A short introduction to Anderson localization

Dirk Hundertmark[*][†]

September 2007

Abstract

We give short introduction to some aspects of the theory of Anderson localization.

1 Introduction

Anderson (1958) published an article where he discussed the behavior of electrons in a dirty crystal. This is the quantum mechanical analogue of a random walk in a random environment. He considered the tight binding approximation, in which the electrons can hop from atom to atom and are subject to an external random potential modeling the random environment and gave some non-rigorous but convincing arguments that in this case such a system should loose all its conductivity properties for large enough disorder, that is, become an insulator. The electrons in such a system become trapped due to the external extensive disorder. This is in sharp contrast to the behavior in ideal crystals which are always conductors.

But what is this supposed to mean precisely? In this note we try to shed some light on the mathematical theory of Anderson localization. Our plan can be summarized as follows,

   I Disordered Matter: The Anderson model of a quantum particle in a random environment.

  II Exponential decay of "correlations" (fractional moments of the resolvent) as a signature for localization.

 III Finite volume criteria for the decay of correlations in a toy model (percolation): Sub-harmonicity is your friend.

 IV Localization at large disorder: The self-avoiding random walk representation

1.1   The Anderson Model

The arena is given by the Hilbert space $l^2(\mathbb{Z}^d)$. On $\mathbb{Z}^d$ we usually consider the Manhattan norm $|x| = |x_1| + \ldots + |x_d|$, but for most results one can equally well use the Euclidian norm. Disordered matter is described by a *random Schrödinger operator*, often called random Hamiltonian,

$$H = H_\omega := H_0 + \lambda V_\omega$$

with H_0 the unperturbed part, coupling constant $\lambda > 0$, and the random potential V_ω which is simply a multiplication operator on $l^2(\mathbb{Z}^d)$ with matrix elements $V_\omega(x) = v_x(\omega)$, where $(v_x(\omega))_{x \in \mathbb{Z}^d}$ is a collection of random variables indexed by $\mathbb{Z}^d$. In Dirac notation

$$V_\omega = \sum_{x \in \mathbb{Z}^d} v_x(\omega) |x\rangle\langle x|.$$

We often use Dirac's notation, since it gives a convenient way to write projections and integral kernels: $|x\rangle = \delta_x$ with δ_x the Kronecker delta function, $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for $y \neq x$, and $|x\rangle\langle x|$ is the Dirac notation for the projection operator $|x\rangle\langle x| = \langle \delta_x, \cdot \rangle \delta_x$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $l^2(\mathbb{Z}^2)$ with the convention that it is linear in the *second* component. For a bounded operator M on $l^2(\mathbb{Z}^d)$, i.e., an infinite matrix M, the expression $M(x, y) = \langle x|M|y\rangle$ denotes the x-y matrix element.

We will often consider the simplest case in which $(v_x)_{x \in \mathbb{Z}^d}$ are independent identically distributed (i.i.d) random variables with (single-site) distribution ρ. In this case the probability space is given by the product space $\Omega = \mathbb{R}^{\mathbb{Z}^d} = \{f : \mathbb{Z}^d \to \mathbb{R}\}$ and the usual product σ-algebra, and the probability measure on Ω is simply the product measure $\mathbb{P} = \otimes_{x \in \mathbb{Z}^d} \rho$. Note, however, that one does not need to consider i.i.d. random variables, the Aizenman-Molchanov approach is rather robust to correlated random potentials, as long as the assumptions on ρ are replaced by suitable uniform assumptions on conditional expectations.

The unperturbed part of the Hamiltonian is given by

$$H_0 = T + V_0.$$

Here V_0 models, e.g., a periodic background potential, T is the kinetic energy, e.g.,

$$T\psi(x) = -\Delta\psi(x) := -\sum_{|e|=1} \psi(x + e), \text{ that is } \langle x|T|y\rangle = T(x, y) = -\delta_{|x-y|,1}$$

is the discrete Laplace, or the (negative) adjacency matrix of $\mathbb{Z}^d$ (nearest neighbor hopping). It is also possible to include a constant magnetic field, in which case the kinetic energy is modified to

$$T(x, y) = -e^{-iA_{x,y}} \delta_{|x-y|,1}.$$

Here $A_{x,y}$ is a *phase function*, i.e., an antisymmetric function of the (oriented) bond $b = \{x, y\}$. Moreover, one can allow for more than nearest neighbor hopping in the kinetic energy T, all that is needed for most of the analysis is a sumability condition of the form

$$\sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} e^{\mu |x-y|} |T(x, y)| < \infty,$$

for some $\mu > 0$. This decay condition can also be weakened a little bit further if one is not interested in *exponential* decay estimates but content with some polynomial decay rate.

The resolvent is given by $G(z) = G_\omega(z) = (H_\omega - z)^{-1}$ and the Green's function, the kernel of the resolvent, is given by $G(x, y; z) = \langle x | G(z) | y \rangle$.

Remark 1.1 i) We mainly focus on an i.i.d. random potential V_ω, a kinetic energy given by $T = -\Delta$ and $H_0 = -\Delta + V_0$ with V_0 periodic or at least bounded.

ii) We will take expectations of "random variables" freely, completely ignoring measurability questions in the spirit of "when it is interesting after integrating then it is usually measurable." Even though sometimes one has to work hard to prove measurability, but in this case one should first be convinced that it is interesting, anyway. For a nice reading concerning measurability and other aspects of random operators see, for example, Kirsch (1989), Kirsch and Metzger (2007), and Pastur (1980).

iii) Originally discussed by Anderson was the case $V_0 = 0$, $\rho(dv) = \frac{1}{2} \chi_{(-1,1)}(x) \, dv$, and no magnetic field, that is, $T = -\Delta$ is the negative adjacency matrix of $\mathbb{Z}^d$.

1.2 What is Anderson localization (for mathematicians)?

Under the conditions described above it is known, see, for example, Pastur (1980) or Kirsch (1989), that the spectrum $\sigma(H_\omega)$ is (almost surely) constant: There is a closed set $\Sigma \subset \mathbb{R}$ such that

$$\sigma(H_\omega) = \Sigma \text{ for } \mathbb{P} \text{ almost all } \omega.$$

Moreover, for the Anderson model on $l^2(\mathbb{Z}^d)$ with an i.i.d. random potential one knows that

$$\Sigma = \sigma(H_0) + \lambda \operatorname{supp}(\rho) \tag{1}$$

where $\operatorname{supp}(\rho)$ is the support of the single-site distribution ρ. But what is the (almost sure) *nature* of the spectrum?

For any self-adjoint operator H in a Hilbert space $\mathcal{H}$ there is a decomposition of the Hilbert space into invariant subspaces,

$$\mathcal{H} = \mathcal{H}_{\mathrm{p}} \oplus \mathcal{H}_{\mathrm{sc}} \oplus \mathcal{H}_{\mathrm{ac}} \tag{2}$$

with

$\mathcal{H}_{\mathrm{p}} =$ subspace corresponding to point spectrum

$\mathcal{H}_{\mathrm{ac}} =$ subspace corresponding to absolutely continuous spectrum

$\mathcal{H}_{\mathrm{sc}} =$ subspace corresponding to singular continuous spectrum

and a decomposition of the spectrum into not necessarily distinct set

$$\sigma(H) = \sigma_p(H) \cup \sigma_{ac}(H) \cup \sigma_{sc}(H). \tag{3}$$

Physical intuition tell us that the states in $\mathcal{H}_p$ are bound states, that is, they should stay in compact regions and the states in $\mathcal{H}_{ac}$ correspond to scattering states, corresponding to transport. This is made precise by the RAGE theorem, due to Ruelle, Amrein, Georcescu, and Enss.

Theorem 1.2 *Under some mild physically reasonable conditions on the Hamiltonian H,*

$$\varphi \in \mathcal{H}_p \iff \lim_{R \to \infty} \sup_{t \in \mathbb{R}} \|\chi_{(|x|>R)} e^{-itH} \varphi\| = 0,$$

$$\varphi \in H_c = \mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \iff \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \|\chi_{(|x| \leq R)} e^{-itH} \varphi\| \, dt = 0.$$

Here $\chi_{(|x| \leq R)}$ is the characteristic function of the ball $\{|x| \leq R\}$ and $\chi_{(|x|>R)}$ the characteristic function the complement of this ball.

Remark 1.3 i) The mild conditions needed in the RAGE theorem are always fulfilled for the Anderson model on $l^2(\mathbb{Z}^d)$, and are generally fulfilled for physically reasonable Schrödinger operators in the continuum $L^2(\mathbb{R}^d)$. Thus the RAGE theorem confirms the intuition that states $\varphi \in \mathcal{H}_p$ correspond to physically bound states in the sense that up to arbitrary small errors, the time evolved state $e^{-itH} \varphi$ does not leave the compact balls $\{|x| \leq R\}$ *uniformly* in time.

ii) The RAGE theorem is somewhat vague for the complement of the bound states. It does not distinguish between the absolutely continuous and the singularly continuous subspaces.

iii) For a nice proof of the RAGE theorem see Hunziker and Sigal (2000).

