15

The Renormalization Group

15.1 Scale dependence in Quantum Field Theory and in Statistical Physics

In earlier chapters we used perturbation theory to compute the N-point functions in field theory, focusing on ϕ^4 theory. There we used a time-honored approach which has been quite successful in theories such as quantum electrodynamics. Already at low orders in this expansion we found difficulties in the form of divergent contributions at every order. In comparison with Quantum Mechanics, this problems may seem unusual and troubling. In Quantum Mechanics Rayleigh-Schrödinger perturbation theory is finite term by term in perturbation theory, although the resulting series may not be convergent, e.g. an expansion in powers of the coupling constant of the non-linear term in an anharmonic oscillator. Instead, in Quantum Field Theory, one generally has divergent contributions at every order in perturbation theory. In order to handle these divergent contributions one introduces a set of effective (or renormalized) quantities. In this approach, the singular behavior is hidden in the relation between bare and renormalized quantities. This approach looks like one is sweeping the problem under a rug and, in fact, it is so.

Quite early on (in 1954), Murray Gell-Mann and Francis Low (Gell-Mann and Low, 1954) proposed to take a somewhat different look at the renormalization process in Quantum Field Theory (QED in their case), by recasting the renormalization of the coupling constant as the solution of a differential equation. The renormalization process tells us that we can change the bare coupling λ and the UV cutoff Λ while, at the same time, keeping the renormalized coupling constant fixed at some value g. This process defines a function, the Gell-Mann-Low beta-function $\beta(\lambda)$,

$$\beta(\lambda) = \Lambda \frac{\partial\lambda}{\partial\Lambda}\Big|_g \tag{15.1}$$

which can be calculated from the perturbation theory diagrams. In fact, integrating this equation amounts to summing over a large class of diagrams at a given order of the loop expansion. The physical significance of this approach, which was subsequently and substantially extended by Bogoliubov and Shirkov , remained obscure for quite some time. Things changed in a fundamental way with the work of Kenneth Wilson in the late 1960's.

Wilson proposed to take a different view on the meaning of the divergencies found in Quantum Field Theory (Wilson, 1983). He argued that these divergent contributions have a physical origin. In a *Classical* Field Theory, the theory is defined by a partial differential equation, such as Maxwell's electrodynamics. In this case, every dimensionful quantity is determined by the dimensionful parameters present in the partial differential equation. In other words, in a classical theory the dimensionful parameters of the equation of motion are the *only* physical scales. In contrast, in a Quantum Field Theory, and in Statistical Mechanics, we cannot isolate a *single* scale as being responsible for the physical behavior of macroscopic quantities. For example, when we consider the one-loop corrections to the 4-point vertex function $\Gamma^{(4)}$, we find that it contains an integral of the form

$$\int^{\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + m_0^2)^2} \tag{15.2}$$

where we introduced a regulator Λ to cutoff the contributions from field configurations with momenta $|p| > \Lambda$. By inspection of this integral, we see that the corrections to the effective (or renormalized) coupling constant has contributions not from a single momentum scale $|p|$ (or length scale $|p|^{-1}$) but from the *entire* range of momenta

$$\xi^{-1} = m_0 \leq |p| \leq \Lambda \tag{15.3}$$

or, equivalently, in term of length scales, from the range of distances

$$a = \Lambda^{-1} \leq |x| \leq \xi = m_0^{-1} \tag{15.4}$$

What the integral is telling us is that we must add up all the contributions within this range. However, on one hand, clearly the contributions at small momenta, $|p| \approx 0$, are important since in that range the denominator of the integrand is smallest and, hence, where the integrand is largest. On the other hand, the contribution on momentum scales close to the cutoff,

$|p| \approx \Lambda$, are even bigger since the phase space is growing by a factor of Λ^D. Thus, we *should* expect divergencies to occur since there is no physical mechanism to stop fluctuations on length scales between a short distance cutoff a and a macroscopic scale ξ from happening. It is important to note that this sensitivity to the definition of the theory at short distances will still be present even if local field theory is replaced at even shorter distances by some other theory, perhaps more fundamental, such as String Theory. At any rate, in all cases, there will remain a sensitivity on the definition of the theory at some short-distance scale used to define the local field theory. We will see shortly that these very same issues arise in the theory of phase transitions in Statistical Physics.

In Quantum Field Theory, in Euclidean space time, the N-point functions are given in terms of the path integral which for a scalar field ϕ has the form

$$Z[J] = \int \mathcal{D}\phi e^{-S_\Lambda[\phi]+\int d^D x J\phi} \tag{15.5}$$

where $S_\Lambda[\phi]$ is the Euclidean action for a theory defined with a regulator $\Lambda = a^{-1}$. To define a theory without a regulator amounts to take the limit $\Lambda \to \infty$ or, equivalently, to take the limit in which short-distance cutoff $a \to 0$. If one regards the short-distance cutoff as a lattice spacing, this is the same as defining a *continuum limit*.

In some sense, the procedure of removing the UV regulator from a field theory is reminiscent to the definition the integral of a real function on a finite interval as the limit of a Riemann sum. However, while in the case of the integral the limit exists, provided the function has bounded variation, in the case of a functional integral the problem is more complex. The problem of defining a field theory without a regulator is then reduced to understand when, and how, it is possible to take such a limit. One can regard the field theory as being defined with an UV cutoff a, e.g., the “lattice spacing”, and define a process of taking the continuum limit as the limit of a sequence of theories defined at progressively smaller UV length scales, say from scale $a = \Lambda^{-1}$, to $\frac{a}{2} = \frac{2}{\Lambda}$, to $\frac{a}{4} = \frac{4}{\Lambda}$, etc. In each step, the number of local degrees of freedom is doubled, the grid on which the degrees of freedom are defined becomes progressively more dense. In the limit of infinitely many such steps, it becomes a continuum, see Fig.15.1. Kenneth Wilson had the deep insight to realize that for this continuum limit of a Quantum Field Theory to exist it is necessary to *tune* the field theory to a particular value of its coupling constants at which the physical length scale of the theory, i.e. the correlation length ξ, should diverge in units of the short-distance UV cutoff a, $\xi \gg a$ (Wilson, 1983). Or, what is the same, we need to find a regime in which

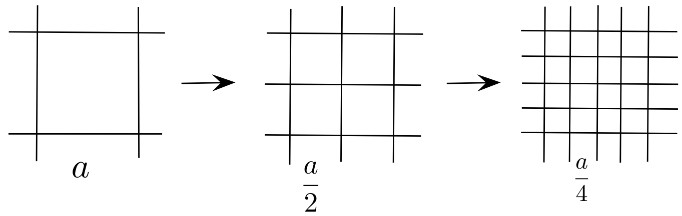


Figure 15.1 Taking the continuum limit: in each step the number of degrees of freedom is doubled, the UV scale a is halved, and the grid becomes progressively denser.

the mass gap $M = \xi^{-1}$ is much smaller than the UV momentum cutoff Λ, $M \ll \Lambda$. Phrased in this fashion, the problem of defining a quantum field theory is reduced to finding a regime in which the physical correlation length is divergent. However, as we saw before, this is the same as the problem of finding a *continuous* phase transition in Statistical Physics!. Thus, the two problems are one and the same thing!.

15.2 RG Flows, Fixed points and Universality

In parallel, but independently from these developments of these ideas in Quantum Field Theory, the problem of the behavior of physical systems near a continuous phase transition was being reexamined and developed by several people, notably by Widom (Widom, 1965) and Fisher (Fisher, 1967), Patashinskii and Pokrovskii (Patashinskii and Pokrovskii, 1966), and, particularly, by Kadanoff (Kadanoff, 1966). Near a continuous phase transition, "second order" in Landau's terminology, certain physical quantities such as the specific heat and the magnetic susceptibility, for a phase transition in a magnet, should generally be divergent as the critical temperature, T_c, is approached. At T_c, the correlation length should be divergent. Using phenomenological arguments, they realized that that these properties can de described by a singular contribution to the free energy density f_{sing}, that has a *scaling form*, i.e. it is a *homogeneous function* of the reduced temperature, $t = (T - T_c)/T_c$ and of the external magnetic field H (in dimensionless units). Implicit in these assumptions was the at the critical temperature T_c, where the correlation length diverges, the system exhibits *scaling* or, what is the same, the system acquired an emergent symmetry: *scale invariance.*

Kadanoff formulated these ideas in terms of the following physically intuitive picture (Kadanoff, 1966). Consider for instance the simplest model of a magnet, the Ising model on a square lattice of spacing a. On each site one has an Ising spin: a degree of freedom that can take only two values, $\sigma = \pm 1$. The partition function of this problem is a sum over all the configurations of spins, $[\sigma]$ weighed by the Boltzmann probability for each configuration at temperature T

$$Z = \sum_{[\sigma]} \exp(-S[\sigma]/T) \tag{15.6}$$

In the Ising model, the Euclidean action $S[\sigma]$ is just the interaction energy of the spins,

$$S[\sigma] = -J \sum_{<i,j>} \sigma(i)\sigma(j) \tag{15.7}$$

where $< i, j >$ are nearest neighboring sites of the lattice, J is an energy scale (the exchange constant) and T is the temperature. In what follows, the energy of the classical statistical mechanical system will be called the "Euclidean action".

In a system with N sites, there are 2^N configurations of spins. The configurations that contribute predominantly at large distance scales, $|x| \gg a$, should be smooth at those scales. In particular, the physics at long distances should not depend on a sensitive way on the physics at short distances. In fact, configurations that vary rapidly at short distances should tend ot average-out, and hence should give small contributions to the long distance behavior. To make this picture concrete, Kadanoff proposed an iterative procedure for the computation of the partition function which will progressive sum over the short-distance degrees of freedom leaving behind an effective, renormalized, theory for the long distance degrees of freedom. This procedure is an example of a renormalization group (RG) transformation.

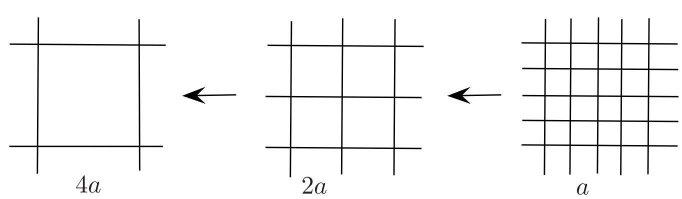


Figure 15.2 Coarse-graining the short distance physics: in each step the number of degrees of freedom is halved, the UV scale a is doubled, and the grid becomes progressively more sparse.

15.2.1 Block-spin Transformations

Beginning with a system with lattice spacing a, we will divide it into a set of block spins so that the new, coarse grained, system will have lattice spacing $2a$. We will then define a new effective spin for each block and compute their effective Hamiltonian by tracing over ("integrating-out") the the rapidly varying configurations at scale a. In effect, this is the same as the procedure for the construction of the continuum field theory shown in Fig.15.1, except that now we run the procedure backwards, from short distances to long distances (see Fig.15.2). To this end, let us divide the system into cells, the block spins, and let $\mathcal{A}$ be one of these cells. We define an effective degree of freedom μ for each cell

$$\mu = \frac{\sum_{i\in\mathcal{A}} \sigma(i)}{||\sum_{i\in\mathcal{A}} \sigma(i)||} \tag{15.8}$$

which represents the average over the configurations $\{\sigma\}$ that vary rapidly on the scale of the cell. Then the configurations on the block spins $\{\mu\}$, defined on a coarse-grained system with a new larger lattice spacing, say $2a$, vary smoothly on short scales but not on long scales.

