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QUARK CONFINEMENT AND TOPOLOGY OF GAUGE THEORIES

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The phenomenon of quark confinement is known to be connected with the restoration of apparently broken gauge symmetry. In this paper we focus on a special mechanism which is responsible for such restoration. The major suggestion is that in the treatment of infrared problems certain classical field trajectories are of paramount importance, which trajectories connect seemingly degenerate vacua thereby eliminating the degeneracy. Initially, we demonstrate this mechanism in certain simple non-gauge models. In order of increasing difficulty, we next examine compact quantum electro-dynamics in 2 + 1 space-time dimensions – essentially a variant of unified models of the Georgi-Glashow type. In this case we prove that charge is confined and show that the force between two charges is independent of the distance between them. For small values of the fine structure constant this force is explicitly evaluated. Finally, we turn to the more realistic case of a 3 + 1 dimensional non-Abelian gauge theory, and analyse the contribution of a single pseudoparticle to the correlation functions. It is proposed that the quantum fluctuations of the pseudoparticle are inessential, and that the one-loop approximation is effectively correct even for the large scale pseudoparticles. The emergent conclusion then is that the renormalized Yang-Mills theory is reduced to the problem of evaluating the effects of those configurations which involve many pseudoparticles, due account being taken of the interactions between them. Some aspects of this last problem are also discussed.

1. Introduction

There is no question that the problem of confinement is one of the most important in present day particle physics. The most plausible candidates for a theory with confinement are commonly believed to be gauge theories of the Yang-Mills type. This opinion is based on the observation [1,2] that in the framework of perturbation theory the effective interaction between two non-singlet particles increases with spatial separation. Extrapolating this increase outside perturbation theory, one presumably gets long-range force between non-singlet (or colored) states.

Recently it has been suggested that this perturbation-theoretical increase may have nothing to do with confinement [3]. The real deep reason for confinement was proposed to be the topological structure of the gauge group. This structure determines whether the gauge symmetry is spontaneously broken and the gauge bosons are Goldstone particles or this symmetry is restored, that is, the vacuum is non-degenerate. In the last case, as has been shown by Wilson [4] (see also ref. [5]) we will have confinement of color in our theory. This statement may be heuristically explained as follows. Gauge invariance with constant phase $\psi \to e^{i\alpha}\psi$ leads to conservation of the total charge. Gauge transformations with a varying phase

$$\psi(x) \to e^{i\alpha(x)}\,\psi(x)$$

will give us conservation of the charge density. But this in turn means that the charged particle cannot move. The only thing which saves the electron from this fatal immobility is the degeneracy of the vacuum in QED, that is, its non-invariance under gauge transformations.

What kind of vacuum shall we have in QED? Is the gauge symmetry broken or not? These are the questions we are going to investigate in this paper. Our main idea is that in cases with superficially broken symmetry there may exist certain classical trajectories which repair the symmetry. The simplest example of this is two symmetrical wells separated by a barrier in quantum mechanics. In this case, if we neglect the small penetration effect we obtain a degenerate vacuum and broken symmetry. Taking into account trajectories connecting the two wells removes this degeneracy. These trajectories will be called pseudoparticles or instantons *.

We shall begin this paper with the discussion of pseudoparticles in several simple models such as quantum mechanics, the Ising model, the xy-model and so on (sect. 2). Then we investigate a gauge theory of the Higgs type in which, superficially, there are two massive vector bosons and one massless photon. In this case a complete investigation of pseudoparticles is possible. After summing over all configurations of pseudoparticles we get confinement in the 2 + 1 dimensional case, and vacuum degeneracy in the 3 + 1 case. The difference between these cases is connected with topology. We would like to stress that the confinement in 2 + 1 dimensions is a highly non-trivial result, since we obtain the effective potential $V(R)$ between two non-singlet states $V(R) \propto R$, and the trivial Coulomb interaction would give us $V(R) \propto \log R$. Our result means that the electric field forms a string connecting two charges. The string is stabilized by the pseudoparticle background. We shall calculate the energy density of the string in this model for small charges, below. In principle, it is also possible to find the resonance spectrum, but this has not yet been done.

In sect. 6 we start examining the pure Yang-Mills field in four dimensions. In this case topology permits the pseudoparticles discovered earlier [6]. We examine the one-particle contribution in the effective potential, discuss the so-called zero-mode problem, speculate about many-particle contributions and the existence or non-existence of the Gell-Mann-Low renormalization group. It is also argued that the quasi-classical approximation might have been exact for renormalized Yang-Mills theory.

* The latter name was proposed by G. 't Hooft, private communication through A. Migdal.

2. Pseudoparticle mechanics. Simple examples with non-gauge symmetries

Let us consider the quantum mechanical problem of a double well potential. The Lagrangian is given by

$$\mathcal{L} = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\mu^2 x^2 - \tfrac{1}{4}\lambda x^4 , \qquad \lambda \ll \mu^2 . \tag{2.1}$$

From a superficial point of view there are two degenerate vacua, corresponding to oscillations of the particle near

$$x_\pm = \pm (\mu^2/\lambda)^{1/2} .$$

If we use perturbation theory,

$$x = -(\mu^2/\lambda)^{1/2} + y , \tag{2.2}$$

and expand in y, then in every finite order the vacuum degeneracy will be preserved. This may be easily checked by noticing that

$$\eta = -\sqrt{\mu^2/\lambda} + \langle y \rangle \neq 0 \tag{2.3a}$$

in every finite order, and that

$$\langle x(0)\, x(t) \rangle \underset{t\to\infty}{\longrightarrow} \eta^2 . \tag{2.3b}$$

Now let us take into account the effect of tunnelling from one well to another, which is non-analytic in the coupling constant. It is convenient to examine correlation functions for imaginary times since

$$\langle x(0)\, x(\tau) \rangle = \sum_n |x_{n0}|^2\, \mathrm{e}^{-(E_n - E_0)\tau} \underset{\tau\to\infty}{\propto} \mathrm{e}^{-\Delta E\cdot\tau} , \qquad \tau = it , \tag{2.4}$$

and ΔE is the energy splitting the two vacua. Another, more heuristic, explanation of the introduction of imaginary time is that we are analyzing barrier penetration of the particle and under barrier the particle lives in imaginary time.

The correlation function is given by the functional integral:

$$\langle x(0)\, x(\tau) \rangle = \frac{\int \mathcal{D}x(\tau)\, \mathrm{e}^{-\mathcal{E}[x]}\, x(0)\, x(\tau)}{\int \mathcal{D}x(\tau)\, \mathrm{e}^{-\mathcal{E}[x]}} ,$$

$$\mathcal{E}[x] = \int_{-\infty}^{+\infty} \mathrm{d}\tau \left\{ \frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 - \tfrac{1}{2}\mu^2 x^2 + \tfrac{1}{4}\lambda x^4 \right\} . \tag{2.5}$$

The quantity $\mathcal{E}$, which is the action for imaginary time, will be called the pseudo-energy. It is obvious from (2.5) that the quantum mechanics of single particles may be considered as the classical statistical mechanics of the one-dimensional string. This fact was first noticed and exploited in the paper by Vaks and Larkin [7].

For $\lambda \ll \mu^2$ one may use a saddle point method for calculation of (2.5). The first, trivial, minimum of $\mathcal{E}$ is

$$x(\tau) = \pm\sqrt{\mu^2/\lambda}$$

and we get correspondingly:

$$\langle x(0)\, x(\tau)\rangle \approx \frac{\mu^2}{\lambda}\left(1 + \mathrm{O}\left(\frac{\lambda}{\mu^3}\right)\right) . \tag{2.6}$$

Secondly, there is another minimum:

$$x_\mathrm{c}(\tau) = \left(\frac{\mu^2}{\lambda}\right)^{1/2} \tanh\frac{\mu(\tau - a)}{\sqrt{2}} . \tag{2.7}$$

For this minimum it is known (see e.g. ref. [8]) that

$$\mathcal{E}[x_\mathrm{c}(\tau)] - \mathcal{E}[\sqrt{\mu^2/\lambda}] = \frac{2\sqrt{2}\mu^3}{\lambda}, \tag{2.8}$$

so at the first sight its contribution will be proportional to $\exp - \{2\sqrt{2}\mu^3/\lambda\}$ and hence will be small. We will show now that this is not the case – in fact the small exponent is multiplied by τ and for large τ is absolutely essential.

In order to perform the functional integration near extremum (2.7) it is necessary to take into account the fact that there exists an infinite number of equivalent extrema with different values of a. Because of this it is not possible to write simply

$$x(\tau) = x_\mathrm{c}(\tau) + \sum \xi_n y_n(\tau) , \tag{2.9}$$

where the y_n satisfy the equation

$$\frac{\delta^2}{\delta x_\mathrm{c}(\tau)\, \delta x_\mathrm{c}(\tau')}\, y_n(\tau')\, \mathrm{d}\tau' = \omega_n^2 y_n(\tau) . \tag{2.10}$$

The problem is that due to time translation symmetry, eq. (2.10) has a zero eigenvalue with eigenfunction

$$y_0(\tau) = \frac{\mathrm{d}}{\mathrm{d}a} x_\mathrm{c}(\tau, a) . \tag{2.11}$$

Because of this fact the oscillations of the corresponding ξ_0 are not harmonic ones. In order to avoid this complication let us introduce a new set of normal coordinates *

$$x(\tau) = x_\mathrm{c}(\tau, a) + \sum_{h \neq 0} \xi_n y_n(\tau - a) . \tag{2.12}$$

* This prescription was first used by Bogolubov and Tyablikov [8] in the polaron problem.

Now we use instead of ξ_0 the coordinate a. It is easy to find the integration measure. The simplest way is to use the method of Faddeev and Popov [9].

