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Mandelstam–'t Hooft Duality in Abelian Lattice Models*

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We derive a precise mathematical connection between the strong-coupling regime of the Abelian lattice gauge theory in four dimensions and a lattice model of a superconductor. This connection relates the quark confinement of the lattice gauge theory to magnetic flux tubes in the superconductor, along lines conjectured earlier by Mandelstam and 't Hooft. We apply this relation to the study of the behavior of physical quantities in the vicinity of the transition of the gauge theory to its quark-confining phase. In the course of our arguments, we develop an analogous connection between models describing a superconductor and a ferromagnet in three dimensions; this analysis reveals a new second-order phase transition in a superconductor which may be interpreted as a Gell-Mann–Low eigenvalue in $(2 + 1)$-dimensional scalar QED.

1. INTRODUCTION

In the past few years, the idea that quarks are permanently confined into charge zero bound states has been elevated from an ad hoc conjecture about force laws to a field-theoretic principle of fundamental significance. This change in outlook has occurred, to a great extent, as a result of the discovery by Wilson [1], Polyakov [2], and Migdal [3] that quark confinement is a natural property of a large class of systems whose underlying symmetry is a local gauge invariance. These authors have made use of the correspondence between Euclidean quantum field theory and statistical mechanics to suggest an analogy between gauge theories and ferromagnets in which the gauge coupling is the counterpart of the temperature. Corresponding to the two possible phases of magnetic systems—a magnetized phase at low temperatures and a disordered phase at high temperatures—one would expect to find in gauge theories two regimes with qualitatively different behaviors. The disordered phase of a ferromagnet corresponds, in the gauge systems, to a strong-coupling phase which naturally exhibits quark confinement.

A second, complementary, point of view on confinement has emerged from a more mechanistic approach proposed by Nielson and Oleson [4] and Tassie [5] and extended by Nambu and Parisi [6]. These authors have imagined quarks to be represented by magnetic monopoles embedded in a superconductor which expels magnetic flux. The

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model provides a very clear picture of quark confinement emerging from a special rigidity of the vacuum state. It allows a simple estimate of the binding potential between two oppositely charged sources: This potential is asymptotically linear, arising as the configurational energy of the Abrikosov flux tube [7] which conveys magnetic field from one monopole to its partner. The model does, however, contain two rather unpleasant features: First, it requires a spontaneous breaking of the local gauge symmetry, a property difficult to incorporate in models with asymptotic freedom. Second, it requires quarks to take the form of exotic objects whose field-theoretic description is not well understood.

Despite these shortcomings, the flux-tube model offers a picture of confinement whose physics can be clearly extracted; this fact has inspired interest in extending the intuitive understand which this model provides to the more formalistic scheme of Wilson and Polyakov. The two pictures of confinement do share the properties of vacuum rigidity and asymptotically linear potentials. However, they differ strikingly in the configuration of gauge fields in the vacuum: A superconductor is a highly ordered structure, while, in the thermodynamic approach, quark confinement is associated with an extreme of disorder. Kogut and Susskind [8] have attempted to bridge this conceptual gap by proposing a phenomenology of quark-confining vacua. Mandelstam [9] and 't Hooft [10] have offered a more profound suggestion; they have speculated that the two disparate confinement mechanisms in fact represent two different views of the same mechanism, related by the duality between electric and magnetic fields.

In order to make these ideas more concretely comprehensible, one must investigate their relation to specific field theory models exhibiting quark confinement. In this paper, we investigate their applicability to one particularly interesting model, the four-dimensional Abelian lattice gauge theory. We will find that the speculation of Mandelstam and 't Hooft emerges, in this system, as a precise mathematical result. By combining this result with arguments—some carefully justifiable, but others frankly speculative—from the phenomenology of superconductors, we will be able to explore the properties of the model in some detail. In particular, we will observe its behavior of the vicinity of the transition, as the gauge coupling is increased, from conventional Maxwell electrodynamics to the confining regime.

It will behelpful, in our investigation, to regard the Abelian lattice gauge theory as a member of a more general class of statistical-mechanical systems based on the invariance group $U(1)$. In these systems, the elementary variables (two-component classical spin vectors $\mathbf{s}_n$ or lattice-link gauge transformations $U_{n\mu}$) are describable as pure phases:

$$s_x + is_y = e^{i\theta_n}, \qquad U_{n\mu} = e^{-iA_{n\mu}}. \tag{1.1}$$

Polyakov [2] has stressed that electrodynamics in which the periodic variable $U_{n\mu}$ is treated as fundamental (as is the case in the lattice formulation) might be expected to exhibit qualitatively different behavior from conventional QED, in which the field A_μ is treated as a fundamental real-valued field. We distinguish the former theory by referring to it as "periodic electrodynamics," or simply PQED. The statistical-mechanical model of interacting two-component spins with a global $U(1)$ invariance is commonly referred to as the XY model.

The XY model, at least in three or more dimensions, exhibits at a critical temperature a phase transition to a magnetized state characterized by $\langle \mathbf{s}_n \rangle \neq 0$; the transition to this state is the subject of an elaborate phenomenology which has been set out, for example, in the review paper of Fisher [11]. For PQED, however, we have only fragments of a phenomenology, and even the existence of phase transitions has been, until recently, only conjectural. In the first papers on the subject, Wilson [1] and Balian *et al.* [12] attempted to estimate the critical coupling—the value of the gauge coupling which marks the transition to a quark-confining phase—by performing self-consistent field calculations which assumed that the weak-coupling, liberated-quark regime was characterized by a gauge magnetization: $\langle U_{n\mu} \rangle \neq 0$. However, Elitzer [13] later proved that rigorously, as a consequence of local gauge invariance, $\langle U_{n\mu} \rangle = 0$ in PQED for all values of the coupling. Elitzer's theorem does not prohibit more subtle forms of ordering (the onset of which might be well estimated by the simple self-consistent scheme), but it does strongly challenge possible conceptions of the nature of the ordered state.

An answer to this challenge arose from a remarkable technique for analyzing $U(1)$-invariant systems, originally devised by Berezinskii [14] and Kosterlitz and Thouless [15] for the study of the two-dimensional XY model and extended to gauge theories by Polyakov [2, 16]. These authors observed that the XY model and PQED allow topologically stable quantized-vortex singularities in their field configurations. (In PQED, the location of such a singularity is gauge invariant.) These singularities interact with one another, and their state of ordering may be seen to determine the thermodynamics of the original model. Polyakov has given a pointed illustration of this idea in an analysis of three-dimensional PQED, in which he extracted directly from the statistical mechanics of the allowed point singularities a weak-coupling calculation of the strength of the linear quark-confining potential. The isolation of the dynamics of singularity structures can be made most clear in lattice models by the use of a special choice of the $U(1)$-invariant lattice action; this choice was first introduced by Villain [17], to treat the two-dimensional XY model, and has been independently rediscovered by Baaquie [18], Polyakov [19], and Savit [20] in the context of gauge theories.

A notable feature of the transformation from the phase variables (1.1) to quantized singularities is that it connects statistical mechanical systems in very different states of excitation: the ordered, low-temperature regime of the XY ferromagnet is carried into a diffuse, high-temperature gas of dislocations; a condensation of the gas of singularities would correspond to thermal disruption of the magnetic order. We may think of the new variables as, in fact, parametrizing the disorder of the original statistical-mechanical system.

The idea of representing a spin model in terms of its disorder variables has a long history in statistical mechanics: The original transformation of this type, applicable to the two-dimensional Ising model, was introduced in 1941 by Kramers and Wannier [21]. In that model, both the spin variables and the disorder variables are constrained to take only the values ± 1, and, in fact, these two parametrizations of the Ising model are described by the same partition function, evaluated at two different temperatures. This self-duality has proved to be a powerful tool in the study of magnetic models with discrete invariance groups [22]. It has also been applied, by Wegner [23] and Balian *et al.* [24], to calculate the critical coupling of the Z_2-invariant lattice gauge theory.

It has recently been stressed by Jose *et al.* [25] that the transformation of $U(1)$-invariant systems from spin variables to quantized vortex singularities represents the generalization to this class of systems of the duality of Kramers and Wannier. In those system which admit point dislocations—the 2-D XY model and 3-D PQED—this duality has indeed offered much new insight, since the statistical mechanics of the dual theory of dislocations is just that of a gas of point particles. Interesting systems of higher dimensionality—including the 3-D XY model and 4-D PQED, which possess integer-valued line singularities—are more difficult to treat, because the integer-valued field enters in a way that cannot be related to a familiar statistical-mechanical model. Polyakov [2] and Banks *et al.* [26] have made a first attempt at the direct analysis of systems of interacting line vortices; they have derived a rough but intuitive picture of the mechanism which yields phase transitions in these two models and the latter group has located, semiquantitatively, the transition points. However, it is very difficult, in their approach, to study the qualitative features of the two phases on either side of the transition and, especially, the detailed behavior of physical parameters in the vicinity of this transition.

In this paper, we describe a reformulation of the spin–dislocation duality which avoids the production of integer-valued fields and, instead, relates systems with fundamental $U(1)$ symmetries to one another. The fact that we stay relatively close to the realm of simple spin systems will allow us to greatly augment the power of the duality transformation by making use of statistical-mechanical intuition. In three dimensions, we will direct our attention to the 3-D XY model coupled gauge invariantly to ordinary (nonperiodic) electromagnetism (essentially, the 3-D superconductor). This model contains two parameters, a temperature T and a gauge coupling e. We will find that the $e \to 0$ limit of this model, the 3-D XY model, is connected by a duality transformation to the $T \to 0$ limit of the model, a rather unusual system which we will refer to as a "frozen superconductor" (FZS). From the existence of a phase transition in the 3-D XY model, we predict in the FZS a novel phase transition which occurs at a critical value of e^2. This transition is of some independent interest, since it corresponds to a Gell-Mann–Low eigenvalue in three-(space-time)-dimensional scalar electrodynamics. In four dimensions, we will find a transformation which relates the 4-D frozen superconductor and 4-D PQED. In the confining phase of PQED, this transformation becomes just the statement of Mandelstem–'t Hooft duality: The transformation (which, in passing, interchanges electric and magnetic fields) connects the disorder of the strong-coupling regime of PQED to the order of a superconducting state. Detailed analysis of this transformation yields a set of scaling laws representing the thermodynamics of PQED near the critical coupling; these scaling laws take a form conventional for magnetic systems, despite the absence of a magnetization variable which might control the local ordering.

