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CURRENT OPERATORS IN THE LOWEST LANDAU LEVEL

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We use Noether's theorem to generate a consistent definition of the current operator for electrons restricted to the lowest Landau level. We exhibit the connection between this current and the Moyal bracket, or W_∞, algebra, and use it to derive the edge-charge algebra for the $\nu = 1/(2n+1)$ FQHE states.

1. Introduction

Most experiments on the fractional quantum Hall effect (FQHE) are carried out in such strong magnetic fields that the inter Landau-level energy-gap, $\hbar\omega_c = \hbar eB/m^*$, is much greater than the energy scale relevant for understanding the observed phenomena. The essential physics of the FQHE results from the rearrangement of the nearly degenerate states in a single Landau level, and it is therefore physically reasonable to retain only the lowest Landau level (LLL) states in any calculation.

Apart from the computational convenience inherent in discarding the larger part of the Hilbert space, restriction to the lowest Landau level is sometimes necessary for deeper reasons. In their classic derivation of the dispersion curve for the magneto-phonons, Girvin, MacDonald and Platzman[1] found it essential to restrict their Hilbert space to the lowest level. If they had not enforced this restriction their variational calculation would have been swamped by the magneto-plasmon, or inter-Landau-level excitations, that saturate the oscillator-strength sum-rule at low k. For the same reason, Read, in his derivation of the long wavelength effective action for the FQHE[2] also works with states entirely in the lowest Landau level.

Use of this restricted Hilbert space is not without its conceptual problems. While diagonalization of the hamiltonian and computation of matrix elements is a straightforward, if non-trivial, task for any finite number of electrons, it would be useful to have a second-quantized LLL field-theory formalism with the same transparent interpretation that makes the various Chern-Simons formalisms[3] so attractive. Unfortunately the lack of completeness of the LLL states in the total Hilbert space renders many expressions non-local and complicates matters considerably. It is not, for example, clear what is the correct expression for such a simple quantity as the LLL current operator. A naive identification of the current with the functional derivative with respect to the gauge field of the LLL projected action leads to the paradox that the electrons cannot move.[1] If this identification were correct, which it is not, we would have thrown the baby (the Hall effect) out with the bathwater.

This paper is devoted to a simple derivation of the correct LLL current operator. We also show how this operator is naturally connected with recent observations[4,5] that Hall effect systems are invariant under the infinite dimensional Lie algebra W_∞. In Sec. 1 we review the elements of field theory in the lowest Landau level. In Sec. 2 we discuss the problems in defining the current operator. In Sec. 3 we resolve these problems by finding the appropriate Lagrangian for non-interacting LLL fermions. In Sec. 4 we show how the LLL current gives rise to the Moyal, or W_∞, algebra, and in Sec. 5 we apply these results to give a direct calculation of the commutators of the edge-charge operators which play a vital role in Wen's theory of edge excitations.[6] Some of the material in this paper has appeared in Ref. 7.

2. Field Theory in the Lowest Landau-Level

In this section, we briefly review the field-theory notation for electrons restricted to lie in the lowest Landau level. To some extent it is a second-quantized version of Ref. 8. The formalism is intended to be self-contained in that it makes no reference to the existence of higher Landau levels. One can regard such a pure LLL system as the low-energy truncation of the full quantum Hilbert space. The truncation becomes exact when either the magnetic field, B, becomes very large, or when the effective mass of the electron, m^*, becomes very small. It is most convenient to think in terms of the latter limit as in this case the electron density at fixed filling-fraction, ν, remains constant. In the sequel we will therefore set $B = 1$ throughout, but keep explicit references to m^* where necessary.

The basic ingredient is the projected electron field operator $\psi = \sum \hat{a}_n \psi_n$ in which the ψ_n are a complete set of normalized LLL wavefunctions. In the symmetric gauge ψ takes the form

$$\psi(z) = \sum_{n=0}^{\infty} \hat{a}_n \frac{1}{\sqrt{2\pi 2^n n!}} z^n e^{-\frac{1}{4}|z|^2} . \tag{2.1}$$

