Geometric and topological aspects of Type IIB D-branes

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Introduction

The discovery of general relativity and quantum theory were remarkable events in the history of theoretical physics. They completely revolutionized our understanding of space, time, matter and energy at very large and very small scales. The simple and intuitive Newtonian pictures of space, time, gravity and pointlike particles were obviously fundamentally flawed and had to be replaced by constructions so astonishingly unintuitive that even the best modern populistic books on these subjects are almost certainly lacking at some fundamental level. The problem being, of course, that there is simply nothing in everyday life that would resemble these phenomena closely enough to be very useful as an explanatory analogue. Thus, if one wishes to truly understand such theories, the only possibility is through mathematics. An ideal plan of work would be to first perform experiments as far as possible, then try to construct a suitably general mathematical structure befitting the physical theory and proceed to investigate it. If the mathematical structure used is general and elaborate enough, one can hope to run into new substructures, not obvious from the original physical theory. Finally, one should attempt to perform an experiment to verify if the new substructure is indeed realized in nature.

String theory is an ongoing attempt to find a consistent mathematical structure unifying both general relativity and quantum theory as substructures, which replaces pointlike elementary particles with small 1-dimensional supersymmetric strings, oscillating in a 10-dimensional spacetime. In addition to strings, superstring theory contains higher-dimensional dynamical extended objects, called D-branes, to which open string endpoints are restricted. An open string propagating in spacetime determines a 2-dimensional superconformal field theory and the boundary conditions the D-brane determines must preserve the superconformal invariance of the theory. This is why D-branes are inherently more complicated objects than simple submanifolds of spacetime.

In the low-energy limit superstring theory is described by an effective field theory, which is 10-dimensional supergravity. It contains several generalized higher-dimensional massless gauge fields, most importantly the RR-fields and the NS-NS B-field. D-branes are the extended objects coupling to the RR-fields. Dirac quantization for the RR-charge (of a D-brane) implies that the charge lifts to a class in some generalized integral cohomology theory. There does not seem to be any absolute way of determining what this generalized cohomology theory should be, but there are strong arguments suggesting that for RR-charges the correct cohomology theory is K-theory, at least when there is no background B-field present.

Turning on a background B-field makes the situation considerably more difficult. The B-field and D-branes are related to each other, because the B-field couples to the 2-dimensional open string worldsheet by higher-dimensional parallel transport and the boundary of the worldsheet couples to the D-brane. There is an anomaly associated to these couplings: the

Freed-Witten anomaly. This anomaly can be cancelled when a certain cohomology equation is satisfied. When the B-field is set to be nontrivial, the Freed-Witten anomaly cancellation condition implies that D-brane charges are classified by twisted K-theory, rather than by K-theory. Finally, twisted K-theory can be defined purely algebraically using operator K-theory, opening up a possibility for extending the definition of RR-charge to noncommutative D-brane worldvolumes in a noncommutative spacetime.

The problem with string theory is that the mathematical structure is incredibly complicated and so far there are no known experiments to verify its reality. One can only continue to investigate the mathematical structure in hopes of running into a possibility for experimental verification. Even if the theory of D-branes would never be verified experimentally, it has already contributed an immense amount of motivation and research topics to pure mathematics, especially to modern geometry, topology and algebra.

The plan of the work

Chapter 1 consists of a brief introduction to the underlying physical theory, followed by a long and comprehensive overview of the rest of the text. This is to make the thesis accessible to a wider audience.

Chapter 2 is the beginning of the main body of the text. It starts with a review of the theory of characteristic classes and Spin structures, followed by the introduction of K-theory (of vector bundles). After that, we explain how the K-theoretic classification of D-brane charge arises. We also take a look at an alternative, and perhaps more intuitive, classification scheme of D-branes in terms of K-homology, the homological dual theory of K-theory. More precisely, we use the geometric model of Baum and Douglas for K-homology.

In chapter 3 we discuss the Freed-Witten anomaly. Most of the chapter deals with introducing the required mathematical machinery, including Deligne cohomology, bundle gerbes, bundle gerbe modules and their connections, higher-dimensional holonomy and higher-dimensional parallel transports. Having done that, we can give a proper mathematical description for all the physical objects contributing to the Freed-Witten anomaly. The chapter concludes with a derivation of the anomaly cancellation condition.

Chapter 4 starts with an introduction to operator K-theory and twisted K-theory, which we use to classify the RR-charge of twisted D-branes, that is, D-branes in a nontrivial background B-field. Our ultimate goal in this chapter is a noncommutative generalization of the D-brane charge formula. Several concepts from geometry and topology need to be translated to the of noncommutative geometry first. This requires the use of KK-theory of Kasparov, a unification of K-theory and K-homology, and the bivariant local cyclic homology theory of Puschnigg, a noncommutative generalization of ordinary homology and cohomology.

The range of mathematical topics treated in this thesis is undeniably quite wide. We provide introductions to the more advanced topics, but the reader is still expected to be familiar with basic topics in geometry, topology and algebra.

Chapter 1

Overview

1.1 Strings and D-branes

In Type IIB superstring theory¹ pointlike elementary particles are replaced with supersymmetric 1-dimensional strings, open or closed, of typical length 10^{-35} m. Thus, they are way beyond what can be seen in particle accelerators. The spacetime, denoted by X throughout the text, is assumed to be a smooth, 10-dimensional² oriented Riemannian³ Spin manifold⁴. All manifolds are locally compact and paracompact. Locally compact spaces can be compactified by adding a "point at infinity". It is a deep result that compact manifolds have the homotopy type of a finite CW-complex. Paracompactness is important for the existence of smooth partitions of unity and good open covers, which makes Čech cohomology particularly well-behaving. We denote by M the worldsheets of both open and closed strings. Thus, M is a 2-dimensional compact oriented manifold with an embedding $\xi: M \to X$. A closed string worldsheet is a closed manifold⁵. An open string worldsheet has a boundary, corresponding to the endpoints of the open string.

The dynamics of the classical string are governed by the famous *Polyakov action* [Pol1]. Canonical quantization can be carried out, at least in flat spacetime, yielding an infinite tower of oscillatory modes, each corresponding to a massless or massive particle. Of course, open and closed strings yield very different sets particles.

The closed string spectrum contains a locally defined symmetric rank 2 tensor field, which can be identified with the *graviton field* (it satisfies the linearized Einstein equation), a locally defined 2-form field known as the NS-NS (Neveu-Schwarz-Neveu-Schwarz) B-field, and a scalar field known as the dilaton, whose vacuum expectation value determines the string coupling constant. Of these, the B-field is the only one we are interested in. They are

¹We will only discuss Type IIB superstring theory, since it is, in some sense, the easiest.

²This is the so-called *critical dimension*, which is required for the superconformal anomaly of the worldsheet superconformal field theory to vanish [Pol1, Joh1].

 $^{^{3}}$ A more physical assumption would be to take X to be pseudo-Riemannian, but we assume that one can always perform Wick rotations in the path integrals when necessary.

⁴A manifold X is Spin if and only if the second Stiefel-Whitney class of its cotangent bundle, $w_2(X) \in H^2(X, \mathbb{Z}_2)$, vanishes, and Spin^c if and only if the third integral Stiefel-Whitney class of its cotangent bundle $W_3(X) \in H^3(X, \mathbb{Z})$ vanishes. A reader unfamiliar with these concepts can, for now, regard them simply as technical conditions defined by the vanishing of the above-mentioned Stiefel-Whitney classes. The assumption that the spacetime should be Spin is, of course, motivated by the physical fact that spinor particles exist.

⁵That is, compact and without boundary.

higher-dimensional gauge fields, with gauge transformations

$$B \mapsto B + d\Lambda$$
,

where Λ is a 1-form, leaving physical states invariant. Instead, the 3-form field H = dB, called the H-flux, is globally defined.

The closed string spectrum contains also other higher p-form fields G_p , the RR (Ramond-Ramond) fields. They are globally defined fields of odd degree (in Type IIB superstring theory). Locally, one can find RR-potentials C_p , where p = 0, 2, 4, 6, 8, such that $G_{p+1} = dC_p$ (in the absence of sources)⁶. Let G denote the polyform of all RR-fields. In the absence of sources, the equations of motion for the RR-fields are

$$d*G=0.$$

Analogously to classical electromagnetism, we can introduce sources for the RR-fields as violations of these equations by an RR-charge density polyform j_e :

$$d*G=j_e$$
.

Such an equation of motion can be obtained from an action of the form

$$\int_X G \wedge *G + \int_X C \wedge j_e.$$

Each homogeneous component of j_e determines a nontrivial class $[j_e^p]$ in compactly supported de Rham cohomology $H_{c,dR}^p(X)$ [BT] of spacetime. The Poincaré duals of these classes are homology cycles, representing certain even-dimensional extended objects.

Open strings are more complicated. First of all, boundary conditions have to be set for the string in such a way that the superconformal invariance of the worldsheet theory is preserved. The open string worldsheet M, embedded into X by $\xi: M \to X$, has a boundary ∂M . Naively, the boundary conditions can be given as a subset of X onto which $\xi(\partial M)$ should be restricted. This is achieved by specifying a p+1-dimensional closed oriented manifold Σ , which we call a Dp-brane worldvolume or simply a D-brane worldvolume, together with a continuous or smooth map $\phi: \Sigma \to X$. It is said that the D-brane wraps the worldvolume Σ . We then require that $\xi(\partial M) \subset \phi(\Sigma)$.

The open string spectrum yields a massless locally defined $\mathfrak{u}(n)$ -valued 1-form field A, living on D-brane worldvolumes. The usual physical interpretation is that n denotes the number of coincident Dp-branes wrapping the same worldvolume. Thus, a single D-brane supports a simple 1-form field. Now, open strings are attached to D-branes, not merely to their worldvolumes. Thus, given a stack of n coincident D-branes, all wrapping Σ , the associated open string states are enriched with new non-dynamical degrees of freedom, since the endpoints are free to attach to any of the n D-branes. These new degrees of freedom, called $Chan-Paton\ factors$, are antihermitian $n \times n$ matrices, elements of $\mathfrak{u}(n)$. There is an anomaly associated to the open string path integral, the $Freed-Witten\ anomaly$, which must be cancelled for the theory to be consistent. Let A_{α} denote the A-field on an open set U_{α}

⁶The RR-fields are actually not all independent. They are related to each other by the duality condition $*G_p = G_{10-p}$. This causes problems in quantization [MW, FH, Val], but since we shall not discuss quantized RR-fields, the reader is advised to simply forget about the duality condition.

of Σ . To cancel the anomaly, on intersections $U_{\alpha} \cap U_{\beta}$ the local representatives A_{α}, A_{β} must satisfy

$$A_{\beta} = g_{\alpha\beta}^{-1} A_{\alpha} g_{\alpha\beta} - i g_{\alpha\beta}^{-1} dg_{\alpha\beta} - \Lambda_{\alpha\beta} , \qquad (1.1)$$

where $g_{\alpha\beta}$ are functions $U_{\alpha\beta} \to U(n)$ and $\Lambda_{\alpha\beta}$ are (real-valued) 1-forms. This resembles the transformation rule of an ordinary U(n) gauge field, which is a connection on a U(n) principal bundle. On triple intersections the "transition functions" $\{g_{\alpha\beta}\}$ satisfy

$$g_{\alpha\beta}g_{\beta\gamma}g_{\alpha\gamma}^{-1} = \zeta_{\alpha\beta\gamma} \,,$$

where $\{\zeta_{\alpha\beta\gamma}\}$ are continuous U(1)-valued functions on $U_{\alpha}\cap U_{\beta}\cap U_{\gamma}$. The right-hand side should obviously be interpreted as $\zeta_{\alpha\beta\gamma}\mathbbm{1}_{n\times n}$. This means that $\{g_{\alpha\beta}\}$ provide the transition data for a principal PU(n):=U(n)/U(1) bundle, the *(projective) Chan-Paton bundle*. The obstruction for lifting a PU(n) bundle to a U(n) bundle is measured by an n-torsion cohomology class in $H^3(\Sigma, \mathbb{Z})$. For the Chan-Paton bundle, this class is denoted by $\beta(\underline{\zeta})$, due to the following simple construction. Notice first, that $\{\zeta_{\alpha\beta\gamma}\}$ determines a Čech 2-cocycle, since

$$\zeta_{\alpha\beta\gamma}\zeta_{\alpha\gamma\delta} = g_{\alpha\beta}g_{\beta\gamma}g_{\alpha\gamma}^{-1}g_{\alpha\gamma}g_{\gamma\delta}g_{\alpha\delta}^{-1} = g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\delta}g_{\alpha\delta}^{-1} = \zeta_{\beta\gamma\delta}g_{\alpha\beta}g_{\beta\delta}g_{\alpha\delta}^{-1} = \zeta_{\beta\gamma\delta}\zeta_{\alpha\beta\delta},$$

which is the cocycle condition

$$\zeta_{\beta\gamma\delta}\zeta_{\alpha\gamma\delta}^{-1}\zeta_{\alpha\beta\delta}\zeta_{\alpha\beta\gamma}^{-1}=1.$$

Thus, it defines a Čech cohomology class $[\underline{\zeta}] \in H^2(\Sigma, \underline{U(1)})$. Here $\underline{U(1)}$ denotes the sheaf [Bry] of continuous functions taking values in U(1). More generally, \underline{G} denotes the sheaf of continuous functions taking values in the Abelian group G. We use the notation $\underline{\zeta}$ to denote the Čech cocycle associated to $\{\zeta_{\alpha\beta\gamma}\}$. In this notation the Čech cocycle condition is written as $\delta(\zeta) = 1$. The exact sequence of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{R}} \longrightarrow \underline{U(1)} \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow \check{H}^2(\Sigma,\underline{\mathbb{R}}) \longrightarrow \check{H}^2(\Sigma,\underline{U(1)}) \stackrel{\beta}{\longrightarrow} H^3(\Sigma,\underline{\mathbb{Z}}) \cong H^3(\Sigma,\mathbb{Z}) \longrightarrow \check{H}^3(\Sigma,\underline{\mathbb{R}}) \longrightarrow \cdots$$

of Čech cohomology groups [Bry]. It is easy to show, by a partition of unity argument, that $H^p(\Sigma, \mathbb{R})$ vanish for p > 0. Thus, the *Bockstein homomorphism* β is an isomorphism, mapping $[\underline{\zeta}]$ to $\beta([\underline{\zeta}]) \in H^3(\Sigma, \underline{\mathbb{Z}}) \cong H^3(\Sigma, \mathbb{Z})^7$. It is easy to see that the vanishing of this class is equivalent to the existence of U(n) lifts $\{\widetilde{g}_{\alpha\beta}\}$, satisfying

$$\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\alpha\gamma}^{-1}=1$$
.

$$\exp(i\delta(\rho)) = \delta(\exp(i\rho)) = \delta(\zeta) = 1.$$

But this is possible if and only if

$$\delta(\rho) = 2\pi n$$
,

for some integer-valued Čech 3-cochain \underline{n} . It is obviously a cocycle due to $\delta^2 = 0$. The image $\beta([\underline{\zeta}]) \in H^3(X, \mathbb{Z})$ is precisely the class $[\underline{n}] \in H^3(X, \mathbb{Z})$.

⁷A more concrete description is given as follows. Writing $\underline{\zeta} = \exp(i\underline{\rho})$, where $\underline{\rho}$ is a real-valued Čech 2-cochain, and using the cocycle condition $\delta(\zeta) = 1$, yields

Thus, they define a U(n) bundle, the *lift* of the PU(n) bundle. The class $\beta([\underline{\zeta}])$ plays a fundamental role in Freed-Witten anomaly cancellation. In conclusion: a D-brane (or rather, a stack of n coincident D-branes) consists of a closed oriented worldvolume manifold Σ , a map $\phi: \Sigma \to X$ and a PU(n) bundle over Σ .

The string spectrum described above can actually be fully obtained only in flat spacetime. We have to assume that the low-energy field content remains the same also in more general spacetimes. Nevertheless, a quantum gravity theory, such as string theory, should be able to produce perturbations to the background metric. Adding a linear perturbation to the metric in the Polyakov action corresponds to turning on an infinite background, or a coherent state, of gravitons [Pol1]. One can similarly consider backgrounds of the other massless string states, in particular, a background B-field⁸. Suppose that various massless background fields have been turned on. By studying string perturbation theory in the low-energy limit, one obtains equations of motion for these background fields. It is then possible to construct a low-energy effective (worldvolume) action, corresponding to these equations of motion. The low-energy effective field theory is the 10-dimensional Type IIB supergravity [Pol1].

1.2 The low-energy effective field theory

We now introduce couplings and sources to the effective field theory. For example, the B-field should couple to a 2-dimensional object, which is the string worldsheet. The coupling should be something like

$$\int_M \xi^* B\,,$$

where $\xi: M \to X$ is the embedding of the open or closed string worldsheet. For an open string worldsheet this integral should be taken with a grain of salt, because when $\partial M \neq \emptyset$, it is not gauge invariant:

$$\int_M \xi^*(B+d\Lambda) = \int_M \xi^* B + \int_{\partial M} \Lambda \,.$$

For a closed string worldsheet this is obviously not a problem, since $\partial M = \emptyset$. This failure of gauge invariance is one of the main aspects of the Freed-Witten anomaly.

Similarly, if C_{p+1} is a p+1-form RR-potential, it should couple to a p+1-dimensional extended object. It was a remarkable discovery by Polchinski [Pol2] that these are precisely the D-branes we described earlier⁹. Now, one might naively anticipate that a Dp-brane should couple to the RR-potentials by

$$\int_{\Sigma} \phi^* C_{p+1} \,,$$

but this turns out to be incorrect: a Dp-brane carries also lower-dimensional RR-charge and thus should couple also to lower-dimensional RR-fields. We now assume that all projective

⁸A background *B*-field makes the theory of D-branes much more complicated. This is because the *B*-field couples to the open string worldsheet and the boundary of the worldsheet couples to the *A*-field, and thus to the D-brane. Therefore, Freed-Witten anomaly cancellation imposes certain geometric and topological requirements for the D-brane.

 $^{^9\}mathrm{As}$ a side note, recall that the worldvolume Σ was not generally assumed to be a submanifold of X. Indeed, it was demonstrated in [ES] that also nonrepresentable homology cycles, that is, homology cycles which are not represented by any nonsingular submanifolds, may still carry RR-charge.

Chan-Paton bundles can all be lifted to U(n) bundles. We call these lifts U(n) Chan-Paton bundles and their associated vector bundles U(n) Chan-Paton vector bundles. This assumption is related to the Freed-Witten anomaly and will later be dropped. Anomaly cancellation arguments can be used to deduce the coupling [CY, El-S]

$$\int_{\Sigma} \phi^* C \wedge \operatorname{ch}(E) \wedge \operatorname{Todd}(\Sigma) \wedge \frac{1}{\phi^* \sqrt{\operatorname{Todd}(X)}} = \int_{X} C \wedge \phi_*(\operatorname{ch}(E) \wedge \operatorname{Todd}(\Sigma)) \wedge \frac{1}{\sqrt{\operatorname{Todd}(X)}},$$

where E is the U(n) Chan-Paton vector bundle of the D-brane, ϕ_* the cohomological Gysin "wrong way" homomorphism, defined by $\phi_* := \operatorname{Pd}_X^{-1} \circ \phi_* \circ \operatorname{Pd}_\Sigma$, where Pd denotes the Poincaré duality isomorphism and ϕ_* (on the right-hand side) the natural push-forward in homology. The double meaning of ϕ_* should not cause problems. Finally, $\operatorname{ch}(E) \in H^{\operatorname{even}}_{\operatorname{dR}}(\Sigma)$ is the Chern character of E, and $\operatorname{Todd}(\Sigma) \in H^{\operatorname{even}}_{\operatorname{dR}}(\Sigma)$ and $\operatorname{Todd}(X) \in H^{\operatorname{even}}_{\operatorname{dR}}(X)$ the Todd classes of the tangent bundles of Σ and X, respectively. These are examples of so-called characteristic classes of vector bundles. It is necessary to assume Σ to be a Spin^c manifold for $\operatorname{Todd}(\Sigma)$ to be defined. Now, while this coupling may look complicated and unnatural, it turns out to be quite natural from the point of view of K-theory. In any case, this coupling suggests that at a classical level the D-brane charge is represented by the compactly supported de Rham cohomology class

$$Q_{\mathbb{R}}(\Sigma, E) := \phi_*(\operatorname{ch}(E) \wedge \operatorname{Todd}(\Sigma)) \wedge \frac{1}{\sqrt{\operatorname{Todd}(X)}} \in H_{c,dR}^{\operatorname{even}}(X).$$

In fact, Todd and ch are classes in $H^{\text{even}}(-,\mathbb{Q})$. Thus we can as well work with rational cohomology instead of de Rham cohomology and define

$$Q_{\mathbb{Q}}(\Sigma, E) := \phi_*(\operatorname{ch}(E) \smile \operatorname{Todd}(\Sigma)) \smile \frac{1}{\sqrt{\operatorname{Todd}(X)}} \in H_c^{\operatorname{even}}(X, \mathbb{Q}), \qquad (1.2)$$

where

$$\smile: H^p(-,\mathbb{Q}) \otimes H^q(-,\mathbb{Q}) \to H^{p+q}(-,\mathbb{Q})$$

is the cup product and $H_c^{\text{even}}(X,\mathbb{Q})$ the compactly supported cohomology [May].

1.3 From tachyon condensation to K-theory

Consider a Dp-brane wrapping a worldvolume $\phi: \Sigma \to X$. An anti-Dp-brane, or a $D\overline{p}$ -brane for short, is one with opposite RR-charge. If Dp-brane and a $D\overline{p}$ -brane wrap the same worldvolume, then obviously the configuration carries no total RR-charge and there should be no conservation laws preventing it from decaying to the closed string vacuum. Unfortunately, the dynamical process through which this happens can not be fully described without a complete quantum theory of D-branes, which we do not have.

It was proposed by Sen in [Sen], that D-brane annihilation can be described by a process called tachyon condensation. The idea is that the coincident $p - \overline{p}$ D-brane pair supports a tachyonic field (with negative mass squared), which indicates an instability in the configuration: the pair is not in a stable potential minimum. The annihilation process can then be described as rolling of the tachyon potential to a stable local minimum. Suppose first that

both the D-brane and the anti-D-brane carry topologically equivalent U(n) Chan-Paton bundles vector bundles E. In this case, the tachyon field is a section of the trivial bundle $E \otimes E^*$ (which obviously admits global nonvanishing sections). Thus, the tachyon field can assume a constant nonzero value everywhere, corresponding to the potential minimum. The resulting energy density precisely cancels that of the original D-brane pair.

Recall that a Dp-brane can carry also lower-dimensional RR-charge. Let us see how this is presented in the tachyon condensation scenario. Consider a configuration of a Dp-brane with U(1) Chan-Paton bundle E and a $D\overline{p}$ -brane with a topologically different U(1) Chan-Paton bundle F, both wrapping the same worldvolume $\phi: \Sigma \to X$. In this case the tachyon is a section of $E \otimes F^*$, which is nontrivial and thus may not admit any global nonvanishing sections. This causes a possible topological obstruction, measured by the nontriviality of the tensor product bundle, for the tachyon field to assume its vacuum expectation value everywhere. As a result, the energy density of the original D-brane pair is not cancelled everywhere. Remarkably, the resulting nonvanishing energy density can be identified with that of a lower-dimensional D-brane.

The D-brane annihilation process is more naturally described in terms of RR-charges. Recall that for a D-brane with worldvolume $\phi: \Sigma \to X$ and U(n) Chan-Paton vector bundle E, the (rational) RR-charge was given by the formula (1.2). The charge on an anti-D-brane wrapping the same worldvolume and carrying a U(n) Chan-Paton bundle vector bundle F should be

$$\overline{Q}_{\mathbb{Q}}(\Sigma, F) := -\phi_*(\operatorname{ch}(F) \smile \operatorname{Todd}(\Sigma)) \smile \frac{1}{\sqrt{\operatorname{Todd}(X)}}.$$

Let us denote the pair by (E, F) and define

$$\operatorname{ch}((E,F)) := \operatorname{ch}(E) - \operatorname{ch}(F) \in H^{\operatorname{even}}(\Sigma,\mathbb{Q}),$$

so that the total RR-charge of the pair (E, F) can be expressed as

$$Q_{\mathbb{Q}}(\Sigma, (E, F)) := Q_{\mathbb{Q}}(\Sigma, E) - \overline{Q}_{\mathbb{Q}}(\Sigma, F) = \phi_*(\operatorname{ch}((E, F)) \smile \operatorname{Todd}(\Sigma)) \smile \frac{1}{\sqrt{\operatorname{Todd}(X)}}.$$
(1.3)

Complex line bundles are completely classified by their first Chern class c_1 , which is a class in $H^2(X,\mathbb{Z})$. For line bundles the Chern character reduces to $ch(E) = 1 + c_1(E)$. Thus, a coincident $p - \overline{p}$ pair with U(1) Chan-Paton bundles E and F, the RR-charge is

$$Q_{\mathbb{Q}}(\Sigma, (E, F)) = \phi_*((c_1(E) - c_1(F)) \smile \operatorname{Todd}(\Sigma)) \smile \frac{1}{\sqrt{\operatorname{Todd}(X)}},$$

which shows exactly how the lower-dimensional RR-charge of the remaining lower-dimensional D-brane depends on the Chan-Paton bundles. Since the class $c_1(E) - c_1(F)$ is of degree 2, the charge $Q_{\mathbb{Q}}(\Sigma, (E, F))$ couples to a D(p-2)-brane. In other words, while there is no Dp-brane charge left after the annihilation process, there might be D(p-2)-brane charge, carried by a D(p-2)-brane whose worldvolume is associated to the solitonic tachyon field. One can similarly consider a coincident $p-\overline{p}$ pair with U(n) and U(m) Chan-Paton vector bundles E and F. The Chern characters of higher rank bundles are polynomials of higher Chern classes, which are all even degree integral cohomology classes, but the lowest degree part is always the rank of the bundle, so that $ch(E) - ch(F) = n - m + \ldots$, where the ellipsis denotes

classes in $H_c^{2k}(X,\mathbb{Q})$, where $k=1,2,\ldots$ Inserting this into the charge formula, we notice that the resulting configuration carries Dp-brane charge if and only if $n \neq m$. It may carry lower-dimensional charge depending on the Chan-Paton bundles.

The above discussion leads directly to the following simple observation [Wit1]. Consider a $p-\overline{p}$ pair with U(n) and U(m) Chan-Paton vector bundles E and F, wrapping the worldvolume $\phi: \Sigma \to X$. If we add to this configuration a pair $p-\overline{p}$ with U(k) Chan-Paton vector bundles (G,G), the Chan-Paton bundles of the original configuration are enhanced to U(m+k) and U(n+k) bundles $E \oplus G$ and $F \oplus G$, respectively. Since the added pair carries no total RR-charge, one would expect the RR-charge of the original configuration to be left invariant. Indeed, since the Chern character satisfies $\operatorname{ch}(E \oplus G) = \operatorname{ch}(E) + \operatorname{ch}(G)$, it follows that

$$ch((E \oplus G, F \oplus G)) = ch(E \oplus G) - ch(F \oplus G)$$
$$= ch(E) + ch(G) - (ch(F) + ch(G))$$
$$= ch((E, F)).$$

Thus, we consider the configurations (E, F) and $(E \oplus G, F \oplus G)$ to be *physically equivalent* and write

$$(E,F) \sim (E \oplus G, F \oplus G)$$
. (1.4)

The equivalence class of a pair (E, F) is denoted by [E] - [F].

Let $\operatorname{Vect}_{\mathbb{C}}(\Sigma)$ denote the Abelian monoid of isomorphism classes of complex vector bundles over the compact manifold Σ . An isomorphism class represented by a vector bundle E is denoted by [E]. Addition in $\operatorname{Vect}_{\mathbb{C}}(\Sigma)$ is defined by the direct sum:

$$[E] + [F] = [E \oplus F].$$

The 0-th K-theory group $K^0(\Sigma)$, often denoted simply by $K(\Sigma)$, is the free Abelian group generated by formal differences of classes of $\operatorname{Vect}_{\mathbb{C}}(\Sigma)$. Thus, the elements of $K^0(\Sigma)$ are formal differences [E] - [F]. Of course, E and F may very well have different rank. Now, comparing

$$[E \oplus G] - [F \oplus G] = [E] + [G] - [F] - [G] = [E] - [F]$$

to the physical equivalence relation (1.4) reveals immediately that physically equivalent Dbrane configurations correspond precisely to K-theory classes of $K^0(\Sigma)$, the correspondence being given by

$$(E,F)/\sim \iff [E]-[F].$$

If Σ is wrapped only by D-branes with Chan-Paton bundle E and no anti-D-branes, the corresponding K-theory class is [E] - [0], where 0 denotes the zero bundle. We denote this simply by [E].

For noncompact manifolds¹⁰ the definition of K^0 is a little different. Suppose that X is noncompact and

$$X^{\infty} := X \coprod \{\infty\}$$

its one-point compactification. The K-theory group $K^0(X)$ is defined as the subgroup $\widetilde{K}^0(X^\infty) \subset K^0(X^\infty)$, generated by pairs of vector bundles isomorphic at the point $\{\infty\}$. In other words, for a noncompact manifold X, $K^0(X)$ contains only differences of vector bundles vanishing asymptotically.

¹⁰Recall that every manifold is locally compact and, thus, admit a one-point compactification by adding a "point at infinity".

1.4 D-branes and K-theory

As we explained above, there is good reason to believe that worldvolume K-theory classifies physically equivalent configurations of D-branes, at least when there is no background B-field. It turns out, that the seemingly obscure rational RR-charge formula (1.3) is extremely natural from the K-theoretic perspective, as was pointed out by Minasian and Moore in the famous paper [MM].

Consider a K-theory class $x \in K^0(\Sigma)$ corresponding to a D-brane configuration wrapping $\phi: \Sigma \to X$. Earlier we extended the Chern character to a map from pairs (E, F) to $H^{\text{even}}(\Sigma, \mathbb{Q})$. It is easy to see that it descends to a well-defined group homomorphism

$$\operatorname{ch}: K^0(\Sigma) \to H^{\operatorname{even}}(\Sigma, \mathbb{Q}),$$

mapping

$$\operatorname{ch}([E] - [F]) := \operatorname{ch}(E) - \operatorname{ch}(F).$$

The rational RR-charge of x is thus given by

$$Q_{\mathbb{Q}}(\Sigma, x) = \phi_*(\operatorname{ch}(x) \smile \operatorname{Todd}(\Sigma)) \smile \frac{1}{\sqrt{\operatorname{Todd}(X)}}.$$
 (1.5)

By the Atiyah and Hirzebruch version of the Grothendieck-Riemann-Roch theorem [AH1, AH2], if Σ is Spin^c, there exists a Gysin "wrong way" homomorphism

$$\phi_!: K^0(\Sigma) \to K^0(X)$$
,

satisfying

$$\operatorname{ch}(\phi_!(x)) \smile \operatorname{Todd}(X) = \phi_*(\operatorname{ch}(x) \smile \operatorname{Todd}(\Sigma)). \tag{1.6}$$

By applying (1.6) to (1.5), we obtain the *Minasian-Moore formula* [MM]

$$Q_{\mathbb{O}}(\Sigma, x) = \operatorname{ch}(\phi_{!}(x)) \smile \sqrt{\operatorname{Todd}(X)} \in H_{c}^{\operatorname{even}}(X, \mathbb{Q}). \tag{1.7}$$

Dirac quantization requires the RR-charge to lift into some generalized integral cohomology theory [Whi, Bro1, Bro2], which in this case seems to be K-theory [Fre1, Fre2]. Thus, the quantized (integral) RR-charge should be a K-theory class of spacetime. From the point of view of (1.7), a natural choice would be

$$Q_{\mathbb{Z}}(\Sigma, x) := \phi_!(x)$$
.

The quantized charge differs fundamentally from the corresponding classical charge in that the quantized charge may contain torsion. Real and rational cohomology are obviously insensitive to torsion due to $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0$.

We have seen how D-branes can be classified by their RR-charge, which is a class in the K-theory of spacetime. This is not a very concrete description, though, considering that D-branes are associated to worldvolume manifolds and U(n) vector bundles¹¹ living on them. It was argued by Periwal in [Per], that a more "natural" description may be given using K-homology, the dual homology theory of K-theory, of spacetime. It admits a purely geometric

¹¹We are still assuming that the PU(n) Chan-Paton bundles lift to U(n) bundles.

definition [Jak1, Jak2, BD, BHS], encoding precisely by the information we have used to define a D-brane: the Spin^c worldvolume manifold Σ , the continuous map $\phi: \Sigma \to X$ and the U(n) Chan-Paton vector bundle $E \to \Sigma$. It turns out, that there is a K-theoretic Poincaré duality isomorphism between K-theory and K-homology, which makes the two classification schemes equally valid. They merely emphasize different physical aspect of D-branes.

1.5 Cancelling the Freed-Witten anomaly

Recall that the B-field is a locally defined 2-form field and that its derivative H := dB, known as the H-flux, is a globally defined 3-form field. By Dirac quantization, the de Rham cohomology class $[H] \in H^3_{\mathrm{dR}}(X)$ lifts to a class $[H] \in H^3(X,\mathbb{Z})$, which may also have a torsion part, unlike its image in $H^3_{\mathrm{dR}}(X)$. The correct mathematical framework for Dirac quantized fields is differential cohomology [Fre1, Fre2, HS]. For example, RR-fields are described by something called differential K-theory [Val]. As was already implicitly stated above, the integral cohomology theory associated to the B-field seems to be the ordinary integral cohomology $H^3(X,\mathbb{Z})$. So, to properly understand the B-field (and the H-flux), we need to know what the associated differential cohomology theory is. It turns out to be something called D-eligne C-cocycle, C a 1-form-valued Čech 1-cochain and C a 2-form-valued 0-cochain on C a satisfying the following relations:

$$d \log g = \delta(\underline{\Lambda}), \qquad d\underline{\Lambda} + \delta(\underline{B}) = 0.$$

The second equation implies $\delta(d\underline{B}) = d\delta(\underline{B}) = 0$, which means that $d\underline{B}$ lifts to a global 3-form H.

The notation above was chosen so as to make the correspondence with D-brane physics obvious: \underline{B} corresponds to the physical B-field, H to the physical H-flux and $\underline{\Lambda}$ to that in (1.1), with the minor modification that the quantities associated to the Deligne cocycle are the corresponding physical quantities multiplied by the complex number i, which is left from taking $d \log$ of \underline{g} . Thus, from now on, we shall use the notation

$$B:=iB_{\rm phys}\,,\quad H:=iH_{\rm phys}\,,\quad \Lambda:=i\Lambda_{\rm phys}\,,\quad A:=iA_{\rm phys}\,.$$

This changes the physical A-field transformation equation (1.1) into

$$A_{\beta} = g_{\alpha\beta}^{-1} A_{\alpha} g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} - \Lambda_{\alpha\beta} . \tag{1.8}$$

It would be convenient to have a geometric interpretation of the *B*-field, just like ordinary gauge fields are geometrically represented by a connection and curvature on a principal bundle. Such a geometric model for the *B*-field, or rather the Deligne cocycle, is given by the theory of bundle gerbes with connection and curving [Mur, BCMMS, MS1].

The relation between the A-field and the B-field can be found by studying the exponential of the open string worldsheet action. It contains the terms

$$\exp(iS_{\text{open}}) \supset \operatorname{pfaff}(D_{\xi}) \cdot \exp\left(\int_{M} \xi^{*}B\right) \cdot \operatorname{Tr}\exp\left(\oint_{\partial M} \xi^{*}A\right),$$

where $\xi: M \to X$ is the embedding of the open string worldsheet and pfaff(\mathcal{D}_{ξ}) the *Pfaffian*, the square root of the determinant, of the worldsheet Dirac operator [FW]. It turns out,

that pfaff(D_{ξ}) is not a well-defined function on the space of embeddings $M \to X$, but rather a section of a certain line bundle. There is also the other problem we mentioned earlier, namely, that the integral of the B-field is not well-defined due to the boundary of M. This ill-definedness of the open string path integral is known as the Freed-Witten anomaly [FW]. To cancel this anomaly, the gauge transformation of the A-field must be such that the trace of its holonomy around ∂M is ill-defined in just a particular way to cancel the ill-definedness of the other two terms. It was shown in [FW, Kap, CJM] that this happens if the A-field transforms as in (1.8) and if the cohomology equation

$$[H]|_{\Sigma} = W_3(\Sigma) + \beta([\zeta]) \tag{1.9}$$

holds. Geometrically the A-field can be interpreted, not as a connection on a bundle, but a connection on an object called a bundle gerbe module [BCMMS, CJM]. In the main body of the text we shall explain what bundle gerbe modules and their connections are and go carefully through the calculations which show how the anomaly is cancelled when the above conditions hold.

Let us now explain the important equation (1.9). Let $\phi: \Sigma \to X$ denote the D-brane worldvolume. First, $[H]|_{\Sigma} := \phi^*[H] \in H^3(\Sigma, \mathbb{Z})$ is the restriction of the H-flux $[H] \in H^3(X, \mathbb{Z})$ onto the worldvolume. Second, the characteristic class $W_3(\Sigma) \in H^3(\Sigma, \mathbb{Z})$ is the so-called third integral Stiefel-Whitney class, which is the obstruction for the existence of a Spin^c structure on Σ^{12} . Finally, $\beta([\underline{\zeta}]) \in H^3(\Sigma, \mathbb{Z})$ is the obstruction for lifting the Chan-Paton bundle from a PU(n) bundle to a U(n) bundle. Recall now, that while developing the K-theoretic classification of D-branes, we assumed $[H]|_{\Sigma} = 0$, $W_3(\Sigma) = 0^{13}$ and that the Chan-Paton bundle was a U(n) vector bundle for its K-theory class to be defined. Obviously (1.9) is satisfied under those conditions.

What happens if, say, $[H]|_{\Sigma} \neq 0$ and there is a single D-brane wrapping Σ ? The Chan-Paton bundle is then automatically a U(1) bundle and thus $\beta([\underline{\zeta}]) = 0$. But now the anomaly cancellation condition implies that such a configuration is not physically possible unless Σ satisfies $W_3(\Sigma) = [H]|_{\Sigma}^{14}$. In this case we can form a K-theory class $[E] \in K^0(\Sigma)$ from the Chan-Paton bundle vector bundle E, but it can not be pushed into $K^0(X)$.

Consider next n D-branes wrapping Σ with a PU(n) Chan-Paton bundle. Suppose that the obstruction class for the U(n) lift is $\beta([\underline{\zeta}]) \neq 0^{15}$. In this case, not even a worldvolume K-theory class can be constructed from the (projective) Chan-Paton bundle.

Finally, consider the case of nontorsion $[H]|_{\Sigma}$. In this case something quite radical must happen for the anomaly to cancel, namely, the torsion degree of $\beta([\underline{\zeta}])$ should somehow be taken to infinity. From the physical point of view, this corresponds to taking the number of D-branes wrapping Σ to infinity in some sense [Wit1, Wit2]. The precise mathematical interpretation is to allow the Chan-Paton bundle to be a principal $PU(\mathcal{H})$ bundle [BM, AS1], where \mathcal{H} is the standard infinite-dimensional separable Hilbert space and $PU(\mathcal{H})$ the quotient of $U(\mathcal{H})$ by its center:

$$PU(\mathcal{H}) := U(\mathcal{H})/U(1)$$
.

 $^{^{12}}$ Again, a reader unfamiliar with Spin^c structures can regard this simply as a technical condition, slightly stronger than orientability.

 $^{^{13}\}Sigma$ had to be Spin^c for the Riemann-Roch theorem.

 $^{^{14}}$ Since $W_3(\Sigma)$ is 2-torsion, also $[H]|_{\Sigma}$ must be 2-torsion. Hence, the image of [H] in de Rham cohomology must necessarily be zero.

¹⁵It is easy to see that $\beta([\underline{\zeta}])$ is necessarily *n*-torsion, implying that $[H]|_{\Sigma}$ must also be *n*-torsion for the anomaly to be cancelled.

Again, the K-theoretic classification fails miserably, since we can not even construct the worldvolume K-theory class.

1.6 Classifying twisted D-branes

We have seen how the K-theoretic classification fails when Σ is not Spin^c or when the Chan-Paton bundle can not be lifted to a U(n) bundle. The key to solving these problems is to use $twisted\ K$ -theory [Ros, Bla, AS1, BM, CW, BCMMS], a modification of K-theory, where U(n)vector bundles are replaced with PU(n) bundles, whose transition functions $\{g_{\alpha\beta}\}$ satisfy a ζ -twisted cocycle condition

$$g_{\beta\gamma}g_{\alpha\gamma}^{-1}g_{\alpha\beta} = \zeta_{\alpha\beta\gamma}$$
.

The direct sum of two such ζ -twisted bundles is clearly again ζ -twisted. Thus, a ζ -twisted PU(n) vector bundle defines a class in the ζ -twisted K-theory $K^0(\Sigma,\zeta)$. The group $K^0(\Sigma,\zeta)$ depends only on the Čech cohomology class $[\zeta] \in H^2(\Sigma,\underline{U(1)}) \cong H^3(\Sigma,\mathbb{Z})$. Therefore, the natural object to twist K-theory with, is not a cocycle but a cohomology class. More generally, K-theory can be twisted by any class in $H^3(\Sigma,\mathbb{Z})$, torsion or nontorsion. For $\sigma \in H^3(\Sigma,\mathbb{Z})$, we denote σ -twisted K-theory by $K^0(\Sigma,\sigma)$. When the twisting class is trivial, twisted K-theory reduces to ordinary "untwisted" K-theory. The definition can be given in several different ways, each useful for certain purposes. The most convenient for us being that of Rosenberg [Ros].

The Freed-Witten anomaly cancellation condition yields¹⁶

$$\beta([\underline{\zeta}]) = [H]|_{\Sigma} + W_3(\Sigma).$$

This means that the PU(n) (or possibly even $PU(\mathcal{H})$) Chan-Paton vector bundle defines a class in $K^0(\Sigma, [H]|_{\Sigma} + W_3(\Sigma))$. For a continuous map $\phi : \Sigma \to X$, there exists a twisted Gysin homomorphism [CW]

$$\phi_!: K^0(\Sigma, [H]|_{\Sigma} + W_3(\Sigma)) \to K^0(X, [H]).$$

This indicates that, in the presence of a nontrivial H-flux, quantized D-brane charge should take values in the [H]-twisted K-theory group of spacetime. In particular, if $\beta([\zeta]) = 0$ but $[H]|_{\Sigma} = W_3(\Sigma) \neq 0$, the class of the Chan-Paton bundle is in (untwisted) worldvolume K-theory, but becomes [H]-twisted when pushed into spacetime.

A twisted version of K-homology and its relation to D-brane theory are developed in [Wan, Sza1]. It correctly extends the K-homological approach to nontrivial B-field backgrounds and non-Spin^c worldvolumes.

D-branes satisfying a nontrivial Freed-Witten anomaly cancellation condition are referred to as $twisted\ D$ -branes. We already explained how (Dirac-quantized physical equivalence classes of) configurations of twisted D-branes can be classified by twisted K-theory or twisted K-homology, instead of ordinary untwisted K-theory or K-homology. In the untwisted case we had a Chern character from K-theory to cohomology, which was used to construct the classical approximation of the quantized RR-charge. In the twisted case there is a $twisted\ Chern$

 $^{^{16}}W_3(\Sigma)$ is 2-torsion.

character from twisted K-theory to twisted (de Rham) cohomology [MS2, MS3, BCMMS, AS2]. Let $\zeta \in H^3_{dR}(X)$ be a closed de Rham 3-form. The ζ -twisted de Rham differential is

$$d_{\zeta} := d - \zeta$$
,

which operates on differential forms $\Omega^{\text{even/odd}}(X)$ by $d_{\zeta}(\omega) = d\omega + \zeta \wedge \omega^{17}$. A direct calculation shows that $d_{\zeta}^2 = 0$. The (periodized) ζ -twisted cohomology groups are defined by

$$H^{\bullet}(X,\zeta) := \ker(\delta_{\zeta})/\operatorname{im}(\delta_{\zeta}),$$

where $\bullet = 0, 1$. Moreover, it turns out that

$$H^{\bullet}(X,\zeta) \cong H^{\bullet}(X,\zeta+d\eta)$$

for any 2-form η , where, on the level of cocycles, the isomorphism is given by

$$\omega \mapsto e^{\eta} \wedge \omega$$
.

Thus, de Rham cohomology is most naturally twisted by the cohomology class $[\zeta] \in H^3_{\mathrm{dR}}(X)$, not the cocycle itself. More generally, if $\sigma \in H^3(X,\mathbb{Z})$ and $\sigma_{\mathrm{dR}} \in H^3_{\mathrm{dR}}(X)$ is image under the natural inclusion, σ -twisted (de Rham) cohomology is defined by

$$H^{\bullet}(X,\sigma) := H^{\bullet}(X,\sigma_{\mathrm{dR}}),$$

where σ_{dR} is the image of σ in $H^3_{dR}(X)$. Now, given a twisting class $\sigma \in H^3(X,\mathbb{Z})$, there exists a σ -twisted Chern character homomorphism

$$\operatorname{ch}_{\sigma}: K^0(X,\sigma) \to H^{\operatorname{even}}(X,\sigma)$$
.

In our physical example $\sigma = [H]$. The twisted Chern character maps the push-forward of the twisted K-theory class of the Chan-Paton vector bundle to (compactly supported) twisted cohomology $H_c^0(X, [H])$. If [H] is pure torsion, then $[H]_{\mathrm{dR}} = 0$ and [H]-twisted cohomology $H^0(X, [H])$ becomes isomorphic to untwisted de Rham cohomology $H^{\mathrm{even}}_{\mathrm{dR}}(X)$. Thus, when [H] is pure torsion, the charge is again a class in untwisted cohomology, given by a generalization of (1.7), where ch is simply replaced with $\mathrm{ch}_{[H]}^{18}$.

1.7 Operator K-theory

Twisted K-theory can be defined in several ways. The most convenient for us is the definitions through operator K-theory (or K-theory of C^* -algebras) [Bla], because it automatically opens up the possibility to generalize the theory of D-branes and their RR-charges to D-branes whose worldvolumes and the spacetime are noncommutative spaces [Con, GVF, Mad, Lan2].

Let A be a C^* -algebra [GVF, Ped], that is, a Banach algebra with involution, satisfying the C^* -norm property

$$||a^*a|| = ||a||^2 \,,$$

¹⁷Clearly, d_{ζ} maps $\Omega^{\text{even}}(X)$ to $\Omega^{\text{odd}}(X)$ and vice versa.

¹⁸Remark, that $\omega \wedge \mu$ is $d_{\sigma_{dR}}$ -closed for any $d_{\sigma_{dR}}$ -closed ω and d-closed $\mu \in \Omega^{\bullet}(X)$.

for all $a \in A$. C^* -algebras have extremely nice spectral properties and their K-theory is particularly well-behaving. Morphisms between C^* -algebras respect the involution and are called *-homomorphisms. C^* -algebras are also at the basis of noncommutative topology and geometry: any locally compact Hausdorff space X is completely characterized by the commutative C^* -algebra of its continuous complex-valued functions vanishing at infinity, $C_0(X)^{19}$. Many important topological and geometric concepts can be expressed purely algebraically, for example vector bundles, differential calculus, Spin^c structures, Dirac operators and de Rham (co)homology, after which the C^* -algebra can be replaced by any, possibly noncommutative, C^* -algebra. Thus, noncommutative spaces are C^* -algebras. Noncommutative (Spin) geometry is a vast subject and treating it in any reasonable detail would take easily a thesis or two. Nevertheless, even readers unfamiliar with the intricate details can easily appreciate the generalization of D-brane theory to the noncommutative (C^* -algebraic) case.

Operator K-theory is a noncommutative generalization of K-theory of vector bundles, defined for a broad class of (topological) algebras, including the so-called local C^* -algebras [Bla] and, naturally, C^* -algebras. The most important examples of local C^* -algebras are the smooth function algebras of manifolds. For any manifold X, there are isomorphisms

$$K_0(C_0^{\infty}(X)) \cong K_0(C_0(X)) \cong K^0(X)$$
,

where K_0 denotes operator K-theory.

Twisted K-theory can now be defined using operator K-theory as follows [Ros]. Let $\sigma \in H^3(X,\mathbb{Z})$ be a twisting cohomology class. There is bijective correspondence between $H^3(X,\mathbb{Z})$ and isomorphism classes of principal $PU(\mathcal{H})$ bundles over X^{20} . The cohomology class corresponding to a principal $PU(\mathcal{H})$ bundle is called its Dixmier-Douady class. Let P be a principal $PU(\mathcal{H})$ bundle with Dixmier-Douady class σ . We denote the C^* -algebra of compact operators on the standard infinite-dimensional separable Hilbert space by \mathcal{K} . Then, since $PU(\mathcal{H}) \cong Aut(\mathcal{K})$, we can form the associated $PU(\mathcal{H})$ vector bundle over X with fibres isomorphic to \mathcal{K} :

$$\mathcal{E}_{\sigma} := P_{\sigma} \times_{PU(\mathcal{H})} \mathcal{K} \to X$$
.

Rosenberg defined σ -twisted K-theory as

$$K^0(X,\sigma) := K_0(C_0(X,\mathcal{E}_\sigma)).$$

A principal $PU(\mathcal{H})$ bundle with trivial Dixmier-Douady class is the projectivization of a principal $U(\mathcal{H})$ bundle [AS1]. The famous theorem of Kuiper [Kui] states that $U(\mathcal{H})$ is contractible (in strong operator topology). Thus, the fibres are all contractible, which means that the principal bundle must trivial. The associated bundle $\mathcal{E}_0 \to X$ is simply the trivial bundle $X \times \mathcal{K}$. The twisted function algebra is then

$$C_0(X, \mathcal{E}_0) = C_0(X, \mathcal{K}) \cong C_0(X) \otimes \mathcal{K}$$
.

K-theory is stable, or Morita invariant, in the sense that

$$K_0(A \otimes \mathcal{K}) \cong K_0(A)$$
,

¹⁹If the space X is compact, the algebra $C_0(X)$ coincides with C(X). Thus, for a compact space, the corresponding C^* -algebra is unital and for a noncompact it is nonunital.

²⁰In other words, the classifying space $BPU(\mathcal{H})$ is a model for the Eilenberg-MacLane space $K(\mathbb{Z},3)$ [AS1].

for any C^* -algebra A. This shows that twisted K-theory reduces to ordinary untwisted K-theory when the twisting class vanishes:

$$K^{0}(X,0) = K_{0}(C_{0}(X,\mathcal{E}_{0})) = K_{0}(C_{0}(X) \otimes \mathcal{K}) \cong K_{0}(C_{0}(X)) \cong K^{0}(X),$$

as would be expected.

There exists also a noncommutative Chern character, which coincides with the twisted Chern character mentioned earlier, when the C^* -algebra is a twisted function algebra. In noncommutative geometry (periodized) de Rham cohomology of a smooth manifold X is given by periodic cyclic homology HP_{\bullet} [Con, GVF] of the smooth function algebra $C_0^{\infty}(X)$:

$$H_{\mathrm{dR}}^{\bullet}(X) \cong \mathrm{HP}_{\bullet}(C_0^{\infty}(X)), \quad \bullet = \mathrm{even/odd}.$$

Similarly, (periodized) de Rham homology is given periodic cyclic cohomology $HP^{\bullet}(C_0^{\infty}(X))$. Let now \mathcal{A} be any local C^* -algebra. The noncommutative version of the Chern character is the Chern-Connes character [GVF]:

$$\operatorname{ch}: K_0(\mathcal{A}) \to \operatorname{HP}_0(\mathcal{A}).$$

We denote by A the C^* -algebra completition of \mathcal{A} . For example $A = C_0(X)$ for $\mathcal{A} = C_0^{\infty}(X)$ and $A = C_0(P \times_{PU(\mathcal{H})} \mathcal{K})$ for $\mathcal{A} = C_0^{\infty}(P \times_{PU(\mathcal{H})} \mathcal{L}^1)$, where P is a principal $PU(\mathcal{H})$ bundle with Dixmier-Douady class σ and $\mathcal{L}^1 \subset \mathcal{K}$ is the ideal of trace class operators. Finally, σ -twisted cohomology is isomorphic to periodic cyclic homology of the twisted smooth function algebra [MS3]. Thus, the diagram

$$K^{0}(X,\sigma) \xrightarrow{\operatorname{def.}} K_{0}(A) \xrightarrow{\cong} K_{0}(A)$$

$$\downarrow_{\operatorname{ch}_{\sigma}} \qquad \qquad \downarrow_{\operatorname{ch}}$$

$$H^{0}(X,\sigma) \xrightarrow{\cong} \operatorname{HP}_{0}(A)$$

commutes.

There exists also noncommutative generalization of K-homology, denoted by $K^0(A)$, for a C^* -algebra A [Kas3, HR]. Twisted K-homology of a topological space can be defined as K-homology of the twisted function algebra.

1.8 Noncommutative generalizations

The C^* -algebraic definitions given above open up hopes for generalizing the Minasian-Moore formula (1.7) to more general noncommutative D-branes in a noncommutative spacetime [BMRS1, BMRS2, Sza2]. Let the C^* -algebra B be a noncommutative D-brane worldvolume and A a noncommutative spacetime. At the level of algebras the map $\phi: \Sigma \to X$ is replaced by a *-homomorphism $\phi: A \to B$. The quantized RR-charge of the noncommutative D-brane B is an operator K-theory class of B, pulled back into the K-theory of A by a noncommutative version of the Gysin homomorphism. The existence of the Gysin homomorphism can be thought of as a noncommutative analogue of the Freed-Witten anomaly cancellation condition. To be able to understand noncommutative Gysin homomorphisms and the relation between K-theory and K-homology (of C^* -algebras), we need to introduce K as K-theory, a bivariant unification of K-theory and K-homology [Kas1, Kas2, Bla, Hig1, Hig2, KT].