Consider the equality

$$\Delta[x] \int_{-\infty}^{+\infty} \mathrm{d}a\, \delta(\; x(\tau + a) y_0(\tau)\, \mathrm{d}\tau) = 1 \,,$$

$$y_0 = \frac{\partial}{\partial a} x_c = \dot{x}_c \,. \tag{2.13}$$

Substituting (2.12) into (2.13) we get

$$\Delta[x] = A^{1/2} + \sum_{n \neq 0} r_n \xi_n \,,$$

$$A = \int_{-\infty}^{+\infty} \mathrm{d}\tau\, \dot{x}_c^2(\tau) \,,$$

$$r_n = \int \mathrm{d}\tau\, \dot{x}_c(\tau) \dot{y}_n(\tau) \,. \tag{2.14}$$

Substituting (2.13) into the functional integral and making a time translation we obtain the expression for the integration measure:

$$\mathcal{D}x(\tau) = \prod_{n \neq 0} \mathrm{d}\xi_n\, \mathrm{d}a (A + \sum r_n \xi_n) \,. \tag{2.15}$$

Of course, for small oscillation (for $\lambda \ll \mu^3$) we may neglect the last term in (2.15), and we get simply

$$\mathcal{D}x(\tau) \approx A \prod_{n \neq 0} \mathrm{d}\xi_n\, \mathrm{d}a \,,$$

$$A = \int \dot{x}_c^2(\tau)\, \mathrm{d}\tau \,. \tag{2.16}$$

Now it is possible to calculate the contribution of one kink to correlation function:

$$\langle x(0) x(\tau) \rangle \approx \frac{\mu^2/\lambda + B \mathrm{e}^{-\mathcal{E}_c} \int \mathrm{d}a\, x_c(0, a)\, x_c(\tau, a)}{1 + B \mathrm{e}^{-\mathcal{E}_c} \int \mathrm{d}a}$$

$$\approx \frac{\mu^2}{\lambda} + B \mathrm{e}^{-\mathcal{E}_c} \int_{-\infty}^{+\infty} \mathrm{d}a (x_c(0, a)\, x_c(\tau, a) - \mu^2/\lambda) \,, \tag{2.17}$$

where

$$B = A\left(\int \prod_n \mathrm{d}\xi_n \mathrm{e}^{-\omega_n \xi_n^2}\right)\left(\int \prod \mathrm{d}\xi_n \mathrm{e}^{-\omega_n^0 \xi_n^2}\right),$$

$$\mathcal{E}_\mathrm{c} = \frac{2\sqrt{2}\mu^3}{\lambda}. \tag{2.18}$$

Here ω_h^0 are the eigenfrequencies for the first, trivial minimum $x_\mathrm{c}^2 = \mu^2/\lambda$. The sum in (2.18) may be expressed through the scattering phase shifts in the kink potential, but it will not be necessary for our purposes. Substituting (2.7) into (2.17) we get

$$\langle x(0)\, x(\tau)\rangle \approx \frac{\mu^2}{\lambda}\,[1 - B\mathrm{e}^{-\mathcal{E}_\mathrm{c}}$$

$$\times \int \mathrm{d}a \left(1 - \mathrm{th}\frac{\mu a}{\sqrt{2}}\, \mathrm{th}\frac{\mu(a-\tau)}{\sqrt{2}}\right) = \frac{\mu^2}{\lambda}[1 - C\mathrm{e}^{-2\sqrt{2}\mu^3/\lambda}\, \tau]\ ,$$

$$C = B \int_{-\infty}^{+\infty} \mathrm{d}x \left(\mathrm{th}\frac{\mu x}{\sqrt{2}}\, \mathrm{th}\frac{\mu(x-1)}{\sqrt{2}} - 1\right). \tag{2.19}$$

So, we have proved the important fact that the one-kink contribution increases with time. It is evident that for $\tau \sim \exp(2\sqrt{2}\mu^3\lambda^{-1})$ many kinks become important. In order to perform the complete calculation let us assume that for very large τ, far separated kinks are essential. This will be shown to be selfconsistent at the end of the calculations. In this case many kinks are also extreme of $\mathcal{E}$ since the kink-kink interaction is proportional to $\mathrm{e}^{-\mu R}$ (where R is distance between kinks). Now, for large times the kink solution has the form

$$x_\mathrm{c}(\tau) \approx \left(\frac{\mu^2}{\lambda}\right)^{1/2} \prod_{j=1}^{N} \mathrm{sgn}(\tau - a_j). \tag{2.20}$$

(We approximated the tanh function by the step function. The accuracy of this will be again $\sim \mathrm{e}^{-\mu R}$.)

$$\mathcal{E}_\mathrm{c} \approx \frac{2\sqrt{2}\mu^3}{\lambda} N. \tag{2.21}$$

Finally we get

$$\langle x(0) x(\tau)\rangle = \frac{\mu^2}{\lambda}\Big(\prod_{N=0}^{\infty} C^N \mathrm{e}^{-2\sqrt{2}\mu^3 N/\lambda}$$

$$\times \int_{a_1<a_2<\ldots<a_N} \mathrm{d}a_1 \ldots \mathrm{d}a_N \prod_j \mathrm{sgn}(\tau - a_j)\Big)$$

$$\times \left(\sum_{N=0}^{\infty} C^N e^{-2\sqrt{2}\mu^3 N/\lambda} \int\limits_{a_1<a_2<\ldots a_N} da_1 \ldots da_N \right)^{-1} = \frac{\mu^2}{\lambda} e^{-\Delta E\cdot\tau} , \tag{2.22}$$

$$\Delta E = C e^{-2\sqrt{2}\mu^3/\lambda} . \tag{2.23}$$

We have obtained the formulae for the level splitting, which of course could have been derived much more simply by use of wave functions in the WKB approximation. Our derivation is important nevertheless because in the case of infinite number of degrees of freedom the only known way to do WKB is by the functional integral method.

In the course of our derivation we learned that repair of broken symmetry ($x \to -x$ symmetry in our case) is produced by special classical trajectories which have a finite action for imaginary time or finite pseudoenergy. These trajectories are called pseudoparticles. The contribution of pseudoparticles for large time scales are also large. The meaning of symmetry restoration due to pseudoparticles is most easily understood in terms of statistics of the one-dimensional string, lying in the double-well potential. Most of the time it lies, say in the left well. However, it may, with small probability, form a number of kinks. The density of kinks per unit length is given by the Boltzmann formula:

$$n \propto e^{-\mathcal{E}_c} = e^{-2\sqrt{2}\mu^3/\lambda} . \tag{2.24}$$

This randomly distributed kink spoils the correlation and creates correlation length R_c of the order

$$R_c = \frac{1}{\Delta E} \sim n^{-1} . \tag{2.25}$$

We discussed this picture since analogous situations exist in many systems containing pseudoparticles.

Now let us analyze less trivial examples of pseudoparticles. In the one-dimensional Ising model at zero temperature there is an infinite correlation length – all spins point in one direction. For arbitrary small but finite temperature we have a finite correlation length due to configurations, shown in fig. 1. The role of a pseudoparticle is played by this configuration of the spins. It is in one to one correspondence with the one-dimensional string model, or with the quantum mechanics of a particle. This analogy was previously noticed in ref. [7].

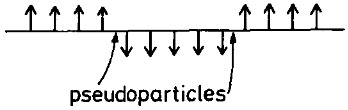


Fig. 1. One-dimensional Ising model at low temperatures.

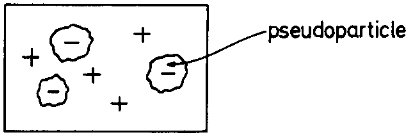


Fig. 2. Two-dimensional Ising model at low temperatures.

In the case of the two-dimensional Ising model the situation is different. At low temperatures there are rare random regions of the reversed spins fig. 2. However, since these random fields are short ranged they do not spoil the correlation and there is long range order. A less trivial example is given by the spin models with continuous symmetries. In the case of the group O(2) such models have been analyzed by Berezinsky [10]. We will give here a short review of his results using our methods in order to stress the influence of pseudoparticles on the long range correlations.

Let us consider a two-dimensional system with partition function

$$Z = \int_{-\pi}^{\pi} \prod_{\bar{x}} \mathrm{d}\varphi_{\bar{x}} \, \exp \beta \prod_{\bar{x},\mu} \cos(\varphi_{\bar{x}} - \varphi_{\bar{x}+\bar{\mu}}) \,, \qquad (\beta \gg 1) \,. \tag{2.26}$$

Here $\boldsymbol{\mu}$ is a lattice vector $\mu = 1, 2$. It was shown in ref. [7] that this model describes not only the low temperature behaviour of planar spin systems but also the properties of planar liquid helium. For very large β the cosine in (2.26) may be approximated by

$$\beta \sum \cos(\varphi_{\bar{x}} - \varphi_{\bar{x}+\bar{\mu}}) \approx \tfrac{1}{2}\beta \int (\nabla\varphi(x))^2 \, \mathrm{d}^2\boldsymbol{x} \,. \tag{2.27}$$

But one must remember in the process of averaging over $\varphi(\boldsymbol{x})$ that $\varphi(\boldsymbol{x})$ is an angular variable. This means that φ need not be continuous; it may have jumps equal to 2π on some lines. Hence the pseudoenergy (2.27) possesses non-trivial extrema, which satisfy the equation:

$$\nabla^2\varphi = 0 \qquad (\mathrm{mod}\ 2\pi) \,. \tag{2.28}$$

The general solution of (2.28) has the form

$$\varphi = \sum_a q_a \, \mathrm{Im} \log(z - z_a) \,, \tag{2.29}$$

where $\{q_a\}$ are integers, $z = x_1 + ix_2$ and $\{z_a\}$ are arbitrary.

This solution corresponds to a number of vertices, fixed at points $\{z_a\}$ with strength q_a. Each of these vertices produces a large range effect according to (2.29). One might think that this effect would spoil the correlation. However that is not the case. The reason is that the pseudoenergy is

$$\mathcal{E} = \tfrac{1}{2}\beta \int (\nabla\varphi)^2 \, \mathrm{d}^2x = \tfrac{1}{2}\beta \sum q_a q_b \log \frac{R}{|z_{ab}|} \,,$$

where R is the size of the system. Hence the only configurations which survive are those with $\Sigma q_a = 0$.

As was shown by Berezinsky, vertices tend to form neutral objects. These neutral objects (dipoles) produce short range random fields and do not influence the correlations. Hence in this case just as in the two-dimensional Ising model, pseudoparticles are not effective.

The case of the non-Abelian symmetry group is more complicated. As an example of such a system one may consider the partition function

$$Z = \int \prod_{x} \mathrm{d}\boldsymbol{n}_x \exp \beta \sum_{\bar{x}\mu} \bar{n}_x \bar{n}_{x+\boldsymbol{\mu}} , \tag{2.30}$$

where $\boldsymbol{n}_x^2 = 1$.