The next step is to define en effective Hamiltonian for the block spins $\{\mu\}$ through a block-spin transformation $T[\mu|\sigma]$ such that

$$\sum_{\{\mu\}} T[\mu|\sigma] = 1 \tag{15.9}$$

For example, one could take

$$T[\mu|\sigma] = \delta\left(\mu - \frac{\sum_{i\in\mathcal{A}} \sigma(i)}{||\sum_{i\in\mathcal{A}} \sigma(i)||}\right) \tag{15.10}$$

Once a specific transformation is chosen, the effective theory of the block spins is obtained by inserting Eq.(15.9) in the partition function:

$$\begin{aligned} Z &= \sum_{\{\sigma\}} e^{-S[\sigma]} \\ &= \sum_{\{\mu\}} \sum_{\{\sigma\}} T[\mu|\sigma]\, e^{-S[\sigma]} \end{aligned} \tag{15.11}$$

which leads to the definition of an effective action, $S_{\text{eff}}[\mu]$ of the block spins

$$e^{-S_{\text{eff}}[\mu]} = \sum_{\{\sigma\}} T[\mu|\sigma]\, e^{-S[\sigma]} \tag{15.12}$$

The partition function of the coarse-grained system is obviously the same as that of the original system

$$Z = \sum_{\{\mu\}} e^{-S_{\text{eff}}[\mu]} \tag{15.13}$$

even though, in general, the actions are not, $S_{\text{eff}}[\mu] \neq S[\sigma]$. As it is apparent from this formal discussion, the form of the effective action S_{eff} depends on the block transformation chosen. Nevertheless, since S_{eff} results from summing over the contributions of a finite subset of degrees of freedom, it is a finite expression that can be written in terms of a set of local operators with effective (renormalized) coefficients.

Thus this coarse-graining procedure maps a system with lattice spacing a (the UV cutoff) with degrees of freedom σ to another system with a larger lattice spacing, say $2a$, for the new degrees of freedom μ, while keeping the partition functions the same. As a result, the two-point correlation function of the coarse-grained degrees of freedom is related to the correlators of the old degrees of freedom by an expression of the form

$$\langle \mu(\boldsymbol{R})\mu(\boldsymbol{R}') \rangle_{S_{\text{eff}}} = \left\langle \frac{\sum_{i\in\mathcal{A}(\boldsymbol{R})} \sigma(i)}{||\sum_{i\in\mathcal{A}(\boldsymbol{R})} \sigma(i)||} \; \frac{\sum_{i\in\mathcal{A}(\boldsymbol{R}')} \sigma(i)}{||\sum_{i\in\mathcal{A}(\boldsymbol{R}')} \sigma(i)||} \right\rangle_S \tag{15.14}$$

Here $\boldsymbol{R}$ and $\boldsymbol{R}'$ are the locations of the two block spins. The right hand side of Eq.(15.14) is computed in a theory with lattice spacing a and the left hand side has lattice spacing $a' = ba$ ($b = 2$ in the above example). Now, if the degrees of freedom σ of the theory of the right-hand-side has correlation length $\xi[\sigma]$, and the correlation length of the coarse-grained degrees of freedom μ is $\xi[\mu]$, since the theories are equivalent then they must be related simply by a change in scale

$$\xi[\mu] = \frac{1}{b}\xi[\sigma] \tag{15.15}$$

simply because the theory has b times fewer degrees of freedom. To account for the change in scale is necessary in oder to be able to compare the units of both theories (i.e. measurements must be with the same ruler!).

15.2.2 RG Flows and Fixed Points

We will now assume that it will always be possible to define a complete set of (conveniently normalized) *local operators*, that we will denote by $O_\alpha[\sigma]$, in terms of which a general action is

$$S[\sigma] = \sum_\alpha h_\alpha O_\alpha[\sigma] \tag{15.16}$$

where h_α are the coupling constants. For example, in the case of an Ising model, the local operators include the identity operator, the spin operator, the nearest-neighbor interactions, next-nearest-neighbor interactions, three and four spin interactions, etc. We will assume that the operators $O_\alpha[\sigma]$ are complete in a sense that we will specify below.

We will also assume that the theory of the corse-grained degrees of freedom μ is defined by the *same* set of local operators $O_\alpha[\mu]$,

$$S_{\text{eff}}[\mu] = \sum_\alpha h_\alpha^{\text{eff}}(b) O_\alpha[\mu] \tag{15.17}$$

where $h_\alpha^{\text{eff}}(b)$ are a set of *renormalized* interactions that depend on $b = a'/a$, the change in the scale of the UV cutoff. The set of operators $\{O_\alpha[\sigma]\}$ is complete in the sense that its elements are the set of possible local operators generated under the RG transformation.

A consequence of this construction is that the coupling constants are no longer fixed but are different at different scales. This result implies that the RG transformation induces a mapping in the space of coupling constants. The repeated action of the RG then can be viewed as a *flow* in this space: the RG flow.

Let us suppose, for the moment, that we have been clever enough to choose a transformation such that the renormalized couplings are related to the old ones by a *homogeneous* transformation of the scale, i.e.

$$h_\alpha^{\text{eff}}(b) = b^{y_\alpha} h_\alpha \tag{15.18}$$

which, implicitly, includes a change in length scale. The exponents y_α describe how the couplings transform under the change of scale from a to ba. We now see that if the exponent $y_\alpha > 0$, then the coupling will be *larger* in the coarse-grained system (with scale ba), $h_\alpha^{\text{eff}} > h_\alpha$. Conversely, if the exponent $y_\alpha < 0$, then the coupling will be *smaller* in the coarse-grained system, $h_\alpha^{\text{eff}} < h_\alpha$. We will say that an operator for which $y_\alpha > 0$ is a *relevant* operator while an operator with $y_\alpha < 0$ is an *irrelevant* operator. The RG flow of the coupling constant h_α is given by the beta-function $\beta(h_\alpha)$, which in this context is defined as

$$\beta(h_\alpha) \equiv (h_\alpha^{\text{eff}}(b) - h_\alpha)/\ln b \tag{15.19}$$

which measures the rate of change of the coupling h_α for a logarithmic change in the UV scale, $\ln b = \ln(a'/a)$. For $b \to 1$, the beta-function reduces to

$$\beta(h_\alpha) = \frac{\partial h_\alpha}{\partial \ln b} = y_\alpha h_\alpha + O(h_\alpha^2) \tag{15.20}$$

which we will call the "tree-level" beta-function. The special case of $y_\alpha = 0$ defines a *marginal* operator and the beta-function is given by higher order terms that we have not yet included.

It is now clear that if we keep repeating this transformation n times, then in the limit $n \to \infty$ the contribution of the irrelevant operators disappears from the renormalized theory. In that limit, the Euclidean action of the effective theory contains only marginal and relevant operators. Following Wilson, we are led to conjecture that there should be theories which are *invariant* under the action of these transformations, called *renormalization group* transformations. Such theories are said to be *fixed points* of the renormalization group, and we will denote their action by S^*. Then, a generic *renormalized* theory will have an Euclidean action of the form

$$S = S^* + \sum_\alpha h_\alpha \int d^D x \, O_\alpha[\sigma(x)] \tag{15.21}$$

which includes the fixed-point action, S^*, plus the set of, marginal and relevant operators.

The existence of theories described by such fixed points of the renormalization group, conjectured by Wilson, is the key to understanding how to define a quantum field theory. A theory described by such a fixed-point action S^* cannot depend on any microscopic scales, such as the UV cutoff (the lattice spacing a). Thus, at a fixed point there no scales left in the theory, and the theory becomes *invariant* under changes scale, i.e. invariant under scale transformations. In order to achieve such a fixed point it is necessary to rescale the lattice spacing ba of the renormalized theory described by S_{eff} back to what is was in the "bare" theory described by S. Hence, all lengths must be divided by b.

We summarize the construction of a renormalization group transformation by the following two steps: a) a block-spin (or coarse-graining) transformation that eliminates a finite fraction of local degrees of freedom, followed by b) a rescaling of all lengths. We should note that an RG transformations is for us to choose and different choices, i.e. different ways of averaging over the short-distance physics, can always be made.

There are, as we noted, two types of fixed points depending on whether the correlation length is zero or infinite. Clearly, the short-distance physics cannot be eliminated from the theory if the fixed point has zero correlation length. We will see that, in spite of this, such fixed points are important since they turn out to label the different possible *phases* (or types of vacua in field theory language) of a theory. One such example is a theory with a global symmetry in its spontaneously broken symmetry phase: the effective field theory of such a phase contains dimensionful quantities determined by the physics at some short-distance scale.

On the other hand, if the fixed point has a divergent (infinite) correlation length, the definition of the theory at short distances should become irrelevant at long distances, and the information on the short-distance physics disappears from a set of dimensionless quantities which, in this sense, become *universal*. This observation leads to the concept of the existence of *universality classes* of theories which are characterized by fixed points with the same universal properties. Once a fixed-point of this type can be reached, the theory has a new, emergent, symmetry: *scale invariance*. We will see shortly that, under most circumstances of physical interest, it can be extended to an larger emergent symmetry: *conformal invariance*. Importantly, a theory at a non-trivial fixed point and its set of marginal and relevant operators *defines* a renormalizable quantum field theory. From this perspective, each fixed pint (and, hence, each universality class) defines a different quantum field theory.

Let us consider now some simple examples of RG flows and fixed points.

We consider first a theory with a fixed point S^* with one *relevant* perturbation:

$$S = S^* + h \int d^D x \, O[\sigma(x)] \tag{15.22}$$

Under the RG with change of scale $b = a'/a$, we obtain

$$S' = S^* + b^y h \int d^D x \, O[\mu(x)] \tag{15.23}$$

with $y > 0$, for a relevant operator. The resulting IR RG flow is shown in Fig.15.2.2. Here the fixed point at $h = 0$ is destabilized by the action of the relevant operator $O[\sigma(x)]$. Since we are flowing to long length scales we will call this an infrared unstable fixed point (IRU), known as a critical fixed point. Conversely, if we reverse the action of the RG and flow instead to short distances, we will say (equivalently!) that this an UV stable fixed point (known as a UV fixed point).

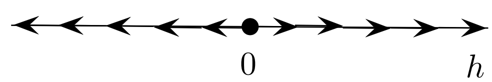


Figure 15.3 IR RG flow for one relevant operator. This is an infrared unstable fixed point (IRU) or, equivalently, an ultra-violet (UV) fixed point.

The opposite flow is obtained if the operator is *irrelevant* and $y < 0$. Now, the fixed point at $h = 0$ is stable in the IR (it is an IR stable fixed point) but unstable in the UV. This RG flow is shown in Fig.15.2.2. Notice that now the theory flows in the IR towards the fixed point at $h = 0$ for all $h \neq 0$. We will call the basin of attraction of the fixed points the set of values of the coupling for which the IR flow is towards the fixed point. In this sense an IR stable fixed point defines a phase of the theory while the IR unstable fixed point describes a continuous phase transition between two phases (to the right and to the left of the fixed point of Fig.15.2.2. The IR flows of the two phases converge to IR stable fixed points located at $h = \pm\infty$, respectively.