Because the three-dimensional situation is somewhat simpler to treat, we will begin our study by examining this case in considerable detail; the four-dimensional transformation may then be developed essentially by analogy. In Section 2 we present a rough analysis of the phase diagram of a superconductor; this discussion will serve as a point of reference for statical-mechanical arguments to be given later. In Section 3, we derive duality relations between the two limiting cases of this model, the 3-D XY model and the 3-D FZS. We will find relations not only for the partition functions, but also for a set of particularly interesting correlation functions. In Section 4, we combine these duality relations with phenomenology associated with the 3-D XY model to paint a rather complete picture of the 3-D FZS. Later in the paper, we will make use of this picture—in conjunction with the strong assumption that, like more familiar features of the phase diagram of a superconductor, its details are qualitatively preserved as one moves from three to four dimensions.

Sections 5 and 6 develop the relation between the 4-D frozen superconductor and the Abelian lattice gauge theory, 4-D PQED. In Section 5, duality relations are derived for the partition function and for a set of correlation functions. In Section 6, these relations are used to translate the properties of the 4-D FZS into a qualitative picture of the two phases of 4-D PQED. The analysis we present here neatly complements the work of Banks, Myerson, and Kogut; our analysis is not able to locate the phase transition in 4-D PQED quantitatively, but it does offer a clear picture of the distinctive properties of the two phases in which the model might be found.

2. Phases of a Lattice Superconductor

In this section, we examine some qualitative features of the XY model and of the XY model coupled to an electromagnetic field. Our intention is to review some well-established aspects of the phenomenology of these models, in order to build intuition that will be useful in our discussion of PQED.

The partition function of the XY model can be written

$$Z_{\mathrm{XY}} = \int_\theta e^{-(1/T)H}, \tag{2.1}$$

where

$$H = -\sum_{n\mu} \mathbf{s}_{n+\hat{\mu}} \cdot \mathbf{s}_n = -\sum_{n\mu} \cos(\theta_{n+\hat{\mu}} - \theta_n). \tag{2.2}$$

In the indicated sum, n runs over lattice sites and $\hat{\mu}$ over elementary vectors on a d-dimensional cubic lattice. The spin variables, $\mathbf{s}_n$ or $e^{i\theta_n}$, may be interpreted as a local magnetization. However, the XY model has also been considered a model of the superfluid transition in He^4; here $\langle e^{i\theta_n}\rangle$ is associated with the wavefunction of the macroscopic condensate. More generally, we expect that, where the lattice theory (2.1) possesses a continuum limit, this model describes a self-interacting two-component scalar field theory. Since a superconductor may be thought of as a superfluid condensation of electrically charged particles, we may model a superconductor by a generalization of (2.1) which includes a gauge-invariant coupling to electromagnetism:

$$Z = \int_{\theta, A} \exp \left[\frac{1}{T} \sum_{n\mu} \cos\left(\theta_{n+\hat{\mu}} - \theta_n - A_{n\mu}\right) - \frac{1}{4e^2} \sum_{n\mu\nu} F_{n\mu\nu}^2 \right]. \tag{2.3}$$

(Here $F_{n\mu\nu}$ is the (discretized) field tensor

$$F_{n\mu\nu} = (A_{n+\hat{\mu}\nu} - A_{n\nu}) - (A_{n+\hat{\nu}\mu} - A_{n\mu}). \tag{2.4}$$

We will follow the convention of defining the gauge field $A_{n\mu}$ to include a factor of the electric charge e; the e^{-2} appears in (2.3) in front of the rescaled $F_{n\mu\nu}^2$.)

To obtain a first estimate of the ordering in these models, we apply "mean-field theory," a standard approximation scheme in which each spin is considered to fluctuate independently in an effective magnetic field produced by the mean ordering of its neighbors [27]. A few details of the analysis are given in the Appendix.

It is well known that the d-dimensional XY model exhibits, in mean-field approximation, two phases; a continuous (second-order) transition to a magnetized phase occurs at a temperature $T_M = d$. These conclusions concerning the existence and nature of the transition are supported by detailed analysis of high-temperature perturbation series. However, one learns from such analyses that mean-field theory consistently overestimates the transition temperature because of its mistreatment of local spin correlations. (For the 3-D XY model, for example, the true critical temperature $T_c = (0.73)\, T_M$ [28].)

Mean-field theory also predicts a phase transition in the model (2.3) with $e^2 \neq 0$. However, for $d \leqslant 4$, the transition is in this case predicted to be of first order, entailing a discontinuous jump to a finite magnetization. (As $e^2 \to 0$, the discontinuity goes smoothly to zero.) This result emerges from the calculation in the following way: In the high-temperature phase, the electromagnetic field is massless, and its infrared fluctuations act to disrupt the spin ordering. When the system is able to acquire a small magnetization, the A field is given a mass by the Higgs mechanism, and its low-momentum fluctuations become less severe. This, in turn, increases the effective spin–spin coupling and renders the state with an infinitesimal ordering unstable with respect to acquisition of a finite magnetization. This argument would be more generally valid if we could be assured that the mass acquired by the A field is (in some sense) large, so that, in the ordered phase, we may consider the longest-wavelength fluctuation of the spin field while ignoring those of the A field. We will discuss this assumption, outside of mean-field theory, at the end of this section. The conclusion that the transition in this model is of first order is already well known from more rigorous renormalization-group analyses of the continuum limit theory—scalar electrodynamics—by Coleman and Weinberg [29] (for $d = 4$) and Halperin *et al.* [30] (for $d = 4 - \epsilon$). Both of these calculations, however, are valid only for sufficiently small e^2, since they treat the coupling to electromagnetism perturbatively.

The global phase diagram predicted by mean-field theory is displayed in Fig. 1. We should note again that, except for the precise location of the line of transitions, this

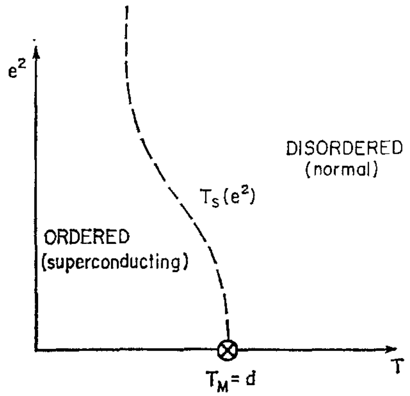

FIG. 1. Phase diagram of the XY model interacting with electromagnetism, as predicted by mean-field theory. The dashed line is a line of first-order phase transitions.

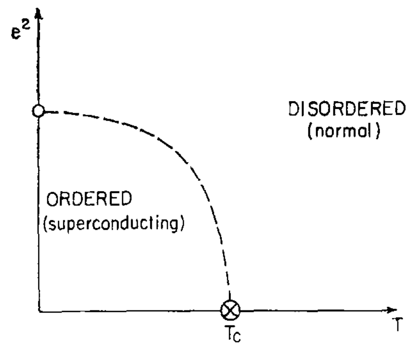

FIG. 2. Conjectured form of the actual phase diagram of the XY model interacting with electromagnetism.

diagram is known to be correct near the axis $e^2 = 0$. However, since mean-field theory characteristically overestimates transition temperatures, we must allow for the possibility that the true phase diagram has the form shown in Fig. 2. In Section 4 we argue that, for $d = 3$, Fig. 2 is in fact the correct alternative.

Having now discussed the global properties of model (2.3), we might next discuss its detailed behavior in the vicinity of the phase transition. It is worth noting, first, that there are no especially long-range correlations in the neighborhood of a first-order transition; this implies, in particular, that points on the first-order line have no interpretation as continuum-limit Euclidean field theories. The only points in the mean-field theory phase diagram of Fig. 1 which do possess such an interpretation are the points in the near neighborhood of the second-order transition at $e^2 = 0$. This latter region has been the subject of much phenomenological analysis by condensed-matter physicists; since we will need to refer to this phenomenology in the course of our discussion, we should pause now to recall some of its features.

It is familiar result that the long-range behavior of correlation functions near T_c leads to the appearance in thermodynamic quantities of expressions nonanalytic in the temperature [11]. Just at the critical temperature, for example, the asymptotic behavior of correlation functions has the scale-invariant form

$$\langle e^{i\theta(x)}e^{-i\theta(0)}\rangle \sim 1/|x|^{d-2+\eta}. \tag{2.5}$$

(In the continuum limit, (2.5) becomes the Green's function of a charged scalar field; $\eta/2$ is its anomalous dimension.) Away from $T = T_c$, this correlation function decays exponentially:

$$\langle e^{i\theta(x)}e^{-i\theta(0)}\rangle \sim \exp[-x/\xi(T)]; \tag{2.6}$$

as $T \to T_c$, the correlation length ξ has a nonanalytic behavior that we may parametrize:

$$\xi(T) \sim |T - T_c|^{-\nu}. \tag{2.7}$$

Comparing (2.6) to continuum-limit expressions for Euclidean Green's functions, we may associate ξ with the mass of the lightest particle created by the charged scalar field:

$$m_H = \xi^{-1}(T). \tag{2.8}$$

It is natural to assume that the asymptotic behavior of other Green's functions is dominated by exchanges of this same light particle. This statement would imply that all correlations scale together with the length $\xi(T)$ as $T \to T_c$, a statement referred to in the statistical mechanics literature as "hyperscaling." Applying this statement to energy–energy correlation functions can be seen to relate the power law with which the specific heat diverges as $T \to T_c$,

$$C_V \sim |T - T_c|^{-\alpha}, \tag{2.9}$$

to the exponent ν, defined in (2.7), and d, the dimensionality of space,

$$d\nu = (2 - \alpha). \tag{2.10}$$

The critical behavior of a superfluid below T_c involves an additional thermodynamic function—the "superfluid density," $\rho_S(T)$ [31]—and a second application of hyperscaling, derived originally by Josephson [32]: Consider an XY model in its ordered state, so that the equilibrium order parameter

$$\Phi(x) = \langle e^{i\theta(x)}\rangle \tag{2.11}$$

is nonvanishing. We may attempt to distort the value of $\Phi(x)$ by applying a magnetic field to the system; it will be useful to consider fields which are position dependent but slowly varying. Define

$$e^{-F[h(n)]} = \int_\theta e^{-H[\theta]+\Sigma h_n \cos\theta_n}. \tag{2.12}$$

The function $F[h(n)]$ is implicitly a function of the $\Phi(n)$ which results from the perturbing field. The Legendre-transformed function $F[\Phi]$ is the Gibbs free energy or, in field theory, the effective potential; its minimum determines the true equilibrium configuration. By the $U(1)$ invariance of the model, the value of $F[\Phi]$ is unchanged when Φ is globally rotated by a phase. If, however, we perform a position-dependent (but slowly varying) rotation of the phase of $\Phi(n)$, the free energy should slightly increase, according to the law

$$F[\Phi(n)] = G[\Phi_0] + \int d^dx\ \tfrac{1}{2}\rho_S(T)(\nabla\theta)^2; \tag{2.13}$$

This relation defines ρ_S. In a superfluid, $\nabla\theta$ is the superfluid velocity, so that ρ_S is naturally proportional to the mass density of the superflow. Josephson has derived from (2.10) the scaling relation

$$\rho_S(T) \sim |\ T_c - T\ |^{(d-2)\nu}, \tag{2.14}$$

where ν is again the exponent in (2.7). This is the same result one would obtain by assuming that ρ_S scaled with $\xi(T)$, to a power which might be determined by dimensional analysis.