(Notice that ψ depends on $\bar{z}$ as well as on z. We write $\psi(z)$ for notational convenience.) The operators $\hat{a}_n$ and $\hat{a}_m^\dagger$, annihilate and create normalized LLL states and so obey the usual Fermi canonical anticommutation relations

$$\{\hat{a}_n, \hat{a}_m^\dagger\} = \delta_{mn} . \tag{2.2}$$

Because LLL wavefunctions are not complete in the total Hilbert space, the fields have unconventional equal-time commutators

$$\{\psi^\dagger(z_1),\, \psi(z_2)\} = \frac{1}{2\pi} e^{-\frac{1}{4}|z_1-z_2|^2} e^{\frac{1}{4}(\bar{z}_1 z_2 - \bar{z}_2 z_1)} = \{z_1|z_2\}\ . \tag{2.3}$$

A bilocal kernel $\{z_1|z_2\}$ has replaced the expected $\delta^2(\mathbf{r}_1 - \mathbf{r}_2)$ function. This kernel is a LLL analogue of the delta function and retains its reproducing property

$$\int d^2 z_1 F(z_1)\{z_1|z_2\} = F(z_2)\ , \tag{2.4}$$

where $F(z)$ is any function of the form $F = f(z)e^{-\frac{1}{4}|z|^2}$, and f is an analytic function of z. The field ψ has this form, so does the kernel, $\{z'|z\}$, as a function of z. The kernel therefore reproduces itself

$$\int d^2 z_2 \{z_3|z_2\}\{z_2|z_1\} = \{z_3|z_1\}\ . \tag{2.5}$$

Being bilocal, the kernel is not gauge invariant. The second exponential factor is a phase and would change if we selected a different gauge.

The operators $\psi^\dagger(z)$ insert electrons in coherent states centered on z. We may use them to make a many-electron state out of any antisymmetric holomorphic function

$$|f\rangle = \int \prod_1^N d^2 z_i f(z_1, \ldots, z_N) e^{-\frac{1}{4}\sum_1^N |z_i|^2} \psi^\dagger(z_1)\ldots\psi^\dagger(z_n)|0\rangle\ . \tag{2.6}$$

The anticommutation relations (2.3) and the reproducing kernel, enable us to express the inner product of two such states as

$$\langle f|g\rangle = N! \int \prod d^2 z_i \overline{f(z_i)} g(z_i) e^{-\frac{1}{2}\sum_i |z_i|^2}\ . \tag{2.7}$$

In particular, defining the state

$$|z_1, \ldots, z_N\rangle = \psi^\dagger(z_1)\ldots\psi^\dagger(z_n)|0\rangle\ , \tag{2.8}$$

we recover the wavefunction description

$$\langle z_1, \ldots, z_N|f\rangle = f(z_1, \ldots, z_N) e^{-\frac{1}{4}\sum_1^N |z_i|^2}\ . \tag{2.9}$$

We obtain the Laughlin wavefunction by taking $f = \prod (z_i - z_j)^{2n+1}$.

3. Charge-Density and Naive Currents

It is natural to define the LLL density operator as

$$\hat{\rho}(z) = \psi^\dagger(z)\psi(z) \tag{3.1}$$

since the integral of $\hat{\rho}(z)$ gives the total fermion number. The action of $\hat{\rho}(z)$ on the state $|f\rangle$ is

$$\hat{\rho}(z)|f\rangle = \sum_i \int \left(\prod d^2 z_i\right)' f(z_1, \ldots, z, \ldots, z_n)|z_1, \ldots, z, \ldots, z_N\rangle \tag{3.2}$$

where the z appears in the i'th slot and the prime on the product indicates that the $d^2 z_i$ factor is to be omitted. The *projected* operator $\hat{\rho}(z)$ therefore acts on the wavefunctions in a manner similar to the *unprojected* density operator $\hat{\rho}_{full}(\mathbf{r}) = \sum_i \delta^2(\mathbf{r} - \mathbf{r}_i)$.