For $\beta \to \infty$ one may again write down an approximate expression:

$$\mathcal{E} = \tfrac{1}{2}\beta \int (\nabla \boldsymbol{n}(x))^2 \, \mathrm{d}^2\boldsymbol{x} . \tag{2.31}$$

This is the action for the non-linear σ-model. The minima of (2.31) were found by Belavin and the author [11]. They have the form

$$\mathrm{tg}\frac{\theta}{2} \mathrm{e}^{i\varphi} = \sum_{n=1}^{q} \frac{\lambda_n}{z - z_n} ,$$

$$\mathcal{E} = 4\pi\beta q \tag{2.32}$$

(λ_n are arbitrary complex numbers).

Other conjugate, pseudoparticles are obtained by the change $z \to z^*$. Though pseudoparticles do not interact with each other it was pointed out by Khoklachev that there may exist some interaction between particles and antiparticles. This interaction law has not been investigated yet.

The important feature which differentiates this non-Abelian system from the others considered above and which make it very close to the Yang-Mills system is that in this model fluctuations near classical trajectories may become large. It was shown by the author [12] that in a non-Abelian planar spin system asymptotic freedom exists. The effective temperature on a length λ scale λ is connected with β by the formula [12]

$$\beta(\lambda) = \beta - \frac{N-2}{2\pi} \log \lambda \tag{2.33}$$

for the O(N) chiral field. Hence, sufficiently large pseudoparticles are live at high temperatures, and fluctuations to their mean field are essential. A very similar puzzle exists in the case of Yang-Mills fields. We shall discuss it at the end of this paper.

3. Compact quantum electrodynamics

It was pointed out by the author [3] that the global structure of the gauge group is of highest importance in discussing infrared properties of the theory. It was shown that two different formulations of QED are possible, compact and non-compact, with identical perturbation series but with completely different physical properties. In order to formulate the idea of compactness of the gauge group in paper [3] the lattice formulation of QED was used at an intermediate stage. There exist, however, much more familiar examples of compact QED, which we shall discuss now.

Our main idea is that the Abelian gauge group of QED will always be compact if it is the subgroup of some simple non-Abelian gauge group such as SU(2). The reason is that the SU(2) group itself is necessarily compact (it is a three-dimensional sphere) and the Abelian subgroup corresponds to some circle on this sphere.

The simplest model is QED in the Georgy-Glashow theory of electromagnetic and weak interactions. The action is given by

$$S = \int \mathrm{d}^d x \left[\frac{1}{4e^2} \boldsymbol{F}_{\mu\nu}^2 + (\nabla_\mu \boldsymbol{\varphi})^2 + \tfrac{1}{4}\lambda(\boldsymbol{\varphi}^2 - \eta^2)^2 \right] ,$$

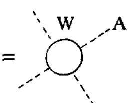
$$\boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + \boldsymbol{A}_\mu \times \boldsymbol{A}_\nu ,$$

$$\nabla_\mu \boldsymbol{\varphi} = \partial_\mu \boldsymbol{\varphi} + \boldsymbol{A}_\mu \times \boldsymbol{\varphi} . \tag{3.1}$$

We shall use Euclidean formulation of the field theory. If the Higgs field $\boldsymbol{\varphi}$ is directed along the third axis, then this model contains the following fields: a photon field A_μ^3, a heavy charged boson field $W_\mu^\pm = (A_\mu^1 \pm iA_\mu^2)$, with $m_\mathrm{W}^2 \simeq e^2\eta^2$, and scalar neutral field, fluctuations of $|\boldsymbol{\varphi}(x)|$, $\sigma(x) = |\boldsymbol{\varphi}(x)| - \eta$, with mass $m_\sigma^2 = 2\lambda\eta^2$. These formula are correct if the coupling λ and e^2 are small enough.

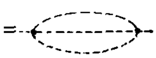
We shall obtain the most interesting results in the case $d = 3$, and so let us start from the consideration of this case (two space, one time dimensions).

First of all let us prove that in the framework of perturbation theory there are no infrared problems. In order to do this we notice that possible infrared dangers come from the photon-photon interaction, which in turn is a consequence of virtual W pair production. The effective Lagrangian describing, say, the four-photon interaction is of the form

$\mathcal{L}_4 =$ 

$$= \frac{e^4}{m_\mathrm{W}} \int \mathrm{d}^3 x \{ C_1 (F_{\mu\nu}^2)^2 + C_2 (e_{\mu\nu\lambda\gamma} F_{\mu\nu} F_{\lambda\gamma})^2 \} . \tag{3.2}$$

It is local in the limit of large photon wave length since W bosons are massive. Now the correction to photon self-energy is given by the diagram

$\Pi =$  . (3.3)

This converges at small momenta since according to (3.2) every vertex is of order (momentum)^4. Because of this large momenta ($\sim m_W$) are important and these give only small $\sim(e^2/m_W)^4$ correction to the renormalization of the charge. The same is true for all other perturbative effects.

Now let us turn to pseudoparticle aspects of this theory. It is well known now that there exist non-trivial minima of (3.1). They were considered in connection with another physical problem by 't Hooft [13] and by the author [14]. In these works a four-dimensional theory was analyzed, and S was not the action but the potential energy. Correspondingly minima of eq. (3.1) were interpreted as real particles. These particles appeared to be magnetic monopoles.

Particles in four dimensions are pseudoparticles in three dimensions. The hedgehog solution of [13,14] will now play an entirely different role.

For $d = 3$ one pseudoparticle solution has the form [13,14]

$$A_\mu^a = a(r)\ \mathcal{E}_{\mu a b}\frac{x_b}{r},$$

$$\varphi^a = \frac{x_a}{r}u(r)\,, \tag{3.4}$$

where $a(r)$ and $u(r)$ are regular functions, and

$$\left.\begin{aligned} a(r) &\approx -\frac{1}{r} \\ u(r) &\approx \eta \end{aligned}\right\} r \gg m_W^{-1}\,. \tag{3.5}$$

The value of S at this minimum is given by

$$S = \frac{m_W}{e^2}\ \epsilon(\lambda/e^2)\,. \tag{3.6}$$

The function $\epsilon(x)$ is slowly varying. It is known [15,16] that $\epsilon(0) = 4\pi$.

It is convenient to make a singular gauge transformation so that the new $\boldsymbol{\varphi}$ field is directed along the third axis. After this is done [17], the solution at large distances from the pseudo-monopole is of the form

$$\varphi^a \approx \eta\delta_{a3}\,, \qquad A_\mu^\pm \approx 0\,,$$

$$F_\mu = e_{\mu\nu\lambda}\,\partial_\nu A_\lambda^3 = \frac{1}{2}\,\frac{x_\mu}{|x|^3} - 2\pi\delta_{\mu 3}\,\theta(x_3)\,\delta(x_1)\,\delta(x_2)\,. \tag{3.7}$$

One easily recognizes in (3.7) the Dirac expression for the strength of the magnetic field produced by a monopole. The last term in (3.7) represents the Dirac string which compensates the total magnetic flux through the closed surface.

Let us imagine now that we are looking for minima of S under the condition that there are several monopoles and antimonopoles, fixed at certain points $\{x_a\}$ far from each other ($|x_{ab}| \gg m_W^{-1}$). Then the solution will surely be given by a superposition of fields (3.7) provided that we are far enough from each monopole:

$$F_\mu = \sum \tfrac{1}{2} q_a \frac{(x - x_a)_\mu}{|x - x_a|^3} - 2\pi\delta_{\mu 3} \sum q_a \theta(x_3 - x_{3a})$$

$$\times\, \delta(x_1 - x_{1a})\, \delta(x_2 - x_{2a})\,, \qquad (q_a = \pm 1)\,. \tag{3.8}$$

The action $S = \int \mathcal{L} \mathrm{d}^3 x$ at this extremum may be calculated as follows. Let us surround each monopole by a sphere of radius such that $m_W^{-1} \ll R \ll |x_{ab}|$. Inside these spheres the solution (3.8) is not valid but the influence of all other monopoles on the given one may be neglected. We thus get the first contribution to the action:

$$S^{(1)} = \sum_a \int_{|\bar{x} - \bar{x}_a| < R} \mathcal{L} \; \mathrm{d}^3 x = \sum_a \frac{m_W}{e^2} \epsilon\left(\frac{\lambda}{e^2}\right) \left(1 + O\left(\frac{1}{m_W R}\right)\right). \tag{3.9}$$

This is the self-pseudoenergy of the monopoles. The second contribution comes from the region outside the spheres, where (3.8) may be used. As usual, singular lines do not contribute to the energy and we obtain the result

$$S^{(2)} = \frac{1}{2e^2} \int_{\substack{\text{(outside} \\ \text{the spheres)}}} (F_\mu - F_\mu^{(\text{sing})})^2 \, \mathrm{d}^3 x$$

$$= \sum_{a \neq b} \frac{q_a q_b}{8e^2} \int \mathrm{d}^3 x \frac{(\boldsymbol{x} - \boldsymbol{x}_a)(\boldsymbol{x} - \boldsymbol{x}_b)}{|\boldsymbol{x} - \boldsymbol{x}_a|^3 \, |\boldsymbol{x} - \boldsymbol{x}_b|^3} + O\left(\frac{1}{e^2 R}\right)$$

$$= \frac{\pi}{2e^2} \sum_{a \neq b} \frac{q_a q_b}{|\boldsymbol{x}_a - \boldsymbol{x}_b|} + O\left(\frac{1}{e^2 R}\right). \tag{3.10}$$

The last term in (3.10) comes from the $a = b$ term in the sum and is negligible in comparison with $S^{(1)}$. Finally we get

$$S = \frac{m_W}{e^2} \epsilon\left(\frac{\lambda}{e^2}\right) \sum q_a^2 + \frac{\pi}{2e^2} \sum_{a \neq b} \frac{q_a q_b}{|\boldsymbol{x}_a - \boldsymbol{x}_b|}\,. \tag{3.11}$$

Let us remark again that (3.11) is true, provided that $m_W |\boldsymbol{x}_a - \boldsymbol{x}_b| \gg 1$. One may ask why we have neglected monopoles with $q > 1$. The reason is that these monopoles will be inessential. In other words they may be considered as the limit of two or more monopoles with $q = 1$ close to each other and these configurations will be proved to be inessential. Only far separated pseudoparticles are important in the infrared region.

