Boson-fermion and fermion-fermion dualities

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(Dated: March 31, 2018)

Abstract

We present exact dualities between relativistic theories of bosons and fermions in 2+1 dimensions

I. FLUX ATTACHMENT

We begin by presenting a formally exact transformation that can be applied to the quantum theory of non-relativistic fermions or bosons.

In the Schrödinger formulation, a system of N particles is described by a wavefunction $\widetilde{\Psi}(\vec{x}_1, \vec{x}_2, \ldots \vec{x}_N)$, obeying an eigenvalue equation

$$H[\vec{\nabla}_i, \vec{x}_i]\, \widetilde{\Psi}(\vec{x}_i) = E\, \widetilde{\Psi}(\vec{x}_i), \tag{1}$$

where the Hamiltonian, H, depends upon the momenta $\vec{\nabla}_i$ and the positions $\vec{x}_i$. For bosons (fermions) the wavefunction $\widetilde{\Psi}$ is a totally symmetric (anti-symmetric) function of its arguments.

Now we perform a "singular gauge transformation" and introduce a new wavefunction via the unitary transformation

$$\Psi(\vec{x}_i) \equiv U^n\, \widetilde{\Psi}(\vec{x}_i) \tag{2}$$

where n is an integer, and U is a unitary transformation given by

$$U = \prod_{i<j} \frac{(z_i - z_j)}{|z_i - z_j|} \tag{3}$$

where $z_i \equiv x_i + i y_i$ are the complex co-ordinates of the particles. Note that for n even (odd) the wavefunction Ψ is a totally symmetric wavefunction for the case where the original $\widetilde{\Psi}$ was a wavefunction for bosons (fermions). For convenience let us now choose Ψ to be bosonic, although almost all of the results below can also easily be extended to the case where Ψ is fermionic.

Note that the unitary transformation in (2) produces a single-valued wavefunction for all integer n. However, U is not well defined when two particles are at the same spatial point. So we will always consider 'hard-core' particles *i.e.* we will assume that Ψ and $\widetilde{\Psi}$ vanish whenever any pair of particle positions come within some small cutoff distance of each other.

We would now like to write down the Schrödinger equation that is obeyed by Ψ, given (1). to obtain this, we need the useful identity

$$\begin{aligned} U^{-1}\, \vec{\nabla}_i\, U &= -i \sum_{j \neq i} \frac{\hat{z} \times (\vec{x}_i - \vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^2} \\ &\equiv -i\, \vec{b}_i. \end{aligned} \tag{4}$$

It is easy to check that the vector $\vec{b}_i$ obeys

$$\begin{aligned} \vec{\nabla}_i \cdot \vec{b}_i &= 0 \\ \vec{\nabla}_i \times \vec{b}_i &= 2\pi \sum_{j \neq i} \delta^2(\vec{x}_i - \vec{x}_j). \end{aligned} \tag{5}$$

Using this unitary transformation, we can deduce the Schrödinger equation obeyed by Ψ:

$$H[\vec{\nabla}_i - i n\, \vec{b}_i, \vec{x}_i]\, \Psi(\vec{x}_i) = E \Psi\, (\vec{x}_i), \tag{6}$$

So in the wavefunction Ψ, each particle i sees a flux tube strength $2\pi n$ attached to every other particle.

We can write the Schrödinger equation (6) in a path integral with the imaginary time Lagrangian

$$\mathcal{L}_\Psi = \Psi^\dagger(x,\tau)\left(\frac{\partial}{\partial\tau} - \frac{(\vec{\nabla} - in\,\vec{b}(\vec{x}) - i\vec{A}(\vec{x}))^2}{2M} - \mu\right)\Psi(x,\tau) + \frac{u}{2}\left|\Psi(x,\tau)\right|^4, \tag{7}$$

where we have introduced simple Hamiltonian with quadratic dispersion with mass M, contact repulsion u, chemical potential, μ, and an external applied magnetic field $\vec{A}$. The 'internal' field $\vec{b}$ now obeys the constraint operator equations

$$\vec{\nabla}\times\vec{b} = 2\pi\,\Psi^\dagger\Psi \tag{8}$$

$$\vec{\nabla}.\vec{b} = 0 \tag{9}$$

at all points in spacetime. Th constraint in (8) can be implemented by introducing a conjugate Lagrange multiplier, nb_τ in the Lagrangian so that (7) becomes

$$\begin{aligned}\mathcal{L}_\Psi = \Psi^\dagger(x,\tau)&\left(\frac{\partial}{\partial\tau} - inb_\tau - \frac{(\vec{\nabla} - in\,\vec{b}(\vec{x}) - i\vec{A}(\vec{x}))^2}{2M} - \mu\right)\Psi(x,\tau) + \frac{u}{2}\left|\Psi(x,\tau)\right|^4 \\ &+ \frac{in}{2\pi}b_\tau\left(\vec{\nabla}\times\vec{b}\right).\end{aligned} \tag{10}$$