One is now tempted to conjecture that the LLL current operator is also given by its usual form,

$$\begin{aligned} \hat{j}_z(z) &= \frac{1}{2m^* i}(\psi^\dagger \nabla_z \psi - (\nabla_z \psi^\dagger)\psi) \\ \hat{j}_{\bar{z}}(z) &= \frac{1}{2m^* i}(\psi^\dagger \nabla_{\bar{z}} \psi - (\nabla_{\bar{z}} \psi^\dagger)\psi) , \end{aligned} \tag{3.3}$$

with the covariant derivatives containing the symmetric gauge A_μ fields

$$\begin{aligned} \nabla_z \psi &= (\partial_z - \bar{z}/4)\psi & \nabla_z \psi^\dagger &= (\partial_z + \bar{z}/4)\psi^\dagger \\ \nabla_{\bar{z}} \psi^\dagger &= (\partial_{\bar{z}} - z/4)\psi^\dagger & \nabla_{\bar{z}} \psi &= (\partial_{\bar{z}} + z/4)\psi . \end{aligned} \tag{3.4}$$

This turns out not to be entirely correct. The operator identities,

$$\nabla_{\bar{z}} \psi = 0 = \nabla_z \psi^\dagger , \tag{3.5}$$

which express the vanishing of the non-diagonal part of the kinetic energy in the LLL states, can be used to simplify (3.3). We see that this form of the current depends only on the density

$$\hat{j}_z(z) = \frac{1}{2m^* i}\partial_z(\psi^\dagger \psi(z)) \qquad \hat{j}_{\bar{z}}(z) = -\frac{1}{2m^* i}\partial_{\bar{z}}(\psi^\dagger \psi(z)) . \tag{3.6}$$

The current in (3.6) is due to the cyclotron motion of the electrons. When the density is uniform, the currents from adjacent cyclotron orbits cancel each other. With a density gradient, such as occurs at the edge of the electron droplet, the cancellation is imperfect, and a net current appears. This current at the boundary is the source of the magnetic moment induced in the 2DEG by the external magnetic field, i.e., Landau diamagnetism. The edge currents become large as $m^* \to 0$ because the diamagnetic response represents the increase in energy of the 2DEG due to the magnetic field, and the LLL electrons posses a common zero-point energy $\frac{1}{2}\omega_c = eB/2m^*$, proportional to the inter-Landau level gaps.

The definition (3.3) contains sensible physics, but not enough. A glance at (3.6) shows that the density gradient current is ***solenoidal***

$$\partial^i \hat{j}_i = 2(\partial_{\bar{z}} \hat{j}_z + \partial_z \hat{j}_{\bar{z}}) = 0 \tag{3.7}$$

If $\hat{\jmath}_i$ were the entire story the divergence-free property, combined with charge conservation, would imply that $\partial_t \hat{\rho} = 0$. This is certainly not correct. The problem with (3.3) is that we have managed to lose the drift currents due to potential gradients i.e., the Hall effect. It is occasionally claimed that one cannot obtain the Hall currents in a purely LLL formalism. This is because, at least in one way of looking at the problem, the currents arise from a a mixing of Landau levels due to the potential perturbation. We hope that the subsequent arguments will lay this fear to rest.

The solution to the problem comes from the observation that the equation of motion for ψ is not obtained by simply replacing the operators in the unprojected equation of motion by their LLL projected versions. The field ψ does not therefore obey the non-relativistic time-dependent Schrödinger equation, and the current (3.3), which is associated with this equation, is no longer appropriate. This is so even when the equation of motion is linear in ψ. Consider an interaction with an external potential V. The kinetic part of the hamiltonian is diagonal in the LLL basis and will be ignored. The hamiltonian becomes simply

$$\hat{H} = \int d^2 z V(z, \bar{z}) \psi^\dagger \psi(z) \ . \tag{3.8}$$

(As with $\psi(z)$, we will frequently suppress the dependence of $V(z, \bar{z})$ on $\bar{z}$, and write "$V(z)$". It should be borne in mind that no analyticity is implied. V is an arbitrary function of position.)

We commute $\hat{H}$ through ψ to find $\partial_t \psi$

$$i\partial_t \psi(z) = \int d^2 z' V(z') \psi(z') \{z'|z\} \ . \tag{3.9}$$

We must use this equation of motion for $\psi(z)$, and its conjugate $\psi^\dagger(z)$, to find $\partial_t \hat{\rho}$. Because of the non-local commutators, non-local terms appear which must be tidied up before they can be interpreted. There are a number of ways to do this. In Ref. 7 we showed that the introduction of an apodized potential

$$\tilde{V}(z_1) = \int \frac{d^2 z}{2\pi} e^{-\frac{1}{2}|z - z_1|^2} V(z) \ , \tag{3.10}$$

allows us to write $\partial_t \hat{\rho}$ in a way that makes it a relatively simple sum of *local* terms

$$\partial_t \hat{\rho}(z) = \frac{1}{i} \sum_{n=1}^{\infty} \frac{2^n}{n!} \left\{ \partial_{\bar{z}}^n \tilde{V}(z, \bar{z}) \partial_z^n \hat{\rho}(z) - \partial_z^n \tilde{V}(z, \bar{z}) \partial_{\bar{z}}^n \hat{\rho}(z) \right\} \ . \tag{3.11}$$