The next problem is to find the proper measure for the $\{\boldsymbol{x}_a\}$ integration. It is an important question for all the kinds of theories we are discussing. Because of it, we interrupt consideration of the above example in order to give a general formalism for taking account of symmetry modes.

4. Zero modes in gauge theories

In sect. 2 we dealt with the zero mode connected with translation symmetry. It is evident that in general the number of different zero modes are equal to the number of conservation laws violated by classical solution. In gauge theories this number is infinite and this creates some difficulties which we are going to resolve.

First of all let us consider small deviations from the equilibrium for the functional (3.1). By representing

$$A_\mu = A_\mu^{\mathrm{cl}} + a_\mu \,, \qquad \varphi = \varphi^{\mathrm{cl}} + \phi \,, \tag{4.1}$$

we obtain a quadratic form in ϕ and a:

$$S = S_{\mathrm{cl}} + S_{\mathrm{II}} \,,$$

$$S_{\mathrm{II}} = \mathrm{tr} \int \mathrm{d}\boldsymbol{x} \left\{ \frac{1}{4e^2} (\nabla_\mu a_\nu - \nabla_\nu a_\mu)^2 + \tfrac{1}{2}(F_{\mu\nu}[a_\mu, a_\nu]) + \tfrac{1}{2}[a_\mu, \varphi_{\mathrm{c}}]^2 + \tfrac{1}{2}(\nabla_\mu \phi)^2 \right.$$
$$\left. + \tfrac{1}{2}(\phi, \mu^2(\varphi_{\mathrm{c}})\phi) + (\varphi_{\mathrm{c}}[\nabla_\mu \phi, a_\mu]) + (\nabla_\mu \varphi_{\mathrm{c}}[a_\mu, \phi]) \right\} . \tag{4.2}$$

Here we have slightly changed the notation, by introducing a matrix instead of a vector form for A_μ and ϕ. Covariant derivatives are defined by $\nabla_\mu X = \partial_\mu X + [A_\mu, X]$.

In order to perform the functional integration it is necessary to find eigenvalues and eigenfunctions of this quadratic form. These quantities are defined by the equations

$$\nabla_\nu(\nabla_\nu a_\mu^{(n)} - \nabla_\mu a_\nu^{(n)}) + [\varphi_{\mathrm{c}}[\varphi_{\mathrm{c}} a_\mu^{(n)}]] - [\varphi_{\mathrm{c}} \nabla_\mu \phi^{(n)}] + [\nabla_\mu \varphi_{\mathrm{c}}, \phi^{(n)}] = -\Omega_n^2 a_\mu^{(n)} \,,$$

$$\nabla_\mu \nabla_\mu \phi^{(n)} - M^2(\varphi_{\mathrm{c}})\phi^{(n)} - [\varphi_{\mathrm{c}}, \nabla_\mu a_\mu^{(n)}] - 2[\nabla_\mu \varphi_{\mathrm{c}}, a_\mu^{(n)}] = -\Omega_n^2 \phi^{(n)} \,. \tag{4.3}$$

There exists a simple but important theorem concerning solutions of (4.3): Every eigenfunction with non-zero eigenvalue $\Omega_n^2 \neq 0$ satisfies automatically the condition

$$\nabla_\mu a_\mu^{(n)} + [\varphi_{\mathrm{c}}, \phi^{(n)}] = 0 \,, \tag{4.4}$$

This statement could have been proved directly from (4.3) but a much more simple proof may be given as follows. Eq. (4.3) always has zero eigenvalue solutions given by the formulae

$$a_\mu^{(0)} = \nabla_\mu \alpha(x) \,, \qquad \phi^{(0)} = [\varphi_{\mathrm{c}}, \alpha] \,, \tag{4.5}$$

since these variations of the fields are simply infinitesimal gauge transformations of A_μ^{cl} and φ_{cl}, which should not change S. The linear operator in (4.3) is self-conjugate and hence zero eigenfunctions should be orthogonal to non-zero ones. Hence,

$$\operatorname{tr}\int (a_\mu^{(n)}\, a_\mu^{(0)} + \phi^{(n)}\phi^{(0)})\,\mathrm{d}\boldsymbol{x} = \operatorname{tr}\int (a_\mu^{(n)}\nabla_\mu\, \mathrm{d}(x) + \phi^{(n)}\,[\varphi_{\mathrm{c}}, \alpha])\,\mathrm{d}\boldsymbol{x} = 0\,. \qquad (4.6)$$

From the arbitrariness of $\alpha(x)$ it follows that $a_\mu^{(n)}$ and $\phi^{(n)}$ satisfy (4.4) QED.

It is clear now that it is very convenient to adopt a special gauge condition, which will be called the "natural gauge", given by:

$$\nabla_\mu^{A^{\mathrm{cl}}} A_\mu + [\varphi^{\mathrm{cl}}, \varphi] = 0\,. \qquad (4.7)$$

Now we are able to eliminate gauge freedom. Let us notice first of all that the Faddeev-Popov determinant in the natural gauge has the form

$$\Delta = \det(\nabla_\mu^{A^{\mathrm{cl}}}\nabla_\mu^{A} + [\varphi^{\mathrm{cl}}\varphi])\,, \qquad (4.8)$$

since the gauge transformation of (4.7) is given by

$$\delta A_\mu = \partial_\mu \alpha + [A_\mu, \alpha] = \nabla_\mu^{A}\alpha\,,$$

$$\delta\varphi \;= [\varphi, \alpha]\,,$$

$$\delta(\nabla_\mu^{A^{\mathrm{cl}}} A_\mu + [\varphi^{\mathrm{cl}}\varphi]) = \nabla_\mu^{A^{\mathrm{cl}}}\nabla_\mu^{A}\alpha + [\varphi^{\mathrm{cl}}[\varphi, \alpha]]\,. \qquad (4.9)$$

The integration measure is given by

$$\text{Measure} = \int \mathcal{D}A_\mu \mathcal{D}\varphi\, \delta(\nabla_\mu^{A\mathrm{cl}} A_\mu + [\varphi^{\mathrm{cl}}, \varphi])\, \det(\nabla_\mu^{A^{\mathrm{cl}}}\,\nabla_\mu^{A} + [\varphi^{\mathrm{cl}}\varphi])\,. \qquad (4.10)$$

Now let us expand

$$A_\mu = A_\mu^{\mathrm{cl}} + \sum \xi_n\, a_\mu^{(n)} + \nabla_\mu^{A^{\mathrm{cl}}}\,\alpha\,,$$

$$\varphi \;= \varphi^{\mathrm{cl}} + \sum \xi_n\, \phi^{(n)} + [\alpha, \varphi^{\mathrm{cl}}]\,. \qquad (4.11)$$

(Here all $a_\mu^{(n)}$ and $\phi^{(n)}$ are assured to satisfy (4.4). The expansion (4.11) is always possible due to completeness of the solutions of (4.3).) It is necessary to express the integration measure through $\mathrm{d}\xi_n$ and $\mathcal{D}\,\alpha(x)$. In order to do this let us find the metric in the functional space of A_μ and φ and use the fact that the volume element is given by the square root of the determinant of the metric. We have

$$(\delta l)^2 = \operatorname{tr}\int \mathrm{d}\boldsymbol{x}((\delta A_\mu)^2 + (\delta\phi)^2) = \sum_n (\delta\xi_n)^2 + (\nabla_\mu^{A^{\mathrm{cl}}}\delta\alpha)^2 + [\delta\alpha, \varphi^{\mathrm{cl}}]^2\,. \qquad (4.12)$$

Hence,

$$\mathcal{D}A_\mu\mathcal{D}\phi = \prod_n \mathrm{d}\xi_n \mathcal{D}\,\alpha \det{}^{1/2}(\nabla_\mu^{A^{\mathrm{cl}}}\nabla_\mu^{A^{\mathrm{cl}}} + [\varphi^{\mathrm{cl}}\varphi^{\mathrm{cl}}])\,, \qquad (4.13)$$

substituting (4.13) and (4.11) into (4.10). Eliminating the δ-function we obtain the final expression

$$\text{measure} = \prod_n \mathrm{d}\xi_n \det(\nabla_\mu^{A^{\mathrm{cl}}} \nabla_\mu^{A} + [\varphi^{\mathrm{cl}}\varphi]) \{\det(\nabla_\mu^{A^{\mathrm{cl}}} \nabla_\mu^{A^{\mathrm{cl}}} + [\varphi^{\mathrm{cl}}\varphi^{\mathrm{cl}})\}^{-1/2} . \tag{4.14}$$

Gauge degrees of freedom are completely eliminated in (4.14). However there are zero modes connected with the translation and other non-gauge symmetries. They can be treated now in the usual fashion analogous to sect. 2. The only new feature is that their eigenfunction must be chosen so that they satisfy (4.4). Let us demonstrate this in the important case of translation symmetry. There are zero modes with eigenfunctions

$$\begin{aligned} \tilde{a}_\mu^{(\lambda,0)} &= \tilde{N}^{-1/2}\, \partial_\lambda A_\mu^{\mathrm{cl}} , \\ \phi^{(\lambda,0)} &= \tilde{N}^{-1/2}\, \partial_\lambda \phi^{\mathrm{cl}} , \end{aligned} \tag{4.15}$$

where $\tilde{N}$ is a normalization constant. However, these functions do not satisfy (4.4) and in order to achieve this it is necessary to perform an infinitesimal gauge transformation simultaneously with a translation. By choosing the gauge function

$$\alpha^{(\lambda)}(x) = -A_\lambda^{\mathrm{cl}} , \tag{4.16}$$

we get

$$\begin{aligned} a_\mu^{(\lambda,0)} &= N^{-1/2} [\partial_\lambda A_\mu^{\mathrm{cl}} - \nabla_\mu A_\lambda^{\mathrm{cl}}] = N^{-1/2} F_{\lambda\mu}^{\mathrm{cl}} , \\ \phi^{(\lambda,0)} &= N^{-1/2} (\partial_\lambda \varphi^{\mathrm{cl}} + [A_\lambda^{\mathrm{cl}}\, \varphi^{\mathrm{cl}}]) = N^{-1/2} \nabla_\lambda^{A^{\mathrm{cl}}} \varphi^{\mathrm{cl}} , \\ N &= \int \{(F_{\lambda\mu}^{\mathrm{cl}})^2 + (\nabla_\lambda \varphi^{\mathrm{cl}})^2\}\, \mathrm{d}\boldsymbol{x} . \end{aligned} \tag{4.17}$$

It is evident that functions (4.17) satisfy condition (4.4). Now it is easy to connect $\xi_0^{(\lambda)}$ and R_λ (where R is the center of mass coordinate). We see that

$$A_\mu^{\mathrm{cl}}(x+R) - \nabla_\mu (R_\sigma A_\sigma) \approx A_\mu^{\mathrm{cl}}(x) + R_\sigma F_{\sigma\mu}^{\mathrm{cl}} . \tag{4.18}$$

Hence,

$$\xi_0^{(\lambda)} \approx N^{1/2} R_\lambda , \tag{4.19}$$

analogously to (2.16).