The integral over b_τ directly reproduces the constraint in (8). Now we notice that all terms in the first line of (10) are invariant under spacetime-dependent gauge transformations in which $b_\mu \equiv (b_\tau, \vec{b})$ transforms as a U(1) gauge field. The last term in (10) is not invariant under such gauge transformations, but we will now show that it does become invariant after we impose the gauge condition in (9). In particular, we note that this last term is almost a Chern-Simons term because

$$\epsilon_{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda = 2b_\tau\left(\vec{\nabla}\times\vec{b}\right) + (b_y\partial_\tau b_x - b_x\partial_\tau b_y) + \text{total derivatives}. \tag{11}$$

Now, we can always solve the constraint (9) in two dimensions by the parameterization $\vec{b} = \hat{z}\times\vec{\nabla}\varrho$ for some scalar field ϱ; inserting this in (11), the second term on the right-hand-side is easily shown to vanish up to total derivatives. Focusing on just the last term in (10), this implies that for the path integral of interest to us we can write

$$\begin{aligned}\int\mathcal{D}b_\mu\,\delta\left(\vec{\nabla}\cdot\vec{b}\right)&\exp\left(-\frac{in}{2\pi}\int d^3x\,b_\tau(\vec{\nabla}\times\vec{b}) + \ldots\right) \\ &= \int\mathcal{D}b_\mu\,\delta\left(\vec{\nabla}\cdot\vec{b}\right)\exp\left(-\frac{in}{4\pi}\int d^3x\,\epsilon_{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda + \ldots\right),\end{aligned} \tag{12}$$

because the difference in the actions vanishes when $\vec{\nabla}\cdot\vec{b} = 0$. In the second form, all terms inside the exponential are invariant under gauge transformations. Now we make the change of variable in the functional integral $\vec{b} \to \vec{b} - \vec{\nabla}\zeta$. The gauge invariance of the action implies that it remains unchanged, while the constraint does require modification:

$$\int \mathcal{D}b_\mu \, \delta\left(\vec{\nabla}\cdot\vec{b}\right) \exp\left(-\frac{in}{4\pi}\int d^3x \, \epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda + \ldots\right)$$
$$= \int \mathcal{D}b_\mu \, \delta\left(\vec{\nabla}\cdot\vec{b} - \vec{\nabla}^2\zeta\right) \exp\left(-\frac{in}{4\pi}\int d^3x \, \epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda + \ldots\right). \tag{13}$$

This equivalence implies that the path integral is independent of the choice of ζ. Hence, we can just integrate over ζ (at the cost of an unimportant overall prefactor and Jacobian), and then the functional integral over b_μ does not have any delta-function constraint. We are left with a conventional path integral over a U(1) gauge field b_μ, and a boson Ψ carrying n units of U(1) charge, with the gauge-invariant action

$$\mathcal{S}_\Psi = \int d^3x \left[\Psi^\dagger(x,\tau)\left(\frac{\partial}{\partial\tau} - in\, b_\tau - \frac{(\vec{\nabla} - in\,\vec{b}(\vec{x}) - i\vec{A}(\vec{x}))^2}{2M} - \mu\right)\Psi(x,\tau) + \frac{u}{2}\left|\Psi(x,\tau)\right|^4 \right.$$
$$\left. + \frac{in}{4\pi}\epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda\right]. \tag{14}$$

So the flux attachment procedure yields a theory of particles minimally coupled to an emergent U(1) gauge field b_μ with a Chern-Simons action.

II. FERMION-BOSON DUALITY I

The flux attachment transformation of Section I has been used extensively to obtain theories of quantum Hall states. The first step in these analyses is invariably a flux averaging procedure: the strongly fluctuating gauge field is expanded about a saddle point in which the gauge flux is replaced by its spatially average value. In recent years, an understanding is emerging that such a flux averaging procedure is can be dangerous, especially when considering the physics within a Landau level: it does lead to mostly correct results, but subtle errors appear when considering higher order corrections [1, 2].