One of the possible definition(s) of Anderson localization is,

Definition 1.4 A random Schrödinger operator H_ω of the type discussed above in section 1.1 has *spectral localization* in an energy interval $[a, b]$ if, with probability one, the spectrum of H_ω in this interval is pure point. That is, if

$$\sigma(H_\omega) \cap [a, b] \subset \sigma_p(H_\omega) \text{ with probability one}$$

The random Schrödinger operator H_ω has *exponential spectral localization* in $[a, b]$ if it has spectral localization in $[a, b]$ and the eigenfunctions corresponding to eigenvalues in $[a, b]$ decay exponentially.

Remark 1.5 Thus exponential spectral localization holds in the interval $[a, b]$ if for almost all ω the random Hamiltonian H_ω has a complete set of eigenvectors $(\varphi_{\omega,n})_{n \in \mathbb{N}}$ in the energy interval $[a, b]$ obeying

$$|\varphi_{\omega,n}(x)| \leq C_{\omega,n} e^{-\mu|x - x_{n,\omega}|} \tag{4}$$

with $\mu > 0$, some finite $C_{\omega,n}$ and $x_{n,\omega}$ the centers of localization.

4

The physical mechanism for localization is the suppression of tunneling over large distances due to de-coherence effect induced by the random potential (as opposed to, say a periodic potential). Spectral localization was for quite some time the only definition used by mathematicians. From a physical point of view, one might prefer to say that Anderson localization holds if there is no transport. But what exactly means transport? A possible way to express this is as follows: take any initial condition φ_0 with compact support and consider

$$\langle x^2 \rangle_{P_{H_\omega \in [a,b]} \varphi_0}(t) := \langle e^{-itH_\omega} P_{H_\omega \in [a,b]} \varphi_0, x^2 e^{-itH_\omega} P_{H_\omega \in [a,b]} \varphi_0 \rangle$$
$$= |||x|e^{-itH_\omega} P_{H_\omega \in [a,b]} \varphi_0||^2 \tag{5}$$

here $P_{H_\omega \in [a,b]}$ is the orthogonal projection onto the spectral subspace of H_ω corresponding to energies in $[a,b]$. That is, we consider only the portion of φ_0 with energy in $[a,b]$. If the electrons with energies in $[a,b]$ move ballistically with average velocity v_{av} then roughly $x(t) \sim v_{av}t$ for large times. Hence $\langle x^2 \rangle_{P_{H_\omega \in [a,b]} \varphi_0}(t)$ will be proportional to t^2 for large t. This can certainly be interpreted as a signature for transport.

On the hand, if the electron in the energy interval $[a,b]$ are *localized* then it should be natural to expect that $\langle x^2 \rangle_{P_{H_\omega \in [a,b]} \varphi_0}(t)$ is bounded uniformly in t. So as long as the spectrum in $[a,b]$ is pure point or at least if exponential spectral localization holds in $[a,b]$ then, with probability one, (5) should be uniformly bounded in t for all suitably localized initial conditions φ_0. But this is *wrong*: It is known, see Simon (1990), that pure point spectrum implies absence of ballistic motion,

$$\lim_{t \to \infty} \frac{\langle x^2 \rangle_{P_{H_\omega \in [a,b]} \varphi_0}(t)}{t^2} = 0$$

for all compactly supported initial conditions φ_0 as soon as the spectrum in $[a,b]$ is pure point. However, del Rio et al. (1996) constructed examples of (non-random) one dimensional Schrödinger operators with pure point spectrum for all energies and exponentially localized eigenfunction for which

$$\limsup_{t \to \pm\infty} \frac{\langle x^2 \rangle_{\varphi_0}(t)}{t^{2-\delta}} = \infty \quad \text{for all } \delta > 0, \tag{6}$$

for a large class of compactly supported initial conditions φ_0. That is, the mean square distance $\langle x^2 \rangle_{\varphi_0}(t)$ can grow *arbitrarily close* to the ballistic motion *even though* the operator has exponential spectral localization. Thus exponential spectral localization in the sense of Definition 1.4 is a-priori not strong enough to restrict the long time dynamics of the system besides by what is given by the RAGE theorem. The main failing has to do with the complete freedom of the constants $C_{\omega,n}$ in (4). When thinking about localization one usually thinks of all the eigenvectors being confined with some typical length scale. If the $C_{\omega,n}$ are allowed to arbitrarily grow, in n, then, in fact, the eigenvectors can be extended over arbitrarily large length scales, possibly leading to transport arbitrarily close to the ballistic motion, even though one has only pure point spectrum. This is nicely discussed in del Rio et al. (1995) with the proofs given in del Rio et al. (1996).

A physically more natural definition for localization than Definition 1.4, one which takes the dynamical properties of the physical system into account, is

Definition 1.6 A random Schrödinger operator has *strong dynamical localization* in an energy interval $[a, b]$ if for any $q > 0$

$$\mathbb{E}\big[\sup_t \||X|^q e^{-itH_\omega} P_{H_\omega \in [a,b]} \varphi\|^2\big] < \infty$$

for all φ with compact support.

So strong dynamical localization holds if for any localized initial condition φ the part of φ with energy in $[a, b]$ (i.e., in the range of the spectral projection $P_{H_\omega \in [a,b]}$) as uniformly in time bounded moments of all orders. In particular, localized initial conditions stay in compact regions for all times up arbitrary small errors. Thus, by the RAGE theorem 1.2 strong dynamical localization in $[a, b]$ implies spectral localization in $[a, b]$, but not vice versa, as the examples in del Rio et al. (1996) show.

1.3 Known (rigorous) results

The following is a heavily personally biased (and fairly incomplete) list of rigorous results for Anderson localization; for example, we will completely disregard all results for continuum random operators.

1. Kunz and Souillard (8081): $d = 1$ and nice ρ: always localization, no matter how small $\lambda > 0$ is. (Extended by Carmona et al. (1987) to general ρ.)

2. Fröhlich and Spencer (1983): Multi-scale Analysis gives

$$\langle x|(H_\omega - E)^{-1}|y\rangle \le A_{\omega,x} e^{-\mu|x-y|}$$

for fixed energy $E \in \sigma(H_\omega)$ and almost all ω. This implies vanishing of the conductivity using the Kubo formula and absence of absolutely continuous spectrum, see Fröhlich and Spencer (1983) for the former and Martinelli and Scoppola (1985) for the latter.)

3. Simon and Wolff (1986): The Fröhlich–Spencer result for fixed $E \in (a, b)$ (and nice ρ) already implies exponential localization in the sense of Definition 1.4:
$$\sigma(H_\omega) \cap (a, b) \subset \sigma_p(H_\omega)$$
and the corresponding eigenfunctions decay exponentially.

4. New approach by Aizenman and Molchanov (1993): Instead of proving point wise bounds for almost all ω, try to prove bounds for *averages*,

$$\tau(x, y; z) := \mathbb{E}[|\langle x|(H_\omega - z)^{-1}|y\rangle|^s]. \tag{7}$$

More precisely, Aizenman and Molchanov showed that these *fractional moments* are exponentially small,

$$\tau(x, y; E + i\varepsilon) \leq Ae^{-\mu|x-y|} \tag{8}$$

for $E \in (a, b)$, uniformly in $\varepsilon \neq 0$ and a suitable (fixed) $0 < s < 1$, in the case of large disorder or extreme energies.

Kunz and Souillard proved, in fact, what we now call strong dynamical localization by showing that $\mathbb{E}[\sup_t |\langle x|e^{itH_\omega}|y\rangle|]$ decays exponentially in the distance $|x - y|$. Their proof is similar to the proof that correlations in the one-dimensional nearest neighbor Ising model decay exponentially at all positive temperatures, see Cycon et al. (1987). One of the central themes of the approach to Anderson localization initiated by Aizenman and Molchanov is a shift in focus: instead of studying the of trying to prove almost sure (in the random potential) decay estimates for the off-diagonal Green's function $G_\omega(x, y; E) = \langle x|(H_\omega - z)^{-1}|y\rangle$ for some energy E *within* the spectrum of the Hamiltonian, one should instead study the *correlation* $\tau(x, y; z)$ given by (7) for $z = E + i\varepsilon$ (uniformly for small $\varepsilon > 0$). As we will see shortly, the reason to look at *fractional moments* is purely technical: to guarantee that τ is well-defined. But first we discuss some consequences of (8). It turns out that the Aizenman-Molchanov criterion is a *very useful* signature for localization. It implies localization in all different manifestations:

Some consequences of the localization criterion (8):

1. *Spectral localization:* H_ω has in (a, b) only pure point spectrum with exponentially decaying eigenfunctions.

