Another look at the SU(2) anomaly

F.R. Klinkhamer [1]
CHEAF/NIKHEF-H, P.O. Box 41882, NL-1009 DB Amsterdam, The Netherlands

Received 23 November 1990

We give a simple derivation of Witten's global SU(2) anomaly.

Witten [1] discovered a global anomaly in SP(n) gauge theories coupled to an odd number of Weyl fermions, the simplest case being that of an SU(2) [=Sp(1)] gauge theory with a single doublet of left-handed fermions. Another derivation, which connects to the chiral U(1) anomaly, has been given by Goldstone [2]. However, doubts as to the validity of these results have been expressed by Banerjee et al. [3]. In this letter we will give an explicit calculation which confirms Witten's result. Our derivation is quite simple as it relates to the well-known *perturbative* non-abelian anomaly of Bardeen [4].

First, we sketch Witten's derivation of the SU(2) anomaly. Consider the effective action of a left-handed fermion ψ_L coupled to a background SU(2) gauge field A:

$$\exp[-W(A)] = \int d\psi\, d\bar{\psi} \exp\left(-\int d^4x\, \bar{\psi}_L i\gamma^\mu(\partial_\mu + A_\mu)\psi_L\right). \tag{1}$$

Witten [1] has argued that under a finite gauge transformation with parameter $U(x)$ in the non-trivial class of $\pi_4[SU(2)] = Z_2$ the Weyl "determinant" changes sign,

$$\exp[-W(A^U)] = (-1)\exp[-W(A)], \tag{2}$$

where $A^U = U(A+d)U^{-1}$. His elegant argument starts from a direct interpolation between A and A^U, for example

$$\mathscr{A} = (1-\tau)A + \tau A^U, \tag{3}$$

where $\tau \in [0,1]$ is like a five-dimensional time. He then establishes the result (2) by use of a mod 2 index theorem [5] and the assumption of adiabaticity (with a new variable τ running from $-\infty$ to ∞). Precisely this last assumption has been criticized in ref. [3] because of the fact that the energy levels of the Dirac operator cross.

Now we turn to our analysis of the problem. It will suffice to establish (2) for a single configuration A, for which we choose $A=0$. Our strategy is to make a different interpolation $\tilde{\mathscr{A}}$ between $A=0$ and $A^U = -dU\,U^{-1}$ which involves gauge transformed fields only. Of course, this is impossible as long as we have SU(2) gauge fields. Instead, we embed the gauge group G = SU(2) in a group $\tilde{G}$, SU(3) say, for which $\pi_4(\tilde{G})$ *is* trivial. Concretely, $U \in SU(2)$ is embedded in $\tilde{G} = SU(3)$ as follows:

$$\tilde{U} = \begin{pmatrix} U_{11} & U_{12} & 0 \\ U_{21} & U_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

and the left-handed fermion $\tilde{\psi}_L$ is in the fundamental representation of SU(3) $(\mathbf{3} = \mathbf{2} + \mathbf{1})$. [For G = Sp($n$), $n > 1$, we take $\tilde{G} = SU(2n) \supset Sp(n)$ and $\tilde{\psi}_L$ in the representation $\mathbf{2n} = \mathbf{2n}$.] The interpolating field (parameter $\rho \in [0,1]$) is then

$$\tilde{\mathscr{A}} = -d\Omega\,\Omega^{-1}, \tag{5}$$

where

$$\Omega(\rho, x) \in SU(3), \quad \Omega(0,x) = 1, \quad \Omega(1,x) = \tilde{U}.$$

[1] KNAW Fellow.