In this section we will present results which have emerged in recent years [3–5], and which are believed to be exactly correct. We will derive these results using the flux attachment procedure of Section I, but will limit the flux-averaging step to the vicinity of a Mott insulator. Density, and hence flux, fluctuations are suppressed in a Mott insulator. Moreover, we will limit ourselves to cases in which the models have an emergent particle-hole symmetry, and so there are no fluctuations in the average flux. Under these circumstances, one can hope the flux averaging step does not introduce any errors.

We will map between the Wilson-Fisher theory of the phase transition in the $D = 3$ XY model, and a model of Dirac fermions. This mapping was originally obtained from a lattice model bosons studied in Ref. [6], and we shall follow their method here. We consider bosons, b_i on the square lattice with a staggered on-site potential:

$$H_b = \epsilon_0 \sum_i \eta_i b_i^\dagger b_i - \sum_{i<j} t_{ij} \left(b_i^\dagger b_j + b_j^\dagger b_i \right) + \frac{U}{2} \sum_i n_i (n_i - 1) \tag{15}$$

where $n_i = b_i^\dagger b_i$ and $\eta_i = +1$ ($\eta_i = -1$) on sublattice A (sublattice B) of the square lattice. Note that $\epsilon_0 \neq 0$ breaks the sublattice symmetry, and there are 2 sites per unit cell. We work at an average density $Q = \langle b_i^\dagger b_i \rangle = 1/2$. At small U the ground state is a superfluid, and large U the ground state is a non-degenerate ('trivial') insulator with all bosons on the B sublattice. The quantum phase transition can be shown, using the same methods as those used for the Hubbard model with $\epsilon_0 = 0$ and $Q = 1$, to be in the universality class of the Wilson-Fisher CFT with $N = 2$ component real fields. We can write partition function for the vicinity of the critical point as a field theory for the complex superfluid order parameter Φ:

$$\mathcal{Z}_b[A] = \int \mathcal{D}\Phi \exp\left(- \int d^3x \, \mathcal{L}_b \right)$$

$$\boxed{\mathcal{L}_b = |(\partial_\mu - i A_\mu)\Phi|^2 + s|\Phi|^2 + u|\Phi|^4 \,.} \tag{16}$$

For future convenience, we have introduced an external U(1) gauge field A_μ which couples minimally to the current associated with the boson number Q.

Now we will analyze the superfluid-insulator transition in H_b in a different manner. We perform the flux attachment transformation, and convert H_b to a model of fermions c_i: this yields a theory of fermions coupled to a dynamical U(1) gauge field b_μ. Up to this point, everything is formally exact. Then we perform a potentially dangerous step: we expand about a saddle point in which we replace the flux of the b_μ field by its average value. As we are in the vicinity of the Mott insulator, where boson number fluctuations are suppressed, we presume that the flux averaging procedure is legitimate.

So let us examine the structure of such a saddle point. As the density of bosons is $Q = 1/2$, the fermions move in an average flux of π per plaquette. The fermion Hamiltonian in such a background takes the form

$$H_f = \epsilon_0 \sum_i \eta_i c_i^\dagger c_i - \sum_{i<j} \left(t_{ij} c_i^\dagger c_j + t_{ij}^* c_j^\dagger c_i \right) \tag{17}$$

Whereas the t_{ij} in H_b were real, now the t_{ij} acquire additional phase factors to account for the average flux. We take first and second neighbor hopping t_1 and t_2 as shown in Fig. 1. We employ a 2 site unit cell, and then the momentum space Hamiltonian is

$$H_f = \sum_{\alpha,\beta} c_\alpha^\dagger(k) M_{\alpha\beta}(k) c_\beta(k) \tag{18}$$