2. *Strong dynamical localization:* Wave-packages corresponding to energies in (a, b) are trapped in finite regions for all times,

$$\mathbb{E}[\sup_t |\langle x|e^{-itH_\omega} P_{H_\omega \in (a,b)}|y\rangle|] \leq \tilde{A}e^{-\tilde{\mu}|x-y|}. \tag{9}$$

3. *No level repulsion:* Minami (1996) showed that (8) implies that the local fluctuation of the energy levels of a multidimensional Anderson model in the energy range (a, b) have a Poisson statistic. The first results for $d = 1$ of this type are due to Molchanov (8081).

4. *Exponential decay of the Fermi Projection kernel:*

$$\mathbb{E}[|\langle x|P_{H_\omega \leq E}|y\rangle|] \leq \hat{A}e^{-\hat{\mu}|x-y|},$$

for some finite $\hat{A}$ and $\hat{\mu} > 0$, see Aizenman and Graf (1998).

Remark 1.7 i) As mentioned before, the RAGE theorem guarantees that strong dynamical localization implies spectral localization. Strong dynamical localization itself follows from the Aizenman-Molchanov criterion (8) and the following

bound on the evolution of states in the random system: if the single site distribution has a density with respect to Lebesgue measure, $\rho(dv) = f(v)dv$, with $f \in L^p$ for some $p > 1$ then

$$\mathbb{E}[\sup_{\|g\|_\infty \leq 1} |\langle x|g(H_\omega) P_{H_\omega \in [a,b]}|y\rangle|] \leq C \liminf_{\varepsilon \to 0} \int_a^b \mathbb{E}[|G(x,y;E+i\varepsilon)|^s] \, dE \quad (10)$$

for all $0 < s < (p-1)/p$. See, for example, Aizenman et al. (2001) or Hundertmark (2000) for a proof of (10). As soon as the Aizenman-Molchanov criterion (8) for localization is fulfilled, the bound (10) clearly implies (9) hence strong dynamical localization according to Definition 1.6 holds in this regime.

ii) The exponential decay of the kernel of the Fermi-Projection plays a central rôle in understanding the plateaus in the quantized Hall effect. As soon as for some $q > 2$ one has

$$\xi_q = \sum_{x \in \mathbb{Z}^2} \mathbb{E}[|\langle 0|P_{H_\omega \leq E}|x\rangle|^q]^{1/q}|x| \leq C < \infty$$

for all $E \in (a, b)$, the Hall conductivity on the interval (a, b) is constant. See Aizenman and Graf (1998) and Bellissard et al. (1994).

1.4 Disclaimer

Finally let us mention what we are *not discussing* in this note. Both the physics and mathematics literature on the Anderson model is huge, leaving us no chance to discuss it fully. Instead we hope that this note wets the appetite of the reader. With this in mind we want to give some, still incomplete, hints to the literature. Since we will stick to the configuration space $\mathbb{Z}^d$, there will be no mentioning of the very nice extension of Aizenman-Molchanov technique to certain random operators on the continuum $\mathbb{R}^d$ given in Aizenman et al. (2006). In addition, we will not discuss at all the so-called multiple-scale approach initiated by Fröhlich and Spencer (1983) nor its often very powerful extensions, see for example Germinet and Klein (2004) and the references therein. Even worse, we will not discuss the beautiful recent results for Anderson localization with Bernoulli random potentials by Bourgain and Kenig (2005) and for Poisson random potentials by Germinet et al. (2005). There are many articles on the multi-scale analysis approach, for a readable book see Stollmann (2001), which, however, does not contain any of the new developments in the multi-scale approach. We will also not discuss at all the results on delocalization in the Anderson model on trees instead of $\mathbb{Z}^d$, see Aizenman et al. (2006), Froese et al. (2007), and Klein (1998). For a nice and readable introduction to the physics of random Schrödinger operators, see, for example, Lifshits et al. (1988).

2　Why fractional moments?

The reason to consider *fractional moments* of the modulus of the Green's function is to guarantee that the expectation in (7) is finite. Since

$$\tau(x, y; z) = \mathbb{E}[|G(x, y; z)|^s].$$

and G is the resolvent of a self-adjoint operator, one always has the bound $\tau(x, y; z) \leq 1/|\Im(z)|$. But can one guarantee that τ is finite for $z = E + i\varepsilon$ uniformly in $\varepsilon \neq 0$? It is here where the fractional moment plus one regularity condition on the single-site distribution ρ enters. To see this in the simplest possible case, we will explicitly show that $\tau(x, x; z)$ is finite.

Write $H = \widetilde{H} + \lambda v_x |x\rangle\langle x|$. Here $|x\rangle\langle x|$ is the Dirac notation for the rank one projection operator onto the one-dimensional subspace generated by $\langle x| = \delta_x$. Thus $\widetilde{H}$ is the Hamiltonian H with $v_x = 0$. The resolvent formula

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} = B^{-1}(B - A)A^{-1}$$

gives

$$(H - z)^{-1} = (\widetilde{H} - z)^{-1} - (H - z)^{-1}\lambda v_x |x\rangle\langle x|(\widetilde{H} - z)^{-1} \tag{11}$$

With the resolvent $G(x, y; z) := \langle x|(H - z)^{-1}|y\rangle$ one has

$$G(x, x; z) = \widetilde{G}(x, x; z) - \lambda v_x G(x, x; z)\widetilde{G}(x, x; z). \tag{12}$$

which is a simple algebraic equation for the complex number $G(x, x; z)$. Solving for $G(x, x; z)$ gives

$$G(x, x; z) = \frac{\widetilde{G}(x, x; z)}{1 + \lambda v_x \widetilde{G}(x, x; z)} = \frac{1}{\beta + \lambda v_x} \tag{13}$$

where $\beta = (\widetilde{G}(x, x; z))^{-1}$ depends on everything but NOT on the potential at site x! In the physics literature β is often called the self-energy. The importance of formula (13) is that although the Green's function is usually a *very* complicated function of the potential V, its diagonal element $G(x, x; z)$ is a very simple fractional function of v_x.

Now assume that the single site distribution ρ is Hölder continuous of order $\alpha \leq 1$, that is,

$$\sup_{E \in \mathbb{R}} \rho([E - \varepsilon, E + \varepsilon]) \leq C\varepsilon^\alpha \tag{14}$$

for all $\varepsilon > 0$. Under this condition $\sup_z \tau(x, y; z)$ is finite for all $0 < s < \alpha$. This is easiest to see for $\tau(x, x; z)$: for all $0 < s < \alpha$

$$\sup_{z \in \mathbb{C}, x \in \mathbb{Z}^d} \tau(x, x; z) \leq C_{s,\alpha} < \infty \tag{15}$$

with $C_{s,\alpha} \leq \frac{1}{1 - s/\alpha} C_H^{s/\alpha} < \infty$. The proof uses heavily the self-energy formula (13). One has

$$\mathbb{E}[|G(x, x; z)|^s |v_x] = \mathbb{E}\Big[\Big|\frac{1}{\beta + \lambda v_x}\Big|^s \Big| v_x\Big] = \lambda^{-s} \int |\widetilde{\beta} + v|^{-s} d\rho(v),$$

9

with $\widetilde{\beta} = \beta/\lambda$, for the conditional expectation of $|G(x,x;z)|^s$ with respect to the random potential at x. Notice that one always has

$$\int |\widetilde{\beta} + v|^{-s} \, d\rho(v) = \int \int_0^{|\widetilde{\beta}+v|^{-s}} 1 \, dt d\rho(v) = \int_0^\infty \int \chi_{(|\widetilde{\beta}+v|^{-s}>t)} \, d\rho(v) dt$$

$$= \int_0^\infty \rho(|\widetilde{\beta} + v|^{-s} > t) \, dt = \int_0^\infty \rho(|\widetilde{\beta} + v| < t^{-s}) \, dt.$$

The assumption that ρ is an α-Hölder continuous probability measure and $0 < s < \alpha$ yields the bound

$$\int_0^\infty \rho(|\widetilde{\beta} + v| < t^{-s}) \, dt \leq \int_0^\infty \max(1, C_H t^{-\alpha/s}) \, dt \leq \int_0^r 1 dt + C_H \int_r^\infty t^{-\alpha/s}) \, dt$$

$$= r + C_H(-1 + \alpha/s)^{-1} r^{1-\alpha/s} = \frac{1}{1 - s/\alpha} C_H^{s/\alpha}$$

for the optimal choice of $r = C_H^{s/\alpha}$. Note that the right hand side here does not depend on $\widetilde{\beta}$ any more. In particular, doing first a conditional expectation with respect to the random variable v_x and then integrating over all random variables, one gets

$$\tau(x,x;z) = \mathbb{E}\Big[\mathbb{E}\big[|G(x,x;z)|^s \big| v_x\big]\Big] \leq \lambda^{-s} \mathbb{E}\Big[\sup_{\widetilde{\beta} \in \mathbb{C}} \int |\widetilde{\beta} + v|^{-s} \, d\rho(v)\Big]$$

$$\leq (1 - s/\alpha)^{-1} C_H^{s/\alpha} \lambda^{-s} \tag{16}$$

which is (15).