The quantity ρ_S provides useful parametrizations of several correlation functions relevant to the superfluid. If we gauge invariantly couple to the XY model (2.1) an external electromagnetic field $\mathbf{B}(x)$, the last term in (2.13) is modified to read

$$\int d^dx\ \tfrac{1}{2}\rho_S(T)(\nabla\theta - \mathbf{B})^2. \tag{2.15}$$

Since such a coupling to an external field may be used to compute the current density,

$$\langle J^\mu(x)\rangle = -\frac{\delta}{\delta B_\mu(x)}(-\log Z[B]), \tag{2.16}$$

Eq. (2.17) indicates that the current–current correlation function is given by

$$\langle J^\mu(x)\ J^\nu(y)\rangle = \frac{\delta^2}{\delta B_\mu(x)\ \delta B_\mu(y)}(-\log Z) \approx \delta^{\mu\nu}\delta(x-y)\ \rho_S(T). \tag{2.17}$$

Equation (2.17) is, however, inadequate in several ways: Besides representing only a long-wavelength limit, it does not properly reflect gauge invariance. A more correct expression which remedies these difficulties is

$$\int d^dx\ e^{-iq\cdot x}\langle J^\mu(x)\ J^\nu(0)\rangle = \left(\delta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)(\rho_S(T) + O(q^2)). \tag{2.18}$$

Equation (2.18) is, of course, just the photon self-energy $\pi_{\mu\nu}(q^2)$; ρ_S appears as the residue of the pole in $\pi(q^2)$ which gives the photon a mass in the Higgs mechanism. We may, in fact, read this mass generation from (2.15):

$$m_W{}^2 = e^2 \rho_S(T). \tag{2.19}$$

The parameter ρ_S may also be related to the behavior of the spin–spin correlation function. According to (2.13), we may regard the low-momentum θ fluctuations as comprising a free massless (Goldstone boson) field $\hat{\theta}$, with propagator

$$\langle \hat{\theta}_q \hat{\theta}_{-q} \rangle = 1/\rho_S(T)\, q^2. \tag{2.20}$$

The spin–spin correlation function then has the long-distance behavior

$$\begin{aligned} \langle e^{i\theta_n} e^{-i\theta_0} \rangle &\underset{|n|\to\infty}{\cong} |\langle e^{i\theta_n} \rangle|^2 \cdot \exp\left[-\int \frac{d^d q}{(2\pi)^d}\, e^{-iq\cdot n} \langle \hat{\theta}_q \hat{\theta}_{-q} \rangle \right] \\ &\cong |\Phi_0|^2 \times \left(1 - \frac{1}{\rho_S(T)} \int \frac{d^d q}{(2\pi)^d} \frac{e^{-iq\cdot n}}{q^2} + \cdots \right). \end{aligned} \tag{2.21}$$

Below T_c, then, the spin–spin correlation function has a power-law falloff; the leading (connected) term corresponds to one-$\hat{\theta}$ exchange and has coefficient ρ_S^{-1}.

For an application of the scaling behavior near $T = T_c$, we might look again at the result quoted above, that, for $e^2 > 0$ and $d \leqslant 4$, the phase transition in model (2.3) is of first order. We would like, in particular, to justify the main assumption of our qualitative argument to this conclusion, that when a superconductor acquires some small ordering, the mass m_W associated with vector field fluctuations is much larger then the mass m_H associated with spin fluctuations.

Since the value of the ratio m_W/m_H is the criterion which distinguishes Type I ($m_W/m_H > 1$) from Type II ($m_W/m_H < 1$) superconductors, the universal applicability of the assumption can surely be questioned. But what if we could find a second-order transition in some regime at some finite value of e^2? Then, from (2.8) and (2.19), we would find as $T \to T_c$

$$\frac{m_W}{m_H} = \frac{e^2 \rho_S(T)}{\xi^{-1}(T)} \sim \frac{|T_c - T|^{[(d-2)/2]\nu}}{|T_c - T|^{\nu}}. \tag{2.22}$$

Asymptotically close to T_c, $m_W/m_H \to \infty$, for $d < 4$, purely as a consequence of hyperscaling. The conclusion that *all* superconductors close to T_c are carried by the renormalization group into the Type I regime is brought out explicitly in the calculations of Coleman and Weinberg and Halperin, Lubensky, and Ma mentioned above.

3. The Duality in three Dimensions

Having reviewed the general features of the XY model, we are ready to begin our study of its duality transformations. In this section we derive a relation connecting two limits of the more general model (2.3) in the specific case of a three-dimensional lattice. The physics contained in these relations is discussed in Section 4.

The starting point for our arguments will be the partition function (2.1) of the 3-D XY model. Our strategy will be to modify this model slightly, in the manner indicated by Villain [17], and then to rearrange the transformed system, using methods similar to those of Jose *et al.* [25].

In model (2.1) the vortices of the θ field are difficult to extract as isolated excitations; since our methodology will depend on the vortex structures, it will be useful to make their appearance more obvious. Villain has noted that (2.1) is, essentially, an integral over peaked periodic functions of the variables θ_n and has suggested replacing the specific function used there by another of the same shape:

$$e^{(1/T)\cos\theta} \rightarrow \sum_{m=-\infty}^{\infty} e^{-(1/2T)(\theta-2\pi m)^2}. \tag{3.1}$$

The two functions given in (3.1) become identical as $T \rightarrow 0$ and differ widely only in the limit of large T. We expect that the replacement (3.1) will not alter the qualitative features of model (2.1) at finite T; it may alter the value of T_c, but it should not alter the detailed character of the phase transition.

We consider, then, the partition function

$$\begin{aligned} Z_{XY} &= \exp[-F(T, e^2 = 0)] \\ &= \prod_n \int_{-\pi}^{\pi} \frac{d\theta_n}{2\pi} \prod_{n\mu} \sum_{m_{n\mu}=-\infty}^{\infty} \times \exp\left[-\frac{1}{2T} \sum_{n\mu} (\theta_{n+\hat{\mu}} - \theta_n - 2\pi m_{n\mu})^2\right]. \end{aligned} \tag{3.2}$$

The integrals over θ_n run from $-\pi$ to π. However, one should note that the integrand of (3.2) is invariant to the integer-valued gauge transformation

$$\theta_n \rightarrow \theta_n + N_n ; \qquad m_{n\mu} \rightarrow m_{n\mu} + N_{n+\hat{\mu}} - N_n . \tag{3.3}$$

We are, then, free to absorb an integer into the variable θ_n, extending its range to $(-\infty, \infty)$, if we apply a gauge constraint such as

$$0 = \sum_{\mu} (m_{n+\hat{\mu}\mu} - m_{n\mu}) \tag{3.4}$$

to the summation over the integer-valued bector field $m_{n\mu}$.

We now introduce an auxiliary field $b_{n\mu}$ by writing

$$\begin{aligned} Z_{XY} = \prod \int \frac{d\theta}{2\pi} \prod_{n\mu} \sum_{m_{n\mu}} &\left(\int_{\infty}^{\infty} db_{n\mu} \left(\frac{T}{2\pi}\right)^{1/2}\right) \\ &\times \exp\left[-\frac{T}{2} \sum_{n\mu} b_{n\mu}^2 + i \sum_{n\mu} b_{n\mu}(\theta_{n+\hat{\mu}} - \theta_n - 2\pi m_{n\mu})\right]. \end{aligned} \tag{3.5}$$

(The normalization factors of $T^{1/2}$ give a contribution to the free energy which is smooth in T; we ignore this and other such contributions.) The integral over θ_n has the form

$$\int_{-\infty}^{\infty} \frac{d\theta_n}{2\pi} \exp\left[i\theta_n \sum_{\mu} (b_{n-\hat{\mu}\mu} - b_{n\mu})\right] = \delta\left[\sum_{\mu} (b_{n-\hat{\mu}\mu} - b_{n\mu})\right]. \tag{3.6}$$

The constraint indicated here is a discretized version of the equation $\nabla \cdot \mathbf{b} = 0$; its most general solution is, similarly, a pure curl. Introducing a second auxiliary field a_n, we can write this solution as

$$b_{n\mu} = \sum_{\nu\sigma} \epsilon_{\mu\nu\sigma}(a_{n-\hat{\sigma}\sigma} - a_{n-\hat{\nu}-\hat{\sigma}\sigma}). \tag{3.7}$$

Note that $b_{n\mu}$ is invariant to the gauge transformation

$$a_{n\sigma} \to a_{n\sigma} + (\lambda_{n+\hat{\sigma}} - \lambda_n) \tag{3.8}$$

for arbitrary λ_n.

Integrating over the delta function (3.6) casts Z into the form

$$Z_{\mathrm{XY}} = \int_a \sum_{m_{n\mu}} \exp\left[-\frac{T}{2} \sum b_{n\mu}^2 - 2\pi i \sum_{n\mu} b_{n\mu} m_{n\mu}\right]. \tag{3.9}$$

A gauge-fixing term is needed to constrain the freedom (3.8), but we omit this term for the sake of clarity. Inserting (3.7) yields

$$Z_{\mathrm{XY}} = \int_a \sum_{m} \exp\left[-\frac{T}{4} \sum_{n\nu\sigma} f_{n\nu\sigma}^2 + 2\pi i \sum_{n\sigma} a_{n\sigma} M_{n\sigma}\right], \tag{3.10}$$

where

$$f_{n\nu\sigma} = (a_{n+\hat{\nu}\sigma} - a_{n\sigma}) - (a_{n+\hat{\sigma}\nu} - a_{n\nu}), \tag{3.11}$$

$$M_{n\sigma} = \sum_{\mu\nu} \epsilon_{\sigma\mu\nu}(m_{n+\hat{\nu}+\hat{\sigma}\mu} - m_{n+\hat{\sigma}\mu}). \tag{3.12}$$

The variables M are invariant to the gauge transformation (3.3) but, themselves, satisfy the constraint

$$(M_{n-\hat{\sigma}\sigma} - M_{n\sigma}) = 0. \tag{3.13}$$

We may exchange the gauge-constrained sum over $m_{n\mu}$ for a sum over $M_{n\sigma}$ subject to (3.13). Integrating (3.10) over $a_{n\mu}$, in Feynman gauge, produces the form for the partition function constructed by Banks *et al.* [26], involving the integer variables $M_{n\sigma}$ interacting through Coulomb potentials.