KK-theory can be defined both axiomatically and explicitly²¹. For now, it is enough

²¹In fact, several different explicit constructions for it are known.

for the reader to think of KK_{\bullet} , $\bullet = 0, 1$ as bivariant bifunctors from pairs of separable C^* -algebras to the category of Abelian groups, contravariant in the first variable and covariant in the second. KK-theory unifies K-theory and K-homology in the following sense:

$$K_0(A) \cong KK_0(\mathbb{C}, A)$$
 (K-theory)
 $K^0(A) \cong KK_0(A, \mathbb{C})$ (K-homology)

Possibly the most important property of KK-theory is the existence of a powerful product structure, the $Kasparov\ product$: for separable C^* -algebras A_1, A_2, B_1, B_2, D , the Kasparov product is an associative, bilinear map

$$\otimes_D: KK_i(A_1, B_1 \otimes D) \times KK_i(D \otimes A_2, B_2) \to KK_{i+i}(A_1 \otimes A_2, B_1 \otimes B_2).$$

It is convenient to restrict the discussion to $nuclear\ C^*$ -algebras, for which there is no ambiguity in the topology of the tensor product C^* -algebras [Lan1].

We are interested in pairs of separable C^* -algebras (A, \widetilde{A}) , for which there exists a fundamental class $\Delta \in KK_d(A \otimes \widetilde{A}, \mathbb{C})$, "invertible", in a certain sense, with respect to the Kasparov product. The pair (A, \widetilde{A}) is called a Poincaré duality (PD) pair. If a C^* -algebra B and its opposite algebra B° form a PD pair (B, B°) , we say that B is a Poincaré duality (PD) algebra. Poincaré duality for a PD pair is defined by taking "cap product" with the fundamental class:

$$KK_{\bullet}(\mathbb{C}, A) \xrightarrow{(-)\otimes_A \Delta} KK_{\bullet+d}(\widetilde{A}, \mathbb{C}).$$

The fundamental class is not unique. In fact, the space of fundamental classes is isomorphic to the group of invertible classes in the ring $KK_0(A, A) \cong KK_0(B, B)$. For example, $(C_0(X), C_0(X, \text{Cliff}(T^*X))$ is a PD pair. Moreover, if X is Spin^c, then C(X) is a PD algebra.

Next we need to introduce a bivariant cyclic homology theory, which is compatible with KK-theory. It is clear that periodic cyclic theory can not be used here, since it does not work well for C^* -algebras and (Kasparov's) KK-theory works only for C^* -algebras. Instead, we need to use the bivariant local cyclic homology of Puschnigg [Pus1, Pus2], denoted by $\mathrm{HL}_{\bullet}(-,-)$, $\bullet=1,2$, which works for both C^* -algebras and for local C^* -algebras. Moreover, it satisfies

$$\mathrm{HL}_{\bullet}(\mathbb{C}, C_0(X, \mathcal{E}_{\sigma}) \cong \mathrm{HL}_{\bullet}(\mathbb{C}, C_0^{\infty}(X, \mathcal{E}_{\sigma})) \cong \mathrm{HP}_{\bullet}(C_0^{\infty}(X, \mathcal{E}_{\sigma})) \cong H_c^{\bullet}(X, \sigma)$$

and an analogous result in homology. There exists a bivariant Chern character, a homomorphism

$$\operatorname{ch}: KK_{\bullet}(A,B) \to \operatorname{HL}_{\bullet}(A,B)$$
.

Restricting to $KK_0(\mathbb{C}, C_0(X, \mathcal{E}_{\sigma}))$ yields the usual twisted Chern character. Given a pair (A, \widetilde{A}) of separable C^* -algebras, for which there exists a certain cyclic fundamental class $\Xi \in \mathrm{HL}_d(A \otimes \widetilde{A}, \mathbb{C})$, we have an analogous notion of Poincaré duality in HL-theory as we had in KK-theory. The pair (A, \widetilde{A}) is called a cyclic Poincaré duality (C-PD) pair. Again, the cyclic fundamental class of a C-PD pair is by no means unique. Any PD pair (A, \widetilde{A}) , with fundamental class Δ , is automatically also a C-PD pair, since $\mathrm{ch}(\Delta)$ is a cyclic fundamental class. However, in general there is no reason to assume that $\Xi = \mathrm{ch}(\Delta)$. The difference between Ξ and $\mathrm{ch}(\Delta)$ is the noncommutative Todd class $\mathrm{Todd}(A) \in \mathrm{HL}_0(A, A)$. A noncommutative Minasian-Moore

formula would require us to take the square root of the Todd class, which is not always possible. We assume for now that $\sqrt{\text{Todd}(A)} \in \text{HL}_0(A, A)$ exists for the spacetime C^* -algebra A.

The final ingredient needed for a noncommutative Minasian-Moore formula is a generalization of the Gysin map. A *-homomorphism $f: A \to B$ is said to be K-oriented, if there exists a class $f! \in KK_d(B, A)$, satisfying certain functorial properties. It can be used to define the noncommutative Gysin homomorphisms

$$f_! := (-) \otimes_B f_! : KK_{\bullet}(\mathbb{C}, B) \to K_{\bullet + d}(\mathbb{C}, A)$$

and

$$f^! := f! \otimes_A (-) : K_{\bullet}(A, \mathbb{C}) \to K_{\bullet + d}(B, \mathbb{C}).$$

One can construct similar Gysin maps in HL-theory, if there exists a class $f* \in \operatorname{HL}_d(B,A)$ with the same functorial properties. Given a suitable noncommutative D-brane B, a suitable noncommutative spacetime A and a *-homomorphism $\phi: A \to B$, the Minasian-Moore formula (1.7) is generalized by

$$Q_{\mathbb{C}}(B,\xi) := \operatorname{ch}(\phi_!(\xi)) \otimes_A \sqrt{\operatorname{Todd}(A)} \in \operatorname{HL}_{\bullet}(\mathbb{C},A),$$

where $\xi \in K_{\bullet}(B)$ is a noncommutative Chan-Paton bundle.

Chapter 2

D-Branes And *K***-Theory**

In this chapter we shall explore the connection between K-theory and D-branes. The spacetime X was assumed to be a smooth oriented Riemannian Spin manifold and the D-brane worldvolume Σ an even-dimensional oriented manifold, together with a continuous map ϕ : $\Sigma \to X$. For now, we assume that Σ is Spin^c , that the quantized H-flux $[H] \in H^3(X,\mathbb{Z})$ vanishes and that Σ carries a U(n) Chan-Paton bundle. These assumptions will be dropped (much) later, after we have discussed the Freed-Witten anomaly.

Before delving into K-theory, we shall take some time to introduce a few mathematical concepts which will be of fundamental importance throughout the text: characteristic classes and Spin/Spin^c structures.

2.1 Universal bundles and characteristic classes

Throughout the text we shall make extensive use of so-called *characteristic classes*, for example the *Chern character*, the *Chern classes*, the *Atiyah-Hirzebruch class* \widehat{A} , the *Stiefel-Whitney classes* and the *Todd class*. It is, therefore, worthwhile to devote a few pages to a discussion of the basic ideas and definitions. More extensive treatments can be found in numerous books, for example [MS4, BT, May, LM, Hus], which are our main references.

Characteristic classes of a vector bundle $E \to X$, real or complex, are cohomology classes of X, which measure the nontriviality of E. There are important differences between real and complex vector bundles, as we shall see. Let us start by introducing *classifying spaces*.

Definition 2.1.1. A classifying space for a Lie group G is a topological space BG equipped with a principal G bundle $EG \to BG$, the universal bundle, whose total space EG is contractible. The space of isomorphism classes of principal G bundles over any manifold X is isomorphic to [X, BG], the space of homotopy classes of continuous maps $X \to BG$. More precisely, every principal G bundle over X is isomorphic to a pullback of $EG \to BG$ by a classifying map $f: X \to BG$. It is a standard result in bundle theory that the isomorphism class of the pullback bundle $f^{-1}EG \to X$ depends only on the homotopy class of f.

Theorem 2.1.2. Classifying spaces, as defined above, always exist.

Proof. Category theoretically the theorem is motivated as follows. The cofunctor

 $\operatorname{Prin}_G(-): \mathbf{hCW} \to \mathbf{Set}$

from the homotopy category of pointed CW-complexes to the category of sets, sending a CW-complex X to the set $\operatorname{Prin}_G(X)$ of isomorphism classes of principal G bundles over X, is representable by the *Brown's representability theorem* [Bro1, Bro2]. An application of the *Yoneda lemma* [Mac] implies the existence of a *universal element* for the functor $\operatorname{Prin}_G(-)$, which establishes the representation. This universal element is precisely the universal bundle $EG \to BG$. The Yoneda lemma yields the bijective correspondence $\operatorname{Prin}_G(X) \cong [X, BG]$ and the pullback construction of definition 2.1.1.

The categorial approach is not very convenient for practical calculations. Milnor [Mil] discovered the following explicit construction. The *join* of two topological spaces X and Y, denoted by $X \star Y$, is the quotient space $X \times Y \times [0,1]/\sim$, where \sim collapses $\{x\} \times Y \times \{0\}$ and $X \times \{y\} \times \{1\}$, for each $x \in X$, $y \in Y$, to points. We denote the (n+1)-fold join $G \star \ldots \star G$ by EG_n . Its points are specified by n-tuples $(t_1, \ldots, t_n) \in [0,1] \times \ldots \times [0,1]$, with $\sum_i t_i = 1$, together with elements $g_i \in G$, $i = 1 \ldots n$, for which $t_i \neq 0$. Denoting the points by (t_1g_1, \ldots, t_ng_n) makes the construction more clear. The topology of EG_n is chosen to be the strongest topology, such that the coordinate functions

$$t_i: EG_n \to [0,1]$$
 and $g_i: t_i^{-1}(0,1] \to G$

are continuous. The *infinite join* $G \star G \star \ldots$ is the direct limit

$$EG := \varinjlim_{n} E_n$$

of topological spaces.

One can show that each EG_n is (n-1)-connected, that is, $\pi_k(EG_n) = 0$ for all $0 \le k \le n-1$. It follows that EG is weakly contractible [May]:

$$\pi_k(EG) = \pi_k(\varinjlim_n EG_n) \cong \varinjlim_n \pi_k(EG_n) = 0,$$

which implies, by Whitehead's theorem [May], that EG is contractible.

The universal bundle is constructed as follows. There is a continuous right action of G on EG, given by

$$(t_1g_1,\ldots,t_ng_n)g:=(t_1g_1g,\ldots t_ng_ng).$$

The base space BG is the space of orbits EG/G and the bundle projection the canonical projection $EG \to EG/G$. Local trivializations and transition functions are given in [Mil], along with the required proofs for continuity. An explicit verification of the universal property is given in [Hus], but it turns out that contractibility is already enough to prove universality $[Ste1]^1$.

Let us now focus on the cases G = U(n) and G = O(n). Let K denote either \mathbb{C} or \mathbb{R} and $Gr_n(K^N)$ the set of n-dimensional K-linear subspaces, that is, n-planes passing through the origin of K^N . A little thought reveals that

$$\operatorname{Gr}_n(\mathbb{C}^N) \cong \frac{U(N)}{U(N-n) \times U(n)}$$

 $\operatorname{Gr}_n(\mathbb{R}^N) \cong \frac{O(N)}{O(N-n) \times O(n)}$,

¹Compare this to the theory of Eilenberg-MacLane spaces, where it is enough to find a space with the correct homotopy properties to immediately conclude that it provides a representation of the cohomology functor [May]. In particular, the n-th Eilenberg-MacLane space must be (n-1)-connected.

because U(n) (resp. O(n)) acts transitively on $\operatorname{Gr}_n(\mathbb{C}^N)$ (resp. $\operatorname{Gr}_n(\mathbb{R}^N)$) and a particular n-plane is left invariant precisely under the subgroups of rotations U(n) (resp. O(n)) of the plane itself and rotations U(N-r) (resp. O(N-r)) of the complementary subspace. The natural inclusions

$$\operatorname{Gr}_n(K^N) \subset \operatorname{Gr}_n(K^{N+N'})$$

allow us to define the direct limit

$$\operatorname{Gr}_n(K^{\infty}) := \varinjlim_{N} \operatorname{Gr}_n(K^N).$$

These are the famous complex and real infinite Grassmann manifolds. There exist certain tautological bundles $E_n(K^N) \to Gr_n(K^N)$, whose fibres at a particular n-plane are the orthonormal frames of the plane itself, acted upon by U(n) or O(n) in the obvious manner. Taking the direct limit over N yields corresponding bundles over the infinite Grassmannians:

$$\operatorname{E}_n(K^{\infty})$$
 $\operatorname{E}_n(K^N)$

$$\downarrow \qquad := \varinjlim_{N} \qquad \downarrow \qquad .$$
 $\operatorname{Gr}_n(K^{\infty})$ $\operatorname{Gr}_n(K^N)$

Proposition 2.1.3.

$$E_n(\mathbb{C}^{\infty})$$
 $EU(n)$ $E_n(\mathbb{R}^{\infty})$ $EO(n)$

$$\downarrow \qquad \cong \qquad \downarrow \qquad \text{and} \qquad \downarrow \qquad \cong \qquad \downarrow \qquad .$$
 $Gr_n(\mathbb{C}^{\infty})$ $BU(n)$ $Gr_n(\mathbb{R}^{\infty})$ $BO(n)$

Proof. The total spaces $E_n(\mathbb{C}^N)$ and $E_n(\mathbb{R}^N)$, consisting of orthonormal frames of n-planes, are diffeomorphic to the *Stiefel manifolds* U(N)/U(N-n) and O(N)/O(N-n). This is reasonably clear, since the subgroups $U(N-n) \subset U(N)$ and $O(N-n) \subset O(N)$ of rotations in the complementary subspace of a particular n-plane are precisely those that leave a frame of the n-plane invariant. The fibrations

$$U(N-1) \longrightarrow U(N) \longrightarrow S^{2N+1}$$

 $O(N-1) \longrightarrow O(N) \longrightarrow S^N$

yield long exact sequences

$$\cdots \longrightarrow \pi_{k+1}(S^{2N+1}) \longrightarrow \pi_k(U(N-1)) \longrightarrow \pi_k(U(N)) \longrightarrow \pi_k(S^{2N+1}) \longrightarrow \cdots$$
$$\cdots \longrightarrow \pi_{k+1}(S^N) \longrightarrow \pi_k(O(N-1)) \longrightarrow \pi_k(O(N)) \longrightarrow \pi_k(S^N) \longrightarrow \cdots$$

in homotopy. Since $\pi_i(S^m) = 0$ for $i \leq m-1$, we obtain the isomorphisms

$$\pi_k(U(N-1)) \cong \pi_k(U(N)) \cong \dots \cong \pi_k(U(N-n)), \quad k \leq 2(N-n)+1$$

 $\pi_k(O(N-1)) \cong \pi_k(O(N)) \cong \dots \cong \pi_k(O(N-n)), \quad k \leq N-n-1.$

On the other hand, the fibrations

$$U(N-n) \hookrightarrow U(N) \longrightarrow U(N)/U(N-n) \cong E_n(\mathbb{C}^N)$$

 $O(N-n) \hookrightarrow O(N) \longrightarrow O(N)/O(N-n) \cong E_n(\mathbb{R}^N)$

induce long exact sequences

$$\cdots \xrightarrow{i_k} \pi_k(U(N)) \longrightarrow \pi_k(\mathcal{E}_n(\mathbb{C}^N)) \longrightarrow \pi_{k-1}(U(N-n)) \xrightarrow{i_{k-1}} \cdots$$

$$\cdots \xrightarrow{j_k} \pi_k(O(N)) \longrightarrow \pi_k(\mathcal{E}_n(\mathbb{R}^N)) \longrightarrow \pi_{k-1}(O(N-n)) \xrightarrow{j_{k-1}} \cdots$$

But we already know, that the inclusion maps

$$i_k: \pi_k(U(N-n)) \to \pi_k(U(N))$$

 $j_k: \pi_k(O(N-n)) \to \pi_k(O(N))$

are isomorphisms for k small enough. Due to exactness, this is possible only if

$$\pi_k(\mathbf{E}_n(\mathbb{C}^N)) = 0, \quad k \le 2(N-n) + 1$$

 $\pi_k(\mathbf{E}_n(\mathbb{R}^N)) = 0, \quad k \le N-n-1.$

The proof is concluded by

$$\pi_k(\mathbf{E}_n(\mathbb{C}^\infty)) = \pi_k(\varinjlim_N \mathbf{E}_n(\mathbb{C}^N)) \cong \varinjlim_N \pi_k(\mathbf{E}_n(\mathbb{C}^N)) = 0$$

$$\pi_k(\mathbf{E}_n(\mathbb{R}^\infty)) = \pi_k(\varinjlim_N \mathbf{E}_n(\mathbb{R}^N)) \cong \varinjlim_N \pi_k(\mathbf{E}_n(\mathbb{R}^N)) = 0,$$

an application of Whitehead's theorem to show that $E_n(\mathbb{C}^{\infty})$ and $E_n(\mathbb{R}^{\infty})$ are contractible, and finally an application of the result in [Ste1], which states that contractibility is enough to imply universality.

Using the "associated vector bundle" construction, the representation theorem for U(n) principal bundles can be translated to the language of U(n) vector bundles: every U(n) vector bundle over X is obtained, up to isomorphism, as the pullback of the associated vector bundle of EU(n) by a map $f: X \to BU(n)$. Obviously, a similar result holds for O(n) vector bundles.

The cohomology groups of classifying spaces are extremely interesting and are the starting point for constructing characteristic classes of bundles.

Definition 2.1.4. The cohomology classes in $H^{\bullet}(BG, \Lambda)$ are called universal characteristic classes². By the representation theorem, any principal G bundle $P \to X$ is isomorphic to the pullback bundle $f^{-1}EG \to X$ by a continuous map $f: X \to BG$. The characteristic classes of P are the pullbacks of the universal characteristic classes by f into $H^{\bullet}(X, \Lambda)$. They are well-defined, since f is unique up to homotopy.

Let us look at some explicit examples.

 $^{^2 \}mathrm{Here} \ \Lambda$ denotes the coefficient ring.

Proposition 2.1.5 ([MS4]). The cohomology ring $H^{\bullet}(BO(n), \mathbb{Z}_2)$ is a freely generated polynomial ring $\mathbb{Z}_2[w_1, \ldots, w_n]$, whose generators

$$w_k \in H^k(BO(n), \mathbb{Z}_2)$$

are called the universal Stiefel-Whitney classes.

For a O(n) vector bundle E over X, represented by a continuous map $f: X \to BO(n)$, the k-th Stiefel-Whitney class of E is

$$w_k(E) := f^* w_k \in H^k(X, \mathbb{Z}_2).$$

We also define $w_0 := 1$, so that $w_0(E) = 1$ for any vector bundle E. The Stiefel-Whitney classes have some remarkable properties. For example, $w_1(E) = 0$ if and only if the vector bundle is orientable³. The second Stiefel-Whitney class $w_2(E)$ is an obstruction for the existence of a Spin structure: the vector bundle E admits a Spin structure if and only if $w_1(E) = w_2(E) = 0$. Finally, the short exact sequence

$$0 \xrightarrow{\hspace*{1cm}} \mathbb{Z} \xrightarrow{\hspace*{1cm}} \mathbb{Z} \xrightarrow{\hspace*{1cm}} \mathbb{Z} \xrightarrow{\hspace*{1cm}} \mathbb{Z}_2 \xrightarrow{\hspace*{1cm}} 0$$

induces a long exact sequence

$$\cdots \longrightarrow H^{k}(X,\mathbb{Z}) \longrightarrow H^{k}(X,\mathbb{Z}_{2}) \xrightarrow{\beta} H^{k+1}(X,\mathbb{Z}) \longrightarrow \cdots$$
 (2.1)

in cohomology, where β is the Bockstein homomorphism. The *integral Stiefel-Whitney classes* are defined by

$$W_k(E) := \beta(w_{k-1}(E)). \tag{2.2}$$

Of particular importance is $W_3(E)$ for the reason that E admits a Spin^c structure if and only if $w_1(E) = W_3(E) = 0$. Readers unfamiliar with Spin^c and Spin structures can think of them as slightly stronger "orientability" conditions, defined as the vanishing of $W_3(E)$ and $w_3(E)$, respectively. The total universal Stiefel-Whitney class is the sum

$$w := w_0 + w_1 + w_2 + \ldots \in H^{\bullet}(BO(n), \mathbb{Z}_2)$$

and the total Stiefel-Whitney class of the O(n) bundle E is

$$w(E) := f^*w \in H^{\bullet}(X, \mathbb{Z}_2)$$
.

Given two real vector bundles E and F, the total Stiefel-Whitney class is additive in the sense that [MS4]

$$w(E \oplus F) = w(E) \smile w(F). \tag{2.3}$$

The second simplest case is that of integer coefficients.

Proposition 2.1.6 ([MS4]). The cohomology ring $H^{\bullet}(BU(n), \mathbb{Z})$ is a freely generated polynomial ring $\mathbb{Z}[c_1, \ldots, c_n]$, whose generators

$$c_k \in H^{2k}(BU(n), \mathbb{Z})$$

are called the universal Chern classes.

³Expressed homotopically, a vector bundle is orientable if and only if the classifying map factors through the simply connected space BSO(n).

For a U(n) vector bundle E represented by a continuous map $f: X \to BU(n)$, the k-th Chern class is

$$c_k(E) := f^*c_k \in H^{2k}(X, \mathbb{Z}).$$

We also define $c_0 := 1$, so that $c_0(E) = 1$ for any vector bundle E. The total universal Chern class is the sum

$$c := c_0 + c_1 + c_2 + \ldots \in H^{\text{even}}(BU(n), \mathbb{Z})$$

and the total Chern class of the bundle E is

$$c(E) := f^*c \in H^{\text{even}}(X, \mathbb{Z})$$
.

Given two complex vector bundles E and F, the total Chern class is additive in the sense that [MS4]

$$c(E \oplus F) = c(E) \smile c(F). \tag{2.4}$$

For a complex vector bundle the Chern classes and the Stiefel-Whitney classes of the underlying real vector bundle are related.

Proposition 2.1.7 ([LM]). There is a ring isomorphism

$$H^{\bullet}(BU(n), \mathbb{Z}_2) \cong \mathbb{Z}_2[\widetilde{c}_1, \dots, \widetilde{c}_n],$$

where \widetilde{c}_k is the mod 2 reduction of c_k . The natural inclusion $BU(n) \to BO(2n)$ induces a homomorphism

$$H^{\bullet}(BO(2n), \mathbb{Z}_2) \to H^{\bullet}(BU(n), \mathbb{Z}_2)$$
,

which maps

$$w_{2k} \mapsto \widetilde{c}_k$$
 and $w_{2k+1} \mapsto 0$.

Hence, the underlying O(2n) vector bundle $E_{\mathbb{R}}$ of a U(n) vector bundle E satisfies $w_{2k}(E_{\mathbb{R}}) \equiv c_k(E) \pmod{2}$ and $w_{2k+1}(E_{\mathbb{R}}) = 0$. In particular, the underlying real vector bundle of any complex vector is orientable⁴ $Spin^c$ structure.

The definition of a characteristic class can be rephrased category theoretically as follows.

Definition 2.1.8. A characteristic class C is a natural transformation [Mac]

$$C: \operatorname{Prin}_G(-) \to H^{\bullet}(-, \Lambda)$$
.

Again, the definition can equally well be given in terms of associated vector bundles.

This definition emphasizes the *naturality* property of characteristic classes: if $Y \to X$ is a continuous map and $E \to X$ a bundle with characteristic class C(E), then $C(f^{-1}E) = f^*C(E)$.

⁴In fact, it is canonically orientable. This is evident from the following alternative proof. Choose a complex basis $\{e_1, \ldots, e_n\}$ for each fibre E_x of E. Then $\{e_1, ie_1, \ldots, e_n, ie_n\}$ is a basis for the 2n-dimensional fibre of the underlying real vector space and defines an orientation for each fibre. The orientation is independent of the complex basis used, because $GL_n(\mathbb{C})$ is connected, allowing for continuous orientation-preserving transformations between any two complex bases.

Let $f: X \to BU(n)$ be the classifying map for a vector bundle $E \to X$ and C a characteristic class $\operatorname{Vect}^n_{\mathbb{C}}(-) \to H^{\bullet}(-,\mathbb{Z})$. Naturality implies

$$C(E) = C(f^{-1}EU(n)) = f^*C(EU(n)).$$

Since $H^{\bullet}(BU(n), \mathbb{Z})$ is generated by $\{c_1, \ldots, c_n\}$, C(EU(n)) is a polynomial $P_C(c_1, \ldots, c_n)$. Thus, C(E) is of the form

$$C(E) = f^* P_C(c_1, \dots, c_n)$$

= $P_C(f^* c_1, \dots, f^* c_n)$
= $P_C(c_1(E), \dots, c_n(E))$,

a polynomial in the Chern classes of E. This proves the following important result.

Proposition 2.1.9. Every characteristic class $C: \operatorname{Vect}_{\mathbb{C}}^k(-) \to H^{\bullet}(-,\mathbb{Z})$ is a polynomial in the Chern classes. Moreover, any characteristic class taking values in either $H^{\bullet}(-,\mathbb{Q})$ or $H^{\bullet}(-,\mathbb{R}) \cong H^{\bullet}_{dR}(-)$ can be expressed as a polynomial in the images of the Chern classes mapped into the respective cohomology theory.

Proposition 2.1.10. The first Chern class $c_1(L) \in H^2(X,\mathbb{Z})$ of a complex line bundle $L \to X$ uniquely classifies the bundle up to isomorphism. Moreover, $c_k(L) = 0$ for $k \ge 2$.

To prove this, we need to know the homotopy groups of S^1 and $S^{\infty} := \underline{\lim} S^n$.

Lemma 2.1.11. *a)*
$$\pi_k(S^1) = 0$$
 for $k \ge 2$. *b)* $\pi_k(S^{\infty}) = 0$ for all k .

Proof. a) Recall from the theory of covering spaces [May] that a map $f: S^k \to S^1$

$$S^{k} \xrightarrow{\widetilde{f}} S^{1}$$

admits a lift $\tilde{f}: S^k \to \mathbb{R}$, if and only if $f_*\pi_1(S^k) \subset p_*\pi_1(\mathbb{R})$. This is obviously the case when $k \geq 2$, since $\pi_1(\mathbb{R}) = 0$ and $\pi_1(S^k) = 0$ for all $k \geq 2$. Therefore, $\pi_k(S^1)$ is the space of homotopy classes of continuous maps $S^k \to S^1$, all of which lift to maps $S^k \to \mathbb{R}$ (for $k \geq 2$), yielding elements in $\pi_k(\mathbb{R})$. The induced map $p_*: \pi_k(\mathbb{R}) \to \pi_k(S^1)$ is surjective (again, for $k \geq 2$). Injectivity follows from a basic result in the theory of covering spaces that the lifts of homotopic maps $S^k \to S^1$ are homotopic [May]. Thus, $\pi_k(S^1) \cong \pi_k(\mathbb{R}) = 0$ for $k \geq 2$.

b) Using $\pi_k(S^n) = 0$ for k < n, we get

$$\pi_k(S^\infty) \cong \varinjlim_n \pi_k(S^n) = 0.$$

Proof of 2.1.10. By proposition 2.1.6, the cohomology ring is freely generated by $c_1 \in H^2(BU(1), \mathbb{Z})$. The classifying space is

$$BU(1) = \operatorname{Gr}_1(\mathbb{C}^{\infty}) = \varinjlim_{N} \frac{U(N)}{U(N-1) \times U(1)}.$$

The N-dimensional complex projective space is the quotient space $\mathbb{C}\mathrm{P}^N:=S^{2N+1}/S^1$. Using

$$U(N)/U(N-1) \cong S^{2N+1}$$

gives

$$\mathbb{C}\mathrm{P}^N \cong \frac{S^{2N+1}}{S^1} \cong \frac{U(N)}{U(N-1) \times U(1)}.$$

Thus, the classifying space BU(1) is simply

$$BU(1) \cong \varinjlim_{N} \mathbb{C}P^{N}$$
,

the *infinite complex projective space*, which we denote by $\mathbb{C}P^{\infty}$. On the other hand, the direct limit of the fibrations

$$S^1 \longrightarrow S^{2N+1} \longrightarrow \mathbb{C}P^N$$

is the fibration

$$S^1 \longrightarrow S^\infty \longrightarrow \mathbb{C}P^\infty$$

which induces a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_k(S^1) \longrightarrow \pi_k(S^\infty) \longrightarrow \pi_k(\mathbb{C}P^\infty) \longrightarrow \pi_{k-1}(S^1) \longrightarrow \cdots$$

This, together with lemma 2.1.11, yields

$$\pi_k(\mathbb{C}\mathrm{P}^{\infty}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$
(2.5)

In other words, $\mathbb{C}P^{\infty}$ is a model for the Eilenberg-MacLane space $K(\mathbb{Z},2)$. This concludes the proof, since both isomorphism classes of complex line bundles over X and classes of $H^2(X,\mathbb{Z})$ are in bijective correspondence with continuous maps $X \to \mathbb{C}P^{\infty} = K(\mathbb{Z},2)$. More precisely, for any two maps $f_1, f_2 : X \to BU(1)$ the pullbacks of the first universal Chern class are equal, $f_1^*c_1 = f_2^*c_1$, if and only if f_1 and f_2 are homotopic, if and only if the bundles $f_1^{-1}EU(1)$ and $f_2^{-1}EU(1)$ are isomorphic.

We now restrict specifically to smooth manifolds. All smooth manifolds admit good open covers [BT], open covers whose elements and their finite intersections are all contractible. Given the transition data for any complex line bundle $L \to X$, it is easy to construct a cohomology class $e(L) \in H^2(X,\mathbb{Z})$, which uniquely determines L up to isomorphism. This is known as the Euler class [BT] and is constructed as follows. Let $\{g_{\alpha\beta}\}$ be the transition functions of L with respect to a good open cover $\{U_{\alpha}\}$. They can be written as

$$g_{\alpha\beta} = \exp\left(i\phi_{\alpha\beta}\right) \,,$$

where $\{\phi_{\alpha\beta}\}$ are real-valued functions. The cocycle condition for $\{g_{\alpha\beta}\}$ becomes

$$g_{\beta\gamma}g_{\alpha\gamma}^{-1}g_{\alpha\beta} = \exp\left(i(\phi_{\beta\gamma} - \phi_{\alpha\gamma} + \phi_{\alpha\beta})\right) = 1,$$

which is satisfied if and only if

$$\phi_{\beta\gamma} - \phi_{\alpha\gamma} + \phi_{\alpha\beta} \in 2\pi\mathbb{Z}$$
.

We denote

$$n_{\alpha\beta\gamma} := -\frac{1}{2\pi} \left(\phi_{\beta\gamma} - \phi_{\alpha\gamma} + \phi_{\alpha\beta} \right) ,$$

which is obviously closed as a Čech cocycle and thus determines a Čech cohomology class $[\underline{n}] \in \check{H}^2(X,\mathbb{Z})^5$. Note that, even though $n_{\alpha\beta\gamma} = -\delta(\phi)_{\alpha\beta\gamma}/(2\pi)$, the class $[\underline{n}] \in \check{H}^2(X,\mathbb{Z})$ is generally nontrivial, since $\phi_{\alpha\beta}/2\pi$ are generally only real numbers, not integers. The Euler class of L is⁶

$$e(L) := [n]$$
.

Theorem 2.1.12. For a complex line bundle $L \to X$ the Euler class and the first Chern class coincide:

$$e(L) = c_1(L) \in H^2(X, \mathbb{Z}).$$

Proof. It is easy to see that e(L) uniquely determines L up to isomorphism and that it behaves naturally under pullbacks. It is thus a characteristic class and as such, by proposition 2.1.9, is a polynomial in the first Chern class. Both are classes in $H^2(X,\mathbb{Z})$ and hence equal up to a constant multiplier, which turns out to be 1 [BT].

We have now established a simple method for computing Chern classes of complex line bundles from transition data. Remarkably, it turns out that many statements concerning Chern classes of higher rank bundles can be proven by merely proving the statements for line bundles, due to the following famous theorem, known as the *splitting principle*.

Theorem 2.1.13 (Splitting principle [May, BT]). Let $E \to X$ be a U(n) vector bundle. There exists a manifold F(E), the flag manifold, and a map $f: F(E) \to X$, such that the pullback bundle $f^{-1}E \to F(E)$ is isomorphic to a sum of complex line bundles

$$f^{-1}E \cong L_1 \oplus \ldots \oplus L_n \longrightarrow F(E)$$

and the induced map $f^*: H^{\bullet}(X, \mathbb{Z}) \to H^{\bullet}(F(E), \mathbb{Z})$ is an injection.

The splitting principle can be used to compute the Chern classes of dual bundles. Denote the dual bundle of a complex line bundle L by L^* . We then have

$$c_1(L) + c_1(L^*) = c_1(L \otimes L^*) = 0,$$

⁵Since we are working on a manifold, both Čech cohomology and de Rham cohomology coincide with singular cohomology with the appropriate coefficients. Hence, the distinction between $H^{\bullet}(X,-)$ and $\check{H}^{\bullet}(X,-)$, and $H^{\bullet}(X,\mathbb{R})$ and $H^{\bullet}_{\mathrm{dR}}(X)$ is purely notational and is merely used to emphasize the method of calculation.

⁶More precisely, we mean the Euler class of the underlying real vector bundle of rank 2.

which can be easily seen from the explicit construction of c_1 for line bundles in terms of the transition functions. Let now E be a U(n) vector bundle. By the splitting principle there exists a flag bundle F(E), such that

$$f^{-1}E \cong L_1 \oplus \ldots \oplus L_n \to F(E)$$

$$f^{-1}E^* \cong L_1^* \oplus \ldots \oplus L_n^* \to F(E).$$

By naturality of the Chern classes and the summation formula (2.4), we get

$$f^*c(E^*) = c(f^{-1}E^*)$$

$$= (1 + c_1(L_1^*)) \smile \dots \smile (1 + c_1(L_n^*))$$

$$= (1 - c_1(L_1)) \smile \dots \smile (1 - c_1(L_n))$$

$$= \sum_i (-1)^i c_i(f^{-1}E)$$

$$= f^* \Big(\sum_i (-1)^i c_i(E) \Big).$$

The map f^* is injective according to the splitting principle. Thus,

$$c_i(E^*) = (-1)^i c_i(E)$$
.

The Chern classes can also be computed purely analytically as we shall next show. Let $E \to X$ be a U(n) vector bundle over a smooth manifold X. Let A be a $\mathfrak{u}(n)$ -valued gauge connection on E and $F = dA + A \wedge A$ the corresponding curvature.

Theorem 2.1.14. The de Rham polyform

$$c(F) := \det\left(1 - \frac{F}{2\pi i}\right). \tag{2.6}$$

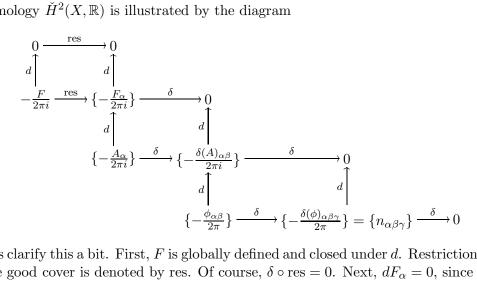
is closed and its de Rham cohomology class is independent of the chosen connection. Moreover, the image of the total Chern character $c(E) \in H^{even}(X, \mathbb{Z})$ in $H^{even}_{dR}(X)$ coincides with [c(F)].

Proof. Let us start by proving the theorem for complex line bundles. If F is the curvature of a U(1) vector bundle L, then

$$c(F) = c_1(F) = -\frac{F}{2\pi i}$$
.

Recall that the Čech–de Rham isomorphism $H_{\mathrm{dR}}^{\bullet}(X) \cong \check{H}^{\bullet}(X,\mathbb{R})$ between de Rham and Čech cohomologies follows from the Čech-de Rham double complex by a "tic-tac-toe" argument [BT]. In particular, the computation of the representative of $[-F/(2\pi i)] \in H_{\mathrm{dR}}^2(X)$ in Čech

cohomology $\check{H}^2(X,\mathbb{R})$ is illustrated by the diagram



Let us clarify this a bit. First, F is globally defined and closed under d. Restriction to open sets of the good cover is denoted by res. Of course, $\delta \circ \text{res} = 0$. Next, $dF_{\alpha} = 0$, since the diagram commutes. Open sets of a good cover are contractible and thus their cohomology vanishes (this is the Poincaré lemma), which implies that $F_{\alpha} = dA_{\alpha}$ for 1-forms A_{α} . Again, commutativity of the diagram implies $d\delta(A)_{\alpha\beta} = 0$. By the Poincaré lemma we have $\delta(A)_{\alpha\beta} = d(i\phi_{\alpha\beta})$, where $\phi_{\alpha\beta}$ are 0-forms (functions). Finally, by commutativity $d\delta(\phi)_{\alpha\beta\gamma} = 0$, which means that $\delta(\phi)_{\alpha\beta\gamma}$ are all constants. We simplify the notation by writing $n_{\alpha\beta\gamma} = -\delta(\phi)_{\alpha\beta\gamma}/(2\pi)$. Clearly, $\{n_{\alpha\beta\gamma}\}\$ determines a nontrivial Cech cohomology class with constant coefficients. When F is the curvature of a connection on a line bundle, remarkable things happen. Namely, $\{A_{\alpha}\}\$ are then the local connection 1-forms, satisfying $\delta(A)_{\alpha\beta} = d\log g_{\alpha\beta}$, where $g_{\alpha\beta}$ are the transition functions. Then $\phi_{\alpha\beta} = -i \log g_{\alpha\beta}$ and $\delta(\phi)_{\alpha\beta\gamma} = -i \log \delta(g)_{\alpha\beta\gamma}$. But since $\{g_{\alpha\beta}\}$ is closed under δ , we get $\delta(\phi)_{\alpha\beta\gamma} \in 2\pi\mathbb{Z}$. Therefore, $n_{\alpha\beta\gamma} = -\delta(\phi)_{\alpha\beta\gamma}/(2\pi)$ are integers and they determine a class $[\underline{n}] \in \check{H}^2(X,\mathbb{Z})^7$. The class $[\underline{n}]$ is precisely the Euler class of the line bundle which we discussed earlier and, by theorem 2.1.12, it coincides with the first Chern class $c_1(L)$. This proves theorem for line bundles.

Consider then a general U(n) vector bundle $E \to X$ with classification map $f: X \to X$ BU(n) and curvature F. The determinant in (2.6) is tricky to deal with, unless F is diagonal. It can always be diagonalized by a unitary matrix A, since $\mathfrak{u}(n)$ consists of complex antihermitian matrices. Because

$$\det\left(1 - \frac{AFA^{-1}}{2\pi i}\right) = \det\left(1 - \frac{F}{2\pi i}\right) \,,$$

we lose nothing by assuming F to be diagonal. Let us continue under this assumption.

By the splitting principle 2.1.13 and naturality of the universal bundle construction, there exists a flag bundle $g: F(E) \to X$, such that

$$f^{-1}E \cong (g \circ f)^{-1} \cong L_1 \oplus \ldots \oplus L_n$$
,

where $L_k \to F(E)$ are line bundles. By naturality and the summation formula (2.4), we get

$$f^*c(E) = c(f^{-1}(E)) = c(L_1 \oplus \ldots \oplus L_n) = c(L_1) \smile \ldots \smile c(L_n)$$
.

⁷In the de Rham picture this property is expressed by fact that the integral of $-F/(2\pi i)$ over any integral homology cycle evaluates to an integer.

We already proved that

$$c(L_k) = 1 + c_1(L_k) = 1 + [c_1(F_k)] = [1 + c_1(F_k)] = \left[1 - \frac{F_k}{2\pi i}\right],$$

where F_k are curvatures of the line bundles L_k , induced by F. Plugging these into the previous formula yields

$$f^*c(E) = \left[1 - \frac{F_1}{2\pi i}\right] \smile \dots \smile \left[1 - \frac{F_n}{2\pi i}\right]$$
$$= \left[\prod_{k=1}^n \left(1 - \frac{F_k}{2\pi i}\right)\right] = \left[\det\left(1 - \frac{f^*F}{2\pi i}\right)\right] = f^*\left[\det\left(1 - \frac{F}{2\pi i}\right)\right],$$

By the splitting principle, the map f^* is injective, which implies

$$c(E) = \left[\det \left(1 - \frac{F}{2\pi i} \right) \right] ,$$

concluding the proof.

The images of $c_k(E)$ in de Rham cohomology are represented by the homogeneous components $c_k(F)$ of c(F),

$$c(F) = c_0(F) + c_1(F) + \ldots + c_n(F), \quad c_k(F) \in \Omega^{2k}(X).$$

From now on, we use the same notation, c(E) and $c_k(E)$, to denote the Chern classes both in integral and de Rham cohomology. It should always be clear from the context which one is used.

Definition 2.1.15. Recall from the proof of theorem 2.1.14 that F can always be diagonalized by a unitary matrix A. Let

$$-\frac{AFA^{-1}}{2\pi i} = \operatorname{diag}(x_1, \dots, x_n),$$

where $\{x_k\}$ are 2-forms, known as the *Chern roots*. Writing out the determinant in c(F) in terms of $\{x_k\}$ yields

$$c(F) = \det\left(1 - \frac{F}{2\pi i}\right) = \prod_{k=1}^{n} (1 + x_k)$$

$$= 1 + \underbrace{\left(\sum_{k=1}^{n} x_k\right)}_{c_1(F)} + \underbrace{\left(\sum_{1 \le k < l \le n} x_k \land x_l\right)}_{c_2(F)} + \dots + \underbrace{\left(\prod_{k=1}^{n} x_k\right)}_{c_n(F)}.$$

Clearly, $c_k(F)$ are simply the elementary symmetric polynomials in the Chern roots. To simplify the notation, we shall make no distinction between x_k and the cohomology class $[x_k]$, denoting both by x_k and referring to both with the name Chern root.

It is a well-known result that all symmetric polynomials can be constructed as polynomials in the elementary symmetric polynomials. In this case, all symmetric polynomials in the Chern roots can be expressed in terms of the Chern classes. Therefore, to construct further characteristic classes, all we need to do is to construct symmetric polynomials in the Chern roots.

The Chern character of a U(n) vector bundle E is the characteristic class

$$\operatorname{ch}(E) := \sum_{k=1}^{n} e^{x_k} = n + \sum_{k=1}^{n} x_k + \frac{1}{2!} \sum_{k=1}^{n} x_k^2 + \dots \in H^{\operatorname{even}}(X, \mathbb{Q}).$$
 (2.7)

The homogeneous components of $\operatorname{ch}(E)$ are denoted $\operatorname{ch}_k(E) \in H^{2k}(X,\mathbb{Q})$. It is a straightforward task to write them out in terms of the Chern classes. The three first are

$$\operatorname{ch}_0(E) = \operatorname{rk}(E)$$

$$\operatorname{ch}_1(E) = c_1(E)$$

$$\operatorname{ch}_2(E) = \frac{1}{2} \left(c_1(E)^2 - 2c_2(E) \right).$$

In terms of a curvature 2-form F of E, the Chern character is

$$\operatorname{ch}(E) = \sum_{k=1}^{n} e^{x_k} = \operatorname{Tr} \exp(\operatorname{diag}(x_1, \dots, x_n)) = \left[\operatorname{Tr} \exp\left(-\frac{F}{2\pi i}\right)\right]. \tag{2.8}$$

For a complex line bundle L, the definition simplifies to

$$ch(L) = 1 + c_1(L) = \left[1 - \frac{F}{2\pi i}\right] \in H^2(X, \mathbb{Q}).$$

Using (2.8), it is a simple calculation to show that [LM]

$$\operatorname{ch}(E \oplus F) = \operatorname{ch}(E) + \operatorname{ch}(F) \tag{2.9}$$

$$\operatorname{ch}(E \otimes F) = \operatorname{ch}(E) \smile \operatorname{ch}(F), \qquad (2.10)$$

for any complex vector bundles E and F over X. The importance of the Chern character becomes clear, when we learn that it is compatible with K-theory under direct sums and tensor products of vector bundles.

The complex Todd class of a U(n) vector bundle E, with Chern roots $\{x_k\}$, is the characteristic class

$$\operatorname{Todd}_{\mathbb{C}}(E) := \prod_{k=1}^{n} \frac{x_k}{1 - e^{-x_k}} \in H^{\operatorname{even}}(X, \mathbb{Q}).$$

The complex Atiyan-Hirzebruch class is the characteristic class

$$\widehat{A}_{\mathbb{C}}(E) := \prod_{k=1}^{n} \frac{x_k/2}{\sinh(x_k/2)} \in H^{4\bullet}(X, \mathbb{Q}).$$

These two are related by

$$\operatorname{Todd}_{\mathbb{C}}(E) = \prod_{k=1}^{n} \frac{x_k}{1 - e^{-x_k}} = \prod_{k=1}^{n} \frac{x_k}{1 - e^{-x_k}} = \prod_{k=1}^{n} \left(\frac{x_k/2}{\sinh(x_k/2)}\right) \smile e^{\sum_k x_k/2}$$
$$= \widehat{A}_{\mathbb{C}}(E) \smile e^{c_1(E)/2}.$$

Since the Chern classes are only defined for complex vector bundles, so are ch, $\hat{A}_{\mathbb{C}}$ and $\operatorname{Todd}_{\mathbb{C}}$. However, given a real vector bundle, we can always complexify the fibres to obtain a complex vector bundles. The Chern roots of the complexified bundle can then be used to define characteristic classes of the original real bundle. For example, let $E \to X$ be a O(n) vector bundle. Its complexification is the product bundle $E \otimes \mathbb{C}$. The splitting pullback of $E \otimes \mathbb{C}$ by $f: F(E \otimes \mathbb{C}) \to X$ is of the form [LM]

$$f^{-1}(E \otimes \mathbb{C}) \cong L_1 \otimes L_1^* \otimes \ldots \otimes L_{|n|} \otimes L_{|n|}^*$$

where L_k are complex line bundles and L_k^* their dual bundles. Since the dual line bundles satisfy $c_1(L^*) = -c_1(L)$, the Chern roots of $E \otimes \mathbb{C}$ are $\{\pm x_1, \pm x_2, \dots, \pm x_{\lfloor x \rfloor}\}$, where $f^*x_k = c_1(L_k)$. The Atiyah-Hirzebruch class of E is the characteristic class

$$\widehat{A}(E) := \prod_{k=1}^{\lfloor n \rfloor} \frac{x_k/2}{\sinh(x_k/2)} \in H^{4\bullet}(X, \mathbb{Q}). \tag{2.11}$$

It is the square root of the complex Todd class of $E \otimes \mathbb{C}$:

$$\operatorname{Todd}_{\mathbb{C}}(E \otimes \mathbb{C}) := \prod_{k=1}^{\lfloor n \rfloor} \frac{-x_k^2}{(1 - e^{-x_k})(1 - e^{x_k})} = \prod_{k=1}^{\lfloor n \rfloor} \frac{-x_k^2}{(e^{x_k/2} - e^{-x_k/2})(e^{-x_k/2} - e^{x_k/2})}$$
$$= \prod_{k=1}^{\lfloor n \rfloor} \frac{x_k^2}{(e^{x_k/2} - e^{-x_k/2})(e^{x_k/2} - e^{-x_k/2})} = \prod_{k=1}^{\lfloor n \rfloor} \left(\frac{x_k/2}{\sinh(x_k/2)}\right)^2 = \widehat{A}(E)^2.$$

The real version of the $Todd\ class$ is more difficult to define and the reason behind the definition can not be explained at this point. In fact, it is only defined for ${\rm Spin}^c$ vector bundles. Let E be an O(n) vector bundle over X, satisfying $w_1(E)=W_3(E)=0$. Exactness of (2.1) and the definition (2.2) imply, that $w_2(E)$ must be the mod 2 reduction of a class $d(E) \in H^2(X,\mathbb{Z})$. Recall from proposition 2.1.7 that if E is a complex vector bundle with underlying real vector bundle $E_{\mathbb{R}}$, then $w_2(E_{\mathbb{R}})$ is the mod 2 reduction of $c_1(E)$. Thus, in the complex case, the first Chern class plays the role of d. This motivates the definition

$$\operatorname{Todd}(E) := \widehat{A}(E) \smile e^{d(E)/2}, \tag{2.12}$$

for a real $Spin^c$ vector bundle E.

2.2 Spin and Spin^c structures

We have already referred several times to Spin and Spin^c vector bundles, which we defined as vector bundles whose Stiefel-Whitney classes satisfy the appropriate vanishing conditions. In this section the reader is acquainted with more intuitive geometric definitions.

Consider first a principal O(n) bundle $P_{O(n)}$ over X. The group O(n) has two connected components, one consisting of matrices with determinant +1 and the other of matrices with determinant -1. Let $\{g_{\alpha\beta}\}$ be the transition functions of $P_{O(n)}$ with respect to a good open cover. We would like to find reductions and lifts of the transition functions to more connected groups than O(n). The first step would be to find a reduction of the structure group to

the connected (0-connected) group SO(n). For this, we need to find a set of SO(n)-valued transition functions $\{g'_{\alpha\beta}\}$, differing from $g_{\alpha\beta}$ by a coboundary. Let

$$t_{\alpha\beta} := \det(g_{\alpha\beta}) \in \{\pm 1\} \cong \mathbb{Z}_2$$
.

Then \underline{t} is a \mathbb{Z}_2 -valued Čech cocycle,

$$\delta(\underline{t}) = \det(\delta(g)) = \det(1) = 1$$

and thus defines a class $[\underline{t}] \in H^1(X, \mathbb{Z}_2)$. If this class is trivial, $t_{\alpha\beta} = s_{\beta}s_{\alpha}^{-1}$ for some \mathbb{Z}_2 -valued functions $\{s_{\alpha}\}$, then we can find O(n)-valued matrices $\{h_{\alpha}\}$, such that det $h_{\alpha} = s_{\alpha}$. These can be used to define SO(n)-valued cocycles (transition functions)

$$g'_{\alpha\beta} := h_{\alpha}g_{\alpha\beta}h_{\beta}^{-1}$$
.

The bundle defined by $\{g'_{\alpha\beta}\}$ is isomorphic to the bundle defined by $\{g_{\alpha\beta}\}$, since the transition functions differ only by a coboundary. Thus, the structure group can be reduced to SO(n) if and only if $[\underline{t}] \in H^1(X, \mathbb{Z}_2)$ vanishes. The bundle $P_{O(n)}$ is then said to be *orientable* and the bundle defined by $\{g'_{\alpha\beta}\}$ is denoted by $P_{SO(n)}$. The class $[\underline{t}]$ can be shown to coincide with $w_1(P_{O(n)})$ by observing that $[\underline{t}]$ respects naturality and that $[\underline{t}] \in H^1(BO(n), \mathbb{Z}) \cong \mathbb{Z}_2$ for the universal bundle $EO(n) \to BO(n)$ produces the nontrivial element (otherwise every n-plane bundle would be orientable). The orientations of $P_{SO(n)}$ over a (generally disconnected) space X are in one-to-one correspondence with $H^0(X, \mathbb{Z}_2)$. Obviously, the correspondence is given by assigning to each connected component of X one of the two possible orientations.

Proposition 2.2.1. A principal O(n) bundle $P_{O(n)} \to X$ is orientable if and only if $w_1(P_{O(n)}) = 0$. The distinct orientations are in bijective correspondence with $H^0(X, \mathbb{Z}_2)$.

The next step in simplifying the structure group would be to find a lift of the SO(n)-valued transition functions to take values in a simply connected (1-connected) structure group. The group SO(n) has a 2-sheeted universal covering space Spin(n) [LM], which fits into the short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \operatorname{Spin}(n) \xrightarrow{\eta} SO(n) \longrightarrow 1$$

where $\eta : \mathrm{Spin}(n) \to SO(n)$ is the fibre map. $\mathrm{Spin}(n)$ is a Lie group with Lie algebra $\mathfrak{spin}(n) \cong \mathfrak{so}(n)$.