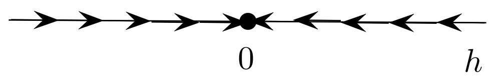


Figure 15.4 IR RG flow for one irrelevant operator. This is an infrared stable fixed point (IRS) or, equivalently, an ultra-violet (UV) unstable fixed point.

We will now consider a theory with two perturbations:

$$S = S^* + h_1 \int d^D x \, O_1[\sigma(x)] + h_2 \int d^D x \, O_2[\sigma(x)] \tag{15.24}$$

We have to consider three cases: a) that both operators are relevant, and hence, $y_1 > 0$ and $y_2 > 0$ (shown in Fig.15.2.2a), b) that both are irrelevant, with $y_1 < 0$ and $y_2 < 0$ (shown in Fig.15.2.2b), and c) that one operator has $y_1 > 0$ and is relevant, and the other has $y_2 < 0$ and is irrelevant (shown in Fig.15.2.2c). While the cases of Figs.15.2.2 a and b are simple extensions of the examples of 1D flows, the flows of the case of Fig.15.2.2c are richer. In this case we see that there is a manifold of values of the coupling constants that flows into the fixed point (driven by the flow of the irrelevant operator) while for other values it flows away from this fixed point (driven now by the relevant operator). In particular the manifold that flows into the fixed point is a separatrix of the flow: systems with bare values to the left and to the right of the separatrix flow to fixed points far away from the fixed point at the origin. Thus, a separatrix of the RG flow is a boundary between two different phases of the system, and it the locus of phase transitions between them.

15.2.3 Simple Examples of Block-Spin Transformations

To make this discussion more concrete we will now do some simple RG transformations, again using the Ising model as an example. The first example

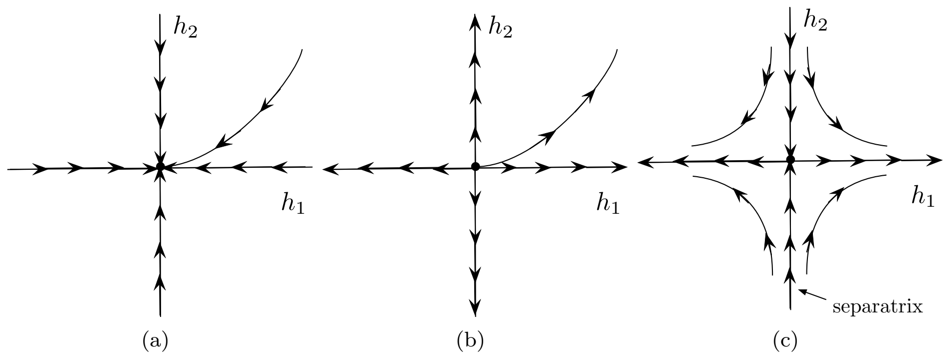


Figure 15.5 IR RG flows with a) for two irrelevant operators, b) two relevant operators, and c) one relevant and one irrelevant operator.

will be the trivial case of the Ising model in one dimension, a chain with N sites (with N even) with periodic boundary conditions. In this case the block spin transformation is quite simple: we will integrate out the spin degrees of freedom on the odd sublattice and the spins on the even sublattice, with $N/2$ sites, are the block spins. This RG transformation is known as decimation. Thus, we begin with the partition function

$$Z = \sum_{\{\sigma\}} e^{-S[\sigma]} = \sum_{\{\sigma\}} e^{\sum_{j=1}^{N} \frac{1}{T}\sigma_j \sigma_{j+1}} \tag{15.25}$$

with where we set the energy scale $J = 1$, such that $\sigma_{N+1} = \sigma_1$. Let us denote the spins on the even sublattice as $\sigma_{2r} \equiv \mu_r$ (with $r = 1, \ldots, N/2$) as the block spins. Thus, we begin with a system with N sites, lattice spacing a, and coupling constant $1/T$. We next integrate-out the degrees of freedom on the odd sites using that

$$\sum_{\sigma_{2r+1}=\pm 1} e^{\frac{1}{T}\sigma_{2r+1}(\mu_r + \mu_{r+1})} = e^{\alpha + \frac{1}{T'}\mu_r \mu_{r+1}} \tag{15.26}$$

where

$$\alpha = \ln 2 + \frac{1}{2}\ln\cosh\left(\frac{2}{T}\right), \qquad T' = \frac{2}{\ln\cosh\left(\frac{2}{T}\right)} \tag{15.27}$$

Therefore, the effective action of the block spins $\{\mu_r\}$ is

$$S' = -\frac{N}{2}\alpha - \frac{1}{T'}\sum_{r=1}^{N/2} \mu_r \mu_{r+1} \tag{15.28}$$

Thus, after the RG transformation we obtain an action with the same form as S (up to a term proportional to the identity operator) but with a renormalized coupling T'. The change in the coupling (the temperature), $\delta T = T' - T$, for low values of $T \to 0$, is given by the beta-function, $\beta(T)$

$$\beta(T) = \frac{\delta T}{\ln 2} = \frac{T^2}{2} + O(T^3) \tag{15.29}$$

where $\ln 2 = \ln(a'/a)$ is the log of the change in the UV scale.

At a fixed point, T^*, we must have $T'^* = T^*$ or, what is the same, $\delta T^* = 0$. This clearly happens only at $T = T^* = 0$. Hence, the only finite fixed point (i.e., aside from $T^* \to \infty$) is at $T^* = 0$. Moreover, since the beta-function of Eq.(15.29) is always positive, the $T^* = 0$ fixed point is unstable and the theory flows to the strong coupling fixed point at $T^* \to \infty$ for all $T > 0$. This RG flow is shown in Fig.15.2.3. Notice that in this case the beta-function does not have a term linear in the coupling constant, T in this case. Thus, in this case the operator perturbing at the fixed point $T^* = 0$ is marginal. Since the coefficient of the leading term (quadratic, in this case) is positive, we will say that this is a marginally relevant perturbation. Thus, in the 1D problem the RG flows in the IR to strong coupling, in this case $T \to \infty$ and there is no phase transition (as it is well known!).

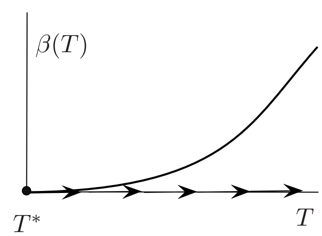


Figure 15.6 The beta-function and the IR RG flow of the one-dimensional Ising model.

We will now look at a less trivial case and consider the Ising model in dimension $d > 1$. This problem is in general not exactly solvable (although it is in $d = 2$ dimensions, see Chapter 14). The naive application of the decimation procedure we just used in 1D will lead to the generation of additional operators in the effective action of the form of next-nearest neighbor interactions, four spin interactions, etc. However, there is a simple and instructive approximate approach, introduced by Migdal (Migdal, 1975a,b)and Kadanoff (Kadanoff, 1977), which makes the physics very apparent. It is a good approximation if the system is almost ordered and hence, for $T \ll 1$. The approximation consists of moving some of the bonds, as shown in Fig.15.2.3, so that some of the spins can be integrated out without generating additional interactions. This done in three steps: 1) one moves first half of the vertical bods to left thus doubling the strength of the remaining interactions on these bonds. Let β_1 and β_2 be initial interactions along the vertical and horizontal bonds, respectively. After these three steps the new interactions in both directions become

$$\beta_1' = \frac{1}{2} \ln \cosh(2\beta_1), \qquad \beta_2' = 2\beta_2 \tag{15.30}$$

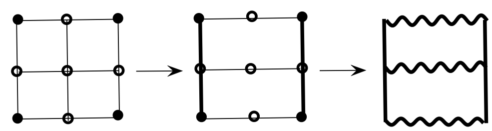


Figure 15.7 The Migdal-Kadanoff RG transformation for the two-dimensional Ising model. Left: the open circles are spins that will be integrated over. Center: the vertical bonds are moved to the left and the strength of the remaining vertical bonds is doubled. Right: the middle spins on the horizontal bonds are integrated out leading to the effective interactions represented by the wiggly lines. For details see the text.

In d dimensions, this leads to a renormalized coupling

$$\frac{1}{T'} = 2^{d-2} \ln \cosh\left(\frac{2}{T}\right) \tag{15.31}$$

where we ignored the artificial asymmetry induced by this procedure. This expression is very similar to the one-dimensional result of Eq.(15.27). We will now perform an analytic continuation in the dimensionality d and find an expression for the beta-function in dimension $d = 1 + \epsilon$. The result is

$$\beta(T) = \frac{T' - T}{\ln 2} = -\epsilon T + \frac{T^2}{2} + O(T^3) \tag{15.32}$$

which is a generalization of Eq.(15.27). This beta-function is shown in Fig.15.2.3.

This beta-function has two zeros, at $T = 0$ and at T^*, the two finite fixed points of tis RG flow. These two fixed points have different character and physical meaning. From the slope of the beta-function of Eq.(15.32) at the IR unstable fixed point at T^*, we find that for a change of scale $b = a'/a = 2$,

$$\delta T' = b^{\epsilon}\, \delta T \tag{15.33}$$

and, hence, that here the exponent is $y = \epsilon = d - 1 > 0$, which implies that the perturbation away from this fixed point is relevant. Conversely, at the

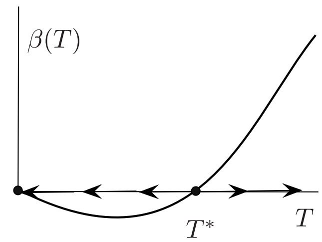


Figure 15.8 The beta-function and the IR RG flow of the $d = 1 + \epsilon$-dimensional Ising model. This RG has two finite fixed points: a) the IR stable fixed point at $T = 0$, and the IR unstable fixed point at $T^* = 2\epsilon$, which labels the phase transition between the disordered phase at $T > T^*$ to the ordered (broken-symmetry) phase at $T < T^*$.

IR stable fixed point at $T = 0$, the exponent is instead $y = -\epsilon$, and the perturbation is now irrelevant. As can be seen in this example, the notion of relevancy or irrelevancy of an operator depends on the fixed point and it is not absolute.

We will now determine the behavior of the correlation length ξ close to the non-trivial fixed point at T^*. After one action of this RG transformation, the correlation length becomes $\xi' = \xi/2$. Hence, after n iterations, we obtain $\xi_n = \xi/2^n$. Since $\delta T' = 2^y \delta T$, then after n iterations it becomes $\delta T^{(n)} = 2^{ny}\delta T$. Thus, we can write

$$2^n = \left(\frac{\delta T^{(n)}}{\delta T}\right)^{1/y} = \frac{\xi}{\xi_n} \tag{15.34}$$

This if we begin the RG flow very close to T^*, such that $\delta T \ll T^*$, after n iterations $\delta T^{(n)} \simeq T^*$ and $\xi_n \simeq a$, the UV cutoff (the lattice spacing). Hence, we deduce that the correlation length ξ must diverge as $\delta T \to 0$ and obey the scaling law

$$\xi(\delta T) = a\left(\frac{T^*}{\delta T}\right)^{\nu} \tag{15.35}$$

where the *critical exponent* ν is given by

$$\nu = \frac{1}{y} = \frac{1}{\epsilon} = \frac{1}{d-1} + O(\epsilon) \tag{15.36}$$

In other words, we now have an example of an IR unstable fixed point and that it has a diverging correlation length (in units of the UV cutoff). hence in a system of this type it should be possible construct a continuum field theory by tuning the theory to the scaling regime of its nontrivial fixed point. This is why fixed points of this type are also known as the so-called UV fixed point.