In this form, by integrating by parts, it is easy to confirm that the total charge is conserved. Also notice that if ρ is uniform, then $\partial_t \rho$ is exactly zero. The flows induced by the potential therefore correspond to possible motions of an incompressible fluid.

For potentials varying slowly on the scale of the magnetic length we need keep only the lowest derivatives[1]

$$\partial_t \hat{\rho} \approx 2i(\partial_z \tilde{V} \partial_{\bar{z}} \hat{\rho} - \partial_{\bar{z}} \tilde{V} \partial_z \hat{\rho}) \ . \tag{3.12}$$

We can view the right hand side of (3.12) as the divergence of a current $\hat{\jmath} = \rho \mathbf{v}$ with $\mathbf{v}$ being the usual drift velocity

$$\begin{aligned} v^x &= +\partial_y \tilde{V} \\ v^y &= -\partial_x \tilde{V} \ . \end{aligned} \tag{3.13}$$

It is therefore certainly possible to find the correct drift motion even when we restrict the basis states to the LLL. It is not yet obvious, however, what is the correct LLL current operator compatible with (3.11).

4. Noether Currents

We now address the problem of finding the correct LLL current operator. This can be done by applying Noether's theorem to the lagrangian which gives rise to the equation of motion for $\psi(z)$ — once we know the lagrangian.

Our problem, then, is reduced to finding a lagrangian whose variation gives (3.9). This equation cannot be directly derived from the naive guess

$$L_0 = \int d^2z \psi^\dagger(z)(i\partial_t - V(z))\psi(z) \ , \tag{4.1}$$

because (4.1) fails to take into account the overcompleteness within the LLL of the states created by $\psi^\dagger(z)$. This overcompletness means, *inter alia*, that we cannot vary ψ, $\psi^\dagger$ independently at different points. Neighbouring fields are linked by the identity

$$\psi(z) = \int d^2z' \psi(z')\{z'|z\} \ . \tag{4.2}$$

To overcome this difficulty we introduce Grassmann valued Lagrange multipliers, $\eta(z)$ and $\eta(z)^\dagger$, to enforce (4.2), and its conjugate, as constraints

$$L = L_0 + \int d^2z \left([\psi^\dagger(z) - \int d^2z' \{z|z'\}\psi^\dagger(z')]\eta(z) + h.c. \right) \ . \tag{4.3}$$

Written in full

$$\begin{aligned} L = \int d^2z \Big(&\psi^\dagger(z)(i\partial_t - V(z))\psi(z) + \psi^\dagger(z) \left[\eta(z) - \int d^2z' \{z'|z\}\eta(z') \right] \\ &+ \left[\eta^\dagger(z) - \int d^2z' \{z'|z\}\eta^\dagger(z') \right] \psi(z) \Big) \ . \end{aligned} \tag{4.4}$$

Variation of this lagrangian with respect to $\eta(z)$ yields (4.2). Variation with respect to $\psi^\dagger(z)$ initially yields

$$(i\partial_t - V(z))\psi(z) = -\eta(z) + \int d^2z'\{z'|z\}\eta(z') \; , \tag{4.5}$$

but application of

$$\int d^2z\{z'|z\}\{z|z''\} = \{z'|z''\} \tag{4.6}$$

shows that this implies

$$\int d^2z(i\partial_t - V(z))\psi(z)\{z|z''\} = 0 \; . \tag{4.7}$$

Finally we use (4.2) to find the equation we want

$$i\partial_t\psi(z'') = \int d^2zV(z)\psi(z)\{z|z''\} \tag{3.9a}$$

The extended L is invariant under the gauge transformation

$$\begin{aligned} &\hat{\psi}(z) \to e^{i\varphi}\hat{\psi}(z) \; , \quad \hat{\psi}^\dagger(z) \to e^{-i\varphi}\hat{\psi}^\dagger(z) \; , \\ &\hat{\eta}(z) \to e^{i\varphi}\hat{\eta}(z) \; , \quad \hat{\eta}^\dagger(z) \to e^{-i\varphi}\hat{\eta}^\dagger(z) \; , \end{aligned} \tag{4.8}$$