Under what conditions can the SO(n)-valued transition functions $\{g_{\alpha\beta}\}$ of $P_{SO(n)}$ be lifted to Spin(n)-valued functions $\{\widetilde{g}_{\alpha\beta}\}$? Of course, the lifting can always be performed locally by simply lifting SO(n) into either one of the two sheets of the covering, but there might be a topological obstruction for patching the local lifts together on triple overlaps. To this end, suppose that $\{\widetilde{g}_{\alpha\beta}\}$ constitutes a local Spin lift. The cocycle condition for $\{g_{\alpha\beta}\}$ requires

$$\eta(\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\alpha\gamma}^{-1}) = \eta(\widetilde{g}_{\alpha\beta})\eta(\widetilde{g}_{\beta\gamma})\eta(\widetilde{g}_{\alpha\gamma}^{-1}) = 1.$$

However, $\ker(\eta) \cong \{\pm 1\} \subset \operatorname{Spin}(n)$, where $-1 \in \operatorname{Spin}(n)$ is a "nontrivial" element not in SO(n). Thus, the elements

$$u_{\alpha\beta\gamma} := \widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\alpha\gamma}^{-1} \in \{\pm 1\} \cong \mathbb{Z}_2$$

form a Čech cocycle $[\underline{u}] \in H^2(X, \mathbb{Z}_2)$. If this cocycle is a coboundary, that is, if

$$u_{\alpha\beta\gamma} = \widetilde{h}_{\alpha\beta}\widetilde{h}_{\beta\gamma}\widetilde{h}_{\alpha\gamma}^{-1}$$

for some $\mathrm{Spin}(n) \ni \{\pm 1\}$ -valued functions $\{\widetilde{h}_{\alpha\beta}\}$, then $\{\widetilde{g}_{\alpha\beta}\}$ (not a cocycle) can be modified into the cocycle $\{\widetilde{g}_{\alpha\beta}\widetilde{h}_{\alpha\beta}^{-1}\}$, determining the transition data for a principal $\mathrm{Spin}(n)$ bundle. Conversely, a global Spin lift obviously yields a trivial class $[\underline{u}] \in H^2(X, \mathbb{Z}_2)$. If the Spin lift does exist, the principal $\mathrm{Spin}(n)$ bundle is denoted by $P_{\mathrm{Spin}(n)}$. The proof that $w_2(E)$ coincides with $[\underline{u}]$ is again done in two steps: first noticing that the construction respects naturality and then arguing that $[\underline{u}]$ of the universal bundle $ESO(n) \to BSO(n)$ produces the nonzero element in $H^2(BSO(n), \mathbb{Z}_2) \cong \mathbb{Z}_2$ (otherwise every SO(n) bundle would have a Spin lift). Consider now another local Spin lift $\{\widetilde{f}_{\alpha\beta}\}$ of $P_{SO(n)}$. The differences $\widetilde{f}_{\alpha\beta}\widetilde{g}_{\alpha\beta}^{-1}$ satisfy

$$\eta(\widetilde{f}_{\alpha\beta}\widetilde{g}_{\alpha\beta}^{-1}) = g_{\alpha\beta}g_{\alpha\beta}^{-1} = 1,$$

which means that

$$\widetilde{f}_{\alpha\beta}\widetilde{g}_{\alpha\beta}^{-1} \in \{\pm 1\} \cong \mathbb{Z}_2$$
.

These differences define a class in $H^1(X, \mathbb{Z}_2)$, which is trivial precisely when the two Spin lifts differ by a cocycle. We have established the following.

Proposition 2.2.2. A principal SO(n) bundle $P_{SO(n)} \to X$ lifts to a Spin(n) bundle $P_{Spin(n)}$ precisely when $w_2(P_{SO(n)}) = 0$. If $w_2(P_{SO(n)}) = 0$, a choice of the Spin lift is called a Spin structure. Distinct Spin structures are in bijective correspondence with $H^1(X, \mathbb{Z}_2)$.

The existence of a Spin^c structure is a slightly stronger condition than orientability, but weaker than the existence of a Spin structure. Let us start again from the principal SO(n) bundle $P_{SO(n)}$ with transition functions $\{g_{\alpha\beta}\}$ and a generally nontrivial Stiefel-Whitney class $w_2(P_{SO(n)})$. The problem was that the local Spin lifts of the SO(n)-valued transition functions could not be made to agree on triple overlaps. However, it is possible to add a suitable extra structure to the problem, which can be tuned to precisely cancel the obstruction for a global lift into a nontrivial 2-sheeted covering. More precisely, instead of trying to lift the SO(n)-valued transition functions of $P_{SO(n)}$ to Spin(n), we augment the SO(n) bundle with a U(1) bundle and try to lift the $SO(n) \times U(1)$ -valued transition functions into a respective nontrivial 2-sheeted covering, which is the $Spin^c$ group

$$\operatorname{Spin}^{c}(n) := \operatorname{Spin}(n) \times_{\mathbb{Z}_{2}} U(1)$$

where $\times_{\mathbb{Z}_2}$ identifies (-1, -1) and (1, 1). Elements of $\operatorname{Spin}^c(n)$ are thus equivalence classes [(g, h)], where $g \in \operatorname{Spin}(n)$ and $h \in U(1)$. The group $\operatorname{Spin}^c(n)$ fits into the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \operatorname{Spin}^c(n) \xrightarrow{\xi} SO(n) \times U(1) \longrightarrow 1$$
,

where the nontrivial element of \mathbb{Z}_2 is mapped into $\mathrm{Spin}^c(n)$ as [(-1,1)] = [(1,-1)]. Let us try to construct a Spin^c lift for a set of transition functions $(\{g_{\alpha\beta}\}, \{h_{\alpha\beta}\})$ of a $SO(n) \times U(1)$ bundle $P_{SO(n)} \times P_{U(1)}$. Locally, again, such a lift obviously exists, but globally there might be a topological obstruction for patching the local liftings together on triple overlaps. We denote

a set of local lifts by $[(\{\widetilde{g}_{\alpha\beta}\}, \{\widetilde{h}_{\alpha\beta}\})]$. The map ξ should map these to the original transition functions:

$$\xi([(\widetilde{g}_{\alpha\beta},\widetilde{h}_{\alpha\beta})]\cdot[(\widetilde{g}_{\beta\gamma},\widetilde{h}_{\beta\gamma})]\cdot[(\widetilde{g}_{\alpha\gamma}^{-1},\widetilde{h}_{\alpha\gamma}^{-1})])=\xi([(\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\alpha\gamma}^{-1},\widetilde{h}_{\alpha\beta}\widetilde{h}_{\beta\gamma}\widetilde{h}_{\alpha\gamma}^{-1})])=1\,.$$

Thus,

$$[(\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\alpha\gamma}^{-1},\widetilde{h}_{\alpha\beta}\widetilde{h}_{\beta\gamma}\widetilde{h}_{\alpha\gamma}^{-1})] \in \{[(1,1)],[(-1,1)]\} \cong \mathbb{Z}_2,$$

determining a class in $H^2(X, \mathbb{Z}_2)$. Clearly, the Spin^c lift exists if and only if this class vanishes (compare to the case of the Spin lift). Remarkably, we now have extra freedom with regards to the Spin lifts $\{\widetilde{g}_{\alpha\beta}\}$, because the equivalence class $[(1,1)] \in \text{Spin}^c(n)$ consists of two points: (1,1) and (-1,-1). Thus, it is enough to have

$$\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\alpha\gamma}^{-1} = u_{\alpha\beta\gamma} = \widetilde{h}_{\alpha\beta}\widetilde{h}_{\beta\gamma}\widetilde{h}_{\alpha\gamma}^{-1}$$

for the Spin^c lift to exist. It is not obvious, though, that such local lifts of the $P_{U(1)}$ bundle exists. The functions $\{\widetilde{g}_{\alpha\beta}\}$ differ from $\{g_{\alpha\beta}\}$ only by signs. This can be expresses in terms of integers $\{n_{\alpha\beta}\}$ as

$$h_{\alpha\beta} = (-1)^{n_{\alpha\beta}} \widetilde{h}_{\alpha\beta} .$$

The cocycle condition yields

$$h_{\alpha\beta}h_{\beta\gamma}h_{\alpha\gamma}^{-1} = (-1)^{n_{\alpha\beta}+n_{\beta\gamma}-n_{\alpha\gamma}}\widetilde{h}_{\alpha\beta}\widetilde{h}_{\beta\gamma}\widetilde{h}_{\alpha\gamma}^{-1} = (-1)^{\delta(n)_{\alpha\beta\gamma}}u_{\alpha\beta\gamma} = 1.$$

Now, $[\delta(\underline{n})]$ is a class in $H^2(X,\mathbb{Z})$ and $[\underline{u}]$ a class in $H^2(X,\mathbb{Z}_2)$. The equation

$$(-1)^{\delta(n)_{\alpha\beta\gamma}}u_{\alpha\beta\gamma} = 1$$

is satisfied if and only if $[\underline{u}] = [\delta(\underline{n})] \mod 2$. Finally, observe that

$$h_{\alpha\beta}h_{\beta\gamma}h_{\alpha\gamma}^{-1} = e^{2\pi i (\delta(n)_{\alpha\beta\gamma}/2)}u_{\alpha\beta\gamma} = e^{2\pi i ((\delta(n)_{\alpha\beta\gamma}/2) + (\delta(n)_{\alpha\beta\gamma} \bmod 2)/2)} \; .$$

The Chern class $c_1(P_{U(1)})$ is thus given by applying the Čech coboundary operator to the expression in the exponential. However, only the half-integer parts of the exponential contribute to the nontriviality of the Chern class. Hence, we can write

$$c_1(P_{U(1)}) = \left[(\delta(\underline{n})/2) + (\delta(\underline{n})/2) \right] = \left[\delta(\underline{n}) \right].$$

This concludes the proof that the Spin^c lift of $P_{SO(n)} \times P_{U(1)}$ exists if and only if

$$w_2(P_{SO(n)}) = [\underline{u}] = c_1(P_{U(1)}) \mod 2.$$
 (2.13)

By the exactness of (2.1), (2.13) is equivalent to

$$W_3(P_{SO(n)}) = \beta(w_2(P_{SO(n)})) = 0.$$

Proposition 2.2.3. A principal SO(n) bundle $P_{SO(n)} o X$ admits a $Spin^c(n)$ lift $P_{Spin^c(n)}$ precisely when $W_3(P_{SO(n)}) = 0$. If $W_3(P_{SO(n)}) = 0$, a choice of the $Spin^c$ lift is called a $Spin^c$ structure. Distinct $Spin^c$ structures of an SO(n) principal bundle are in bijective correspondence with $2H^2(X,\mathbb{Z}) \oplus H^1(X,\mathbb{Z}_2)$. The first term comes from the freedom of tensoring the $P_{U(1)}$ bundle (satisfying $c_1(P_{U(1)}) = w_2(P_{SO(n)})$ mod 2) by $\widetilde{P}_{U(1)} \otimes \widetilde{P}_{U(1)}$, where $\widetilde{P}_{U(1)}$ is any principal U(1) bundle⁸. The second term comes from the freedom of choosing the lifts as in the Spin case.

⁸Namely, then $c_1(P_{U(1)} \otimes \widetilde{P}_{U(1)} \otimes \widetilde{P}_{U(1)}) \mod 2 = w_2(P_{SO(n)}).$

Note that any bundle with a Spin structure carries a canonical Spin^c structure, given by the trivial class in $H^2(X,\mathbb{Z})$, corresponding to a trivial U(1) bundle. Also, the underlying SO(2n) bundle of any SU(n) bundle is canonically Spin^c by proposition 2.1.7.

We have discussed only principal bundles so far, but there are analogous concepts for vector bundles through the associated bundle construction. More precisely, we call an O(n) or SO(n) vector bundle orientable, Spin or Spin^c, if their frame bundles are orientable, Spin or Spin^c, respectively.

Definition 2.2.4. A manifold is said to be orientable/ $Spin/Spin^c$, if its tangent bundle is orientable/ $Spin/Spin^c$.

2.3 K-theory of vector bundles

In this section we discuss the basic concepts of K-theory (of vector bundles). A reader interested in more comprehensive expositions is referred to [Ati, Kar, LM, GVF]. Our treatment is quite minimalistic, containing only the basics needed to understand the rest of the thesis.

K-theory can be defined for a wide class of topological spaces. We have no need for very general kinds of spaces as we are only interested in K-theory of the spacetime and the D-brane worldvolume, which are both manifolds. Thus, it is more than enough to suppose that the spaces are locally compact Hausdorff topological spaces of the homotopy type of a CW-complex. However, we shall start by restricting to compact Hausdorff spaces with the homotopy type of a finite CW-complex (for example, compact manifolds).

Let $\operatorname{Vect}_{\mathbb{C}}(X)$ denote the set of isomorphism classes of complex⁹ vector bundles over the compact space X. For vector bundles E and F, their classes in $\operatorname{Vect}_{\mathbb{C}}(X)$ are denoted by [E] and [F]. Defining an operation

$$[E] + [F] := [E \oplus F],$$

turns $\operatorname{Vect}_{\mathbb{C}}(X)$ into an Abelian monoid, with identity given by the 0-dimensional bundle.

Definition 2.3.1. The K-theory group K(X) is the G-rothendieck group of $\text{Vect}_{\mathbb{C}}(X)$. Explicitly, it is the quotient of the free Abelian group generated by $\text{Vect}_{\mathbb{C}}(X)$ by the subgroup of elements of the form $[E \oplus F] - [E] - [F]$. In other words, it is the free Abelian group of formal differences [E] - [F] of isomorphism classes of vector bundles over X. K-theory classes are called V-rotated V-rotated

$$\operatorname{rk}([E]-[F]) := \operatorname{rk}(E) - \operatorname{rk}(F)\,,$$

is the virtual rank of [E] - [F]. The class $[E] - [0] \in K(X)$ is denoted by [E].

The Grothendieck group K(X) can also be axiomatically characterized by the following universal property. If G is any other group and $f : \text{Vect}_{\mathbb{C}}(X) \to G$ any monoid homomorphism,

 $^{^9}K$ -theory can as well be defined for real vector bundles. K-theory for real vector bundles is typically referred to as KR-theory. Just as complex K-theory classifies D-branes in Type II superstring theory, KR-theory classifies D-branes in Type I superstring theory. KR-theory is also more complicated than K-theory and thus the K-theoretic aspects are more pronounced in Type I theory. We recommend [Val, OS] for readers interested in the real case and its relation to D-brane physics.

then there exists a unique group homomorphism $g:K(X)\to G$, such that the diagram

$$\operatorname{Vect}_{\mathbb{C}}(X) \xrightarrow{E \mapsto [E]} K(X) \tag{2.14}$$

commutes. By universality, if such a group K(X) exists (and it does by the above standard construction), then it is necessarily unique.

Proposition 2.3.2. If X is a finite disjoint union of compact spaces $X = \coprod_k X_k$, then the K-theory group decomposes into a direct sum

$$K(X) \cong \bigoplus_k K(X_k)$$
.

There is nothing to prove really, since vector bundles over X are completely described by their restrictions onto $\{X_k\}$.

Proposition 2.3.3 ([LM]). For any vector bundle E over X, there exists a complementary bundle E^{\perp} , such that $E \oplus E^{\perp} \cong \mathbb{1}_n$, where $\mathbb{1}_n$ denotes the trivial bundle of rank n.

By the above proposition, any class [E] - [F] can be written as

$$[E] - [F] = [E] - [F \oplus F^{\perp}] + [F^{\perp}] = [E \oplus F^{\perp}] - [\mathbb{1}_n].$$

In other words, every class of K(X) is of the form $[E] - [\mathbb{1}_n]$, for some vector bundle E and non-negative integer n.

Proposition 2.3.4. K-theory defines a cofunctor from the homotopy category of "suitable spaces", for example compact Hausdorff spaces, to the category of Abelian groups. A morphism $f: X \to Y$ gets mapped into a morphism $f^*: K(Y) \to K(X)$, given explicitly by

$$f^*([E] - [F]) := [f^{-1}E] - [f^{-1}F].$$

Since the isomorphism classes of pullback bundles depend only on the homotopy class of the map inducing the pullback, the domain of the cofunctor is, indeed, the homotopy category. It follows immediately that homotopy equivalent spaces have isomorphic K-theory.

Definition 2.3.5. For vector bundles E and E' over X, the tensor product $E \otimes E'$ is again a vector bundle over X. This allows us to define a \mathbb{Z} -linear map, the *cup product*,

$$\smile: K(X) \otimes K(X) \to K(X)$$
 (2.15)

by

$$([E] - [F]) \smile ([E'] - [F']) := [E \otimes E'] - [E \otimes F'] - [F \otimes E'] + [F \otimes F']$$
$$= [(E \otimes E') \oplus (F \otimes F')] - [(E \otimes F') \oplus (F \otimes E')].$$

The exterior product is a map

$$\otimes: K(X) \otimes K(Y) \to K(X \times Y), \tag{2.16}$$

defined as follows. Note first, that for vector bundles $E \to X$ and $F \to Y$ there is a tensor product bundle $E \otimes F \to X \times Y$. Let E and F be vector bundles over X, and E' and F' over Y. The projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ induce maps π_X^* and π_Y^* in K-theory. To simplify the definition, we write

$$E \boxtimes E' := \pi_X^{-1} E \otimes \pi_Y^{-1} E'$$

and

$$[E]\boxtimes [E']:=\pi_X^*[E]\otimes \pi_Y^*[E']=[E\boxtimes E']\in K(X\times Y)\,.$$

Using this notation, the exterior product is

$$([E] - [F]) \otimes ([E'] - [F']) := [E] \boxtimes [F'] - [E] \boxtimes [F'] - [F] \boxtimes [E'] + [F] \boxtimes [F']$$
$$= [(E \boxtimes E') \oplus (F \boxtimes F')] - [(E \boxtimes F') \oplus (F \boxtimes E')].$$

For Y = X, the combined map $K(X) \otimes K(X) \to K(X \times X) \to K(X)$, where the second arrow is induced by the diagonal inclusion $X \hookrightarrow X \times X$, coincides with (2.15).

The cup product turns K(X) into a ring, with multiplicative identity given by the class [1], the class of the trivial line bundle.

The K-group of a one-point space $\{\infty\}$ is $K(\{\infty\}) \cong \mathbb{Z}$, where the integer corresponding to a virtual bundle is simply its virtual rank. Let X^{∞} be a compact pointed space, with basepoint ∞ . The inclusion $i: \{\infty\} \hookrightarrow X$ induces a map

$$i^*: K(X^\infty) \to K(\{\infty\}) \cong \mathbb{Z}$$
,

which maps a virtual bundle to the virtual rank of its restriction onto the connected component of the basepoint.

Definition 2.3.6. The subset

$$\widetilde{K}(X^{\infty}) := \ker(i) \subset K(X^{\infty})$$

is an ideal without identity. It is called the *reduced K-theory* group of X^{∞} . It fits into the naturally split exact sequence

$$0 \longrightarrow \widetilde{K}(X^{\infty}) \longrightarrow K(X) \xrightarrow{i^*} K(\{\infty\}) \cong \mathbb{Z} \longrightarrow 0$$

The splitting yields the isomorphism

$$K(X^{\infty}) \cong \widetilde{K}(X^{\infty}) \oplus \mathbb{Z}$$
.

For a connected pointed space¹⁰ X^{∞} , $\widetilde{K}(X^{\infty})$ consists of virtual bundles [E] - [F], with $\mathrm{rk}([E] - [F]) = 0$. If X is a non-pointed compact space, we can always augment a disjoint basepoint ∞ . The resulting pointed space is

$$X^{\infty} := X \coprod \{\infty\}.$$

$$\operatorname{rk}: K(X) \to \bigoplus_{1}^{n} \mathbb{Z},$$

which assigns an integer, the virtual rank of the restriction of the virtual bundle, to each component.

¹⁰If X has $n < \infty$ connected components, the virtual rank function is

Clearly, if X is non-pointed, we have

$$\widetilde{K}(X^{\infty}) \cong K(X)$$
.

Let (X,Y) be a compact pair of spaces, $\varnothing \neq Y \subset X$. Collapsing Y to a point ∞ yields the canonically pointed space X/Y (with basepoint ∞). For $Y=\varnothing$, we set $X/\varnothing:=X^\infty=X$ II $\{\infty\}$.

Definition 2.3.7. The relative K-theory group of the pair (X,Y) is

$$K(X,Y) := \widetilde{K}(X/Y)$$
.

Intuitively, relative K-theory is the K-theory of X, with all contribution from Y removed. It is built from virtual bundles over X that vanish over Y. The need to use reduced K-theory can, thus, be understood as the requirement of having bundles of equal rank forming the virtual bundle, for otherwise it would be impossible for them to cancel each other over Y. For a pointed space X^{∞} with basepoint ∞ , we recover reduced K-theory from relative K-theory as X^{1}

$$\widetilde{K}(X^{\infty}) = K(X, \{\infty\}).$$

Remark also, that

$$K(X,\varnothing) = \widetilde{K}(X^{\infty}) \cong K(X)$$
,

K-theory actually provides a multitude of functors, K^{-n} , where $n \in \mathbb{N}^{12}$. Before defining the higher K-functors, let us quickly recall a few standard constructions from topology of pointed spaces [Whi, May]. Let X and Y be pointed spaces with basepoints x_0 and y_0 . Their wedge sum $X \vee Y$ is obtained by forming the disjoint union and collapsing the basepoints together¹³:

$$X \vee Y := X \coprod Y/\{x_0, y_0\}$$
.

There is a natural inclusion $X \vee Y \hookrightarrow X \times Y$ whose image is $X \times \{y_0\} \cup \{x_0\} \times Y$. If we collapse $X \vee Y \subset X \times Y$ into a point, we obtain the *smash product*,

$$X \wedge Y := X \times Y/X \vee Y$$
.

Lemma 2.3.8.

$$S^n \cong S^1 \wedge \ldots \wedge S^1$$

Proof. First, rewrite $X \wedge S^1$ using $S^1 \cong I/\{0,1\}$ as

$$X \wedge S^{1} = \frac{X \times I}{X \times \{0, 1\} \cup \{x_{0}\} \times I}.$$
 (2.17)

Next, consider the diagram

$$S^{n} \times I \xrightarrow{\iota} S^{n+1}$$

$$(2.18)$$

$$(S^{n} \times I)/(S^{n} \times \{0,1\} \cup \{x_{0}\} \times I)$$

¹¹Remark, that these are all standard constructions in (generalized) (co)homology.

¹²The purpose of the minus sign in K^{-n} is to emphasize the cohomological nature of K-theory.

¹³This is the coproduct in the category of pointed spaces.

where ι is the obvious (continuous) inclusion $S^n \times I \to S^{n+1}$ combined with a continuous deformation of all the basepoints into a single point. It is obviously surjective but not injective, since the subspace $S^n \times \{0,1\} \cup \{x_0\} \times I$ gets mapped to a single (base)point. Thus, passing to the quotient space (2.17) yields the natural bijection $\tilde{\iota}$. The diagram (2.18) obviously commutes.

Let $V \subset S^{n+1}$ be open. Then $\iota^{-1}(V) = \pi^{-1} \circ \widetilde{\iota}^{-1}(V) \subset S^n \times I$ is open. A set $U \subset (S^n \times I)/(S^n \times \{0,1\} \cup \{x_0\} \times I)$ is open in the quotient topology precisely when $\pi^{-1}(U)$ is open in $S^n \times I$. Therefore, $\widetilde{\iota}^{-1}(V)$ is open, which proves the continuity of $\widetilde{\iota}$.

Let then $W \subset (S^n \times I)/(S^n \times \{0,1\} \cup \{x_0\} \times I)$ be closed. Its image under $\tilde{\iota}$ is

$$\widetilde{\iota}(W) = \widetilde{\iota} \circ \pi \circ \pi^{-1}(W) = \iota \circ \pi^{-1}(W)$$
,

where $\pi^{-1}(W)$ is closed by the quotient topology. Since $S^n \times I$ is compact, $\pi^{-1}(W)$ is compact. Its image under the continuous map ι is again compact. Finally, since S^{n+1} is Hausdorff, all compact subsets of S^{n+1} are closed. Thus, $\tilde{\iota}$ is closed, which concludes the proof.

The smash product of any pointed space X with S^1 is called the *reduced suspension* of X and is denoted by

$$\Sigma X := S^1 \wedge X$$
.

The n-fold reduced suspension is then

$$\Sigma^n X := S^1 \wedge \ldots \wedge S^1 \wedge X \cong S^n \wedge X.$$

Theorem 2.3.9 ([Ati, Kar]). The K-functor we have discussed above is actually the 0-th K-functor K^0 . The higher K-functors K^{-n} , $n \in \mathbb{N}$, are also contravariant functors from the homotopy category of spaces to the category of Abelian groups. For a compact pointed space X^{∞} , we set

$$\widetilde{K}^{-n}(X^{\infty}) := \widetilde{K}(\Sigma^n X^{\infty}),$$

for a compact pair (X,Y) the higher relative K-theory groups are

$$K^{-n}(X,Y) := \widetilde{K}^{-n}(X/Y) = \widetilde{K}(\Sigma^n(X/Y)) \,.$$

For a generally non-pointed space X we define

$$K^{-n}(X) := K^{-n}(X, \varnothing) = \widetilde{K}^{-n}(X^{\infty}).$$
 (2.19)

These are all standard constructions in (generalized) cohomology theory.

The exterior product (2.16) restricts to a product [Ati]

$$\otimes: \widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y), \qquad (2.20)$$

which is enough to generalize the product structures to relative and higher K-theory. Furthermore, choosing Y = X and taking the pullback by the diagonal inclusion $X \hookrightarrow X \times X$, yields corresponding generalizations of (2.15). For example,

$$\smile: K^{-n}(X) \otimes K^{-m}(X) \to K^{-n-m}(X)$$

is defined by first using (2.19), then (2.20) to obtain a class in $\widetilde{K}(\Sigma^n(X^{\infty}) \wedge \Sigma^m(X^{\infty}))$, then using the fact that \wedge is commutative to obtain a class in $\widetilde{K}(\Sigma^{n+m}(X^{\infty} \wedge X^{\infty}))$, then taking

the pullback by the diagonal map to obtain a class in $\widetilde{K}(\Sigma^{n+m}X^{\infty})$ and finally using (2.19) again to obtain a class in $K^{-n-m}(X)$.

It should not come as a surprise that K-theory satisfies all of the Eilenberg-Steenrod axioms except the dimension axiom, thus determining a generalized cohomology theory.

The failure of the dimension axiom is seen as follows. If $\{\infty\}$ is considered as a non-pointed space, then $\{\infty\}^{\infty} = S^0$, the 0-sphere, a set of two points. It is easy to see that $\Sigma S^0 \cong S^1$. Thus, we have

$$K^{-1}(\{\infty\}) = \widetilde{K}(\Sigma S^0) = \widetilde{K}(S^1),$$

and more generally,

$$K^{-n}(\{\infty\}) = \widetilde{K}(S^n).$$

It turns out, that $\widetilde{K}(S^{\text{even}}) \cong \mathbb{Z}$, which is contrary to the dimension axiom, that all higher cohomology groups of a point should vanish. However, all the other Eilenberg-Steenrod axioms do hold. For example, associated to a compact pair (X,Y) there is a long exact sequence

$$\cdots \longrightarrow K^{-2}(Y) \longrightarrow K^{-1}(X,Y) \longrightarrow K^{-1}(X)$$

$$\longrightarrow K^{-1}(Y) \longrightarrow K^{0}(X,Y) \longrightarrow K^{0}(X) \longrightarrow K^{0}(Y)$$

$$(2.21)$$

induced by the *Puppe sequence* [May]. If X^{∞} has a basepoint $\{\infty\}$, (2.21) yields the naturally split exact sequences

$$0 \longrightarrow K^{-n}(X^{\infty}, \{\infty\}) \cong \widetilde{K}^{-n}(X^{\infty}) \longrightarrow K^{-n}(X^{\infty}) \longmapsto K^{-n}(\{\infty\}) \longrightarrow 0$$

which imply the isomorphisms

$$K^{-n}(X^{\infty}) \cong \widetilde{K}^{-n}(X^{\infty}) \oplus K^{-n}(\{\infty\}) \cong \widetilde{K}^{-n}(X^{\infty}) \oplus \widetilde{K}(S^n). \tag{2.22}$$

It is a general result in cohomology theory, that a reduced cohomology theory for pointed spaces determines a relative cohomology theory for pairs of non-pointed spaces and $vice\ versa$ [May]. In the case of K-theory, $\widetilde{K}^{-\bullet}$ is the reduced cohomology theory corresponding to $\widetilde{K}^{-\bullet}$. It is convenient to restrict the spaces to be CW-complexes. Otherwise, we would have to make sure that the inclusions of the basepoints are cofibrations [May]. In particular, this is the case for manifolds.

Generalized cohomology and homology theories, such as K-theory, admit always a homotopic definition in terms of an Ω -spectrum (a loop spectrum)¹⁴ [Bro1, Whi]. Unfortunately, we can not discuss the general principles here, as it would take us too far from the topic. Instead, we treat only the special case of K-theory. It should not be surprising, considering the homotopic classification of vector bundles, that (the Ω -spectrum of) K-theory should be somehow closely related to $\{BU(n)\}$. However, K-theory involves considering vector bundles of all ranks, which suggests that all the $\{BU(n)\}$ spaces should be taken into account. Thus, the limit

$$BU := \varinjlim_{n} BU(n)$$
,

taken over the natural inclusions $BU(n) \subset BU(n+1)$ is expected to play a role. We shall not prove the following, even though the proof is fairly short and easy.

¹⁴Just like ordinary cohomology and homology can be defined using the Eilenberg-MacLane spectrum.

Theorem 2.3.10 ([May]). Give \mathbb{Z} the discrete topology and choose the basepoint of $BU \times \mathbb{Z}$ to reside in the component $BU \times \{0\}$. All homotopies are understood to be in the category of pointed spaces (they preserve the basepoint). Then, for any compact pointed space X^{∞} , there exists an isomorphism

$$\widetilde{K}(X^{\infty}) \cong [X^{\infty}, BU \times \mathbb{Z}].$$
 (2.23)

If X^{∞} is connected, this is equivalent to

$$\widetilde{K}(X^{\infty}) \cong [X^{\infty}, BU].$$

For a non-pointed compact space we first augment a basepoint and then apply the pointed definition. If X is non-pointed, compact and X^{∞} denotes the space with a disjoint basepoint added, then there is an isomorphism

$$K(X) \cong [X^{\infty}, BU \times \mathbb{Z}].$$
 (2.24)

The ring structure of K-theory can also be expressed in terms of the homotopic definition, but we shall not go into that. The reader is referred to [Whi, May] for related results in a more general context.

Remarkably, it turns out that only two of all the K-theory groups are actually distinct. This is the famous $Bott\ periodicity\ theorem$. Detailed proofs can be found in [Ati, Kar] or in any standard treatise on K-theory. There are several more or less equivalent formulation of this important and classical result. We shall present some of them, in an attempt to clarify how the K-theoretic version is related to the others. We start from the classical result of Bott [Bot], concerning homotopy groups of U(n), and work our way towards the K-theoretic version.

Using the natural inclusions $U(n) \hookrightarrow U(n+1)$, we can define the limit

$$U := \underset{n}{\varinjlim} U(n)$$
,

the "infinite union" of unitary groups. The result of Bott concerned the higher homotopy groups of U^{15} .

Theorem 2.3.11 (Bott periodicity [Bot]).

$$\pi_k(U) \cong \pi_{k+2}(U)$$
.

The long exact sequence of homotopy groups, induced by the fibration

$$U(n) \longrightarrow EU(n) \longrightarrow BU(n)$$

is

$$\cdots \longrightarrow \pi_k(EU(n)) \longrightarrow \pi_k(BU(n)) \longrightarrow \pi_{k-1}(U(n)) \longrightarrow \pi_{k-1}(EU(n)) \longrightarrow \cdots,$$

¹⁵Recall that for big enough n, $\pi_k(U(n))$ depends only on k.

By exactness and $\pi_k(EU(n)) = \pi_{k-1}(EU(n)) = 0$, there are isomorphisms¹⁶

$$\pi_k(BU(n)) \cong \pi_{k-1}(U(n))$$
.

Passing to the limit $n \to \infty$ and applying Theorem 2.3.11 yields

$$\pi_k(BU) \cong \pi_{k-1}(U) \cong \pi_{k+1}(U) \cong \pi_{k+2}(BU)$$
.

For based CW-complexes X, Y the suspension and loop space functors are adjoint in the sense that

$$[\Sigma X, Y] \cong [X, \Omega Y]. \tag{2.25}$$

Thus,

$$\pi_k(BU) \cong [S^{k+2}, BU] = [\Sigma S^{k+1}, BU] \cong [S^k, \Omega^2 BU] = \pi_k(\Omega^2 BU),$$

for all $k \geq 1$. Still, $\pi_0(BU) = 0$ and $\pi_0(\Omega^2 BU) \cong \pi_2(BU) \cong \mathbb{Z}$, but replacing $\pi_k(BU)$ with $\pi_k(BU \times \mathbb{Z})$ does not affect any of the higher homotopy groups, since we are working with basepoint preserving homotopies and we have

$$\pi_k(BU \times \mathbb{Z}) \cong \pi_k(\Omega^2 BU)$$
.

Furthermore, an application of (2.25) yields

$$\pi_k(\Omega^2 BU) = [S^k, \Omega^2 BU] \cong [S^{k+2}, BU] \cong [S^{k+2}, BU \times \mathbb{Z}] \cong \pi_k(\Omega^2 (BU \times \mathbb{Z}))$$
.

We have obtained a weak homotopy equivalence $BU \times \mathbb{Z} \cong \Omega^2(BU \times \mathbb{Z})$, which implies a homotopy equivalence by Whitehead's theorem¹⁷.

Theorem 2.3.12 (Bott periodicity). There are homotopy equivalences

$$BU \times \mathbb{Z} \cong \Omega^2 BU \cong \Omega^2 (BU \times \mathbb{Z})$$
.

The K-theoretic version follows now easily. If X^{∞} is a compact pointed space,

$$\begin{split} \widetilde{K}^{-n}(X^{\infty}) &= \widetilde{K}(\Sigma^{n}X^{\infty}) \cong [\Sigma^{n}X^{\infty}, BU \times \mathbb{Z}] \\ &\cong [\Sigma^{n-2}X^{\infty}, \Omega^{2}(BU \times \mathbb{Z})] \cong [\Sigma^{n-2}X^{\infty}, BU \times \mathbb{Z}] \\ &\cong \widetilde{K}^{-n+2}(X^{\infty}) \,. \end{split}$$

By (2.22), we immediately obtain a similar result in (unreduced) K-theory.

Theorem 2.3.13 (Bott periodicity). There are canonical isomorphisms

$$K^{-n}(X) \cong K^{-n+2}(X).$$

$$\pi_k(BU(n)) \cong \operatorname{Vect}^n_{\mathbb{C}}(S^k) \cong [S^{k-1}, U(n)] = \pi_{k-1}(U(n)).$$

¹⁶This can also be understood more straightforwardly as follows: $\pi_k(BU(n)) \cong \operatorname{Vect}^n_{\mathbb{C}}(S^k)$, but on the other hand, such vector bundles are uniquely classified by the homotopy class of the *clutching function* at the equator, which is homotopy equivalent to S^{k-1} . Thus, we have

 $^{^{17}}BU$ is a CW-complex, since the direct limit of CW-complexes is again a CW-complex [May].

Using Bott periodicity, the sequence (2.21) can be extended infinitely to the right, by defining (for $n \in \mathbb{Z}$)

$$K^n := \begin{cases} K^0 & \text{for } n \text{ even,} \\ K^{-1} & \text{for } n \text{ odd.} \end{cases}$$

From now on, we shall talk only about K^0 and K^1 , and define

Definition 2.3.14.

$$K^{\bullet}(X) := K^{0}(X) \oplus K^{1}(X) ,$$

and similarly for relative and reduced K-theory.

An immediate consequence of the Bott periodicity theorem is that the long exact sequence (2.21) truncates to a 6-term cyclic exact sequence

$$K^{0}(X,Y) \longrightarrow K^{0}(X) \longrightarrow K^{0}(Y)$$

$$\uparrow \qquad \qquad \downarrow$$

$$K^{1}(Y) \longleftarrow K^{1}(X) \longleftarrow K^{1}(X,Y)$$

Let us take another look at (2.22). It would be very nice to know what the groups $\widetilde{K}(S^n)$ are.

Proposition 2.3.15.

$$K^n(\{\infty\}) \cong \widetilde{K}(S^n) \cong \begin{cases} \mathbb{Z} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$
 (2.26)

Proof. By Bott periodicity, it is clear that $\widetilde{K}(S^n)$ depends only on whether n is even or odd. Also, since $\widetilde{K}(S^2) \cong K^{-2}(\{\infty\}) \cong K(\{\infty\}) \cong \mathbb{Z}$, the only thing we need to do is to find $\widetilde{K}(S^1)$.

We use the various results that came up when discussing Bott periodicity:

$$\widetilde{K}(S^1) = [S^1, BU \times \mathbb{Z}] \cong [S^1, BU] = \pi_1(BU).$$

There is a fibration

$$S^{2n-1} \longrightarrow BU(n-1) \longrightarrow BU(n)$$
,

with associated long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_1(S^{2n-1}) \longrightarrow \pi_1(BU(n-1)) \longrightarrow \pi_1(BU(n)) \longrightarrow \pi_0(S^{2n-1}) \longrightarrow \cdots$$

But if $n \geq 2$, then $\pi_1(S^{2n-1}) = \pi_0(S^{2n-1}) = 0$ and there are isomorphisms

$$\pi_1(BU(n)) \cong \pi_1(BU(n-1)) \cong \ldots \cong \pi_1(BU(1)).$$

Since $BU(1) = \mathbb{C}P^{\infty}$, we see from (2.5) that

$$\pi_1(BU(n)) = 0.$$

Therefore,

$$\pi_1(BU) \cong \varinjlim_n \pi_1(BU(n)) = 0,$$

concluding the proof.

A very different kind of proof can be given by a clever application of the theory of Clifford modules [ABS], where the periodicity can be understood to follows from the periodicity of complex Clifford algebras. We shall not go into that, however. \Box

By (2.22)

$$K^n(X^{\infty}) \cong \begin{cases} \widetilde{K}^n(X^{\infty}) \oplus \mathbb{Z} & \text{for } n = 0, \\ \widetilde{K}^n(X^{\infty}) & \text{for } n = 1, \end{cases}$$
 (2.27)

for any compact pointed space X^{∞} .

Recall the definition of the Chern character of a vector bundle (2.7) and its simple behavior under direct sums and tensor products of vector bundles, equations (2.9), (2.10). In particular, (2.9) implies that

$$\operatorname{ch}:\operatorname{Vect}_{\mathbb{C}}\to H^{\operatorname{even}}(X,\mathbb{Q})$$

is a monoid homomorphism. Thus, by the universal property of K-theory mentioned in definition 2.3.1, it has a unique extension into a group homomorphism

$$\operatorname{ch}: K(X) \to H^{\operatorname{even}}(X, \mathbb{Q}),$$
 (2.28)

which maps a virtual bundle [E] - [F] to $\operatorname{ch}(E) - \operatorname{ch}(F)$. It is easy to see from (2.10) that (2.28) is actually a ring homomorphism¹⁸. For a pointed space X^{∞} , restricting ch to reduced K-theory classes yields classes in reduced cohomology $\widetilde{H}(X^{\infty}, \mathbb{Q})$:

$$\mathrm{ch}(\widetilde{K}(X^\infty))\subset \widetilde{H}^{\mathrm{even}}(X^\infty,\mathbb{Q}):=\ker(H^{\mathrm{even}}(X^\infty,\mathbb{Q})\to H^{\mathrm{even}}(\{\infty\},\mathbb{Q})\,.$$

For classes in $K^1(X)$ the Chern character maps

$$\mathrm{ch}: K^1(X) = \widetilde{K}(\Sigma X^\infty) \to \widetilde{H}^{\mathrm{even}}(\Sigma X^\infty, \mathbb{Q}) \cong \widetilde{H}^{\mathrm{odd}}(X^\infty, \mathbb{Q}) \cong H^{\mathrm{odd}}(X, \mathbb{Q}) \,.$$

It is a famous result, that the Chern character becomes an isomorphism between rationalized K-theory and cohomology:

$$\operatorname{ch}: K^{\bullet}(X) \otimes \mathbb{Q} \xrightarrow{\cong} H^{\bullet}(X, \mathbb{Q}).$$

This means that the difference between K-theory and cohomology with rational coefficients is contained in the torsion subgroup.

Let us now get rid of the requirement for compactness and generalize to locally compact spaces. Of course, we are primarily interested in manifolds, which are always locally compact spaces of the homotopy type of a CW-complex. Locally compact spaces always admit one-point compactifications. In particular, if the original space is a manifold, then the compactified space is of the homotopy type of a finite CW-complex. For locally compact spaces X and Y with one-point compactifications X^{∞} and Y^{∞} , morphisms $X \to Y$ are continuous maps that

 $^{^{18}}$ This is the fundamental reason why the Chern character will be so important to us: it is compatible with the algebraic structure of K-theory. The appearance of the Chern character in the D-brane charge formula is already, by itself, strong evidence that K-theory should be expected to play a role in classifying D-branes.

are restrictions of continuous basepoint preserving maps $X^{\infty} \to Y^{\infty}$. For example, proper $maps^{19}$ are morphisms between locally compact spaces, but every morphism between locally compact spaces is not a proper map. However, considering proper maps is enough for our applications, because any continuous map from a compact space to a Hausdorff space, such as the inclusion of the D-brane $\phi: \Sigma \to X$, is a proper.

Suppose that X is a locally compact space with one-point compactification X^{∞} . We define K-theory for X as in (2.19):

$$K^{\bullet}(X) := \widetilde{K}^{\bullet}(X^{\infty}). \tag{2.29}$$

This definition extends K-theory into the category of locally compact spaces as a *compactly* supported (generalized) cohomology theory, in the following sense.

Proposition 2.3.16 ([Kar]). Let X be a locally compact and

$$\ldots \subset X_k \subset X_{k+1} \subset \ldots$$

an inductive family of open subspaces of X, such that every compact subset of X is contained in at least one X_k . Then

$$K^{\bullet}(X) \cong \varinjlim_{k} K^{\bullet}(X_{k}).$$

From the physical point of view, the compactly supported definitions can be understood as "finite energy assumptions" for physical objects. More precisely, if the energy or charge density of some physical object is described by a (generalized) cohomology theory, then the restriction to the compactly supported subgroup is equivalent to excluding classes which are "infinitely nontrivial", corresponding to infinitely extended nonzero energy or charge densities.

The Chern character generalizes without problems to the noncompact case. Its image lies in *compactly supported cohomology* (compare with (2.29) and proposition 2.3.16) [May]:

$$H^{\bullet}_{c}(X,G):=\ker(H^{\bullet}(X^{\infty})\to H^{\bullet}(\{\infty\})\cong G)\,,$$

where G is some Abelian coefficient group. Thus, the Chern character is a homomorphism

$$\operatorname{ch}: K^{\bullet}(X) \to H_c^{\bullet}(X, \mathbb{Q}),$$

where, on the right-hand side, $H_c^{\bullet}(X,\mathbb{Q})$ denotes \mathbb{Z}_2 -periodized compactly supported cohomology.

We conclude this section by presenting the Atiyah-Hirzebruch version of the Grothendieck-Riemann-Roch theorem [AH1, AH2]. Let Σ and X be smooth oriented manifolds, Σ compact and X generally noncompact. By the contravariant functoriality of cohomology theories, a smooth map $\phi: \Sigma \to X$, which is necessarily also proper, induces natural maps

$$\phi^*: H_c^{\bullet}(X, \mathbb{Z}) \to H^{\bullet}(\Sigma, \mathbb{Z}), \qquad \phi^*: K^{\bullet}(X) \to K^{\bullet}(\Sigma).$$

However, in D-brane theory we need to be able to send K-theory and cohomology classes from Σ to X. There is a general recipe for obtaining such "wrong way" functoriality, at least when the spaces are of a particular kind. Let us recall some basic constructions of algebraic topology.

¹⁹A map is *proper*, if inverse images of compact sets are compact.

First of all, we restrict to manifolds with an additional structure known as an *orientation*. For a closed²⁰ n-dimensional manifold Σ , an orientation (more precisely, we are referring to a \mathbb{Z} -orientation) is a homology class $[\Sigma] \in H_n(\Sigma, \mathbb{Z})$, whose restriction to $H_n(\Sigma, \Sigma)$ $\{x\} \cong \mathbb{Z}$ for any $x \in \Sigma$, is a generator. Taking the cap product with the fundamental class determines an isomorphism between homology and cohomology groups:

$$\operatorname{Pd}_{\Sigma}(-) := (-) \frown [\Sigma] : H^{k}(\Sigma, \mathbb{Z}) \xrightarrow{\cong} H_{n-k}(\Sigma, \mathbb{Z}),$$

the *Poincaré duality isomorphism*. For a noncompact manifold X, fundamental classes can be defined only locally. A choice of local fundamental classes, consistent at overlaps²¹, determines an *orientation* for the noncompact manifold. There is a Poincaré duality isomorphism in the noncompact case also, but it is between homology and compactly supported cohomology:

$$\operatorname{Pd}_X(-): H_c^k(X, \mathbb{Z}) \xrightarrow{\cong} H_{n-k}(X, \mathbb{Z}),$$

where

$$H_c^k(X,\mathbb{Z}) := \ker \left(H^k(X^\infty, \mathbb{Z}) \to H^k(\{\infty\}, \mathbb{Z}) \right)$$
.

Compactly supported cohomology can be defined also for more general (than locally compact) spaces, but the above definition suffices for our purposes. These definitions of orientability coincide with orientability of the tangent bundle, which, again, was equivalent to vanishing of the Stiefel-Whitney class $w_2(\Sigma)$ [Hus]. The definitions also generalize straightforwardly to (co)homology with coefficients in an Abelian ring R, or an R-module²² [May]. If a manifold is \mathbb{Z} -orientable, then it is also orientable for other coefficients, most importantly \mathbb{R} and \mathbb{Q} . If a manifold is R-orientable and not \mathbb{Z} -orientable (defined completely analogously to \mathbb{Z} -orientability), then $R = \mathbb{Z}_2$. Thus, every manifold is \mathbb{Z}_2 -orientable. If a manifold is R-orientable, then there is a Poincaré duality isomorphism between cohomology and homology with coefficients in an R-module π :

$$\operatorname{Pd}_X(-): H_c^k(X,\pi) \xrightarrow{\cong} H_{n-k}(X,\pi).$$

Definition 2.3.17. Let X and Y be orientable locally compact manifolds and $f: X \to Y$ a morphism. The Gysin "wrong way" homomorphism,

$$f_*: H_c^{\bullet}(X, \mathbb{Z}) \to H_c^{\bullet + \dim(Y) - \dim(X)}(Y, \mathbb{Z}),$$
 (2.30)

is defined by

$$f_*: H_c^{\bullet}(X, \mathbb{Z}) \xrightarrow{\operatorname{Pd}_X} H_{\dim(X) - \bullet}(X) \xrightarrow{f_*} H_{\dim(X) - \bullet}(Y) \xrightarrow{\operatorname{Pd}_Y^{-1}} H_c^{\bullet + \dim(Y) - \dim(X)}(X, \mathbb{Z}).$$
 (2.31)

The map f_* is the natural push-forward in homology. Functoriality follows immediately from the definition.

One immediately starts to wonder if some sort of generalized orientations could be defined for generalized cohomology and homology theories, which would imply the existence of corresponding Poincaré duality type isomorphisms. Turns out that the answer is "yes" [Whi, HW]. We shall focus on the special case of K-theory and K-orientability.

²⁰That is, compact without boundary.

²¹This is nontrivial, since $H_n(\Sigma, \Sigma \setminus \{x\}) \cong \mathbb{Z}$ has two generators

 $^{^{22}}$ The (local) fundamental classes are always classes in homology with coefficients in R, not in an R-module.

Theorem 2.3.18 ([ABS]). A manifold X is K-orientable, if and only if it is $Spin^c$.

We would next like to present a Poincaré duality isomorphism between K-theory and a "dual" generalized homology theory. It is not at all obvious that such a homology theory should even exist. However, turns out that all generalized cohomology theories do come with a natural dual homology theory, which admits a simple homotopic definition in terms of the Ω -spectrum of the cohomology theory [Whi]. For K-theory, the dual homology theory is called (spectral) K-homology and is denoted by $K_{\bullet}(-)$, $\bullet = 0, 1$. We shall soon discuss K-homology in great detail and present a geometric definition, equivalent to the homotopic one. For now, it is enough to know that it exists, it has natural covariant functoriality, and is dual to K-theory in the same sense as ordinary cohomology is dual to ordinary homology.

Theorem 2.3.19 ([Whi]). For a K-orientable manifold X of dimension n, there exists a K-theoretic Poincaré duality isomorphism

$$\operatorname{Pd}_{X}^{K}: K^{\bullet}(X) \xrightarrow{\cong} K_{n-\bullet}(X).$$
 (2.32)

K-theoretic Poincaré duality immediately yields a K-theoretic Gysin homomorphism, completely analogously to definition 2.3.17.

Definition 2.3.20. Let X and Y be K-orientable locally compact manifolds and $f: X \to Y$ a morphism. The K-theoretic Gysin "wrong way" homomorphism,

$$f_!: K^{\bullet}(X) \to K^{\bullet + \dim(Y) - \dim(X)}(Y),$$
 (2.33)

is defined by

$$f_!: K^{\bullet}(X) \xrightarrow{\operatorname{Pd}_X^K} K_{\dim(X) - \bullet}(X) \xrightarrow{f_*} K_{\dim(X) - \bullet}(Y) \xrightarrow{(\operatorname{Pd}_Y^K)^{-1}} K^{\bullet + \dim(Y) - \dim(X)}(X).$$
 (2.34)

The map f_* is the natural push-forward in K-homology. Again, functoriality follows immediately from the definition.

Orientations are usually not unique. For example, a connected manifold always admits exactly two distinct \mathbb{Z} -orientations. Let X be a manifold with fixed \mathbb{Z} - and K-orientations. The K-orientation induces a \mathbb{Z} -orientation through a homological Chern character homomorphism [Jak1]

$$\operatorname{ch}: K_{\bullet}(X) \to H_{\bullet}(X, \mathbb{Q})$$
,

where the right-hand side denotes \mathbb{Z}_2 -periodized homology. This map is analogous to the cohomological Chern character, for example, it is an isomorphism modulo the torsion subgroup. The K-orientation is, analogously to the \mathbb{Z} -orientation, given by local K-homology classes. Taking the homological Chern character yields local classes in rational cohomology, which constitute a \mathbb{Q} -orientation. Since they are in the image of ch, they lift to a \mathbb{Z} -orientation. However, the orientation induced by the K-orientation might be different from the original orientation. This causes a *commutativity defect* between the Gysin maps and Chern characters. If X and Y are K-oriented locally compact manifolds, $d = \dim(Y) - \dim(X)$ and $f: X \to Y$ a morphism, the diagram

$$K^{\bullet}(X) \xrightarrow{f_!} K^{\bullet+d}(Y)$$

$$\downarrow \text{ch} \qquad \qquad \downarrow \text{ch}$$

$$H_c^{\bullet}(X, \mathbb{Q}) \xrightarrow{f_*} H_c^{\bullet+d}(Y, \mathbb{Q})$$

$$(2.35)$$

does not commute. The Todd-class measures the difference between the original \mathbb{Z} -orientation and the \mathbb{Z} -orientation induced by the K-orientation²³. The Grothendieck-Riemann-Roch theorem tells us precisely how (2.35) needs to be modified to restore commutativity.

Theorem 2.3.21 ((Atiyah-Hirzebruch-)Grothendieck-Riemann-Roch [AH1, AH2]). When X and Y are as above, the diagram

$$K^{\bullet}(X) \xrightarrow{f_!} K^{\bullet+d}(Y)$$

$$\text{Todd}(X) \smile \text{ch} \qquad \qquad \downarrow \text{Todd}(Y) \smile \text{ch}$$

$$H^{\bullet}_c(X, \mathbb{Q}) \xrightarrow{f_*} H^{\bullet+d}_c(Y, \mathbb{Q})$$

commutes. More generally, the theorem holds if $W_3(X) \neq 0$ and $W_3(Y) \neq 0$ as long as

$$W_3(f) := \beta(w_2(X) - f^*w_2(Y)) = 0$$
,

which is equivalent to the vector bundle $TX \oplus f^{-1}TY$ being $Spin^{c24}$. The map f is said to be K-oriented. Of course, if f is K-oriented, but X and Y are not, the Gysin maps require more careful thought. However, in our applications such a situation does not arise.

2.4 D-branes and K-theory

We have now enough mathematical machinery at our disposal to start exploring the D-brane charge. The necessary physical background was explained in chapter 1 along with two strong arguments in favor of the K-theoretic classification scheme. In this section we go through these arguments again, putting our new understanding of characteristic classes and K-theory to use. The role of K-theory in D-brane theory was first realized by Minasian and Moore [MM] and the basics were worked out in [Wit1]. A nice overview is also given in [Wit2].

It was explained in section 1.1 that quasi-classically²⁵ a D-brane in Type IIB superstring theory is described by an even-dimensional closed worldvolume manifold Σ carrying a principal PU(n) bundle²⁶, the Chan-Paton bundle, together with a continuous map $\phi: \Sigma \to X$, where X is the 10-dimensional Riemannian spacetime manifold. Both Σ and X are oriented in Type

$$1 + w_1(TX \oplus f^{-1}TY) + w_2(TX \oplus f^{-1}TY) + \dots = (1 + w_2(TX) + \dots) \smile (1 + w_2(f^{-1}TY) + \dots),$$

where we have used the assumption that X and Y are orientable, together with proposition 2.2.1. Comparing classes of equal degree on both sides reveals

$$w_2(TX \oplus f^{-1}TY) = w_2(TX) + w_2(f^{-1}TY) = w_2(TX) + f^*w_2(TY) = w_2(X) + f^*w_2(Y) = w_2(X) - f^*w_2(Y)$$
.

²³This statement will be made precise later.

²⁴This is easy to show. From (2.3), we have

²⁵The complete quantum theory of D-branes is, as of yet, unknown. However, a quasi-classical model (of the D-brane charge) can be obtained through Dirac quantization.

 $^{^{26}}$ Locally, such a bundle arises as the projectivization of a principal U(n) bundle, but globally there may be an obstruction to gluing the U(n) lifts together. Thus, not every PU(n) bundle arises as the projectivization of a U(n) bundle [AS1]. We shall return to this later in more detail.

II theories, unoriented in Type I theory, and X is assumed to be Spin, since spinor particles obviously exist in reality²⁷.