15.2.4 The Wilson-Fisher momentum-shell RG

We will now look at a different type of RG transformation: the momentum shell RG introduced by Kenneth Wilson and Michael Fisher (Wilson and Fisher, 1972). Instead of blocks of local degrees of freedom, we will focus on momentum space and we will integrate-out progressively the high-momentum modes of the field configurations (Wilson and Kogut, 1974). Here we will focus on the case of the Euclidean ϕ^4 theory, which is a equivalent to the Landau-Ginzburg theory of phase transitions. The action in D Euclidean dimensions is

$$S = \int d^D x \left[\frac{1}{2} \left(\partial_\mu \phi\right)^2 + \frac{t}{2} \Lambda^2 \phi^2 + \frac{u}{4!} \Lambda^\epsilon \phi^4 \right] \tag{15.37}$$

where we have defined the mass m_0^2 and the coupling constant λ in terms of the UV cutoff Λ as $m_0^2 = t\Lambda^2$ and $\lambda = u\Lambda^\epsilon$, where t and u are dimensionless and $\epsilon = 4 - D$.

This approach begins by splitting the field into slow and fast components, denoted by $\phi_<$ and $\phi_>$, respectively,

$$\phi(x) = \phi_<(x) + \phi_>(x) \tag{15.38}$$

For a theory with UV cutoff Λ (which we will assume is imposed using a smooth cutoff procedure), we will split the momentum space into a (thin) shell of momenta $b\Lambda < |p| < \Lambda$ (with $b < 1$) and the rest, $|p| < b\Lambda$, (as shown in Fig.15.2.4). The fast fields $\phi_>$ have support only on the momentum shell while the slow fields $\phi_<$ have support on the rest of the momentum sphere:

$$\phi_<(x) = \int_{|p|<b\Lambda} \frac{d^D p}{(2\pi)^D}\, \phi(p) e^{ip.x} \tag{15.39}$$

$$\phi_>(x) = \int_{b\Lambda<|p|<\Lambda} \frac{d^D p}{(2\pi)^D}\, \phi(p) e^{ip.x} \tag{15.40}$$

Written in terms of the fast and slow fields the action of Eq.(15.37) takes the form

$$S[\phi] = S_<[\phi_< +] S_>[\phi_>] + S_{\text{int}}[\phi_<, \phi_>] \tag{15.41}$$

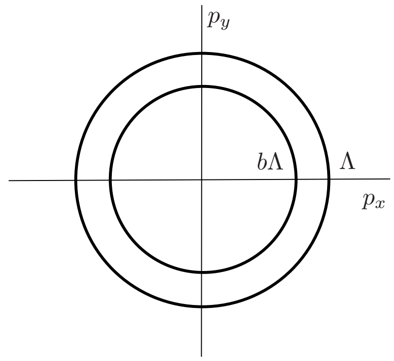


Figure 15.9 The momentum shell, $b\Lambda < |p| < \Lambda$, with $b < 1$.

where we defined

$$S_<[\phi_<] \equiv \int d^D x \left[\frac{1}{2} \left(\partial_\mu \phi_<\right)^2 + \frac{t}{2}\Lambda^2 \phi_<^2 + \frac{u}{4!}\Lambda^\epsilon \phi_<^4 \right], \tag{15.42}$$

$$S_<[\phi_>] \equiv \int d^D x \left[\frac{1}{2} \left(\partial_\mu \phi_>\right)^2 + \frac{t}{2}\Lambda^2 \phi_>^2 + \frac{u}{4!}\Lambda^\epsilon \phi_>^4 \right], \tag{15.43}$$

$$S_{\text{int}}[\phi_<, \phi_>] \equiv \int d^D x \; \frac{u}{4!}\Lambda^\epsilon \left[4\phi_<^3 \phi_> + 6\phi_<^2 \phi_>^2 + 4\phi_< \phi_>^3 \right]. \tag{15.44}$$

Notice that the fast and slow fields do not mix at the quadratic (free field) level since

$$\begin{aligned}\int d^D x \frac{1}{2}\left[(\partial_\mu \phi)^2 + t\Lambda^2 \phi^2 \right] &= \int_{|p|<\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{2}(p^2 + t\Lambda^2)\phi(p)\phi(-p) \\ &= S_<[\phi_<] + S_>[\phi_>] \end{aligned} \tag{15.45}$$

The next step to obtain an effective action for the slow fields (with UV cutoff $b\Lambda$) that we will denote by $S^{b\Lambda}_{\text{eff},<}[\phi_<]$. We will do this by integrating out the fast fields $\phi_>$ as follows,

$$\begin{aligned} Z &= \int \mathcal{D}\phi \; e^{-S^\Lambda[\phi]} = \int \mathcal{D}\phi_< \mathcal{D}\phi_> \; e^{-S^\Lambda[\phi_>, \phi_<]} \\ &= \int \mathcal{D}\phi_< \; e^{-S^{b\Lambda}_<[\phi_<]} \int \mathcal{D}\phi_> \; e^{-S^\Lambda_>[\phi_>] - S^\Lambda_{\text{int}}[\phi_> \,, \phi_<]} \end{aligned} \tag{15.46}$$

which defines the effective action $S^{b\Lambda}_{\text{eff},<}[\phi_<]$ of the slow fields in a theory with cutoff $b\Lambda$:

$$e^{-S^{b\Lambda}_{\text{eff},<}[\phi_<]} \equiv e^{-S^\Lambda_<[\phi_<]} \int \mathcal{D}\phi_> \; e^{-S^\Lambda_>[\phi_>] - S^\Lambda_{\text{int}}[\phi_>, \phi_<]} \tag{15.47}$$

The path integral in this expression now has to be computed. This will do using perturbation theory in u, which is equivalent to working at one loop order in the fluctuations if the fast fields. To this end we write the action of the fast fields as a sum of a massive free fast field action and an interaction term,

$$S_>^\Lambda[\phi_>] = S_{0,>}^\Lambda[\phi_>] + S_{\text{int},>}^\Lambda[\phi_>] \tag{15.48}$$

We find

$$\int \mathcal{D}\phi_>\ e^{-S_>^\Lambda[\phi_>]-S_{\text{int}}^\Lambda[\phi_>,\phi_<]} = Z_{0,>} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(-\frac{u\Lambda^\epsilon}{4!}\right)^n I_n[\phi_<] \tag{15.49}$$

where $Z_{0,>} = \exp(-F_>^0)$ is the partition function for free fast fields defined in the momentum shell, and where de define $I_n[\phi_<]$ as

$$\begin{aligned} I_n[\phi_<] = \int_{\{x_j\}} \Big\langle \prod_{j=1}^{n} \Big[&\phi_>^4(x_j) + 4\phi_>^3(x_j)\phi_<(x_j) + 6\phi_>^2(x_j)\phi_<^2(x_j) \\ &+4\phi_>(x_j)\phi_<^3(x_j) + \phi_<^4(x_j) \Big]\Big\rangle_{0,>} \end{aligned} \tag{15.50}$$

where $\langle \dots \rangle_{0,>}$ represents an expectation value on the free fast fields. We will use below that, symmetry, $\langle \phi_>^{2k+1}(x) \rangle_{0,>} = 0$, which applies to the expectation values of similar expectation values of odd numbers of fast field operators.

To first (lowest) order in u we find

$$I_1 = \int d^D x \left[\langle \phi_>^4(x) \rangle_{0,>} + 6 \langle \phi_>^2(x) \rangle_{0,>} \phi_<^2(x) \right] \tag{15.51}$$

Similarly, we find that I_2 is given by the expression

$$\begin{aligned} I_2[\phi_<] = \int d^D x_1 \int d^D x_2 \Big[&\langle \phi_>^4(x_1)\phi_>^4(x_2) \rangle_{0,>} + 12 \langle \phi_>^4(x_1)\phi_>^2(x_2) \rangle_{0,>} \phi_<^2(x_2) \\ &+ 16 \langle \phi_>^3(x_1)\phi_>^3(x_2) \rangle_{0,>} \phi_<(x_1)\phi_<(x_2) \\ &+ 36 \langle \phi_>^2(x_1)\phi_>^2(x_2) \rangle_{0,>} \phi_<^2(x_1)\phi_<^2(x_2) \\ &+ 16 \langle \phi_>(x_1)\phi_>(x_2) \rangle_{0,>} \phi_<^3(x_1)\phi_<^3(x_2) \\ &+ 32 \langle \phi_>^3(x_1)\phi_>(x_2) \rangle_{0,>} \phi_<(x_1)\phi_<^3(x_2) \Big] \end{aligned} \tag{15.52}$$

Using these results we find that, up to cubic order in the dimensionless coupling constant u, the effective action $S^{b\Lambda}_{\text{eff},<}[\phi_<]$ is

$$S^{b\Lambda}_{\text{eff},<}[\phi_<] = S^{b\Lambda}_<[\phi_<] + \left(\frac{u\Lambda^\epsilon}{4!}\right) I_1 - \frac{1}{2!}\left(\frac{u\Lambda^\epsilon}{4!}\right)^2 \left(I_2 - I_1^2\right) + O(u^3) \quad (15.53)$$

where I_1 and I_2 are given in Eq.(15.51) and Eq.(15.52), respectively. The explicit form of the effective action is

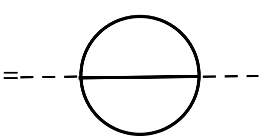
$$\begin{aligned}
S^{b\Lambda}_{\text{eff},<}[\phi_<] &= \int d^Dx \Big[\frac{1}{2}\left(\partial_\mu \phi_<\right)^2 + \frac{t}{2}\Lambda^2 \phi_<^2 + \frac{u}{4!}\Lambda^\epsilon \phi_<^4\Big] \\
&+ \Big[F^0_> + \left(\frac{u\Lambda^\epsilon}{4!}\right)\int d^Dx \langle \phi_>^4(x)\rangle_{0,>} - \frac{1}{2}\left(\frac{u\Lambda^\epsilon}{4!}\right)^2 \int_{x_1,x_2} \langle :\phi_>^4(x_1)::\phi_>^4(x_2):\rangle_{>,0}\Big] \\
&+ \Big\{ \left(\frac{u\Lambda^\epsilon}{4!}\right)\int d^Dx\ 6\langle \phi_>^2(x)\rangle_{>,0}\phi_<^2(x) \\
&- \frac{1}{2}\left(\frac{u\Lambda^\epsilon}{4!}\right)^2 \int_{x_1,x_2} \Big[36\langle :\phi_>^2(x_1)::\phi_>^2(x_2):\rangle_{>,0}\phi_<^2(x_1)\phi_<^2(x_2) \\
&+ 12\langle :\phi_>^4(x_1)::\phi_>^2(x_2):\rangle_{>,0}\phi_<^2(x_2) + 16\langle \phi_>(x_1)\phi_>(x_2)\rangle_{>,0}\phi_<^3(x_1)\phi_<^3(x_2) \\
&+ 16\langle \phi_>^3(x_1)\phi_>^3(x_2)\rangle_{>,0}\phi_<(x_1)\phi_<(x_2) + 32\langle \phi_>^3(x_1)\phi_>(x_2)\rangle_{>,0}\phi_<(x_1)\phi_<^3(x_2)\Big] \\
&+ O(u^3)\Big\} \qquad (15.54)
\end{aligned}$$

where $F^0_> = -\ln Z_{0,>}$, and where we used the notation $:A: = A - \langle A\rangle_{0,>}$ for the normal ordered operator A with respect to the fast free field theory.