In the tree approximation the integration measure is of the form

$$\text{measure} = N^{d/2}\, \mathrm{d}\boldsymbol{R} . \tag{4.20}$$

This expression will be sufficient for sect. 5. It is important, however, to have a general expression for the measure beyond the tree approximation. In order to get it we shall again use the Faddeev-Popov method. From the preceding discussion it is clear that the most reasonable subsidary conditions should be of the form

$$\psi(x) = \nabla_\mu^{A\,\mathrm{cl}} A_\mu + [\varphi_{\mathrm{cl}}, \varphi] = 0 \,,$$

$$\chi_\lambda = \int \mathrm{d}\boldsymbol{x} \{(A_\mu - A_\mu^{\mathrm{cl}}) F_{\mu\lambda}^{\mathrm{cl}} + (\varphi - \varphi^{\mathrm{cl}}) \nabla_\lambda \varphi^{\mathrm{cl}}\} = 0 \,. \tag{4.21}$$

The second condition will eliminate translational freedom. In order to find the Faddeev-Popov determinant, let us perform an infinitesimal gauge transformation with a phase $\alpha(x) - R_\lambda A_\lambda$, and a translation R_λ. We get

$$\delta\psi(x) = \nabla_\mu^{A\mathrm{cl}} \nabla_\mu^{A} \alpha + R_\lambda \nabla_\mu^{A\mathrm{cl}} F_{\lambda\mu} \,,$$

$$\delta\chi_\nu = -\int \nabla_\mu^{A} F_{\mu\nu}^{\mathrm{cl}} + R_\lambda \int F_{\lambda\mu} F_{\mu\nu}^{\mathrm{cl}} \,. \tag{4.22}$$

The determinant should be taken from the operator on the right-hand side of (4.22). It may be expressed in the usual way through ghost fermi fields. In the one-loop approximation, the difference between ∇^A and $\nabla^{\bar{A}}$ may be neglected and in this important case we obtain from (4.22) and (4.14)

$$\text{measure} = \mathcal{N}^{d/2} \, \mathrm{d}\boldsymbol{R} \prod_{n \neq 0} \mathrm{d}\xi_n \sqrt{\det(\nabla_\mu^{A\mathrm{cl}} \nabla_\mu^{A\mathrm{cl}} + [\varphi^{\mathrm{cl}}[\varphi^{\mathrm{cl}})} \,, \tag{4.23}$$

where the product over n runs only over non-zero modes. If the classical solution violates some other non-gauge symmetries, such as rotational dilatational and so on, the only correction to (4.23) would be (in the one-loop approximation) the addition of corresponding collective coordinate integrations with proper normalization constants. Examples of this will be given below. Only normalizable zero modes should be taken into account.

5. Solution of compact QED

Now we return to the problem of sect. 3. Let us now find the one-monopole contribution to the functional integral in the one-loop approximation. This is given by the expression

$$Z_1 = \int \mathcal{N}^{3/2} \, \mathrm{d}\boldsymbol{R} \, \det{}^{1/2} \{\nabla_\mu^{A\,\mathrm{cl}} \nabla_\mu^{A\mathrm{cl}} + [\varphi^{\mathrm{cl}}[\varphi^{\mathrm{cl}}\} \left\{\exp - \left(\sum_{n \neq 0} \log \frac{\Omega_n}{\Omega_n^0}\right)\right\} m_{\mathrm{W}}^3 \, \mathrm{e}^{-S_{\mathrm{cl}}}$$

$$= \int \frac{m_{\mathrm{W}}^{7/2}}{e} \alpha\left(\frac{\lambda}{e^2}\right) \mathrm{e}^{-\epsilon(\lambda/e^2)\, m_{\mathrm{W}}/e^2} \, \mathrm{d}\boldsymbol{R} \,, \tag{5.1}$$

where α is some function which can be calculated in principle.

For widely separated monopoles we get from (5.1) and (3.11)

$$Z = \sum_{\substack{N \\ \{q_a\}}} \frac{\zeta^N}{N!} \int \prod_{j=1}^{N} \mathrm{d}\boldsymbol{R}_j \exp\left\{-\frac{\pi}{2e^2} \sum_{a \neq b} \frac{q_a q_b}{R_{ab}}\right\}, \tag{5.2}$$

where

$$\zeta = \frac{m_{\mathrm{W}}^{7/2}}{e}\, \alpha(\lambda/e^2)\, \mathrm{e}^{-(m_{\mathrm{W}}/e^2)\,\epsilon(\lambda/e^2)} . \tag{5.3}$$

This is the partition function for the Coulomb gas. The correlation functions of Higgs-Yang-Mills system are, we see, directly connected with the correlation functions of this gas. Hence it is necessary to have an efficient method for calculation of these latter functions. Now we shall obtain a functional representation for them. Let us rewrite (5.2) in the form

$$Z = \int \mathcal{D}\chi(x)\, \mathrm{e}^{-(\pi e^2/2) \int (\nabla\chi)^2} \sum_N \sum_{\{q_a = \pm 1\}} \frac{\zeta^N}{N!} \int \mathrm{d}\boldsymbol{R}_1 \ldots \mathrm{d}\boldsymbol{R}_N\, \mathrm{e}^{i\Sigma q_a \chi(\boldsymbol{R}_a)}$$

$$= \int \mathcal{D}\chi(\boldsymbol{x})\, \mathrm{e}^{(-\pi e^2/2) \int (\nabla\chi)^2 \mathrm{d}\boldsymbol{x}} \sum \frac{\zeta^N}{N!} \int \mathrm{d}\boldsymbol{R} (\mathrm{e}^{i\chi(R)} + \mathrm{e}^{-i\chi(R)})^N$$

$$= \int \mathcal{D}\chi(x) \exp\{-\tfrac{1}{2}\pi e^2\} \int ((\nabla\chi)^2 - M^2 \cos\chi) . \tag{5.4}$$

Here,

$$M^2 = \frac{4\zeta}{\pi e^2} . \tag{5.5}$$

The functional integral (5.4) supplies us with the desired diagrammatic expansion. However the effective coupling, g, in (5.4) is exponentially small. It may be estimated from the coefficient of χ^4, in units of M. We get

$$g \sim \frac{M^2}{e} \propto e^{-\epsilon m_{\mathrm{W}}/e^2} \ll 1 .$$

This result could have been anticipated since it corresponds to the condition for validity of the Debye or mean field approximation. For this to hold it is necessary that in the Debye volume, of order $M^{-3} \sim \exp\{-3m_{\mathrm{W}}\epsilon/e^2\}$ there is a large number of particles, so that the fluctuations of the sum of their individual fields may be neglected. But according to the Boltzman formula, the density of particles is given by

$$n \propto \mathrm{e}^{-m_{\mathrm{W}}\epsilon/e^2} . \tag{5.6}$$

Hence the criterion for the mean field approximation is

$$nM^{-3} \propto \mathrm{e}^{m_{\mathrm{W}}\epsilon/e^2} \gg 1 , \tag{5.7}$$

which is the same as before.

Now let us calculate certain correlation functions. We shall concern ourselves only with gauge invariant quantities. As an intermediate step it is convenient to have an expression for the generating functional for the charge density of the plasma. After simply repeating the derivation of (5.4) we get

$$\langle e^{i\int \rho(x)\,\eta(x)\,dx}\rangle = \frac{Z[\eta(x)]}{Z[0]}\,, \tag{5.5}$$

where

$$\rho(x) = \sum_a q_a \delta(x - x_a)\,, \tag{5.6}$$

$$Z[\eta] = \int \mathcal{D}\chi \exp[-\tfrac{1}{2}\pi e^2] \int dx \{(\nabla(\chi - \eta))^2 - 2M^2 \cos\chi\}\,. \tag{5.7}$$

The simplest correlation functions for our problem are those of the operator

$$H_\mu(x) = \epsilon_{\mu\nu\lambda}\,\boldsymbol{\varphi} \boldsymbol{F}_{\nu\lambda}(x)\,\frac{1}{m_\mathrm{W}}\,. \tag{5.8}$$

At large distances in the singular gauge $\varphi_1 = \varphi_2 = 0$ this is just the electromagnetic field strength.