A few assumptions must be made at this point. First, we assume that the D-brane worldvolume Σ is K-oriented (a Spin^c manifold). Second, we assume that PU(n) Chan-Paton bundles are always projectivizations of U(n) bundles. Their associated U(n) vector bundles are typically also referred to as Chan-Paton bundles. Recall from section 1.2 that the coupling of a D-brane with worldvolume Σ and U(n) Chan-Paton (vector) bundle $E \to \Sigma$ to the RR-fields is of the form²⁸

$$\int_{X} C \wedge \phi_{*}(\operatorname{ch}(E) \wedge \operatorname{Todd}(\Sigma)) \wedge \frac{1}{\sqrt{\operatorname{Todd}(X)}}, \qquad (2.36)$$

where $C = C_0 + C_2 + ...$ is the total RR-potential polyform. The integral is understood to evaluate the 10-form part of the even-degree polyform and discard everything else. Such a coupling implies that the total RR-charge (density) of the D-brane is the cohomology class

$$Q_{\mathbb{R}}(\Sigma, E) := \phi_*(\operatorname{ch}(E) \wedge \operatorname{Todd}(\Sigma)) \wedge \frac{1}{\sqrt{\operatorname{Todd}(X)}} \in H_{c,dR}^{\operatorname{even}}(X), \qquad (2.37)$$

where ϕ_* is the Gysin map in cohomology. Since both Σ and X are even-dimensional, ϕ_* does not change the degree of the class $\operatorname{ch}(E) \wedge \operatorname{Todd}(\Sigma)$ in periodized cohomology. All the quantities appearing in (2.37) take naturally values in rational cohomology, which allows us to work as well with

$$Q_{\mathbb{Q}}(\Sigma, E) := \phi_*(\operatorname{ch}(E) \smile \operatorname{Todd}(\Sigma)) \smile \frac{1}{\sqrt{\operatorname{Todd}(X)}} \in H_c^{\operatorname{even}}(X, \mathbb{Q}). \tag{2.38}$$

Remark, that the Gysin map shifts the degree of ch(E) by $dim(X) - dim(\Sigma) = 10 - (p+1) = 9 - p$. Since $\sqrt{Todd(X)} = 1 + \dots$ and $ch(E) = rk(E) + \dots$, where the ellipsis denotes classes of higher (even) degree, we can write (2.38) out as

$$Q_{\mathbb{Q}}(\Sigma, E) := \underbrace{\operatorname{rk}(E)\phi_{*}(1)}_{\in H_{c}^{9-p}(X, \mathbb{Q})} + \dots$$

The first term $\operatorname{rk}(E)\phi_*(1)$ pairs up with the RR-potential C_{p+1} in (2.36) to yield a class in $H^{10}_{\operatorname{c,dR}}(X)$, which can then be integrated. The result is the Dp-brane charge of the D-brane, which is the higher degree charge the D-branes carries. Hence, the RR-charge actually dictates the dimensionality of the D-brane: one can think of a Dp-brane as a D-brane with highest nonzero RR-charge of degree p (coupling to C_{p+1}). The class $\operatorname{rk}(E)\phi_*(1)$ represents the image in spacetime of the Poincaré dual the fundamental class of Σ . The rest of the cohomology

$$\frac{1}{\sqrt{\widehat{A}(X)}} \stackrel{(2.11)}{:=} \prod_{k} \sqrt{\frac{\sinh(x_{k}/2)}{x_{k}/2}} = \prod_{k} \sqrt{1 + \frac{(x_{k}/2)^{2}}{3!} + \dots} = \prod_{k} \left(1 + \frac{1}{2} \left(\frac{(x_{k}/2)^{2}}{3!} + \dots \right) + \dots \right).$$

²⁷To be absolutely precise, the existence of a Spin structure is inferred from the existence of *uncharged* spinor particles, since only a Spin^c structure is required for U(1)-charged spinors.

²⁸Square roots and inverse square roots of Todd(X) (which coincides with $\widehat{A}(X)$ by (2.12) since X is Spin) are defined by a straightforward expansion:

classes in $Q_{\mathbb{Q}}(\Sigma, E)$ are of lower degree and thus they pair up with RR-potentials of higher degree to yield integrable 10-forms.

Minasian and Moore remarked in [MM], that by considering E rather as a K-theory class $[E] \in K^0(\Sigma)$ than simply a vector bundle allows us to apply the Grothendieck-Riemann-Roch theorem 2.3.21 to (2.38), to get

$$Q_{\mathbb{Q}}(\Sigma, [E]) := \operatorname{ch}(\phi_{!}([E])) \smile \sqrt{\operatorname{Todd}(X)} \in H_{c}^{\operatorname{even}}(X, \mathbb{Q}). \tag{2.39}$$

Recall from section 1.3, that an anti-D-brane for a D-brane with RR-charge $Q_{\mathbb{Q}}(\Sigma, [F])$ is a D-brane carrying the opposite RR-charge

$$\overline{Q}_{\mathbb{Q}}(\Sigma, [F]) := -Q_{\mathbb{Q}}(\Sigma, [F]).$$

Using ch(-[F]) = -ch(F) in the Minasian-Moore formula (2.39) yields

$$\overline{Q}_{\mathbb{Q}}(\Sigma, [F]) = Q_{\mathbb{Q}}(\Sigma, -[F]).$$

Thus, a configuration of n D-branes with U(n) Chan-Paton bundle E and m anti-D-branes with U(m) Chan-Paton bundle F, wrapping the same worldvolume Σ , carries total RR-charge

$$Q_{\mathbb{Q}}(\Sigma, [E]) + \overline{Q}_{\mathbb{Q}}(\Sigma, [F]) = Q_{\mathbb{Q}}(\Sigma, [E]) - Q_{\mathbb{Q}}(\Sigma, [F]) = \operatorname{ch}(\phi_{!}([E] - [F])) \smile \sqrt{\operatorname{Todd}(X)}.$$

Hence, the K-theoretic definition for RR-charge easily generalizes to configurations of coincident D-branes and anti-D-branes. It makes sense to denote the total RR-charge of such a configuration by $Q_{\mathbb{Q}}(\Sigma, [E]-[F])$. Concluding, the total RR-charge of a stack of D-branes with Chan-Paton bundle E and a stack of anti-D-branes with Chan-Paton bundle F, all wrapping the same worldvolume Σ , is given by the Minasian-Moore formula

Definition 2.4.1 (Minasian-Moore [MM]).

$$Q_{\mathbb{Q}}(\Sigma, x) := \operatorname{ch}(\phi_!(x)) \smile \sqrt{\operatorname{Todd}(X)} \in H_c^{\operatorname{even}}(X, \mathbb{Q}), \qquad (2.40)$$

where $x = [E] - [F] \in K^0(\Sigma)$.

The above discussion was purely classical. In quantum theory we need to impose a *Dirac* quantization condition on the charges. The general principles are explained in [Fre1, Fre2]. The idea is that Dirac quantization corresponds to lifting the charges from rational (or real) cohomology to some generalized cohomology theory with integral coefficients. Unfortunately, there seems to be no way of determining absolutely what the generalized cohomology theory should be, but one can try to guess it by investigating the structure of the classical theory. Suppose $h^{\bullet}(-)$ is a generalized cohomology theory with integral coefficients. The Dirac quantization condition is the requirement for the existence of a natural transformation

$$ch: h^{\bullet}(-) \to H^{\bullet}(-, \mathbb{Q}), \qquad (2.41)$$

the "Chern character", such that the classical charge lies in the image lattice of ch. Finally, we postulate that the quantized charge actually lives in the integral cohomology theory, not merely in the image in $H^{\bullet}(-,\mathbb{Q})$. The distinction being that the integral cohomology theory is sensitive to torsion effects, whereas rational cohomology is obviously not. Thus, the

Dirac quantized theory might contain charge configurations which are unstable in the classical theory, but are nevertheless stable in the quantum theory due to topological torsion effects. In classical electromagnetism the integral cohomology theory is simply $H^{\bullet}(-,\mathbb{Z})$ with the "Chern character" being the natural inclusion. The hypothesis is that for RR-charges the integral cohomology theory is K-theory and the "Chern character" is the modified Chern character

$$Ch(-) := ch(-) \smile \sqrt{Todd(X)}$$
.

The classical fields should also be Dirac quantized by lifting them to some kind of integral cohomology theory. Recall, again, the case of electromagnetism, where classically the electromagnetic field is given by a field strength 2-form F. Dirac quantization then implies that F is really the curvature of a connection on a principal U(1) bundle. However, this geometric description somewhat obscures the general principle, namely, principal U(1) bundles with connection provide a geometric model for a certain cohomology class in *Deligne* cohomology [Bry, Joh2]. Deligne cohomology, on the other hand, turns out to be an example of a differential cohomology theory, which seems to be the correct mathematical framework for Dirac quantized fields [Fre1, Fre2, HS, Val]. The idea with differential cohomology is that it captures both global topological data and local differential geometric data of the field and combines them into a single package. Both are needed, since it is the differential geometric representative of the field that enters the path integral, but on the other hand, it ignores global topological torsion effects. Another example of differential cohomology is given by the B-field, which, purely classically, is described by the 3-form field strength H. Dirac quantization lifts H to a class in a differential cohomology theory, which turns out to be a certain class in Deligne cohomology. However, the Deligne class of the B-field is of higher rank than that of the electromagnetic field and is slightly more complicated. In particular, it can not be geometrically represented by a principal bundle with connection, but by a much more complicated object: a bundle gerbe with connection and curving [Mur, BCMMS, Joh2, Ste2, Gaj, MS1]. For RR-fields the correct differential cohomology theory seems to be differential K-theory. A nice exposition of differential K-theory, along with applications to D-brane theory, is given in [Val].

From the point of view of the Minasian-Moore formula (2.40) it seems natural that the generalized cohomology theory for RR-charges should be K-theory. More precisely, that the quantized RR-charge of a D-brane configuration corresponding to a K-theory class $x \in K^0(\Sigma)^{29}$ should be

$$Q_{\mathbb{Z}}(\Sigma, x) := \phi_!(x) \in K^0(X). \tag{2.42}$$

The natural transformation (2.41) from K-theory to rational cohomology, which physically associates to a quantized RR-charge its classical approximation, is clearly given by the *modified Chern character*

$$Ch(-) := ch(-) \smile \sqrt{Todd(X)}$$
.

Recall that the role of the class $\operatorname{Todd}(X)$ was to cure the commutativity defect in (2.35), arising from the difference between the fixed orientation of X and the orientation induced by the K-orientation. This is further emphasized by the following isometric pairing formula. Suppose X is compact. There is a natural bilinear pairing $(-,-)_K: K^0(X) \otimes_{\mathbb{Z}} K^0(X) \to \mathbb{Z}$,

²⁹The class x being constructed as in (2.40).

given by

$$([E], [F])_K := \operatorname{index}(\mathcal{D}_{E \otimes F}),$$

with the obvious bilinear extension to classes of a more general kind. Similarly, there is a natural bilinear pairing $(-,-)_H: H_c^{\bullet}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} H_c^{10-\bullet}(X,\mathbb{Q}) \to \mathbb{Q}$, defined as the obvious evaluation pairing of the cup product of the cup product of the two classes with the fundamental class:

$$(\eta, \theta)_H := \langle \eta \smile \theta, [X] \rangle$$
.

Proposition 2.4.2 (Isometric pairing formula). The modified Chern character is an isometry with respect to the pairings in K-theory and cohomology.

Proof. The proof is a straightforward application of the Atiyah-Singer index theorem [LM]:

$$([E], [F])_{K} = \operatorname{index}(\mathcal{D}_{E \otimes F}) \stackrel{\text{A-S}}{=} \langle \operatorname{ch}(E \otimes F) \smile \operatorname{Todd}(X), [X] \rangle$$

$$\stackrel{(2.10)}{=} \left\langle \left(\operatorname{ch}(E) \smile \sqrt{\operatorname{Todd}(X)} \right) \smile \left(\operatorname{ch}(F) \smile \sqrt{\operatorname{Todd}(X)} \right), [X] \right\rangle$$

$$= \left(\operatorname{Ch}([E]), \operatorname{Ch}([F]) \right)_{H}.$$

The bilinear extension to more general K-theory classes is trivial.

In conclusion, merely the presence of the Chern character in the coupling of D-branes to RR-fields suggests that K-theory might somehow be related to RR-charge. This hypothesis is strongly supported by the Minasian-Moore formula, which expresses the coupling in terms of a spacetime K-theory class constructed from the Chan-Paton bundle. These arguments were developed further in [Wit1], along with an explanation of the relation to an earlier observation by Sen [Sen] that a bound state of n coincident Dp-branes and n $D\overline{p}$ -branes carrying U(n)Chan-Paton bundles E and F, respectively, contains a tachyonic mode in its open string spectrum. The appearance of a tachyon indicates that the configuration has not yet reached a stable minimum of potential energy and is thus unstable. Unfortunately, the dynamical process of the tachyon field rolling to a stable minimum can not be described rigorously without complete quantum theory, which we do not have. However, given a particular tachyon field it is possible to figure out what the final state of the tachyon field is after the rolling process has finished (recall section 1.3). The tachyon field is a section of the bundle $E \otimes F^*$. When E and F are not isomorphic, this bundle is nontrivial, causing the tachyon field to run into a topological obstruction when trying to reach its vacuum expectation value globally. Instead, it collapses into a topologically stable solitonic "vortex" configuration, with nonzero energy density and RR-charge smeared over some subspace of Σ . Remarkably, the remaining energy density and RR-charge correspond precisely to that of a lower dimensional D-brane. This phenomenon is reflected by the Minasian-Moore formula as

$$Q_{\mathbb{Q}}(\Sigma, [E] - [F]) = \phi_*((\operatorname{ch}(E) - \operatorname{ch}(F)) \smile \operatorname{Todd}(\Sigma)) \smile \frac{1}{\sqrt{\operatorname{Todd}(X)}}$$

$$= \phi_*(\underbrace{(\operatorname{rk}(E) - \operatorname{rk}(F) + c_1(E) - c_1(F) + \dots)}_{=0} \smile \operatorname{Todd}(\Sigma)) \smile \frac{1}{\sqrt{\operatorname{Todd}(X)}}$$

$$= \underbrace{\phi_*(c_1(E) - c_1(F))}_{\in H_c^{11-p}(X)} + \dots,$$

where the ellipsis denotes cohomology classes of higher degree. Thus, the highest RR-potential such a configuration can couple to is C_{p-1} . The class $\phi_*(c_1(E) - c_1(F))$ obviously represents the image in spacetime homology of the Poincaré dual of the fundamental class of the residual D(p-2)-brane in Σ .

The K-theoretic classification extends also to Type IIA and Type I D-branes: Type IIA D-branes are classified by $K^1(X)$ and Type I D-branes by $real\ K$ -theory. We shall not discuss these, but refer the interested reader to [Wit1, Hor, Val, OS].

There is still one big problem left, namely, everything above relied heavily on the two assumptions that the Chan-Paton bundle is the projectivization of a U(n) bundle and that the K-theoretic Gysin map exists, which required $\phi: \Sigma \to X$ to be K-oriented. Since X is Spin (and hence also Spin^c), K-orientability of ϕ is equivalent to Σ being Spin^c . But on the other hand, there does not seem to be any obvious physical reason why either one of these assumptions should hold. Before trying to solve these problems, we take a look at an alternative, perhaps more natural, description of D-branes in terms of K-homology.

2.5 Geometric K-homology

Recall that reduced K-theory could be homotopically defined for a compact pointed manifold X^{∞} , as

$$\widetilde{K}^n(X^\infty) := [X^\infty, \mathbb{K}_n],$$

where the spectrum \mathbb{K}_n is

$$\mathbb{K}_n \cong \begin{cases} BU \times \mathbb{Z} & \text{for } n \text{ even,} \\ U := \varinjlim_k U(k) & \text{for } n \text{ odd.} \end{cases}$$

The associated reduced homology theory, call it *(reduced) spectral K-homology*, is given by [Whi]

$$\widetilde{K}_n(X^{\infty}) := \varinjlim_k \pi_{n+k}(X^{\infty} \wedge \mathbb{K}_k).$$

More generally, for any (locally compact) manifold X, we set

$$K_n(X) := \widetilde{K}_n(X^{\infty}).$$

There is then a Poincaré duality isomorphism between K-theory and spectral K-homology [Whi], whenever the manifold is K-oriented. A compact K-oriented manifold has a (global) fundamental K-homology class, which establishes the Poincaré duality by a homotopically defined cap product with the K-theory classes. The homotopic definitions are not very intuitive, however, and it would be convenient to have a geometric definition for K-homology³⁰. It would then be interesting to see how the Poincaré dual of the quantized RR-charge of a D-brane configuration, which is a spacetime K-theory class, is represented geometrically in K-homology. The geometric definition for K-homology was initially introduced by Baum and Douglas in [BD]. A more general approach to geometric representations for homology theories was developed in [Jak1, Jak2].

 $^{^{30}}$ The geometric definition (in terms of vector bundles) for K-theory is easy to deduce from the spectrum. For K-homology the task seems to be much more difficult.

Geometric K-homology cycles, or K-cycles for short, on a compact manifold X are represented by triples (Σ, x, ϕ) , where Σ is a closed Spin^c manifold, $x \in K^0(X)$ and $\phi : \Sigma \to X$ is a continuous map. Two such triples (Σ_1, x_1, ϕ_1) and (Σ_1, x_2, ϕ_2) are isomorphic if there exists a diffeomorphism $f : \Sigma_1 \to \Sigma_2$ such that the Spin^c structures are compatible under f^* , $x_1 = f^*x_2$, and the diagram

$$\begin{array}{c}
\Sigma_1 \xrightarrow{f} \Sigma_2 \\
\downarrow^{\phi_2} \\
X
\end{array}$$

commutes. Consider then the free Abelian group generated by the isomorphism classes, quotiented by the subgroup of classes of the form

$$(\Sigma_1 \coprod \Sigma_1, x, \phi) - (\Sigma_1, x|_{\Sigma_1}, \phi|_{\Sigma_1}) - (\Sigma_2, x|_{\Sigma_2}, \phi|_{\Sigma_2}). \tag{2.43}$$

The resulting Abelian group is the group of K-cycles, with addition given by the disjoint union. Henceforth, we shall make no distinction between a triple (Σ, x, ϕ) and the K-cycle it represents. The group of K-cycles is naturally \mathbb{Z}_2 -graded, given by the parity of the dimension of Σ . The sets of K-cycles with Σ of even and odd dimension are denoted by $\Gamma^0(X)$ and $\Gamma^1(X)$, respectively.

To obtain K-homology, the group of K-cycles needs to be quotiented by three equivalence relations.

Spin^c **bordism:** Let (Σ_1, x_1, ϕ_1) and (Σ_2, x_2, ϕ_2) be two *K*-cycles on *X*. They are said to be *bordant*,

$$(\Sigma_1, x_1, \phi_1) \sim_b (\Sigma_2, x_2, \phi_2),$$

if there exists a "K-cycle" (Σ, x, ϕ) , where W is now a compact Spin^c manifold with boundary $\partial \Sigma$, interpolating between the two K-cycles in the sense that

$$(\partial \Sigma, x|_{\partial \Sigma}, \phi|_{\partial \Sigma}) \cong (\Sigma_1, x_1, \phi_1) + (-\Sigma_2, x_2, \phi_2).$$

Here $-\Sigma_2$ denotes Σ_2 with inverse Spin^c structure³¹.

Direct sum: For K-cycles (Σ, x_k, ϕ) , k = 1, 2, we set

$$(\Sigma, x_1 + x_2, \phi) \sim_s (\Sigma, x_1, \phi) + (\Sigma, x_2, \phi).$$

Vector bundle modification: Let (Σ, x, ϕ) be a K-cycle and $F \to \Sigma$ a real Spin^c vector bundle of even rank³². If $\mathbb{1} \to \Sigma$ is the trivial real line bundle, the (Spin^c) vector bundle $\pi : F \oplus \mathbb{1} \to \Sigma$ has odd rank fibres. Choosing a Riemannian metric on $F \oplus \mathbb{1}$ yields a split exact sequence

$$0 \longrightarrow \pi^{-1}(F \oplus 1) \longrightarrow T(F \oplus 1) \rightleftarrows \pi^{-1}(T\Sigma) \longrightarrow 0$$

³¹Recall that a Spin^c structure on Σ is described by a class $c_1(P_{U(1)}) \in H^2(\Sigma, \mathbb{Z})$, whose mod 2 reduction is $w_2(\Sigma)$. The inverse Spin^c structure is the one corresponding to the class $c_1(P_{U(1)}^*) = -c_1(P_{U(1)})$.

 $^{^{32}}$ Remark, that Σ is not necessarily connected and F can very well have different (but still even) rank on different components.

of vector bundles over $F \oplus \mathbb{1}$ and the direct sum decomposition

$$T(F \oplus 1) \cong \pi^{-1}(F \oplus 1) \oplus \pi^{-1}(T\Sigma)$$
.

Finally, (2.3) implies (using $w_1(F \oplus 1) = w_1(T\Sigma) = 0$) that

$$w_2(T(F \oplus 1)) = w_2(\pi^{-1}(E \oplus 1)) + w_2(\pi^{-1}(T\Sigma)).$$

Since both $w_2(\pi^{-1}(E \oplus 1))$ and $w_2(\pi^{-1}(T\Sigma))$ are mod 2 reductions of integral cohomology classes, so is their sum. Thus, $T(F \oplus 1)$ is a Spin^c vector bundle with a canonical choice for the Spin^c structure. Choosing a smooth metric on $F \oplus 1$ enables us to define the associated *unit sphere bundle*,

$$\mathbb{S}(F \oplus \mathbb{1}) \to \Sigma$$
,

whose fibres are the (even-dimensional) unit spheres of the fibres of $F \oplus \mathbb{1}$. The Spin^c structure of $T(F \oplus \mathbb{1})$ restricts to a Spin^c structure for $T\mathbb{S}(F \oplus \mathbb{1})$, turning $\mathbb{S}(F \oplus \mathbb{1})$ into a Spin^c manifold. The bundle $\mathbb{S}(F \oplus \mathbb{1}) \to \Sigma$ admits a nowhere vanishing constant section σ , sending each point of Σ to $0 \oplus 1$. We set

$$(\Sigma, x, \phi) \sim_v (\mathbb{S}(F \oplus, \mathbb{1}), \sigma_!(x), \phi \circ \pi)$$
.

This concludes the construction of the necessary equivalence relations. All three are obviously compatible with the grading $\Gamma^{\bullet}(X)$. Geometric K-homology is now defined as follows³³.

Definition 2.5.1 ([Jak1, BD, BHS, RS]). The geometric K-homology groups of X are the Abelian groups

$$K^g_{\bullet}(X) = \Gamma^{\bullet}(X)/\sim$$
,

where \sim is the equivalence relation generated by \sim_b , \sim_s and \sim_v . The equivalence class of a K-cycle (Σ, x, ϕ) is denoted $[\Sigma, x, \phi]$.

The inverse of a class $[\Sigma, E, \phi]$ is $[-\Sigma, E, \phi]$, where $-\Sigma$ denotes Σ with inverse Spin^c structure. The manifold $\Sigma \coprod -\Sigma$ is the entire boundary of a higher-dimensional manifold and, thus, bordant to the empty manifold \varnothing . The zero element is the class $[\varnothing, 0, 0]$.

A continuous map $f: X \to Y$ induces a homomorphism of Abelian groups

$$f_*: K^g_{\bullet}(X) \to K^g_{\bullet}(Y)$$

by

$$f_*[\Sigma, x, \phi] := [\Sigma, x, f \circ \phi].$$

Homotopic maps induce the same map in K^g_{\bullet} . This follows easily from bordism, since vector bundles, and hence K-theory classes of Σ extend to K-theory classes of $\Sigma \times [0,1]$. Hence, K^g_{\bullet} are covariant functors from the homotopy category of finite CW-complexes, or some other suitable category containing compact manifolds, to the category of Abelian groups.

It is not at all easy to verify that K^g_{\bullet} actually satisfy the axioms of a 2-periodic (generalized) homology theory. It is, however, an immediate consequence of the following result.

³³Our definition is the one given in [Jak1], but it is isomorphic to that of [BD, BHS, RS]. The proof is not difficult [Val].

Theorem 2.5.2 ([Jak1, Val], see also [BHS, Kas3]). For any smooth manifold X of the homotopy type of a finite CW-complex, spectral K-homology is isomorphic to geometric K-homology:

$$K^g_{\bullet}(X) \cong K_{\bullet}(X)$$
.

In particular, this holds for compact manifolds.

For a compact pointed manifold X^{∞} the collapsing map $X^{\infty} \to \{\infty\}$ induces a map $K^g_{\bullet}(X^{\infty}) \to K^{\bullet}(\{\infty\})$.

Definition 2.5.3. The reduced geometric K-homology groups are

$$\widetilde{K}^g_{\bullet}(X^{\infty}) := \ker(K^g_{\bullet}(X^{\infty}) \to K^{\bullet}(\{\infty\})).$$

Like the K-theory groups, the K-homology groups of X^{∞} decompose as

$$K^g_{\bullet}(X^{\infty}) \cong \widetilde{K}^g_{\bullet}(X^{\infty}) \oplus K^g_{\bullet}(\{\infty\}).$$

For the one-point space $\{\infty\}$ we have (for example, by Poincaré duality and (2.26))

$$K_n^g(\{\infty\}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ 0 & \text{for } n = 1. \end{cases}$$

Thus (compare to (2.27)), we have

$$K_n^g(X^\infty) \cong \begin{cases} \widetilde{K}_n^g(X^\infty) \oplus \mathbb{Z} & \text{for } n = 0, \\ \widetilde{K}_n^g(X^\infty) & \text{for } n = 1. \end{cases}$$

Definition 2.5.4. For any (locally compact) manifold X, define

$$K^g_{\bullet}(X) := \widetilde{K}^g_{\bullet}(X^{\infty}).$$

The cap product, defined homotopically in spectral K-homology [Whi], translates in the geometric picture to a map

$$\frown: K^n(X) \otimes K^g_{\bullet}(X) \to K^g_{\bullet-n}(X)$$

with a surprisingly simple form [Jak1, RS].

Definition 2.5.5. For $y \in K^0(X)$ and $[\Sigma, x, \phi] \in K^g(X)$, set

$$y \frown [\Sigma, x, \phi] := [\Sigma, \phi^* y \smile x, \phi] \in K^g_\bullet(X) \,,$$

where \smile is the cup product (2.15) in K-theory.

For $z \in K^1(X)$, the cap product $z \frown [\Sigma, x, \phi]$ is given as follows. First, take the cup product $\phi^*z \smile x \in K^1(\Sigma)$. The product manifold $\Sigma \times S^1$ is Spin^c , which allows us to push $\phi^*z \smile x$ into $K^0(\Sigma \times S^1)$ using the Gysin map $\sigma_!$ induced by the map $\sigma(p) = (p, 1)$. We set

$$z \frown [\Sigma, x, \phi] := [\Sigma \times S^1, \sigma_!(\phi^*z \smile x), \phi \circ \operatorname{pr}_\Sigma] \in K^g_{\bullet+1}(X) \,.$$

For a closed K-oriented n-dimensional manifold X the Poincaré duality isomorphism (2.32) is given by taking the cap product with the K-homology fundamental class. In the geometric picture this class is particularly easy to describe [Jak2]. It is

$$[X] := [X, [1], \mathrm{id}_X].$$

Poincaré duality is then given explicitly by

$$K^{\bullet}(X) \ni y \stackrel{\operatorname{Pd}_{X}^{K}}{\longmapsto} y \frown [X] = \begin{cases} [X, y, \operatorname{id}_{X}] \in K_{n}^{g}(X) & \text{for } \bullet = 0, \\ [X \times S^{1}, \sigma_{!}(y), \operatorname{pr}_{X}] \in K_{n-1}^{g}(X) & \text{for } \bullet = 1. \end{cases}$$

$$(2.44)$$

For noncompact boundaryless manifolds a K-orientation determines only local fundamental classes, yielding local Poincaré duality isomorphisms. These can then be patched together into a global isomorphism³⁴.

We already mentioned earlier that there exists a homological Chern character, a natural transformation from spectral K-homology to periodized rational homology. It has a simple description in the geometric picture. For a class $[\Sigma, x, \phi] \in K^g_{\bullet}(X)$ it is given by [Jak1, RS]

$$\operatorname{ch}([\Sigma, x, \phi]) := \operatorname{Pd}_X(\operatorname{ch}(\phi_!(x)) \smile \operatorname{Todd}(X)) \in H_{\bullet}(X, \mathbb{Q}). \tag{2.45}$$

Chern characters in K-theory and K-homology are compatible with cap products in such a way that the diagram

$$K^{\bullet}(X) \otimes K_{n}^{g}(X) \xrightarrow{\frown} K_{n-\bullet}^{g}(X)$$

$$\begin{array}{c} \text{ch} \otimes \text{ch} \\ \end{array} \downarrow \text{ch}$$

$$H_{c}^{\bullet}(X, \mathbb{Q}) \otimes H_{n}(X, \mathbb{Q}) \xrightarrow{\frown} H_{n-\bullet}(X, \mathbb{Q})$$

commutes. The ordinary cohomology and homology rings are considered to be \mathbb{Z}_2 -periodized.

2.6 D-branes and K-homology

It was first suggested by Periwal in [Per] that D-branes may be more "naturally" described by K-homology than by K-theory. In this section we explain why this is the case. It turns out that geometric K-homology provides an intuitive "worldvolume" description of D-branes, as opposed to the somewhat abstruse K-theoretic "RR-charge" description.

We claim that a $p - \overline{p}$ D-brane configuration³⁵ corresponding to a K-theory class $x \in K^0(\Sigma)$ (wrapping $\phi: \Sigma \to X$) is described by the geometric K-homology class $[\Sigma, x, \phi] \in K_0^g(X)^{36}$. Or more precisely, the K-homology class corresponds to an equivalence class of physically equivalent D-brane configurations, which can be obtained from each other by a smooth physical processes. For example, nucleating a $p - \overline{p}$ pair with isomorphic Chan-Paton bundles should not change the physical equivalence class. For a noncompact spacetime manifold we defined K-homology using relative K-homology of the one-point compactification.

³⁴Compare to the similar result in ordinary cohomology [May].

³⁵We are still working under the assumptions that the Chan-Paton bundles are projectivizations of U(n) bundles, that D-brane worldvolumes are Spin^c and that there is no background B-field.

³⁶Recall that since we are dealing with Type IIB theory, Σ is even-dimensional.

This is, again, the requirement that physical D-brane configurations should not be "infinitely nontrivial", which ensures that no infinite charges or energies are taken into account³⁷.

Consider a D-brane wrapping $\phi: \Sigma \to X$ with U(n) Chan-Paton bundle E. The corresponding K-homology class is then $[\Sigma, [E], \phi]$. To motivate this, we need to find physical interpretations for the various equivalence relations in the definition of K-homology [RS, Sza1]. Addition of K-homology classes (as in (2.43)) means simply that several disconnected D-branes can be treated by a single K-homology class.

Direct sum: If two D-branes with U(n) and U(m) Chan-Paton bundles are set to be coincident, we would expect to see the Chan-Paton bundles merge into a U(n+m) bundle. This is precisely what the direct sum equivalence relation does. For D-branes $[\Sigma, [E], \phi]$ and $[\Sigma, [F], \phi]$ it implies

$$[\Sigma, [E], \phi] + [\Sigma, [F], \phi] = [\Sigma, [E \oplus F], \phi].$$

Spin^c **bordism:** The Spin bordism relation is also very easy to interpret physically. It states that a smooth deformation of a D-brane and its Chan-Paton bundle does not change the K-homology class. For example, if a D-brane can be smoothly deformed to the vacuum, its K-homology class, and thus also its RR-charge, vanishes.

Vector bundle modification: This one is more complicated to interpret. Physically, it represents the *Myers' dielectric effect* [Mye, Sza1, RS]. The following simple example is given in [RS]. Consider a D(-1)-brane³⁸ [$\{\infty\}$, [1], ι] $\in K_0^g(X)$. Choose F in the definition of vector bundle modification to be the trivial bundle $\{\infty\} \times \mathbb{R}^{2n}$, with $n \geq 1$. The unit sphere bundle $\mathbb{S}(F \oplus \mathbb{I})$ is simply $\{\infty\} \times S^{2n} \cong S^{2n}$. Therefore, by vector bundle modification,

$$[\{\infty\}, [\mathbb{1}], \iota] = [S^{2n}, \sigma_!([\mathbb{1}]), \epsilon],$$

where ϵ collapses S^{2n} into the point in X and σ is the constant section given in the definition of vector bundle modification. This is the dielectric effect: the D(-1)-brane "blows up" into a collection of spherical D(2n - 1)-branes.

The K-theoretic and K-homological descriptions of D-branes should obviously be related to each other by Poincaré duality. Let us consider, for simplicity the case of a closed space-time manifold X. Recall the explicit form of the Poincaré duality isomorphism from (2.44). If $\phi_!(x) \in K^0(X)$ is the K-theory class describing the D-brane configuration, then the corresponding K-homology class should be $\operatorname{Pd}_X^K(\phi_!(x)) = [X, \phi_!(x), \operatorname{id}_X]$. However, our physical interpretation associated to the D-brane the class $[\Sigma, x, \phi]$. Luckily, these two are in fact equivalent [Jak2]. This is easy to verify in the special case when $\phi: \Sigma \to X$ is an embedding and X is connected. The normal bundle $N\Sigma$ of Σ in X has rank $n = 10 - \dim(\Sigma)$. Since $n = 10 - \dim(\Sigma)$ is even $n = 10 - \dim(\Sigma)$. Since $n = 10 - \dim(\Sigma)$ is even $n = 10 - \dim(\Sigma)$. The total space of the unit sphere bundle $\mathbb{S}(F \oplus \mathbb{I})$ is 10-dimensional and embeds into X.

³⁷We are measuring the D-brane charge with respect to that of the vacuum.

 $^{^{38}}$ A D(-1)-brane, also known as a *D-instanton*, is simply a 0-dimensional pointlike D-brane localized in spacetime.

³⁹Remember that $\dim(\Sigma)$ is even in Type IIB theory.

⁴⁰This follows immediately from $TX \cong T\Sigma \oplus N\Sigma$, (2.3) and that X and Σ are both Spin^c.

Such an embedding is actually a diffeomorphism [RS]. Using the notation in the definition of vector bundle modification, we immediately get (by vector bundle modification and bordism)

$$[\Sigma, x, \phi] = [\mathbb{S}(F \oplus \mathbb{1}), \sigma_!(x), \phi \circ \pi] = [X, \phi_!(x), \mathrm{id}_X].$$

The Minasian-Moore formula (2.40) can also be expressed purely in terms of homological quantities by employing the homological Chern character (2.45):

$$Q_{\mathbb{Q}}(\Sigma, x) = \operatorname{ch}(\phi_!(x)) \smile \sqrt{\operatorname{Todd}(X)} = \operatorname{ch}(\phi_!(x)) \smile \operatorname{Todd}(X) \smile \frac{1}{\sqrt{\operatorname{Todd}(X)}}$$
$$= \operatorname{Pd}_X^{-1} \circ \operatorname{ch}([\Sigma, x, \phi]) \smile \frac{1}{\sqrt{\operatorname{Todd}(X)}}.$$

Definition 2.6.1 (Minasian-Moore). The homological classical RR-charge of a D-brane configuration $[\Sigma, x, \phi] \in K_0^g(X)$ is

$$\begin{split} \widehat{Q}_{\mathbb{Q}}(\Sigma, x) := & \operatorname{Pd}(Q_{\mathbb{Q}}(\Sigma, x)) \\ = & \operatorname{Pd}_{X} \left[\operatorname{Pd}_{X}^{-1} \circ \operatorname{ch}([\Sigma, x, \phi]) \smile \frac{1}{\sqrt{\operatorname{Todd}(X)}} \right] \in H_{\operatorname{even}}(X, \mathbb{Q}) \,. \end{split}$$

Chapter 3

The Freed-Witten Anomaly

The goal of this chapter is to drop the assumptions used in the previous chapter, that

- a) Chan-Paton bundles lift to U(n) bundles,
- **b)** D-brane worldvolumes are Spin^c,
- c) there is no background B-field.

Obviously, the first two assumptions are crucial for the K-theoretic and K-homological classification schemes discussed above. If a) is not satisfied, we can not form K-theory classes from the Chan-Paton bundles. If a) is satisfied but b) is not, we would not be able to describe the D-branes in terms of spacetime quantities, since Gysin maps would fail to exist. The last condition is more mysterious and is required by Freed-Witten anomaly cancellation.

Suppose that we have a nonzero locally defined background B-field. Let us first take a look at closed strings. The worldsheet of a closed string is a closed surface M, embedded into spacetime by $\xi: M \to X$. The B-field couples to the worldsheet as follows [Pol1]:

$$\exp(iS_{\mathrm{closed}}) \supset \exp\left(\int_{M} \xi^* B\right)$$
.

The exponential of the integral is called the *holonomy* of B over M. We shall employ the shorthand notation

$$hol(M,B) := \exp\left(\int_M \xi^* B\right) \tag{3.1}$$

for holonomy. Since any two local representatives of the B-field differ by a de Rham coboundary $d\Lambda$, the integrals of ξ^*B over M are invariant under gauge transformations $B \mapsto B + d\Lambda$:

$$\exp\left(\int_{M} \xi^{*}(B+d\Lambda)\right) = \exp\left(\int_{M} \xi^{*}B\right) \cdot \exp\left(\left(\int_{M} d\xi^{*}\Lambda\right) \stackrel{\text{Stokes}}{=} \exp\left(\int_{M} \xi^{*}B\right) \,.$$

Still, hol(M, B) is not automatically a well-defined function on the space of maps, Map(M, X). We demand that

$$\int_{M} \xi^* B$$

is well-defined on Map(M, X) up to an integral multiple of $2\pi i$, which leads to the usual Dirac quantization condition that the de Rham class $[H]_{dR}$, evaluated on any closed homology cycle,

must be integral [Alv]. As was briefly explained in section 2.4, we postulate that the Dirac quantized H-flux should, more correctly, take values in $H^3(X,\mathbb{Z})$ [Fre1, Fre2]. The distinction is that a class $[H] \in H^3(X,\mathbb{Z})$ may contain torsion parts, whereas its image in $H^3_{dR}(X)$ is necessary nontorsion (for the obvious reason that $\mathbb{Z}_n \otimes \mathbb{R} = 0$)¹.

We are actually more interested in open strings than closed strings. The worldsheet of an open string is a compact oriented surface, again denoted by M, but this time with boundary ∂M . Again, we denote its embedding into spacetime by $\xi: M \to X$. The B-field couples again to M, just like in (3.1), but now the integral is no longer gauge independent:

$$\int_{M} \xi^{*}(B + d\Lambda) = \int_{M} \xi^{*}B + \oint_{\partial M} \xi^{*}\Lambda \neq \int_{M} \xi^{*}B.$$

This is a major problem, because it means that the path integral is ill-defined.

Recall from *Introduction And Overview*, that D-branes could be thought of as boundary conditions for open strings. This manifests in the open string path integral as a coupling of the boundary, ∂M (corresponding to the open string endpoints), to the D-brane. More precisely, recall that the D-brane carries a Lie algebra valued locally defined 1-form "gauge" field A. The boundary ∂M couples to A by [Pol1]

$$\operatorname{Tr} \operatorname{hol}(\partial M, A) := \operatorname{Tr} \exp \left(\oint_{\partial M} \xi^*(A) \right)$$
.

Now, if A is a gauge connection on a principal U(n) bundle, $\operatorname{Tr} \operatorname{hol}(\partial M, A)$ is the well-defined parallel transport of A around ∂M [Nak]. However, these are the kind of assumptions we want to get rid of, but it is unclear if the path integral is well-defined otherwise.

There is a third difficulty associated to the fermionic part of the worldsheet path integral, namely the *Pfaffian* of the worldsheet Dirac operator [FW],

$$\operatorname{pfaff}(D_{\xi}) = \exp\left(\int d\psi \, \bar{\psi} i D_{\xi} \psi\right),\,$$

is ill-defined as a function on the space Map(M, X).

Concluding, the open string path integral contains the terms

$$\exp(iS_{\text{open}}) \supset \operatorname{pfaff}(\mathcal{D}_{\xi}) \cdot \operatorname{hol}(M, B) \cdot \operatorname{Tr} \operatorname{hol}(\partial M, A), \qquad (3.2)$$

each of which is possibly ill-defined. This is called the *Freed-Witten anomaly* [FW]. In the next sections, we shall explain how each of these terms can be given a rigorous definition and how the ill-defined parts precisely cancel each other, when a certain cohomological condition,

$$[H]|_{\Sigma} = W_3(\Sigma) + \beta([\underline{\zeta}]),$$

is satisfied [FW, Kap, CJM]. Here $[H]|_{\Sigma}$ is the pullback of [H] onto Σ and $\beta([\underline{\zeta}]) \in H^3(X, \mathbb{Z})$ is the obstruction for lifting the PU(n) Chan-Paton bundle into a U(n) bundle. For example, if $[H]|_{\Sigma} = 0$ and the Chan-Paton bundle is a U(n) bundle, then the worldvolume Σ must

¹Recall also from section 2.4 that the correct mathematical formalism to describe the Dirac quantized B-field should be a differential cohomology theory, in this case one associated to $H^{\bullet}(-,\mathbb{Z})$ [Fre1, Fre2]. It happens to be isomorphic to something called the (diagonal) *Deligne cohomology* [HS], which we introduce soon.

necessarily be Spin^c , just like we assumed above. Because $W_3(\Sigma)$ is 2-torsion and $\beta([\underline{\zeta}])$ n-torsion, $[H]|_{\Sigma}$ must also be pure torsion. This means that the classical H-flux 3-form must vanish, when restricted onto Σ . The K-theoretic classification of D-branes in the presence of nontorsion $[H]|_{\Sigma}$ was presented as a puzzle in [Wit1]. To cancel the anomaly, we would somehow have to take the torsion degree of $\beta([\underline{\zeta}])$ to infinity. This would correspond to taking the number of D-branes to infinity. It was finally explained in [BM] that the correct way to take this limit is to allow the Chan-Paton bundles to be an infinite-dimensional projective bundle, a $PU(\mathcal{H}) := U(\mathcal{H})/U(1)$ bundle, where \mathcal{H} denotes the usual infinite-dimensional separable Hilbert space and U(1) the center of $U(\mathcal{H})$.

3.1 Deligne cohomology

We shall start with a brief review of the basics of *sheaf theory*, *sheaf cohomology* and *sheaf hypercohomology*, which are used to define Deligne cohomology. We then explain, how the obscure sheaf hypercohomology groups can be computed relatively easily using *Čech hypercohomology*. Our treatment is based on [Bry], but we tend to make more use of the language of category theory than [Bry] does. We omit the proofs, since they would take too many pages to present and also, because the material in this section is not used much in the rest of the text.

Let $\mathbf{Set}^{\mathbf{o}}(X)$ denote the category of open sets of X, with morphisms the inclusions, and \mathbf{Ab} the category of Abelian groups.

Definition 3.1.1 ([Bry]). A presheaf A of Abelian groups on X is a contravariant functor²

$$A: \mathbf{Set^o}(X) \to \mathbf{Ab}$$
.

For an inclusion $V \subset U$ of open sets of X, we denote the homomorphism $A(V \subset U) : A(U) \to A(V)$ by

$$A(U) \ni \alpha_U \mapsto \alpha_U|_V \in A(V)$$

and call it the restriction map. We also assume that $A(\emptyset) = 0$.

Let $\mathcal{V}_U = \{V_i\}$ be an open covering of $U \in \mathbf{Set^o}(X)$. A family of elements $\{\alpha_i \in A(V_i)\}$ is said to be *compatible*, if $\alpha_i|_{V_i \cap V_j} = \alpha_j|_{V_i \cap V_j}$ for all i, j. A compatible family is said to satisfy the *glueing condition*, if there exists a unique $\alpha_U \in U$, such that $\alpha_i = \alpha_U|_{V_i}$ for all i.

Definition 3.1.2. The presheaf A is a *sheaf*, if every compatible family of elements satisfies the glueing condition.

If A and B are (pre)sheaves $\mathbf{Set^o}(X) \to \mathbf{Ab}$, a morphism of (pre)sheaves is a natural transformation $\phi: A \to B$. In other words, it associates to every $U \in \mathbf{Set^o}(X)$ a homomorphism $\phi(U): A(B) \to B(U)$, such that the diagram

$$A(U) \xrightarrow{\phi(U)} B(U)$$

$$|_{V} \downarrow \qquad \qquad \downarrow |_{V}$$

$$A(V) \xrightarrow{\phi(V)} B(V)$$

 $^{^{2}}$ We are only interested in presheaves and sheaves of Abelian groups. However, in most of the following definitions, the category \mathbf{Ab} can easily be replaced by any other category.

commutes. (Pre)sheaves (of Abelian groups) on X, together with their morphisms, form a category (**Pre**)**Sheaf**(X). For $Y \in \mathbf{Set^o}(X)$, there is a cofunctor

$$|_{Y}: (\mathbf{Pre})\mathbf{Sheaf}(X) \to (\mathbf{Pre})\mathbf{Sheaf}(Y)$$
,

the restriction functor, which assigns to a (pre)sheaf $A \in (\mathbf{Pre})\mathbf{Sheaf}(X)$ the (pre)sheaf $A|_Y$, defined by $A|_Y(V) = A(V)$, for $V \in \mathbf{Set}^{\mathbf{o}}(Y)$.

The stalk of a (pre)sheaf A at $x \in X$ is the direct limit

$$A_x := \varinjlim_{U \ni x} A(U)$$
.

For any open set $V \ni x$ of X, the direct limit provides a natural restriction map

$$|_x: A(V) \to A_x$$
.

Elements of A(U) are called *(local) sections* of A over U. Elements of A(X) are the *global sections* of A. The *local section functor* is the bifunctor

$$\Gamma(-,-): \mathbf{Set}^{\mathbf{o}}(X) \times (\mathbf{Pre}) \mathbf{Sheaf}(X) \to \mathbf{Ab}$$

defined by

$$\Gamma(U,A) := A(U)$$
.

Restricting to U=X yields the global section functor, $\Gamma(X,-)$. A section $\alpha_U \in A(U)$ is continuous at $x \in U$, if there exists an open set $V \subset U$, such that $x \in V$ and $\alpha_U|_V = \alpha_U|_x$. A section of A(U) is continuous, if it is continuous at every $x \in U$. The continuous section functor,

$$\widetilde{\Gamma}(-,-):\mathbf{Set^o}(X) imes(\mathbf{Pre})\mathbf{Sheaf}(X) o\mathbf{Ab}\,,$$

assigns to the pair $(U, A) \in \mathbf{Set^o}(X) \times (\mathbf{Pre})\mathbf{Sheaf}(X)$ the subgroup of continuous sections of A(U).

Definition 3.1.3 ([Bry]). Let A be a presheaf. Its associated sheaf, or sheafification, is a sheaf \widetilde{A} , given by

$$\widetilde{A}(U) := \widetilde{\Gamma}(U, A)$$
.

A sheaf A is also a presheaf and its sheafification \widetilde{A} coincides A [Bry].

Let $\phi: A \to B$ be a morphism of sheaves (of Abelian groups) on X. Its kernel, $\ker(\phi)$, is a sheaf on X, given by

$$\ker(\phi)(U) := \ker(\phi(U) : A(U) \to B(U)),$$

for $U \in \mathbf{Set^o}(X)$. Let P be the assignment

$$P(U) := \operatorname{im} \left(\phi(U) : A(U) \to B(U) \right).$$

Then P defines a presheaf on X. The image of ϕ is

$$\operatorname{im}(\phi) := \widetilde{P}$$
.

These definitions enable us to talk about exact sequences of sheaves, complexes of sheaves and cohomology sheaves. These are defined as one would expect. A sequence

$$\cdots \longrightarrow A^{n-1} \xrightarrow{\phi^{n-1}} A^n \xrightarrow{\phi^n} A^{n+1} \longrightarrow \cdots$$

of sheaves (and morphisms of sheaves) is exact, if $\ker(\phi^{k+1}) = \operatorname{im}(\phi^k)$ at each step. A complex of sheaves, $(K^{\bullet}, d^{\bullet})$, is a sequence of sheaves

$$\cdots \longrightarrow K^{n-1} \xrightarrow{d^{n-1}} K^n \xrightarrow{d^n} K^{n+1} \longrightarrow \cdots$$

where $d^{j+1} \circ d^j = 0$, for all j.

A sheaf K on X is *injective*, if for any morphism of sheaves $\phi: A \to K$ and an injective morphism³ $\iota: A \to B$, there exists a morphism $\psi: B \to K$, such that

$$A \xrightarrow{\iota} B$$

$$\phi \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad$$

commutes. An injective resolution of a sheaf A is a complex $(I^{\bullet}, d^{\bullet})$ of injective sheaves, together with a monomorphism $i_A : A \to I^0$, such that the sequence

$$0 \longrightarrow A \xrightarrow{i} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \longrightarrow \cdots$$

is exact. The category of sheaves on X has enough injectives in the sense that every sheaf A on X admits an injective resolution [Bry].

The global section functor $\Gamma(X,-)$ is left-exact: for an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of sheaves, the sequence

$$0 \longrightarrow \Gamma(X, A) \longrightarrow \Gamma(X, B) \longrightarrow \Gamma(X, C)$$
(3.3)

is exact. Since **Sheaf**(X) has enough injectives, (3.3) can be extended canonically to the right into a long exact sequence

$$0 \longrightarrow \Gamma(X,A) \longrightarrow \Gamma(X,B) \longrightarrow \Gamma(X,C)$$

$$\longrightarrow R^{1}\Gamma(X,A) \longrightarrow R^{1}\Gamma(X,B) \longrightarrow R^{1}\Gamma(X,C)$$

$$\longrightarrow R^{2}\Gamma(X,A) \longrightarrow R^{2}\Gamma(X,B) \longrightarrow R^{2}\Gamma(X,C) \longrightarrow \cdots$$

where

$$R^j\Gamma(X,-):\mathbf{Sheaf}(X)\to\mathbf{Ab}$$

 $^{^{3}}$ That is, a morphism with ker = 0.

are the right derived functors of $\Gamma(X, -)$, defined as follows. For a sheaf F on X, take any injective resolution $F \xrightarrow{i_F} I^{\bullet}$. Applying $\Gamma(X, -)$, yields a complex $\Gamma(X, I^{\bullet})$ of Abelian groups. The right derived functors $R^{\bullet}\Gamma(X, -)$ send the sheaf F to the cohomology groups of this complex:

$$R^{\bullet}\Gamma(X,F) := \frac{\ker\left(\Gamma(X,I^{\bullet}) \to \Gamma(X,I^{\bullet+1})\right)}{\operatorname{im}\left(\Gamma(X,I^{\bullet-1}) \to \Gamma(X,I^{\bullet})\right)}.$$

Definition 3.1.4 ([Bry]). The *sheaf cohomology* groups of a sheaf A on X, are the Abelian groups

$$H^{\bullet}(X,A) := R^{\bullet}\Gamma(X,A)$$
.

It is far from clear that the sheaf cohomology groups are well-defined, for example, that they are independent of the injective resolutions used. The reader is referred to [Bry] for a proof.

Next we want to define *sheaf hypercohomology*, which is a generalization of sheaf cohomology for sequences of sheaves. We need first the concept of an *injective resolution of a complex* of sheaves on X. We assume, that all complexes of sheaves are bounded below, meaning that they are of the form

$$K^0 \xrightarrow{d_K} K^1 \xrightarrow{d_K} K^2 \xrightarrow{d_K} \cdots$$

We refer the reader to [Bry] for a proof of the following. Given a bounded below complex of sheaves $(K^{\bullet}, d_K)^4$, there exists a double complex

$$\begin{array}{c|c}
\vdots & \vdots & \vdots \\
\delta & & \delta \\
I^{0,2} \xrightarrow{d} I^{1,2} \xrightarrow{d} I^{2,2} \xrightarrow{d} \cdots \\
\hline
\delta & & \delta \\
I^{0,1} \xrightarrow{d} I^{1,1} \xrightarrow{d} I^{2,1} \xrightarrow{d} \cdots \\
\hline
\delta & & \delta \\
I^{0,0} \xrightarrow{d} I^{1,0} \xrightarrow{d} I^{2,0} \xrightarrow{d} \cdots \\
\hline
\downarrow i_0 & & \downarrow i_1 \\
K^0 \xrightarrow{d_K} K^1 \xrightarrow{d_K} K^2 \xrightarrow{d_K} \cdots
\end{array}$$

called an injective resolution of (K^{\bullet}, d_K) , where

- a) the complex $(I^{p,\bullet}, \delta)$ is an injective resolution of K^p ,
- **b)** the complex $\operatorname{im}(d) \cap I^{p,\bullet}$ is an injective resolution of $\operatorname{im}(d_K) \cap K^p$,
- c) the complex $\ker(d) \cap I^{p,\bullet}$ is an injective resolution of $\ker(d_K) \cap K^p$, and finally
- d) the complex $\underline{H}^{p,\bullet}$ of horizontal cohomology sheaves is an injective resolution of $\underline{H}^p(K^{\bullet})$.

 $^{^4}$ We drop the upper indices from the maps d_K^{ullet} to simplify notation.

Definition 3.1.5 ([Bry]). Let (K^{\bullet}, d_K) be as above with an injective resolution $I^{\bullet \bullet}$. The sheaf hypercohomology group $H^j(X, K^{\bullet})$ is the j-th cohomology group of the total complex

$$\left(\bigoplus_{p+q=j} \Gamma(X, I^{p,q}), D := \delta + (-1)^q d\right).$$

Again, it is far from clear that this definition is unambiguous. The proof that it indeed is, is given in [Bry].