so there exists a conserved Noether current. To find the current let us first write the non-local lagrangian L in a local form at the price of introducing an infinite number of derivatives of the fields. Integrating (4.4) in z', and then integrating by parts in z, we find that the lagrangian can be written in the form,

$$\begin{aligned} L = \int d^2z \Bigg\{ & \hat{\psi}^\dagger(z)(i\partial_t - V(z))\hat{\psi}(z) \\ & - \sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n!}\Bigg[\left(\partial_z + \frac{\bar{z}}{4}\right)^n(\hat{\psi}^\dagger(z))\left(\partial_{\bar{z}} + \frac{z}{4}\right)^n\hat{\eta}(z) \\ & + \left(\partial_z + \frac{\bar{z}}{4}\right)^n\hat{\eta}^\dagger(z)\left(\partial_{\bar{z}} + \frac{z}{4}\right)^n(\hat{\psi}(z))\Bigg]\Bigg\} \; . \end{aligned} \tag{4.9}$$

Consider now an infinitesimal gauge transformation

$$\begin{aligned} &\delta\hat{\psi}(z) = i\varepsilon\hat{\psi}(z) \; , \qquad \delta\hat{\eta}(z) = -i\varepsilon\hat{\eta}(z) \; , \\ &\delta\hat{\psi}^\dagger(z) = i\varepsilon\hat{\psi}^\dagger(z) \; , \quad \delta\hat{\eta}^\dagger(z) = -i\varepsilon\hat{\eta}^\dagger(z) \; . \end{aligned} \tag{4.10}$$

From the general expression,

$$j^{\mu} = \sum_{a} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^m \left[\partial_{\mu_1} \ldots \partial_{\mu_m} \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu_1} \ldots \partial_{\mu_n} \partial_{\mu} \phi_a)} \right) \right] \delta(\partial_{\mu_{m+1}} \ldots \partial_{\mu_n} \phi_a)$$
$$+ h.c. \,, \tag{4.11}$$

for the Noether currents in a field theory involving arbitrarily high derivatives of scalar (Bose- or Dirac-like) fields ϕ_a and ϕ_a^*, invariant under the infinitesimal transformation taking $\phi \to \phi + \delta\phi$ and $\phi^* \to \phi^* + \delta\phi^*$, after eliminating non-dynamical gradients of $\eta(z)$, $\eta^{\dagger}(z)$, we find that the time component of the current is the naive expression

$$\begin{aligned} \hat{\jmath}_0(z) &= \hat{\psi}^{\dagger}(z)\hat{\psi}(z) \\ &= \hat{\rho}(z) \,. \end{aligned} \tag{4.12a}$$

The space components, on the other hand, are given by

$$\hat{\jmath}^{\bar{z}}(z) = -i \sum_{n=1}^{\infty} \frac{2^n}{n!} \partial_{\bar{z}}^{n-1} \left[\hat{\rho}(z) \partial_z^n V(z) \right] \,, \tag{4.12b}$$

and

$$\hat{\jmath}^{z}(z) = i \sum_{n=1}^{\infty} \frac{2^n}{n!} \partial_{z}^{n-1} \left[\hat{\rho}(z) \partial_{\bar{z}}^n V(z) \right] \,. \tag{4.12c}$$

Here indices are raised and lowered via the metric $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$, so

$$\hat{\jmath}_z = \frac{1}{2}\hat{\jmath}^{\bar{z}}, \qquad \hat{\jmath}_{\bar{z}} = \frac{1}{2}\hat{\jmath}^{z} \,. \tag{4.13}$$

The charge conservation law takes the form

$$\partial_t \hat{\jmath}_0(z) + \partial_z \hat{\jmath}^z(z) + \partial_{\bar{z}} \hat{\jmath}^{\bar{z}}(z) = 0 \,. \tag{4.14}$$

The conservation law leads immediately to the equation of motion for $\hat{\rho}$ in the form obtained in Ref. 4

$$\partial_t \hat{\rho} = i \sum_{n=1}^{\infty} \frac{2^n}{n!} \left[\partial_{\bar{z}}^n (\hat{\rho} \partial_z^n V) - \partial_z^n (\hat{\rho} \partial_{\bar{z}}^n V) \right] \,. \tag{4.15}$$

This form is easily shown to be equivalent to (3.11), but has the advantage that no smoothed potential is needed.