We will now write the expression for the effective action for the slow fields in terms of a set of local operators of the form $\phi_<^n$ and derivatives, e.g. $\partial_i\partial_j\phi_<^n$, etc. We can do that since the kernels that appear in Eq.(15.54) involve only the correlators of the fast fields which are rapidly decaying functions of the coordinates (provided the cutoff function used to restrict the fields to the momentum shell are smooth).

Let us consider for example the contribution

$$16\int_{x_1,x_2} \langle \phi_>^3(x_1)\phi_>^3(x_2)\rangle_{>,0}\phi_<(x_1)\phi_<(x_2) =$$



$$= 96\int dx_1^D dx_2^D\, \phi_<(x_1)\phi_<(x_2)\left[G_{0,>}(x_1-x_2)\right]^3 \qquad (15.55)$$

where the internal momenta of the Feynman diagram of the right hand side are in the momentum shell (the full lines) and the external momenta (the broken lines) are outside the momentum shell. Here, $G_{0,>}(x_1 - x_2)$ is the

Euclidean propagator of the fast free fields

$$G_{0,>}(x_1 - x_2) = \int_{b\Lambda<|p|<\Lambda} \frac{d^D p}{(2\pi)^D} \frac{e^{ip\cdot(x_1-x_2)}}{p^2 + t\Lambda^2} \tag{15.56}$$

We will assume that the cutoff implied by the restriction to the momentum shell is smooth enough so that this propagator decays exponentially with distance. Under these assumptions, it is legitimate perform inside the integral shown in Eq.(15.55) a Taylor expansion of the slow field $\phi_<(x_2)$ about the coordinate x_1:

$$\phi_<(x_2) = \phi_<(x_2 + \ell) = \phi_<(x_1) + \ell_i \partial_i \phi_<(x_1) + \frac{1}{2}\ell_i \ell_j \partial_i \partial_j \phi_<(x_1) + \ldots \tag{15.57}$$

We can then write the integral in right hand side of Eq.(15.55) as

$$\begin{aligned} &\int dx_1^D dx_2^D \; \phi_<(x_1)\phi_<(x_2) \; \left[G_{0,>}(x_1 - x_2)\right]^3 = \\ &= \int d^D x \; \phi_<^2(x) \int d^D \ell \; \left[G_{0,>}(\ell)\right]^3 - \frac{1}{D}\left[\int d^D \ell \; \ell^2 \left[G_{0,>}(\ell)\right]^3\right] \frac{1}{2}\left(\partial_\mu \phi_<(x)\right)^2 + \ldots \end{aligned} \tag{15.58}$$

Therefore, the contribution of Eq.(15.55) is part of the mass renormalization and of the wave function renormalization.

Notice that reducible diagrams cannot may appear in the effective actions for the slow fields since they vanish automatically as the internal momenta are inside the momentum shell and the external momenta are not (as in the example shown in Fig.15.2.4).

We conclude that, up to this order in an expansion in powers in the dimensionless coupling constant u, the effective action of the slow fields for a theory with UV cutoff $b\Lambda$ becomes

$$S_{\text{eff},<}^{b\Lambda}[\phi_<] = \int d^D x \Big[\frac{A}{2}\left(\partial_\mu \phi_<\right)^2 + B\phi_<^2(x) + C\phi_<^4(x) + \text{const.} + \ldots \tag{15.59}$$

where $\ldots$ denotes operators with higher powers of $\phi_<$ and/or higher derivatives (only even powers and even derivatives will appear by symmetry). The coefficients A, B and C are

$$
\begin{aligned}
A =& 1 + 48\left(\frac{u\Lambda^{\epsilon}}{4!}\right)^2 \frac{1}{D} \int d^D\ell \left[G_{0,>}(\ell)\right]^3 \ell^2 + O(u^3) \\
B =& \frac{t}{2}\Lambda^2 + 6\left(\frac{u\Lambda^{\epsilon}}{4!}\right) G_{0,>}(0) - 72\left(\frac{u\Lambda^{\epsilon}}{4!}\right)^2 G_{0,>}(0) \int d^D\ell \left[G_{0,>}(\ell)\right]^2 \\
&- 48\left(\frac{u\Lambda^{\epsilon}}{4!}\right)^2 \int d^D\ell \left[G_{0,>}(\ell)\right]^3 + O(u^3) \\
C =& \frac{u\Lambda^{\epsilon}}{4!} - 36\left(\frac{u\Lambda^{\epsilon}}{4!}\right)^2 \int d^D\ell \left[G_{0,>}(\ell)\right]^2 - 48\left(\frac{u\Lambda^{\epsilon}}{4!}\right)^2 G_{0,>}(0) \int d^D\ell\, G_{0,>}(\ell) \\
&+ O(u^3)
\end{aligned}
\tag{15.60}
$$

Here we ignored the constant terms, which are a contribution to the renormalization of the identity operator.

The remaining integrals can be done quite easily. In the limit in which the shell is very thin (compared to the cutoff scale Λ), i.e. $b = e^{-\delta s} \to 1^-$ as $\delta s \to 0^+$, the integrals become

$$
\begin{aligned}
G_{0,>}(0) &= \int_{\text{shell}} \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + t\Lambda^2} = \frac{\Lambda^{D-2}}{1+t} \frac{S_D}{(2\pi)^D} \delta s \\
\int d^D\ell\, G_{0,>}(\ell) &= \int_{\text{shell}} \frac{d^D p}{(2\pi)^D} \frac{(2\pi)^D \delta^D(p)}{p^2 + t\Lambda^2} = 0 \\
\int d^D\ell \left[G_{0,>}(\ell)\right]^2 &= \frac{\Lambda^{D-4}}{(1+t)^2} \frac{S_D}{(2\pi)^D} \delta s \\
\int d^D\ell \left[G_{0,>}(\ell)\right]^3 &= \frac{\Lambda^{2D-6}}{(1+t)^3} \frac{S_D}{(2\pi)^D} \delta s \\
\int d^D\ell \left[G_{0,>}(\ell)\right]^3 \ell^2 &= \Lambda^{2D-8} \left[\frac{2D}{(1+t)^4} - \frac{8}{(1+t)^5}\right] \frac{S_D}{(2\pi)^D} \delta s
\end{aligned}
\tag{15.61}
$$

where

$$
S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{15.62}
$$

is the area of the hypersphere, and $\Gamma(x)$ is the Gamma function.

The last step is to define a rescaled field

$$
\phi' = Z_{\phi}^{-1/2} \phi_< \tag{15.63}
$$

which we readily identify with the wave function renormalization, and to

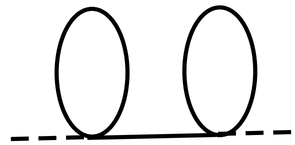


Figure 15.10 Reducible Feynman diagrams with internal momenta in the momentum shell vanish identically by momentum conservation.

rescale the coordinates,

$$x' = bx \tag{15.64}$$

so that the cutoff scale in the new rescaled coordinates in back to Λ. The resulting action for the (rescaled) field ϕ' in the rescaled coordinates x' is

$$\begin{aligned} S'(\phi') =& S^{b\Lambda}_{\text{eff},<}(\phi_<) \\ =& \int d^D x' b^{-D} \Big[\frac{A}{2} Z_\phi b^2 \left(\partial'_\mu \phi'\right)^2 + B Z_\phi \phi'^2 + C Z_\phi^2 \phi'^4 \Big] \end{aligned} \tag{15.65}$$

and we will require that the renormalization conditions

$$1 = b^{2-D} A Z_\phi \tag{15.66}$$

$$\frac{t'}{2}\Lambda^2 = b^{-D} B Z_\phi \tag{15.67}$$

$$\frac{u'}{4!}\Lambda^\epsilon = b^{-D} C Z_\phi^2 \tag{15.68}$$

be satisfied. By keeping only the leading corrections to t and u, we obtain

$$Z_\phi = b^{D-2} A^{-1} = b^{D-2}\left(1 + O(u^2))]\right) \tag{15.69}$$

$$t' = b^{-2}\left[t + \frac{1}{2}(u - ut)\frac{S_D}{(2\pi)^D}\delta s + O(u^2, t^2)\right] \tag{15.70}$$

$$u' = b^{-\epsilon}\left[u - \frac{3}{2}u^2 \frac{S_D}{(2\pi)^D}\delta s + O(u^3 t^2)\right] \tag{15.71}$$

Notice that, at this leading order, the wave function renormalization is trivial. Also notice the important fact that all dependence from the momentum cutoff scale Λ has cancelled exactly. This is a generic feature of RG transformations. Finally, these equations simplify if we absorb the phase space factors in a new dimensionless coupling constant v

$$v = u\frac{S_D}{(2\pi)^D} \tag{15.72}$$

The (Gell-Mann-Low) beta-functions for the dimensionless couplings t and v are defined to be

$$\beta_t = \frac{dt}{ds} = a\frac{dt}{da} = -\Lambda\frac{dt}{d\Lambda}$$
$$\beta_v = \frac{dv}{ds} = a\frac{dv}{da} = -\Lambda\frac{dv}{d\Lambda} \tag{15.73}$$

Using the results we just derived, we find that the beta-functions are

$$\beta_t = 2t + \frac{v}{2} - \frac{1}{2}vt + \dots \tag{15.74}$$
$$\beta_v = \epsilon v - \frac{3}{2}v^2 + \dots \tag{15.75}$$

to leading orders in the couplings t and v, and in $\epsilon = 4 - D$. Therefore, this *perturbative* RG transformation, is accurate only to order ϵ, and is the beginning of a series in powers of ϵ: the ϵ-expansion (Wilson and Kogut, 1974).

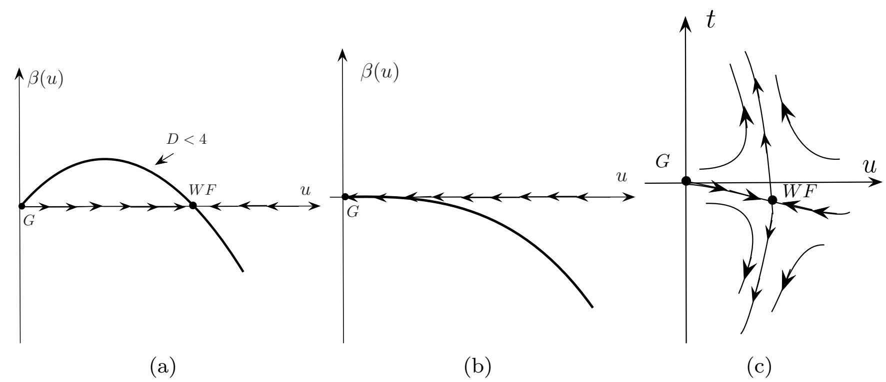


Figure 15.11 RG beta-function for the coupling constant v in a) $D < 4$ dimensions and b) In D=4 dimensions. The full RG flow is shown in c). Here t is the dimensionless mass, $m^2 = t\Lambda^2$, and u is the dimensionless coupling constant, $\lambda = u\Lambda^\epsilon$, and WF is the Wilson-Fisher fixed point.