Remark 2.1 i) ii) In the model originally studied by Anderson ρ is given by $\rho(dv) = \frac{1}{2}\chi_{[-1,1]}(v) \, dv$. In this case $C_{s,1} = C_s = (1-s)^{-1}$.

ii) The key to showing that $\tau(x,x;z)$ is bounded uniformly in $z \in \mathbb{C}$ and $x \in \mathbb{Z}^d$ was the representation (13). To see that also $\tau(x,y;z)$ is bounded uniformly in $z \in \mathbb{C}$ and $x, y \in \mathbb{Z}^d$ one can argue similarly using a rank-two perturbation argument. Write

$$H = \widetilde{H} + \lambda v_x |x\rangle\langle x| + \lambda v_y |y\rangle\langle y|,$$

that is, $\widetilde{H}$ is now the H with the potential at sites x and y put to zero. The resolvent equation yields

$$G = (H - z)^{-1} = \widetilde{G} - G\big(\lambda v_x |x\rangle\langle x| + \lambda v_y |y\rangle\langle y|\big)\widetilde{G} \tag{17}$$

where we put $\widetilde{G} = (\widetilde{H} - z)^{-1}$ and suppressed the dependence on z. Equation (17) is an equation for G on the whole Hilbert-space $l^2(\mathbb{Z}^d)$, but one can restrict it to the two-dimensional subspace spanned by the two vectors $|x\rangle$ and $|y\rangle$: defining the 2×2 matrices

$$B = \begin{pmatrix} G(x,x) & G(x,y) \\ G(y,x) & G(y,y) \end{pmatrix} \text{ and } \widetilde{B} = \begin{pmatrix} \widetilde{G}(x,x) & \widetilde{G}(x,y) \\ \widetilde{G}(y,x) & \widetilde{G}(y,y) \end{pmatrix}$$

10

equation (17) gives

$$B = \widetilde{B} - B \begin{pmatrix} \lambda v_x & 0 \\ 0 & \lambda v_y \end{pmatrix} \widetilde{B}. \tag{18}$$

Hence

$$B \left(\mathbf{1}_{2\times 2} + \lambda \begin{pmatrix} v_x & 0 \\ 0 & v_y \end{pmatrix} \widetilde{B} \right) = \widetilde{B}$$

which can be solved as

$$B = \widetilde{B} \left(\mathbf{1}_{2\times 2} + \lambda \begin{pmatrix} v_x & 0 \\ 0 & v_y \end{pmatrix} \widetilde{B} \right)^{-1} = \left(\Theta + \lambda \begin{pmatrix} v_x & 0 \\ 0 & v_y \end{pmatrix} \right)^{-1} \tag{19}$$

with $\Theta = \widetilde{B}^{-1}$. Note that here all matrix inverses are for 2×2 matrices. As in the rank-one case, the 2×2 matrix Θ depends on everything BUT the potential v_x and v_y and the off-diagonal Green's function $G(x, y; z)$ is seen to be

$$G(x, y; z) = \left(\Theta + \lambda \begin{pmatrix} v_x & 0 \\ 0 & v_y \end{pmatrix}, \right)^{-1}_{1,2} \tag{20}$$

that is, $G(x, y; z)$ can be identified as the off-diagonal matrix element of the 2×2 matrix $\left(\Theta + \lambda \begin{pmatrix} v_x & 0 \\ 0 & v_y \end{pmatrix} \right)^{-1}$. A similar, and not much more complicated, argument to the one leading from (13) to (16) now shows that $\sup_{x,y\in\mathbb{Z}^d, z\in\mathbb{C}\setminus\mathbb{R}} \tau(x, y; z)$ is finite as long as the single-site distribution of the random potential is Hölder continuous, for details, see Aizenman and Molchanov (1993) or Aizenman et al. (2001).

3   Finite-volume criteria

A main extension of the original Aizenman-Molchanov approach to localization was that one can develop *finite-volume* criteria for the exponential decay of the correlations τ, see Aizenman et al. (2001). These bounds are related to similar bounds in statistical mechanics. One restricts the random operator $H = -\Delta + V_\omega$ to a finite box $\Lambda = \{x \in \mathbb{Z}^d : |x_j| < L \text{ all } j = 1, \ldots, d\}$, and puts $G_\Lambda(z) = (H_\Lambda - z)^{-1}$ considered as an operator (or infinite matrix) in $l^2(\Lambda)$. Somewhat loosely speaking, the finite volume criteria say that as soon as

$$\sup_{v\in\partial\Lambda} \mathbb{E}[|G_\Lambda(0, v; z)|^s] \text{ is small enough} \tag{21}$$

for all $z = E + i\varepsilon$, $E \in [a, b]$, $\varepsilon > 0$, then there exist constants $A < \infty, \mu > 0$ such that

$$\sup_{\varepsilon>0} \tau(x, y; E + i\varepsilon) = \sup_{\varepsilon>0} \mathbb{E}[|G(x, y; E + i\varepsilon)|^s] \leq A e^{-\mu|x-y|} \tag{22}$$

for all $E \in [a, b]$. Thus in this case the correlation $\tau(x, y; E + i\varepsilon)$ decays exponentially on all of $\mathbb{Z}^d$ and the Aizenman-Molchanov criterion for localization is fulfilled, yielding strong dynamical localization etc., as discussed in the introduction. Of course, the question is what "small enough" in (21) mean precisely

and, when made precise, whether this criterion is fulfilled in physical interesting situations. We are deliberately vague at this point. Let us only mention that (21) (in its precise formulation) is a physically very natural condition which is often quite easily seen to be true in the relevant cases. See Aizenman et al. (2001) for a more precise formulation of such a finite volume criterium for Anderson localization and Aizenman et al. (2000) for a discussion of the physical implications of these criteria in the case of Anderson localization.

Here we would like to take the opportunity and discuss finite volume criteria for the decay of correlations in their original setting, namely statistical mechanics. There the type of argument we are going to give is known as Simon-Lieb inequalities Simon (1980); Lieb (1980). In fact, we will discuss the simplest possible case, namely *finite volume criteria* for percolation in the spirit of Aizenman and Newman (1984).

3.1   Analogy with percolation

Consider independent site percolation on $\mathbb{Z}^d$. A site $x \in \mathbb{Z}^d$ is occupied with probability p and empty with probability $1 - p$. We usually visualize this by drawing occupied sites as small solid black circles and empty cites as white circles. Recall that two sites x, y in $\mathbb{Z}^d$ are nearest neighbors if $|x - y| = \sum_{j=1}^d |x_j - y_j| = 1$.

Definition 3.1 Two point $x, y \in \mathbb{Z}^d$ are *connected*, in short $x \longleftrightarrow y$, if both x and y are occupied cites and one can hop from x to y via a sequence of occupied nearest neighbor sites.

The connectivity τ is given by the probability that x and y are connected,

$$\tau(x, y) := \mathbb{P}\{x \longleftrightarrow y\}.$$

The situation in Definition 3.1 is sketched in Fig. 1. Note that, with a slight abuse of notation, we denote the connectivity with the same symbol as the correlation in the Anderson model. The point being that one can rather easily deduce finite-volume criteria for the connectivity in percolation: Let $\Lambda = \Lambda_L$ be the centered cube

$$\Lambda_L = \{x \in \mathbb{Z}^d : |x_j| < L \text{ for all } j = 1, \ldots, d\};$$

we will always choose $L \notin \mathbb{N}$. Note that Λ has an inner and an outer boundary

$$\partial_- \Lambda = \{x \in \Lambda : \exists\, v \notin \Lambda, |x - v| = 1\},$$
$$\partial_+ \Lambda = \{v \notin \Lambda : \exists\, x \in \Lambda, |x - v| = 1\}.$$

For a subset $B \subset \mathbb{Z}^d$ we denote by $|B|$ the "volume" of B, that is, the number of elements in B and with $\tau(x, B) = \mathbb{P}(x \longleftrightarrow B)$ the probability that x is connected to some point in B. Of course, on can restrict percolation to, possibly finite, subsets A of $\mathbb{Z}^d$. In this case we denote by $\tau_A(x, y) = \mathbb{P}(x \overset{\text{in } A}{\longleftrightarrow} y)$ the probability that x and y are connected by a path in A and similarly for $\tau_A(x, B)$.
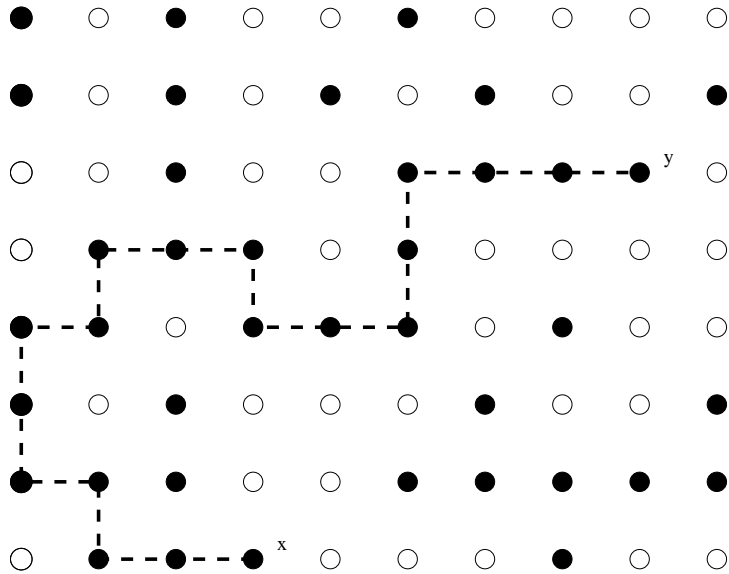
Figure 1: Two points x and y connected by a path vie nearest neighbor occupied sites. The percolation with $p = 1/2$ was simulated by tossing a coin 80 times.