5. Moyal Brackets

If we define

$$\hat{\rho}(f) = \int d^2z f(z, \bar{z}) \hat{\rho}(z) \,, \tag{5.1}$$

we can use (4.16) to find the rate of change $\partial_t\hat{\rho}(f_1)$ produced by using a second operator $\hat{\rho}(f_2)$ of this form as hamiltonian.

Since this is equivalent to the original commutator expression for the equation of motion, after an integration by parts, we have the identity[4]

$$[\hat{\rho}(f_1),\, \hat{\rho}(f_2)] = \hat{\rho}(\{f_1,\, f_2\}) \ . \tag{5.2}$$

Here the symbol $\{f_1,\, f_2\}$ denotes a form of the *Moyal Bracket*[9,10,11] of the functions f_1 and f_2,

$$\{f_1,\, f_2\} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} \left(\partial_z^n f_1 \partial_{\bar{z}}^n f_2 - \partial_z^n f_2 \partial_{\bar{z}}^n f_1\right) \ . \tag{5.3}$$

We hope the notation $\{f_1, f_2\}$ for this deformed Poisson bracket will not cause confusion with the conventional Poisson bracket, or with the symbol $\{z_1|z_2\}$ which denotes the bilocal kernel.

By virtue of the Jacobi identity for the commutator on the LHS of (5.2), the Moyal bracket satisfies the Jacobi identity,

$$\{f_1,\, \{f_2,\, f_3\}\} + \{f_2,\, \{f_3,\, f_1\}\} + \{f_3,\, \{f_1,\, f_2\}\} = 0 \ , \tag{5.4}$$

and is the only deformation of the Poisson bracket to do so.[11] The Moyal bracket thus induces a Lie algebra on C^∞ functions $f : \mathbf{R}^2 \to \mathbf{C}$. This algebra is not a conventional Poisson algebra as it does not posses the derivation property, $\{f,\, gh\} \neq \{f,\, g\}h + g\{f,\, h\}$.

There are slightly different forms for the Moyal bracket in the literature, depending on the rule selected for factor-ordering the operator being quantized (see Ref. 11 for a discussion). In the present case ordering is necessary because the LLL projected z and $\bar{z}$ do not commute. On first quantized states the function z acts as a creation operator and $\bar{z}$ as an annihilation operator.[8] The LLL operator $\hat{\rho}(f)$ implicitly corresponds to a first quantized anti-normal-ordered operator where the $\bar{z}$ annihilation operators are represented by $2\frac{d}{dz}$ and stand to the left of the creation operators.

The Lie algebra of functions, and the isomorphic algebra of operators $\hat{\rho}(f)$, are discussed in more detail in Refs. 4 and 5. There they are called W_∞ after the usage established in string theory.[12] Since the f_i act as hamiltonians inducing incompressible flows on the electron gas, it is not surprising that W_∞ is a deformation of the classical Lie algebra w_∞ of the group of area-preserving diffeomorphisms.

The Moyal bracket may be evaluated on the complete set of functions[13]

$$f_{\mathbf{p}} = \exp(i\mathbf{p}\cdot\mathbf{x}) = \exp(p_z z + p_{\bar{z}}\bar{z}) \ , \tag{5.5}$$

where $p_z = \frac{1}{2}(p_x - ip_y)$, $p_{\bar{z}} = \frac{1}{2}(p_x + ip_y)$. Writing the resulting operators as $\hat{\rho}_{\mathbf{p}_1}$ *etc.* we find

$$[\hat{\rho}_{\mathbf{p}_1}, \hat{\rho}_{\mathbf{p}_2}] = -2i \sin \frac{1}{2}(\mathbf{p}_1 \times \mathbf{p}_2) e^{-\mathbf{p}_1 \cdot \mathbf{p}_2/2} \hat{\rho}_{\mathbf{p}_1 + \mathbf{p}_2} \ . \tag{5.6}$$

Here $\mathbf{p} \times \mathbf{q} = p_1 q_2 - p_2 q_1$. In Ref. 1 the fourier components of the density operator in its first-quantized form were shown to obey these commutation relations.