In $D < 4$ dimensions, the RG equations, Eq.(15.74) and Eq.(15.75) define an RG flow. In $D < 4$ dimensions this RG flow has two fixed points: a) the free-field (or Gaussian) fixed point at $t = u = 0$, and b) the non-trivial fixed point, the Wilson-Fisher fixed point, at $u^* = \frac{2}{3}\epsilon$ and $t^* = -\frac{\epsilon}{6}$. The RG flow of the coupling constant 4 is shown in Fig.15.2.4 a and b. The full RG flow is shown in Fig.15.2.4c. If $D < 4$, the free-field fixed point is completely unstable in the IR. On the other hand, the Wilson-Fisher fixed point, which only exists if $D < 4$, is bistable: there is a trajectory from the free field fixed point flows into the Wilson-Fisher fixed point, and another trajectory that in the IR flows away from the Wilson-Fisher fixed point towards the $t \to \pm\infty$ regimes, where the theory describes a symmetric phase (for $t \to +\infty$) and a broken symmetry phase (for $t \to -\infty$). Hence, for $D < 4$ the free field fixed point has two relevant operators (with coupling constant t and u) whereas the Wilson-Fisher fixed point has one irrelevant and one relevant operator. Notice that the Wilson-Fisher fixed point is at a finite value of $t^* < 0$, which reflects the mass renormalization at one-loop level. On the other hand, as $D \to 4$ dimensions the Wilson-Fisher fixed point coalesces with the free field fixed point, which becomes the only remaining fixed point. As a result, in $D = 4$ dimensions the free field fixed point is bistable in the IR: it is *marginally stable* in the coupling constant u and unstable in t.

The RG flow for the behavior determines the behavior of the correlation length ξ. By dimensional analysis we can always write ξ in the form

$$\xi = \Lambda^{-1} f(t, u) \tag{15.76}$$

where $f(t, u)$ is a dimensionless function of t and u. Consider now two physical systems defined with different values of of the UV scale Λ but with the *same* value of the physical scale ξ. Since ξ is fixed, we can write

$$0 = \frac{\partial \xi}{\partial \Lambda} = -\Lambda^{-2} f(t, u) + \Lambda^{-1} \frac{\partial f}{\partial \Lambda} \tag{15.77}$$

However, $f(t, u)$ does not depend explicitly on Λ but it depends implicitly on the scale through the RG flow. Hence,

$$\frac{\partial f}{\partial \Lambda} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial \Lambda} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial \Lambda} \tag{15.78}$$

By using the definitions of the beta-functions we find that the function $f(t, u)$ must obey the partial differential equation

$$f + \frac{\partial f}{\partial u} \beta_u + \frac{\partial f}{\partial t} \beta_t = 0 \tag{15.79}$$

This is an example of a *Callan-Symanzik equation*.

We will solve Eq.(15.79) for the unstable trajectory of the Wilson-Fisher fixed point. Close to the fixed point we can work with the linearized flow expressed in terms of the variables x and y,

$$x = 4(t - t^*) + \left(1 - \frac{\epsilon}{6}\right)(v - v^*), \qquad y = v - v^* \tag{15.80}$$

where (v^*, t^*) is the location of the Wilson-Fisher fixed point. The linearized beta-functions are

$$\beta_x = \left(2 - \frac{\epsilon}{3}\right) x + \dots, \qquad \beta_y = -\epsilon y + \dots \tag{15.81}$$

In these coordinates the fixed point is at $(0, 0)$. Since the flow in y is irrelevant (it flows into the fixed point), we will set $y = 0$ and focus on the flow away from the fixed point along x.

The Callan-Symanzik equation to be solved now is

$$0 = f + \frac{\partial f}{\partial x}\beta_x \tag{15.82}$$

The general solution of this equation is

$$\ln f = \text{const} - \int \frac{dx}{\beta_x} \tag{15.83}$$

For a flow beginning at some value $x_0 \to 0$ and ending at some value x, of $O(1)$, this solution is

$$f(x) = f(x_0)\exp\left[-\int_{x_0}^{x} \frac{dx'}{\beta_x(x')}\right] \tag{15.84}$$

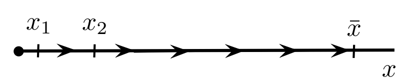


Figure 15.12 RG flows along the scaled variable x with two initial values, x_1 and x_2, and the same final value $\bar{x}$.

Let us consider two flows along x with different starting points, x_1 and x_2, and with the same final point $\bar{x}$, with the *same* UV cutoff Λ_0 (shown in Fig.15.2.4). The value of the correlation length ξ at x_1 and x_2 is

$$\xi(x_1) = \Lambda_0^{-1} f(x_1), \qquad \xi(x_2) = \Lambda_0^{-1} f(x_2) \tag{15.85}$$

where

$$\begin{aligned} f(x_1) =& f(\bar{x})\exp\left(\int_{x_1}^{\bar{x}} \frac{dx}{\beta(x)}\right) \\ f(x_2) =& f(\bar{x})\exp\left(\int_{x_2}^{\bar{x}} \frac{dx}{\beta(x)}\right) \end{aligned} \tag{15.86}$$

Hence, we find

$$\xi(x_1) = \xi(x_2)\exp\left(\int_{x_1}^{x_2} \frac{dx}{\beta(x)}\right) \tag{15.87}$$

Suppose now that x_1 is very close to the fixed point ($x_1 \to 0$) and that x_2 is far away enough that $\xi(x_1) \simeq a$ (the short distance cutoff). Using te linearized beta-function, we find that

$$\xi(x_1) \simeq a \exp\left(\int_{x_1}^{x_2} \frac{dx}{(2 - \frac{\epsilon}{3})x} \right) = a \left| \frac{x_1}{x_2} \right|^{-\nu} \tag{15.88}$$

where

$$\nu = \frac{1}{\beta'(0)} = \frac{1}{2 - \frac{\epsilon}{3}} = \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2) \tag{15.89}$$

Hence, the critical exponent ν of the correlation length is determined by the slope of the beta-function at the Wilson-Fisher fixed point, which at the present level of approximation is $\beta'(0) = 2 - \frac{\epsilon}{3} + O(\epsilon^2)$.

15.3 General Properties of a Fixed Point Theory

We see that at a fixed point of a renormalization group transformation the theory has a new emergent symmetry: scale-invariance. This is a new, emergent, symmetry that is operative at length scales long compared to a short-distance cutoff a (the UV regulator) but short compared to the linear size L of the system (the IR regulator). No other scales are present at the fixed point. In Chapter 21 we will see that in most cases of interest in quantum field theory scale invariance can be extended to a larger emergent symmetry, conformal invariance, i.e. invariance under conformal coordinate transformations $x' = f(x)$ such that angles between vectors are preserved. There we will show that most of the statements that we will make in this section follow from conformal invariance.

In a scale-invariant theory the action (or, more precisely, the partition function) is invariant under scale transformations, $x \mapsto x' = \lambda x$. In such a theory, the expectation values of the observables transform under dilations as homogenous functions. A function $F(x)$ is a homogenous function of degree k if $F(\lambda x) = \lambda^k F(x)$.

It follows that at, a general fixed point whose action we will denote by S^*, the correlators of a physical observable $\mathcal{O}(x)$ obey power laws of the form

$$\langle \mathcal{O}(x)\mathcal{O}(y) \rangle^* = \frac{\text{const.}}{|x-y|^{2\Delta_{\mathcal{O}}}} \tag{15.90}$$

Here $\langle A \rangle^*$ denotes the expectation value of the observable A at the fixed point S^*. The quantity $\Delta_{\mathcal{O}}$ is called the *scaling dimension* of the local operator $\mathcal{O}$. Hence, the operator $\mathcal{O}$ has units of

$$[\mathcal{O}] = \ell^{-\Delta_{\mathcal{O}}} \tag{15.91}$$

where ℓ is a length scale.

The scaling dimensions of the observables is one of the universal properties that define a fixed point. A universal quantity here means a quantity whose value is independent of the short-distance definition of the theory, i.e. of the particular definition in the UV. Clearly, dimensionful quantities are by definition not universal since a microscopic scale is needed to define the units and different choices will lead to different values. Hence, only dimensionless quantities, such as the scaling dimensions, can be universal.

In a general fixed point, the scaling dimensions of the operators must be positive real numbers so that the correlators of local observables obey cluster decomposition, and must decay (as a power law in this case) at large separations. In exceptional cases, the scaling dimensions take rational values. This is the case in scale-invariant free field theories and also in integrable systems such as the minimal models of $D = 2$ dimensional conformal field theories that will be discussed in Chapter 21. There we will show that a theory with conformal invariance has the following properties:

1) The theory has a conformally-invariant vacuum state $|0\rangle$.
2) It has a set of operators, that we will denote by $\{\phi_j\}$, called quasi-primary, which transforms irreducibly under conformal transformations

$$\phi_j(x) \mapsto \left|\frac{\partial x'}{\partial x}\right|^{\Delta_j/D} \phi_j(x') \tag{15.92}$$

where

$$J = \left|\frac{\partial x'}{\partial x}\right| \tag{15.93}$$

is the Jacobian of the conformal transformation, D is the space-time dimension, and the real numbers $\{\Delta_j\}$ are the scaling-dimensions of the operators. The scaling dimensions are the quantum numbers that label the representations of the conformal group.
3) Under a conformal transformation the correlators of the fields transform as

$$\langle \phi_1(x_1)\dots\phi_N(x_N)\rangle^* = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/D} \dots \left|\frac{\partial x'}{\partial x}\right|_{x=x_N}^{\Delta_N/D} \langle \phi_1(x_1')\dots\phi_N(x_N')\rangle^* \tag{15.94}$$

4) All other operators in the theory are linear combinations of the quasi-primary fields and their derivatives.

The symmetries of the fixed point strongly constrain the behavior of the correlators. Translation and rotation (or Lorentz) invariance require that the

N-point functions depend only on the pairwise distances, $|x_{12}| = |x_1 - x_2|$. Scale invariance then requires that they depend only on scale-invariant ratios, $|x_{12}|/|x_{34}|$. In addition, conformal invariance requires that they depend only on cross ratios $|x_{12}||x_{34}|/(|x_{13}||x_{24}|)$.

Translation, rotation and scale invariance restrict the form of the two-point functions, $\langle \phi_1(x_1)\phi_2(x_2)\rangle^*$, to have the form

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle^* = \frac{C_{12}}{|x_{12}|^{\Delta_1+\Delta_2}} \tag{15.95}$$

However conformal invariance further restricts the form of the two point functions, requiring that $C_{12} = 0$ if $\Delta_1 \neq \Delta_2$. Hence

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle^* = \begin{cases} \dfrac{C_{12}}{|x_{12}|^{2\Delta}}, & \Delta = \Delta_1 = \Delta_2 \\ 0, & \Delta_1 \neq \Delta_2 \end{cases} \tag{15.96}$$

Hence, primary fields with different scaling dimensions (i.e. in different representations) are "orthogonal" to each other. Clearly, by a suitable redefinition of the operators we can always set the non-vanishing coefficient $C_{12} = 1$.