The variables $M_{n\sigma}$ possess a direct physical interpretation, one which we have noted already in the Introduction: $M_{n\sigma}$ represents the local vortex strength of the superfluid velocity. We may compute the current, in the Villain representation of the XY model, by coupling (3.2) to an external vector field through the replacement

$$(\theta_{n+\hat{\mu}} - \theta_n) \to (\theta_{n+\hat{\mu}} - \theta_n - B_{n\mu}) \tag{3.14}$$

and applying definition (2.16): the result is the operator

$$J_n{}^\mu = (1/T)(\theta_{n+\hat{\mu}} - \theta_n - 2\pi m_{n\mu}). \tag{3.15}$$

The curl of this field is just proportional to (3.12). Equation (3.13) is the familiar statement that the strength of a quantized line vortex is conserved along its length.

Our final manipulations will involve a rearrangement of the constrained sum over the $M_{n\sigma}$. We reintroduce a field θ by writing

$$\delta\left(\left[\sum_\sigma (M_{n-\delta\sigma} - M_n)\right], 0\right) = \int_{-\infty}^{\infty} \frac{d\theta_n}{2\pi} \exp\left[-i\theta_n \sum_\sigma (M_{n-\delta\sigma} - M_{n\sigma})\right]. \tag{3.16}$$

Inserting (3.16) into (3.10) allows us to remove constraint (3.13). Since we expect that spin configurations involving enormous values of the local vorticity are relatively rare, we might also introduce into (3.10) a convergence factor

$$1 = \lim_{t\to 0} \exp\left[-\frac{t}{2}\sum_{n\mu} M_{n\mu}^2\right].$$

We must now perform a sum over M of the form

$$\lim_{t\to 0}\left\{\sum_{M_{n\sigma}} \exp\left[+2\pi i \sum_{n\sigma} a_{n\sigma} M_{n\sigma} - i\sum_{n\sigma}\theta_n(M_{n-\delta\sigma} - M_{n\sigma}) - \frac{t}{2}\sum_{n\sigma} M_{n\sigma}^2\right]\right\}. \tag{3.17}$$

Using the identity

$$\sum_{M=-\infty}^{\infty} e^{i\phi M} e^{-t/2M^2} = \left(\frac{2\pi}{t}\right)^{1/2} \sum_{m=-\infty}^{\infty} e^{-(1/2t)(\phi-2\pi m)^2} \tag{3.18}$$

and the rescaling

$$A_{n\sigma} = 2\pi a_{n\sigma} \tag{3.19}$$

we may set Z_{XY} in a final form

$$Z_{XY} = \lim_{t\to 0} \int_A \int_\theta \sum_{m_{n\mu}} \exp\left[-\frac{1}{2t}\sum_{n\sigma}(\theta_{n+\delta} - \theta_n - A_{n\sigma} - 2\pi m_{n\sigma})^2\right]$$
$$\times \exp\left[-\left(\frac{T}{(2\pi)^2}\right)\frac{1}{4}\sum_{n\nu\sigma} F_{n\nu\sigma}^2\right], \tag{3.20}$$

where $F_{n\nu\sigma}$ is given by (2.4). We have cast the partition function into the Villain transcription of model (2.3) in its zero temperature limit. The free energy $F(T, e^2)$ of this Villain model thus obeys the relation

$$F(T_1, e_1{}^2 = 0) = F(T_2 = 0, e_2{}^2 = (2\pi)^2/T_1), \tag{3.21}$$

up to an additive term smooth in T_1. Equation (3.21) is just the duality relation claimed in the Introduction, a relation between the 3-D XY model and the 3-D FZS.

Let us now generalize this procedure slightly to derive duality relations for correlation functions. Consider. for example, the spin–spin correlation function of the XY model:

$$\langle e^{i\theta_N} e^{-i\theta_{N'}} \rangle_{\mathrm{XY}} = \frac{1}{Z_{\mathrm{XY}}} \int_\theta e^{i\theta_N - i\theta_{N'}} e^{-H[\theta]}. \tag{3.22}$$

If one introduces the auxiliary field $b_{n\mu}$ and then integrates over θ, one finds, instead of (3.6), the constraint equation

$$\sum_\mu (b_{n+\hat{\mu}\mu} - b_{n\mu}) = \delta_{n,N} - \delta_{n,N'}. \tag{3.23}$$

One solution to this equation is the integer-valued field $L_{n\mu}$ defined by $L_{n\mu} = \pm 1$, directed along a path on the lattice leading from N to N'; $L_{n\mu} = 0$ on all other links. (Any path from N to N' may be chosen.) The general solution to the inhomogeneous equation (3.22) is, then

$$b_{n\mu} = \left[\sum_{\nu\sigma} \epsilon_{\mu\nu\sigma}(a_{n-\hat{\sigma}\sigma} - a_{n-\hat{\nu}-\hat{\sigma}\sigma})\right] + L_{n\mu}. \tag{3.24}$$

Expression (3.11) must be modified to include $L_{n\mu}$, but this field drops out of the manipulations involving $M_{n\sigma}$: the only effect of inserting (3.24) into the last term of (3.9) is to produce an extra contribution

$$\exp\left[-2\pi i \sum_{n\mu} L_{n\mu} m_{n\mu}\right] = 1,$$

since $L_{n\mu}$ and $m_{n\mu}$ are integers. Our final expression for the correlation function is

$$\begin{aligned} \langle e^{i\theta_N} e^{-i\theta_{N'}} \rangle_{\mathrm{XY}} = \frac{1}{Z_{\mathrm{XY}}} \lim_{t\to 0} \Bigg\{ \int_A \int_\theta \sum_m \exp\Bigg[&-\frac{1}{2t} \sum_{n\sigma} (\theta_{n+\hat{\sigma}} - \theta_n - A_{n\sigma} - 2\pi m_{n\sigma})^2 \\ &- \frac{T}{(2\pi)^2} \frac{1}{4} \sum_{n\nu\sigma} \hat{F}^2_{n\nu\sigma} \Bigg] \Bigg\}, \end{aligned} \tag{3.25}$$

where

$$\hat{F}_{n\nu\sigma} = F_{n\nu\sigma} + \sum_\mu \epsilon_{\nu\sigma\mu}(2\pi L_{n+\hat{\nu}+\hat{\sigma}\mu}). \tag{3.26}$$

The integer-valued addition to the field tensor $F_{\nu\sigma}$ has just the structure of a Dirac string[33] connecting a pair of oppositely charged magnetic monopoles located at points N and N'. The relation between the monopole charge g_2 and the new electric charge $e_2 = 2\pi/T^{1/2}$ is just the Dirac condition

$$e_2 g_2 = 2\pi. \tag{3.27}$$

Integrating over vector potentials $A_{n\sigma}$ in the presence of this string is equivalent to integrating over vector potentials superposed on the background field of a pair of external monopoles. For later convenience, we represent the free energy of the FZS in the presence of a monopole pair by $F(T_2 = 0, e_2) + \Delta F_M(e_2, (N - N'))$; the duality relation for the spin–spin correlation function then reads

$$\langle e^{i\theta_N} e^{-i\theta_{N'}} \rangle_{XY} = (\exp[-\Delta F_M(e_2, N - N')])_{FZS}. \tag{3.28}$$

A reciprocal relation, connecting spin–spin correlation functions in the FZS to monopole correlations in the XY model, cannot be so carefully constructed. The natural starting point for such an analysis would be the gauge-invariant correlation functions

$$G_P(N, N') = \langle e^{i\theta_N} (\textstyle\prod_P e^{-iA_{n\mu}})\, e^{-i\theta_{N'}} \rangle_{FZS}. \tag{3.29}$$

(P, in this expression, is a path leading from N to N'.) However, in the $T \to 0$ limit which defines the FZS, these correlation functions freeze to

$$G_P(N, N') = 1 \qquad \text{(identically!)}. \tag{3.30}$$

A less trivial relation is obtained by considering the non-gauge-invariant object $\langle e^{i\theta_N} e^{-i\theta_{N'}} \rangle_{FZS}$. Working in some particular gauge, we may relate this function to vortex free energies in the XY model by returning from (3.20) to (3.27) and performing the integrals over θ_n. The result of this procedure is

$$\langle e^{i\theta_N} e^{-i\theta_{N'}} \rangle_{FZS} = Z'_{XY}/Z_{XY}, \tag{3.31}$$

where Z_{XY} is the partition function of the XY model in the form (3.10) and Z'_{XY} is given by the same expression but with constraint (3.13) on the sum over $M_{n\sigma}$ replaced by

$$\sum_\sigma (M_{n-\delta\sigma} - M_{n\sigma}) = \delta_{n,N'} - \delta_{n,N}. \tag{3.32}$$

Z' is, then, the partition function of the XY model in the presence of operators at N and N' which create and annihilate vortex lines. If we express the added free energy associated with the presence of these operators as $\Delta F_V(T_1, (N - N'))$, we can rewrite (3.31) as

$$\langle e^{i\theta_N} e^{-i\theta_{N'}} \rangle_{FZS} = (\exp[-\Delta F_V(T_1, (N - N'))])_{XY}. \tag{3.33}$$

We can define current–current correlation functions by introducing an external vector field as in (3.14) and applying definition (2.16). The substitution (3.14) adds to the exponent in (3.5) an extra term:

$$-i \sum_{n\mu} b_{n\mu} B_{n\mu} = -i \sum_{n\mu\nu\sigma} \tfrac{1}{2} \epsilon_{\mu\nu\sigma} (2\pi F_{n-\hat{\nu}-\delta\nu\sigma})\, B_{n\mu}. \tag{3.34}$$

We may then compute

$$\begin{aligned}\langle J_N{}^{\mu} J_{N'}^{\nu}\rangle_{\mathrm{XY}} &= -\frac{\delta^2}{\delta B_{N\mu}\,\delta B_{N'\nu}} \log Z \Big|_{B=0} \\ &= (2\pi)^2 \,\langle (\tfrac{1}{2}\epsilon_{\mu\sigma\lambda} F_{N-\hat{\sigma}-\hat{\lambda}\sigma\lambda})(\tfrac{1}{2}\epsilon_{\nu\rho\tau} F_{N'-\hat{\rho}-\hat{\tau}\rho\tau})\rangle_{\mathrm{FZS}} \,. \end{aligned} \tag{3.35}$$

Current–current correlations in the XY model become magnetic field correlations in the FZS. Reciprocally, one can relate vortex–vortex correlations in the XY model to current–current correlations in the FZS. Applying the shift (3.14) to the partition function (3.20) (where it appears as a shift of $A_{n\sigma}$ in the first term only) and returning the resultant expression to the stage (3.10),

$$\langle J_N{}^{\mu} J_{N'}^{\nu}\rangle_{\mathrm{FZS}} = -\frac{\delta^2}{\delta B_{N\mu}\,\delta B_{N'\nu}} \log Z \Big|_{B=0} = (2\pi)^2 \,\langle M_{N\mu} M_{N'\nu}\rangle_{\mathrm{XY}} \,. \tag{3.36}$$

Through Eqs. (3.21), (3.28), (3.33), (3.35), and (3.36), we have now demonstrated a rather complete connection between the two disparate singular limits of the general model (2.3).