The finite volume criterion for percolation with the probably simplest proof is

Theorem 3.2 *Let Λ be a centered box in $\mathbb{Z}^d$ and $b = b_\Lambda := |\partial_+\Lambda|\tau_\Lambda(0, \partial_-\Lambda)$. Then*

$$\tau(x,y) \leq b^{-1}b^{|x-y|} \tag{23}$$

for all $x, y \in \mathbb{Z}^d$.

Note that this criterion predicts exponential decay of the connectivity as soon as $b_\Lambda < 1$. That is, strong enough finite-volume decay implies exponential decay of the connectivity τ in the infinite volume $\mathbb{Z}^d$. Since the expression b_λ is computed in a *finite volume* Λ, one can, in principle, give it to the friendly neighborhood computational physicist in order to check on a computer if $b_\Lambda < 1$ for some maybe large box Λ.

Moreover, the criterion given in Theorem 3.2 is not only sufficient for exponential decay of τ but also *necessary*. Indeed, the volume of the boundary, $|\partial_+\lambda|$, grows at a polynomial rate in the side-length of Λ and since the origin is in the interior of the box Λ, it is connected to $\partial_-\Lambda$ within the box Λ, if and only if they are connected, $\tau_\Lambda(0, \partial_-\Lambda) = \tau(0, \partial_-\Lambda)$. Thus, as soon as τ on $\mathbb{Z}^d$ decays with some exponential rate, one will have $b_\Lambda < 1$ for all large enough boxes Λ.

The proof of Theorem 3.2 is surprisingly simple and was the main driving force in the search of the finite volume criteria in Aizenman et al. (2001).

Proof: Let $\Lambda = \Lambda_L$ be a centered cube of side length L and $\Lambda(x) = \Lambda + x$. Assume that x and y are so far from each other that $y \notin \Lambda(x)$ and that $x \longleftrightarrow y$.

13

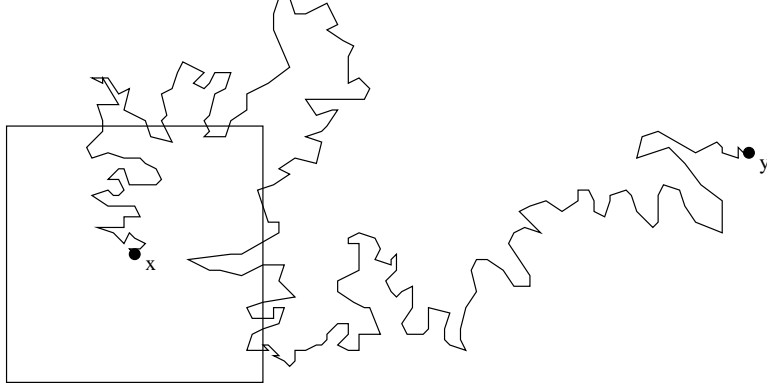This situation is sketched in Fig. 2. Since $x \longleftrightarrow y$, there must be a path within

Figure 2: The event that x and y are connected. Note that the path has to cross the box $\Lambda(x)$ at least once.

$\Lambda(x)$ from x to the (inner) boundary $\partial_-\Lambda(x)$ and a path within the complement of $\Lambda(x)$ from the (outer) boundary $\Lambda_+(x)$ to y, see Fig. 3.

Thus, as long as $y \notin \Lambda(x)$,

$$
\begin{aligned}
\tau(x,y) = \mathbb{P}(x \longleftrightarrow y) \\
\leq \mathbb{P}(x \overset{\text{in } \Lambda(x)}{\longleftrightarrow} \partial_-\Lambda(x),\ \partial_-\Lambda(x) \longleftrightarrow \partial_+\Lambda(x),\ \text{and } \partial_+\Lambda(x) \overset{\text{in } \Lambda(x)^c}{\longleftrightarrow} y) \\
\leq \mathbb{P}(x \overset{\text{in } \Lambda(x)}{\longleftrightarrow} \partial_-\Lambda(x) \text{ and } \partial_+\Lambda(x) \overset{\text{in } \Lambda(x)^c}{\longleftrightarrow} y)
\end{aligned}
\tag{24}
$$

by dropping the restriction that the inner boundary has to be connected to the outer boundary by some path of connected sites in $\mathbb{Z}^d$, see Fig. 3.
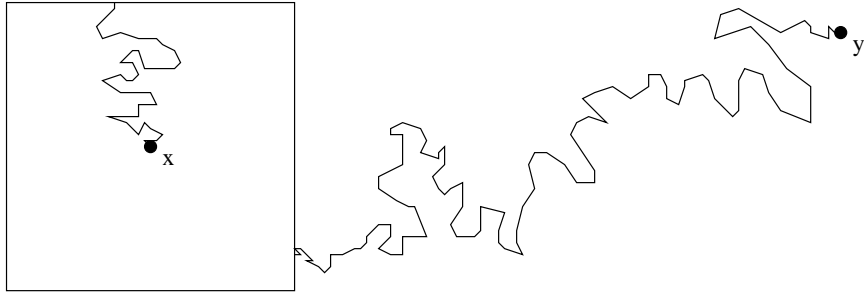

Figure 3: The upper bound in equation (24): x is connected to the boundary of $\Lambda(x)$ within $\Lambda(x)$ and y is connected to the boundary of $\Lambda(x)$ within $\Lambda(x)^c$.

Since the events "x is connected to $\partial_-\Lambda(x)$ within $\Lambda(x)$" and "$\partial_+\Lambda(x)$ is connected to y within the complement of $\Lambda(x)$" are *independent*, the probability

14

in (24) factorizes into a product and we arrive at

$$\tau(x,y) \leq \mathbb{P}(x \overset{\text{in } \Lambda(x)}{\longleftrightarrow} \partial_-\Lambda(x)) \, \mathbb{P}(\partial_+\Lambda(x) \overset{\text{in } \Lambda(x)^c}{\longleftrightarrow} y)$$
$$\leq \tau_{\Lambda(x)}(x, \partial_-\Lambda(x)) \, \tau_{\Lambda(x)^c}(\partial_+\Lambda(x), y)$$

By translation invariance, $\tau_{\Lambda(x)}(x, \partial_-\Lambda(x)) = \tau_\Lambda(0, \partial_-\Lambda)$. Moreover we have the simple monotonicity ,

$$\tau_{\Lambda(x)^c}(\partial_+\Lambda(x), y) = \mathbb{P}(\partial_+\Lambda(x) \overset{\text{in } \Lambda(x)^c}{\longleftrightarrow} y) \leq \mathbb{P}(\partial_+\Lambda(x) \longleftrightarrow y)$$
$$= \tau(\partial_+\Lambda(x), y) \leq \sum_{v \in \partial_+\Lambda(x)} \tau(v, y).$$

Thus for all $y \notin \Lambda(x)$, τ obeys the bound

$$\tau(x,y) \leq \frac{b_\Lambda}{|\partial_+\Lambda(x)|} \sum_{v \in \partial_+\Lambda(x)} \tau(v, y), \tag{25}$$

which, if $b_\Lambda < 1$ says for fixed y, $\tau(x,y)$ is a *subharmonic* function of $x \notin \Lambda(y)$.

There are many ways to see that subharmonic function have a tendency to decay exponentially. In the case at hand, the possibly easiest way is to iterate (25), which can be done at least $|x - y| - 1$ times, and then use the a-priori bound $\tau(x,y) \leq 1$. This gives

$$\tau(x,y) \leq b_\Lambda^{|x-y|-1}$$

which is (23). $\qquad \square$

Remark 3.3 i) One should notice that the above proof did not need the underlying lattice to be given by Z^d. In fact, it does not need to be a lattice at all; the proof works for percolation on arbitrary graphs $\mathcal{G}$. A similar argument as given after Theorem 3.2 then shows that the condition $b_\Lambda < 1$ is not only *sufficient* for the exponential decay of τ but also *necessary* as long as the growth of the surface volume $|\partial_+\Lambda_L|$ of large boxes Λ_L in the graph $\mathcal{G}$ is *sub-exponential*.

ii) This type of idea seems to go back at least to Hammersley (1957) in the case of percolation.

iii) Using the van den Berg–Kesten inequalities for percolation one can improve on Theorem 3.2, see Aizenman and Newman (1984).