The form of the three-point functions is also completely determined by translation, rotation, scale and conformal invariance. In Chapter 21 we will show that the three-point function has the form (Polyakov, 1974)

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle^* = \frac{C_{123}}{|x_{12}|^{\Delta_{12}}|x_{23}|^{\Delta_{23}}|x_{31}|^{\Delta_{31}}} \tag{15.97}$$

where $\Delta_{ij} = \Delta_i + \Delta_j - \Delta_k$ (with $i, j, k = 1, 2, 3$). Provided the primary fields are normalized so that the coefficient of the two point function is $C_{ij} = \delta_{ij}$, the coefficients C_{ijk} of the three-point functions are universal numbers which further characterize the fixed point. We will see that they play a key role.

These symmetries also restrict the form of the higher point functions but not as completely. For instance, the four-point function must have the form

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle^* = F\left(\frac{|x_{12}||x_{34}|}{|x_{13}||x_{24}|}\right) \prod_{i<j} \frac{1}{|x_{ij}|^{\Delta_i+\Delta_j-\bar{\Delta}/3}} \tag{15.98}$$

where $\bar{\Delta} = \sum_{i=1}^4 \Delta_i$. The function $F(z)$, where $z = |x_{12}||x_{34}|/(|x_{13}||x_{24}|)$ is the cross ratio of the coordinates of the four operators, is a universal scaling function. Similar, but more complicated expressions, apply for the higher-order N-point functions (with $N > 4$).

15.4 The Operator Product Expansion

Let us consider an N-point function of a theory at a general fixed point $\langle \phi_1(x_1) \dots \phi_N(x_N) \rangle^*$. Let us now consider the case in which two of the operators, say $\phi_1(x_1)$ and $\phi_2(x_2)$ approach each other and, hence, they closer to each other than to the insertions of any of the other operators, i.e. $a \ll |x_{12}| \ll |x_{ij}|$, where $i = 1, 2$ and $j = 3, \dots, N$ (but still farther apart than the value of a UV cutoff a). The form of the N-point functions in a fixed point theory (c.f. Eq.(15.98)) tells us that the N point function will be singular as the coordinates of any pair of fields approach each other, and that, furthermore, the singularity is determined by the scaling dimensions of the fields involved.

This property suggests that this behavior should be the same as if the product of these two fields is replaced by some suitable combination of all the operators in the theory. This observation suggests that, inside expectation values, the product of fields can be replaced by an expansion involving all other fields. The precise statement is as follows. Given that the set of operators $\{\phi_j\}$, and their descendants (involving derivatives) are a complete set, it follows that the product of two such fields, say $\phi_i(x)$ and $\phi_j(y)$ at short distances should be equivalent to a suitable linear combination of the fields $\{\phi_k\}$ and their descendants (Kadanoff, 1969; Wilson, 1969; Polyakov, 1970):

$$\lim_{y \to x} \phi_i(x) \phi_j(y) = \lim_{y \to x} \sum_k \frac{\widetilde{C}_{ijk}}{|x - y|^{\Delta_i + \Delta_j - \Delta_k}} \, \phi_k \left(\frac{x + y}{2} \right) \tag{15.99}$$

This statement is the content of the *Operator Product Expansion* (OPE). Notice that this operator identity is a weak identity as it is only meant to hold inside an expectation value of the fixed point theory. Also notice that the only operators ϕ_k that will contribute to the singular behavior must have scaling dimensions such that $\Delta_k \leq \Delta_i + \Delta_j$.

We will now see that the coefficients of the OPE are the same as the coefficients of the three-point functions. To this effect let us consider the three point function $\langle \phi_i(x) \phi_j(y) \phi_l(z) \rangle^*$, and use the OPE of Eq.(15.99) to write

$$\lim_{y \to x} \langle \phi_i(x) \phi_j(y) \phi_l(z) \rangle^* = \lim_{y \to x} \sum_k \frac{\widetilde{C}_{ijk}}{|x - y|^{\Delta_i + \Delta_j - \Delta_k}} \, \langle \phi_k \left(\frac{x + y}{2} \right) \phi_l(z) \rangle^* \tag{15.100}$$

We can now use the expressions for the two-point functions of Eq.(15.96) to find that

$$\lim_{y\to x}\langle\phi_i(x)\phi_j(y)\phi_l(z)\rangle^* = \sum_k \widetilde{C}_{ijk} \lim_{y\to x} \frac{1}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}} \langle\phi_k\left(\frac{x+y}{2}\right)\phi_l(z)\rangle^*$$
$$= \widetilde{C}_{ijl} \lim_{y\to x} \frac{1}{|x-y|^{\Delta_i+\Delta_j-\Delta_l}} \frac{1}{|x-z|^{2\Delta_l}} \qquad (15.101)$$

Using the expression for the three-point functions, Eq.(15.97), we readily find that

$$\widetilde{C}_{ijk} = C_{ijk} \qquad (15.102)$$

In other terms, the coefficients of the OPE are the same as the coefficients of the three-point functions (for normalized operators).

These results is also interpreted as meaning that at a fixed point the operators (or fields) as they approach each other they satisfy an *fusion algebra* which is conventionally denoted as

$$[\phi_i]\cdot[\phi_j] = \sum_k C_{ijk}[\phi_k] \qquad (15.103)$$

Here we did not write down explicitly the possible multiplicities of these fusion channels. We will see below that the fusion algebra encodes the properties of the one-loop renormalization group beta functions in the neighborhood of the fixed point.

15.5 Simple Examples of Fixed Points

We will now discuss several simple examples of fixed points in free field theories.

15.5.1 Free massless scalar field

Let $\phi(x)$ be a free massless scalar field in D Euclidean dimensions. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 \qquad (15.104)$$

From dimensional analysis and the condition that action be dimensionless we see that the Lagrangian has units of $[\mathcal{L}] = \ell^{-D}$. Hence the free field ϕ has units of $[\phi] = \ell^{-(D-2)/2}$. Thus, the scaling dimension of the free field $\phi(x)$ is

$$\Delta_\phi = \frac{1}{2}(D-2) \qquad (15.105)$$

which is consistent with the fact that the two-point function of a free scalar field is

$$\langle \phi(x)\phi(y)\rangle = \frac{\text{const.}}{|x-y|^{2\Delta_\phi}} \tag{15.106}$$

where $2\Delta_\phi = D-2$.

In a free field it is trivial to find the scaling dimensions of all other operators. Thus, the operator $\phi^n(x)$ has dimension

$$\Delta_n \equiv \Delta[\phi^n] = n\Delta_\phi = \frac{n}{2}(D-2) \tag{15.107}$$

which is consistent with the two point function of the normal-ordered operator $:\phi^n(x):$,

$$\langle :\phi^n(x)::\phi^n(y):\rangle = \frac{\text{const.}}{|x-y|^{2\Delta_n}} \tag{15.108}$$

where here normal ordering means $:A:= A - \langle A\rangle$. Another equivalent way to see that Eq.(15.108) is correct is to use Wick's theorem and show that the correlator of ϕ^n reduces to a sum of products of n two-point correlators.

These results are also consisten with a simple dimensional analysis of the Lagrangian. Indeed, under a scale transformation $x \mapsto \lambda x$ the scalar field transforms as $\phi(\lambda x) \mapsto \lambda^{-\Delta_\phi}\phi(x)$, where $\Delta_\phi = \frac{1}{2}(D-2)$, which leaves the action invariant.

15.5.2 Free massless Dirac field

The Lagrangian of a free massless Dirac field in D dimensions is

$$\mathcal{L} = \bar\psi(x) i\gamma^\mu \partial_\mu \psi(x) \tag{15.109}$$

where $\psi(x)$ is a spinor Fermi field, with two components in $D=2,3$ dimensions, four components in $D=4,5$ dimensions, etc, and γ_μ are the Dirac matrices in D dimensions.

By dimensional analysis, we see that the Dirac field has units of $[\psi] = \ell^{-(D-1)/2}$. Hence, its scaling dimension is $\Delta_\psi = \frac{1}{2}(D-1)$, and the two-point function is

$$\langle \bar\psi(x)\psi(y)\rangle = \frac{\text{const.}}{|x-y|^{2\Delta_\psi}} \tag{15.110}$$

Similarly, the scaling dimension of a composite operator $(\bar\psi\psi)^n$ is

$$\Delta((\bar\psi\psi)^n) = 2n\Delta_\psi = n(D-1) \tag{15.111}$$

Thus, the current operator $j_\mu = \bar\psi\gamma_\mu\psi$ has scaling dimension $D-1$.

15.5.3 Gauge Theories

The Lagrangian for a gauge theory is

$$\mathcal{L} = \frac{1}{4g^2}\mathrm{tr}F_{\mu\nu}^2 \tag{15.112}$$

where the field strength is $F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative. From the fact that the gauge field is a one-form it follows that it has units of $[A] = \ell^{-1}$, and hence its scaling dimension is $\Delta[A] = 1$. Notice that here the coupling constant is in the prefactor of the Lagrangian. Thus the field strength has scaling dimension $\Delta[F] = 2$. The action is scale-invariant if we assign the coupling constant units of $[g] = (D-4)/2$.

15.6 Perturbing a Fixed Point Theory

We will now consider a theory close to a fixed point. We will use perturbation theory in powers of the coupling constants of the theory. we will use this framework to derive the general form of the renormalization group beta functions close to a general fixed point theory. This approach is known as conformal perturbation theory. It is a generalization of the Kosterlitz renormalization group (Kosterlitz, 1974) used to describe the Kosterlitz-Thouless phase transition in two-dimensional Classical Statistical Mechanics (Kosterlitz and Thouless, 1973). An insightful discussion of this renormalization group is presented in Cardy's book (Cardy, 1996), whose approach we will follow closely.

Let S^* be the action of a fixed point theory in D Euclidean dimensions . Let δS be a set of general perturbations around this fixed point parametrized by the complete set of primary fields $\{\phi_j\}$ of the fixed point. The total action S is

$$\begin{aligned} S =& S^* + \delta S \\ =& S^* + \sum_j \int dx^D \; g_j a^{\Delta_j - D} \phi_j(x) \end{aligned} \tag{15.113}$$

where $\{g_j\}$ are the dimensionless coupling constants, the short distance scale a is the UV regulator, and $\{\Delta_j\}$ are the scaling dimensions of the fields $\{\phi_j\}$.