Using sheaf hypercohomology, we can finally define (smooth) Deligne cohomology. Let X be a smooth manifold. We denote by $\underline{U(1)}$ the sheaf of U(1)-valued functions on X and by $\underline{\Omega}^p$ the sheaf of differential p-forms on \overline{X} .

Definition 3.1.6. Let \mathcal{D}^q be the complex of sheaves

$$\mathcal{D}^q := U(1) \xrightarrow{d \log} \underline{\Omega}^1 \xrightarrow{d} \cdots \xrightarrow{d} \underline{\Omega}^q$$

Deligne cohomology groups are the sheaf hypercohomology groups $H^{\bullet}(X, \mathcal{D}^q)$.

In fact, the original definition of Deligne cohomology is given by a different complex of sheaves [Bry]. However, the definition given above yields the same groups and is much more tractable. The original definition would be required, for example, to define the cup product, but we have no need for such advanced cohomological properties of Deligne cohomology. It is enough for us to have a model for the cocycles and be able to compute the groups, as easily as possible. Unfortunately, computing Deligne cohomology groups still seems to be an extremely difficult task. Even normal sheaf cohomology groups are very difficult to compute, since one has to deal with abstruse injective resolutions. Luckily, for such well-behaving spaces as smooth manifolds, there is a beautiful way around these difficulties.

Let A be a sheaf (of Abelian groups) on a smooth manifold X, with a good open cover $\mathcal{U} = \{U_i\}$. To simplify the notation, we write

$$U_{i_0...i_p} := U_{i_0} \cap \ldots \cap U_{i_p}.$$

As usual, the Čech complex $(C^{\bullet}(\mathcal{U}, A), \delta)$ is the complex of Abelian groups

$$C^{p}(\mathcal{U}, A) = \prod_{i_0 \dots i_p} A(U_{i_0 \dots i_p}),$$

together with the Čech coboundary operator $\delta: C^p(\mathcal{U}, A) \to C^{p+1}(\mathcal{U}, A)$ given by

$$\delta(\underline{\alpha})_{i_0...i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j (\alpha_{i_0...i_{j-1}i_{j+1}...i_{p+1}})|_{U_{i_0...i_{p+1}}},$$

for $\underline{\alpha} \in C^p(\mathcal{U}, A)$.

Definition 3.1.7. The Čech cohomology groups of \mathcal{U} with coefficients in the sheaf A, are the cohomology groups $\check{H}^{\bullet}(\mathcal{U}, A)$ of the Čech complex. They certainly seem to depend on the open cover \mathcal{U} . However, it is a classical result that on such a well-behaving space as a smooth manifold, Čech cohomology is independent of the good open cover used [Bry].

Let $C^p(\mathcal{U}, A)$ be the sheaf that associates to an open set $V \in \mathbf{Set}^{\mathbf{o}}(X)$ the group $C^p(\mathcal{U}|_V, A|_V)$ These form, in an obvious way, a complex of sheaves $(C^{\bullet}(\mathcal{U}, A), \delta)$. Moreover, the sequence

$$0 \longrightarrow A \xrightarrow{i} \mathcal{C}^0(\mathcal{U}, A) \xrightarrow{\delta} \mathcal{C}^1(\mathcal{U}, A) \xrightarrow{\delta} \mathcal{C}^2(\mathcal{U}, A) \xrightarrow{} \cdots$$

of sheaves, where $i:A\to \mathcal{C}^0(\mathcal{U},A)$ is the natural morphism, is exact. It is called the $\check{C}ech\ resolution$ of A. If the sheaves $\mathcal{C}^{\bullet}(\mathcal{U},A)$ were injective, the $\check{C}ech\ resolution$ would be an injective resolution and would provide an extremely convenient way of computing sheaf cohomology. Unfortunately, they are *not* injective. In any case, we can form the complex $\Gamma(X,\mathcal{C}^{\bullet}(\mathcal{U},A))$ of groups. Its cohomology group are simply the $\check{C}ech$ cohomology groups $\check{H}^{\bullet}(\mathcal{U},A)$. There is no a priori reason to suspect that $\check{C}ech$ cohomology would have anything to do with sheaf cohomology. However, the best possible thing happens anyway.

Proposition 3.1.8 ([Bry]). Čech cohomology groups are isomorphic to sheaf cohomology groups:

$$\check{H}^{\bullet}(\mathcal{U},A) \cong H^{\bullet}(X,A)$$
.

This changes the very difficult problem of computing sheaf cohomology straight from the definition into the relatively simple combinatorial problem of computing Čech cohomology. To be able to compute Deligne cohomology, we need an analogous result for sheaf hypercohomology.

Simply put, Čech hypercohomology is the total cohomology of the natural extension of the Čech resolution to a complex of sheaves, just as sheaf hypercohomology was an extension of sheaf cohomology to a complex of sheaves. Let (K^{\bullet}, d_K) be a sequence of sheaves on X. Each K^p admits a Čech resolution $\mathcal{C}^{\bullet}(\mathcal{U}, K^p)$, which can be patched together to form a double complex of sheaves. Applying the global sector functor yields a double complex of groups

Definition 3.1.9 ([Bry]). The Čech hypercohomology groups $\check{H}^{\bullet}(\mathcal{U}, K^{\bullet})$ of (K^{\bullet}, d_K) , are the total cohomology groups of (3.4), with respect to the coboundary operator

$$D := \delta + (-1)^q d_K, (3.5)$$

where q denotes the Čech index. Again, a result in [Bry] states that, at least for manifolds, Čech hypercohomology groups are independent of the good open cover used.

Finally, there is the generalization of proposition 3.1.8 to hypercohomology.

Proposition 3.1.10 ([Bry]). Čech hypercohomology groups are isomorphic to sheaf hypercohomology groups:

$$\check{H}^{\bullet}(\mathcal{U}, K^{\bullet}) \cong H^{\bullet}(X, K^{\bullet}).$$

Applying proposition 3.1.10 to the definition 3.1.6 immediately yields the following alternative definition.

Definition 3.1.11. The Deligne cohomology groups $H^{\bullet}(X, \mathcal{D}^q)$ are isomorphic to the total cohomology groups of the double complex

$$\begin{array}{c} \vdots \\ \delta \\ \end{array} \begin{array}{c} C^3(\mathcal{U}, \underline{\Omega}^1) \xrightarrow{-d} \cdots \xrightarrow{-d} C^3(\mathcal{U}, \underline{\Omega}^q) \\ \vdots \\ C^2(\mathcal{U}, \underline{U(1)}) \xrightarrow{d \log} C^2(\mathcal{U}, \underline{\Omega}^1) \xrightarrow{d} \cdots \xrightarrow{d} C^2(\mathcal{U}, \underline{\Omega}^q) \\ \vdots \\ \delta \\ C^1(\mathcal{U}, \underline{U(1)}) \xrightarrow{-d \log} C^1(\mathcal{U}, \underline{\Omega}^1) \xrightarrow{-d} \cdots \xrightarrow{-d} C^1(\mathcal{U}, \underline{\Omega}^q) \\ \vdots \\ \delta \\ C^0(\mathcal{U}, \underline{U(1)}) \xrightarrow{d \log} C^0(\mathcal{U}, \underline{\Omega}^1) \xrightarrow{d} \cdots \xrightarrow{d} C^0(\mathcal{U}, \underline{\Omega}^q) \end{array}$$

with respect to the differentials written therein. The alternating signs arise from (3.5). For example, a part of a cocycle might read

We are particularly interested in diagonal Deligne cohomology groups $H^p(X, \mathcal{D}^p)$. For

example, consider the complex

$$\begin{array}{ccc}
\vdots & & \vdots \\
\delta & & & \delta \\
C^2(\mathcal{U}, \underline{U(1)}) \xrightarrow{d \log} C^2(\mathcal{U}, \underline{\Omega}^1) \\
& & & \delta \\
C^1(\mathcal{U}, \underline{U(1)}) \xrightarrow{d \log} C^1(\mathcal{U}, \underline{\Omega}^1) \\
& & & \delta \\
C^0(\mathcal{U}, \underline{U(1)}) \xrightarrow{d \log} C^0(\mathcal{U}, \underline{\Omega}^1)
\end{array}$$

We denote the set of p-fold intersections of open sets of the cover \mathcal{U} by $X_{\mathcal{U}}^{[p]}$. Note, that $X_{\mathcal{U}}^{[1]}$ is simply the set \mathcal{U} . We shall use the notation $X_{\mathcal{U}} := X_{\mathcal{U}}^{[1]}$. Now, a cohomology class in $H^1(X, \mathcal{D}^1)$ is represented by a pair

$$(\underline{g},\underline{A}) \in \underline{U(1)}(X_{\mathcal{U}}^{[2]}) \oplus \underline{\Omega}^1(X_{\mathcal{U}}),$$

satisfying

$$\delta(g)_{\alpha\beta\gamma} = g_{\alpha\beta}g_{\alpha\gamma}^{-1}g_{\beta\gamma} = 1$$
 and $\delta(\underline{A})_{\alpha\beta} = A_{\beta} - A_{\alpha} = d\log g_{\alpha\beta}$.

In other words, a cocycle $(\underline{g},\underline{A})$ can be geometrically interpreted as a principal U(1) bundle $P \to X$, with transition functions $\{g_{\alpha\beta}\}$ and connection $\{A_{\alpha}\}$, or equivalently as the associated complex line bundle with connection. The coboundaries are pairs

$$(\delta(\underline{f})_{\alpha\beta}, d\log f_{\alpha}) = (f_{\beta}f_{\alpha}^{-1}, d\log f_{\alpha}).$$

Adding such terms to $(\underline{g},\underline{A})$ corresponds precisely to performing a bundle isomorphism and a corresponding gauge transformation. Hence, elements of $H^1(X,\mathcal{D}^1)$ correspond to isomorphism classes of complex hermitian lines bundles with connection. Recall that the first Chern class of the line bundle is the image of g under the Bockstein homomorphism

$$\beta: H^1(X, \underline{U(1)}) \to H^2(X, \mathbb{Z})$$

and that its image under the inclusion $H^2(X,\mathbb{Z}) \to H^2_{\mathrm{dR}}(X)$ coincides with the cohomology class of $d\underline{A}$. The first Chern class of this line bundle is called the *Dixmier-Douady class* of the Deligne class.

3.2 Bundle gerbes

It will become clear during this chapter that a Dirac quantized B-field is described by a class in $H^2(X, \mathcal{D}^2)^5$. Dirac quantized electromagnetic fields, which are cohomologically classes in $^1(X, \mathcal{D}^1)$, correspond geometrically to principal U(1) bundles with connection. Likewise,

⁵More correctly, it is a class in a differential cohomology theory, which happens to coincide with $H^2(X, \mathcal{D}^2)$, but we have no need for such a general point of view.

having a geometric model for classes of $H^2(X, \mathcal{D}^2)$ provides new insight to the case of the B-field.

There exist several useful realizations of $H^2(X, \mathcal{D}^2)$. One of them is the theory of gerbes developed by Giraud [Bry, Moe, Hit]. However, Giraud's gerbes are not very well suited for our purpose and we shall instead employ a bundle theoretic realization by Murray [Mur], called bundle gerbes⁶ with bundle gerbe connection and curving. Our references on bundle gerbes are [Mur, BCMMS, CJM, Joh2, Ste2].

Let X and Y be smooth manifolds, $\mathcal{U} = \{U_{\alpha}\}$ be a good open cover of X, $\pi: Y \to X$ a smooth, locally split map, with local sections $s_{\alpha}: U_{\alpha} \to X$. The p-fold fibre product $Y^{[p]}$ is defined as

$$Y^{[p]} = \{(y_1, \dots, y_p) \in Y \times \dots \times Y : \pi(y_1) = \dots = \pi(y_p)\}.$$

We also need the maps $\pi_i: Y^{[p]} \to Y^{[p-1]}$ which simply omit the *i*-th element:

$$\pi_i(y_1,\ldots,y_p) = (y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_p).$$

Definition 3.2.1 ([Mur]). Let $\pi: Y \to X$ be a locally split map and $L \to Y^{[2]}$ a complex line bundle, with a bundle isomorphism

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \xrightarrow{\cong} L_{(y_1,y_3)}$$
,

called the bundle gerbe product. Note, that the isomorphism is between L and the restriction of $L \otimes L \to Y^{[2]} \times Y^{[2]}$ over

$$Y^{[2]} \circ Y^{[2]} = \{(y_1, y_2, y_3, y_4) \in Y^{[2]} \times Y^{[2]} | y_2 = y_3\}.$$

The product is required to be associative, whenever triple product are defined. It is shown in [Mur] that the existence of the product necessarily implies $L_{(y,y)} \cong \mathbb{C}$ and $L_{(y_1,y_2)} \cong L^*_{(y_2,y_1)}$, where L^* is denotes dual bundle. The triple (L,Y,X) is called a U(1) bundle gerbe and is represented diagrammatically by

$$\downarrow \\
Y^{[2]} \xrightarrow{\pi_1} Y \\
\downarrow \\
X$$

To keep the notations simple, we denote (L, Y, X) by (L, Y) or L, when the omitted objects are clear from the context.

Let (L, Y, X) be a bundle gerbe, $Y' \to X'$ locally split and $\phi: X' \to X$ a map with a lift

$$Y' \xrightarrow{\widehat{\phi}} Y .$$

$$\downarrow \qquad \qquad \downarrow$$

$$X' \xrightarrow{\phi} X$$

⁶More precisely, U(1) bundle gerbes. However, we call them simply bundle gerbes, since all bundle gerbes we are interested in are of this type.

Now, $\widehat{\phi}$ induces a map $\widehat{\phi}^{[2]}: Y'^{[2]} \to Y^{[2]}$, which can be used to pull back $L \to Y^{[2]}$ into $(\widehat{\phi}^{[2]})^{-1}L \to Y'^{[2]}$. The bundle gerbe $\phi^{-1}L := ((\widehat{\phi}^{[2]})^{-1}(L), Y', X')$ is called the *pullback of the bundle gerbe L*. There are two important special cases of this construction. The first is that we are given a map $\phi: X' \to X$ and take as Y' the pullback $\phi^{-1}(Y)$. The second is the case where $\phi: X \to X$ is the identity map. Let (L, Y) be a bundle gerbe. We can then define another bundle gerbe (L^*, Y) , called the *dual of* (L, Y), where L^* is the dual of L. If (L, Y, X) and (J, Z, X) are two bundle gerbes, there is a *product bundle gerbe* $(L \otimes J, Y \times_{\pi} Z)$, where $Y \times_{\pi} Z := \{(y, z) \in Y \times Z : \pi_Y(y) = \pi_Z(z)\}$.

Let $J \to Y$ be a complex line bundle. We denote by $(\delta(J), Y)$ the bundle gerbe $(\pi_1^{-1}(J) \otimes \pi_2^{-1}(J)^*, Y)$, where $\delta(J)_{(y_1, y_2)} = J_{y_2} \otimes J_{y_1}^*$, with the natural pairing

$$\delta(J)_{(y_1,y_2)} \otimes \delta(J)_{(y_2,y_3)} = J_{y_2} \otimes J_{y_1}^* \otimes J_{y_3} \otimes J_{y_2}^* \to J_{y_3} \otimes J_{y_1}^* = \delta(J)_{(y_1,y_3)},$$

as the bundle gerbe product.

Let $f: Y \to Z$ be a smooth map, $f^{[2]}: Y^{[2]} \to Z^{[2]}$ the induced map on the fibre products and $g: L \to J$ a bundle morphism, covering $f^{[2]}$. The pair (g, f) defines a bundle gerbe morphism $(L, Y) \to (J, Z)$, if it commutes with the bundle gerbe products⁷. If f and g are isomorphisms, (g, f) is called a bundle gerbe isomorphism. A trivial bundle gerbe is a bundle gerbe (L, Y), which is isomorphic to some $(\delta(J), Y)$. The line bundle J together with the isomorphism $(L, Y) \cong (\delta(J), Y)$ is called a trivialization of (L, Y).

Bundle gerbes are completely classified, up to a certain equivalence relation, by $H^3(X,\mathbb{Z})$, just like isomorphism classes of complex line bundles are classified by $H^2(X,\mathbb{Z})$. Unfortunately, bundle gerbe isomorphism is not the correct equivalence relation. Instead, we require the concept of *stable isomorphism*. Two bundle gerbes, say L and J, are said to be stably isomorphic, if there exists a complex line bundle K, such that $L \cong J \otimes \delta(K)$. The bundle K is referred to as the stable isomorphism. It is easy to see that any two stable isomorphisms, K and K, between bundle gerbes K and K, must be related by $K = H \otimes \pi^{-1}E$, where $K \to K$ is a complex line bundle.

Proposition 3.2.2 ([Mur]). The group of stable isomorphism classes of bundle gerbes over X, with group operator the product of bundle gerbes, is isomorphic to $H^3(X,\mathbb{Z})$. The cohomology class corresponding to a particular bundle gerbe, or its stable equivalence class, is referred to as the Dixmier-Douady class.

The Dixmier-Douady class of a bundle gerbe (L, Y, X) is constructed as follows. Let $\mathcal{U} = \{U_{\alpha}\}$ be a good open cover of X, with local sections $s_{\alpha} : U_{\alpha} \to Y$ of $Y \to X$. On double intersections $U_{\alpha\beta}$, these can be combined into sections $(s_{\alpha}, s_{\beta}) : U_{\alpha\beta} \to Y^{[2]}$, which can then be used to pull $L \to Y^{[2]}$ back onto each $U_{\alpha\beta}$, yielding a set of U(1) vector bundles

$$L_{\alpha\beta} := (s_{\alpha}, s_{\beta})^{-1} L \to U_{\alpha\beta}$$
.

Because \mathcal{U} is a good cover, $U_{\alpha\beta}$ are all contractible and hence $L_{\alpha\beta}$ are trivial bundles. They admit global sections $\sigma_{\alpha\beta}: U_{\alpha\beta} \to L_{\alpha\beta}$. The bundle gerbe product $L_{(\alpha,\beta)} \otimes L_{(\beta,\gamma)} \to L_{(\alpha,\gamma)}$ induces a bundle isomorphism $L_{\alpha\beta} \otimes L_{\beta\gamma} \to L_{\alpha\gamma}$, which can be used to patch $\sigma_{\alpha\beta}$ together on triple overlaps:

$$\sigma_{\alpha\beta}\otimes\sigma_{\beta\gamma}=\sigma_{\alpha\gamma}g_{\alpha\beta\gamma}\,,$$

⁷More generally, a bundle gerbe morphism can be defined between bundle gerbes over different base spaces, in which case we would need a third map between the base spaces, with fibre preserving lift f. However, we have no need for such general morphisms.

where $g_{\alpha\beta\gamma}$ are functions $U_{\alpha\beta\gamma} \to U(1)$. On quadruple overlaps

$$\sigma_{\alpha\beta}\otimes\sigma_{\beta\gamma}\otimes\sigma_{\gamma\delta}=\sigma_{\alpha\gamma}\otimes\sigma_{\gamma\delta}g_{\alpha\beta\gamma}=\sigma_{\alpha\gamma}g_{\alpha\beta\gamma}g_{\alpha\gamma\delta}\,,$$

but on the other hand

$$\sigma_{\alpha\beta}\otimes\sigma_{\beta\gamma}\otimes\sigma_{\gamma\delta}=\sigma_{\alpha\beta}\otimes\sigma_{\beta\delta}g_{\beta\gamma\delta}=\sigma_{\alpha\gamma}g_{\alpha\beta\delta}g_{\beta\gamma\delta}.$$

Thus, we conclude that $\{g_{\alpha\beta\gamma}\}$ satisfy the Čech cocycle condition

$$g_{\alpha\beta\gamma}^{-1}g_{\alpha\beta\delta}g_{\alpha\gamma\delta}^{-1}g_{\beta\gamma\delta} = 1,$$

and thus define an element $[g] \in H^2(X, U(1))$.

The exact sequence of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{R}} \longrightarrow U(1) \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow H^2(X,\underline{\mathbb{R}}) \longrightarrow H^2(X,\underline{U}(1)) \xrightarrow{\ \beta \ } H^3(X,\underline{\mathbb{Z}}) \cong H^3(X,\mathbb{Z}) \longrightarrow H^3(X,\underline{\mathbb{R}}) \longrightarrow \cdots$$

of cohomology groups. It is easy to show using a partition of unity argument that $H^p(X, \mathbb{R})$ vanish for p > 0 [Bry] and, hence, that the Bockstein homomorphism

$$\beta: H^2(X, \underline{U(1)} \to H^3(X, \mathbb{Z})$$
(3.6)

is actually an isomorphism⁸.

The Dixmier-Douady class of (L, Y), which we denote by d(L) or d(L, Y), is the image of $[\underline{g}]$ under (3.6). We sometimes also call $[\underline{g}]$ the Dixmier-Douady class, hoping this will not cause any confusion. The Dixier-Douady class satisfies [Mur]

$$d(L^*) = -d(L)$$
 and $d(L \otimes J) = d(L) + d(J)$.

It also behaves naturally under pullbacks of bundle gerbes, namely

$$d(\phi^{-1}L, Z) = \phi^* d(L, Y).$$

 $Lifting\ bundle\ gerbes$ are one of the most important examples of bundle gerbes. Consider the central extension

$$U(1) \longrightarrow \widehat{G} \stackrel{\pi}{\longrightarrow} G \tag{3.7}$$

$$\exp(i\delta(\rho)) = \delta(\exp(i\rho)) = \delta(g) = 1$$
,

but this is possible if and only if

$$\delta(\rho) = 2\pi \underline{n}$$
,

for an integer-valued Čech 3-cochain \underline{n} . It is obviously a cocycle due to $\delta^2 = 0$. The image $\beta([\underline{g}]) \in H^3(X, \mathbb{Z})$ is precisely the class $[\underline{n}] \in H^3(X, \mathbb{Z})$.

⁸A more concrete description is given as follows. Writing $\underline{g} = \exp(i\underline{\rho})$, where $\underline{\rho}$ is a real-valued Čech 2-cochain, and using the cocycle condition $\delta(\underline{g}) = 1$, yields

of Lie groups. We ask, what the obstruction for lifting a principal G bundle $P \to X$ into a principal \widehat{G} bundle is? In other words, given a principal G bundle P, with transition functions $\{g_{\alpha\beta}\}$, exactly when is it possible to find a principal \widehat{G} bundle, with transition functions $\{\widehat{g}_{\alpha\beta}\}$, such that $\pi(\widehat{g}_{\alpha\beta}) = g_{\alpha\beta}$? First, consider the central extension (3.7) as a principal U(1) bundle $\widehat{G} \to G$. Next, define a map $g: P^{[2]} \to G$ by $p_1g(p_1, p_2) = p_2$ and pull $\widehat{G} \to G$ back into a principal U(1) bundle over $P^{[2]}$. The fibre of the pullback over a point (p_1, p_2) is $\{\widehat{g} \in \widehat{G}: p_1\pi(\widehat{g}) = p_2\}$. Finally, form the associated complex line bundle of the U(1) principal bundle. The product structure on \widehat{G} induces the bundle gerbe product for the associated line bundle. The resulting bundle gerbe is called the lifting bundle gerbe of $P \to X$. It is trivial precisely when the aforementioned lift exists.

We are particularly interested in lifting bundle gerbes of central extensions

$$U(1) \longrightarrow U(n) \longrightarrow PU(n)$$

and

$$U(1) \longrightarrow U(\mathcal{H}) \longrightarrow PU(\mathcal{H})$$
,

where \mathcal{H} is the infinite-dimensional separable Hilbert space, $U(\mathcal{H})$ the unitary group and $PU(\mathcal{H}) := U(\mathcal{H})/U(1)$ the projective unitary group. In D-brane theory, a stack of n D-branes carries a PU(n) Chan-Paton bundle. As we have already demonstrated, the possibility of lifting the PU(n) bundle into a U(n) bundle is crucial for the K-theoretic classification. As we already briefly mentioned in the beginning of this chapter, the infinite-dimensional case arises from the Freed-Witten anomaly cancellation condition, when the H-flux restricts to a nontorsion element on the D-brane worldvolume.

3.3 Bundle gerbe connections

The reason for the introduction of bundles gerbes, was to obtain a purely geometric description of $H^2(X, \mathcal{D}^2)$, classes of which are represented by cocycles

$$(g,\underline{\Lambda},\underline{B}) = (\{g_{\alpha\beta\gamma}\},\{\Lambda_{\alpha\beta}\},\{B_{\alpha}\}) \in (C^2(\mathcal{U},U(1)),C^1(\mathcal{U},\underline{\Omega}^1),C^0(\mathcal{U},\underline{\Omega}^2)).$$

As the notation suggests, the first entry on the left-hand side is the Dixmier-Douady class. We need to add more structure to bundle gerbes to obtain a model for the full Deligne cohomology class.

Definition 3.3.1 ([Mur]). A bundle gerbe connection is a connection ∇ on $L \to Y^{[2]}$, with the additional requirement that it commutes with the bundle gerbe product in the following sense. For any section $t_{12} \otimes t_{23} \in \Gamma(Y^{[2]} \circ Y^{[2]}, L \otimes L)$, the connection satisfies

$$\nabla(t_{12} \otimes t_{23}) = \nabla t_{12} \otimes 1 + 1 \otimes \nabla t_{23} = \nabla t_{13}, \tag{3.8}$$

where $t_{13} = t_{12} \otimes t_{23}$ is given by the bundle gerbe product.

It is not immediately obvious that bundle gerbe connections exist, but [Mur] provides the following simple argument. Let $J \to Y$ be a U(1) bundle and $\delta(J)$ the corresponding trivial bundle gerbe. It is a classical result that J admits a connection ∇ inducing the tensor product connection $\nabla \otimes 1 + 1 \otimes \nabla^*$ on the bundle $\pi_1^{-1}(J) \otimes \pi_2^{-1}(J)^* \to Y^{[2]}$, where ∇^* denotes the dual

connection. It is straightforward to verify that it commutes with the bundle gerbe product in the sense of (3.8). This can be extended to nontrivial bundle gerbes by a simple partition of unity argument. For each open set U_{α} of X with a local section s_{α} of Y, we denote by Y_{α} the open subset $\pi^{-1}(U_{\alpha}) \subset Y$. Any bundle gerbe, say (L, Y, X), restricted to Y_{α} , is trivial. We already argued that there exist bundle gerbe connections for trivial bundle gerbes. By choosing a partition of unity subordinate to the open covering $\{U_{\alpha}\}$ and pulling it back onto Y, we can form a partition of unity for $Y^{[2]}$ subordinate to $\{U_{\alpha}^{[2]}\}$, which can finally be used to patch together the local bundle gerbe connections to form a bundle gerbe connection for L.

The curvature 2-form of a bundle gerbe connection ∇ is denoted by F_{∇} .

Proposition 3.3.2 ([Mur]). The complex

$$0 \longrightarrow \Omega^{q}(X) \xrightarrow{\pi^{*}} \Omega^{q}(Y) \xrightarrow{\delta} \Omega^{q}(Y^{[2]}) \xrightarrow{\delta} \Omega^{q}(Y^{[3]}) \xrightarrow{\delta} \cdots$$
 (3.9)

where $\delta(\eta) := \sum_{i=1}^{p} (-1)^{i} \pi_{i}^{*}(\eta)$, is exact everywhere.

Since $\delta(F_{\nabla}) = 0$, exactness of (3.9), there exists a 2-form $f \in \Omega^2(Y)$, such that

$$F_{\nabla} = \delta(f) = \pi_2^*(f) - \pi_1^*(f)$$
.

The 2-form f is called the *curving* for the bundle gerbe connection ∇ . The choice of a curving is obviously not unique. Any two curvings, f_1 and f_2 , for the same connection ∇ , differ by a pullback of a 2-form on X. Choosing a curving f for ∇ , we have

$$\delta(df) = d\delta(f) = dF_{\nabla} = 0$$
.

Exactness of (3.9) implies $df = \pi^*(\omega)$. The 3-form $\omega \in \Omega^3(X)$ is called the 3-curvature and $\omega/2\pi i$ the Dixmier-Douady form of the pair (∇, f) . The Dixmier-Douady form is closed, because $\pi^*(d\omega) = d(\pi^*\omega) = d^2f = 0$. In fact, the de Rham cohomology class of ω coincides with the image of the Dixmier-Douady class under the inclusion $H^3(X, \mathbb{Z}) \to H^3(X, \mathbb{R})$ [Mur].

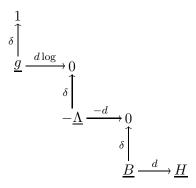
Using s_{α} to pull $f \in \Omega^{2}(Y)$ back onto X, yields 1-forms $B_{\alpha} := s_{\alpha}^{*}(f)$. Recall that the trivial line bundles $L_{\alpha\beta} \to U_{\alpha\beta}$ were the pullbacks of L by (s_{α}, s_{β}) . The bundle gerbe connection ∇ is also pulled back by (s_{α}, s_{β}) to form connections $\nabla_{\alpha\beta}$ on $L_{\alpha\beta}$, which define a set of 1-forms $\Lambda_{\alpha\beta} \in \Omega^{1}(Y^{[2]})$ by

$$\nabla_{\alpha\beta}\sigma_{\alpha\beta} = \Lambda_{\alpha\beta} \otimes \sigma_{\alpha\beta} \,. \tag{3.10}$$

We then need to extend the concept of stable isomorphism to bundle gerbes with connection and curving. Recall, how a bundle gerbe (L,Y) was said to be trivial if it was induced by a line bundle $J \to Y$. Now, consider the line bundle $J \to Y$ equipped with a connection ∇ and curvature F. As before, we obtain a trivial bundle gerbe $(\delta(J),Y)$ but this time we also obtain a trivial bundle gerbe connection $\delta(\nabla) := \pi_2^*(\nabla) - \pi_1^*(\nabla)$ and its corresponding trivial curving F. Two bundle gerbes with connection and curving are called stably isomorphic, if they differ by a trivial bundle gerbe with connection and curving.

The connection between bundle gerbes and Deligne cohomology is the following.

Theorem 3.3.3 ([Mur]). Given a bundle gerbe with connection and curving $(L, Y, X; \nabla, f)$, the triple $(g, -\underline{\Lambda}, \underline{B})$ is a Deligne cocycle:



Moreover, the mapping

$$(L, Y, X; \nabla, f) \mapsto [g, -\underline{\Lambda}, \underline{B}]$$

is an isomorphism between the group of stable isomorphism classes of bundle gerbes with connection and curving and the Deligne cohomology group $H^2(X, \mathcal{D}^2)$.

Proof. We already showed that $\delta(\underline{g}) = 1$. According to (3.10),

$$(\nabla_{\alpha\beta} \otimes 1 + 1 \otimes \nabla_{\beta\gamma})\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma} = (\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma}) \otimes \sigma_{\alpha\gamma}g_{\alpha\beta\gamma}. \tag{3.11}$$

On the other hand, bundle gerbe connections were required to commute with the product in the sense of (3.8), which induces similar commutativity conditions for $\{\nabla_{\alpha\beta}\}$ and thus implies

$$(\nabla_{\alpha\beta} \otimes 1 + 1 \otimes \nabla_{\beta\gamma})\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma} = \nabla_{\alpha\gamma}(\sigma_{\alpha\gamma}g_{\alpha\beta\gamma})$$

$$= dg_{\alpha\beta\gamma} \otimes \sigma_{\alpha\gamma} + \Lambda_{\alpha\gamma} \otimes \sigma_{\alpha\gamma}g_{\alpha\beta\gamma}$$

$$= (d \log g_{\alpha\beta\gamma} + \Lambda_{\alpha\gamma}) \otimes \sigma_{\alpha\gamma}g_{\alpha\beta\gamma}.$$
(3.12)

Comparing (3.11) and (3.12) yields the transformation property

$$\Lambda_{\beta\gamma} - \Lambda_{\alpha\gamma} + \Lambda_{\alpha\beta} = d \log g_{\alpha\beta\gamma}$$
.

Finally, recall that $F_{\nabla} = \pi_2^* f - \pi_1^* f$. The sections $(s_{\alpha}, s_{\beta}) : U_{\alpha\beta} \to Y$ can be used to pull F_{∇} back to curvature 2-forms

$$F_{\alpha\beta} = (s_{\alpha}, s_{\beta})^* F_{\nabla} = (s_{\alpha}, s_{\beta})^* (\pi_2^* f - \pi_1^* f) = s_{\alpha}^* f - s_{\beta}^* f = B_{\alpha} - B_{\beta},$$

for the line bundles $L_{\alpha\beta}$. Using $F_{\alpha\beta} = d\Lambda_{\alpha\beta}$, we finally get

$$d\Lambda_{\alpha\beta} = B_{\alpha} - B_{\beta} .$$

This proves the first claim, that $(\underline{g}, -\underline{\Lambda}, \underline{B})$ is a Deligne cocycle. For the second part of the proof, we refer the reader to [MS1].

3.4 Bundle gerbe modules

In this section we shall describe yet more extra structure for bundle gerbes, namely bundle gerbe modules and bundle gerbe module connections. These were originally introduced in [BCMMS], which is our reference for what follows⁹.

Definition 3.4.1 ([BCMMS]). Let (L, Y, X) be a bundle gerbe and $E \to Y$ a U(n) vector bundle. Suppose there exists a bundle isomorphism

$$\phi: L \otimes \pi_1^{-1} E \xrightarrow{\cong} \pi_2^{-1} E \,, \tag{3.13}$$

which is compatible with the bundle gerbe product in the sense of the commutative diagram

$$L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \otimes E_{y_3} \xrightarrow{\operatorname{id} \otimes \phi} L_{(y_1,y_2)} \otimes E_{y_2}$$

$$\downarrow \qquad \qquad \downarrow \phi$$

$$L_{(y_1,y_3)} \otimes E_{y_3} \xrightarrow{\phi} E_{y_1}$$

Then bundle E is called a (finite rank) bundle gerbe module. The bundle gerbe (L, Y) is said to act on E by (3.13).

There is a natural concept of isomorphism for bundle gerbe modules. Two bundle gerbe modules are said to be isomorphic, if they are isomorphic as vector bundles and the isomorphism commutes with the action ϕ . If E and F are bundle gerbe modules for L, the direct sum $E \oplus F$ is also a bundle gerbe module for L and thus defines a monoid structure on the set Mod(L) of isomorphism classes of (finite rank) bundle gerbe modules for L.

Proposition 3.4.2 ([BCMMS]). A stable isomorphism of (L,Y) and (J,Y) induces a semi-group isomorphism $Mod(L) \cong Mod(J)$. The isomorphism is not canonical, however, and a change in the stable isomorphism by tensoring with $\pi^{-1}K^{10}$, results in a change of the isomorphism by composition with the endomorphism of Mod(J) induced by tensoring with $\pi^{-1}K$.

Proposition 3.4.3. If (L,Y) has a bundle gerbe module $E \to Y$ of finite rank n, then the Dixmier-Douady class d(L) is n-torsion, that is, n d(L) = 0.

Proof. If $\operatorname{rk} E=1$, then clearly E is a trivialization of L. If $\operatorname{rk} E=n$, then $L^{\otimes n}$ acts on the rank 1 bundle $\Lambda^n E$. Thus $L^{\otimes n}$ is necessarily trivial, and using the properties of the Dixmier-Douady class we immediately obtain $d(L^{\otimes n})=n\,d(L)=0$.

Definition 3.4.4 ([BCMMS]). A connection ∇^E on a (finite rank) bundle gerbe module $E \to Y$ for a bundle gerbe (L,Y) with connection ∇ , is called a bundle gerbe module connection, if ∇^E and ∇ are compatible in such a way that the induced connections on $L \otimes \pi_1^{-1}E$ and $\pi_2^{-1}E$ are equal under the isomorphism ϕ , defining the module structure. It can be shown using a partition of unity argument, somewhat similar to the one used to show that bundle gerbe connections exist, that also bundle gerbe module connections always exist [BCMMS].

 $^{^9}$ The main result in [BCMMS] was the formulation of twisted K-theory in terms of bundle gerbe modules. We discuss twisted K-theory only at the end of this chapter.

¹⁰Recall that any two stable isomorphisms differ necessarily by a tensor product with the pullback of a line bundle $K \to X$.

It would be interesting to see what bundle gerbe modules look like locally. Let $(L, Y; \nabla, f)$ be a bundle gerbe with connection and curving, with associated Deligne class $[\underline{g}, -\underline{\Lambda}, \underline{B}]$. Let $\{s_{\alpha}\}$ be local sections $s_{\alpha}: U_{\alpha} \to Y$ of $Y \to X$. A bundle gerbe module $E \to Y$ of rank n can be pulled back into trivial bundles $E_{\alpha} = s_{\alpha}^{-1}(E)$. The isomorphism (3.13) descends into a bundle isomorphism

$$L_{\alpha\beta}\otimes E_{\beta} \xrightarrow{\cong} E_{\alpha}$$
.

Let $\{\delta_{\alpha}: U_{\alpha} \to E_{\alpha}\}$ be global sections of $\{E_{\alpha}\}$. The above isomorphisms then define a set $\{h_{\alpha\beta}\}$ of $n \times n$ matrices by

$$\sigma_{\alpha\beta}\otimes\delta_{\beta}=\delta_{\alpha}h_{\alpha\beta}\,,$$

which satisfy a g-twisted cocycle condition. This is easily verified by considering a section

$$\sigma_{\alpha\beta}\otimes\sigma_{\beta\gamma}\otimes\delta_{\gamma}\in\Gamma(X,L_{\alpha\beta}\otimes L_{\beta\gamma}),$$

which can be simplified using the bundle gerbe product and the bundle gerbe module structure in two ways which, ultimately, must yield equal results. Namely,

$$\sigma_{\alpha\beta} \otimes \sigma_{\beta\gamma} \otimes \delta_{\gamma} = \sigma_{\alpha\gamma} g_{\alpha\beta\gamma} \otimes \delta_{\gamma} = \delta_{\alpha} h_{\alpha\gamma} g_{\alpha\beta\gamma} ,$$

but on the other hand

$$\sigma_{\alpha\beta}\otimes\sigma_{\beta\gamma}\otimes\delta_{\gamma}=\sigma_{\alpha\beta}\otimes\delta_{\beta}h_{\beta\gamma}=\delta_{\alpha}h_{\alpha\beta}h_{\beta\gamma}.$$

In order for these to be equal, $\{h_{\alpha\beta}\}$ must satisfy the g-twisted cocycle condition

$$h_{\alpha\beta}h_{\beta\gamma}h_{\alpha\gamma}^{-1} = g_{\alpha\beta\gamma}. \tag{3.14}$$

We denote the pullback connections $s_{\alpha}^{-1}(\nabla^{E})$ by ∇_{α}^{E} and define

$$\nabla^E_{\alpha}\delta_{\alpha} = A_{\alpha} \otimes \delta_{\alpha} .$$

A standard calculation using the definition of a bundle gerbe module connection reveals that on double intersections $U_{\alpha\beta}$, $\{A_{\alpha}\}$ and $\{A_{\beta}\}$ must be related by

$$A_{\beta} = h_{\alpha\beta}^{-1} A_{\alpha} h_{\alpha\beta} + h_{\alpha\beta}^{-1} dh_{\alpha\beta} - \Lambda_{\alpha\beta} . \tag{3.15}$$

While the \underline{g} -twisted transition functions $\{h_{\alpha\beta}\}$ do not define a U(n) bundle, their images in PU(n) do define a PU(n) bundle $P \to X$. Let us now form the lifting bundle gerbe for $P \to X$ and denote it by (J,P). Its Dixmier-Douady class is the obstruction for lifting the PU(n) bundle P into a U(n) bundle, which is simply the failure of \underline{g} being a coboundary. However, the image of $[\underline{g}]$ under the isomorphism $H^2(X,\underline{U(1)}) \xrightarrow{\cong} H^3(X,\mathbb{Z})$ is precisely the Dixmier-Douady class of the original bundle gerbe we started with. Therefore, the Dixmier-Douady class of (J,P) is the same as that of the original bundle gerbe (L,Y). Thus, (J,P) and (L,Y) are stably isomorphic. The lifting bundle gerbe admits a trivial bundle gerbe module $P \times \mathbb{C}^{n+1}$. The isomorphism of proposition 3.4.2 relates the two bundle gerbe modules. Different stable isomorphisms between (L,Y) and (J,P) yield different bundle gerbe modules for (L,Y), but by proposition 3.4.2, they differ only by a tensor product with the pullback of a line bundle over X. Repeating the construction of the projective bundle for any of these bundle gerbe modules results in PU(n) bundles isomorphic to $P \to X$. We have proved the following.

¹¹Recall, how the lifting bundle gerbe was constructed: the fibre of J at a point $(p_1, p_2) \in P^{[2]}$ is $\{g \in U(n) : p_1\pi(g) = p_2\}$, where $\pi: U(n) \to PU(n)$ is the natural projection. The bundle gerbe acts on the module $P \times \mathbb{C}^n$ simply by the natural action of U(n) on \mathbb{C}^n .

Proposition 3.4.5 ([BCMMS]). Let Lin(X) denote the group of isomorphism classes of line bundles over X and $Proj(X, \sigma)$ the set of isomorphism classes of finite rank projective bundle with Dixmier-Douady class σ . If (L, Y) is a bundle gerbe over X with $d(L) = \sigma$, then the above construction defines a bijection

$$\frac{\operatorname{Mod}(L)}{\operatorname{Lin}(X)} \to \operatorname{Proj}(X, \sigma).$$

According to proposition 3.4.3, a bundle gerbe with a nontorsion Dixmier-Douady class can not have a finite rank bundle gerbe module. The obvious solution is to allow for bundle gerbe modules to be bundles of infinite-dimensional Hilbert spaces. The natural topology of $U(\mathcal{H})$ as the structure group of a principal $U(\mathcal{H})$ bundle is the strong operator topology [HJJS]. In principal $PU(\mathcal{H})$ bundles $PU(\mathcal{H})$ is endowed with the quotient topology of $U(\mathcal{H})/U(1)$. Then, $PU(\mathcal{H})$ acts continuously on \mathcal{K} by conjugation with unitary operators. Moreover, $Aut(\mathcal{K}) \cong PU(\mathcal{H})$ [CRM].

Lemma 3.4.6. Isomorphism classes of principal $PU(\mathcal{H})$ bundles, over X are in bijective correspondence with classes of $H^3(X,\mathbb{Z})$.

Proof. A famous theorem by Kuiper [Kui] states that $U(\mathcal{H})$ is contractible in the strong operator topology. The central extension

$$U(1) \longrightarrow U(\mathcal{H}) \longrightarrow PU(\mathcal{H})$$

defines a principal U(1) bundle with contractible total space. The long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_2(U(\mathcal{H})) \longrightarrow \pi_2(PU(\mathcal{H})) \longrightarrow \pi_1(U(1)) \longrightarrow \pi_1(U(\mathcal{H})) \longrightarrow \cdots$$

yields the isomorphisms $\pi_2(PU(\mathcal{H})) \cong \pi_1(U(1)) \cong \mathbb{Z}$ and $\pi_k(PU(\mathcal{H})) = 0$, for all $k \neq 2$. Hence, $PU(\mathcal{H})$ is a model for the Eilenberg-MacLane space $K(\mathbb{Z},2)$ [May]. Applying the long exact sequence of homotopy groups to the universal $PU(\mathcal{H})$ bundle $EPU(\mathcal{H}) \to BPU(\mathcal{H})$ and using the contractibility of $EPU(\mathcal{H})$, yields the isomorphisms

$$\pi_{k+1}(BPU(\mathcal{H})) \cong \pi_k(PU(\mathcal{H})) = \pi_k(K(\mathbb{Z},2)).$$

This proves that $BPU(\mathcal{H})$ is a model for $K(\mathbb{Z},3)$. Therefore,

$$H^3(X,\mathbb{Z}) \cong [X,K(\mathbb{Z},3)] = [X,BPU(\mathcal{H})],$$

which proves the claim. The class of $H^3(X,\mathbb{Z})$ corresponding to a $PU(\mathcal{H})$ bundle is called the *Dixmier-Douady class* of the bundle.

Let (L,Y) be a bundle gerbe, with possibly nontorsion Dixmier-Douady class d(L). By lemma 3.4.6, there is a $PU(\mathcal{H})$ bundle $P \to X$, with lifting bundle gerbe (J,P), such that $d(J) = d(L)^{12}$. Now, let $E \to P$ be a bundle of infinite-dimensional separable Hilbert spaces \mathcal{H} . There is an associated principal $U(\mathcal{H})$ bundle $U(E) \to P$, whose fibre over $p \in P$ is the group of isomorphisms $E_p \cong \mathcal{H}$. For $f \in U(E)_p$, the right action of $u \in U(\mathcal{H})$ is given by $fu := f \circ u$.

¹²Remark, that this approach works even when d(L) is torsion.

Definition 3.4.7 ([BCMMS]). Suppose that the structure group of U(E) reduces to $U_1 \subset U(\mathcal{H})$, the subgroup of unitary operators of the form 1+ trace class. We denote the principal U_1 subbundle of U(E) by $R \to P$. If $u \in U(\mathcal{H})$, we denote its projectivization by $[u] \in PU(\mathcal{H})$. If $p_1, p_2 \in P$ and $p_1[u] = p_2$, then by the construction of the lifting bundle gerbe, $u \in J_{p_1,p_2}$. We then require that the map $U(E)_{p_1} \to U(E)_{p_2}$, sending f to ufu^{-1} , preserves R. That is, for any $f \in R_{p_1}$ and $u \in J_{p_1,p_2}$, $ufu^{-1} \in R_{p_2}$. The bundle $E \to P$ is then called a U_1 bundle gerbe module. A stable isomorphism between (L,Y) and (J,P) can then be used to carry the U_1 bundle gerbe modules for (J,P) to (L,Y). The monoid of U_1 bundle gerbe modules for L is denoted by $Mod_{U_1}(L)$.

Finally, one can define bundle gerbe module connections for E in analogy with the finite dimensional case.

We shall now briefly comment on how all the mathematical constructions discussed so far relate to the theory of D-branes. Our goal is to make the following claim precise and show that it is consistent with the physical theory of D-branes, as far as we have discussed it.

Claim. Suppose $[H]|_{\Sigma}$ is torsion. The Dirac quantized background B-field is a bundle gerbe (L_B, Y, X) with connection and curving. The A-field, living on the D-brane worldvolume Σ , is a (finite rank) bundle gerbe module connection for a certain bundle gerbe on Σ . If $[H]|_{\Sigma}$ is nontorsion, the A-field is a U_1 bundle gerbe module connection.

In fact, one does not even need such complicated objects as bundle gerbes are to treat the case of torsion $[H]|_{\Sigma}$. In [Kap], Kapustin solved the anomaly cancellation problem in the torsion case, using the theory of Azumaya algebras and Azumaya algebra modules, in place of bundle gerbes and bundle gerbe modules. However, the approach of Kapustin can not be extended to treat the case of nontorsion $[H]|_{\Sigma}$. The bundle gerbal approach was first presented in [CJM], where the anomaly cancellation problem was finally solved also for the nontorsion case.

We shall now start to make sense of the holonomy terms in (3.2).

3.5 Holonomy

Let us start by recalling the usual definition of holonomy for U(1) line bundles with connection. We wish to think of isomorphism classes of line bundles with connection as Deligne classes in $H^1(X, \mathcal{D}^1)$ and formulate holonomy as "Deligne cohomologically" as possible, in hopes of finding a generalization to higher diagonal Deligne classes, in particular to $H^2(X, \mathcal{D}^2)$. Our references are [Joh2] and [CJM].

Let $\gamma: S^1 \to X$ be a closed curve on X, for which $\gamma(0) = \gamma(1) = x$ and P a principal U(1) bundle, with connection 1-form A. Then, for any point p of the fibre P_x , there exists a unique solution $\widetilde{\gamma}$ of the parallel transport equation [Nak], with $\widetilde{\gamma}(1) = p$, that is, a unique horizontal lift of the curve γ , starting at p. It turns out that

$$\widetilde{\gamma}(1) = \widetilde{\gamma}(0) \cdot \mathrm{hol}(\gamma)$$

defines a mapping hol: $LX \to U(1)$, which depends only on the bundle and the connection. Let us take a closer look at it. The curve γ may be broken into parts $\{\gamma([t_i, t_{i+1}])\}$, such that $\gamma([t_i, t_{i+1}]) \subset U_i$, where $\{U_i\}$ is a good open cover, with local sections $\{s_i\}$, trivializing the connection. Now, it is well known [Nak] that the parallel transport of s_i along $\gamma|_{[t_i, t_{i+1}]}$ maps

$$s_i(\gamma(t_i)) \mapsto s_i(\gamma(t_{i+1})) \exp\left(-\int_{t_i}^{t_{i+1}} \gamma^*(A_i)\right).$$

Combining two such maps yields

$$\begin{split} s_i(\gamma(t_i)) &\mapsto s_i(\gamma(t_{i+1})) \, \exp\left(-\int_{t_i}^{t_{i+1}} \gamma^* A_i\right) \\ &= s_{i+1}(\gamma(t_{i+1})) \, \exp\left(-\int_{t_i}^{t_{i+1}} \gamma^* A_i\right) g_{t_{i+1}t_i}(\gamma(t_{i+1})) \\ &\mapsto s_{i+1}(\gamma(t_{i+2})) \, \exp\left(-\int_{t_i}^{t_{i+1}} \gamma^* A_i - \int_{t_{i+1}}^{t_{i+2}} \gamma^* A_{i+1}\right) g_{t_{i+1}t_i}(\gamma(t_{i+1})) \,. \end{split}$$

It is straightforward to see that the holonomy along the entire closed curve is

$$\operatorname{hol}(\gamma) = \prod_{i} \exp\left(-\int_{t_i}^{t_{i+1}} \gamma^* A_i\right) \cdot \prod_{i} g_{t_{i+1}t_i}(\gamma(t_{i+1})). \tag{3.16}$$

Let us now take a different approach. Denote by $[\underline{g},\underline{A}] \in H^1(X,\mathcal{D}^1)$ the Deligne class of the line bundle with connection. Given the curve $\gamma: S^1 \to X$, we can pull the line bundle back onto S^1 . The first Chern class of the pullback bundle is a class in $H^2(S^1,\mathbb{Z})$. However, $H^2(S^1,\mathbb{Z})=0$, and so the pullback bundle is trivial. This is not to say that the pullback of the Deligne class, $[\gamma^*(\underline{g}), \gamma^*(\underline{A})]$, with respect to the good open cover $\gamma^{-1}(\mathcal{U})=\{\gamma^{-1}(U_\alpha)\}$ of S^1 , would be zero, but only that it is of the form $[\delta(\underline{h}), \gamma^*(\underline{A})]$. Diagrammatically, this is expressed by

$$\begin{array}{ccc}
1 & & \\
\delta & & \\
\delta(\underline{h}) & \xrightarrow{-d \log} 0 \\
\delta & & \delta \\
\underline{h} & & \gamma^* \underline{A}
\end{array}$$

where

$$-\gamma^* A_\beta + d \log h_\beta = -\gamma^* A_\alpha + d \log h_\alpha.$$

In other words, $-\gamma^*A_{\alpha} + d\log h_{\alpha}$ is a globally defined 1-form on S^1 and, as such, can be integrated. It is easy to see that the difference between two such 1-forms, $-\gamma^*(A_{\alpha}) + d\log h_{\alpha}$ and $-\gamma^*(\widetilde{A}_{\alpha}) + d\log \widetilde{h}_{\alpha}$, constructed from the same Deligne class, is $2\pi i$ times an integral 1-form. Therefore, the exponential

$$\exp \int_{S^1} \left(-\gamma^* A_\alpha + d \log h_\alpha \right) \tag{3.17}$$

takes values in S^1 .

To evaluate (3.17), we need to break the integral into a sum of integrals over open sets of S^1 in such a way, that the integrals of A_{α} and $d \log h_{\alpha}$ become independently defined. Let τ be a triangulation of S^1 , consisting of vertices v and edges e, such that the image under γ of each edge is contained within at least one U_{α}^{13} . For an edge e, we denote by $U_{\rho(e)}$ the open set in which $\gamma(e)$ is contained, that is, $e \subset \gamma^{-1}(U_{\rho(e)})$. Here $\rho : \tau \to \mathcal{I}$ is an index map, mapping elements of the triangulation into the index set of the good open cover \mathcal{U} of X. The integral (3.17) becomes

$$\prod_{e} \exp\left(-\int_{e} \gamma^{*} A_{\rho(e)} + \int_{e} d \log h_{\rho(e)}\right)
= \prod_{e} \exp\left(-\int_{e} \gamma^{*} A_{\rho(e)} + \log h_{\rho(e)}(v_{e}^{1}) - \log h_{\rho(e)}(v_{e}^{0})\right), \quad (3.18)$$

where v_e^0 and v_e^1 are the bounding vertices of e. On the other hand, using

$$\log(\gamma^* g_{\alpha\beta}) = \log h_{\beta} - \log h_{\alpha} \,,$$

the difference

$$\log h_{\rho(e)}(v_e^1) - \log h_{\rho(e)}(v_e^0)$$

becomes

$$\log(\gamma^* g_{\rho(e)\rho(v_e^0)})(v_e^0) - \log(\gamma^* g_{\rho(e)\rho(v_e^1)})(v_e^1) - \log h_{\rho(v_e^0)}(v_e^0) + \log h_{\rho(v_e^1)}(v_e^1).$$

Summing over all edges makes the last two terms cancel each other. Hence, (3.18) becomes

$$\prod_{e} \exp\left(-\int_{e} \gamma^{*} A_{\rho(e)} + \log(\gamma^{*} g_{\rho(e)\rho(v_{e}^{0})})(v_{e}^{0}) - \log(\gamma^{*} g_{\rho(e)\rho(v_{e}^{1})})(v_{e}^{1})\right)
= \prod_{e} \exp\left(-\int_{e} \gamma^{*} A_{\rho(e)}\right) \cdot \prod_{e} g_{\rho(e)\rho(v_{e}^{0})}(\gamma(v_{e}^{0})) g_{\rho(e)\rho(v_{e}^{1})}^{-1}(\gamma(v_{e}^{1}))
= \prod_{e} \exp\left(-\int_{e} \gamma^{*} A_{\rho(e)}\right) \cdot \prod_{v \in e} g_{\rho(e)\rho(v)}^{-1}(\gamma(v)),$$
(3.19)

where the product of $g_{\rho(e)\rho(v)}^{-1}(\gamma(v))$ over vertices should be understood as indicated by the previous form.