3.2 Some consequences from finite volume criteria

As already mentioned in Remark 3.3, the finite-volume criterion in Theorem 3.2 is a sufficient and for a large class of graphs also necessary condition for the exponential decay of the connectivity. For percolation on graphs it is known that there is a critical probability $0 < p_c < 1$ such that for $p < p_c$ $\tau(x,y)$ decays exponentially in the distance $|x - y|$ and for $p > p_c$ it does not. This

is also related to the occurrence of an infinite connected cluster above p_c with probability one, Grimmett (1999).

The finite-volume criteria turn out to be a useful tool there. For example, they yield an algorithm to compute p_c: for $p > p_c$,

$$\liminf_{L \to \infty} b_{\Lambda_L} \geq 1$$

while for $p < p_c$ there exists at least one box Λ with $b_\Lambda < 1$. In particular, this yields lower bounds on p_c for graphs for which the precise value is not known.

The finite volume criteria can also be used to give painless proofs of the following not necessarily obvious facts,

1. Exponential decay of the connectivity is stable under small perturbations of parameters (For example, variation of p or slight deformations of the underlying graph) for all graphs for which the volume of boxes λ_L grows sub-exponentially in L.

2. Fast power law decay $\Rightarrow$ exponential decay. (For graphs in which the surface volume growth of boxes is polynomially bounded.)

3. At critical percolation, the connectivity cannot decay too fast.

Indeed, that the exponential decay of the connectivity is stable under small perturbations of the parameters is not all clear since one might be at a phase-transition point. That this is not the case is due to the finite-volume criteria. To show 1, assume that τ decays exponentially. Then, for some finite box Λ one must have $b_\Lambda < 1$. Since b_Λ is computed in a *finite volume*, it depends continuously on the parameters, hence wiggling them a little bit will still result in $b_\Lambda < 1$, hence the connectivity will still decay exponentially, by Theorem 3.2.

To show 2 one argues similarly, if τ decays so fast that it beats the growth of the surface volume $|\partial_+\Lambda|$, then b_Λ will be less than 1 for all large enough boxes Λ and hence τ must decay exponentially by Theorem 3.2.

Finally, for 3, note that τ does not decay exponentially for all $p > p_c$. Hence for $p = p_c$ we must have

$$1 \leq b_\Lambda = |\partial_+\Lambda|\tau(0, \partial_-\Lambda)$$

for all centered boxes Λ. Otherwise, by the first fact, one would have exponential decay of the connectivity for all p slightly above p_c, which contradicts the definition of the critical probability. Thus

$$\tau(0, \partial_-\Lambda) \geq \frac{1}{|\partial_+\Lambda|}$$

for all centered boxes Λ.

In a similar fashion, the finite volume criteria for Anderson localization give rise to the stability results for the exponential decay of the fractional moments of the Green's function analogously to the stability results (1)–(3) for percolation

above. In particular, the exponential decay of the fractional moments is stable under small perturbation of external fields, like an external (periodic) potential or an external magnetic field. For more discussions of this, see Aizenman et al. (2000).

4 Localization for large disorder: a simple proof

Our discussion of the finite volume criteria for Anderson localization has been, deliberately, somewhat vague. In contrast to this we would like to give a full and, we think, rather simple proof of Anderson localization which yields, in addition, very easily checkable assumptions with explicit bounds.

Theorem 4.1 *Consider the random operator $H = H_\omega = -\Delta + V_\omega$ on $l^2(\mathbb{Z}^d)$. Let the single side distribution ρ of the random potential at site 0, say, be such that*

$$C_s = \sup_{\beta \in \mathbb{C}} \int |\beta - v|^{-s} \, d\rho(v) < \infty.$$

for some $0 < s < 1$. Then for all $\lambda^s > (2d-1)C_s$, the exponential bound

$$\sup_{z \in \mathbb{C}\backslash\mathbb{R}} \mathbb{E}[|G(x, y; z)|^s] \leq A_{d,\lambda} e^{-\mu(d,\lambda)|x-y|} \tag{26}$$

holds. Here

$$A_{d,\lambda} = \frac{2d(2d-1)C_s\lambda^{-s}}{(2d-1)^2[1 - (2d-1)C_s\lambda^{-s}]} \tag{27}$$

and

$$\mu(d,\lambda) = -\ln\left(\lambda^s/((2d-1)C_s)\right) > 0. \tag{28}$$

Remark 4.2 i) As the proof will show, the conclusion of the Theorem remains valid even for highly correlated random potentials $V = (v_j)_{j\in\mathbb{Z}^d}$ as long as a suitable bound of the form

$$C_s = \sup_{\beta \in \mathbb{C}, j \in \mathbb{Z}^d} \mathbb{E}[|\beta + v_j|^{-s} \,|\, v_j] < \infty \tag{29}$$

for the fractional moments of the *conditional* expectations of the potential at site j holds.

ii) For the original Anderson model, $\rho(dv) = \frac{1}{2}\chi_{[-1,1]}(v)\,dv$. In this case $C_s = 1/(1-s)$ and one has localization at all energies as soon as

$$\lambda > (1-s)^{-1/s}(2d-1)^{1/s}$$

for some $0 < s < 1$.

4.1 The self-avoiding random walk representation

The observation which leads to a simple and straightforward proof of Anderson localization for large disorder is the following self-avoiding walk (SAW) representation for the off-diagonal Green's function. That such a representation holds is not necessarily new, but that it holds for all complex energies off the real axis seems to be.

Lemma 4.3 (The Self-Avoiding Walk Representation) *Let $B \subset \mathbb{Z}^d$ be finite, $G_B(z) = (H_B - z)^{-1}$ and $G_B(x, y; z) = \langle x | G_B(z) | y \rangle$. Then*

$$G_B(x, y; z) = \sum_{\substack{w:SAW\ in\ B \\ x \longleftrightarrow y}} \prod_{j=0}^{|w|} G_{B_j}(w(j), w(j); z) \tag{30}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$. Here w is a self-avoiding random walk connecting $x = w(0)$ and y in B, $|w|$ is the length of the walk, and the sets $B_j = B_j(w)$ are recursively defined by $B_0 = B$ and $B_{j+1} = B_j \setminus \{w(j)\}$.

Remark 4.4 Given a self-avoiding path w in B the sets B_j are given by $B_j = B \setminus \{x, w(1), w(2), \ldots w(j-1)\}$ for $j = 1, \ldots |w|$. Thus they are a nested shrinking sequence of subsets of B depending on the self-avoiding walk only up to time-step $j - 1$. In particular, given a self-avoiding path w, the resolvent G_{B_j} does not depend any more on the potential at the previously visited places $x, w(1), \ldots, w(j-1)$. This makes the representation (30) very powerful.

It is crucial for the application we have in mind that the representation (30) in terms of a self-avoiding random walk is valid *for all $z \in \mathbb{C} \setminus \mathbb{R}$* and not only for complex z with a large enough imaginary part. Nevertheless, we will deduce Lemma 4.3 from a perturbative result which a-priori is valid only for complex energies far up in the complex plane.

Lemma 4.5 (The random walk representation) *Let $B \subset \mathbb{Z}^d$ be an arbitrary subset and $G_B = (H_B - z)^{-1}$ as above. Then*

$$G_B(x, y; z) = \sum_{\substack{w:\ RW\ in\ B \\ x \longleftrightarrow y}} \prod_{j=0}^{|w|} \frac{1}{\lambda V(w(j)) - z} \tag{31}$$

for all $\Im(z)$ large enough.

Proof: Recall the resolvent formula $\frac{1}{A} - \frac{1}{B} = \frac{1}{B}(B - A)\frac{1}{A}$. Using this with the choice $A = H_B - z$ and $B = \lambda V - z$ yields

$$\frac{1}{H_B - z} = \frac{1}{-\Delta_B + \lambda V - z} = \frac{1}{\lambda V - z} + \frac{1}{\lambda V - z} \Delta_B \frac{1}{-T_0 + \lambda V - z}$$

where Δ_B is the adjacency matrix of the graph $\mathbb{Z}^d \cap B$. Iterating the above gives

$$\frac{1}{H_B - z} = \frac{1}{\lambda V - z} \sum_{n \geq 0} \left(\Delta \frac{1}{\lambda V - z} \right)^n \tag{32}$$

which is, of course, a Neumann series and converges for large enough $|\Im(z)|$. More precisely, since the operator norm $\|\Delta\| = 2d$ and V is real valued, we need $\Im(z) > 2d$ to guarantee convergence of the right hand side of (32).