4. Properties of the Three-Dimensional Frozen Superconductor

In this section we apply the exact relations we have just derived to explore the properties of the more unfamiliar limit of the lattice superconductor (2.3), the limit $T \to 0$ which defines the FZS. Our arguments will make strong use of the phenomenological theory reviewed in Section 2.

An obvious starting point is Eq. (3.21), the statement that the free energy of the 3-D XY model at temperature T is just equal to the free energy of the 3-D FZS at a coupling e given by

$$e^2 = (2\pi)^2/T. \tag{4.1}$$

(In this section, whenever we use the symbols T and e^2, it should be understood that they are related by (4.1).) As we have indicated in (2.9), the second-order phase transition of the 3-D XY model is visible as a singularity in the free energy. The FZS must, then, possess a second-order phase transition as a function of its coupling e^2, at the point

$$e_{\mathrm{c}}{}^2 = (2\pi)^2/T_{\mathrm{c}} \,. \tag{4.2}$$

At $e_{\mathrm{c}}{}^2$, the free energy exhibits the nonanalytic behavior

$$\frac{\partial^2}{\partial(e^2)^2} F_{\mathrm{FZS}}(e^2) \sim |\, e^2 - e_{\mathrm{c}}{}^2 \,|^{-\alpha}, \tag{4.3}$$

with the same exponent α observed in the XY model. This behavior is quite inconsistent with the mean-field theory phase diagram shown in Fig. 1 and lends support to the alternative sketched in Fig. 2. This phase diagram suggests that the transition in the FZS involves the destruction of superconductivity by the quantum fluctuations of the A field. The large-e^2 phase of the FZS would, in this picture, be a theory of massive charged scalar mesons coupled to conventional electromagnetism. Of course, such a picture is at this point purely conjectural; it is based on a smooth extrapolation of our understanding of the small-e^2 region across the phase diagram to the $T = 0$ axis. However, we should be able to subject this picture to stringent tests by combining the duality relations for correlation functions with the known behavior of the XY model. (In following our discussion of these relations, the reader might find it helpful to refer to Fig. 3, which shows pictorially the correspondences between the phases of the two models.)

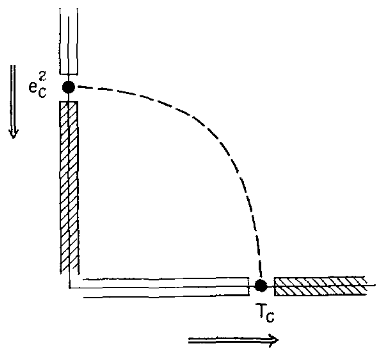

FIG. 3. A pictorial representation of the duality in three dimensions. Shaded regions are dual to one another; unshaded regions are similarly related. The arrows point in the direction of increasing T.

We consider first the connection (3.28) between the correlation functions of the XY model and the free-energy increment acquired by the FZS in the presence of magnetic sources. Below T_c, the correlation functions of the XY model have the form (2.21); this predicts for the FZS:

$$\Delta F_M(e^2, (N - N')) = -2 \log | \Phi_0(T)| - \frac{1}{\rho_S(T)} \frac{1}{4\pi \mid N - N' \mid} + \cdots . \quad (4.4)$$

The leading term has the interpretation of a monopole self-energy. This is a finite constant times the inverse of the lattice spacing (which, in our units, is taken to be 1). The second term represents the longest-range interaction of the poles. It is a Coulomb potential, as we would expect from charges interacting with normal electrodynamics. In the high-temperature phase the spin–spin correlation functions are exponentially damped, as indicated in (2.6). Then

$$\Delta F_M(e^2, (N - N')) \cong (\text{constant}) \times \mid N - N' \mid \qquad (e^2 < e_c^2). \quad (4.5)$$

This linear increase of free energy with separation is just what one would expect from a superconducting phase, in which the monopoles would be connected by an Abrikosov flux tube. Just at $T = T_c$, according to (2.5), the monopoles are confined by logarithmic potentials.

Our information about the $F_{\mu\nu}$–$F_{\mu\nu}$ correlation functions of the FZS tells a similar story. For $T < T_c$, the current–current correlation function of the XY model contains a pole of the form (2.18). Thus, using (3.35),

$$\int d^3q\, e^{-iq\cdot x}\langle\epsilon_{\mu\lambda\sigma}F^{\lambda\sigma}(x)\,\epsilon_{\nu\tau\rho}F^{\tau\rho}(0)\rangle_{\mathrm{FZS}} \underset{q^2\to 0}{\sim} (2\pi)^2\,\rho_S(T)(\delta^{\mu\nu} - q^\mu q^\nu/q^2) \qquad (e^2 > e_c^2), \tag{4.6}$$

which is just the prediction of normal electrodynamics. Note that the superfluid density of the XY model appears here as a wavefunction renormalization

$$Z_3(e^2) = (2\pi)^2\,\rho_S(T). \tag{4.7}$$

Comparing (4.4) and (4.6), we see that electric and magnetic charges are renormalized in just such a way as to preserve the Dirac condition (3.27).

When $T > T_c$, the pole (4.6) and, therefore, the massless photon of the FZS, disappears. In this phase, the current–current correlation function in coordinate space is exponentially decaying for large $| N - N' |$. The correspondence between statistical mechanics and Euclidean field theory tells us that, for T near T_c,

$$\langle J^\mu(x)\,J^\nu(0)\rangle_{\mathrm{XY}} \sim e^{-(2m)x}, \tag{4.8}$$

where m is the mass defined by (2.8). The field-strength correlations in the FZS then behave as

$$\langle\epsilon_{\mu\lambda\sigma}F^{\lambda\sigma}(x)\,\epsilon_{\nu\tau\rho}F^{\tau\rho}(0)\rangle_{\mathrm{FZS}} \sim e^{-(2m)x} \qquad (e^2 < e_c^2), \tag{4.9}$$

from which we deduce that the photon has indeed acquired a mass, as we would expect in a superconductor. It is possible that $(2m)$ is just this acquired mass m_W; alternatively, if the FZS contains a scalar meson with mass $m < \frac{1}{2}m_W$, the exponential decay (4.9) would arise from a two-particle cut. The question may be decided by examining in more detail the form of the monopole-pair free energy (4.5). The original definition (2.8) of m implies that

$$e^{-\Delta F_M(N-N')} \sim e^{-m|N-N'|}. \tag{4.10}$$

Returning again from statistical mechanics to field theory, and viewing the monopoles as sources in 3-D space-time, the result (4.10) indicates the presence of a scalar meson (whose world-line is the flux tube) with mass m [34]. Evidently, this meson becomes the lightest particle in the theory as $e^2 \to e_c^2$.

A more subtle test of our understanding of this system is the prediction, from duality,

$$m \sim | e_c^2 - e^2 |^\nu, \tag{4.11}$$

where ν is related to α through (2.10). At one level, this relation can simply be claimed as a consequence of hyperscaling. However, we must not forget that, besides m^{-1}, there is another correlation length in the theory—that determined by the spin correlation functions. It is interesting that a plausible model of the phase transition in the FZS allows us to understand the connection between these two lengths in a formalism which treats the flux tube as a soliton in the scalar field condensate. We must ask, first, why this transition can be continuous at all; we have argued in Section 2 that a transition at $e^2 \neq 0$ must be of first order. The explanation must come from the constraint of the $T \to 0$ limit. From (3.30) we see that the intrinsic correlation length ξ of the scalar field becomes infinite in this limit, invalidating the argument presented in (2.22). The gauge-variant correlation function (3.33) may have a nontrivial form, but only as a result of the spin field fluctuating just in opposition to A field fluctuations (or vice versa). We conclude that a phenomenological description of the phase transition should include only one fluctuating field; we take this to be the charged scalar (spin) field $\phi(x)$ and assume that this field has correlations which scale as $(e_c^2 - e^2)^{-\nu}$. A rather complete scaling theory of distortions of the condensate for a charged scalar field near its transition point has been constructed by Fisher *et al.* [35]. We may apply their formalism directly to the description of a flux tube joining a monopole–antimonopole pair by the use of the following construction: Choose a gauge in which the Dirac string lies along a straight line joining the magnetic poles. The scalar field expectation value may be chosen purely real, but it must have a node on the string. For large separation of the poles, and away from the endpoints, we might expect

$$\langle \phi(x) \rangle = \Phi_0(e^2)\, S(r_\perp \cdot (e_c^2 - e^2)^\nu), \tag{4.12}$$

where $r_\perp$ is the radial distance from the string, $\Phi_0(e^2)$ is the equilibrium field expectation value, and $S(\rho) \to 1$ as $\rho \to \infty$. The ansatz (4.12) is consistent with the phenomenological free energy of Fisher *et al.* [31]; the factors of $(e_c^2 - e^2)$ may be scaled out and combine to produce the dependence (4.11).