More generally, whenever we write a product over edges and vertices, we always assume it to be an *oriented product*, meaning that we automatically take into account both endpoints of the (oriented) edge in such a way, that the value of the function at the initial vertex is taken inverted.

We have obtained an expression for (3.17), which depends only on the Deligne cocycle, τ and the index map ρ . It is proved in [Joh2] that it is also independent of the chosen triangulation and the index map and thus yields, for a given line bundle with connection, a well-defined map from the loop space LX into S^1 . Remark also, that (3.16) is just a special case of (3.19), which gives us hope of generalizing the concept of holonomy to higher Deligne

 $^{^{13}}$ Such a triangulation always exists, because S^1 is compact and thus has a finite Lebesgue number. We simply choose the edges to be of length less than the Lebesgue number.

cohomology, in particular to $H^2(X, \mathcal{D}^2)$. If the connection happened to be globally defined, the holonomy would simplify into

$$hol(\gamma) = \exp\left(-\int_{S^1} \gamma^* A\right) = \exp\left(-\int_{\gamma} A\right).$$

Even when the connection is not globally defined, we denote $hol(\gamma)$ heuristically by

$$\exp\left(-\int_{\gamma}A\right). \tag{3.20}$$

In [Alv], the starting point is to make sense of integrals such as (3.20). It might sound surprising that one then necessarily obtains the data for a Deligne class. This is explained by the fact that a suitable holonomy function, together with a generalized gauge field strength define a so-called *Cheeger-Simons differential character*, the set of which forms a group isomorphic to $H^p(X, \mathcal{D}^p)$ [CS2, Bry]. We have nothing more to say about differential characters, but one should still keep in mind the deep meaning of the holonomy function.

Let us recapitulate the above discussion on holonomy along a closed curve γ . We expressed our line bundle with connection as $[\underline{g},\underline{A}] \in H^1(X,\mathcal{D}^1)$ and pulled it back into a class $[\gamma^*\underline{g},\gamma^*\underline{A}] \in H^1(S^1,\mathcal{D}^1)$. Then we remarked that the pullback bundle is trivial, which was a central aspect of the construction, since the trivialization was used to construct a globally defined 1-form, which could be then integrated over S^1 . It was finally argued that the exponential of the integral is a well-defined function on the loop space LX, depending only on the Deligne class of the line bundle with connection and that this function coincided with the classical notion of holonomy along a closed curve. Also, there was nothing in the Deligne theoretic construction that would have prevented us from using any closed one-dimensional manifold in place of S^1 . Only the dimensionality and closedness mattered.

Let now $[\underline{g}, -\underline{\Lambda}, \underline{B}] \in H^2(X, \mathcal{D}^2)$ be a Deligne class, corresponding to a bundle gerbe with connection and curving. Let M be a 2-dimensional closed surface and $\xi: M \to X$ a smooth map. Stable isomorphism classes of bundle gerbes were classified by their Dixmier-Douady class in $H^3(X, \mathbb{Z})$. Since $H^3(M, \mathbb{Z}) = 0$, all bundle gerbes over M are trivial. This means that the pullback $[\xi^*\underline{g}, -\xi^*\underline{\Lambda}, \xi^*\underline{B}]$ can be written in the form $[\delta(\underline{h}), -\xi^*\underline{\Lambda}, \xi^*\underline{B}]$. Diagrammatically, it is

$$\begin{array}{ccc}
 & \uparrow & \\
\delta & \uparrow & \\
\delta & b & -\xi^* \underline{\Lambda} & -d \\
 & & \xi^* \underline{\underline{H}}
\end{array}$$

from which we can immediately read

$$d\log(h_{\beta\gamma}h_{\alpha\gamma}^{-1}h_{\alpha\beta}) = \xi^*\Lambda_{\beta\gamma} - \xi^*\Lambda_{\alpha\gamma} + \xi^*\Lambda_{\alpha\beta},$$

which is just the Čech cocycle condition:

$$\delta(d\log \underline{h} - \xi^* \underline{\Lambda}) = 0.$$

By exactness, there exists $\underline{m} \in C^0(\xi^{-1}(\mathcal{U}), \underline{\Omega}^1)$, such that

$$d\log h_{\alpha\beta} - \xi^* \Lambda_{\alpha\beta} = m_\beta - m_\alpha. \tag{3.21}$$

Taking the de Rham differential on both sides, yields

$$\xi^* d\Lambda_{\alpha\beta} = dm_{\alpha} - dm_{\beta}.$$

On the other hand, since $d\Lambda_{\alpha\beta} = B_{\alpha} - B_{\beta}$, we get

$$\xi^* B_{\beta} - dm_{\beta} = \xi^* B_{\alpha} - dm_{\alpha} .$$

Hence, the 2-form $\xi^* B_{\alpha} - dm_{\alpha}$ is globally defined and can be integrated over M. Indeed,

$$hol(\xi) = \exp \int_{M} (\xi^* B_{\alpha} - dm_{\alpha})$$
 (3.22)

provides the correct generalization of holonomy for bundle gerbes with connection and curving. By choosing a triangulation τ of M, consisting of faces f, edges e and vertices v, such that the image under ξ of each object in τ is contained in an element of the open cover \mathcal{U} of X, the integral may be computed in local parts. It is straightforward computation to show that

$$\operatorname{hol}(\xi) = \prod_{f} \exp \int_{f} \xi^{*} B_{\rho(f)} \cdot \prod_{e \subset f} \exp \left(- \int_{e} \xi^{*} \Lambda_{\rho(f)\rho(e)} \right) \cdot \prod_{v \subset e \subset f} g_{\rho(f)\rho(e)\rho(v)}(\xi(v)), \quad (3.23)$$

where the product of $g_{\rho(f)\rho(e)\rho(v)}(\xi(v))$ over vertices is understood the same way as in (3.19). It can also be shown that $\text{hol}(\xi)$ depends only on the Deligne class [Joh2]. If \underline{B} defines a global 2-form B, also $d\underline{m}$ must define a global 2-form dm and thus, by Stoke's theorem and $\partial M = \emptyset$, the integral (3.22) simplifies into

$$\exp \int_M \xi^* B = \exp \int_{\xi} B.$$

In general, the above integral is ill-defined, but we still use it heuristically to denote the holonomy. We could alternatively have started with the above integral and attempt to make sense of it, in the spirit of [Alv], only to arrive at a Deligne cocycle and (3.23).

An even more general perspective at holonomy is provided by the exact sequence [Gaj]

$$0 \longrightarrow \Omega^p(X)_{(c,0)} \longrightarrow \Omega^p(X) \stackrel{\iota}{\longrightarrow} H^p(X,\mathcal{D}^p) \stackrel{d}{\longrightarrow} H^{p+1}(X,\mathbb{Z}) \longrightarrow 0$$

where $\Omega^p(X)_{(c,0)}$ is the set of closed p-forms, with $2\pi i$ times integral periods, ι maps $\rho \in \Omega^p(X)$ to the Deligne class $[1,0,\ldots,0,\{\rho|_{U_\alpha}\}]$ and d denotes taking the Dixmier-Douady class. Consider the pullback of a Deligne class $[\underline{g},\underline{\omega}^1,\ldots,\underline{\omega}^p]$ by $\xi:M\to X$, where M is p-dimensional. Because $H^{p+1}(M,\mathbb{Z})=0$, the exact sequence yields an element $\rho\in\Omega^p(M)$ such that

$$[1,0,\ldots,0,\{\rho|_{U_{\alpha}}\}] = [\xi^*\underline{g},\xi^*\underline{\omega}^1,\ldots,\xi^*\underline{\omega}^p].$$

We may now define

$$hol(\xi) := \exp \int_M \rho$$
.

This expression is independent of the chosen p-form because, by exactness, any two such p-forms ρ and $\tilde{\rho}$ representing the same Deligne class, satisfy

$$\int_{M} (\rho - \widetilde{\rho}) \in 2\pi i \mathbb{Z}. \tag{3.24}$$

It is straightforward to verify that for $H^1(X, \mathcal{D}^1)$ and $H^2(X, \mathcal{D}^2)$, the global forms $\gamma^* A_\alpha - d \log h_\alpha$ and $\xi^* B_\alpha - d m_\alpha$ are candidates for ρ . The reader may wonder why there is a sign difference between this definition for the holonomy of $H^1(X, \mathcal{D}^1)$ and the one we developed earlier, starting from the parallel transport equation. This is purely a notational choice really. It turns out that to cancel the Freed-Witten anomaly, we need this sign difference if we want to have $\text{hol}(\partial M, A)$ in the path integral. If we would not insert the extra minus sign here, we would have to do it somewhere else, but this seems to be a convenient choice. Ignoring the extra sign, all the holonomy functions defined above coincide with (3.24).

3.6 Higher-dimensional parallel transports

Let M be a p-dimensional compact manifold. We denote by $\operatorname{Map}(M,X)$ the infinite-dimensional manifold of smooth maps $M \to X$, equipped with the compact-open smooth topology [Hir]. For a triangulation τ of M, consisting of simplices σ and an index map ρ ,

$$V_{(\tau,\rho)} = \{ f \in \mathrm{Map}(M,X) : f(\sigma) \subset U_{\rho(\sigma)}, \sigma \in \tau \}$$

is an open set in Map(M, X). The sets $V_{(\tau, \rho)}$ form an open cover \mathcal{V} of Map(M, X). In the previous section we explained, how the ill-defined integral

$$\exp \int_M \xi^* B \,,$$

appearing in the closed string path integral, should be correctly understood in terms of holonomy of a Deligne class. The holonomy integral, as developed in the previous section, works only for closed surfaces. However, the worldsheet of an open string, denote it by M, is a compact manifold with boundary ∂M .

Let us start by considering a lower dimensional example. Let I = [0,1] and $\gamma: I \to X$ an open curve. Given a line bundle L with connection, holonomy around a closed curve could be defined as parallel transport around it. It is natural to expect that parallel transport provides the correct generalization of holonomy for open curves. Unlike the holonomy around a closed curve, parallel transport along an open curve is not a U(1)-valued function, because there is no canonical way to identify the fibres over $\gamma(0)$ and $\gamma(1)$, but rather a linear map $L_{\gamma(0)} \to L_{\gamma(1)}$, or alternatively an element of the vector space $L^*_{\gamma(0)} \otimes L_{\gamma(1)}$. Consider the line bundle $(\operatorname{ev}_0^{-1} L)^* \otimes (\operatorname{ev}_1^{-1} L) \to \operatorname{Map}(I,X)$, where ev_t is the evaluation map $\gamma \mapsto \gamma(t)$. The fibre over $\gamma \in \operatorname{Map}(I,X)$ is precisely $L^*_{\gamma(0)} \otimes L_{\gamma(1)}$ and, therefore, we may think of parallel transport as a section of this bundle. It turns out that there is a higher-dimensional generalization of parallel transport and it is a section of a trivial line bundle over $\operatorname{Map}(M,X)$. Moreover, this

line bundle is the pullback by the restriction map $r: \operatorname{Map}(M, X) \to \operatorname{Map}(\partial M, X)$ of a line bundle over $\operatorname{Map}(\partial M, X)$. We shall illustrate, following [Joh2], how this line bundle can be constructed starting from the local formula for holonomy, which can then be generalized to the higher-dimensional case.

Consider two open curves, γ_0 and γ_1 , with the same endpoints. We denote by $\gamma_0 \star \gamma_1^{-1}$ the composition of γ_0 and the inverse of γ_1 , which is then a closed curve. It is clear that the composition of parallel transport along γ_0 and the inverse of parallel transport along γ_1 must be equal to $\text{hol}(\gamma_0 \star \gamma_1^{-1})$. In other words, if we denote parallel transport along a curve γ by $t(\gamma)$,

$$t(\gamma_0)t(\gamma_1)^{-1} = \text{hol}(\gamma_0 \star \gamma_1^{-1}).$$
 (3.25)

We could attempt to define t using either (3.17) or (3.19), by replacing S^1 with I, for example

$$t(\gamma) \stackrel{?}{:=} \exp \int_{I} (-\gamma^* A_{\alpha} + d \log h_{\alpha}),$$

which obviously satisfies (3.25). However, unlike in the case of holonomy, the integral depends on the choice of the trivialization $\{h_{\alpha}\}$ and only the combination yielding the holonomy is independent of it. It is explained in [Joh2], why this approach becomes very inconvenient in the higher-dimensional case. Instead, we shall take (3.19) as our starting point and define the parallel transport map

$$t_{(\tau,\rho)}(\gamma) := \prod_{e} \exp\left(-\int_{e} \gamma^* A_{\rho(e)}\right) \cdot \prod_{v \in e} g_{\rho(e)\rho(v)}^{-1}(\gamma(v)),$$

where the edges e, vertices v and the index map ρ constitute a triangulation τ of I. In this case the definition depends on the triangulation, which is why we have indicated it with a subscript. In other words, it is only well-defined locally on open sets $V_{(\tau,\rho)} \subset \operatorname{Map}(I,X)$. We may restate (3.25) using $\{t_{(\tau,\rho)}\}$ as

$$t_{(\tau_0,\rho_0)}(\gamma_0)t_{(\tau_1,\rho_1)}(\gamma_1)^{-1} = \text{hol}(\gamma_0 \star \gamma_1^{-1}), \tag{3.26}$$

if and only if $\gamma_0 \in V_{(\tau_0,\rho)}$, $\gamma_1 \in V_{(\tau_1,\rho_1)}$ and the index maps satisfy $\rho_0(v_b) = \rho_1(v_b)$ for the boundary vertices v_b , because only then the triangulations (τ_0, ρ_0) and (τ_1, ρ_1) combine to form a triangulation $(\tau_0 \cup \tau_1, \rho_0 \cup \rho_1)$ of S^1 . Suppose that $\gamma \in V_{(\tau_0,\rho_0)} \cap V_{(\tau_1,\rho_1)}$, where $\rho_0(v_b) = \rho_1(v_b)$ for the boundary vertices. Then we have

$$t_{(\tau_0,\rho_0)}(\gamma) = \mathrm{hol}(\gamma \star \gamma^{-1}) t_{(\tau_0,\rho_0)}(\gamma) = t_{(\tau_1,\rho_1)}(\gamma) t_{(\tau_0,\rho_0)}^{-1}(\gamma) t_{(\tau_0,\rho_0)}(\gamma) = t_{(\tau_1,\rho_1)}(\gamma) \,,$$

which means that we are free to alter the triangulation of I from (τ_0, ρ_0) to (τ_1, ρ_1) , as long as γ is also in $V_{(\tau_1, \rho_1)}$ and the images of the boundary vertices under the index maps stay intact.

The local functions $t_{(\tau,\rho)}: V_{(\tau,\rho)} \to U(1)$ can be used to construct a trivial line bundle over Map(I,X), with transition functions $t_{(\tau_i,\rho_i)}^{-1}t_{(\tau_j,\rho_j)}$ on $V_{(\tau_i,\rho_i)}\cap V_{(\tau_j,\rho_j)}$ and natural flat connection, given by the local 1-forms $\{d \log t_{(\tau,\rho)}\}$. For $\gamma \in V_{(\tau_i,\rho_i)} \cap V_{(\tau_j,\rho_j)}$, the transition

functions are

$$\begin{split} t_{(\tau_{i},\rho_{i})}^{-1}(\gamma)t_{(\tau_{j},\rho_{j})}(\gamma) &= t_{(\tau_{i},\rho_{i})}^{-1}(\gamma)t_{(\tau_{i},\rho_{j})}(\gamma) \\ &= \prod_{e \in \tau_{i}} \exp \int_{e} \gamma^{*}(A_{\rho_{i}(e)} - A_{\rho_{j}(e)}) \cdot \prod_{v \subset e \in \tau_{i}} g_{\rho_{j}(e)\rho_{j}(v)}^{-1}(\gamma(v))g_{\rho_{i}(e)\rho_{i}(v)}(\gamma(v)) \\ &= \prod_{e \in \tau_{i}} \exp \int_{e} \gamma^{*}d\log g_{\rho_{j}(e)\rho_{i}(e)} \cdot \prod_{v \subset e \in \tau_{i}} g_{\rho_{j}(e)\rho_{j}(v)}^{-1}(\gamma(v))g_{\rho_{i}(e)\rho_{i}(v)}(\gamma(v)) \\ &= \prod_{v \subset e \in \tau_{i}} g_{\rho_{j}(e)\rho_{i}(e)}(\gamma(v)) \cdot \prod_{v \subset e \in \tau_{i}} g_{\rho_{j}(v)\rho_{j}(e)}(\gamma(v))g_{\rho_{i}(e)\rho_{i}(v)}(\gamma(v)) \\ &= \prod_{v \subset e \in \tau_{i}} g_{\rho_{j}(e)\rho_{i}(v)}(\gamma(v))g_{\rho_{j}(v)\rho_{j}(e)}(\gamma(v)) \\ &= \prod_{v \subset e \in \tau_{i}} g_{\rho_{j}(v)\rho_{i}(v)}(\gamma(v)) \\ &= \prod_{v \subset e \in \tau_{i}} g_{\rho_{j}(v)\rho_{i}(v)}(\gamma(v)) \\ &= \prod_{v \subset e \in \tau_{i}} g_{\rho_{j}(v)\rho_{i}(v)}(\gamma(v)) , \end{split}$$

but since these depend only on the boundary of I, they may be understood as pullbacks of transition functions $\{\mathfrak{g}_{(\tau_i,\rho_i)(\tau_j,\rho_j)}\}$ of a (generally nontrivial) line bundle over $\operatorname{Map}(\partial I,X)$ by the restriction map $r:\operatorname{Map}(I,X)\to\operatorname{Map}(\partial I,X)$. In other words, we can write

$$r^*\mathfrak{g}_{(\tau_i,\rho_i)(\tau_j,\rho_j)} = t_{(\tau_i,\rho_i)}^{-1} t_{(\tau_j,\rho_j)}.$$

One can also compute explicit expressions for the local connections $\{d \log t_{(\tau,\rho)}\}$. It turns out that they are of the form $r^*\mathfrak{A}_{(\tau,\rho)}$, where $\{\mathfrak{A}_{(\tau,\rho)}\}$ are local 1-forms on $\operatorname{Map}(\partial I,X)$, which patch up into a connection for the line bundle given by $\{\mathfrak{g}_{(\tau_i,\rho_i)(\tau_j,\rho_j)}\}$. We may then write these as a Deligne class $\mathfrak{T}[\underline{g},\underline{A}]=[\underline{\mathfrak{g}},\underline{\mathfrak{A}}]\in H^1(\operatorname{Map}(\partial I,X),\mathcal{D}^1)$, called the transgression of $[\underline{g},\underline{A}]$. The map $\mathfrak{T}:H^1(X,\overline{\mathcal{D}}^1)\to H^1(\partial I,\mathcal{D}^1)$ is called the transgression homomorphism.

The above procedure works also in higher-dimensional cases. Let us start with a bundle gerbe with connection and curving, with corresponding Deligne class $[\underline{g}, -\underline{\Lambda}, \underline{B}] \in H^2(X, \mathcal{D}^2)$. Recall the local formula (3.23) for holonomy over a closed surface. Consider a surface M, with boundary ∂M . The simplest example is, of course, the 2-dimensional disc B^2 , with boundary S^1 . Let $\xi \in \operatorname{Map}(M, X)$ and define the parallel transport map

$$t_{(\tau,\rho)}(\xi) := \prod_f \exp \int_f \xi^* B_{\rho(f)} \cdot \prod_{e \subset f} \exp \left(- \int_e \xi^* \Lambda_{\rho(f)\rho(e)} \right) \cdot \prod_{v \subset e \subset f} g_{\rho(f)\rho(e)\rho(v)}(\xi(v)) \,,$$

where faces f, edges e and vertices v constitute a triangulation τ of M and ρ is the associated index map. As expected, $t_{(\tau,\rho)}$ is not globally defined on $\operatorname{Map}(M,X)$, but only locally on the open sets $V_{(\tau,\rho)} \subset \operatorname{Map}(M,X)$. If $\xi_1, \xi_2 \in \operatorname{Map}(M,X)$ are such that the images of the boundaries coincide, they may be glued together to form the connected sum $\xi_1 \# \xi_2 : M \# M \to X$, oriented by the first factor. For example, if $M = B^2$, then $M \# M \cong S^2$. This generalizes the composition of paths, since $\gamma_1 \star \gamma_2^{-1} = \gamma_1 \# \gamma_2$. The equivalent of (3.25) for surfaces would be

$$t_{(\tau_0,\rho_0)}(\xi_0)t_{(\tau_1,\rho_1)}(\xi_1)^{-1} = \text{hol}(\xi_0 \# \xi_1),$$

if and only if $\xi_0 \in V_{(\tau_0,\rho_0)}$, $\xi_1 \in V_{(\tau_1,\rho_1)}$, $\rho_0(e_b) = \rho_1(e_b)$ and $\rho_0(v_b) = \rho_1(v_b)$, where v_b and e_b denote vertices and edges belonging to the boundary ∂M . Analogously to the lower-dimensional case, for such ρ_0, ρ_1 and $\xi \in V_{(\tau_0,\rho_0)} \cap V_{(\tau_1,\rho_1)}$, we have

$$t_{(\tau_0,\rho_0)}(\xi) = t_{(\tau_1,\rho_1)}(\xi)$$
.

The local functions $\{t_{(\tau,\rho)}\}$ can be used to construct a trivial line bundle over Map(M,X), with transition functions $t_{(\tau_i,\rho_i)}^{-1}t_{(\tau_j,\rho_j)}$ on $V_{(\tau_i,\rho_i)}\cap V_{(\tau_j,\rho_j)}$ and with a natural flat connection. Computing an explicit expression for the transition functions is a straightforward task, performed in [Joh2]. The result is

$$\prod_{e \subset \partial M} \exp\left(-\int_{e} \xi^* \Lambda_{\rho_i(e)\rho_j(e)}\right) \cdot \prod_{v \subset e \subset \partial M} g_{\rho_i(e)\rho_i(v)\rho_j(v)}^{-1} g_{\rho_i(e)\rho_j(e)\rho_j(e)}(\xi(v)), \tag{3.27}$$

which depends only on the boundary ∂M . Therefore, if $r: \operatorname{Map}(M,X) \to \operatorname{Map}(\partial M,X)$ denotes the restriction map, we can write

$$r^*\mathfrak{g}_{(\tau_i,\rho_i)(\tau_j,\rho_j)} = t_{(\tau_i,\rho_i)}^{-1} t_{(\tau_j,\rho_j)}.$$

Moreover, the natural flat connection for $\{r^*\mathfrak{g}_{(\tau,\rho)}\}$ depends also only on the boundary ∂M . We denote it by $r^*\mathfrak{A}$. The *transgression* of $[\mathfrak{g},\underline{\mathfrak{A}}]$ is

$$\mathfrak{T}[\underline{g},-\underline{\Lambda},\underline{B}]:=[\underline{\mathfrak{g}},\underline{\mathfrak{A}}]\in H^1(\operatorname{Map}(\partial M,X),\mathcal{D}^1)\,.$$

When $\partial M = S^1$, Map $(\partial M, X)$ is simply the loop space LX.

The transgression of a cohomology class F in $H^3(X,\mathbb{Z})$ is defined by pulling back the class onto $\partial M \times \operatorname{Map}(\partial M, X)$ with the evaluation map

$$\operatorname{ev}: \partial M \times \operatorname{Map}(\partial M, X) \to X$$

and integrating over ∂M . The result is a class in $H^2(\operatorname{Map}(\partial M, X), \mathbb{Z})$, denoted by $\mathfrak{T}(F)$.

The transgression map for Deligne cohomology classes is homomorphic in the following sense [GT, Joh2, CJM]. For a class $\chi \in H^2(X, \mathcal{D}^2)$, we denote the line bundle over $\operatorname{Map}(\partial M, X)$, obtained through transgression, by $\mathfrak{T}(\chi)$. For $\chi_1, \chi_2 \in H^2(X, \mathcal{D}^2)$, the line bundles $\mathfrak{T}(\chi_1)$, $\mathfrak{T}(\chi_2)$ and $\mathfrak{T}(\chi_1 + \chi_2)$ satisfy

$$\mathfrak{T}(\chi_1 + \chi_2) \cong \mathfrak{T}(\chi_1) \otimes \mathfrak{T}(\chi_2),$$

$$\mathfrak{T}(-\chi) \cong \mathfrak{T}(\chi)^*,$$

$$\mathfrak{T}(0) \cong U(1) \times \operatorname{Map}(\partial M, X).$$

Also, for any $\chi \in H^2(X, \mathcal{D}^2)$, the first Chern class $c_1(\mathfrak{T}(\chi))$ coincides with the transgression $\mathfrak{T}(d(L_{\chi}))$ of the Dixmier-Douady class of the bundle gerbe L_{χ} , corresponding to the Deligne cohomology class χ .

Transgression can be performed also for higher Deligne cocycles. Let M be a compact manifold of dimension m, with or without boundary. In [GT] the authors gave an explicit expression for a transgression map $C^p(X, \mathcal{D}^q) \to C^{p-m}(X, \mathcal{D}^{q-m})$. When q = p and M is of dimension p+1 with a p-dimensional boundary ∂M , one can construct a line bundle with connection over $\operatorname{Map}(\partial M, X)$ using the transgression map, such that the pullback bundle by the restriction $r: \operatorname{Map}(M, X) \to \operatorname{Map}(\partial M, X)$ is trivial [GT]. In [Joh2] the construction is explained using similar reasoning as in the cases p=1,2. There are also other ways of approaching the higher-dimensional parallel transport construction, but there is no point in discussing them here. Instead, we refer the interested reader to [Joh2].

3.7 Cancelling the Freed-Witten anomaly

We have now developed enough mathematical machinery to understand the Freed-Witten anomaly. We shall start by recalling how the anomaly arises and then proceed to show, using the above constructions, how it cancels when the D-brane and the B-field, or its restriction to the worldvolume, are related in a particular way. Our references for this section are [Joh2], [CJM] and [Kap].

Since endpoints of open strings are attached to a D-brane $\phi: \Sigma \to X$, we are actually interested, not in Map(M,X), but rather in the submanifold Map $_{\Sigma}(M,X)$ of maps, satisfying

$$\xi(\partial M) \subset \phi(\Sigma)$$
.

Trivial line bundles over $\operatorname{Map}(M,X)$ restrict to trivial line bundles over $\operatorname{Map}_{\Sigma}(M,X)$. It follows that the above constructions for holonomy and parallel transport generalize to yield similar constructions for $\operatorname{Map}_{\Sigma}(M,X)$.

Recall that the fermionic worldsheet path integral (3.2) contained three problematic terms: the Pfaffian pfaff(\mathcal{D}_{ξ}) of the worldsheet Dirac operator, the "holonomy" of B over the worldsheet M, which we now understand to be a higher-dimensional parallel transport, that is, a section of a trivial line bundle over $\operatorname{Map}_{\Sigma}(M,X)$, and finally the "trace of holonomy" of A around the boundary ∂M , which we have not yet discussed.

It was shown in [FW], that pfaff := pfaff(\mathcal{D}_{ξ}) is also not a well-defined function on $\operatorname{Map}_{\Sigma}(M,X)$, but rather a section of the line bundle

$$r_{\Sigma}^{-1}(\mathfrak{T}[\underline{w},0,0]) \to \operatorname{Map}_{\Sigma}(M,X)$$
,

where \underline{w} is a representative of the image of the second Stiefel-Whitney class of Σ under the inclusion $H^2(\Sigma, \mathbb{Z}_2) \to H^2(\Sigma, U(1))$ and $r_{\Sigma} : \operatorname{Map}_{\Sigma}(M, X) \to \operatorname{Map}(\partial M, \Sigma)$ is the restriction map. We denote the bundle gerbe corresponding to $[\underline{w}, 0, 0]$ by L_w . The Pfaffian is, therefore, a section of $r_{\Sigma}^{-1}(\mathfrak{T}(L_w))$. We remind the reader that $\mathfrak{T}(L_w)$ is by no means trivial. It is the pullback that is trivial and its trivialization is given by the parallel transports.

There are basically three cases to consider, of which the first and simplest, treated in [FW], is when the *B*-field bundle gerbe restricted to Σ , $L_B|_{\Sigma}$, satisfies

$$d(L_B|_{\Sigma}) = d(L_w) \in H^3(\Sigma, \mathbb{Z}).$$
(3.28)

Using $d(L_{w^{-1}}) = d(L_w^*) = -d(L_w)^{14}$, equation (3.28) becomes

$$d(L_w \otimes L_B|_{\Sigma}) = d(L_w^* \otimes L_B^*|_{\Sigma}) = 0,$$

which implies that the transgression $\mathfrak{T}(L_w^* \otimes L_B^*|_{\Sigma})$ is a trivial line bundle over $\operatorname{Map}(\partial M, \Sigma)$, whose pullback onto $\operatorname{Map}_{\Sigma}(M, X)$ is then also trivial. The bundle gerbe $L_w \otimes L_B|_{\Sigma}$ corresponds

¹⁴This follows from $d(L_w)$ being 2-torsion.

to the Deligne cocycle $(\underline{gw}, -\underline{\Lambda}, \underline{B})^{15}$, which fits into

$$\frac{\underline{g}\underline{w}}{\delta} = \delta(\underline{h}) \xrightarrow{d \log} 0$$

$$\delta \uparrow \qquad \qquad \delta \uparrow$$

$$\underline{h} \qquad \qquad -\underline{\Lambda} \xrightarrow{-d} 0$$

$$\delta \downarrow \underline{B}$$

From the diagram we can immediately read that

$$(gw)_{\alpha\beta\gamma} = h_{\beta\gamma}h_{\alpha\gamma}^{-1}h_{\alpha\beta}$$

and (compare to (3.21))

$$A_{\beta} = A_{\alpha} + d \log h_{\alpha\beta} - \Lambda_{\alpha\beta} \,, \tag{3.29}$$

for some $\underline{A} \in C^0(\mathcal{U}|_{\Sigma}, \underline{\Omega}^1)$. We already know that the transgression line bundle is trivial. This may be explicitly seen by substituting the above relations into the formula for the transition functions (3.27). Indeed, the transition functions become

$$\begin{split} \prod_{e \subset \partial M} \exp \int_{e} & (-\xi^{*} d \log h_{\rho_{i}(e)\rho_{j}(e)} + \xi^{*} A_{\rho_{j}(e)} - \xi^{*} A_{\rho_{i}(e)}) \\ & \cdot \prod_{v \subset e \subset \partial M} h_{\rho_{i}(e)\rho_{i}(v)}^{-1} h_{\rho_{i}(e)\rho_{j}(v)} h_{\rho_{i}(v)\rho_{j}(v)}^{-1} h_{\rho_{j}(e)\rho_{j}(v)} h_{\rho_{i}(e)\rho_{j}(v)}^{-1} h_{\rho_{i}(e)\rho_{j}(v)} h_{\rho_{i}(e)\rho_{j}(v)} (\xi(v)) \\ & = \prod_{e \subset \partial M} \exp \int_{e} (\xi^{*} A_{\rho_{j}(e)} - \xi^{*} A_{\rho_{i}(e)}) \cdot \prod_{v \subset e \subset \partial M} h_{\rho_{i}(e)\rho_{i}(v)}^{-1} h_{\rho_{j}(e)\rho_{j}(v)} (\xi(v)) = \Gamma_{i}^{-1}(\xi) \Gamma_{j}(\xi) \,, \end{split}$$

where

$$\Gamma_i(\xi) = \prod_{e \subset \partial M} \exp \int_e \xi^* A_{\rho_i(e)} \cdot \prod_{v \subset e \subset \partial M} h_{\rho_i(e)\rho_i(v)}(\xi(v))$$

are local functions trivializing the transgression bundle $\mathfrak{T}(L_w \otimes L_B|_{\Sigma})$. Finally, their inverses,

$$\Gamma_i^{-1}(\xi) = \prod_{e \subset \partial M} \exp\left(-\int_e \xi^* A_{\rho_i(e)}\right) \cdot \prod_{v \subset e \subset \partial M} h_{\rho_i(e)\rho_i(v)}^{-1}(\xi(v)),$$
(3.30)

provide a local trivialization for $\mathfrak{T}(L_w^* \otimes L_B^*|_{\Sigma})$.

Let us now analyze \underline{A} a bit. The transformation property (3.29) is reminiscent of ordinary gauge transformations. Indeed, if $\underline{\Lambda} = 0$, it would be of the same form as a gauge transformation of a connection on a bundle, with transition functions \underline{h} , although in this case \underline{h} satisfies merely a \underline{gw} -twisted cocycle condition. Hence, one can heuristically regard A as "almost" a

¹⁵More correctly, we should write $(\underline{g}|\underline{\Sigma}\underline{w}, -\underline{\Lambda}|\underline{\Sigma}, \underline{B}|\underline{\Sigma})$. To simplify the notation, we leave out the obvious restriction signs and hope this will not cause confusion.

U(1) gauge connection on a projective bundle. Ignoring the slightly different meanings of the mathematical objects involved, we notice that (3.30) and (3.19) are quite alike, which motivates the definition

$$\operatorname{hol}(\partial M, A) := r_{\Sigma}^{-1}(\Gamma_i^{-1}).$$

The holonomy of B, or rather the parallel transport, is a trivializing section of $r_{\Sigma}^{-1}(\mathfrak{T}(L_B|_{\Sigma}))$. Thus, the product

$$\operatorname{pfaff} \cdot \operatorname{hol}(M, B) \cdot \operatorname{hol}(\partial M, A)$$

is the difference of two trivializing sections and, as such, a well-defined function on $\operatorname{Map}_{\Sigma}(M, X)$, taking values in the trivial complex line

$$r_{\Sigma}^{-1}(\mathfrak{T}(L_w)) \otimes r_{\Sigma}^{-1}(\mathfrak{T}(L_B|_{\Sigma})) \otimes r_{\Sigma}^{-1}(\mathfrak{T}(L_w^* \otimes L_B^*|_{\Sigma})).$$

We now proceed to make sense of the second case, originally explained by Kapustin [Kap], where $d(L_B|_{\Sigma}) \neq d(L_w)$ are such that

$$d(L_B|_{\Sigma}) - d(L_w) = d(L_B|_{\Sigma}) + d(L_w) \in \text{Tor } H^3(\Sigma, \mathbb{Z}).$$
(3.31)

Because $d(L_w)$ is torsion, (3.31) is equivalent to

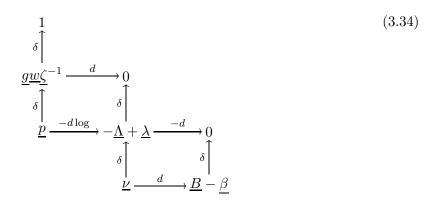
$$d(L_B|_{\Sigma}) = [H]|_{\Sigma} \in \text{Tor } H^3(\Sigma, \mathbb{Z}). \tag{3.32}$$

We obviously need some new way to trivialize pfaff \cdot hol(M, B). For definiteness, suppose that $d(L_B|_{\Sigma}) + d(L_w) = d(L_B|_{\Sigma} \otimes L_w)$ is n-torsion. The bundle gerbe $L_B|_{\Sigma} \otimes L_w$ corresponds to the Deligne cocycle $(\underline{gw}, -\underline{\Lambda}, \underline{B})$. By (3.32), the global 3-form H is exact, which means that there exists a global 2-form B, such that H = dB. Since $d(B_{\alpha} - B) = 0$, exactness implies $B_{\alpha} = B + d\mu_{\alpha}$, where μ_{α} is a 1-form. The difference $B_{\alpha} - d\mu_{\alpha}$ is, therefore, a global 2-form. Combining this with $B_{\alpha} - B_{\beta} = d\Lambda_{\alpha\beta}$, yields

$$\mu_{\beta} = \mu_{\alpha} + d \log q_{\alpha\beta} - \Lambda_{\alpha\beta} \,, \tag{3.33}$$

where $q_{\alpha\beta}$ is a U(1)-valued function.

Let L_{ζ} be a bundle gerbe, represented by the Deligne cocycle $(\underline{\zeta}, -\underline{\lambda}, \underline{\beta})$, such that L_{ζ} is stably isomorphic to $L_B|_{\Sigma} \otimes L_w$. The product $L_p := L_B|_{\Sigma} \otimes L_w \otimes L_{\zeta}^*$ is then trivial and represented by the trivial Deligne cocycle $(\delta(\underline{p}), \delta(\underline{\nu}) - d \log \underline{p}, \underline{\nu})$. Expressed diagrammatically, the trivialization looks like



Since

$$d(L_{\zeta}) = \beta([\zeta]) = [H]|_{\Sigma} + W_3(\Sigma), \qquad (3.35)$$

 L_{ζ} is an *n*-torsion bundle gerbe. As such, it admits a finite rank bundle gerbe module with connection, locally represented by $\{A_{\alpha}\}$, satisfying (see (3.15))

$$A_{\beta} = h_{\alpha\beta}^{-1} A_{\alpha} h_{\alpha\beta} + h_{\alpha\beta}^{-1} dh_{\alpha\beta} - \lambda_{\alpha\beta} , \qquad (3.36)$$

where

$$h_{\alpha\beta}h_{\beta\gamma}h_{\alpha\gamma}^{-1} = \zeta_{\alpha\beta\gamma}.$$

From (3.34) we can read off the relation

$$\lambda_{\alpha\beta} = \nu_{\beta} - \nu_{\alpha} - d\log p_{\alpha\beta} + \Lambda_{\alpha\beta}. \tag{3.37}$$

Combining (3.37) with (3.33) yields

$$\lambda_{\alpha\beta} = \nu_{\beta} - \mu_{\beta} - \nu_{\alpha} + \mu_{\alpha} - d\log(pq^{-1})_{\alpha\beta}. \tag{3.38}$$

Inserting (3.38) into (3.36), defining

$$\widetilde{A}_{\alpha} := A_{\alpha} + \nu_{\alpha} - \mu_{\alpha}$$
 and $\widetilde{h}_{\alpha\beta} := (hpq^{-1})_{\alpha\beta}$,

and using

$$\widetilde{h}_{\alpha\beta}^{-1} d\widetilde{h}_{\alpha\beta} = h_{\alpha\beta}^{-1} dh_{\alpha\beta} + d\log(pq^{-1})_{\alpha\beta} ,$$

yields

$$\widetilde{A}_{\beta} = \widetilde{h}_{\alpha\beta}^{-1} \widetilde{A}_{\alpha} \widetilde{h}_{\alpha\beta} + \widetilde{h}_{\alpha\beta}^{-1} d\widetilde{h}_{\alpha\beta} , \qquad (3.39)$$

a formula, which looks precisely like an ordinary gauge transformation. Of course, $\{\tilde{h}_{\alpha\beta}\}$ is still only a twisted cocycle:

$$\widetilde{h}_{\alpha\beta}\widetilde{h}_{\beta\gamma}\widetilde{h}_{\alpha\gamma}^{-1} = (\zeta\delta(p)\delta(q)^{-1})_{\alpha\beta\gamma}.$$

Let us now briefly comment on the physical interpretations of the above constructions. We interpret $\{A_{\alpha}\}$ as the locally defined A-field and $\{h_{\alpha\beta}\}$ as the U(n) lifts of the transition functions of the Chan-Paton bundle. The obstruction for the Chan-Paton bundle lifting into a U(n) bundle is

$$[\zeta \delta(p)\delta(q)^{-1}] = [\zeta] \in H^2(\Sigma, U(1)),$$

or equivalently, its image $\beta([\underline{\zeta}]) \in H^3(\Sigma, \mathbb{Z})$, where $\beta : H^2(\Sigma, \underline{U(1)}) \xrightarrow{\cong} H^3(\Sigma, \mathbb{Z})$ is the Bockstein isomorphism (3.6).

The Freed-Witten anomaly can now be cancelled as follows. The path integral must, of course, contain the usual sections of $r_{\Sigma}^{-1}(\mathfrak{T}(L_B|_{\Sigma}))$ and $r_{\Sigma}^{-1}(\mathfrak{T}(L_w))$, coming from hol(M,B) and pfaff. Next, we need local sections of $r_{\Sigma}^{-1}(\mathfrak{T}(L_p^*)) = r_{\Sigma}^{-1}(\mathfrak{T}(L_B^*|_{\Sigma} \otimes L_w^* \otimes L_{\zeta}))$, but these are easy to obtain, since the transgressed bundle is trivial. Inserting the data of L_p into (3.27), gives trivializing sections Γ_i^{-1} of $\mathfrak{T}(L_p^*)$. We shall drop the index i for notational convenience and simply write Γ^{-1} to denote the pullback $r_{\Sigma}^*(\Gamma_i^{-1})$, keeping in mind that it is a trivializing section, not a function on $\mathrm{Map}_{\Sigma}(M,X)$. The only ingredient that is missing, is a trivializing section of $r_{\Sigma}^{-1}(\mathfrak{T}(L_{\zeta}^*))$. Kapustin showed in [Kap] that it is given by the "trace of holonomy" of the A-field.

Recall first the Abelian holonomy formula (3.19). In the non-Abelian case there is also a parallel transport equation, but the solution is not as simple to write as in the Abelian case. This is basically because the exponentials

$$\exp\left(-\int_e \xi^* \widetilde{A}_{\rho(e)}\right)$$

are ambiguous, due to the noncommutativity of U(n). The correct interpretation is to expand the exponential and perform the products of integrals in path-order. This is customarily indicated by placing a \mathcal{P} in front of the exponential. However, it should suffice to think of ¹⁶

$$\mathcal{P}\exp\left(-\int_{e}\xi^{*}\widetilde{A}_{\rho(e)}\right)$$

as the solution of the non-Abelian parallel transport equation along e, if we use the well-known result that these quantities transform as [PS, Gil, Nak]

$$\mathcal{P}\exp\left(-\int_{e} \xi^* \widetilde{A}_{\rho_j(e)}\right) = \widetilde{h}_{\rho_i(e)\rho_j(e)}^{-1}(\xi(v_e^1)) \cdot \mathcal{P}\exp\left(-\int_{e} \xi^* \widetilde{A}_{\rho_i(e)}\right) \cdot \widetilde{h}_{\rho_i(e)\rho_j(e)}(\xi(v_e^0))$$

under

$$\widetilde{A}_{\rho_j(e)} = \widetilde{h}_{\rho_i(e)\rho_j(e)}^{-1} \widetilde{A}_{\alpha} \widetilde{h}_{\rho_i(e)\rho_j(e)} + \widetilde{h}_{\rho_i(e)\rho_j(e)}^{-1} d\widetilde{h}_{\rho_i(e)\rho_j(e)} \,.$$

To make the following calculations a bit shorter, we shall use the shorthand notation

$$\mathcal{W}_{\rho(e)}^{\widetilde{A}}(\xi) = \mathcal{P} \exp\left(-\int_{e} \xi^* \widetilde{A}_{\rho(e)}\right).$$

Let us, for the sake of simplicity, assume that $\partial M \cong S^1$. We define

$$\operatorname{Tr} \operatorname{hol}(S^{1}, A)_{(\tau, \rho)} := \operatorname{Tr} \left[\prod_{e \subset S^{1}} \widetilde{h}_{\rho(e)\rho(v_{e}^{1})}^{-1} \cdot \mathcal{W}_{\rho(e)}^{\widetilde{A}} \cdot \widetilde{h}_{\rho(e)\rho(v_{e}^{0})} \right] \cdot \prod_{e \subset S^{1}} \exp \int_{e} \xi^{*}(\nu - \mu)_{\rho(e)} \cdot \prod_{v \subset e \subset S^{1}} (pq^{-1})_{\rho(e)\rho(v)}, \quad (3.40)$$

where we have suppressed the arguments of $\widetilde{h}_{\alpha\beta}$, p and q, as they are unambiguous, and also left out the argument ξ of $\mathcal{W}_{\rho(e)}^{\widehat{A}}$. Comparing $\operatorname{Tr} \operatorname{hol}(S^1,A)_{(\tau_i,\rho_i)}$ and $\operatorname{Tr} \operatorname{hol}(S^1,A)_{(\tau_j,\rho_j)}$ on overlaps $V_{(\tau_i,\rho_i)} \cap V_{(\tau_j,\rho_j)}$, is a tedious task. The trace is the difficult part:

$$\begin{split} &\operatorname{Tr}\left[\prod_{e\subset S^1}\widetilde{h}_{\rho_j(e)\rho_j(v_e^1)}^{-1}\cdot \mathcal{W}_{\rho_j(e)}^{\widetilde{A}}\cdot \widetilde{h}_{\rho_j(e)\rho_j(v_e^0)}\right]\\ &=\operatorname{Tr}\left[\dots \mathcal{W}_{\rho_j(e_{k+1})}^{\widetilde{A}}\cdot \widetilde{h}_{\rho_j(e_{k+1})\rho_j(v_{e_{k+1}}^0)}\cdot \widetilde{h}_{\rho_j(e_k)\rho_j(v_{e_k}^1)}^{-1}\cdot \mathcal{W}_{\rho_j(e_k)}^{\widetilde{A}}\dots\right]\\ &=\operatorname{Tr}\left[\dots \mathcal{W}_{\rho_i(e_{k+1})}^{\widetilde{A}}\cdot \widetilde{h}_{\rho_i(e_{k+1})\rho_j(e_{k+1})}\widetilde{h}_{\rho_j(e_{k+1})\rho_j(v_{e_{k+1}}^0)}\cdot \widetilde{h}_{\rho_j(e_k)\rho_j(v_{e_k}^1)}^{-1}\widetilde{h}_{\rho_i(e_k)\rho_j(e_k)}^{-1}\cdot \mathcal{W}_{\rho_j(e_k)}^{\widetilde{A}}\dots\right] \end{split}$$

 $^{^{16}}$ Note that we are using \widetilde{A} here, since it has the correct U(n) gauge transformation property.

$$\begin{split} &= \operatorname{Tr} \left[\dots \mathcal{W}_{\rho_{i}(e_{k+1})}^{\widetilde{A}} \cdot \widetilde{h}_{\rho_{i}(e_{k+1})\rho_{j}(v_{e_{k+1}}^{0})} \cdot \widetilde{h}_{\rho_{i}(e_{k})\rho_{j}(v_{e_{k}}^{1})}^{-1} \cdot \mathcal{W}_{\rho_{j}(e_{k})}^{\widetilde{A}} \cdot \dots \right] \\ &\cdot \prod_{k} \left(\zeta \delta(p) \delta(q)^{-1} \right)_{\rho_{i}(e_{k+1})\rho_{j}(e_{k+1})\rho_{j}(v_{e_{k+1}}^{0})} \left(\zeta \delta(p) \delta(q)^{-1} \right)_{\rho_{i}(e_{k})\rho_{j}(e_{k})}^{-1} \dots \right] \\ &= \operatorname{Tr} \left[\dots \mathcal{W}_{\rho_{i}(e_{k+1})}^{\widetilde{A}} \cdot \widetilde{h}_{\rho_{i}(e_{k+1})\rho_{i}(v_{e_{k+1}}^{0})} \widetilde{h}_{\rho_{i}(v_{e_{k+1}}^{0})\rho_{j}(v_{e_{k+1}}^{0})} \cdot \widetilde{h}_{\rho_{i}(v_{e_{k+1}}^{0})\rho_{j}(v_{e_{k+1}}^{0})} \cdot \widetilde{h}_{\rho_{i}(v_{e_{k+1}}^{0})\rho_{j}(v_{e_{k+1}}^{0})} \cdot \widetilde{h}_{\rho_{i}(e_{k})\rho_{j}(v_{e_{k+1}}^{0})} \cdot \widetilde{h}_{\rho_{i}(e_{k})\rho_{i}(v_{e_{k+1}}^{0})\rho_{j}(v_{e_{k+1}}^{0})} \cdot \widetilde{h}_{\rho_{i}(e_{k})\rho_{j}(v_{e_{k+1}}^{0})\rho_{j}(v_{e_{k+1}}^{0})} \cdot \widetilde{h}_{\rho_{i}(e_{k})\rho_{i}(v_{e_{k+1}}^{0})\rho_{j}(v_{e_{k+1}}^{0})} \cdot \widetilde{h}_{\rho_{i}(e_{k})\rho_{i}(v_{e_{k+1}}^{0})\rho_{j}(v_{e_{k+1}}^{0})} \cdot \widetilde{h}_{\rho_{i}(e_{k})\rho_{i}(v_{e_{k+1}}^{0})} \cdot \widetilde{h}_{\rho_{i}(e_{k})\rho_{i}(e_{k+1})\rho_{i}(e_{k+1})\rho_{i}(e_{k+1})\rho_{i}(e_{k+1})\rho_{i}(e_{k+1})\rho_{i}(e_{k+1})\rho_{i$$

Using this result, we can easily compute

$$\begin{split} & \text{Tr} \left[\prod_{e \subset S^1} \widetilde{h}_{\rho_j(e)\rho_j(v_e^1)}^{1} \cdot \mathcal{W}_{\rho_j(e)}^{\tilde{A}} \cdot \widetilde{h}_{\rho_j(e)\rho_j(v_e^0)} \right] \cdot \prod_{e \subset S^1} \exp \int_e \xi^*(\nu - \mu)_{\rho_j(e)} \cdot \prod_{v \subset e \subset S^1} (pq^{-1})_{\rho_j(e)\rho_j(v)} \\ & = \text{Tr} \left[\prod_{e \subset S^1} \widetilde{h}_{\rho_i(e)\rho_i(v_e^1)}^{1} \cdot \mathcal{W}_{\rho_i(e)}^{\tilde{A}} \cdot \widetilde{h}_{\rho_i(e)\rho_i(v_e^0)} \right] \cdot \prod_{e \subset S^1} \exp \int_e \xi^*(\nu - \mu)_{\rho_i(e)} \cdot \prod_{v \subset e \subset S^1} (pq^{-1})_{\rho_i(e)\rho_i(v)} \\ & \cdot \prod_{e \subset S^1} \exp \int_e \xi^* \left((\nu - \mu)_{\rho_j(e)} - (\nu - \mu)_{\rho_i(e)} \right) \cdot \prod_{v \subset e \subset S^1} (pq^{-1})_{\rho_j(e)\rho_j(v)} (p^{-1}q)_{\rho_i(e)\rho_i(v)} \\ & \cdot \prod_{v \subset e \subset S^1} \left(\langle \delta(p)\delta(q)^{-1} \right)_{\rho_i(e)\rho_i(v)\rho_j(v)} \left(\langle \delta(p)\delta(q)^{-1} \right)_{\rho_i(e)\rho_j(e)\rho_j(v)} \right] \cdot \prod_{v \subset e \subset S^1} \exp \int_e \xi^*(\nu - \mu)_{\rho_i(e)} \cdot \prod_{v \subset e \subset S^1} (pq^{-1})_{\rho_i(e)\rho_i(v)} \\ & \cdot \prod_{v \subset e \subset S^1} \widetilde{h}_{\rho_i(e)\rho_i(v_e^1)}^{-1} \cdot \mathcal{W}_{\rho_i(e)}^{\tilde{A}} \cdot \widetilde{h}_{\rho_i(e)\rho_i(v_e^0)} \right] \cdot \prod_{e \subset S^1} \exp \int_e \xi^*(\nu - \mu)_{\rho_i(e)} \cdot \prod_{v \subset e \subset S^1} (pq^{-1})_{\rho_i(e)\rho_i(v)} \\ & \cdot \prod_{e \subset S^1} \exp \int_e \xi^* \left((\nu - \mu)_{\rho_j(e)} - (\nu - \mu)_{\rho_i(e)} \right) \cdot \prod_{v \subset e \subset S^1} \zeta_{\rho_i(e)\rho_i(v)\rho_j(v)} \left(\delta(p)\delta(q)^{-1} \right)_{\rho_i(e)\rho_j(v)}^{-1} \\ & \cdot \prod_{v \subset e \subset S^1} \widetilde{h}_{\rho_i(e)\rho_i(v_e^1)}^{-1} \cdot \mathcal{W}_{\rho_i(e)}^{\tilde{A}} \cdot \widetilde{h}_{\rho_i(e)\rho_i(v_e^0)}^{-1} \right] \cdot \prod_{v \subset e \subset S^1} \exp \int_e \xi^*(\nu - \mu)_{\rho_i(e)} \cdot \prod_{v \subset e \subset S^1} (pq^{-1})_{\rho_i(e)\rho_j(v)} \\ & \cdot \prod_{e \subset S^1} \exp \int_e \xi^* \left((\nu - \mu)_{\rho_j(e)} - (\nu - \mu)_{\rho_i(e)} \right) \cdot \prod_{v \subset e \subset S^1} \zeta_{\rho_i(e)\rho_i(v)\rho_j(v)} \left(\delta(p)^{-1} \delta(q) \right)_{\rho_i(e)\rho_j(v)} \\ & \cdot \prod_{e \subset S^1} \widetilde{h}_{\rho_i(e)\rho_i(v_e^1)}^{-1} \cdot \mathcal{W}_{\rho_i(e)}^{\tilde{A}} \cdot \widetilde{h}_{\rho_i(e)\rho_i(v_e^0)}^{-1} \right) \cdot \prod_{v \subset e \subset S^1} \zeta_{\rho_i(e)\rho_i(v)\rho_j(v)} \left(\delta(p)^{-1} \delta(q) \right)_{\rho_i(e)\rho_j(v)} \\ & \cdot \prod_{v \subset e \subset S^1} \widetilde{h}_{\rho_i(e)\rho_i(v_e^1)}^{-1} \cdot \mathcal{W}_{\rho_i(e)}^{\tilde{A}} \cdot \widetilde{h}_{\rho_i(e)\rho_i(v_e^0)}^{-1} \right) \cdot \prod_{v \subset e \subset S^1} \zeta_{\rho_i(e)\rho_i(v)\rho_j(v)} \left(\delta(p)^{-1} \delta(q) \right)_{\rho_i(e)\rho_j(v)} \\ & \cdot \prod_{v \subset e \subset S^1} \widetilde{h}_{\rho_i(e)\rho_i(v_e^1)}^{-1} \cdot \mathcal{W}_{\rho_i(e)}^{\tilde{A}} \cdot \widetilde{h}_{\rho_i(e)\rho_i(v_e^0)}^{-1} \right) \cdot \prod_{v \subset e \subset S^1} \zeta_{\rho_i(e)\rho_i(v)\rho_i(v)} \left(\delta(p)^{-1} \delta(q) \right)_{\rho_i(e)\rho_i(v)} \\ & \cdot \prod_{v \subset e \subset S^1} (pq^{-1}q)_{\rho_i(e)\rho_i(e)} \cdot \mathcal{W}_{\rho_i(e)}^{\tilde{A}} \cdot \widetilde{h}_{\rho_i(e)\rho_i(v_e^0)}^{-1$$

Hence, $\operatorname{Tr} \operatorname{hol}(S^1, A)_{(\tau, \rho)}$ transforms as

$$\operatorname{Tr} \operatorname{hol}(S^{1}, A)_{(\tau_{j}, \rho_{j})} = \operatorname{Tr} \operatorname{hol}(S^{1}, A)_{(\tau_{i}, \rho_{i})} \cdot \prod_{e \subset S^{1}} \exp \int_{e} \xi^{*} ((\nu - \mu)_{\rho_{j}(e)} - (\nu - \mu)_{\rho_{i}(e)})$$
$$\cdot \prod_{v \subset e \subset S^{1}} \zeta_{\rho_{i}(e)\rho_{i}(v)\rho_{j}(v)} \zeta_{\rho_{i}(e)\rho_{j}(e)\rho_{j}(v)}^{-1} \cdot \prod_{v \subset e \subset S^{1}} (p^{-1}q)_{\rho_{i}(e)\rho_{j}(e)}.$$

On the other hand, inserting the data of L_{ζ} into (3.27), gives

$$\prod_{e \subset S^{1}} \exp\left(-\int_{e} \xi^{*} \lambda_{\rho_{i}(e)\rho_{j}(e)}\right) \cdot \prod_{v \subset e \subset S^{1}} \zeta_{\rho_{i}(e)\rho_{i}(v)\rho_{j}(v)}^{-1} \zeta_{\rho_{i}(e)\rho_{j}(e)\rho_{j}(v)} \zeta_{\rho_{i}(e)\rho_{j}(e)\rho_{j}(v)}$$

$$\stackrel{(3.38)}{=} \exp \int_{e} \xi^{*} \left((\nu - \mu)_{\rho_{i}(e)} - (\nu - \mu)_{\rho_{j}(e)}\right) \cdot \prod_{v \subset e \subset S^{1}} (pq^{-1})_{\rho(e_{i})\rho(e_{j})}$$

$$\cdot \prod_{v \subset e \subset S^{1}} \zeta_{\rho_{i}(e)\rho_{i}(v)\rho_{j}(v)}^{-1} \zeta_{\rho_{i}(e)\rho_{j}(e)\rho_{j}(v)}$$

as the transition functions of $\mathfrak{T}(L_{\zeta})$, but these are simply inverses of the transition functions of $\operatorname{Tr} \operatorname{hol}(S^1,A)_{(\tau,\rho)}$. Therefore, $\operatorname{Tr} \operatorname{hol}(S^1,A)$ defines a trivializing section of the dual $\mathfrak{T}(L_{\zeta}^*)$, which is precisely what we wanted. We denote the pullback $r_{\Sigma}^*(\operatorname{Tr} \operatorname{hol}(S^1,A))$ onto $\operatorname{Map}_{\Sigma}(M,X)$ also by $\operatorname{Tr} \operatorname{hol}(S^1,A)$. The above computations can easily be generalized to the case where ∂M consists of finitely many circles, in which case the transgressed bundles over $\operatorname{Map}_{\Sigma}(\partial M,X)$ should be understood as tensor products of bundles transgressed over $\operatorname{Map}_{\Sigma}(S^1,X)$. Then $\operatorname{Tr} \operatorname{hol}(\partial M,A)$ becomes a tensor product of $\operatorname{Tr} \operatorname{hol}(S^1,A)$ over all the boundary circles.