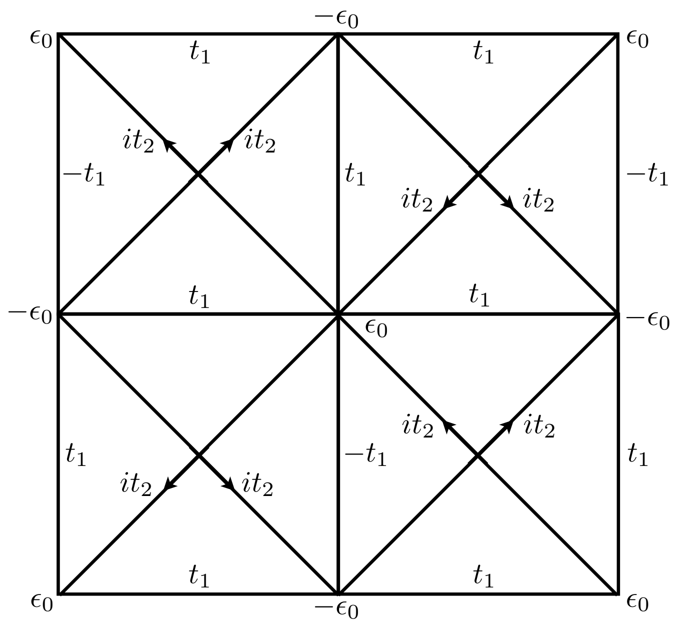

FIG. 1. Unit cell of the saddle-point Hamiltonian, H_f, for the fermions

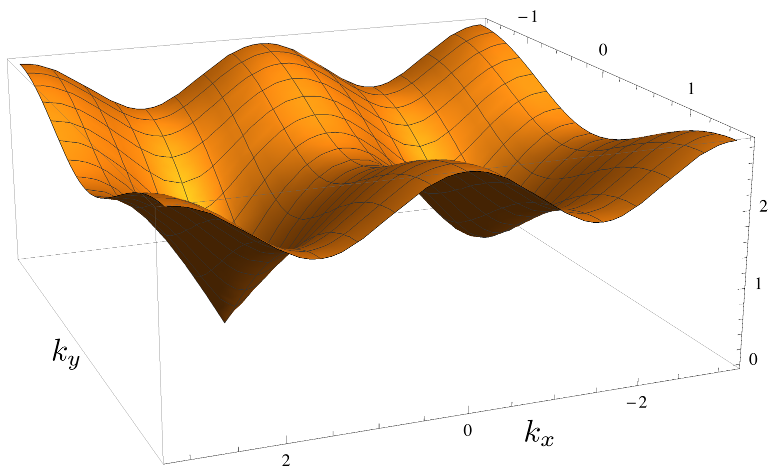

FIG. 2. The upper band of the Hamiltonian in Eq. (18) for $t_1 = 1$, $t_2 = 0.1$, $\epsilon_0 = 0.4$. There is a massless Dirac node at $(-\pi/2, 0)$, while that at $(\pi/2, 0)$ remains massive.

where $\alpha, \beta = A, B$, and

$$M = \begin{pmatrix} \epsilon_0 - 2t_2 \sin(k_x + k_y) - 2t_2 \sin(k_x - k_y) & -2t_1 \cos(k_x) - 2it_1 \sin(k_y) \\ -2t_1 \cos(k_x) + 2it_1 \sin(k_y) & -\epsilon_0 + 2t_2 \sin(k_x + k_y) + 2t_2 \sin(k_x - k_y) \end{pmatrix} . \quad (19)$$

The band structure obtained from M is shown in Fig. 2. For $t_2 = \epsilon_0 = 0$, this Hamiltonian has Dirac nodes at $\vec{k} = (\pm\pi/2, 0)$. We focus on the vicinities of these points by writing $\vec{k} = (\pm\pi/2 + q_x, q_y)$ and expand for small q_x, q_y. We also introduce Pauli matrices τ^x, τ^y, τ^z in the sublattice space.

Then we can write the Hamiltonian as

$$H_f = c^\dagger \left[\pm 2t_1 q_x \tau^x + 2t_2 q_y \tau^y + (\epsilon_0 \mp 4t_2)\tau^z \right] c \tag{20}$$

This is the Hamiltonian of two species of two-component Dirac fermions with masses $\epsilon_0 \pm 4t_2$. We introduce the relativistic fermion fields

To map to the superfluid-insulator transition of bosons, it turns out that we have to consider the phase transition in which the mass of one of the Dirac fermions changes sign, while the mass of the other Dirac fermion remains non-zero: this corresponds to a change in the Chern number of the band structure by unity. So let us consider the regime $\epsilon_0 \approx 4t_2$ and focus first on the Dirac fermion with the light mass $m = \epsilon_0 - 4t_2$. We introduce relativistic notation by defining $\psi = c$, $\bar{\psi} \equiv \psi^\dagger \tau^z$, and $\gamma^\mu = (\tau^z, -\tau^y, \tau^x)$, and then the imaginary time Lagrangian corresponding to the light mass Dirac fermion in H_f is

$$\mathcal{L}_f = \bar{\psi}\gamma^\mu \partial_\mu \psi + m\bar{\psi}\psi \,, \tag{21}$$

where we have absorbed the Fermi velocity $2t_1$ by rescaling time.