In the quasiclassical approximation, $H_\mu(x)$ is connected with the charge density as follows:

$$H_\mu(x) = \tfrac{1}{2} \int d^3y\, \frac{(x-y)_\mu}{|\boldsymbol{x}-\boldsymbol{y}|}\,\rho(y)\,,$$

$$H_\mu(k) = \frac{2\pi k_\mu}{k^2}\,\rho(k)\,. \tag{5.9}$$

Using formula (5.7) we get

$$\langle \rho(k)\,\rho(-k)\rangle = k^2 - k^4 \langle \chi(k)\,\chi(-k)\rangle\,,$$

$$\langle \rho(k_1) \ldots \rho(k_N)\rangle = (-)^N \prod_{j=1}^{N} k_j^2 \langle \chi(k_1) \ldots \chi(k_N)\rangle\,. \tag{5.10}$$

Now the correlation function of the H-field is given by

$$\langle H_\mu(k)\,H_\nu(-k)\rangle = \langle H_\mu(k)\,H_\nu(-k)\rangle^{(0)} + \frac{k_\mu k_\nu}{k^4}\,\langle \rho(k)\,\rho(-k)\rangle\,. \tag{5.11}$$

The first term in (5.11) is the bare (that is without monopoles) Green function of the H-field. It was the form

$$\langle H_\mu H_\nu \rangle^{(0)} = \frac{e^2\eta^2}{m_W^2 k^2}(k^2\delta_{\mu\nu} - k_\mu k_\nu) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\,. \tag{5.12}$$

Its singularity at $k = 0$ reflects the existence of the massless photon in this approximation. Using (5.10) and the previous comment about the small coupling of the χ-field we get

$$\langle \rho(k)\,\rho(-k)\rangle = k^2 - \frac{k^4}{M^2 + k^2} = \frac{M^2k^2}{M^2+k^2}\,. \tag{5.13}$$

From formulae (5.11), (5.12) and (5.13) we obtain

$$\langle H_\mu(k)\,H_\nu(-k)\rangle = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \frac{k_\mu k_\nu}{k^2}\frac{M^2}{M^2+k^2} = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+M^2}\,. \tag{5.14}$$

This formula implies that there are no massless particles in the theory and instead we have a massive scalar particle with a small mass M. Analogously we find the result

$$\langle H_{\mu_1}(k_1)\,...\,H_{\mu_j}(k_j)\rangle_{\text{conn}} = k_{\mu_1}\,...\,k_{\mu_j}\prod_j \frac{1}{k_j^2+M^2}\,, \qquad (\textstyle\sum k_j = 0)\,. \tag{5.15}$$

The qualitative explanation of the above result is the following. In our system there is a finite density of pseudoparticles with long-range interaction and their random fields spoil the correlation.

In order to show directly that there are electric strings in our theory let us calculate the correlation function

$$F[C] = \mathrm{e}^{-W[C]} = \langle \mathrm{e}^{i\oint A_\mu^3 \mathrm{d}x_\mu}\rangle\,, \tag{5.16}$$

which was introduced in ref. [4]. To see the meaning of this correlation function, let us introduce two very heavy charged particles in our theory, and consider the process shown in fig. 1. The transition amplitude for this process (in which particles are created, then separated by a distance R, remain there for a very long time T, and

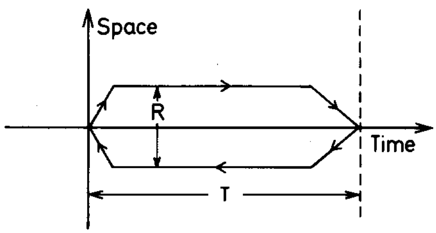


Fig. 3. Integration contour for (5.16).

then annihilate) is given on the one hand by $F[C]$ where C is calculated from fig. 3, and on the other hand by

$$F[C] = \mathrm{e}^{-iE(R)(-iT)} = \mathrm{e}^{-E(R)T} . \tag{5.17}$$

Here $E(R)$ is the energy of our system. In (5.17) it was taken into account that we are using an imaginary time formulation of the theory. Formula (5.17) is true for heavy sources since they are approximately static. It is clear now that the calculation of F will give us the interaction law between two heavy charged particles.

The calculation is easily performed since

$$F[C] = \langle \mathrm{e}^{i\oint A_\mu^3 \mathrm{d}x_\mu} \rangle \approx \langle \mathrm{e}^{i\int_{(S)} H_\mu \mathrm{d}S_\mu} \rangle , \tag{5.18}$$

and we know from (5.9) that this can be written

$$F[C] = \langle \mathrm{e}^{i\int \eta(x)\rho(x)\mathrm{d}x} \rangle , \tag{5.19}$$

where

$$\eta(x) = \int_{(S)} \mathrm{d}\boldsymbol{S}_y \cdot \frac{(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3} .$$

Since the field $\eta(x)$ is strong enough we can not neglect non-linearities in (5.7). Rather, $F[C]$ is given by

$$F[C] = \exp - \{\tfrac{1}{2}\pi e^2 \int (\nabla(\chi_{\mathrm{cl}} - \eta))^2 - 2M^2 \cos \chi_{\mathrm{cl}}\} \,\mathrm{d}^3\boldsymbol{x} , \tag{5.20}$$

where χ_{cl} is determined from the non-linear Debye equation

$$\nabla^2(\chi_{\mathrm{cl}} - \eta) = M^2 \sin \chi_{\mathrm{cl}} . \tag{5.21}$$

Fluctuation corrections to this field are again exponentially small.

Let us assume that the contour C is planar and is placed in the xy plane. Then, eq. (5.21) takes the form

$$\nabla^2 \chi_{\mathrm{cl}} = 2\pi\delta^1(z)\,\theta_S(xy) + M^2 \sin \chi_{\mathrm{cl}} ,$$

$$\theta_S(xy) = \begin{cases} 1 & x, y \in S \\ 0 & \text{otherwise} \end{cases} . \tag{5.22}$$

Far from the boundaries of the contour, eq. (5.22) is essentially one dimensional (χ_{cl} depends only on z) and has the solution

$$\chi_{\mathrm{cl}}(z) = \begin{cases} 4 \operatorname{arctg}(\mathrm{e}^{-mz}), & z > 0 \\ -4 \operatorname{arctg}(\mathrm{e}^{mz}) , & z < 0 \end{cases} . \tag{5.23}$$

Substituting (5.23) into (5.20), we obtain

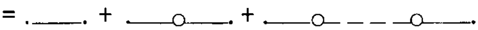
$$F[C] = \mathrm{e}^{-\gamma S} ,$$

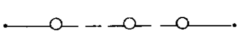
$$E(R) = \gamma R ,$$

$$\gamma = \tfrac{1}{2}\pi e^2 M \int_{-\infty}^{+\infty} \mathrm{d}y((\chi'' - \eta'')(x - y) + 2M^2 \cos \chi(y)) . \qquad (5.24)$$

The result (5.24) implies that between two fixed charges there exists an electric string with energy density γ.

Now several comments are in order. First it is useful to understand the results we have obtained, using a special diagram technique. Namely, it is possible to represent

$\langle H_\mu(x) H_\nu(y) \rangle =$ 

$+$  $+ \ldots .$ (5.25)

Here we denote the free H-field operator by a solid line, pseudoparticles by white circles and Coulomb interaction between pseudoparticles by dashed lines. It is possible to draw more complicated diragrams containing both solid and dashed lines, but all of them are small for small momenta and charges.

From (5.25) and (5.11) we see the crucial difference between pseudoparticle and instanton contributions to correlation functions. The second is purely transverse and the first is longitudinal, due to the fact that the quantity $\partial_\mu H_\mu$ measures the density of topological charge. The existence of these two contributions makes possible cancellation of singularities at zero momentum.

We proved that the potential between two charges grows linearly. It is evident that there should exist an infinite resonance spectrum in our theory, and it is tempting to find it directly by considering the correlation functions for operators of higher spin. Unfortunately in our approximation these correlation functions contain only scalar particle thresholds, and resonances should appear in higher order approximation, so the resonance problem remains to be solved even in our model.

6. Non-Abelian gauge theory. One-instanton contribution

In this section we calculate the one pseudoparticle contribution to the partition and different correlation functions of non-Abelian gauge theory. All the calculations will be made for the SU(2) gauge group, but this limitation is not important.

Let us start by giving the results of ref. [6]. The action has the form

$$S = \frac{1}{4g_0^2} \int \sum (F^a_{\mu\nu})^2 \, \mathrm{d}^4 x ,$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i f_{abc} A^b_\mu A^c_\nu . \qquad (6.1)$$

All fields $A_\mu(x)$ should be purely longitudinal at infinity:

$$A_\mu(x) \to g^{-1}(x)\, \partial_\mu g(x) , \tag{6.2}$$

where g is the gauge group matrix and hence to each possible field A_μ there corresponds a mapping of the boundary of our space onto the gauge group. Due to this remark all possible fields A_μ can be subdivided into different topologically irreducible classes corresponding to topologically irreducible mappings. Each class is characterized by the integer

$$q = \frac{1}{8\pi^2} \int e_{\mu\nu\lambda\gamma} F^a_{\mu\nu} F^a_{\lambda\gamma} \, \mathrm{d}^4 x . \tag{6.3}$$

The instanton with $q = 1$ has the form

$$A^a_\mu = \frac{C_{a\mu\nu} x_\nu}{x^2 + \lambda^2} , \tag{6.4}$$

where the $C_{a\mu\nu}$ are constants easily extracted from ref. [6], and λ is an arbitrary scale parameter. The action calculated for this solution is given by

$$S[A] = \frac{8\pi^2}{g_0^2} \equiv \int \epsilon(x)\, \mathrm{d}x ,$$

$$\epsilon(x) = \frac{1}{g_0^2} \frac{\lambda^2}{(x^2 + \lambda^2)^2} . \tag{6.5}$$

Now let us proceed to the one-loop calculation of the instanton contribution. The only essentially new feature of the problem compared with the one above is the existence of the dilational zero mode. If we represent A_μ in the form

$$A_\mu = \bar{A}_\mu + a_\mu , \tag{6.6}$$

then according to (4.2),

$$S = S_{\mathrm{cl}} + S_{\mathrm{II}} ,$$

$$S_{\mathrm{II}} = \frac{1}{4g_0^2} \operatorname{tr} \int (\nabla_\mu a_\nu - \nabla_\nu a_\mu)^2 + 2(\bar{F}_{\mu\nu} [a_\mu a_\nu]) , \tag{6.7}$$

and again we conclude that all non-zero modes automatically satisfy the condition

$$\nabla^{\bar{A}}_\mu a^{(\mathrm{n})}_\mu = 0 . \tag{6.8}$$

By applying the same condition to the non-gauge zero modes we get for the dilatational mode

$$a_\mu^{(\mathrm{d})} \propto (1 + x_\lambda \partial_\lambda)\bar{A}_\mu - \nabla_\mu \alpha = x_\lambda \bar{F}_{\lambda\mu} ,$$

$$\alpha = x_\lambda \bar{A}_\lambda . \tag{6.9}$$

This dilatational mode is normalizible since $\bar{F} \propto x^{-4}$ and should be taken into account. There is also a conformal mode, with eigenfunction

$$a_\mu^{(c,\lambda)} = (x^2 \delta_{\lambda\gamma} - 2x_\lambda x_\gamma)\bar{F}_{\gamma\mu} , \tag{6.10}$$

but it is not normalizible and we should not worry about it.