Below $e^2 = e_c^2$, then, we have found it at least consistent to model the FZS as a theory of a self-interacting charged scalar field, with the field $A_\mu(x)$ considered as a constraint variable. It is worth noting that this description looks rather unnatural for the strong-coupling phase. Whereas in the disordered phase of a conventional magnetic system all spin–spin correlations fall off exponentially, in the large-e^2 phase of the FZS

$$\langle e^{i(\theta_{N+\hat{\mu}} - \theta_N)} e^{-i(\theta_{N'+\hat{\nu}} - \theta_{N'})} \rangle \equiv \langle e^{iA_{N\mu}} e^{-iA_{N'\nu}} \rangle \tag{4.13}$$

decays only as a power law; if A_μ is treated as approximately free, (4.15) decays as $|N - N'|^{-1}$. Apparently, it is the A field which is the more fundamental field in the strong-coupling phase. We have not succeeded in constructing a suitable effective action for the region just above e_c^2. (A specific, annoying detail is our lack of understanding of the critical behavior of the renormalization factor (4.7):

$$Z_3(e^2) \sim |e^2 - e_c^2|^\nu.) \tag{4.14}$$

Despite our failure to find a detailed understanding of the phase transition in the FZS, we have, at least, established the qualitative nature of the two phases. It is of some interest to apply this information, using the duality relations (3.33) and (3.36), to study a relatively unfamiliar feature of the 3-D XY model—the disposition of vortices in its two phases. In the large-e^2, low-T regime, the connection between the spin correlations of the FZS and the vortex free energy of the XY model is quite familiar: Since this is the disordered phase of the FZS, the spin correlation function is just the propagator of a charged scalar field. But a charged particle in 3-D space-time has an infrared divergent self-energy due to the long range of the (logarithmic) Coulomb potential. An isolated vortex has, similarly, a divergent line tension:

$$\begin{aligned}\langle e^{i\theta_N} e^{-i\theta_{N'}}\rangle_{\mathrm{FZS}} &= e^{-\Delta F_{\mathrm{V}}(N-N')}\\ &= \exp[-C \mid N-N' \mid \log \mid N-N' \mid] \qquad (T < T_{\mathrm{c}}). \end{aligned} \tag{4.15}$$

When $e^2 < e_{\mathrm{c}}^2$, however, we are in the ordered phase of the FZS, so that

$$\langle e^{i\theta}{}_{-}e^{-i\theta_{N'}}\rangle_{\mathrm{FZS}} \xrightarrow[|N-N'|\to\infty]{} \mid \Phi_0(e^2)|^2 \tag{4.16}$$

and $\Delta F_{\mathrm{V}}(N-N')$ approaches asymptotically a constant, independent of $\mid N-N' \mid$. This makes precise the criterion used by Banks *et al.* [26] to locate T_{c} in the system of vortices, that T_{c} marks the point of appearance of vortices of infinite length.

From the connection (3.36) and the result that the FZS exhibits normal electrodynamics for $e^2 > e_{\mathrm{c}}^2$, we deduce that the vortex–vortex correlation function $\langle M_\mu(x)\, M_\nu(0)\rangle_{\mathrm{XY}}$ contains no pole of the form (2.18) at $q^2 = 0$. At first sight, this seems odd, since the small-T expansion of the partition function Z_{XY} (easily generated by integrating over $a_{n\mu}$ in (3.10) and then perturbing about $M_{n\sigma} = 0$) produces small loops on which $M_{n\sigma} \neq 0$. (A leading configuration contriburing to $\langle M_{N\mu} M_{N'\nu}\rangle$ is pictured in Fig. 4a.) These loops interact through dipole potentials, which behave at small q as $(q_\mu q_\nu/q^2)$. However, if N and N' are distant, the local dipole moments at these points are uncorrelated, and cancellations among different configurations of loops cut down this contribution by two more powers of q. This cancellation leaves a residual long-range interaction, so that $\langle J_\mu J_\nu\rangle_{\mathrm{FZS}}$ must still be weakly singular at $q^2 = 0$. But such a singularity is familiar in electrodynamics; a graph contributing to a small-q^2 cut is shown in Fig. 5. It is remarkable, though, that such nonleading singularities disappear entirely in the superconducting phase; otherwise, the photon propagator would not be analytic for $q^2 < (2m)^2$. In the phase $e^2 < e_{\mathrm{c}}^2$, $T > T_{\mathrm{c}}$, then, all long-range terms disappear from $\langle M_\mu(x)\, M_\nu(0)\rangle$ except a term of the form (2.18), corresponding to interactions of correlated local dipole moments:

$$(2\pi)^2 \langle M_\mu(x)\, M_\nu(0)\rangle \underset{|x|\to\infty}{\sim} \rho_{\mathrm{S,FZS}}(e^2) \left(\frac{3x^\mu x^\nu - \delta^{\mu\nu} x^2}{4\pi x^5}\right) \qquad (T > T_{\mathrm{c}}). \tag{4.17}$$

In this section, then, we have examined the phenomenology of a novel second-order phase transition located by taking a singular limit of lattice scalar electrodynamics. In Section 6 we find the four-dimensional version of this system relevant to the study

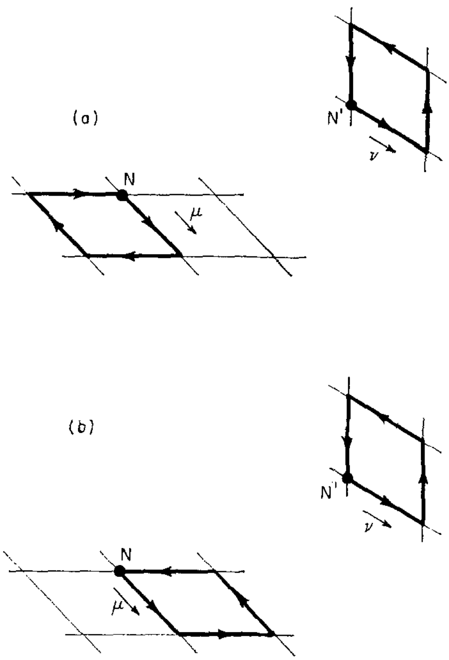

FIG. 4. Two leading contributions to $\langle M_{N\mu} M_{N'\nu} \rangle_{XY}$ at small T. $M_{n\sigma} = 1$ on the heavy lines, directed along the arrows; $M_{n\sigma} = 0$ elsewhere.

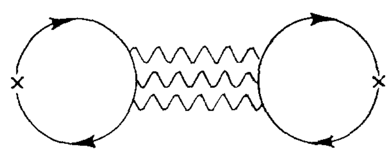

FIG. 5. A familiar diagram which produces a cut singularity, starting at $q^2 = 0$, in the current-current correlation function.

of quark confinement; its three-dimensional analog is, then, worth studying to gain a qualitative feel for the nature of the special limit. But our results are of interest from a very different point of view. We have located, in three-dimensional Euclidean space-time, a second-order phase transition to a phase with the properties of quantum electrodynamics at finite gauge coupling. The long-range correlations characteristic of a second-order phase transition should allow us to define a continuum limit theory in the region just above the point $e^2 = e_c^2$; this theory would be a formulation of scalar electrodynamics which was well defined even in the extreme ultraviolet domain. In the language of the renormalization group, the fixed associated with this second-order phase transition would be an ultraviolet-stable fixed point for scalar electrodynamics. It is at this point, of course, that we severely regret our inability to construct an effective action describing this phase transition; we are only able to hold out our formal result as indicating a possible new approach to the short-distance behavior of electrodynamics.

What happens to our picture of the FZS when the dimensionality of space is increased from 3 to 4? The duality relation (3.21) is a peculiar property of three dimensions; we cannot call upon it when $d = 4$. However, as we have remarked earlier, the more familiar features of the phase diagram of system (2.3), the existence of a T_c and a line of first-order transitions at finite e^2, are qualitatively unchanged as one makes this jump in dimensionality. In the remainder of this paper we will assume that the phase structure of the FZS also remains qualitatively unchanged between three and four dimensions; although this seems a relatively conservative assumption, it is not an a priori justifiable one. In terms of the discussion of the previous paragraph, this assumption is equivalent to the statement that the ultraviolet-stable fixed point of scalar electrodynamics which we have already located in three dimensions is present also in four dimensions. We prove in Section 6, however, that the existence of this fixed point in four dimensions is equivalent to the existence of a second-order transition to a quark-confining state in 4-D PQED.

5. The Duality in Four Dimensions

In this section we describe a duality transformation which relates the 4-D FZS and the 4-D PQED. The proof of this duality will essentially follow step by step the argument we have given in Section 3 for three-dimensional $U(1)$-invariant systems; for this reason, our discussion will be brief. Before beginning, however, it will be useful to review some features of PQED which will enter our analysis.

The Wilson–Polyakov formulation of electrodynamics on a lattice is defined through the partition function

$$Z = \int_A \exp\left[-\frac{1}{4g^2}\sum_{\mu\nu}(U_{n\mu}U_{n+\hat{\mu}\nu}U^{\dagger}_{n+\hat{\nu}\mu}U^{\dagger}_{n\nu} + \text{c.c.})\right], \tag{5.1}$$

in which $U_{n\mu} = \exp[-iA_{n\mu}]$ should be considered the fundamental dynamical variable. $A_{n\mu}$ appears only as the phase of $U_{n\mu}$ and is integrated over the domain $(-\pi, \pi)$. The coupling g^2 plays the role of the temperature. We may rewrite the exponent in (5.1) as

$$\frac{1}{2g^2}\sum_{n,\mu<\nu}\cos(F_{n\mu\nu}), \tag{5.2}$$

where $F_{\mu\nu}$ is the field tensor (2.4). If we now apply Villain's transcription (3.1), the partition function becomes

$$\begin{aligned} Z &= \exp[-F_{\text{PQED}}(g^2)] \\ &= \left(\prod_{n\mu}\int_{-\pi}^{\pi}\frac{dA_{n\mu}}{2\pi}\right)\left(\prod_{n,\mu<\nu}\sum_{m_{n\mu\nu}=-\infty}^{\infty}\right)\exp\left[-\frac{1}{2g^2}\sum_{n,\mu<\nu}(F_{n\mu\nu}-2\pi m_{n\mu\nu})^2\right]. \end{aligned} \tag{5.3}$$

As we noted in the Introduction, 4-D PQED has been conjectured to possess two phases, a strong-coupling phase exhibiting quark confinement and a weak-coupling phase with conventional Maxwell electrodynamics. It is not possible, however, to characterize the weak-coupling phase by a nonzero expectation value of a symmetry-forbidden quantity; we must seek a more subtle description of its ordering. A first criterion, introduced by Wilson [1], makes use of the expectation value of a gauge-invariant product of $U_{n\mu}$'s following a closed path P on the lattice. Wilson has proposed that the behavior

$$\left\langle \prod_{(n\mu)\in P} e^{-iA_{n\mu}} \right\rangle \sim \exp[-(\text{area enclosed by } P)/\Xi] \tag{5.4}$$

characterizes the strong-coupling phase of PQED, just as (2.6) characterizes the high-temperature phase of the XY model. He has shown that the behavior (5.4) predicts asymptotically linear confining potentials between charged sources.