Let us now move beyond the saddle point approximation, and include fluctuations by including the dynamical U(1) gauge field b_μ and its Chern-Simons term. We also couple the fermions to the external U(1) gauge field A_μ. So we now obtain a theory of Dirac fermions coupled to a U(1) gauge field:

$$\mathcal{L}_f = \bar{\psi}\gamma^\mu(\partial_\mu - ib_\mu - iA_\mu)\psi + m\bar{\psi}\psi + \frac{i}{4\pi}\epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda \,. \tag{22}$$

We have chosen the integer $n = 1$ in the Chern-Simons term: this turns out to be necessary to properly reproduce the time-reversal invariant physics of the bosonic side of the fermion-boson duality we will shortly describe.

We need one additional important ingredient. It turns out the Dirac fermions with the heavy mass $M = \epsilon_0 + 4t_2$ cannot entirely be neglected. We label this fermion ψ_h, and we have a corresponding Lagrangian

$$\mathcal{L}_{fh} = \bar{\psi}_h\gamma^\mu(\partial_\mu - ib_\mu - iA_\mu)\psi_h + M\bar{\psi}_h\psi_h \,. \tag{23}$$

Because M is large, we can safely integrate out the ψ_h: this induces an effective action for the gauge field $b_\mu + A_\mu$. The important term is obtained exactly at the one loop level: it is a Chern-Simons term 'at level -1/2':

$$\mathcal{L}_{b+A} = -\frac{i\,\mathrm{sgn}(M)}{8\pi}\epsilon_{\mu\nu\lambda}(b_\mu + A_\mu)\partial_\nu(b_\lambda + A_\lambda) \tag{24}$$

The negative sign is from the opposite chirality between the light and heavy fermions. Combining Eqs. (22) and (24), and redefining $b_\mu \to b_\mu - A_\mu$, we have the continuum fermion theory in its final form

$$\boxed{\mathcal{L}_f = \bar{\psi}\gamma^\mu(\partial_\mu - ib_\mu)\psi + m\bar{\psi}\psi + \frac{i}{8\pi}\epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda - \frac{i}{2\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu b_\lambda + \frac{i}{4\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \,.} \tag{25}$$

We can now present the final statement of the boson-fermion duality. The fermionic partition function

$$\mathcal{Z}_f[A] = \int \mathcal{D}\psi \mathcal{D}b_\mu \exp\left(-\int d^3x\, \mathcal{L}_f\right) \tag{26}$$

with the Lagrangian in Eq. (25) equals the bosonic partition function $\mathcal{Z}_b[A]$ in Eq. (16). In the bosonic theory, we tune across the superfluid-insulator quantum critical point by changing the parameter s. In the fermionic theory we tune across the critical point where the sign of m changes.

To verify this duality, let us confirm that the bosonic and fermionic theories match on the two sides of the quantum critical point.

First consider the case $s > s_c$. Here the boson Φ is gapped, and the effective action of A_μ is just a conventional Maxwell action. In the fermionic theory, this corresponds to the case $m > 0$. We integrate out ψ with this mass using Eq. (24) but with the opposite sign; this yields the effective Lagrangian for the gauge fields

$$\frac{i}{4\pi}\epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda - \frac{i}{2\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu b_\lambda + \frac{i}{4\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda = \frac{i}{4\pi}\epsilon_{\mu\nu\lambda} (b_\mu - A_\mu)\partial_\nu (b_\lambda - A_\lambda)\,. \tag{27}$$

If we now integrate over b_μ, it appears that the effective action for A_μ vanishes. More precisely, the Chern-Simons term in A_μ cancels, and higher order terms (which we have ignored) will yield a Maxwell action for A_μ. This matches with the bosonic answer.

Next consider the case $s < s_c$. Here the boson Φ is condensed, and A_μ is Higgsed *i.e.* we are in a Meissner phase were A_μ flux vanishes. In the In the fermionic theory, this corresponds to the case $m < 0$: we integrate out ψ with this mass, and now the effective Lagrangian for the gauge fields is

$$-\frac{i}{2\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu b_\lambda + \frac{i}{4\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda\,. \tag{28}$$

The integral over b_μ yields a delta function in the A_μ flux, which is the expected answer in the Meissner phase.

III. FERMION-BOSON DUALITY II

We obtain our second exact duality by introducing another external gauge field C_μ, and adding the term

$$\frac{i}{2\pi}\epsilon_{\mu\nu\lambda} C_\mu \partial_\nu A_\lambda - \frac{i}{4\pi}\epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda\,, \tag{29}$$

to both sides of the duality in Section II. Then we promote the gauge field A_μ to a dynamical gauge field, and integrate over it. In keeping with our notational convention, we map $A_\mu \to a_\mu$, and then perform the integration.