Let Z be the path integral for this theory which we will formally denote by

$$\begin{aligned} Z =& \operatorname{tr} \exp(-S^* - \delta S) \\ =& \operatorname{tr} \exp\left(-S^* - \sum_j \int d^D x \; g_j \; a^{\Delta_j - D} \phi_j(x)\right) \\ =& Z^* \times \left\langle \exp\left(-\sum_j \int d^D x \; g_j \; a^{\Delta_j - D} \phi_j(x)\right)\right\rangle^* \end{aligned} \tag{15.114}$$

where $Z^* = \operatorname{tr} \exp(-S^*)$ is the partition function of the fixed point theory. Here, as before, we represented expectation values in the fixed point theory by the notation

$$\langle A \rangle^* = \frac{1}{Z^*} \operatorname{tr}\left(A \exp(-S^*)\right) \tag{15.115}$$

Our next step is to write the expansion of the partition function Z in powers of the coupling constants $\{g_j\}$:

$$\frac{Z}{Z^*} = \left\langle \exp\left(-\sum_j \int d^D x \; g_j \; a^{\Delta_j - D} \phi_j(x)\right)\right\rangle^*$$

$$= 1 - \sum_j \int d^D x \; g_j \; a^{\Delta_j - D} \langle \phi_j(x) \rangle^* \tag{15.116}$$

$$+ \frac{1}{2!} \sum_{j,k} \int d^D x_j \int d^D x_k \; g_j g_k \; a^{\Delta_j + \Delta_k - 2D} \langle \phi_j(x_j) \phi_k(x_k) \rangle^* \tag{15.117}$$

$$- \frac{1}{3!} \sum_{j,k,l} \int d^D x_j \int d^D x_k \int d^D x_l \; g_j g_k g_l \; a^{\Delta_j + \Delta_k + \Delta_l - 3D} \langle \phi_j(x_j) \phi_k(x_k) \phi_l(x_l) \rangle^* \tag{15.118}$$

$$+ \dots$$

At a formal level, we can reinterpret this expression as the partition function of a gas of particles of different types in a Grand Canonical Ensemble, with each species labeled by j at coordinates x_j. In this perspective, the coupling constants g_j play the role of the the fugacity of each type of particle and the interactions are given by the (negative of) the logarithm of the correlators at the fixed point. Since S^* is a fixed point action, the integrals of the correlators present in Eqs.(15.116), (15.117) and (15.118) will generally contain IR divergencies. We will assume that the theory is in a box of linear size L to cutoff the IR behavior. But, precisely because S^* is a fixed point action and hence is scale invariant, we will need essentially their short distance UV singular behavior which is controlled by the OPE. We will control the short-distance singularities by cutting off the integrals at some short distance a. Hence the span if the integrals will be the range $L \gg |x_j - x_k| \gg a$. In other words, the "particles" of this gas has a "hard-core" a.

We will analyze this expansion using the following renormalization group transformation. We will attempt to change the UV cuttoff a by some small amount $\delta a = a\delta\ell$ (with $\delta\ell \ll 1$) and then compute the change of the action needed to compensate for that change, while requiring that the full partition function Z be left unchanged in a box of fixed linear size L. Instead of integrating out modes in momentum-shells as we did in Section 15.2.4, we will integrate out contributions to the integrals appearing in Eqs.(15.116), (15.117) and (15.118) in a shell in real space comprised between the UV cutoff a and the dilated cutoff ba, with $b = 1 + \delta\ell$. We will proceed as follows:

We rescale the UV cutoff $a \mapsto ba$, with $b = 1 + \delta\ell > 1$. Hence, we are increasing the UV cutoff. We will then need to rescale the coupling constants $\{g_j\}$ to compensate for this change, while keeping Z unchanged. How do we do this? The UV cutoff a appears in three places

a) in factors of the form $a^{\Delta_j - D}$, where Δ_j is the scaling dimension of the operators,
b) as a UV cutoff of the integrals,
c) through the dependence of the integrals on the IR cutoff L, which by scale invariance must enter in the form L/a.

If the relative change of UV scale is small, i.e. $\delta\ell \ll 1$, we will need to look only for linear changes in $\delta\ell$. Since these factors always appear multiplying the dimensionless coupling constants g_j,

$$g_j a^{\Delta_j - D} \mapsto g_j a^{\Delta_j - D} b^{\Delta_j - D} \tag{15.119}$$

we can compensate this change by a change in the coupling constants,

$$g_j \mapsto g_j b^{D-\Delta_j} = g_j(i + (D - \Delta_j)\delta\ell) \tag{15.120}$$

Hence, we will change the coupling constants by

$$g_j \mapsto g_j + (D - \Delta_j) g_j \delta\ell \tag{15.121}$$

Next, we compute the result of a change the cutoff in the integrals. This amounts to computing the contributions to the integrals in the shell in *real space* $a(1+\delta\ell) > |x_i - x_j| > a$ (instead of a shell in momentum space). Thus we split the integrals as follows

$$\int_{|x_i-x_j|>a(1+\delta\ell)} = \int_{|x_i-x_j|>a} - \int_{a(1+\delta\ell)>|x_i-x_j|>a} \tag{15.122}$$

The application of this analysis to the terms linear in the coupling constants, Eq.(15.116), yields a purely numerical contribution to the partition function Z (and hence amount to a renormalization of the identity operator). the contribution of the terms quadric in the coupling constants, Eq.(15.117), can be computed using the OPE. This is legitimate since we will look at contributions only inside the infinitesimal shell in real space. Using the OPE we find that this contribution is

$$\begin{aligned}
&\frac{1}{2!}\sum_{j,k}\sum_{l} C_{jkl}\; g_j g_k \; a^{\Delta_l-\Delta_j-\Delta_k} \int_{\text{shell}} d^D x_j d^D x_k \langle \phi_l((x_j+x_k)/2)\rangle^* a^{\Delta_j+\Delta_k-2D} \\
=&\frac{1}{2!}\sum_{j,k}\sum_{l} C_{jkl}\; g_j g_k \; a^{\Delta_l-D} \int d^D x \langle \phi_l(x)\rangle^* S_D \delta\ell
\end{aligned} \tag{15.123}$$

where

$$S_D = \frac{2\pi^D}{\Gamma(D/2)} \tag{15.124}$$

is the area of the hypersphere in D dimensions and $\Gamma(x)$ is the Euler Gamma function. The "one-loop" contribution, Eq.(15.123), can be compensated by a change in the coupling constants

$$g_l \mapsto g_l - \frac{1}{2} S_D \sum_{j,k} g_j g_k C_{jkl} \delta\ell \tag{15.125}$$

In fact, the structure of the OPEs implies that similar changes happen at all orders in the expansion.

Finally, there is the dependence in a due to the dependence in L. However, in order to keep the system unchanged the IR cutoff L must be held fixed. Hence this dependence is trivial.

We now collect all the changes in the coupling constants in the differential equation

$$\frac{dg_l}{d\ell} = (D-\Delta_l) g_l - \sum_{k,l} C_{jkl} g_j g_k + \dots \tag{15.126}$$

where we absorbed a factor of $S_D/2$ in a (multiplicative) redefinition of the coupling constants. In summary, we derived the beta-functions for *all* the coupling constants using conformal perturbation theory to order one-loop. As we can see, the beta functions depend on the data of the fixed point:

the scaling dimensions Δ_j and the coefficients of the OPEs of the primary fields. This a general result for all fixed point theories.

15.7 Example of OPEs: ϕ^4 theory

We will now apply the ideas of the preceding sections to the free field fixed point of ϕ^4 theory. The Euclidean lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + ta^{-2}\phi^2 + ua^{D-4}\phi^4 + ha^{-(1+D/2)}\phi \tag{15.127}$$

where we defined the mass m^2, the coupling constant λ, and the symmetry breaking field J in terms the dimensionless couplings, t, u and h,

$$m^2 = ta^{-2}, \quad \lambda = ua^{D-4}, \quad J = ha^{-(1+D/2)} \tag{15.128}$$

Let us now define the normal ordered composite operators

$$:\phi^2 := \phi^2 - \langle \phi^2 \rangle, \qquad :\phi^4 := \phi^4 - 3\langle \phi^2 \rangle \phi^2, \qquad \text{etc} \tag{15.129}$$

In what follows we will use the notations $\phi_n =: \phi^n :$. Clearly, $\phi_1 = \phi$ since $\langle \phi \rangle^* = 0$ at the free field fixed point.

Let us begin by computing the OPE of two $\phi \equiv \phi_1$ fields. Using our results we find

$$\lim_{y\to x} \phi_1(x)\phi_1(y) = \lim_{y\to x} :\phi(x)::\phi(y): = \lim_{y\to x}\left[\frac{2}{|x-y|^{D-2}} + :\phi^2((x+y)/2): + \dots\right] \tag{15.130}$$

Notice that the first term, proportional to the identity operator, is most singular term in the expansion, and that the next term is finite (as $|x-y| \to 0$). The next possible operator would be $:(\partial\phi)^2:$. However, the coefficient will vanish as $|x-y|^2$. hence in the limit it does not contribute. Only singular and finite terms will be kept in the expansion.

We will record this equation as the OPE of two ϕ_1 fields

$$[\phi_1]\cdot[\phi_1] = [1] + [\phi_2] \tag{15.131}$$

where 1 denotes the identity field.

Next we compute the OPE of a ϕ_1 field (i.e. the field ϕ itself) and the $\phi_2 \equiv: \phi(x)^2 :$ field. Using Wick's theorem we find

$$[\phi_1]\cdot[\phi_2] = 2[\phi_1] + [\phi_3] \tag{15.132}$$

where $\phi_3 =: \phi(x)^3 :$.

Next we compute the OPE of two $\phi_2 =: \phi^2 :$ fields. Again, using Wick's theorem, we obtain

$$\lim_{y\to x} : \phi^2(x) :: \phi^2(y) := \lim_{y\to x} \left[\frac{2}{|x-y|^{2(D-2)}} + \frac{4}{|x-y]^{D-2}} : \phi^2((x+y)/2) : + \dots \right] \tag{15.133}$$

which we will record as

$$[\phi_2] \cdot [\phi_2] = 2 + 4[\phi_2] + [\phi_4] \tag{15.134}$$

The following OPEs are derived in a similar fashion

$$[\phi_1] \cdot [\phi_4] = 4[\phi_3] \tag{15.135}$$

$$[\phi_2] \cdot [\phi_4] = 12[\phi_2] + 8[\phi_4] \tag{15.136}$$

$$[\phi_4] \cdot [\phi_4] = 24 + 96[\phi_2] + 72[\phi_4] \tag{15.137}$$

Notice that we have neglected possible descendant operators, such as $: (\partial_\mu\phi)^2 :$, $: (\partial^2\phi)^2 :$, $: (\partial_\mu\phi)^4 :$, etc. These higher derivative operators are typically lead irrelevant operators. This is however not always the case. For instance in the OPE of $: \phi^2 :$ with itself the operator $: (\partial_\mu\phi)^2 :$ can appear. By matching scaling dimensions we see that the prefactor of $: (\partial_\mu\phi)^2 :$ must be proportional to $|x-y|^{-(D-4)}$, which is singular if $D > 4$ and has a logarithmic divergence if $D = 4$. These contributions, however, enter in higher order loops (as we will see in Chapter 16).

Using these results we can now write the beta functions for ϕ^4 theory. With the identification $g_1 = h$ (the symmetry breaking field), $g_2 = t$ (the dimensionless mass squared), and $g_4 = u$ (the coupling constant), we find that their one-loop beta functions are

$$\frac{dh}{d\ell} = \left(\frac{D}{2} + 1\right) h - 4ht + \dots \tag{15.138}$$

$$\frac{dt}{d\ell} = 2t - 4h^2 - 4t^2 - 24tu - 96u^2 + \dots \tag{15.139}$$

$$\frac{du}{d\ell} = (4-D)u - t^2 - 16tu - 72u^2 + \dots \tag{15.140}$$

For $D = 4 - \epsilon$ these beta functions have two fixed points: a) the trivial fixed point at $h^* = t^* = u^* = 0$, and b) the non-trivial Wilson-Fisher fixed point at $h^* = 0$, $t^* = O(\epsilon^2)$, and $u^* = \frac{\epsilon}{72} + O(\epsilon^2)$. The critical exponent ν of the correlation length is the inverse of the slope of the beta function for t at the Wilson-Fisher fixed point. We obtain, $\nu = \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2)$, consistent with what we found before. Also notice that since $t^* = O(\epsilon^2)$, the anomalous dimension of the field ϕ is zero at one-loop order. We will see in Chapter 16 that it is non-zero at two-loop order.