As promised, all interpolating fields (5) are pure gauge (but in a larger group though). Now it is very simple to calculate the change in the fermion effective action (1) with fermions $\tilde{\psi}_L$ and gauge fields $\tilde{\mathscr{A}}$: $W(-\mathrm{d}\tilde{U}\tilde{U}^{-1})-W(\tilde{0})$ is the cumulative effect of the perturbative non-abelian anomaly [4]. Furthermore, for fields of the form (4), which are essentially in SU(2), we have $W(-\mathrm{d}\tilde{U}\tilde{U}^{-1})-W(\tilde{0})=W(-\mathrm{d}UU^{-1})-W(0)$. Using the results and notation of ref. [6] we find [with $W(0)\equiv 0$]

$$W(-\mathrm{d}UU^{-1}) = \mathrm{i}\left(\frac{-\mathrm{i}}{24\pi^2}\right)\int_0^1 \delta\rho \int_{S^4} \omega_4^1(-\partial_\rho\Omega\,\Omega^{-1}, -\mathrm{d}\Omega\,\Omega^{-1}) , \tag{6}$$

where the perturbative anomaly is given by

$$\omega_4^1(\tilde{v},\tilde{A}) = \mathrm{tr}[\tilde{v}d(\tilde{A}\,\mathrm{d}\tilde{A}+\tfrac{1}{2}\tilde{A}^3)] .$$

Since the fields at $\rho=0$ do not depend on $x\in S^4$ the integral in (6) is really over a five-dimensional ball with S^4 as its boundary and we have (D is the exterior derivative on the five-dimensional manifold)

$$W(-\mathrm{d}UU^{-1}) = \mathrm{i}\left(\frac{-\mathrm{i}}{240\pi^2}\right)\int \mathrm{tr}[(-\mathrm{D}\Omega\,\Omega^{-1})^5] = \mathrm{i}\pi \quad (\mathrm{mod}\ 2\pi\mathrm{i}) , \tag{7}$$

which confirms the existence of the global anomaly (2). Actually, the right-hand side of (7) was already calculated by Witten [7] in the context of Wess–Zumino actions; what we have shown here is how this term *arises*.

We make three further comments:

(1) Note that if U were in the trivial class of $\pi_4[\mathrm{SU}(2)]$ the interpolation $\tilde{\mathscr{A}}$ would essentially be in SU(2) and the integral in (7) would vanish. By the same argument it follows [7] that the effective action (7) is a homotopy invariant over SU(2).

(2) It may be instructive to compare how the Weyl "determinant" moves from 1 to -1 in Witten's derivation and ours: for Witten's interpolation (3) the determinant stays real and passes through zero, while for our interpolation (5) the determinant runs along the unit circle in the complex plane.

(3) Having established the global SU(2) anomaly we can turn the argument around and try to find groups $\tilde{\mathrm{G}}$ with anomalies. It turns out, perhaps not unexpectedly, that we only recover the known perturbative non-abelian anomalies [4].

To summarize, we have given yet another derivation of Witten's SU(2) anomaly. This should lay to rest the doubts expressed in ref. [3]. It is interesting to note that we now have three derivations of the global SU(2) anomaly based on, respectively, the mod 2 index theorem, the chiral U(1) anomaly and the perturbative non-abelian anomaly.

We thank J.W. van Holten and R.H. Rietdijk for discussions. This research has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences (KNAW).

Note added. After we submitted this letter we became aware of an earlier paper by Elitzur and Nair [8] that uses the same embedding trick to establish Witten's anomaly. In view of the recent claims [3] for the absence of the anomaly it may be worthwhile to present (once more) this different derivation.

References

[1] E. Witten, Phys. Lett. B 117 (1982) 324.
[2] J. Goldstone, quoted by R. Jackiw, in: Relativity, groups and topology II (Les Houches, 1983), eds. B. DeWitt and R. Stora (North-Holland, Amsterdam, 1984);
see also S. de Alwis, Phys. Rev. D 32 (1985) 2837;
P. Nelson and L. Alvarez-Gaumé, Commun. Math. Phys. 99 (1985) 103.
[3] H. Banerjee, G. Bhattacharya and R. Banerjee, Z. Phys. C 45 (1989) 253.
[4] W. Bardeen, Phys. Rev. 184 (1969) 1848.
[5] M. Atiyah and I. Singer, Ann. Math. 93 (1971) 139.
[6] B. Zumino, in: Relativity, groups and topology II (Les Houches, 1983), eds. B. DeWitt and R. Stora (North-Holland, Amsterdam, 1984).
[7] E. Witten, Nucl. Phys. B 223 (1983) 433.
[8] S. Elitzur and V. Nair, Nucl. Phys. B 243 (1984) 205.