If we restrict ourselves to momenta with integer values the $f_{\mathbf{p}}$ are well-defined on the torus. We can define

$$M_{\mathbf{n}} = e^{-\frac{1}{4}\mathbf{n}^2} \hat{\rho}_{\mathbf{n}} \tag{5.7}$$

and find the infinite discrete classical "sine" algebra[14]

$$[M_{\mathbf{n}}, M_{\mathbf{m}}] = -2i \sin \left(\frac{1}{2} \mathbf{n} \times \mathbf{m} \right) M_{\mathbf{m}+\mathbf{n}} \ . \tag{5.8}$$

This algebra is isomorphic to the Lie algebra of $U(\infty)$.

In Ref. 5 the Moyal bracket is evaluated on the basis set $z^{n+1}\bar{z}^{m+1}$. This gives rise to a superficially different, but entirely equivalent form for the structure constants.

6. Edge-Charges

As an application of the density operator algebra, although one involving w_∞ rather than the full quantum W_∞, we now derive the commutator for the edge-charges of the Laughlin $\nu = 1/(2n+1)$ states. In this way we provide a direct route to the result first postulated by Wen in his "Chiral Luttinger Liquid" theory of edge excitations.[6]

When the functions f_i from the preceeding section are slowly varying on the scale of the magnetic length, we can approximate

$$\{f_1, f_2\} = (-2)(\partial_z f_1 \partial_{\bar{z}} f_2 - \partial_z f_2 \partial_{\bar{z}} f_1) = -i(\partial_x f_1 \partial_y f_2 - \partial_x f_2 \partial_y f_1) \ . \tag{6.1}$$

Up to a trivial factor this is the conventional Poisson bracket.

Consider now the boundary of a Hall droplet. We will take the undisturbed edge to be coincident with the x axis. Assume the region $y > 0$ to be empty and the region $y < 0$ to be occupied by an incompressible quantum Hall phase at filling fraction ν. In the same spirit as [15] we define f-smeared edge-charge operators

$$\hat{\jmath}(f) = \int dx \hat{\jmath}(x) f(x) = \int dx dy f(x) g(y) \psi^\dagger \psi(x, y) \ . \tag{6.2}$$

Here $g(y)$ is a cut-off function which is unity over some interval $[-\Lambda, \Lambda]$ wide enough to accommodate any fluctuations in the position of the edge. It tends smoothly to zero near $y = \pm\Lambda$ over a distance greater than a magnetic length. These edge-charges both measure the amplitude of the distortion of the droplet edge and create the distortions when acting on the undeformed edge to create coherent states.[15,16,17]

From (6.1)

$$\begin{aligned}[\hat{\jmath}(f_1), \hat{\jmath}(f_2)] &= i\int dxdy\,((\partial_x f_1)f_2 - (\partial_x f_2)f_1)\, g\frac{\partial g}{\partial y}\psi^\dagger\psi \\ &= i\int dxdy\,((\partial_x f_1)f_2 - (\partial_x f_2)f_1)\,\frac{1}{2}\frac{\partial g^2}{\partial y}\psi^\dagger\psi \\ &= \frac{i}{2}\int dx(f'_1 f_2 - f'_2 f_1)\hat{\rho}(-\Lambda)\ .\end{aligned} \tag{6.3}$$

In the last line we have noted that the y integration receives contributions only from the regions where g is changing. These regions are far away from the surface of the droplet, and in them deformation of the edge should not affect the density of the incompressible state. Thus it is reasonable to expect that the operator $\psi^\dagger\psi(-\Lambda)$ may be replaced by its expectation value $\nu/2\pi$.

The commutator of the edge-charges is therefore

$$[\hat{\jmath}(x), \hat{\jmath}(x')] = -i\frac{\nu}{2\pi}\partial_x\delta(x - x')\ . \tag{6.4}$$

Our argument leading to (6.4) is by no means rigorous. It is shown in Ref. 5 that W_∞ does not preserve the essential property of the $\nu = 1/(2n+1)$ Laughlin states — that the wavefunction vanishes at short distance as $(z - z')^{2n+1}$ — but the fact that the edge-charges generate deformations corresponding to incompressible flows suggests that (6.4) should be a good approximation.