Now we claim that (32) is nothing but (31) in disguise. Indeed, taking the x, y-matrix element of (32) gives

$$G_B(x, y; z) = \langle x| \frac{1}{H_B - z} |y\rangle = \frac{1}{\lambda V(x) - z} \sum_{n \geq 0} \langle x| \big(\Delta_B \frac{1}{\lambda V - z}\big)^n |y\rangle.$$

Note that

$$\langle x| \big(\Delta_B \frac{1}{\lambda V - z}\big)^n |y\rangle = \sum_{\substack{\text{paths } w \text{ of length } n \\ \text{in } B, \, x \leftrightsquigarrow y}} \prod_{j=1}^{n} \frac{1}{\lambda V(w(j)) - z}.$$

Setting $w(0) = x$, $w(n) = y$, and $|w| = $ the length of the path, we arrive at the *random walk representation* (31). $\qquad\square$

At first sight it might seem that the random walk representation is just a simple rewriting of a particular Neumann series for the resolvent G_B and does not necessarily deserve its own name. This is not true, however, since giving it a *new name* can drastically change the *emphasis*: the key for the proof of the self-avoiding random walk representation is the observation that every random walk leads to a self-avoiding random walk by *deleting loops*. By the random walk representation, for any set $C \subset \mathbb{Z}$, the Green function on the diagonal is given by summing over all loops of a random walk within the set C,

$$G_C(x, x; z) = \sum_{\substack{w: \, RW \text{ in } C \\ x \leftrightsquigarrow x}} \prod_{j=0}^{|w|} \frac{1}{\lambda V(w(j)) - z} \tag{33}$$

To re-sum the loops, let

$$n_x(w) := \inf\{n : \, w(j) \neq x \text{ for all } j > n\},$$

that is, $n_x(w)$ is the last time the path w visited the point x.

Cut the path w from the random walk representation (31) into two parts, $w = (w_1, \overline{w})$, where w_1 runs from 0 up to time $n_x(w)$ ($|w_1| = n_x(w)$) and $\overline{w}$ runs from $n_x(w) + 1$ up to time $|w|$. In particular,

$$|w| = |w_1| + |\overline{w}| + 1.$$

for the lengths of the combined paths.

From (31) one infers

$$G_B(x, y; z) = \sum_{\substack{w_1: RW \text{ in } B \\ x \leftrightsquigarrow x}} \prod_{j=0}^{|w_1|} \frac{1}{\lambda V(w_1(j)) - z} \sum_{\substack{x' \in B: |x' - x| = 1 \\ \overline{w}: RW \text{ in } B, x' \leftrightsquigarrow y \\ \overline{w} \text{ never visits } x}} \prod_{j=0}^{|\overline{w}|} \frac{1}{\lambda V(\overline{w}(j)) - z}.$$

$$\tag{34}$$

Using (33), the first factor is just $G_B(x, x; z)$, and appealing to the random walk representation (31) once more, one sees that the second factor is the resolvent of the operator $H_{B \setminus \{x\}}$, summed over the nearest neighbors of x,

$$\sum_{\substack{\overline{w}:\text{ in } B, x' \leftrightsquigarrow y \\ \overline{w} \text{ never visits } x}} \prod_{j=0}^{|w|} \frac{1}{\lambda V(\overline{w}(j)) - z} = \langle x' | \frac{1}{(H-z)|_{B \setminus \{x\}}} | y \rangle = G_{B \setminus \{x\}}(x', y)$$

Thus (34) can be rewritten as

$$G_B(x, y; z) = G(x, x; z) \sum_{\substack{x_1 \in B \\ |x_1 - x| = 1}} G_{B \setminus \{x\}}(x_1, y). \tag{35}$$

Of course, iterating (35) yields

$$G_B(x, y; z) = G(x, x; z) \sum_{\substack{x_1 \in B \\ |x_1 - x| = 1}} G_{B \setminus \{x\}}(x_1, x_1; z) \sum_{\substack{x_2 \in B \setminus \{x\} \\ |x_2 - x_1| = 1}} G_{B \setminus \{x, x_1\}}(x_2, y; z)$$

$$= \ldots = \sum_{\substack{w:\text{ SAW in } B \\ x \leftrightsquigarrow y}} \prod_{j=0}^{|w|} G_{B_j}(w(j), w(j); z)$$

with $B_0 = B$ and $B_{j+1} = B_j \setminus \{w(j)\}$, which nearly finishes the proof of Lemma 4.3, except that so far we only know that (30) holds as long as $\Im(z)$ is large enough.

To finish the proof of Lemma (4.3), let B be a finite set. We know that the resolvent is an analytic operator-valued function for z in the complex upper half-plane. In particular, $G_B(x, y; z)$ is an analytic function on the upper half-plane and so are all the factors $G_{B_j}(w(j), w(j); z)$ in the right hand side of (30). The punchline is that although in a *finite* set B there are infinitely many different random walk of arbitrary length, there are only *finitely many self-avoiding* random walks. Thus for finite B, the right hand side of (30) is a finite sum of a finite product of analytic functions, hence it is also analytic on the upper half-plane. Since by the above both sides of (30) agree for z with a large enough imaginary part, by analyticity they must agree on the whole complex upper half-plane. A similar arguments holds for the lower half-plane. This concludes the proof of the self-avoiding random walk representation.

4.2 Proof of localization at large disorder

In this section we use the self-avoiding random walk representation to give a straightforward proof of Anderson localization for large disorder. In some sense this proof makes precise Anderson's original heuristical argument, which uses second order perturbation theory.

Before we fully embark on the proof of Theorem 4.1, let us first note that it is enough to prove this bound for the Green function restricted to some

finite set $B \subset \mathbb{Z}^d$ as long as the bounds are uniform in B. This follows from the strong resolvent convergence of H_B to H as $B \to \mathbb{Z}^d$ and Fatou's lemma, $\mathbb{E}[|G(x, y; z)|^s] \leq \liminf_{B \to \mathbb{Z}^d} \mathbb{E}[|G_B(x, y; z)|^s]$. Secondly, note that for $0 < s \leq 1$ the bound

$$|\sum_{j=1}^{n} \alpha_j|^s \leq \sum_{j=1}^{n} |\alpha_j|^s \tag{36}$$

for all complex numbers $\alpha_j \in \mathbb{C}$ hold. This is one of the fundamental observations in the original Aizenman-Molchanov proof. By induction, it is enough to consider the case $n = 1$. In this case,

$$|\alpha_1 + \alpha_2|^s \leq (|\alpha_1| + |\alpha_2|)^s = \frac{|\alpha_1|}{(|\alpha_1| + |\alpha_2|)^{1-s}} + \frac{|\alpha_2|}{(|\alpha_1| + |\alpha_2|)^{1-s}}$$

$$\leq \frac{|\alpha_1|}{(|\alpha_1|)^{1-s}} + \frac{|\alpha_2|}{(|\alpha_2|)^{1-s}} = |\alpha_1|^s + |\alpha_2|^s$$

by dropping the term $|\alpha_2|$ in the first denominator, respectively $|\alpha_1|$ in the second denominator.

Now let $B \subset \mathbb{Z}^d$ be an arbitrary finite set and $0 < s < 1$. We claim that

$$\mathbb{E}[|G_B(x, y; z)|^s] \leq \sum_{\substack{w:\text{SAW}, \\ x \longleftrightarrow y}} (C_s/\lambda^s)^{|w|+1} \tag{37}$$

which is exponentially small in the distance $|x - y|$ (note that the right hand side does not depend on the set B anymore). Indeed, organizing the summation over the self-avoiding random walks according to their lengths one has

$$\sum_{\substack{w:\text{SAW}, \\ x \longleftrightarrow y}} (C_s/\lambda^s)^{|w|+1} = \sum_{n \geq 0} (C_s \lambda^{-s})^{n+1} \sum_{\substack{w:\text{SAW}, \\ x \longleftrightarrow y \\ |w|=n}} 1.$$

Of course, in order to connect x with y the length n of the walk must be at least $|x - y|$. In this case, since in the first step a self avoiding walk has $2d$ of the neighbors to choose and at most $2d - 1$ from then on, one has the general bound

$$\sum_{\substack{w:\text{SAW}, \\ x \longleftrightarrow y \\ |w|=n}} 1 \leq 2d(2d-1)^{n-1}.$$

This gives

$$\sum_{\substack{w:\text{SAW}, \\ x \longleftrightarrow y}} (C_s/\lambda^s)^{|w|+1} \leq \sum_{n \geq |x-y|} (C_s \lambda^{-s})^{n+1} 2d(2d-1)^{n-1}$$

$$= \frac{2d}{(2d-1)^2} \frac{[(2d-1)C_s \lambda^{-s}]^{|x-y|+1}}{1 - (2d-1)C_s \lambda^{-s}}$$

which is the right hand side of (26).

