1(\phi)[TW_1(\phi)T^{-1}]$ is very close to the operator $W_1(\phi)$. We now make this argument precise.

Note that $\langle W_1(-\phi)R_1(\phi)[TW_1(\phi)T^{-1}]\rangle=\langle W_2(-\phi)W_1(-\phi)R_1(\phi)[TW_1(\phi)T^{-1}]W_2(\phi)\rangle$. Using previous results, we know that $|\Psi_0^\dagger W_2(-\phi)W_1(-\phi)-\Psi_0^\dagger R(-\phi)|\leq c_2(\phi)+c_3(\phi)$. Now, consider $[TW_1(\phi)T^{-1}]W_2(\phi)\Psi_0$. Let us define a new twisted Hamiltonian, $\mathcal{H}'_{\theta_1,\theta_2}$, in which the twist in boundary conditions is by angle θ_1 between $x=1$ and $x=2$ and by angle θ_2 between $x=L/2$ and $x=L/2+1$. Specifically, $\mathcal{H}'_{\theta_1,\theta_2}=R_1(-\theta_1)\mathcal{H}'_{\theta_1,\theta_2}R_1(\theta_1)$. Define $W_1'(\phi)=\Theta\exp[-\int_0^\phi\mathrm{d}\theta\int_0^\infty\mathrm{d}\tau\exp[-(\tau\Delta E)^2/(2q)][\tilde{w}'^{+}_{1,\theta}(i\tau)-\mathrm{H.c.}]]$, where $w'_{1,\theta}=\partial_\theta\mathcal{H}'_{\theta,0}$ and where the time evolution is defined using the Hamiltonian $\mathcal{H}'_{1;\theta}\equiv T\mathcal{H}_{1;\theta}T^{-1}$. Crucially, $W_1'(\phi)=TW_1(\phi)T^{-1}$. Similarly, define W' in analogy to the definition of W, using $w'_\theta=\partial_\theta\mathcal{H}'_{\theta,-\theta}$.

Then, following the same steps as above, we can show that $|W'(\phi)\Psi_0 - R_1(-\phi)R(\phi)\Psi_0| \leq c_2(\phi)$ and $||W_1'(\phi)W_2(\phi) - W'(\phi)|| \leq c_3(\phi)$ so that $|[TW_1(\phi)T^{-1}]W_2(\phi)\Psi_0 - R_1(-\phi)R(\phi)\Psi_0| \leq c_2(\phi) + c_3(\phi)$. Thus, $|\langle W_2(-\phi)W_1(-\phi)R_1(\phi)[TW_1(\phi)T^{-1}]W_2(\phi)\rangle - 1| \leq 2[c_2(\phi) + c_3(\phi)]$, and thus $|\langle\Psi_n|T|\Psi_n\rangle - z_0 \exp[i2\pi n\rho(V/L)]| \leq e_n$ as desired. Note that in this derivation, the vanishing of the phase factors $\langle\partial_\theta'[R_1(-\theta')R(\theta')]\rangle$ and $Z_{00} = \langle\partial_{\theta'}R(\theta')\rangle$ is crucial; we chose the factor of $\prod_{j,x_j=1}\exp[-i\theta\rho]$ in the definition of R_1 to make this phase factor vanish and it is this factor that led to the expectation value of T for state Ψ_n.

Discussion. – The case of general filling fraction has been previously considered in one dimension, and used to study magnetization plateaus [10]. It was shown that gapped states could only exist at integer filling fraction, while at filling fraction $\rho = p/q$ with p, q coprime there were found to be at least q low-energy states. We have shown very similar behavior in higher dimensions for $\rho(V/L) = p/q$: if ΔE remains non-vanishing and $\epsilon \to 0$ sufficiently rapidly as $L \to \infty$, then we find that the states Ψ_n, for $n = 0, \ldots, q-1$ provide q degenerate low-energy states. The nature of these q states is not known *a priori*. They may correspond to either discrete symmetry breaking or to topological order. If *all* local operators, not just the specific ones considered in the proof above, have matrix elements between the low-lying states which vanish as $L \to \infty$, then we identify this as topological order. If some local operator has non-vanishing matrix elements between the local states, then there is long-range order in that operator [9], and we identify this as symmetry breaking. It is a limitation of the proof that the ability to construct low-energy states depends on the width V/L of the system. We conjecture that if $\rho = p/q$ then one can still find q low-lying states for arbitrary, sufficiently large, V/L, but we are not able to prove this.

The technique shown in this paper is very general, and can be used in any case in which one can identify a quantized conserved charge, including cases of spins transforming under higher $SU(N)$ groups. Comparing to [11], the technique here creates a "global vison" excitation; if instead the flux is inserted along a finite line with endpoints, this technique can create a local "vison" excitation in two-dimensional spin systems as will be discussed elsewhere.

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