On the bosonic side of the duality we obtain the partition function

$$\mathcal{Z}_b[C] = \int \mathcal{D}\Phi \mathcal{D}a_\mu \exp\left(-\int d^3x\, \mathcal{L}_b\right)$$

$$\boxed{\mathcal{L}_b = |(\partial_\mu - ia_\mu)\Phi|^2 + s|\Phi|^2 + u|\Phi|^4 - \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}C_\mu\partial_\nu a_\lambda\,.} \tag{30}$$

On the fermionic side we obtain

$$\mathcal{Z}_f[C] = \int \mathcal{D}\psi\mathcal{D}b_\mu\mathcal{D}a_\mu \exp\left(-\int d^3x\,\mathcal{L}_f\right) \tag{31}$$

where

$$\mathcal{L}_f = \bar{\psi}\gamma^\mu(\partial_\mu - ib_\mu)\psi + m\bar{\psi}\psi + \frac{i}{8\pi}\epsilon_{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda - \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu b_\lambda + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}C_\mu\partial_\nu a_\lambda\,. \tag{32}$$

From the path integral over a_μ, we obtain the constraint $b_\mu = C_\mu$. So the final form of the fermionic theory is

$$\mathcal{Z}_f[C] = \int \mathcal{D}\psi \exp\left(-\int d^3x\,\mathcal{L}_f\right) \tag{33}$$

where

$$\boxed{\mathcal{L}_f = \bar{\psi}\gamma^\mu(\partial_\mu - iC_\mu)\psi + m\bar{\psi}\psi + \frac{i}{8\pi}\epsilon_{\mu\nu\lambda}C_\mu\partial_\nu C_\lambda\,.} \tag{34}$$

IV. FERMION-FERMION DUALITY

Finally, we can combine the fermion-boson dualities in Sections II and III to obtain a new fermion-fermion duality.

We begin by adding the following term to the right hand side of Eq. (34)

$$-\frac{i}{8\pi}\epsilon_{\mu\nu\lambda}C_\mu\partial_\nu C_\lambda + \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu C_\lambda \tag{35}$$

where A_μ is a new background gauge field. Then we integrate over C_μ in the fermionic partition function Eq. (33). Changing notation with $C_\mu \to b_\mu$, we obtain the fermionic theory

$$\mathcal{Z}_f[A] = \int \mathcal{D}\psi\mathcal{D}b_\mu \exp\left(-\int d^3x\,\mathcal{L}_f\right) \tag{36}$$

where

$$\boxed{\mathcal{L}_f = \bar{\psi}\gamma^\mu(\partial_\mu - ib_\mu)\psi + m\bar{\psi}\psi + \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu b_\lambda\,.} \tag{37}$$

Now let us apply the same transformation to the bosonic theory in Eq. (30). Then we obtain

$$\mathcal{Z}_b[A] = \int \mathcal{D}\Phi\mathcal{D}a_\mu\mathcal{D}b_\mu \exp\left(-\int d^3x\,\mathcal{L}_b\right)$$

$$\mathcal{L}_b = |(\partial_\mu - ia_\mu)\Phi|^2 + s|\Phi|^2 + u|\Phi|^4 - \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}b_\mu\partial_\nu a_\lambda - \frac{i}{8\pi}\epsilon_{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda + \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu b_\lambda\,. \tag{38}$$

Integrating over b_μ in Eq. (38) we obtain

$$\mathcal{L}_b = |(\partial_\mu - ia_\mu)\Phi|^2 + s|\Phi|^2 + u|\Phi|^4 + \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda + \frac{i}{8\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda\,. \tag{39}$$

Comparing with a time-reversed partner of the duality in Section III (see Ref. [4] for details) we find finally that the theory is equivalent to the fermionic theory of Section III.

$$\mathcal{Z}_f[A] = \int \mathcal{D}\psi \exp\left(-\int d^3x\,\mathcal{L}_f\right) \tag{40}$$

where

$$\boxed{\mathcal{L}_f = \bar{\psi}\gamma^\mu(\partial_\mu - iA_\mu)\psi + m\bar{\psi}\psi\,.} \tag{41}$$

It should be noted that the fermionic fields ψ and the masses m in the dual theories in Eq. (37) and (41) are not the same. The fermionic fields are related by the attachment of a doubled 'monopole' [4], while the masses have opposite signs.

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