For higher fractions there is always more than one branch of edge excitations, and effectively more than one incompressible fluid. There is an edge-charge algebra, with generators $\hat{\jmath}_i(x)$, for each branch, but a sum-rule[6] for the coefficients of the $\partial_x\delta(x - x')$ requires that the commutator of the *total* edge-charge density $\hat{\jmath}_{tot} = \sum_i \hat{\jmath}_i$ to be still given by (6.4).

The important feature of (6.4) is the proportionality of the RHS to the filling fraction. This proportionality normalizes the currents and is a crucial ingredient in the theory of the edge-waves in the FQHE.[6] Wen originally obtained the normalization by an indirect argument. He worked backwards from the the energy-cost of deforming the edge of a $\nu = 1/(2n+1)$ droplet, and the known velocity of the edge-waves, to deduce the relation between the edge displacement field $y(x)$ and its conjugate momentum. For completness we will use (6.4) to proceed in the other direction.

When there is a confining electric field E at the edge of the droplet, the energy due to a displacement of the edge by a distance y must be

$$\mathcal{E} = \int dx\int_0^y dyEy\langle\hat{\rho}\rangle = \int dx\frac{1}{2}Ey^2\langle\hat{\rho}\rangle\ . \tag{6.5}$$

Setting $\hat{\jmath}(x) = \langle\hat{\rho}\rangle y(x)$ we take this as our hamiltonian

$$H_{\mathrm{edge}} = \int dx \frac{1}{2}\frac{E}{\langle\hat{\rho}\rangle}\hat{\jmath}^2(x) \ . \tag{6.6}$$

On using (6.4) to commute H_{edge} through $\hat{\jmath}(x)$, the $\langle\hat{\rho}\rangle$ factors cancel and yield the equation of motion

$$\partial_t\hat{\jmath}(x) = i[H_{\mathrm{edge}}, \hat{\jmath}(x)] = E\partial_x\hat{\jmath}(x) \ . \tag{6.7}$$

The edge-waves therefore (remember $B = 1$) propagate at $v = E/B$. We know that this is correct because the edge-waves are simply lumps of charge being advected along with all the other electrons at the local Hall drift velocity E/B.

Knowing the normalization of the currents shows that the vertex operator $\exp i\varphi$, where $\varphi(x) = \frac{1}{\langle\hat{\rho}\rangle}\int_{-\infty}^{x}\hat{\jmath}(x)dx$, obeys

$$[\hat{\jmath}(x), e^{i\varphi(x')}] = -i\delta(x - x')e^{i\varphi(x)} \ , \tag{6.8}$$

and has the correct commutation relations to be the bosonized edge-electron field. This in turn yields the equal-time correlator for the edge-electrons

$$\langle 0|\psi^\dagger(x)\psi(x')|0\rangle = \frac{i}{(x - x')^{1/\nu}} \ . \tag{6.9}$$

which is the key result in Ref. 6. Finite size scaling arguments[18] confirm that this power is consistent with an occupancy density of $1/2n + 1$.

Conclusion and Discussion

We have shown that the lowest Landau level current operator is not given by its naive form, but instead contains infinitely many derivatives of both the field operators and the scalar potential. This current is most easily found by applying Noether's theorem to an appropriate Lagrangian. It is thus possible to describe the motion of electrons without recourse to mixing with higher Landau levels.

The current operator we construct is intimately related to the W_∞, or Moyal, algebra whose relevance to the lowest Landau level QHE was pointed out in Ref 4 and in Ref. 5. This large algebra is a quantum deformation of the algebra of area-preserving maps which takes into account the finite magnetic length. It has a sub-algebra which reduces to the U(1) Kac-Moody algebra which has been used to characterize the edge excitations of the QHE. In previous work,[15,16,17] the quantum mechanics of the edge-waves have been discussed in terms of coherent states (equivalently co-adjoint orbits) of the associated $LU(1)$ loop groups. The edge-waves could also be quantized as coherent states or co-adjoint orbits of the W_∞ algebra. Indeed this may turn out to be the best language in which to address large amplitude edge-waves. A program with similar mathematical content, although with a rather different physical motivation, is to be found in the analysis of $c = 1$ string theory in Ref. 19.

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