We can now put all the pieces together to construct a well-defined function

pfaff · hol
$$(M, B)$$
 · Tr hol $(\partial M, A)$ · Γ^{-1}

over the space $\operatorname{Map}_{\Sigma}(M,X)$. Thus, the Freed-Witten anomaly is cancelled, when $[H]|_{\Sigma} \in \operatorname{Tor} H^3(\Sigma,\mathbb{Z})$, whenever the cohomology equation

$$[H]|_{\Sigma} = W_3(\Sigma) + \beta([\zeta]) \tag{3.41}$$

holds.

Even if $[H]|_{\Sigma} \neq \text{Tor } H^3(\Sigma, \mathbb{Z})$, by allowing the A-field to be a U_1 bundle gerbe module connection, the anomaly can still be cancelled when (3.41) holds. The interested reader is referred to [CJM] for a treatment of the nontorsion case. We shall not discuss it.

Chapter 4

Twisted D-Branes And Noncommutative Generalizations

By a twisted D-brane, we mean a D-brane in the presence of a nonvanishing background B-field. We start this chapter by extending the K-theoretic classification to twisted D-branes. This is done by replacing K-theory with twisted K-theory [Ros, Bla, AS1, BM, CW, BCMMS]. The Chan-Paton bundle of a stack of n coincident D-branes, is generally a PU(n), or even $PU(\mathcal{H})$, bundle. Just as a configuration of D-branes with U(n) Chan-Paton bundles yields a K-theory class $x \in K^0(\Sigma)$, a configuration of D-branes with projective bundles yields a twisted K-theory class of Σ . To express the charge in terms of spacetime quantities, we need a Gysin map in twisted K-theory. More precisely, we need a generalization of the concept of K-orientability, which is equivalent to the existence of twisted fundamental classes in a dual twisted K-homology [Wan, Sza1] theory. Twisted K-theory and twisted K-homology can be defined using K-theory and K-homology of C^* -algebras [Kas3, Ros, HR]. This takes us automatically to the world of noncommutative geometry [Con, GVF, Mad, Lan2], since K-theory and K-homology of C^* -algebras are the noncommutative generalizations of K-theory and K-homology of topological spaces.

The philosophy behind noncommutative topology is that any locally compact space X can be completely described by its algebra $C_0(X)$ of continuous functions $X \to \mathbb{C}$, vanishing at infinity. Clearly $C_0(X)$ is a commutative C^* -algebra¹. It turns out that a number of geometric concepts for X can be described as purely algebraic constructions for $C_0(X)$. In particular, Spin^c-manifolds can be described purely algebraically using objects called *spectral triples* [Con, GVF]. Having obtained an algebraic definition for the geometric objects of interest, we can easily change the C^* -algebra to any other, generally noncommutative, C^* -algebra, resulting in what is called noncommutative geometry. Unfortunately, we can not discuss noncommutative geometry in detail in this thesis, as it would take a huge number of pages to even get through the basics.

Having described twisted K-theory and K-homology in terms of C^* -algebras, we can easily replace the C^* -algebras of the twisted D-branes (and of the spacetime) with more general noncommutative C^* -algebras. To proceed, we need a unification of K-theory and K-homology of C^* -algebras. Such a bivariant K-theory is the KK-theory of Kasparov [Kas1,

¹We assume the reader is familiar with the basics of C^* -algebras. A friendly introduction is given in [GVF]. A more extensive treatment can be found in [Ped].

Kas2, Bla, KT]. Our goal is to make sense of the Minasian-Moore formula (2.40) in the noncommutative case [BMRS1, BMRS2, Sza2].

4.1 Classifying twisted D-branes

Let A be a C^* -algebra. We always assume our C^* -algebras to be separable, although this is not required for most of the definitions. From the point of view of noncommutative topology, separability of the function algebra translates to metrizability of the topological space [GVF]. The C^* -algebras are *not* assumed to be unital in general. A topological space is compact if and only if its function algebra is unital. Allowing the C^* -algebras to be nonunital, translates to allowing the topological spaces to be locally compact².

The C^* -algebra A can be made into a unital C^* -algebra A^+ by adjoining a unit as follows. Define $A^+ := A \oplus \mathbb{C}$, multiplication

$$(a, \lambda) \cdot (b, \mu) := (ab + \lambda b + \mu a, \lambda \mu).$$

In the commutative picture, this corresponds to one-point compactification of the locally compact space. If A is already unital, A^+ is not isomorphic to A^3 . It is also useful to define

$$\widetilde{A} := \begin{cases} A & \text{if } A \text{ is unital,} \\ A^+ & \text{if } A \text{ is nonunital.} \end{cases}.$$

Recall that an *idempotent* of a C^* -algebra A is an element $e \in A$, such that $e^2 = e$. A projection $p \in A$ is a self-adjoint idempotent: $p^* = p^2 = p$. Two projections, p and q, are said to be unitarily equivalent, if $q = upu^*$, for some unitary element $u \in \widetilde{A}$. Unitary equivalence defines an equivalence relation \sim_u . The set of unitary equivalence classes of projections of A is denoted by $\mathcal{P}(A)$

Let $M_n(A)$ denote the C^* -algebra $A \otimes M_n(\mathbb{C})$ of $n \times n$ matrices, with entries in A. There are natural inclusions, $M_n(A) \hookrightarrow M_{n+1}(A)$, given by diagonal embeddings $a \mapsto \operatorname{diag}(a,0)$, for $a \in M_n(A)$. Taking the limit of n to infinity, yields the C^* -algebra of compact operators on \mathcal{H} , tensored with A:

$$A \otimes \mathcal{K} := \varinjlim_{n} M_n(A)$$
.

Intuitively, it is the C^* -algebra of infinite A-valued matrices, with a finite number of nonzero entries

Let $V(A) := \mathcal{P}(A \otimes \mathcal{K})$. For elements $[p], [q] \in V(A)$, define

$$[p]+[q]:=[p\oplus q]\,,$$

where

$$p \oplus q := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} .$$

This makes V(A) an Abelian monoid, with identity [0]. It is Abelian, since

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \sim_u \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} \,.$$

 $^{^{2}}$ Of course, if the C^{*} -algebra is noncommutative, it does not represent any actual topological space.

³Compare this to adjoining a disjoint point to a compact space.

V(A) is stable, meaning that

$$V(A \otimes \mathcal{K}) = \mathcal{P}((A \otimes \mathcal{K} \otimes \mathcal{K})) = \mathcal{P}(A \otimes \mathcal{K}) = V(A), \qquad (4.1)$$

where we have used $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$.

Definition 4.1.1. Let A be a unital C^* -algebra. The *(operator)* K-theory group $K_0(A)$ is the Grothendieck group of V(A). In other words, it is the Abelian group generated by formal differences [p] - [q].

The simplest nontrivial C^* -algebra is \mathbb{C} . If P is a projection in $M_n(\mathbb{C})$, then, as a self-adjoint matrix, it is diagonalizable by a unitary matrix $U: D = UPU^*$, where D is diagonal. Now, $P^2 = U^*DUU^*DU = U^*D^2U$, but on the other hand $P^2 = P = U^*DU$. Thus, D is also a projection and all its entries must necessarily be 1. Hence, all projections of rank n belong to the same unitary equivalence class. Thus, $V(\mathbb{C}) \cong \mathbb{N}_0$ and $K_0(\mathbb{C}) \cong \mathbb{Z}$.

Lemma 4.1.2. For C^* -algebras A_1 and A_2 ,

$$K_0(A_1 \oplus A_2) \cong K_0(A_1) \oplus K_0(A_2)$$
. (4.2)

Proof. Let $\pi_k: A_1 \oplus A_2 \to A_k$ denote the projections and $i_k: A_k \to A_1 \oplus A_2$ the inclusions. The split exact sequence

$$0 \longrightarrow A_1 \xrightarrow{i_1} A_1 \oplus A_2 \xleftarrow{\pi_2} A_2 \longrightarrow 0$$

induces a split exact sequence [Bla]

$$0 \longrightarrow K_0(A_1) \xrightarrow{(i_1)_*} K_0(A_1 \oplus A_2) \xrightarrow{(\pi_2)_*} K_0(A_2) \longrightarrow 0$$

in K-theory. The commutative diagram

$$0 \longrightarrow K_0(A_1) \xrightarrow{(i_1)_*} K_0(A_1 \oplus A_2) \xrightarrow{(\pi_2)_*} K_0(A_2) \longrightarrow 0$$

$$\cong \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow (\pi_1)_* \oplus (\pi_2)_* \qquad \cong \downarrow \qquad \qquad \cong \downarrow$$

$$0 \longrightarrow K_0(A_1) \longrightarrow K_0(A_1) \oplus K_0(A_2) \longrightarrow K_0(A_2) \longrightarrow 0$$

has exact rows. Since all the vertical arrows, except the one in the middle, are isomorphisms, the *five lemma* [BT] implies that also the middle one is an isomorphism. \Box

Definition 4.1.3. Let A be a nonunital C^* -algebra. The natural map $\pi: A^+ \to \mathbb{C}$ induces a homomorphism $\pi_*: K_0(A^+) \to K_0(\mathbb{C}) \cong \mathbb{Z}$. Reduced operator K-theory of A^+ is defined as the subgroup

$$\widetilde{K}_0(A^+) := \ker(\pi_*) \subset K_0(A^+)$$
.

For the nonunital C^* -algebra A we define

$$K_0(A) := \widetilde{K}_0(A^+).$$

If A is unital, (4.2) implies

$$K_0(A^+) = K_0(A \oplus \mathbb{C}) \cong K_0(A) \oplus K_0(\mathbb{C}) \cong K_0(A) \oplus \mathbb{Z},$$

with

$$\ker(\pi_*) = K_0(A) \oplus 0 \cong K_0(A).$$

Thus, the definition

$$K_0(A) := \widetilde{K}_0(A^+)$$

for both unital and nonunital C^* -algebras.

As a Grothendieck group, K_0 satisfies a similar universality property as K^0 for vector bundles (see (2.14)). Let A and B be C^* -algebras and $f:A\to B$ a *-homomorphism. There is an induced homomorphism $f_*:V(A)\to V(B)$, which descends to a homomorphism $f_*:K_0(A)\to K_0(B)$. Moreover, homotopic maps $f,g:A\to B$ induce the same map $f_*=g_*$ in K-theory [Bla]. It follows from (4.1), that K-theory is stable, or $Morita\ invariant$, meaning that

$$K_0(A \otimes \mathcal{K}) \cong K_0(A)$$
. (4.3)

Proposition 4.1.4. Let \mathbf{SC}^* be the category of separable C^* -algebras. Operator K-theory defines a covariant functor $K_0: \mathbf{SC}^* \to \mathbf{Ab}$.

For a C^* -algebra A, we define the suspension C^* -algebra

$$\Sigma A := C_0(\mathbb{R}) \otimes A. \tag{4.4}$$

Definition 4.1.5. Higher operator K-functors K_n , $n \in \mathbb{N}$, are covariant functors $K_n : \mathbf{SC}^* \to \mathbf{Ab}$. For a C^* -algebra A, we set

$$K_n(A) := K_0(\Sigma^n A) = \widetilde{K}_0((\Sigma^n A)^+).$$

Theorem 4.1.6 (Bott periodicity [Bla]). There are canonical isomorphisms

$$K_n(A) \cong K_{n+2}(A)$$
.

Definition 4.1.7. By Bott periodicity, there are only two inequivalent operator K-functors: K_0 and K_1 . Thus, we write

$$K_{\bullet}(A) := K_0(A) \oplus K_1(A)$$
.

To understand the relation between operator K-theory and K-theory of vector bundles, we need the following basic result of noncommutative geometry. Let X be compact, $E \to X$ be a U(n) vector bundle and $\Gamma(E) := C(X, E)$ the group of continuous sections of E. It is a module over the C^* -algebra C(X) and is compatible with direct sum and tensor product of vector bundles E, F over X:

$$\Gamma(E) \oplus \Gamma(F) \cong \Gamma(E \oplus F)$$
 and $\Gamma(E) \otimes_{C(X)} \Gamma(F) \cong \Gamma(E \otimes F)$.

Finally, Γ sends dual bundles to dual modules:

$$\Gamma(E^*) \cong \operatorname{Hom}_{C(X)}(\Gamma(E), C(X))$$
.

It is proved in [GVF] that $\Gamma(E)$ is a projective module of rank n, meaning that there exists a projection $p \in C(X)^n$, such that $\Gamma(E) \cong pC(X)^n$.

Theorem 4.1.8 (Serre-Swan [GVF]). The functor Γ , from $\mathbf{Vect}_{\mathbb{C}}(X)$ to the category of finite rank projective modules over C(X), is an equivalence of categories. Vector bundles of rank n correspond to projective modules of rank n.

The Serre-Swan theorem provides the noncommutative generalization of finite rank vector bundles as finite rank projective modules of C^* -algebras.

Corollary. Let X^{∞} be a compact pointed space and $E \to X^{\infty}$ a U(n) vector bundle, corresponding to the projective module $pC(X^{\infty})^n$, the projection p defines a class $[p] \in K_0(C(X^{\infty}))$. The Serre-Swan theorem induces the isomorphism

$$K^0(X^\infty) \cong K_0(C(X^\infty))$$
.

Using $C(\Sigma X^{\infty}) \cong \Sigma(C(X^{\infty}))$, we get

$$K^{\bullet}(X^{\infty}) \cong K_{\bullet}(C(X^{\infty}))$$
.

If X is locally compact, there is a similar result, but with the function algebra replaced by the algebra $C_0(X)$ of functions vanishing at infinity:

$$K^{\bullet}(X) \cong K_{\bullet}(C_0(X))$$
.

K-theory can also be defined for more general algebras, for example local C^* -algebras [Bla]. Most importantly, the smooth function algebra $C_0^{\infty}(X)$ is a local C^* -algebra. The algebra $C_0(X)$ is the C^* -norm completition of $C_0^{\infty}(X)$. K-theory is invariant under such norm completitions [Bla]:

$$K_{\bullet}(C_0^{\infty}(X)) \cong K_{\bullet}(C_0(X))$$
.

Rosenberg [Ros] defines twisted K-theory by modifying the algebra $C_0(X)$ as follows. Let $\sigma \in H^3(X,\mathbb{Z})$ and $P_{\sigma} \to X$ a principal $PU(\mathcal{H})$ bundle⁴, with Dixmier-Douady class σ . By lemma 3.4.6, P_{σ} is unique up to isomorphism. Using the action of $PU(\mathcal{H})$ on \mathcal{K} , we can form the associated bundle

$$\mathcal{E}_{\sigma} := P_{\sigma} \times_{PU(\mathcal{H})} \mathcal{K} \to X$$
,

with fibres isomorphic to \mathcal{K} .

Definition 4.1.9 ([Ros]). The σ -twisted K-theory groups of X are defined by

$$K^{\bullet}(X,\sigma) := K_{\bullet}(C_0(X,\mathcal{E}_{\sigma})).$$

They are independent of the representative of the isomorphism class of $PU(\mathcal{H})$ bundles with Dixmier-Douady class σ .

Untwisted K-theory can be recovered by setting $\sigma = 0$. Then $\mathcal{E}_0 = X \times \mathcal{K}$, $C_0(X, \mathcal{E}_0) = C_0(X, \mathcal{K}) \cong C_0(X) \otimes \mathcal{K}$ and

$$K^{\bullet}(X,0) = K_{\bullet}(C_0(X) \otimes \mathcal{K}) \stackrel{(4.3)}{\cong} K_{\bullet}(C_0(X)) = K^{\bullet}(X).$$

⁴Recall that as a structure group of a principal $PU(\mathcal{H})$ bundle, $PU(\mathcal{H})$ is endowed with the quotient topology of $U(\mathcal{H})/U(1)$, where $U(\mathcal{H})$ has the strong operator topology.

Twisted K-theory, with a torsion twisting class $\sigma \in \text{Tor } H^3(X,\mathbb{Z})$, can be thought of as K-theory of σ -twisted vector bundles. Recall proposition 3.4.5, which explained the relationship between finite rank bundle gerbe modules for a bundle gerbe with Dixmier-Douady class σ , and finite rank projective bundles with Dixmier-Douady class σ over the base space X. Thus, it is not very surprising that σ -twisted K-theory can be defined in terms of bundle gerbe modules.

Definition 4.1.10 ([BCMMS]). Let (L, Y) be a bundle gerbe over X with Dixmier-Douady class $\sigma \in \text{Tor } H^3(X, \mathbb{Z})$. We define $K_{\text{bg}}(X, \sigma)$ to be the Grothendieck group of Mod(L). It is, of course, independent of the choice of the bundle gerbe (L, Y).

We shall not prove the following.

Proposition 4.1.11 ([BCMMS]). For any torsion twisting class $\sigma \in \text{Tor } H^3(X,\mathbb{Z})$, there is an isomorphism

$$K_{bq}(X,\sigma) \cong K^0(X,\sigma)$$
.

If σ is nontorsion, we have to use U_1 bundle gerbe modules instead⁵.

Definition 4.1.12. Let (L, Y) be a bundle gerbe over X with nontorsion Dixmier-Douady class $\sigma \in H^3(X, \mathbb{Z})$. We define $K_{U_1}(X, \sigma)$ to be the Grothendieck group of $\operatorname{Mod}_{U_1}(L)$. It is, of course, independent of the choice of the bundle gerbe (L, Y).

Proposition 4.1.13 ([BCMMS]). For any nontorsion twisting class $\sigma \in H^3(X,\mathbb{Z})$, there is an isomorphism

$$K_{U_1}(X,\sigma) \cong K^0(X,\sigma)$$
.

Twisted K-theory can also be defined homotopically by modifying the spectrum of K-theory. We have no need for this approach so we shall not discuss it. An extensive treatment is given in [AS1].

Let $\phi: \Sigma \to X$ be a twisted D-brane with PU(n) Chan-Paton bundle P. Let $\{h_{\alpha\beta}\}$ be U(n) lifts of the transition functions of P, satisfying

$$h_{\alpha\beta}h_{\beta\gamma}h_{\alpha\gamma}^{-1} = \zeta_{\alpha\beta\gamma}.$$

Freed-Witten anomaly cancellation requires that

$$\beta([\underline{\zeta}]) = [H]|_{\Sigma} + W_3(\Sigma).$$

Thus, P is a $[H]|_{\Sigma} + W_3(\Sigma)$ -twisted principal PU(n) bundle. The trivial rank n bundle gerbe module for its lifting bundle gerbe defines a class

$$[P] \in K_{\text{bg}}(X, [H]|_{\Sigma} + W_3(\Sigma)) \cong K^0(X, [H]|_{\Sigma} + W_3(\Sigma)).$$

It is clear how coincident anti-D-branes can be taken into account.

If $[H]|_{\Sigma}$ is nontorsion, the Chan-Paton bundle P is described by a U_1 bundle gerbe module, which looks locally like a $[H]|_{\Sigma} + W_3(\Sigma)$ -twisted principal U_1 bundle P, or a $PU(\mathcal{H})$ bundle,

⁵The reason why $U(\mathcal{H})$ bundle gerbe modules are not interesting is that, while such a module would indeed yield a $PU(\mathcal{H})$ bundle over X, it is the only principal $PU(\mathcal{H})$ bundle with that precise Dixmier-Douady class. Thus, all $U(\mathcal{H})$ bundle gerbe modules are actually stably isomorphic.

whose transition functions lift to $U_1 \subset U(\mathcal{H})$, satisfying a $[H]|_{\Sigma} + W_3(\Sigma)$ -twisted cocycle condition. We fix a basepoint for the compact space Σ . Then, the U_1 bundle gerbe module determines a class

$$[P] \in K_{U_1}(\Sigma, [H]|_{\Sigma} + W_3(\Sigma)) \cong K^0(\Sigma, [H]|_{\Sigma} + W_3(\Sigma)).$$

Again, coincident anti-D-branes can be taken into account in the obvious way.

The goal is to obtain a generalization of the Minasian-Moore formula (2.40) to the twisted case. What we need, is a twisted K-orientability condition to be able to define a Gysin map from $K^{\bullet}(\Sigma, [H]|_{\Sigma} + W_3(\Sigma))$ into twisted K-theory of spacetime. We shall later discuss K-orientability in a much more general context, when we have the powerful machinery of KK-theory at our disposal. Hence, we now only present the result of [CW].

Theorem 4.1.14 ([CW]). Let $f: X \to Y$ be a continuous map⁶, $\sigma \in H^3(Y, \mathbb{Z})$,

$$W_3(f) := \beta(w_2(X) - f^*w_2(Y))$$

and $d := \dim(Y) - \dim(X)$. Then, there exists a natural Gysin homomorphism

$$f_!: K^{\bullet}(X, \sigma|_{\Sigma} + W_3(f)) \to K^{\bullet + d}(Y, \sigma)$$
.

If both X and Y are both K-orientable (Spin^c) and $\sigma = 0$, then $f_!: K^{\bullet}(X) \to K^{\bullet+d}(Y)$ coincides with definition 2.3.20.

Theorem 4.1.14 implies that there exists a Gysin homomorphism

$$\phi_!: K^0(\Sigma, [H]|_{\Sigma} + W_3(\Sigma)) \to K^0(X, [H]).$$

Consider now a configuration of twisted D-branes wrapping $\phi : \Sigma \to X$, with twisted K-theory class $x \in K^0(\Sigma, [H]|_{\Sigma} + W_3(\Sigma))$. The quantized RR-charge of this D-brane configuration is the [H]-twisted K-theory class

$$Q_{\mathbb{Z}}(\Sigma, x) := \phi_!(x) \in K^0(X, [H]),$$

generalizing (2.42).

Next we need a twisted generalization of the Chern character. The ordinary Chern character admits a purely algebraic description, which allows a generalization to noncommutative algebras, thus yielding the *Chern-Connes character*, which takes values in something called periodic cyclic homology, $HP_{\bullet}(-)$ [Con, GVF, Cun]. Periodic cyclic homology is the noncommutative generalization of (periodized) de Rham cohomology in the sense that for a compact manifold X, we have

$$H_{\mathrm{dR}}^{\bullet}(X) \cong \mathrm{HP}_{\bullet}(C^{\infty}(X))$$
,

where \bullet = even/odd. Similarly, *periodic cyclic cohomology* generalizes de Rham homology of X. Periodic cyclic (co)homology degenerates for C^* -algebras. That is why we used the

⁶In [CW] the authors restrict to differentiable maps. However, all differentiable maps between manifolds are homotopic to continuous maps [BT] and the Gysin map depends only on the homotopy class of the map. Hence, the result of [CW] generalizes to continuous maps.

smooth function algebra instead. For a local C^* -algebra \mathcal{A} the Chern-Connes character is a homomorphism

$$\operatorname{ch}: K_{\bullet}(\mathcal{A}) \to \operatorname{HP}_{\bullet}(\mathcal{A}).$$
 (4.5)

Let $P \to X$ be a principal $PU(\mathcal{H})$ bundle with Dixmier-Douady class σ . The algebra $C_0^{\infty}(X, P \times_{PU(\mathcal{H})} \mathcal{L}^1)$, where $\mathcal{L}^1 \subset \mathcal{K}$ is the ideal of trace class operators of \mathcal{H} , is a local C^* -algebra, with C^* -norm completition $C_0(X, \mathcal{E}_{\sigma})$. The generalized Hochschild-Connes-Kostant-Rosenberg theorem of Mathai and Stevenson [MS3] states that $\operatorname{HP}_{\bullet}(C_0^{\infty}(X, P \times_{PU(\mathcal{H})} \mathcal{L}^1))$ is isomorphic to σ -twisted (de Rham) cohomology. Luckily, twisted cohomology is very easy to define.

Definition 4.1.15 ([MS2, MS3, BCMMS, AS2]). Let ζ be a closed de Rham 3-form. The ζ -twisted de Rham differential is

$$d_{\zeta} := d - \zeta$$
,

which operates on differential forms $\Omega^{\text{even/odd}}(X)$ by $d_{\zeta}(\omega) = d\omega - \zeta \wedge \omega^{7}$. It is easy to see that $d_{\zeta}^{2} = 0$:

$$d_{\zeta}^{2}\omega = d^{2}\omega - d(\zeta \wedge \omega) - \zeta \wedge d\omega + \zeta \wedge \zeta \wedge \omega = 0 - d\zeta \wedge \omega + \zeta \wedge d\omega - \zeta \wedge d\omega + 0 = 0.$$

The (periodized) ζ -twisted de Rham cohomology groups are

$$H^{\bullet}(X,\zeta) := \ker(d_{\zeta})/\operatorname{im}(d_{\zeta}),$$

where $\bullet = \text{even/odd}$. Moreover, adding an exact 3-form $d\eta$ to ζ leaves the twisted de Rham cohomology group invariant:

$$H^{\bullet}(X, \zeta + d\eta) \cong H^{\bullet}(X, \zeta)$$
.

On the level of cocycles, the isomorphism is given by

$$\omega \mapsto e^{\eta} \wedge \omega$$
.

It is easy to see to be well defined, for example

$$(d - (\zeta + d\eta))(e^{\eta} \wedge \omega) = d\eta \wedge e^{\eta} \wedge \omega + e^{\eta} \wedge d\omega - (\zeta + d\eta) \wedge (e^{\eta} \wedge \omega)$$
$$= e^{\eta} \wedge d\omega - \zeta \wedge e^{\eta} \wedge \omega$$
$$= e^{\eta} \wedge (d\omega - \zeta \wedge \omega) = 0.$$

Thus, de Rham cohomology is most naturally twisted by the cohomology class $[\zeta] \in H^3_{dR}(X)$. More generally, if $\sigma \in H^3(X,\mathbb{Z})$, σ -twisted (de Rham) cohomology is defined by

$$H^{\bullet}(X,\sigma) := H^{\bullet}(X,\sigma_{\mathrm{dR}}),$$

where σ_{dR} is the image of σ in $H^3_{dR}(X)$.

If $\mu \in \Omega^{\bullet}(X)$ is d-closed and ω represents a class in $H^{\bullet}(X, \sigma)$, then $d_{\sigma_{\mathrm{dR}}}(\omega \wedge \mu) = 0$. This action makes $H^{\bullet}(X, \sigma)$ into a $H^{\bullet}(X)$ module.

⁷Clearly, d_{ζ} maps $\Omega^{\text{even}}(X)$ to $\Omega^{\text{odd}}(X)$ and vice versa.

For $\sigma \in H^3(X,\mathbb{Z})$, the twisted Chern character is a homomorphism

$$\operatorname{ch}_{\sigma}: K^{\bullet}(X, \sigma) \to H_{c}^{\bullet}(X, \sigma),$$
 (4.6)

where $H_c^{\bullet}(X, \sigma)$ denotes compactly supported σ -twisted cohomology, defined by restricting to compactly supported differential forms. It can be defined as a special case of (4.5), where $\mathcal{A} = C_0^{\infty}(X, P \times_{PU(\mathcal{H})} \mathcal{L}^1)$. Another definition was given in [BCMMS] using bundle gerbe module connections. We take a slightly different approach, using local representations on X. Suppose that $\sigma \in \text{Tor } H^3(X, \mathbb{Z})$. Let (L, Y) be a bundle gerbe with connection and curving, with Dixmier-Douady class σ . Then L admits a finite rank bundle gerbe module with connection. If L is represented by the Deligne cocycle $(\underline{g}, -\underline{\Lambda}, \underline{B})$, then the connection had the local representation (3.15). By (3.33), we can rewrite the transformation as

$$A_{\beta} = h_{\alpha\beta}^{-1} A_{\alpha} h_{\alpha\beta} + h_{\alpha\beta}^{-1} dh_{\alpha\beta} + \mu_{\beta} - \mu_{\alpha} - d \log q_{\alpha\beta}.$$

Then, writing $\check{A}_{\alpha} := A_{\alpha} - \mu_{\alpha}$ and $\check{h}_{\alpha\beta} := (hq^{-1})_{\alpha\beta}$, yields

$$\breve{A}_{\beta} = \breve{h}_{\alpha\beta}^{-1} \breve{A}_{\alpha} \breve{h}_{\alpha\beta} + \breve{h}_{\alpha\beta}^{-1} \, d\breve{h}_{\alpha\beta} \, .$$

Now, $\check{F}_{\alpha} := d\check{A}_{\alpha} + \check{A}_{\alpha} \wedge \check{A}_{\alpha}$ behaves precisely like a curvature 2-form does, even though $\{\check{h}_{\alpha\beta}\}$ is not quite a cocycle. Hence, we can use it as the curvature in analytic formula (2.8) for the Chern character.

Definition 4.1.16. Suppose that $\sigma \in \text{Tor } H^3(X,\mathbb{Z})$. By proposition 4.1.11, any class $x \in K^{\bullet}(X,\Sigma)$ can be equivalently represented as the formal difference of two finite rank bundle gerbe modules. Let A_1 and A_2 be any bundle gerbe module connections for these two modules. Using the above construction, we can form the "curvatures" \check{F}_1 and \check{F}_2 from A_1 and A_2 . The σ -twisted Chern character of x is

$$ch_{\sigma}(x) := \left[\operatorname{Tr}\exp\left(-\frac{\breve{F}_1}{2\pi i}\right)\right] - \left[\operatorname{Tr}\exp\left(-\frac{\breve{F}_2}{2\pi i}\right)\right] \in H_{\mathrm{dR}}^{\mathrm{even}}(X).$$

Remark, that since σ is torsion in the above, $\sigma_{dR} = 0$ and so $H^{\bullet}(X, \sigma) = H^{\bullet}_{dR}(X)$. Thus, definition 4.1.16 is consistent with (4.6). The nontorsion case requires more careful thought. The interested reader is referred to [BCMMS, MS2, MS3].

Now that we have the twisted Chern character at our disposal, generalizing the Minasian-Moore formula (2.40) is easy. Let $x \in K^0(\Sigma, [H]|_{\Sigma} + W_3(\Sigma))$ describe a configuration of twisted D-branes, wrapping $\phi : \Sigma \to X$. The real, or classical, RR-charge of the configuration is given by the following twisted Minasian-Moore formula.

Definition 4.1.17 (Minasian-Moore).

$$Q_{\mathbb{R}}(\Sigma, x) := \operatorname{ch}_{[H]}(\phi_!(x)) \wedge \sqrt{\operatorname{Todd}(X)} \in H^{\operatorname{even}}(X, [H]).$$

The next logical step would be to define a corresponding twisted K-homology theory and describe D-branes using it. Twisted K-homology can be defined in different ways. There is C^* -algebraic definition of spectral K-homology, often referred to as analytic K-homology [HR, Kas3, BHS]. More precisely, for a compact manifold X, analytic K-homology of C(X) is isomorphic to $K_{\bullet}(X) \cong K_{\bullet}^{\text{geom}}(X)$. Then, σ -twisted K-homology is simply defined as analytic K-homology of the σ -twisted function algebra $C(X, \mathcal{E}_{\sigma})$. Just like untwisted K-homology, twisted K-homology admits a geometric description [Wan, Sza1]. We shall not discuss twisted K-homology here, but instead jump straight to KK-theory, which yields both K-theory and K-homology as special cases, thus providing a unified treatment of both cases.

4.2 KK-theory

In KK-theory, we need to impose some restrictions on the C^* -algebras we use, namely σ -unitality and separability. A σ -unital C^* algebra is one with a countable approximate identity. In particular, every separable C^* -algebra is σ -unital. Thus, for simplicity, we restrict to the category of separable C^* -algebras. Furthermore, we often need to take tensor products of C^* -algebras and it is well-known that, in general, the tensor product algebra admits several inequivalent C^* -norms. For nuclear C^* -algebras the tensor product works particularly well: if either one of the algebras in the tensor product is nuclear, the projective and injective cross norms on the product algebra, and hence all cross norms, are equivalent [Lan1]. In other words, the tensor product of any C^* -algebra with a nuclear C^* -algebra is a C^* -algebra in a unique way. To make life easier, we assume C^* -algebras to be nuclear whenever tensor products are discussed. This is not so strict a requirement, as nuclear C^* -algebras are quite abundant. For example, all commutative, finite dimensional and type I C^* -algebras are nuclear.

Our main references for this section are the books [Bla, KT]. We also take this opportunity to mention the papers [Kas1, Kas2] by Kasparov and the excellent papers [Hig1, Hig2] by Higson.

Definition 4.2.1 ([Bla, KT]). Let $B = (B, \|\cdot\|)$ be a C^* -algebra. A pre-Hilbert B-module is a complex vector space E, which is also a right B-module with a B-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to B$. The inner product is B-antilinear in the first and B-linear in the second argument:

- i) $\langle x, yb \rangle = \langle x, y \rangle b$,
- ii) $\langle x, y \rangle * = \langle y, x \rangle$,
- iii) $\langle x, x \rangle > 0$, that is, $\langle x, x \rangle = a^*a$ for some $a \in B$,
- iv) $\langle x, x \rangle = 0$ if and only if x = 0,

where $x, y \in E$ and $a, b \in B$ are arbitrary.

A *Hilbert B-module* is a pre-Hilbert *B*-module which is complete with respect to the norm $||x|| = ||\langle x, x \rangle||^{1/2}$.

Let E_1 and E_2 be Hilbert *B*-modules and assume $T: E_1 \to E_2$ to be *adjointable*, meaning that there exists a map $T^*: E_2 \to E_1$, satisfying $\langle x, Ty \rangle = \langle T^*x, y \rangle$. Then *T* is automatically a *B*-module map:

$$\langle y, T(x_1 + x_2 b) \rangle = \langle T^* y, x_1 + x_2 b \rangle = \langle T^* y, x_1 \rangle + \langle T^* y x_2 \rangle b = \langle y, T x_1 + \langle T x_2 \rangle b \rangle$$

for all $x_1, x_2 \in E_1, y \in E_2$ and $b \in B$. It is also easy to show that T and T^* are bounded and, in fact, $||T|| = ||T^*||$. The set of adjointable maps $E_1 \to E_2$ is denoted by $\mathcal{L}_B(E_1, E_2)$. If $E_1 = E_2 = E$, we write simply $\mathcal{L}_B(E) = \mathcal{L}_B(E, E)$. It can easily be shown that $\mathcal{L}_B(E)$ is a C^* -algebra with respect to the operator norm [KT]. For any $x, y \in E$, the operators

$$\Theta_{x,y}(z) = x \langle y, z \rangle$$

in E are elements of $\mathcal{L}_B(E)$. The norm completion of their linear span yields an ideal $\mathcal{K}_B(E)$ of $\mathcal{L}_B(E)$, which generalizes the ideal of compact operators on a separable Hilbert space.

Definition 4.2.2 ([Bla, KT]). Let A and B be C^* -algebras. An odd Kasparov (A, B)-module⁸ is a triple $\mathcal{E} = (E, \phi, F)$, where

- i) E is a countably generated Hilbert B-module,
- ii) $\phi: A \to \mathcal{L}_B(E)$ is a *-representation and
- iii) $F \in \mathcal{L}_B(E)$ is a self-adjoint operator, satisfying $(F^2 1)\phi(a) \in \mathcal{K}_B(E)$ and $[F, \phi(a)] \in \mathcal{K}_B(E)$.

Likewise, an even Kasparov (A, B)-module is a triple $\mathcal{E} = (E, \phi, F)$, where

- i) $E = E^+ \oplus E^-$ is a countably generated \mathbb{Z}_2 -graded Hilbert B-module,
- ii) $\phi: A \to \mathcal{L}_B(E)$ is a *-representation, such that $\phi(a)$ is an even degree map for all $a \in A$, and
- iii) $F \in \mathcal{L}_B(E)$ is a self-adjoint operator of odd degree, satisfying $(F^2 1)\phi(a) \in \mathcal{K}_B(E)$ and $[F, \phi(a)] \in \mathcal{K}_B(E)$.

A Kasparov module, even or odd, satisfying $(F^2 - 1)\phi(a) = [F, \phi(a)] = 0$ for all $a \in A$ is said to be degenerate. Two Kasparov (A, B)-modules are isomorphic, when there exists an isomorphism of the Hilbert modules, compatible, in the obvious way, with the representations ϕ and the operators F. The sets of isomorphism classes of even and odd Kasparov (A, B)-modules are denoted by $\mathbb{E}_0(A, B)$ and $\mathbb{E}_1(A, B)$, respectively. Similarly, the sets of even and odd degenerate Kasparov (A, B)-modules are denoted by $\mathbb{D}_0(A, B)$ and $\mathbb{D}_1(A, B)$.

Both $\mathbb{E}_0(A, B)$ and $\mathbb{E}_1(A, B)$ are Abelian monoids. The addition is given by the direct sum of Kasparov modules:

$$(E_1, \phi_1, F_1) \oplus (E_2, \phi_2, F_2) = (E_1 \oplus E_2, \phi_1 \oplus \phi_2, F_1 \oplus F_2),$$

where, in the even case, the grading of $E_1 \oplus E_2$ is given by

$$(E_1 \oplus E_2)^+ \oplus (E_1 \oplus E_2)^- = (E_1^+ \oplus E_2^+) \oplus (E_1^- \oplus E_2^-).$$

Definition 4.2.3 ([Bla, KT]). Let $\mathcal{E}_1 = (E_1, \phi_1, F_1)$ and $\mathcal{E}_2 = (E_2, \phi_2, F_2)$ be even (resp. odd) Kasparov (A, B)-modules. If $U : E_2 \to E_1$ is a graded (resp. ungraded) unitary isomorphism, such that $F_2 = U^*F_1U$ and $\phi_2 = U^*\phi_1U$, then we call \mathcal{E}_1 and \mathcal{E}_2 unitarily equivalent and write $\mathcal{E}_1 \approx_u \mathcal{E}_2$. If there exists a family $\mathcal{F}_t = (E, \phi, F_t)$, $t \in [0, 1]$ of even (resp. odd) Kasparov (A, B)-modules, for which $t \mapsto F_t$ is norm-continuous, $\mathcal{E}_1 \approx_u \mathcal{F}_0 = (E, \phi, F_0)$ and $\mathcal{E}_2 \approx_u \mathcal{F}_1 = (E, \phi, F_1)$, we call \mathcal{E}_1 and \mathcal{E}_2 operator homotopic. Furthermore, if \mathcal{E}_1 and \mathcal{E}_2 are operator homotopic modulo direct sum by even (resp. odd) degenerate modules, that is, there exist degenerate even (resp. odd) Kasparov (A, B)-modules \mathcal{F}_1 and \mathcal{F}_2 such that $\mathcal{E}_1 \oplus \mathcal{F}_1$ is operator homotopic to $\mathcal{E}_2 \oplus \mathcal{F}_2$, then we write $\mathcal{E}_1 \sim_{\text{oh}} \mathcal{E}_2$.

 $^{^8}KK$ -theory is often defined using Kasparov modules with both C^* -algebras and Hilbert modules \mathbb{Z}_2 -graded. We have used this "simplified" approach for two reasons. First, it is perhaps slightly easier to comprehend and second, it fits more naturally to our purposes.

Definition 4.2.4 ([Bla, KT]). For C^* -algebras A and B, the KK-theory groups $KK_0(A, B)$ and $KK_1(A, B)$ are defined as

$$KK_0(A,B) = \mathbb{E}_0(A,B)/\sim_{\mathrm{oh}}$$
 and $KK_1(A,B) = \mathbb{E}_1(A,B)/\sim_{\mathrm{oh}}$.

We omit the proof of the following proposition. It is a straightforward application of the definitions, though somewhat tedious, if written out in full detail.

Proposition 4.2.5. Both $KK_0(A, B)$ and $KK_1(A, B)$ are Abelian groups, with addition induced by the direct sum of Kasparov modules. All degenerate even (resp. odd) Kasparov (A, B)-modules belong to same equivalence class, which is the zero element of $KK_0(A, B)$ (resp. $KK_1(A, B)$). If (E, ϕ, F) is an even Kasparov (A, B)-module, its inverse in $KK_0(A, B)$ is given by $(E^{\circ}, \phi, -F)$, where E° denotes E with opposite grading. If (E, ϕ, F) is odd, the inverse in $KK_1(A, B)$ is given by $(E, \phi, -F)$.

Definition 4.2.6. We write

$$KK_{\bullet}(A,B) := KK_0(A,B) \oplus KK_1(A,B)$$
.

The lower index is always assumed to be in \mathbb{Z}_2^9 .

A key aspect of KK-theory is the existence of a powerful product structure on KK_{\bullet} . It is notoriously difficult to define, however, and even the proof for existence is far beyond the scope of our treatment. In fact, the original proof by Kasparov was nonconstructive and extremely complicated [Kas1]. For a simpler constructive proof, see [KT] or [Bla]. In any case, the proof is too long to be presented here.

Theorem 4.2.7 ([Bla, KT]). For separable C^* -algebras A, B, C, there exists a bilinear associative composition product

$$\otimes_C: KK_i(A,C) \times KK_i(C,B) \to KK_{i+j}(A,B)$$
.

The composition product is not the most general product structure in KK-theory. Let $\alpha \in KK_i(A, B)$ be represented by the Kasparov (A, B)-module (E, ϕ, F) . The dilation of x by the C^* -algebra C is a class in $KK_i(A \otimes C, B \otimes C)$, represented by the Kasparov $(A \otimes C, B \otimes C)$ -module $(E \otimes C, \phi \otimes \mathrm{id}_C, F \otimes \mathrm{id}_C)$. Dilation can be used, together with the composition product, to define the Kasparov product

$$\otimes_D: KK_i(A_1, B_1 \otimes D) \times KK_i(D \otimes A_2, B_2) \to KK_{i+i}(A_1 \otimes A_2, B_1 \otimes B_2)$$
.

For $x_1 \in KK_i(A_1, B_1 \otimes D)$ and $x_2 \in KK_i(D \otimes A_2, B_2)$, the Kasparov product is given by

$$x_1 \otimes_D x_2 = (x_1 \otimes 1_{A_2}) \otimes_{B_2 \otimes D \otimes A_2} (x_2 \otimes 1_{B_1}).$$

For $D = \mathbb{C}$, it reduces to the exterior product

$$\otimes = \otimes_{\mathbb{C}} : KK_i(A_1, B_1) \times KK_i(A_2, B_2) \to KK_{i+i}(A_1 \otimes A_2, B_1 \otimes B_2).$$

 $^{^9}KK$ -theory can actually be defined more generally than what we have presented [Kas1], in which case this follows from a KK-theoretic version of Bott periodicity.

Theorem 4.2.8 ([Kas2]). The Kasparov product

$$\otimes_D: KK_i(A_1, B_1 \otimes D) \times KK_i(D \otimes A_2, B_2) \to KK_{i+1}(A_1 \otimes A_2, B_1 \otimes B_2)$$

satisfies the following properties:

- a) It is bilinear, contravariantly functorial in A_1 and A_2 , and covariantly functorial in B_1 and B_2 .
- **b)** If $f: D_1 \to D_2$, $x_1 \in KK_i(A_1, B_1 \otimes D_1)$ and $x_1 \in KK_j(D_2 \otimes A_2, B_2)$, then $(\mathrm{id}_{B_1} \otimes f)_*(x_1) \otimes_{D_2} x_2 = x_1 \otimes_{D_1} (f \otimes \mathrm{id}_{A_2})^*(x_2)$.
- c) It is associative in the sense that if $x_1 \in KK_i(A_1, B_1 \otimes D_1), x_2 \in KK_j(D_1 \otimes A_2, B_2 \otimes D_2)$ and $x_3 \in KK_k(D_2 \otimes A_3, B_2)$, then

$$(x_1 \otimes_{D_1} x_2) \otimes_{D_2} x_3 = x_1 \otimes_{D_1} (x_2 \otimes_{D_2} x_3).$$

- d) If $x_1 \in KK_i(A_1, B_1 \otimes D_1 \otimes D)$ and $x_2 \in KK_j(D \otimes D_2 \otimes A_2, B_2)$, then $x_1 \otimes_D x_2 = (x_1 \otimes 1_{D_2}) \otimes_{D_1 \otimes D \otimes D_2} (1_{D_1} \otimes x_2)$.
- e) If $x_1 \in KK_i(A_1, B_1 \otimes D)$ and $x_2 \in KK_j(D \otimes A_2, B_2)$, then

$$(x_1 \otimes_D x_2) \otimes 1_{D_1} = (x_1 \otimes 1_{D_1}) \otimes_{D \otimes D_1} (x_2 \otimes 1_{D_1}).$$

f) The exterior product is commutative, that is, for $x_1 \in KK_i(A_1, B_1)$ and $x_2 \in KK_j(A_2, B_2)$ one has

$$x_1 \otimes x_2 = x_2 \otimes x_1$$
.

g) The class $1_{\mathbb{C}} \in KK_0(\mathbb{C}, \mathbb{C})$ is a unit for the product:

$$1_{\mathbb{C}} \otimes x = x \otimes 1_{\mathbb{C}} = x,$$

for any $x \in KK_i(A, B)$.

Remarkably, KK-theory can also be described purely axiomatically [Hig1, Bla] as follows. There exists an additive category¹⁰ \mathbf{KK} , with objects separable C^* -algebras. The groups of morphisms $\mathbf{Hom}(-,-)$ are denoted by KK(-,-). Composition of morphisms in \mathbf{KK} defines a bilinear map

$$KK(A, D) \times KK(D, B) \to KK(A, B)$$
,

called the *composition product*. There exists a functor $KK : \mathbf{SC}^* \to \mathbf{KK}$, where \mathbf{SC}^* denotes the category of separable C^* -algebras and *-homomorphisms, sending each object to itself and satisfying the following properties:

$$(g_1+g_2)\circ (f_1+f_2)=g_1\circ f_1+g_1\circ f_2+g_2\circ f_1+g_2\circ f_2$$

for any $f_1, f_2 \in \mathbf{Hom}(\mathbf{a}, \mathbf{b})$ and $g_1, g_2 \in \mathbf{Hom}(\mathbf{b}, \mathbf{c})$.

 $^{^{10}}$ By an additive category, we mean a category **C**, where every **Hom**-set is equipped with an Abelian group structure and the composition is bilinear in the sense that

i) Stability: KK sends the stabilization maps $A \to A \otimes \mathcal{K}$ and $B \to B \otimes \mathcal{K}$, given by tensoring with a rank-one projection, to isomorphisms $KK(A, A \otimes \mathcal{K})$ and $KK(B, B \otimes \mathcal{K})$ in such a way that the composition product yields group isomorphisms

$$KK(A, B) \cong KK(A, B \otimes \mathcal{K}) \cong KK(A \otimes \mathcal{K}, B)$$
.

- ii) Homotopy invariance: If $f_0, f_1 : A \to B$ are homotopic homomorphisms, meaning that there exists a mediating homomorphism $g : A \to C([0,1],B)$ such that $\operatorname{ev}_0 \circ g = f_0$ and $\operatorname{ev}_1 \circ g = f_1$, then $KK(f_1) = KK(f_2)$. It follows immediately that if A and C are homotopy equivalent, that is, there exist homomorphisms $f : A \to C$ and $g : C \to A$, such that $f \circ g$ and $g \circ f$ are homotopic to the identity homomorphisms, then $KK(C,B) \cong KK(A,B)$, where the isomorphism is given by composition with KK(f) and KK(g). Similarly, if B and D are homotopy equivalent, then $KK(A,D) \cong KK(A,B)$.
- iii) Split exactness: KK preserves split exact sequences. It follows that, for the split exact sequence

$$0 \longrightarrow J \longrightarrow D \Longrightarrow D/J \longrightarrow 0$$

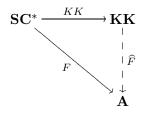
of separable C^* -algebras and *-homomorphisms and arbitrary separable C^* -algebras A and B, the sequences

$$0 \longrightarrow KK(A, J) \longrightarrow KK(A, D) \longrightarrow KK(A, D/J) \longrightarrow 0$$

$$0 \longrightarrow KK(D/J, B) \xrightarrow{\longrightarrow} KK(D, B) \longrightarrow KK(J, B) \longrightarrow 0$$

are split exact.

Suppose now that **A** is another additive category and $F: \mathbf{SC}^* \to \mathbf{A}$ a functor satisfying the above three properties, that is, stabilization maps induce isomorphisms, homotopic *-homomorphisms induce the same morphism and split exact sequences induce split exact sequences. The category \mathbf{KK} is then uniquely determined by the universal property that such an F always factors uniquely through \mathbf{KK} . More precisely, there exists a unique functor $\hat{F}: \mathbf{KK} \to \mathbf{A}$, such that the diagram



commutes.

Theorem 4.2.9. For any separable C^* -algebras A and B, the group of morphisms KK(A, B) in **KK** is given by $KK_0(A, B)$. $KK_1(A, B)$ can then be defined axiomatically by

$$KK_1(A, B) \cong KK_0(\Sigma A, B) \cong KK_0(A, \Sigma B)$$
,

where Σ is the suspension functor (4.4).

Proof. The proof is essentially contained in [Hig1]. See also [Bla] for axiomatic KK-theory.