In order to exclude translational and dilatational modes we shall use two supplementary conditions by the method of Faddeev and Popov:

$$1 = S_{\mathrm{cl}}^{+5} \int_0^\infty \mathrm{d}\lambda^2 \int \mathrm{d}\boldsymbol{R}\, \delta(\int \epsilon(x)(\boldsymbol{x} - \boldsymbol{R})\,\mathrm{d}\boldsymbol{x})\, \delta(\int \epsilon(\boldsymbol{x})\, [(\boldsymbol{x} - \boldsymbol{R})^2 - \lambda^2]\, \mathrm{d}\boldsymbol{x}), \tag{6.11}$$

where $\epsilon(x)$ is the action density. These two δ-functions imply the orthogonality of dilatational and translational modes. Substituting the identity (6.11) into the functional integral and making the variable changes

$$\boldsymbol{x} \to \lambda \boldsymbol{x} + \boldsymbol{R} ,$$

$$A_\mu \to \lambda A_\mu(\lambda \boldsymbol{x} + \boldsymbol{R}) , \tag{6.12}$$

we find

$$Z = S_{\mathrm{cl}}^{-5} \int \frac{\mathrm{d}\lambda}{\lambda^5} \int \mathrm{d}\boldsymbol{R} \int \mathcal{D} A_\mu \delta(\nabla_\mu^{A^{\mathrm{cl}}} A_\mu) \det(\nabla_\mu^{A^{\mathrm{cl}}} \nabla_\mu^A)$$

$$\times\, \delta(\int \epsilon(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\boldsymbol{x})\, \delta(\int \epsilon(\boldsymbol{x})(\boldsymbol{x}^2 - 1)\,\mathrm{d}\boldsymbol{x}) \exp\{-\int \epsilon(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}\} . \tag{6.13}$$

The meaning of δ-functions containing ϵ in (6.13) becomes apparent if we make the expansion

$$A_\mu = A_\mu^{\mathrm{cl}} + a_\mu ,$$

$$\epsilon = \epsilon^{\mathrm{cl}} + F_{\mu\nu}^{\mathrm{cl}} \nabla_\mu a_\nu . \tag{6.14}$$

If the scale and position of the instanton are chosen so that

$$\int \epsilon(\boldsymbol{x})\, \boldsymbol{x} \mathrm{d}\boldsymbol{x} = 0 ,$$

$$\int \epsilon(\boldsymbol{x})(\boldsymbol{x}^2 - 1)\, \mathrm{d}\boldsymbol{x} = 0 , \tag{6.15}$$

then the arguments of the δ-functions reduce to the following:

$$\int \epsilon(x) x_\lambda \mathrm{d}x \approx \int x_\lambda \bar{F}_{\mu\nu} \nabla_\mu a_\nu \mathrm{d}x = -\int F^{\mathrm{cl}}_{\lambda\nu} a_\nu \,\mathrm{d}x = 0 \,,$$

$$\int \epsilon(x)(x^2-1)\,\mathrm{d}x \approx \int x^2 F^{\mathrm{cl}}_{\mu\nu} \nabla_\mu a_\nu \mathrm{d}x \approx -\int x_\lambda F^{\mathrm{cl}}_{\lambda\nu} a_\nu \mathrm{d}x = 0 \,, \tag{6.16}$$

and the δ-functions give us the conditions of orthogonality of $a_\mu(x)$ to zero modes.

The next step is to use the expression to show

$$Z = \frac{S^5_{\mathrm{cl}}}{N_{\mathrm{t}}^2 N_{\mathrm{d}}^{1/2}} \,\mathrm{e}^{-S_{\mathrm{cl}}} \int \mathrm{d}R \;\frac{\mathrm{d}\lambda}{\lambda^5} \prod_{n\neq 0} \mathrm{d}\xi_n \,\det{}^{1/2}(\nabla^{A\,\mathrm{cl}}_\mu \nabla^{A\,\mathrm{cl}}_\mu)\,\mathrm{e}^{-\Sigma_n \Omega_n^2 \xi_n^2} \,. \tag{6.17}$$

Here N_{d} and N_{t} are normalization constants for dilatations and translations,

$$\int F_{\lambda\nu} F_{\lambda'\nu} \mathrm{d}x = N_{\mathrm{t}} \delta_{\lambda\lambda'} \,, \qquad \int (x_\lambda F_{\lambda\nu})^2 \,\mathrm{d}x = N_{\mathrm{d}} \,,$$

and Ω_n^2 are eigenfrequencies of S_{II}. The resulting integral over ξ_n depends on λ due to the following phenomenon. The determinants in the integral (6.17) are ultraviolet divergent and should be cut off at some scale a. After rescaling of the fields this scale becomes equal to a/λ, and determinants become λ-dependent. This dependence can be traced since it is connected with the ultraviolet divergences of the corresponding determinants. Due to one-loop renormalizibility the dependence on a must cancel after replacing $1/g_0^2$ by

$$\frac{1}{g^2(\lambda)} \equiv \frac{1}{g_0} - \tfrac{11}{3} C_2 \frac{1}{8\pi^2} \log \frac{\lambda}{a} . \tag{6.18}$$

This comment implies that

$$Z = \int \mathrm{d}R \int \frac{\mathrm{d}\lambda}{\lambda^5} g^{-5}(\lambda)\, C(g^2(\lambda)) \exp\{-8\pi^2/g^2(\lambda)\} \,, \tag{6.19}$$

and C can be expanded in g^2 for $g^2 \to 0$. The value $C(0)$ is calculable due to the O(5) properties of the instanton. This calculation was performed in collaboration with A.A. Belavin and will be published elsewhere.

The origin of the g^{-5} dependence in (6.19) is that $S_{\mathrm{cl}} \propto g^{-2}$ and N_{t}, $N_{\mathrm{d}} \propto g^{-2}$. So

$$\frac{S_{\mathrm{cl}}}{N_{\mathrm{d}} N_{\mathrm{t}}^{1/2}} \propto g^{-5} \,.$$

This formula is correct only for λ such that $g^2(\lambda)$ is small and hence we cannot find the contribution of a very large instanton (recall that λ is the instanton scale). Now let us calculate the correlation functions of some operator $f(x)$ which is gauge invariant and which has some scale dimension Δ. The recipe for calculating $\langle f(x_1) f(x_2)\rangle$

is as follows. Let us introduce the classical value of $f(x)$, $f_{\rm cl}(x)$ equal to $f(x)$ calculated for the instanton solution. Then, recalling all the scale and translation transformations we have done in the course of the derivation of (6.19), we get

$$\langle f(x_1)f(x_2)\rangle = \langle f(x_1)f(x_2)\rangle_0 + \int \mathrm{d}R \int \frac{\mathrm{d}\lambda}{\lambda^5} C(g^2(\lambda))\, g^{-5}(\lambda)$$

$$\times\, \mathrm{e}^{-8\pi^2/g^2(\lambda)} \left\{ f_{\rm cl}\left(\frac{x_1 - R}{\lambda}\right) f_{\rm cl}\left(\frac{x_2 - R}{\lambda}\right) \lambda^{-2\Delta} - (f(x_1)f(x_2))|_{A=0} \right\}. \quad (6.20)$$

Here $\langle f(x_1)f(x_2)\rangle$ is the free value of this correlation function. The second term in the integrand is connected with the $Z^{(1)}$ correction to the partition function. Using (6.18) we may write

$$\mathrm{e}^{-8\pi^2/g^2(\lambda)} = \mathrm{e}^{-8\pi^2/g^2} (\mu\lambda)^a\,, \qquad a = \tfrac{1}{3}(11N - 2N_{\rm F})\,, \quad (6.21)$$

for the SU(N) group. Here μ is the normalization point, $g^2 = g^2(1/\mu)$ is the coupling at this point and $N_{\rm F}$ is the number of flavors of the quarks. As an important example let us calculate the pseudoenergy correlation function. According to ref. [6],

$$\epsilon_{\rm cl}(x) = \tfrac{1}{4}F_{\mu\nu}^2 = \frac{1}{(1+x^2)^4}\,, \quad (6.22)$$

and has scale dimension $\Delta = 4$. Hence

$$\langle \epsilon(0)\, \epsilon(r)\rangle^{(0)} = \frac{\text{const}}{|r|^8}\,. \quad (6.23)$$

Substituting (6.22) into (6.20) we get

$$\langle \epsilon(0)\, \epsilon(r)\rangle = C(0)\, g^{-5}\, \mathrm{e}^{-8\pi^2/g^2} \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^5} (\mu\lambda)^{11N/3}\, I(\lambda, r)\,, \quad (6.24)$$

$$I(\lambda, r) = \lambda^8 \int \frac{\mathrm{d}^4R}{((r-R)^2 + \lambda^2)^4 (R^2 + \lambda^2)^4}\,. \quad (6.25)$$

(We consider the case $N_{\rm F} = 0$ for simplicity.) The function $I(\lambda, r)$ has the obvious properties

$$I(\lambda, r) \propto \begin{cases} \lambda^4 r^{-8} & r \gg \lambda \\ \lambda^{-4} & \lambda \gg r \end{cases}. \quad (6.26)$$

The behaviour of the resulting contribution is different for $N = 2$ and $N > 2$. In the first case the integral converges at $\lambda \sim r$ and we get

$$\langle \epsilon(0)\, \epsilon(r)\rangle^{(1)} = \text{const.}\, r^{-8}\, \mathrm{e}^{-8\pi^2/g^2} (\mu r)^{22/3} = \text{const.}\, r^{-8}\, \mathrm{e}^{-8\pi^2/g^2(r)}\,. \quad (6.27)$$

(Assuming $g^2(r) \ll 1$).