A second distinction between the two phases has been suggested by Polyakov [36], who has shown how to define an analog of the superfluid density for the weak-coupling phase of PQED. Consider modifying the action (5.3) in a manner analogous to (3.14), by introducing an external field $G_{n\mu\nu}$ residing on plaquettes:

$$F_{n\mu\nu} \to (F_{n\mu\nu} - G_{n\mu\nu}). \tag{5.5}$$

A supercurrent is now defined by

$$\langle J_N^{\mu\nu} \rangle = (-\delta/\delta G_{N\mu\nu})(-\log Z). \tag{5.6}$$

In the Villain transcription, the current $J^{\mu\nu}$ is given by

$$J_N^{\mu\nu} = (1/g^2)(F_N^{\mu\nu} - 2\pi m_{N\mu\nu}). \tag{5.7}$$

As in the 3-D case, $m_{N\mu\nu} \neq 0$ will be thermodynamically suppressed when g^2 is small; then $J_N^{\mu\alpha}$ reduces to the familiar electromagnetic field strength, The current–current correlation function

$$\langle J_N^{\mu\nu} J_{N'}^{\lambda\sigma} \rangle = \frac{\delta^2}{\delta G_N^{\mu\nu}\, \delta G_{N'}^{\lambda\sigma}} (-\log Z)\Big|_{G=0} \tag{5.8}$$

is required (in a continuum theory), by the antisymmetry and conservation of $J_N^{\mu\nu}$, to take the form

$$\int d^4x\, e^{-iq\cdot x} \langle J^{\mu\nu}(x)\, J^{\lambda\sigma}(0) \rangle$$
$$= \left(\frac{q^\mu q^\lambda}{q^2} \delta^{\nu\sigma} - \frac{q^\nu q^\lambda}{q^2} \delta^{\mu\sigma} - \frac{q^\mu q^\sigma}{q^2} \delta^{\nu\lambda} + \frac{q^\nu q^\sigma}{q^2} \delta^{\mu\lambda} \right) \times P(q^2). \tag{5.9}$$

Polyakov defines, by analogy to (2.18),

$$\rho_S(g^2) = P\,(q^2 = 0, g^2). \tag{5.10}$$

A nonzero value of $\rho_S(g^2)$ indicates the existence of a massless vector meson—a photon, indicative of Maxwell electrodynamics. At small g^2, ignoring $m_{n\mu\nu}$, $\rho_S(g^2) \to 1/g^2$; strong-coupling expansions give the result $\rho_S(g^2) = 0$.

Having now discussed some of the systematics of PQED, let us concentrate on the four-dimensional case and subject the action to our duality transformation. As before, we extend the integrals over the phase variables to $(-\infty, \infty)$, using here the gauge invariance:

$$A_{n\mu} \to A_{n\mu} + N_{n\mu}\,; \qquad m_{n\mu\nu} \to m_{n\mu\nu} + (N_{n+\hat{\mu}\nu} - N_{n\nu} - N_{n+\hat{\nu}\mu} + N_{n\mu}). \tag{5.11}$$

Now introduce an auxiliary field $g_{n\mu\nu}$ by analogy to (3.5). Integrating over $A_{n\mu}$ yields the constraint equation

$$\sum_{\nu} (g_{n\mu\nu} - g_{n-\hat{\nu}\mu\nu}) = 0. \tag{5.12}$$

The general solution to this equation is

$$g_{n\mu\nu} = \sum_{\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} f_{n-\hat{\lambda}-\hat{\sigma}\lambda\sigma}\,, \tag{5.13}$$

where $f_{n\mu\nu}$ is given, in terms of a second auxiliary field $a_{n\mu}$, by (3.11). Expression (5.13) possesses a gauge invariance under (3.8). Z may now be reduced to the form

$$Z = \int_a \sum_{m_{n\mu\nu}} \exp\left[-\frac{g^2}{2} \sum_{n,\mu<\nu} f_{n\mu\nu}^2 - 2\pi i \sum_{n,\mu<\nu,\lambda<\sigma} (\epsilon_{\mu\nu\lambda\sigma} f_{n-\hat{\lambda}-\hat{\sigma}\lambda\sigma})\, m_{n\mu\nu}\right]. \tag{5.14}$$

The last term in the exponent can be rewritten

$$+2\pi i \sum_{n\sigma} a_{n\sigma} M_{n\sigma}\,, \tag{5.15}$$

where

$$M_{n\sigma} = \tfrac{1}{2} \sum_{\mu\nu\lambda} \epsilon_{\sigma\lambda\mu\nu} (m_{n+\hat{\sigma}+\hat{\lambda}\mu\nu} - m_{n+\hat{\sigma}\mu\nu}). \tag{5.16}$$

The integer-valued field $M_{n\sigma}$ satisfies the conservation law (3.13). If we remove this constraint on the sum over $M_{n\sigma}$ by introducing an auxiliary field θ_n, we recover a partition function analogous to (3.20):

$$\begin{aligned} Z = \lim_{t\to 0} \int_A \int_\theta \exp\left[-\frac{1}{2t} \sum_{n\sigma} (\theta_{n+\hat{\sigma}} - \theta_n - A_{n\sigma} - 2\pi m_{n\sigma})^2\right] \\ \times \exp\left[-\frac{g^2}{(2\pi)^2} \frac{1}{4} \sum_{n\nu\sigma} F_{n\nu\sigma}^2\right]. \end{aligned} \tag{5.17}$$

We have, then, found a duality relation between PQED and the FZS in four dimensions:

$$F_{\text{PQED}}(g^2) = F_{\text{FZS}}(e^2 = (2\pi)^2/g^2) \tag{5.18}$$

(up to an additive contribution smooth in g^2).

It is worth briefly recalling the physical significance of the variables $M_{n\sigma}$, first noted in the original paper of Polyakov [2]. Using expression (5.7) for the current $J^{\mu\nu}$:

$$\frac{1}{g^2} M_{n\sigma} = \frac{1}{2} \sum_{\mu\nu\lambda} \epsilon_{\sigma\lambda\mu\nu}(J^{\mu\nu}_{n+\hat{\sigma}+\hat{\lambda}} - J^{\mu\nu}_{n+\hat{\sigma}}). \tag{5.19}$$

If we take $(g^2 J^{\mu\nu})$ to be the generalization of $F^{\mu\nu}$ to the region $g^2 \sim 1$ of PQED, (5.19) is seen to be a discretized version of the equation

$$M_\sigma(x) = \epsilon_{\sigma\lambda\mu\nu}\partial_\lambda F^{\mu\nu}(x); \tag{5.20}$$

$M_{n\sigma}$ may, then, be interpreted as the charge current of a gas of magnetic monopoles. The reader should note that these monopoles need not be imposed upon PQED; like the 't Hooft–Polyakov monopoles of the $O(3)$ gauge model [37], they arise naturally from the nonlinearities of the original theory. It is also of some interest that (5.7) has just the form of Dirac's proposed field tensor for monopole theories [33], containing a conventional $F^{\mu\nu}$ plus quantized sheet singularites.

Duality relations may also be constructed for correlation functions, again following the methods of Section 3. Instead of the spin–spin correlation (3.22), we might examine Wilson's loop-integral correlation functions $\langle \prod_P e^{-iA_{n\mu}} \rangle$. Introducing the auxiliary field $g_{n\mu\nu}$ and integrating over $A_{n\mu}$, one finds instead of (5.12) the constraint equation

$$\left(\sum_\nu g_{n\mu\nu} - g_{n-\nu\mu\nu}\right) = - \sum_{(n'\mu')\in P} \delta_{n\mu,n'\mu'}\,. \tag{5.21}$$

Choose an integer-valued solution $L_{n\mu\nu}$; this will take the form of an array of quantized strings tiling the loop P. The manipulations following (3.24) then produce the expression

$$\left\langle \prod_{(n\mu)\in P} e^{-iA_{n\mu}} \right\rangle = \frac{1}{Z} \lim_{t\to 0} \int_A \int_\theta \sum_m \exp\left[-\frac{1}{2t}\sum_{n\sigma}(\theta_{n+\hat{\sigma}} - \theta_n - A_{n\sigma} - 2\pi m_{n\sigma})^2\right]$$

$$\times \exp\left[-\frac{g^2}{(2\pi)^2}\frac{1}{4}\sum_{n\nu\sigma} \hat{F}^2_{n\nu\sigma}\right],$$

where

$$\hat{F}_{n\nu\sigma} = \left(F_{n\nu\sigma} + \tfrac{1}{2}\sum_{\mu\rho} \epsilon_{\nu\sigma\mu\rho} L_{n+\hat{\nu}+\hat{\sigma}\mu\rho}\right). \tag{5.23}$$

This can be abbreviated

$$\Big\langle \prod_{(n\mu)\in P} e^{-iA_{n\mu}} \Big\rangle_{\text{PQED}} = \exp[-\varDelta F_{\text{M}}[P]], \tag{5.24}$$

where $\varDelta F_{\text{M}}[P]$ is the free-energy increment in the FZS associated with a magnetic charge current moving along the path P.

The other three duality relations are even more trivial to generalize. The proof of (3.33) goes through in this 4-D situation in precisely the same way, to produce

$$\langle e^{+i\theta_N} e^{-i\theta_{N'}} \rangle_{\text{FZS}} = \exp[-\varDelta F_{\text{V}}(g^2, N - N')], \tag{5.25}$$

where $\varDelta F_{\text{V}}(N - N')$ is the free-energy increment in PQED associated with a vortex (monopole current) source and sink. Relation (3.35) becomes

$$\begin{aligned} \langle J_N^{\mu\nu} J_{N'}^{\lambda\sigma} \rangle_{\text{PQED}} &= \frac{\delta^2}{\delta G_N^{\mu\nu}\, \delta G_{N'}^{\lambda\sigma}} (-\log Z) \Big|_{G=0} \\ &= (2\pi)^2 \left\langle (\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta} F_{n-\hat{\alpha}-\hat{\beta}\alpha\beta})(\tfrac{1}{2}\epsilon_{\lambda\sigma\gamma\delta} F_{n-\hat{\gamma}-\hat{\delta}\gamma\delta}) \right\rangle_{\text{FZS}} . \end{aligned} \tag{5.26}$$

Its reciprocal relation (3.36) becomes

$$\langle J_N{}^{\mu} J_{N'}^{\nu} \rangle_{\text{FZS}} = (2\pi)^2 \langle M_{N\mu} M_{N'\nu} \rangle_{\text{PQED}} . \tag{5.27}$$

In Eqs. (5.18) and (5.24)–(5.27), we have extended the full duality which we had found in three dimensions to this four-dimensional situation.

6. Properties of Four-Dimensional Periodic QED

In this section, we discuss the qualitative features of PQED, using the duality relations just derived. Of course, duality alone gives no direct information about the properties of a given model; it can only relate one model to another, perhaps equally opaque. We should begin, then, by noting the little that is already known about the two models we consider here, the 4-D PQED and the 4-D FZS. The estimates of Polyakov [2], Kogut *et al.* [38], and Banks *et al.* [26] indicate that PQED has a phase transition at a finite coupling $g^2 \approx 2$. For the FZS we have no direct information, but we do have the intuitive arguments directed in Section 4 to its three-dimensional counterpart. On this basis, we might argue that if the 4-D FZS possesses a phase transition—as it must, by duality—then the phase diagram of Fig. 2 applies in four dimensions and the transition in the FZS is of second order. Let us assume this conclusion and derive the consequences for PQED.