The functor KK sends a *-homomorphisms $f:A\to B$ to the equivalence class $[f]_{KK}=KK(f)\in KK_0(A,B)$, given by the Kasparov (A,B)-module (B,f,0), where B itself is considered as a trivially graded Hilbert B-module, with inner product $\langle b_1,b_2\rangle=b_1^*b_2$. Functoriality of KK implies that for *-homomorphisms $f:A\to B$ and $g:B\to C$, the composition $g\circ f$ satisfies

$$[g \circ f]_{KK} = [f]_{KK} \otimes_B [g]_{KK} \in KK_0(A, C).$$

In this sense, classes of $KK_0(A, B)$ can be regarded as generalized *-homomorphisms between A and B. Remark also, that for any separable C^* -algebra A, the group $KK_0(A, A)$ admits a unital ring structure, with multiplication given by the composition product and multiplicative identity induced by the identity morphism id : $A \to A$.

4.3 Poincaré duality in noncommutative geometry

From now on, until the end of the text, we are more or less tacitly referring to [BMRS1, BMRS2, Sza2, CS1]. We basically follow [BMRS1, BMRS2].

In this section our primary goal is to explain the noncommutative Poincaré duality theorem [Con], which describes the precise relation between K-theory and K-homology. This is where KK-theory is needed.

Definition 4.3.1. Let A be a separable C^* -algebra. K-theory and K-homology are defined as special cases of KK-theory:

$$K_{\bullet}(A) := K_{\bullet}(\mathbb{C}, A)$$
 (K-theory)
 $K^{\bullet}(A) := K_{\bullet}(A, \mathbb{C})$ (K-homology)

It is far from clear that the above definition of K-theory is equivalent to that of definition 4.1.5. The proof can be found in [Bla].

The composition product specializes to various homomorphisms and isomorphisms in, and between, K-theory and K-homology, provided that suitable mediating classes exist. For example, any $\alpha \in KK_d(A, B)$ provides homomorphisms

$$(-) \otimes_A \alpha : K_{\bullet}(A) = KK_{\bullet}(\mathbb{C}, A) \to KK_{\bullet+d}(\mathbb{C}, B) = K_{\bullet+d}(B),$$

$$\alpha \otimes_B (-) : K^{\bullet}(B) = KK_{\bullet}(B, \mathbb{C}) \to KK_{\bullet+d}(A, \mathbb{C}) = K_{\bullet+d}(A).$$

Definition 4.3.2. If $\alpha \in KK_d(A, B)$ is invertible with respect to the composition product, that is, if there exists a class $\beta \in KK_{-d}(B, A)$, such that

$$\alpha \otimes_B \beta = 1_A \in KK_0(A, A)$$

and

$$\beta \otimes_A \alpha = 1_B \in KK_0(B, B) \,,$$

then we write $\beta = \alpha^{-1}$ and call A and B (strongly¹¹) KK-equivalent.

¹¹There is also a weaker pointwise notion of KK-equivalence given in [BMRS2].

Let A and B be KK-equivalent separable C^* -algebras and $\alpha \in KK_d(A, B)$ invertible. Then

$$KK_{\bullet}(\mathbb{C}, A) \xrightarrow{(-)\otimes_{A}\alpha} KK_{\bullet+d}(\mathbb{C}, B) \xrightarrow{(-)\otimes_{B}\alpha^{-1}} KK_{\bullet}(\mathbb{C}, A),$$
$$KK_{\bullet}(B, \mathbb{C}) \xrightarrow{\alpha\otimes_{B}(-)} KK_{\bullet+d}(A, \mathbb{C}) \xrightarrow{\alpha^{-1}\otimes_{A}(-)} K_{\bullet}(B, \mathbb{C}),$$

but on the other, by associativity, we have

$$((-) \otimes_A \alpha) \otimes_B \alpha^{-1} = (-) \otimes_A (\alpha \otimes_B \alpha^{-1}) = (-) \otimes_A 1_A,$$

$$\alpha^{-1} \otimes_A (\alpha \otimes_B (-)) = (\alpha^{-1} \otimes_A \alpha) \otimes_B (-) = 1_B \otimes_B (-).$$

Thus, $K_{\bullet}(A) \cong K_{\bullet+d}(B)$ and $K^{\bullet}(B) \cong K^{\bullet+d}(A)$.

There is a universal coefficient theorem for KK-theory, which we are going to need later. It tells us to what extent $KK_{\bullet}(A, B)$ can be computed from $K_{\bullet}(A)$ and $K_{\bullet}(B)$.

Theorem 4.3.3 (Universal coefficient theorem [Bla]). The sequence

$$0 \longrightarrow \operatorname{Ext}_{\mathbb{Z}}^{1}(K_{\bullet+1}(A), K_{\bullet}(B)) \longrightarrow KK_{\bullet}(A, B) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(K_{\bullet}(A), K_{\bullet}(B)) \longrightarrow 0$$

is exact for all separable C^* -algebras B if and only if A is KK-equivalent to a commutative separable C^* -algebra. We then say that A satisfies the universal coefficient theorem (UCT) for KK-theory.

The class of algebras satisfying the UCT is quite abundant. In fact, it is stated in [Mey] that, at the time it was written, no example of a separable nuclear C^* -algebra not satisfying the UCT is known.

A Hilbert (A, B)-bimodule is a Hilbert B-module with a left A-module structure and an A-valued inner product, which is compatible with the B-valued inner product in the obvious sense that $\langle x, y \rangle_A z = x \langle y, z \rangle_B$, for all $x, y, z \in E$. Alternatively, one can think of a Hilbert (A, B)-bimodule as a Hilbert $A^{\circ} \otimes B$ -module, where A° denotes the opposite algebra for A^{12} .

Definition 4.3.4. Two C^* -algebras, A and B, are (strongly) Morita equivalent, if there exists a full Hilbert A-module E, such that $\mathcal{K}_A(E) \cong B$. A Hilbert A-module E is full, if the norm closure of $\langle E, E \rangle$ is the whole of B. It can be shown that A and B are Morita equivalent if and only if there exists a Hilbert (A, B)-bimodule E and a Hilbert (B, A)-bimodule E^{\vee} , such that $E \otimes_B E^{\vee} \cong A$ and $E^{\vee} \otimes_A E \cong B$. E and E^{\vee} are called equivalence bimodules. For a proof, see [GVF]. We denote Morita equivalence of A and B by $A \stackrel{\text{Mor}}{\sim} B$.

If A is unital, we associate to the map $\Theta_{a,b}$ the element $\Theta_{a,b}(1) = ab^* \in A$. Conversely, to an element $a \in A$ we associate the map $\Theta_{a,1}$. Taking the norm closure yields the *-isomorphism $\mathcal{K}_A(A) \cong A$, which proves that A is Morita equivalent to itself. In the nonunital case one has to work with approximate units [GVF]. Moreover, as the name indicates, Morita equivalence determines an equivalence relation. Another important example of Morita equivalence is the following.

Proposition 4.3.5 ([GVF]). A C^* -algebra A is Morita equivalent to its stabilization $A \otimes \mathcal{K}$.

The opposite algebra A° of A has the same elements as A, but with the product reversed. For $x \in A$, we denote the corresponding element in A° by x° . The product of $x^{\circ}, y^{\circ} \in A^{\circ}$ is given by $x^{\circ}y^{\circ} = (yx)^{\circ}$.

Proposition 4.3.6. If $A \stackrel{Mor}{\sim} B$, then A is KK-equivalent to B.

Proof. Denote the equivalence bimodules by E (an (A,B)-bimodule) and E^{\vee} (a (B,A)-bimodule). Let $\alpha \in KK_0(A,B)$ and $\beta \in KK_0(B,A)$ be classes represented by the Kasparov modules $(E, \mathrm{id}_A, 0)$ and $(E^{\vee}, \mathrm{id}_B, 0)$, respectively. Both E and E^{\vee} are taken trivially graded. The composition product $\alpha \otimes_B \beta$ is represented by the Kasparov (A,A)-module

$$(E \otimes_B E^{\vee}, \mathrm{id}_A \otimes \mathrm{id}_{E^{\vee}}, 0) \stackrel{\mathrm{Mor}}{=} (A, \mathrm{id}_A, 0).$$

Likewise, $\beta \otimes_A \alpha$ is represented by

$$(E^{\vee} \otimes_A E, \mathrm{id}_B \otimes \mathrm{id}_E, 0) \stackrel{\mathrm{Mor}}{=} (B, \mathrm{id}_B, 0).$$

Thus, we have $\alpha \otimes_B \beta = 1_A \in KK_0(A, A)$ and $\beta \otimes_A \alpha = 1_B \in KK_0(B, B)$, proving the KK-equivalence of A and B.

Suppose that there exist classes

$$\Delta \in KK_d(A \otimes A^{\circ}, \mathbb{C}) = K^d(A \otimes A^{\circ})$$

and

$$\Delta^{\vee} \in KK_{-d}(\mathbb{C}, A \otimes A^{\circ}) = K_{-d}(A \otimes A^{\circ}),$$

satisfying

$$\Delta^{\vee} \otimes_{A^{\circ}} \Delta = 1_A \in KK_0(A, A)$$
 and $\Delta^{\vee} \otimes_A \Delta = (-1)^d \, 1_{A^{\circ}} \in KK_0(A^{\circ}, A^{\circ})$.

The C^* -algebra A is called a $(strong^{13})$ Poincar'e duality algebra $(PD \ algebra)$. The K-homology class Δ is called the $fundamental\ class$ for A and Δ^{\vee} its inverse. The fundamental class can be used to construct Poincar'e fundamental fundamental

$$K_{\bullet}(A) = KK_{\bullet}(\mathbb{C}, A) \xrightarrow{(-) \otimes_{A} \Delta} KK_{\bullet + d}(A^{\circ}, \mathbb{C}) = K^{\bullet + d}(A^{\circ}) = K^{\bullet + d}(A),$$

$$K^{\bullet}(A) = K^{\bullet}(A^{\circ}) = KK_{\bullet}(A^{\circ}, \mathbb{C}) \xrightarrow{\Delta^{\vee} \otimes_{A^{\circ}}(-)} KK_{\bullet - d}(\mathbb{C}, A) = K_{\bullet - d}(A).$$

Poincaré duality algebras are, unfortunately, quite scarce. Instead, we introduce a slightly weaker notion of Poincaré duality, where the duality isomorphism is between K-theory and K-homology of generally different C^* -algebras.

Definition 4.3.7. A pair (A, B) of separable C^* -algebras is called a *(strong) Poincaré duality pair (PD pair)*, if there exist classes $\Delta \in KK_d(A \otimes B, \mathbb{C}) = K^d(A \otimes B)$ and $\Delta^{\vee} \in KK_{-d}(\mathbb{C}, A \otimes B) = K_{-d}(A \otimes B)$, satisfying

$$\Delta^{\vee} \otimes_B \Delta = 1_A \in KK_0(A, A)$$
 and $\Delta^{\vee} \otimes_A \Delta = (-1)^d 1_B \in KK_0(B, B)$.

The class Δ is called a fundamental class for the PD pair (A, B) and Δ^{\vee} its inverse.

¹³Again, one can give also a weaker pointwise definition.

Given a PD pair (A, B) with fundamental class $\Delta \in K^d(A \otimes B)$, there are Poincaré duality isomorphisms

$$K_{\bullet}(A) = KK_{\bullet}(\mathbb{C}, A) \xrightarrow{(-) \otimes_{A} \Delta} KK_{\bullet + d}(B, \mathbb{C}) = K^{\bullet + d}(B),$$

$$K^{\bullet}(B) = KK_{\bullet}(B, \mathbb{C}) \xrightarrow{\Delta^{\vee} \otimes_{B}(-)} KK_{\bullet - d}(\mathbb{C}, A) = K_{\bullet - d}(A).$$

The complete form of the Kasparov product can be employed to define even more general Poincaré duality isomorphisms. Let C and D be any separable C^* -algebras. Then there are isomorphisms

$$KK_{\bullet}(C, A \otimes D) \xrightarrow{(-) \otimes_A \Delta} KK_{\bullet + d}(C \otimes B, D),$$

 $KK_{\bullet}(B \otimes C, D) \xrightarrow{\Delta^{\vee} \otimes_B (-)} KK_{\bullet - d}(C, A \otimes D).$

It is easy to see that, given a PD pair (A, B), there exists an isomorphism $KK_{\bullet}(A, A) \cong KK_{\bullet}(B, B)$. In particular, the multiplicative groups of their invertible classes are isomorphic.

Let X be a compact oriented manifold with continuous function algebra C(X). The pair $(C(X), C(X, \text{Cliff}(T^*X)))$, where $\text{Cliff}(T^*X)$ is the *Clifford algebra bundle* of the cotangent bundle [GVF], is a PD pair. The fundamental class is the K-homology class of the Dirac operator on $\text{Cliff}(T^*X)$, represented by the Kasparov $(C(X) \otimes C(X, \text{Cliff}(T^*X), \mathbb{C})$ -module

$$\left(L^2(X, \operatorname{Cliff}(T^*X)), \rho, \frac{D\!\!\!\!/}{\sqrt{1+D\!\!\!\!\!/^2}}\right),$$

where $L^2(X, \operatorname{Cliff}(T^*X))$ is the Hilbert space of square integrable sections of $\operatorname{Cliff}(T^*X)$ and the action ρ is given by Clifford multiplication [HR, Con]. The Dixmier-Douady class of $\operatorname{Cliff}(T^*X)$ is $W_3(X)$ [GVF]. Thus, Poincaré duality yields isomorphisms $K^{\bullet}(X, W_3(X)) \cong K_{\bullet}(X)$ and $K_{\bullet}(X, W_3(X)) \cong K^{\bullet}(X)$. If X is Spin^c , then $W_3(X) = 0$ and we have the usual Poincaré duality isomorphisms between (untwisted) K-theory and K-homology and K-homology

$$KK(C(X, \mathcal{E}_{\sigma}) \otimes C(X, \mathcal{E}_{-\sigma} \otimes W_3(X)), \mathbb{C}) = KK(C(X) \otimes C(X, W_3(X)) \otimes \mathcal{K}, \mathbb{C})$$

 $\cong KK(C(X) \otimes C(X, W_3(X)), \mathbb{C}).$

Thus, also $(C(X, \mathcal{E}_{\sigma}), C(X, \mathcal{E}_{-\sigma} \otimes W_3(X)))$ are PD pairs.

Generally, a PD pair (A, B) may have several different fundamental classes. For example, if $\Delta \in K^d(A \otimes B)$ is a fundamental class and $\lambda \in KK_0(A, A)$ is invertible, then $\lambda \otimes_A \Delta \in K^d(A \otimes B)$ is another fundamental class, with inverse $\Delta^{\vee} \otimes_A \lambda^{-1} \in K_{-d}(A \otimes B)$. This is easy to see using the properties of the Kasparov product, listed in theorem 4.2.8:

$$(\Delta^{\vee} \otimes_A \lambda^{-1}) \otimes_A (\lambda \otimes_A \Delta) = \Delta^{\vee} \otimes_A (\lambda^{-1} \otimes_A \lambda) \otimes_A \Delta$$
$$= \Delta^{\vee} \otimes_A 1_A \otimes_A \Delta$$
$$= \Delta^{\vee} \otimes_A \Delta$$
$$= (-1)^d 1_B.$$

The other direction requires more sophisticated use of associativity:

$$(\Delta^{\vee} \otimes_{A} \lambda^{-1}) \otimes_{B} (\lambda \otimes_{A} \Delta) = (\Delta^{\vee} \otimes_{A \otimes B} (\lambda^{-1} \otimes 1_{B})) \otimes_{B} ((\lambda \otimes 1_{B}) \otimes_{A \otimes B} \Delta)$$

$$= ((\Delta^{\vee} \otimes_{A \otimes B} (\lambda^{-1} \otimes 1_{B})) \otimes 1_{A})$$

$$\otimes_{B \otimes A \otimes A} (((\lambda \otimes 1_{B}) \otimes_{A \otimes B} \Delta) \otimes 1_{A})$$

$$= ((\Delta^{\vee} \otimes 1_{A}) \otimes_{A \otimes B \otimes A} (\lambda^{-1} \otimes 1_{B} \otimes 1_{A}))$$

$$\otimes_{B \otimes A \otimes A} (((\lambda \otimes 1_{B} \otimes 1_{A}) \otimes_{A \otimes B \otimes A} (\Delta \otimes 1_{A})))$$

$$= (\Delta^{\vee} \otimes 1_{A}) \otimes_{A \otimes B \otimes A} ((\lambda^{-1} \otimes 1_{B} \otimes 1_{A}) \otimes_{B \otimes A \otimes A} ((\lambda \otimes 1_{B} \otimes 1_{A}))$$

$$\otimes_{A \otimes B \otimes B} (\Delta \otimes 1_{A}))$$

$$= (\Delta^{\vee} \otimes 1_{A}) \otimes_{A \otimes B \otimes A} ((\lambda^{-1} \otimes_{A} \lambda) \otimes 1_{B} \otimes 1_{A})$$

$$\otimes_{A \otimes B \otimes A} (\Delta \otimes 1_{A}))$$

$$= (\Delta^{\vee} \otimes 1_{A}) \otimes_{A \otimes B \otimes A} ((\lambda^{-1} \otimes_{A} \lambda) \otimes 1_{B} \otimes 1_{A})$$

$$\otimes_{A \otimes B \otimes A} (\Delta \otimes 1_{A}))$$

$$= (\Delta^{\vee} \otimes 1_{A}) \otimes_{A \otimes B \otimes A} (\Delta \otimes 1_{A})$$

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$$= (\Delta^{\vee} \otimes 1_{A}) \otimes_{A \otimes A} (\Delta^{\vee} \otimes 1_{A})$$

$$= (\Delta^{\vee} \otimes 1_{A})$$

Conversely, if $\Delta_1, \Delta_2 \in K^d(A \otimes B)$ are fundamental classes, then $\Delta_1^{\vee} \otimes_B \Delta_2 \in KK_0(A, A)$ is invertible, with inverse $(-1)^d \Delta_2^{\vee} \otimes_B \Delta_1 \in KK_0(A, A)$. The proof is again a straightforward computation, so we omit it.

Proposition 4.3.8. For a PD pair (A, B) the space of fundamental classes is isomorphic to the group of invertible elements in the ring $KK_0(A, A) \cong KK_0(B, B)$.

4.4 Local bivariant cyclic homology

It would be useful to have some sort of a bivariant Chern character from KK-theory to some bivariant cohomology theory. A first guess for the target would be a bivariant extension of periodic cyclic homology. However, HP $_{\bullet}$ is not suited for this purpose, since it degenerates on C^* -algebras and Kasparov's KK-theory works only for C^* -algebras. There are two possible solutions. Either we replace Kasparov's KK-theory with some other bivariant K-theory, which is defined for a wider class of algebras, or we use some other kind of cyclic theory as the target. Indeed, the bivariant K-theory of Cuntz [KT] is defined for local C^* -algebras and, thus, is compatible with periodic cyclic homology. Since we want to use Kasparov's theory to describe D-branes, we need to find some replacement for periodic cyclic theory.

The bivariant cyclic homology theory we are looking for, is the *local bivariant cyclic homology* of Puschnigg [Pus1, Pus2, Cun]. Due to its complexity, we will not give an explicit definition, but rather take a more or less axiomatic approach.

The following theorem is a collection of results proven in [Pus1, Pus2].

Theorem 4.4.1. There exist bivariant bifunctors $\operatorname{HL}_{\bullet}(-,-)$, $\bullet = 1, 2$ from a suitable category of topological algebras, containing the category of separable C^* -algebras as a subcategory, to the category of complex vector spaces, satisfying:

i) $\mathrm{HL}_{\bullet}(-,-)$ is contravariant in the first variable and covariant in the second variable.

- ii) It is split exact, homotopy invariant and satisfies excision in both variables.
- iii) It is invariant under passing to smooth subalgebras of Banach algebras with the metric approximation property. All nuclear C*-algebras have the metric approximation property [CE].
- iv) For (nuclear) separable C*-algebras, there exists a bilinear associative composition prod-

$$\otimes_D : \operatorname{HL}_i(A_1, B_1 \otimes D) \times \operatorname{HL}_i(D \otimes A_2, B_2) \to \operatorname{HL}_{i+i}(A_1 \otimes A_2, B_1 \otimes B_2),$$

which naturally reduces into an exterior product

$$\otimes : \operatorname{HL}_{i}(A_{1}, B_{1}) \times \operatorname{HL}_{i}(A_{2}, B_{2}) \to \operatorname{HL}_{i+j}(A_{1} \otimes A_{2}, B_{1} \otimes B_{2})$$

and into a composition product

$$\otimes_D : \mathrm{HL}_i(A,D) \times \mathrm{HL}_i(D,B) \to \mathrm{HL}_{i+i}(A,B)$$
.

v) For separable C*-algebras A, B, there exists a natural bivariant Chern character

$$\operatorname{ch}: KK_{\bullet}(A,B) \to \operatorname{HL}_{\bullet}(A,B)$$
,

which is compatible with the product structures. For example, if $x \in KK_i(A, B)$ and $y \in KK_i(B, C)$, then

$$\operatorname{ch}(x) \otimes_B \operatorname{ch}(y) = \operatorname{ch}(x \otimes_B y). \tag{4.7}$$

vi) Any *-homomorphism $f: A \to B$ of separable C^* -algebras induces a class $[f]_{\rm HL} \in {\rm HL}_0(A,B)$. If also $g: B \to C$ is a *-homomorphism, then $[g \circ f]_{\rm HL} = [f]_{\rm HL} \otimes_B [g]_{\rm HL}$ and ${\rm ch}([f]_{KK}) = [f]_{\rm HL}$. In particular, ${\rm HL}_0(A,A)$ is a ring with multiplicative identity

$$1_A = [\mathrm{id}_A]_{\mathrm{HL}} = \mathrm{ch}([\mathrm{id}_A]_{KK}) \in \mathrm{HL}_0(A, A)$$
.

vii) For any smooth manifold X there exists an isomorphism

$$\operatorname{HL}_{\bullet}(\mathbb{C}, C_0(X)) \stackrel{iii)}{\cong} \operatorname{HL}_{\bullet}(\mathbb{C}, C_0^{\infty}(X)) \cong \operatorname{HP}_{\bullet}(C_0^{\infty}(X)),$$

and therefore

$$\operatorname{HL}_0(\mathbb{C}, C_0(X)) \cong H_{dR}^{even}(X)$$
 and $\operatorname{HL}_1(\mathbb{C}, C_0(X)) \cong H_{dR}^{odd}(X)$.

It follows immediately from the above properties that invertibility in KK-theory carries over to HL-theory under ch: if $x \in KK_d(A, B)$ and $y \in KK_{-d}(B, A)$ are such that

$$x \otimes_B y = 1_A \in KK_0(A, A)$$
 and $y \otimes_A x = 1_B \in KK_0(B, B)$,

then

$$\operatorname{ch}(x) \otimes_B \operatorname{ch}(y) = 1_A \in \operatorname{HL}_0(A, A)$$
 and $\operatorname{ch}(y) \otimes_A \operatorname{ch}(x) = 1_B \in \operatorname{HL}_0(B, B)$.

Furthermore, there is an analogue for KK-equivalence in HL-theory, called HL-equivalence. It is easy to see that KK-equivalence implies HL-equivalence. The converse is not true, though, basically because HL-theory yields vector spaces over \mathbb{C} , which are insensitive to torsion.

Definition 4.4.2. A pair (A, B) of separable C^* -algebras is called a *cyclic Poincaré duality* pair $(C\text{-}PD \ pair)$, if there exist $\Xi \in \operatorname{HL}_d(A \otimes B, \mathbb{C})$ and $\Xi^{\vee} \in \operatorname{HL}_{-d}(\mathbb{C}, A \otimes B)$, such that

$$\Xi^{\vee} \otimes_B \Xi = 1_A \in \mathrm{HL}_0(A, A)$$
 and $\Xi^{\vee} \otimes_A \Xi = (-1)^d 1_B \in \mathrm{HL}_0(B, B)$.

We call the class Ξ a cyclic fundamental class for the pair (A, B) and Ξ^{\vee} its inverse.

As in KK-theory, Poincaré duality for a C-PD pair (A, B) with cyclic fundamental class Ξ , yields isomorphisms

$$\operatorname{HL}_{\bullet}(C, A \otimes D) \xrightarrow{(-) \otimes_A \Xi} \operatorname{HL}_{\bullet + d}(C \otimes B, D),$$

 $\operatorname{HL}_{\bullet}(B \otimes C, D) \xrightarrow{\Xi^{\vee} \otimes_B (-)} \operatorname{HL}_{\bullet - d}(C, A \otimes D).$

By choosing $D = C = \mathbb{C}$, we get

$$\operatorname{HL}_{\bullet}(A) := \operatorname{HL}_{\bullet}(\mathbb{C}, A) \xrightarrow{(-) \otimes_{A} \Xi} \operatorname{HL}_{\bullet + d}(B, \mathbb{C}) = \operatorname{HL}^{\bullet + d}(B),$$

$$\operatorname{HL}^{\bullet}(B) := \operatorname{HL}_{\bullet}(B, \mathbb{C}) \xrightarrow{\Xi^{\vee} \otimes_{B}(-)} \operatorname{HL}_{\bullet - d}(\mathbb{C}, A) = \operatorname{HL}_{\bullet - d}(A).$$

Again, it follows easily that $\mathrm{HL}_{\bullet}(A,A) \cong \mathrm{HL}_{\bullet}(B,B)$, which restricts to an isomorphism between the multiplicative groups of their invertible elements.

If (A, B) is a PD pair with fundamental class $\Delta \in K^d(A \otimes B)$, then (A, B) is also a C-PD pair¹⁴ with cyclic fundamental class $\operatorname{ch}(\Delta) \in \operatorname{HL}^d(A \otimes B)$ (and inverse $\operatorname{ch}(\Delta)^{\vee} = \operatorname{ch}(\Delta^{\vee})$). It follows from (4.7) that the diagram

$$K_{\bullet}(A) \xrightarrow{\Delta \otimes_{A}(-)} K^{\bullet+d}(B)$$

$$\downarrow \text{ch}$$

$$\text{HL}_{\bullet}(A) \xrightarrow{\text{ch}(\Delta) \otimes_{A}(-)} \text{HL}^{\bullet+d}(B)$$

commutes. It should be noted that $\operatorname{ch}(\Delta)$ is generally not the only possible choice for a cyclic fundamental class for (A, B). In fact, the space of cyclic fundamental classes is isomorphic to the space of invertible elements of $\operatorname{HL}_0(A, A) \cong \operatorname{HL}_0(B, B)$ [BMRS2]. If $\Xi \in \operatorname{HL}^d(A \otimes B)$ is any cyclic fundamental class for (A, B) with inverse Ξ^{\vee} , then

$$\Xi = \lambda \otimes_A \operatorname{ch}(\Delta)$$
 and $\Xi^{\vee} = \operatorname{ch}(\Delta^{\vee}) \otimes_A \lambda^{-1}$,

for some invertible class $\lambda \in \mathrm{HL}_0(A, A)$.

Let X be a compact oriented manifold of dimension d. For the C-PD algebra C(X), we obtain the classical cyclic fundamental class $\Xi \in \mathrm{HL}^d(C(X) \otimes C(X))$ as the image of the class $[\varphi_X] \in \mathrm{HP}^{\bullet}(C^{\infty}(X))$ of the cyclic d-cocycle

$$\varphi_X(f^0, f^1, \dots, f^d) = \frac{1}{d!} \int_X f^0 df^1 \wedge \dots \wedge df^d,$$

for $f^i \in C^{\infty}(X)$, under the homomorphism

$$m^* : \mathrm{HP}^{\bullet}(C^{\infty}(X)) \cong \mathrm{HL}^{\bullet}(C(X)) \to \mathrm{HL}^{\bullet}(C(X) \otimes C(X)),$$

 $^{^{14}}$ The converse is, of course, not true, due to the usual problem with torsion.

where the arrow is induced by the product map $m: C(X) \otimes C(X) \to C(X)$, $f \otimes g \mapsto fg$. The class Ξ , therefore, corresponds to the classical fundamental class [X] and Poincaré duality in HL-theory agrees with the classical one.

We conclude this section with the universal coefficient theorem for HL-theory. For the proof, see [Mey].

Theorem 4.4.3 (Universal coefficient theorem for HL-theory). Let A be a separable C^* -algebra satisfying the UCT. There exist natural isomorphisms

$$\mathrm{HL}_{\bullet}(A,B) \cong \mathrm{Hom}_{\mathbb{C}}(\mathrm{HL}_{\bullet}(A),\mathrm{HL}_{\bullet}(B)) \cong \mathrm{Hom}_{\mathbb{C}}(K_{\bullet}(A) \otimes_{\mathbb{Z}} \mathbb{C},\mathrm{HL}_{\bullet}(B))$$
.

If also B satisfies the UCT, then

$$KK_{\bullet}(A, B) \otimes_{\mathbb{Z}} \mathbb{C} \cong \operatorname{Hom}_{\mathbb{Z}}(K_{\bullet}(A), K_{\bullet}(B)) \otimes_{\mathbb{Z}} \mathbb{C}$$
.

It also follows easily from the universal coefficient theorems that if both A and B satisfy the UCT and the groups $K_{\bullet}(A)$ are finitely generated, then

$$\mathrm{HL}_{\bullet}(A,B) \cong KK_{\bullet}(A,B) \otimes_{\mathbb{Z}} \mathbb{C}$$
.

4.5 Todd classes and Gysin homomorphisms

Let $\underline{\mathbf{SC}}^*$ denote the category of separable C^* -algebras, which form a PD pair with some other separable C^* -algebra. We remind that, within this category, we restrict to nuclear algebras whenever tensor products are involved (practically always), for which we do not have to worry about the topology of the tensor product C^* -algebra. Let A be an object of $\underline{\mathbf{SC}}^*$ and denote by \widetilde{A} the algebra, for which (A, \widetilde{A}) is a PD pair. For a PD algebra A, the canonical choice $\widetilde{A} = A^{\circ}$ is always assumed.

Definition 4.5.1. Let $A \in \underline{\mathbf{SC}}^*$ and $\Delta \in K^d(A \otimes \widetilde{A})$ the fundamental class for (A, \widetilde{A}) . Then (A, \widetilde{A}) is also a C-PD pair, with cyclic fundamental class $\Xi \in \mathrm{HL}^d(A \otimes \widetilde{A})$. The *Todd class* of A is defined to be the difference between the classes Ξ and $\mathrm{ch}(\Delta)^{15}$:

$$\operatorname{Todd}(A) = \operatorname{Todd}_{\Delta,\Xi}(A, \widetilde{A}) := \Xi^{\vee} \otimes_{\widetilde{A}} \operatorname{ch}(\Delta) \in \operatorname{HL}_0(A, A).$$

It is invertible with inverse

$$\operatorname{Todd}(A)^{-1} = \operatorname{Todd}_{\Delta,\Xi}(A,\widetilde{A})^{-1} := (-1)^d \operatorname{ch}(\Delta^{\vee}) \otimes_{\widetilde{A}} \Xi \in \operatorname{HL}_0(A,A).$$

Remark, that $Todd(A) = 1_A$, when $\Xi = ch(\Delta)$.

It is proved in [BMRS1] that $\underline{\mathbf{SC}}^*$ is not too restrictive a category of noncommutative spaces: any algebra in \mathbf{SC}^* , which satisfies the universal coefficient theorem for KK-theory and has finitely generated K-theory, is also in $\underline{\mathbf{SC}}^*$.

The definition of $\operatorname{Todd}(A)$ involves a choice for the fundamental classes. Recall that if $\Delta_1 \in K^d(A \otimes \widetilde{A})$ and $\Xi_1 \in \operatorname{HL}^d(A \otimes \widetilde{A})$ are any other fundamental classes for the pair (A, \widetilde{A}) , they are necessarily of the form $\Delta_1 = \lambda_{KK} \otimes_A \Delta$ and $\Xi_1 = \lambda_{\operatorname{HL}} \otimes_A \Xi$, with $\lambda_{KK} \in KK_0(A, A)$ and $\lambda_{\operatorname{HL}} \in \operatorname{HL}_0(A, A)$. In [BMRS2] it is shown, using associativity of the Kasparov product, that

$$\operatorname{Todd}_{\Delta_1,\Xi_1}(A,\widetilde{A}) = \operatorname{ch}(\lambda_{KK}) \otimes_A \operatorname{Todd}_{\Delta,\Xi}(A,\widetilde{A}) \otimes_A \lambda_{\operatorname{HL}}^{-1}.$$

Let X be a compact complex manifold 16 . Then C(X) is a PD algebra with fundamen-

¹⁵Recall the discussion before theorem 2.3.21.

 $^{^{16}}$ Recall that complex manifolds are always Spin^c.

tal class given by the Dolbeault (Dirac) operator and the cyclic fundamental class given by the orientation homology cycle. It follows from the universal coefficient theorem that $\mathrm{HL}_0(C(X),C(X))\cong\mathrm{End}(H^\bullet(X,\mathbb{Q}))$. Under this isomorphism, $\mathrm{Todd}(C(X))$ corresponds to $(-) \smile \mathrm{Todd}(X)$, the cup product with the complex Todd class $\mathrm{Todd}_{\mathbb{C}}(X) \in H^\bullet(X,\mathbb{Q})$. If X is a real compact oriented manifold, there is an analogous result, with $\mathrm{Todd}(C(X))$ corresponding to the cup product with the usual Todd class $\mathrm{Todd}(X)$.

Next we want develop a noncommutative generalization for the Gysin "wrong way" homomorphisms. Recall that if A and B are separable C^* -algebras and $f:A\to B$ a *homomorphism, then f induces homomorphisms

$$f_*: K_{\bullet}(A) \to K_{\bullet}(B)$$
 and $f^*: K^{\bullet}(B) \to K^{\bullet}(A)$.

The corresponding Gysin maps would be

$$f_!: K_{\bullet}(B) \to K_{\bullet+d}(A)$$
 and $f^!: K^{\bullet}(A) \to K^{\bullet+d}(B)$.

If A and B happen to be PD algebras, the Gysin maps can be defined, in the spirit of definitions 2.30 and 2.3.20, as compositions

$$f_! := \operatorname{Pd}_{KK}^{-1} \circ (f^{\circ})_* \circ \operatorname{Pd}_{KK}$$
 and $f^! := \operatorname{Pd}_{KK} \circ (f^{\circ})^* \circ \operatorname{Pd}_{KK}^{-1}$.

The maps $(f^{\circ})_*: K_{\bullet}(A^{\circ}) \to K_{\bullet}(B^{\circ})$ and $(f^{\circ})^*: K^{\bullet}(B^{\circ}) \to K^{\bullet}(A^{\circ})$ are induced by $f^{\circ}: A^{\circ} \to B^{\circ}$, $f^{\circ}(a^{\circ}) = f(a)^{\circ}$, for $a \in A$. This is not very satisfactory however, since PD algebras are too restrictive to be useful. We shall instead consider a certain subcategory of SC^* , in which well-behaving Gysin maps exist.

Definition 4.5.2. Let $\mathbf{SC}_{\mathbf{K}}^*$ be a subcategory of \mathbf{SC}^* , consisting of separable C^* -algebras and $(strongly)^{17}$ K-oriented *-homomorphisms, such that there exists a contravariant functor $!: \mathbf{SC}_{\mathbf{K}}^* \to \underline{KK}$, sending

$$\mathbf{SC}^*_{\mathbf{K}} \ni (A \xrightarrow{f} B) \to f! \in KK_d(B, A)$$
,

with the following properties:

- i) For any C^* -algebra A in $\mathbf{SC_K^*}$, the identity homomorphism $\mathrm{id}_A:A\to A$ and the zero homomorphism $0_A:A\to 0$ are K-oriented, with $\mathrm{id}_A!=1_A\in KK_0(A,A)$ and $0_A!=0\in KK_0(0,A)$.
- ii) If $(A \xrightarrow{f} B) \in \mathbf{SC}_{\mathbf{K}}^*$, then also $(A^{\circ} \xrightarrow{f^{\circ}} B^{\circ}) \in \mathbf{SC}_{\mathbf{K}}^*$.
- iii) If A and B are PD algebras in $\mathbf{SC}_{\mathbf{K}}^*$, with fundamental classes $\Delta_A \in K^{d_A}(A \otimes A^{\circ})$ and $\Delta_B \in K^{d_B}(B \otimes B^{\circ})$ and $(A \xrightarrow{f} B) \in \mathbf{SC}_{\mathbf{K}}^*$, the class $f! \in KK_d(B, A)$ is determined by

$$f! = (-1)^{d_A} \Delta_A^{\vee} \otimes_{A^{\circ}} [f^{\circ}]_{KK} \otimes_{B^{\circ}} \Delta_B,$$

with $d = d_A - d_B$.

 $^{^{17}}$ There is also a weaker definition of K-orientation but we shall not discuss it here. See [BMRS2] for further details.

Let $A \xrightarrow{f} B \xrightarrow{g} C$ be K-oriented *-homomorphisms. Functoriality of $!: \mathbf{SC}^*_{\mathbf{K}} \to \underline{KK}$ requires that $(g \circ f)! = f! \otimes_B g!$. It is also easy to see that the third property above is compatible with functoriality. Suppose A, B, C are PD algebras, with fundamental classes $\Delta_A \in K^{d_A}(A \otimes A^\circ)$, $\Delta_B \in K^{d_B}(B \otimes B^\circ)$ and $\Delta_C \in K^{d_C}(C \otimes C^\circ)$, and f, g K-oriented *-homomorphisms as above. It is then a straightforward computation to show that

$$f! \otimes_B g! = \left((-1)^{d_A} \Delta_A^{\vee} \otimes_{A^{\circ}} [f^{\circ}]_{KK} \otimes_{B^{\circ}} \Delta_B \right) \otimes_B \left((-1)^{d_B} \Delta_B^{\vee} \otimes_{B^{\circ}} [g^{\circ}]_{KK} \otimes_{C^{\circ}} \Delta_C \right)$$
$$= \left((-1)^{d_A} \Delta_A^{\vee} \otimes_{A^{\circ}} [(g \circ f)^{\circ}]_{KK} \otimes_{C^{\circ}} \Delta_C \right).$$

One simply has to write out the Kasparov products in terms of dilations and composition products and apply associativity.

Definition 4.5.3. Suppose $(A \xrightarrow{f} B) \in \mathbf{SC}_{\mathbf{K}}^*$ and $f! \in KK_d(B, A)$. Noncommutative versions of the Gysin "wrong way" homomorphisms in K-theory and K-homology are defined as follows:

$$f_! := (-) \otimes_B f_! : K_{\bullet}(B) \to K_{\bullet+d}(A),$$

$$f_! := f_! \otimes_A (-) : K^{\bullet}(A) \to K^{\bullet+d}(B).$$

We could now almost state a noncommutative version of the Grothendieck-Riemann-Roch theorem, but we are still missing Gysin maps in HL-theory. Due to the algebraic similarity of KK-theory and HL-theory, the definition of K-oriented maps can be directly carried over to HL-theory to define HL-oriented maps.

Definition 4.5.4. Let $\mathbf{SC}_{\mathrm{HL}}^*$ be a subcategory of \mathbf{SC}^* , consisting of separable C^* -algebras and $(strongly)^{18}$ HL-oriented *-homomorphisms, such that there exists a contravariant functor $*: \mathbf{SC}_{\mathrm{HL}}^* \to \underline{\mathrm{HL}}^{19}$, sending

$$\mathbf{SC}^*_{\mathrm{HL}} \ni (A \xrightarrow{f} B) \to f * \in \mathrm{HL}_d(B, A),$$

with the following properties:

- i) For any C^* -algebra A in $\mathbf{SC}^*_{\mathrm{HL}}$ the identity homomorphism $\mathrm{id}_A:A\to A$ and the zero homomorphism $0_A:A\to 0$ are HL-oriented with $\mathrm{id}_A*=1_A\in\mathrm{HL}_0(A,A)$ and $0_A*=0\in\mathrm{HL}_0(0,A)$.
- ii) If $(A \xrightarrow{f} B) \in \mathbf{SC}^*_{\mathrm{HL}}$, then also $(A^{\circ} \xrightarrow{f^{\circ}} B^{\circ}) \in \mathbf{SC}^*_{\mathrm{HL}}$.
- iii) If A and B are PD algebras in $\mathbf{SC}^*_{\mathrm{HL}}$ with fundamental classes $\Xi_A \in \mathrm{HL}^{d_A}(A \otimes A^{\circ})$ and $\Xi_B \in \mathrm{HL}^{d_B}(B \otimes B^{\circ})$ and $(A \xrightarrow{f} B) \in \mathbf{SC}^*_{\mathrm{HL}}$, the class $f * \in \mathrm{HL}_d(B, A)$ is determined by

$$f* = (-1)^{d_A} \Xi_A^{\vee} \otimes_{A^{\circ}} [f^{\circ}]_{\mathrm{HL}} \otimes_{B^{\circ}} \Xi_B,$$

with
$$d = d_A - d_B$$
.

 $^{^{18}}$ There is again a weaker definition, analogous to the KK-theoretic one.

 $^{^{19}}$ The category <u>HL</u> is analogous to the category <u>KK</u>.

Definition 4.5.5. Suppose $(A \xrightarrow{f} B) \in \mathbf{SC}^*_{\mathrm{HL}}$ and $f! \in \mathrm{HL}_d(B, A)$. The noncommutative Gysin "wrong way" homomorphisms in HL-theory are defined as follows:

$$f_! := (-) \otimes_B f_! : \mathrm{HL}_{\bullet}(B) \to \mathrm{HL}_{\bullet+d}(A),$$

 $f_! := f_! \otimes_A (-) : \mathrm{HL}^{\bullet}(A) \to \mathrm{HL}^{\bullet+d}(B).$

Suppose that $f: A \to B$ is both K-oriented and HL-oriented. Then there exists both Gysin classes $f! \in KK_d(B,A)$ and $f* \in \operatorname{HL}_d(B,A)$. Another class in $\operatorname{HL}_d(B,A)$ can be obtained from f! as $\operatorname{ch}(f!) \in \operatorname{HL}_d(B,A)$. By the properties of the Chern character, the diagrams

$$K_{\bullet}(B) \xrightarrow{f_!} K_{\bullet+d}(A) \qquad K^{\bullet}(A) \xrightarrow{f^!} K^{\bullet+d}(B)$$

$$\downarrow \operatorname{ch} \qquad \qquad \operatorname{ch} \downarrow \qquad \qquad \downarrow \operatorname{ch}$$

$$\operatorname{HL}_{\bullet}(B) \xrightarrow{(-) \otimes_B \operatorname{ch}(f!)} \operatorname{HL}_{\bullet+d}(A) \qquad \operatorname{HL}^{\bullet}(A) \xrightarrow{\operatorname{ch}(f!) \otimes_A(-)} \operatorname{HL}^{\bullet+d}(B)$$

obviously commute. Replacing ch(f!) with f*, yields the diagrams

$$K_{\bullet}(B) \xrightarrow{f_!} K_{\bullet+d}(A) \qquad K^{\bullet}(A) \xrightarrow{f^!} K^{\bullet+d}(B)$$

$$\downarrow \text{ch} \qquad \downarrow \text{ch} \qquad \downarrow \text{ch}$$

$$\text{HL}_{\bullet}(B) \xrightarrow{f_*} \text{HL}_{\bullet+d}(A) \qquad \text{HL}^{\bullet}(A) \xrightarrow{f^*} \text{HL}^{\bullet+d}(B)$$

which, in general, do *not* commute. When A and B are PD algebras, we can compare ch(f!) and f* explicitly and find the necessary corrections to be made to restore commutativity.

Theorem 4.5.6 (Noncommutative Grothendieck-Riemann-Roch theorem). For any PD algebras A and B, the relation

$$\operatorname{ch}(f!) = (-1)^{d_B} \operatorname{Todd}(B) \otimes_B (f*) \otimes_A \operatorname{Todd}(A)^{-1}, \tag{4.8}$$

between ch(f!) and f*, holds. Also, the diagrams

$$K_{\bullet}(B) \xrightarrow{f_{!}} K_{\bullet+d}(A)$$

$$(-1)^{d_{B}} \operatorname{ch}(-) \otimes_{B} \operatorname{Todd}(B) \downarrow \operatorname{ch}(-) \otimes_{A} \operatorname{Todd}(A)$$

$$\operatorname{HL}_{\bullet}(B) \xrightarrow{f_{*}} \operatorname{HL}_{\bullet+d}(A)$$

and

$$K^{\bullet}(A) \xrightarrow{f!} K^{\bullet+d}(B)$$

$$(-1)^{d_B} \operatorname{Todd}(A)^{-1} \otimes_A \operatorname{ch}(-) \downarrow \qquad \qquad \downarrow \operatorname{Todd}(B)^{-1} \otimes_B \operatorname{ch}(-)$$

$$\operatorname{HL}^{\bullet}(A) \xrightarrow{f^*} \operatorname{HL}^{\bullet+d}(B)$$

commute.

Proof. Equation (4.8) is proven in [BMRS2] by using

$$f! = (-1)^{d_A} \Delta_A^{\vee} \otimes_{A^{\circ}} [f^{\circ}]_{KK} \otimes_{B^{\circ}} \Delta_B$$
$$f* = (-1)^{d_A} \Xi_A^{\vee} \otimes_{A^{\circ}} [f^{\circ}]_{HL} \otimes_{B^{\circ}} \Xi_B$$

and theorem 4.2.8. The diagrams are obtained easily from (4.8):

$$\operatorname{ch}(f_{!}(\xi)) \otimes_{A} \operatorname{Todd}(A) = \operatorname{ch}(\xi \otimes_{B} f_{!}) \otimes_{A} \operatorname{Todd}(A)$$

$$= \operatorname{ch}(\xi) \otimes_{B} \operatorname{ch}(f_{!}) \otimes_{A} \operatorname{Todd}(A)$$

$$= \operatorname{ch}(\xi) \otimes_{B} (-1)^{d_{B}} \operatorname{Todd}(B) \otimes_{B} f_{*}$$

$$= (-1)^{d_{B}} f_{*}(\operatorname{ch}(\xi) \otimes_{B} \operatorname{Todd}(B)),$$

for any $\xi \in K_{\bullet}(B)$, and

$$\operatorname{Todd}(B)^{-1} \otimes_B \operatorname{ch}(f^!(\eta)) = \operatorname{Todd}(B)^{-1} \operatorname{ch}(f! \otimes_A \eta)$$

$$= \operatorname{Todd}(B)^{-1} \operatorname{ch}(f!) \otimes_A \operatorname{ch}(\eta)$$

$$= (-1)^{d_B} f * \otimes_A \operatorname{Todd}(A)^{-1} \otimes_A \operatorname{ch}(\eta)$$

$$= (-1)^{d_B} f^*(\operatorname{Todd}(A)^{-1} \otimes_A \operatorname{ch}(\eta)),$$

for any $\eta \in K^{\bullet}(A)$.

4.6 The Minasian-Moore formula

We are almost ready to present the noncommutative generalization of the Minasian-Moore formula

$$Q_{\mathbb{Q}}(\Sigma, x) := \operatorname{ch}(\phi_!(x)) \smile \sqrt{\operatorname{Todd}(X)} \in H^{\operatorname{even}}(X, \mathbb{Q}).$$

We expect to obtain the correct generalization by replacing $\operatorname{Todd}(X)$ with the Todd class of a noncommutative spacetime A, the untwisted K-theory class with a class $\xi \in K_0(B)$, where B is the noncommutative worldvolume. Naturally, the algebra A must be in $\operatorname{\underline{SC}}^*$ for the Todd class to exist and the continuous map $\phi: A \to B$ must be K-oriented²⁰.

There is still a slight technical problem, namely it is not clear that a square root of $\operatorname{Todd}(A)$ exists. However, if the algebra A satisfies the UCT, then $\operatorname{HL}_{\bullet}(A,A) \cong \operatorname{End}_{\mathbb{C}}(\operatorname{HL}_{\bullet}(A), \operatorname{HL}_{\bullet}(A))$. If $\dim_{\mathbb{C}}(\operatorname{HL}_{\bullet}(A))$ is finite, say n, then $\operatorname{Todd}(A)$ is in $\operatorname{GL}_n(\mathbb{C})$.

Lemma 4.6.1. Every matrix M in $GL_n(\mathbb{C})$ admits a square root $\sqrt{M} \in GL_n(\mathbb{C})$.

Proof. The matrix M can be brought to the Jordan canonical form J by conjugation with an invertible matrix P, that is,

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{pmatrix} = P^{-1}MP, \quad \text{with} \quad J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}.$$

Note, that in the noncommutative case the direction of the map $\phi: A \to B$ is from the spacetime algebra to the worldvolume algebra, whereas in the commutative case it points from the worldvolume to spacetime.

The complex numbers λ_i are nonzero since M is invertible. On the other hand,

$$J_i = \lambda_i \begin{pmatrix} 1 & \lambda_i^{-1} & & & \\ & 1 & \ddots & & \\ & & \ddots & \lambda_1^{-1} \\ & & & 1 \end{pmatrix} = \lambda_i (1 + K_i),$$

where the matrices K_i are nilpotent. The square roots $\sqrt{1+K_i}$ are thus given by finite Taylor polynomials. Therefore, also the square roots $\sqrt{J_i} = \sqrt{\lambda_i} \sqrt{1+K_i}$ of the Jordan blocks exist. The square root of J is given by

$$\sqrt{J} = \begin{pmatrix} \sqrt{J_1} & & \\ & \ddots & \\ & & \sqrt{J_p} \end{pmatrix} .$$

We can now choose $\sqrt{M} = P\sqrt{J}P^{-1}$.

The lemma establishes that, under the given assumptions, $\sqrt{\text{Todd}(A)}$ exists. Still, the square root matrix constructed above is not unique. It is not known if, or under which circumstances, a canonical choice for $\sqrt{\text{Todd}(A)}$ can be made [BMRS2]. We shall ignore the possible ambiguity and proceed as if a canonical choice has been made.

Definition 4.6.2 (Minasian-Moore). Let $A \in \underline{SC}^*$ be the noncommutative spacetime and $B \in SC^*$ the noncommutative D-brane worldvolume. Suppose that A satisfies the UCT and $\dim_{\mathbb{C}}(\mathrm{HL}_{\bullet}(A))$ is finite. Let $\phi: A \to B$ be a K-oriented *-homomorphism²¹. Let $\xi \in K_{\bullet}(B)$ classify the configuration of noncommutative D-branes wrapping B. The quantized RR-charge of the noncommutative D-brane is

$$Q_{\mathbb{Z}}(B,\xi) := \phi_!(\xi) \in K_{\bullet}(A)$$

and the classical RR-charge by

$$Q_{\mathbb{C}}(B,\xi) := \operatorname{ch}(\phi_!(\xi)) \otimes_A \sqrt{\operatorname{Todd}(A)} \in \operatorname{HL}_{\bullet}(A).$$

One last thing to consider is a noncommutative generalization of the isometric pairing formula, proposition 2.4.2. Suppose that A and B are as in definition 4.6.2. Poincaré duality yields a pairing

$$(-,-)_K: K_{\bullet}(A) \otimes_{\mathbb{Z}} K_{-\bullet-d}(\widetilde{A}) \to \mathbb{Z},$$

given by

$$(\alpha, \beta)_K := (\alpha \otimes \beta) \otimes_{A \otimes \widetilde{A}} \Delta \in KK_0(\mathbb{C}, \mathbb{C}) = \mathbb{Z}.$$

If X is Spin^c and $A = \widetilde{A} = C(X)$ with fundamental "Dirac" class Δ , the noncommutative pairing reduces to the classical index pairing appearing in proposition 2.4.2. If both A and \widetilde{A} satisfy the UCT and both $K_{\bullet}(A)$ and $K_{\bullet}(\widetilde{A})$ are finitely generated, then it follows from the UCT that the pairing $(-,-)_K$ is nondegenerate (modulo torsion). The PD pair (A,\widetilde{A}) is

 $^{^{-21}}$ In [BMRS2] the theory is worked out also for weakly K-oriented maps, which generalizes the definition slightly.

also a C-PD pair with cyclic fundamental class Ξ , possibly different from $\operatorname{ch}(\Delta)$. There is a pairing

$$(-,-)_H: \mathrm{HL}_{\bullet}(A) \otimes_{\mathbb{C}} \mathrm{HL}_{-\bullet-d}(\widetilde{A}) \to \mathbb{C}$$
,

given by

$$(x,y)_H := (x \otimes y) \otimes_{A \otimes \widetilde{A}} \Xi.$$

It is nondegenerate if the UCT holds and coincides with the classical pairing appearing in proposition 2.4.2 when $A = \tilde{A} = C(X)$, with X an oriented manifold, and when Ξ is the cyclic "orientation" fundamental class.

Proposition 4.6.3 (Isometric pairing formula [BMRS1]). Suppose that A and \widetilde{A} satisfy the UCT and $\operatorname{HL}_{\bullet}(A)$ and $\operatorname{HL}_{\bullet}(\widetilde{A})$ are finite dimensional vector spaces. The modified Chern character

$$\operatorname{Ch} := \operatorname{ch}(-) \otimes_A \sqrt{\operatorname{Todd}(A)} : K_{\bullet}(A) \to \operatorname{HL}_{\bullet}(A),$$

appearing in the noncommutative Minasian-Moore formula, is an isometry with respect to the inner products $(-,-)_K$ and $(-,-)_H$:

$$(\alpha,\beta)_K = \left(\operatorname{Ch}(\alpha), \operatorname{Ch}(\beta)\right)_H = \left(\operatorname{ch}(\alpha) \otimes_A \sqrt{\operatorname{Todd}(A)}, \operatorname{ch}(\beta) \otimes_{\widetilde{A}} \sqrt{\operatorname{Todd}(\widetilde{A})}\right)_H.$$

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