For $N > 2$ the integral is divergent at large λ and the best we can do is to cut it off at the scale λ_*, such that $g^2(\lambda_*) \sim 1$. In this case:

$$\langle \epsilon(0)\, \epsilon(r) \rangle^{(1)} \approx \text{const} \cdot \lambda_*^{-8} (\mu\lambda_*)^{11N/3}\, \mathrm{e}^{-8\pi^2/g^2} \sim \lambda_*^{-8} \,. \tag{6.28}$$

So in both cases we got power law corrections to the correlation function behaving as

$$\frac{\langle \epsilon(0)\, \epsilon(r) \rangle^{(1)}}{\langle \epsilon(0)\, \epsilon(r) \rangle^{(0)}} \sim \begin{cases} (r/\lambda_*)^{22/3} & N = 2 \\ (r/\lambda_*)^8 & N > 2 \end{cases} . \tag{6.29}$$

Power-like corrections give us a hint that there are massive particles in the theory. However the experience of sect. 5 shows that the instanton interaction is very important and should be taken into account properly. There are both conceptual and technical difficulties in considering this effect. We shall try to overcome some of them in the next section.

7. Superposition of instantons

Every classical solution of the field equations can be characterized by a number of conserved quantities such as the energy-momentum tensor, the dilatation current, the isospin current etc. All types of instantons described above correspond to all these quantities being zero. In the Yang-Mills case this fact is a consequence of the identity

$$\theta_{\mu\nu} = \tfrac{1}{4} \mathrm{tr}\, \{(F_{\mu\lambda} - \bar{F}_{\mu\lambda})(F_{\nu\lambda} + \bar{F}_{\nu\lambda}) + (\mu \leftrightarrow \nu)\} \tag{7.1}$$

(here $\theta_{\mu\nu}$ is the energy-momentum tensor, and $\bar{F}_{\mu\nu} = \frac{1}{2} e_{\mu\nu\lambda\gamma} F_{\lambda\gamma}$). Since the one instanton solution satisfies the equation

$$F_{\mu\nu} = \bar{F}_{\mu\nu} \tag{7.2}$$

we see that

$$\theta_{\mu\nu} = 0 \,. \tag{7.3}$$

The corollary is also true: the condition (7.3) implies (7.2).

This result is of course connected with the fact that the instanton trajectory connects two different vacuum states. One may think that such solutions are important only in the vacuum degeneracy problem. However this is not so. In order to analyze the question let us consider different solutions and estimate their contributions to the functional integral.

Let us search for a solution of the form

$$A_\mu^a = r^{-2} [1 - \varphi(r)]\; C_{a\mu\nu} x_\nu \,. \tag{7.4}$$

(Here $C_{a\mu\nu}$ are the same as in (6.4).) After some calculations, using formulas of ref. [6] we find

$$S = \frac{1}{4g_0^2}\int F_{\mu\nu}^2 = \frac{3\pi^2}{g_0^2}\int_{-\infty}^{+\infty} \mathrm{d}\xi \left[\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right)^2 + (1-\varphi^2)^2\right]. \tag{7.5}$$

The solution of ref. [6] corresponds to the $\varphi(\xi) = -\tanh \xi$ solution of (7.5). Our problem now is to consider other, many kink solutions of (7.5). It is clear that they will correspond to an instanton placed inside a larger anti-instanton and so on. Though the intuitive definition of such configurations is obvious for largely separated kinks and it is also obvious that the interaction energy of the kinks behaves like $\mathrm{e}^{-2\xi} = r^{-2}$, it is necessary to give a rigorous definition of such superpositions. In order to do this let us introduce a cut-off in ξ-space by assuming that $-L < \xi < L$, with the periodicity condition $\varphi(L) = \varphi(-L)$. Each classical solution corresponds to the motion of a classical particle in the potential $U = -(1-\varphi^2)^2$. (ξ is the time, and φ the coordinate.) There are different periodic motions with different values of the "energy" of the particle. It is straightforward to see that this "energy" is just a dilatation charge:

$$\mathcal{D} = \int x_\lambda \theta_{\lambda\mu} \mathrm{d}\sigma^\mu = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right)^2 - (1-\varphi^2)^2 . \tag{7.6}$$

Our solution represents one soliton inside another, something similar to a spherical wave on water. There exist also solutions with $P_\mu = \int \theta_{\mu\nu} \mathrm{d}\sigma^\nu \neq 0$; many separated solitons. All these configurations should be taken into account. I suspect that such solutions can be obtained from the $\mathcal{D} \neq 0$ solution by conformal transformations, but this has not been demonstrated yet. So, the summary of this section is rather disappointing – we have not succeeded in completely classifying the relevant classical configurations. But the hope of doing this remains.

8. Quantum fluctuations and negative temperatures

Up to now we have done various calculations in the WKB approximation without thinking much about quantum corrections. Now the reader may think that our procedure is hopeless since usual perturbation theory gives corrections of order $g^2(r)$ and the instanton contribution is of order $\mathrm{e}^{-8\pi^2/g^2}$. Both formulas are valid for $g^2 \ll 1$ only and in this region the second contribution is negligible. It seems that we must somehow handle the strong coupling limit, and for strong coupling there is at the first sight no way of differentiating between the instanton and the usual contributions. In this section we shall explain why the situation seems not to be so severe and why it is reasonable to sum up instanton contributions without summing perturbation theory.

Let us notice first that for arbitrary coupling it is possible to define what is meant by the instantons contributions. In the one-instanton case it is done by limiting the integration region in the functional integral to those fields for which

$$\int F_{\mu\nu}\bar{F}_{\mu\nu}\,\mathrm{d}^4x = 1\ , \tag{8.1}$$

or by imposing the boundary condition

$$A_\mu(x) \xrightarrow[x\to\infty]{} g_q^{-1}(x)\,\partial_\mu g_q(x)\ , \tag{8.2}$$

where $g_q(x)$ is some topologically non-trivial SU(2) matrix such that

$$\oint \mathrm{d}^3\sigma_\alpha\, e_{\alpha\beta\gamma\delta}\,\mathrm{tr}(L_\beta L_\gamma L_\delta) = q\ ,$$

$$L_\alpha = g_q^{-1}\,\partial_\alpha g_q\ . \tag{8.3}$$

A definition of the several instanton sector is a little bit more complicated. For widely separated instantons the boundary condition in the corresponding part of the functional integral is given by

$$g = \prod_a g_{q_a}(x - x_a)\ , \tag{8.4}$$

$$A_\mu \to L_\mu = g^{-1}\,\partial_\mu g\ . \tag{8.5}$$

The no instanton sector is defined by the condition $A_\mu \to 0$. It is clear that the latter definitions are valid only for non-overlapping instantons. Hopefully only such configurations will be important.

Now let us describe the possibility of avoiding the strong coupling problem. Let us define a "naive" β-function as one obtained by functional integration over the zero instanton sector. The corresponding coupling $g^2(\lambda)$ satisfies the Gell-Mann-Low equation

$$\lambda\frac{\mathrm{d}g^2(\lambda)}{\mathrm{d}\lambda} = \beta^{(0)}(g) = \beta_4 g^4 + \beta_6 g^6 + \dots\ . \tag{8.6}$$

This series is of course an asymptotic one, and the zero-instanton condition defines the way it is summed. There are arguments, which will not be presented here, that show that the zero-instanton or "naive" β-function is obtained by a Borel summation of the perturbation series.

Now let us assume that $\beta^{(0)}$ has no zeros and increases so that

$$\int^\infty \frac{\mathrm{d}g'^2}{\beta(g'^2)} < \infty\ . \tag{8.7}$$

In this case the naive coupling $g(\lambda)$ tends to infinity at

$$\lambda_* = \mu \exp \int_g^\infty \frac{dg'}{\beta(g')} , \tag{8.8}$$

and for $\lambda > \lambda_*$, $g^2(\lambda)$ becomes negative. This is not a paradox yet since $g^2(\lambda)$ does not take into account instantons and hence has no direct physical meaning. For $\lambda > \lambda^*$, $g^2(\lambda)$ increases with λ, and as $\lambda \to \infty$ we have

$$g^2(\lambda) \xrightarrow[\lambda\to\infty]{} -\frac{11}{3}\frac{8\pi^2 N}{\log(\lambda/\mu)} . \tag{8.9}$$

Formula (8.9) means that the effective temperature of the instanton gas is negative and instantons will be created in order to stabilize the system. Their contributions are now proportional to $\exp + \{8\pi^2/|g^2|\}$ and are large in comparison with the naive contributions.

The situation described is reminiscent of the phenomenon found in scalar massless QED by Coleman and Weinberg [18]. They noticed that the naive calculation of the scalar field coupling leads to attraction due to photon exchange. Attraction of Bose particles means instability and leads to the rearrangement of the vacuum. In the resulting vacuum there is non-vanishing classical scalar field, which produces repulsion between the bosons.

We hope that the same stabilization happens in our theory due to multiple instanton production.

There is also another argument that above the relation between instantons and confinement. If we take large bare coupling constant (which is possible in the lattice theory) then, as was shown by Wilson [4], confinement indeed exists in every order in g_0^{-2}. The possibility of expanding in g_0^{-2} is just a consequence of the compactness of the gauge group. But terms like $e^{-8\pi^2 g_0^2}$ in the weak coupling limit are also a reflection of the same compactness [notice, that $8\pi^2$ is precisely the volume of SU(2)]. Since there is no confinement without compactness, instantons should play a crucial role in all this. A crude model of the phenomenon just described is given by the Migdal recursion formula [19].

So, our main point may be formulated as follows. We hope that in the functional integral of Yang-Mills theory, when the bare charge tends to zero and when we are calculating some long range correlation function, only certain field configurations are important. In this case the hopeless problem of integration over all possible fields is greatly reduced to the reasonably complicated problem of summation over classical configurations. This kind of summation was demonstrated for 2 + 1 QED, but up to now I can only speculate on how to do this in 3 + 1 Yang-Mills theory.

I had a lot of useful discussions with so many people that it is almost impossible to list them all. My special gratitude is due to V.N. Gribov, who proposed the "tunneling interpretation" of pseduoparticles, to A.A. Belavin with whom each point of this paper was discussed up to exhaustion and who contributed many important comments and to A.A. Migdal for stimulating discussions of quantum fluctuations. I benefitted also from stimulating conversation with L.D. Faddeev about properties of topological charges.

When the manuscript was finished I got papers [20,21], devoted to the similar problems. There exist certain overlaps between the present paper and the above mentioned ones. But it seems to me that complicated things are worth repeating several times using different words.

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