The duality relations (5.18), (5.24), and (5.26) may be put to use by simply running backward the arguments we applied to the analogous three-dimensional expressions.

Equation (5.18) displays a correspondence between couplings in PQED and the FZS:

$$g^2 = (2\pi)^2/e^2. \tag{6.1}$$

The correspondence between thermodynamic phases is sketched in Fig. 6. The relation predicts a specific heat anomaly of the form (2.9) in PQED, with the specific heat exponent α of the 4-D FZS. Through (5.24), the presence of a massless photon in the strong-coupling phase of the FZS implies the existence of a similar massless particle in the weak-coupling phase of PQED. In fact, it implies that $\rho_S(g^2)$, as defined by (5.10), is nonzero in this phase. In the superconducting phase of the FZS, and, therefore, in the strong-coupling phase of PQED, the field–field correlation function has no zero-mass singularities. The Wilson loop-integral correlation function may be estimated by the use of (5.24); in the weak-coupling phase of PQED, it has the behavior expected from Maxwell electrodynamics. However, in the superconducting phase of the FZS, Abrikosov flux tubes will surround the Dirac strings which tile the path P, so that the loop correlations in the strong-coupling phase of PQED will exhibit the behavior (5.4). The area law (5.4) and the absence of massless gauge excitations assure us that the strong-coupling phase of PQED is quark confining. The connection between this confining phase and a superconductor, including the exchange of electric and magnetic fields implied in (5.26), is just that conjectured by Mandelstam and 't Hooft.

Let us now turn our attention to the behavior of PQED just in the vicinity of $g^2 = g_c^2$. It is well known that most statistical-mechanical models have computable critical behavior at $d = 4$. since the fixed points of the renormalization group which govern their phase transitions move to zero coupling as $d \to 4$ [39]. However, as we have already remarked, scalar QED possesses no fixed point for $e^2 \neq 0$ which is in the weak-coupling domain near $d = 4$. The fixed point which governs the phase transition in the FZS, then, may not be easily examined in this limit; we have not been able to calculate the exponents, such as α, which parametrize its critical behavior.

We may, however, apply our scaling theory of the FZS near $e^2 = e_c^2$ to find scaling laws for various quantities in PQED. As in Section 4, our results will, for the most part, be applications of hyperscaling. From the exponential decay of $\langle J^{\mu\nu}(x)\, J^{\lambda\alpha}(0)\rangle$ in PQED for $g^2 > g_c^2$, we may extract a correlation length $\xi(g^2)$; the mass $m_G = \frac{1}{2}\xi^{-1}(g^2)$

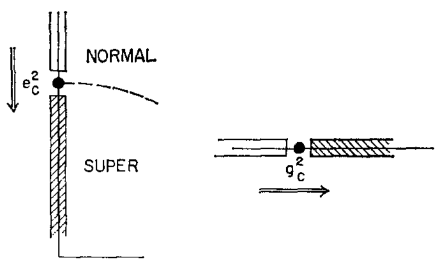

FIG. 6. A pictorial representation of the duality in four dimensions, using the conventions of Fig. 3.

is the mass of the lightest pure gauge excitation. (Strong-coupling estimates of Kogut *et al.* [38] indicate that this particle is a scalar meson.) From Section 4, we know that the dual quantity in the FZS has a correlation length which scales as $| e_c^2 - e^2 |^{-\nu}$, where ν is connected to α through (2.10). The duality relation (5.26) thus extends the hyperscaling relation (2.10) to PQED.

A similar connection may be made for $g^2 < g_c^2$ using relation (5.25). Unlike its counterpart for $d = 3$, the spin–spin correlation function of the FZS in $d = 4$ is free of infrared divergences and may be expected to exhibit an esponential decay. Using hyperscaling in the FZS for $e^2 > e_c^2$, and duality, we learn that in the weak-coupling phase of PQED

$$\Delta F_V \sim m_M(g^2) | N - N' |, \tag{6.2}$$

where, as $g^2 \to g_c^2$,

$$m_M(g^2) \sim | g_c^2 - g^2 |^{\nu}. \tag{6.3}$$

Since the vortices of PQED may be interpreted as the world-lines of magnetic monopoles, (6.3) is a scaling relation for the monopole mass. A more direct (and more naive) application of hyperscaling to PQED would give this same result, as well as the scaling law

$$\rho_S(g^2) \sim | g_c^2 - g^2 |^{(d-2)\nu}. \tag{6.4}$$

Finally, we might discuss the scaling of the string tension $\Xi^{-1}(g^2)$, defined by (5.4). By duality, this is related to the free energy associated with a planar flux-tube structure in the FZS. Applying the formalism of Fisher *et al.* [35], just as we did in Section 4, we reach the expected result

$$\Xi^{-1}(g^2) \sim (g^2 - g_c^2)^{2\nu}. \tag{6.5}$$

A conventional picture of pure gauge excitations is that they are closed strings, as illustrated in Fig. 7. We are amused to note that the energy of a closed flux tube, of radius comparable to its intrinsic thickness, does scale to zero at the same rate as the mass m_G when $g^2 \to g_c^2$.

The complete list of hyperscaling relations of 4-D PQED is displayed in Fig. 8. m_G and m_M represent, again, the masses of gauge excitations and magnetic monopoles, respectively. All exponents ν obey (2.10). Our diagram may be seen to correspond closely to one conjectured earlier by 't Hooft [10].

The results displayed in Fig. 8 exhaust the consequences of duality for 4-D PQED.

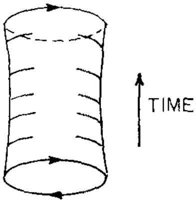

FIG. 7. A gauge-field excitation viewed as a closed string, propagating between current sources.

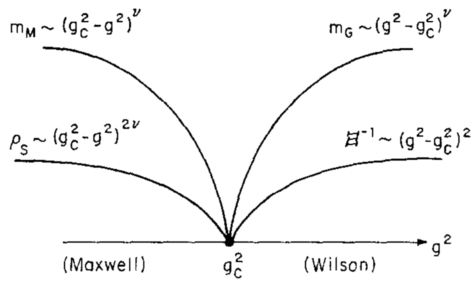

FIG. 8. Scaling behavior of the parameters of 4D PQED in the vicinity of the critical coupling.

The duality has led us to a very powerful conclusion—the complete identification of a strong-coupling quark-confining gauge theory with a superconductor, along lines suggested earlier by Mandelstam and 't Hooft. Our precise connection between the two models has allowed us to explore the behavior of physical quantities near the onset of quark confinement. But it is disappointing that we cannot do more. We still lack a detailed phenomenological theory of the phase transition in PQED, one that would allow direct arguments about the critical behavior, rather than the circuitous arguments which we have, of necessity, presented here.

One aspect of our analysis—the close connections we have displayed between the XY model and PQED—may provide a clue as to the nature of this theory, and we end our discussion with a conjecture to this point: The continuum limit of the XY model—the two-component scalar field—may be described by one of two distinct field theories, a nonlinear and a linear sigma model. The nonlinear sigma model offers a convenient description of the ordered (superfluid) phase, but it is the linear sigma model which offers the best view of the phase transition between the ordered and the disordered phase. This fact is easily understood: The nonlinear sigma model contains a constraint that the magnitude of $\rho_S(T)$ be fixed and nonzero, while, in the linear sigma model, ρ_S becomes a dynamical variable proportional to $\langle\sigma(x)\rangle$. The conventional formulation of gauge theories is the analog of the nonlinear sigma model: the dynamical variables $A_\mu(_N)$ are phase variables, while $\rho_S(g^2)$ has no fundamental dynamical role. Can one write a gauge theory in a formulation in which $\rho_S(g^2)$ becomes a dynamical variable? Such a theory would surely offer a direct picture of a phase transition to quark confinement; it is likely that it would offer a clearer representation of a confining phase and, in so doing, answer a few of the new questions which the analysis of this paper has posed.

APPENDIX: MEAN-FIELD THEORY

Although the estimation of T_c in the XY model by mean-field theory is a standard exercise [27], the mean-field-theory analysis of the full model (2.3) has not, to our knowledge, appeared in the literature. For completeness, we provide a few details of this analysis here.

Mean-field theory involves the estimation of a partition function by the inequality

$$Z = \int e^{-H} \geqslant \left(\int e^{-H_0}\right) e^{-\langle H-H_0\rangle_0}, \tag{A1}$$

where $\langle\ \rangle_0$ denotes an expectation value computed with H_0. The standard analysis of the XY model uses the Hamiltonian of uncorrelated spins in a background magnetic field

$$H_0 = -h \sum_n \cos\theta_n . \tag{A2}$$

The parameter h is adjusted to optimize inequality (A1). A solution $h \neq 0$ implies that $\langle e^{i\theta_n}\rangle \neq 0$. Differentiating (A1) with respect to h yields the equation

$$h = \frac{2d}{T}\frac{I_1(h)}{I_0(h)}, \tag{A3}$$

where I_0 and I_1 are Bessel functions; the analysis of this equation is carried out, for example, in Ref. [12].

To treat model (2.3), we have carried out the variational estimate (A1) in Landau gauge ($\nabla \cdot \mathbf{A} = 0$), using

$$H_0 = -h \sum \cos\theta_n + \frac{1}{4e^2}\sum F_{n\mu\nu}^2 + \frac{m^2}{2e^2}\sum A_{n\mu}^2 . \tag{A4}$$

We find the coupled equations

$$\begin{aligned} h &= \frac{2d}{T}\frac{I_1(h)}{I_0(h)} A, \\ m^2 &= \frac{e^2}{2d}\frac{I_1(h)}{I_0(h)} h, \\ A &= \exp\left[-\frac{e^2}{2}\frac{(d-1)}{d}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(\nabla(k)+m^2)}\right], \end{aligned} \tag{A5}$$

where $\nabla(k) = \sum_\nu (e^{ik_\nu} - 1)^2$ and the integral over k_μ extends over the range $(-\pi, \pi)$. The system of equations (A5) may be solved graphically by standard methods; the optimal values of h and m change discontinuously from zero across the first-order line in Fig. 1.

Note added in proof. Since submitting this paper, we have received several interesting preprints on the same topic. The connection between 4-D PQED and a Higgs theory has been derived independently by Stone and Thomas (Cambridge University preprints, 1978). The XY model coupled to *periodic* QED has been studied using the Villain representation by Einhorn and Savit (FERMILAB-Pub-77/97-THY and -77/105-THY, 1977).

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