21

It remains to prove (37). Applying (36) to the self-avoiding random walk representation from Lemma 4.3 and taking the expectation with respect to the random potential yields the bound

$$\mathbb{E}[|G_B(x,y;z)|^s] \leq \sum_{\substack{w:\text{SAW in } B \\ x \longleftrightarrow y}} \mathbb{E}[\prod_{j=0}^{|w|} |G_{B_j}(w(j),w(j);z)|^s]. \tag{38}$$

We evaluate the expectation on the right hand side of (38) successively with the help of conditional expectations with respect to the random potential visited along the path of the self-avoiding walk w: Take first the expectation with respect to $v(x) = v(w(0))$ and note that the only Green's function which depends on on $v(w(0))$ is $G_{B_0}(w(0),w(0);z)$. Thus

$$\mathbb{E}[\prod_{j=0}^{|w|} |G_{B_j}(w(j),w(j);z)|^s| \, v(w(0))]$$
$$= \mathbb{E}[|G_{B_0}(w(0),w(0);z)|^s|v(w(0))] \prod_{j=1}^{|w|} |G_{B_j}(w(j),w(j);z)|^s \tag{39}$$

Recalling the rank-one perturbation formula (13) one can bound the conditional expectation on the right hand side of (39) simply by C_s/λ^s. Thus

$$\mathbb{E}[\prod_{j=0}^{|w|} |G_{B_j}(w(j),w(j);z)|^s| \, v(w(0))] \leq \frac{C_s}{\lambda^s} \prod_{j=1}^{|w|} |G_{B_j}(w(j),w(j);z)|^s \tag{40}$$

Now take the conditional expectation of (40) with respect to $v(w(1))$ and note that all factors $G_{B_j}(w(j),w(j);z)$ with $j \geq 2$ can again be taken out of the expectation since they do not depend on $v(w(1))$. Again one uses the a-priori bound $\mathbb{E}[|G_{B_1}(w(1),w(1);z)|^s|v(w(1))] \leq C_s/\lambda^s$ to see

$$\mathbb{E}[\prod_{j=0}^{|w|} |G_{B_j}(w(j),w(j);z)|^s| \, v(w(0)),v(w(1))] \leq \left(\frac{C_s}{\lambda^s}\right)^2 \prod_{j=2}^{|w|} |G_{B_j}(w(j),w(j);z)|^s \tag{41}$$

Iterating this procedure $|w| + 1$ times yields the bound

$$\mathbb{E}[\prod_{j=0}^{|w|} |G_{B_j}(w(j),w(j);z)|^s|] \leq \left(\frac{C_s}{\lambda^s}\right)^{|w|+1}$$

which together with (38) gives (37). This ends the proof of Theorem 4.1

References

Aizenman, M., A. Elgart, S. Naboko, J. H. Schenker, and G. Stolz (2006). Moment analysis for localization in random Schrödinger operators. *Invent. Math. 163*(2), 343–413.

Aizenman, M. and G. M. Graf (1998). Localization bounds for an electron gas. *J. Phys. A 31*(32), 6783–6806.

Aizenman, M. and S. Molchanov (1993). Localization at large disorder and at extreme energies: an elementary derivation. *Comm. Math. Phys. 157*(2), 245–278.

Aizenman, M. and C. M. Newman (1984). Tree graph inequalities and critical behavior in percolation models. *J. Statist. Phys. 36*(1-2), 107–143.

Aizenman, M., J. H. Schenker, R. M. Friedrich, and D. Hundertmark (2000). Constructive fractional-moment criteria for localization in random operators. *Phys. A 279*(1-4), 369–377. Statistical mechanics: from rigorous results to applications.

Aizenman, M., J. H. Schenker, R. M. Friedrich, and D. Hundertmark (2001). Finite-volume fractional-moment criteria for Anderson localization. *Comm. Math. Phys. 224*(1), 219–253. Dedicated to Joel L. Lebowitz.

Aizenman, M., R. Sims, and S. Warzel (2006). Fluctuation-based proof of the stability of ac spectra of random operators on tree graphs. In *Quantum graphs and their applications*, Volume 415 of *Contemp. Math.*, pp. 1–14. Providence, RI: Amer. Math. Soc.

Anderson, P. W. (1958, Mar). Absence of diffusion in certain random lattices. *Phys. Rev. 109*(5), 1492–1505.

Bellissard, J., A. van Elst, and H. Schulz-Baldes (1994). The noncommutative geometry of the quantum Hall effect. *J. Math. Phys. 35*(10), 5373–5451. Topology and physics.

Bourgain, J. and C. E. Kenig (2005). On localization in the continuous Anderson-Bernoulli model in higher dimension. *Invent. Math. 161*(2), 389–426.

Carmona, R., A. Klein, and F. Martinelli (1987). Anderson localization for Bernoulli and other singular potentials. *Comm. Math. Phys. 108*(1), 41–66.

Cycon, H. L., R. G. Froese, W. Kirsch, and B. Simon (1987). *Schrödinger operators with application to quantum mechanics and global geometry* (Study ed.). Texts and Monographs in Physics. Berlin: Springer-Verlag.

del Rio, R., S. Jitomirskaya, Y. Last, and B. Simon (1995, Jul). What is localization? *Phys. Rev. Lett. 75*(1), 117–119.

del Rio, R., S. Jitomirskaya, Y. Last, and B. Simon (1996). Operators with singular continuous spectrum. IV. Hausdorff dimensions, rank one perturbations, and localization. *J. Anal. Math. 69*, 153–200.

Froese, R., D. Hasler, and W. Spitzer (2007). Absolutely continuous spectrum for the Anderson model on a tree: a geometric proof of Klein's theorem. *Comm. Math. Phys. 269*(1), 239–257.

Fröhlich, J. and T. Spencer (1983). Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Comm. Math. Phys. 88*(2), 151–184.

Germinet, F., P. Hislop, and A. Klein (2005). On localization for the Schrödinger operator with a Poisson random potential. *C. R. Math. Acad. Sci. Paris 341*(8), 525–528.

Germinet, F. and A. Klein (2004). A characterization of the Anderson metal-insulator transport transition. *Duke Math. J. 124*(2), 309–350.

Grimmett, G. (1999). *Percolation* (Second ed.), Volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Berlin: Springer-Verlag.

Hammersley, J. M. (1957). Percolation processes: Lower bounds for the critical probability. *Ann. Math. Statist. 28*, 790–795.

Hundertmark, D. (2000). On the time-dependent approach to Anderson localization. *Math. Nachr. 214*, 25–38.

Hunziker, W. and I. M. Sigal (2000). The quantum N-body problem. *J. Math. Phys. 41*(6), 3448–3510.

Kirsch, W. (1989). Random Schrödinger operators. A course. In *Schrödinger operators (Sønderborg, 1988)*, Volume 345 of *Lecture Notes in Phys.*, pp. 264–370. Berlin: Springer.

Kirsch, W. and B. Metzger (2007). The integrated density of states for random Schrödinger operators. In *Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday*, Volume 76 of *Proc. Sympos. Pure Math.*, pp. 649–696. Providence, RI: Amer. Math. Soc.

Klein, A. (1998). Extended states in the Anderson model on the Bethe lattice. *Adv. Math. 133*(1), 163–184.

Kunz, H. and B. Souillard (1980/81). Sur le spectre des opérateurs aux différences finies aléatoires. *Comm. Math. Phys. 78*(2), 201–246.

Lieb, E. H. (1980). A refinement of Simon's correlation inequality. *Comm. Math. Phys. 77*(2), 127–135.

Lifshits, I. M., S. A. Gredeskul, and L. A. Pastur (1988). *Introduction to the theory of disordered systems*. A Wiley-Interscience Publication. New York: John Wiley & Sons Inc. Translated from the Russian by Eugene Yankovsky [E. M. Yankovskiĭ].

Martinelli, F. and E. Scoppola (1985). Remark on the absence of absolutely continuous spectrum for d-dimensional Schrödinger operators with random potential for large disorder or low energy. *Comm. Math. Phys. 97*(3), 465–471.

Minami, N. (1996). Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. *Comm. Math. Phys. 177*(3), 709–725.

Molchanov, S. A. (1980/81). The local structure of the spectrum of the one-dimensional Schrödinger operator. *Comm. Math. Phys. 78*(3), 429–446.

Pastur, L. A. (1980). Spectral properties of disordered systems in the one-body approximation. *Comm. Math. Phys. 75*(2), 179–196.

Simon, B. (1980). Correlation inequalities and the decay of correlations in ferromagnets. *Comm. Math. Phys. 77*(2), 111–126.

Simon, B. (1990). Absence of ballistic motion. *Comm. Math. Phys. 134*(1), 209–212.

Simon, B. and T. Wolff (1986). Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians. *Comm. Pure Appl. Math. 39*(1), 75–90.

Stollmann, P. (2001). *Caught by disorder*, Volume 20 of *Progress in Mathematical Physics*. Boston, MA: Birkhäuser Boston Inc. Bound states in random media.