16

The Perturbative Renormalization Group

In this chapter we will reexamine the renormalized perturbation theory discussed in Chapters 12 and 13 from the perspective of the Renormalization Group. We will discuss ϕ^4 theory, the $O(N)$ non-linear sigma model , and Yang-Mills gauge theories.

16.1 The Perturbative Renormalization Group

We begin by setting up the perturbative renormalization group to ϕ^4 theory. In Sec.13.2 we used renormalized perturbation theory to show that, to two-loop order, the theory can be made finite by defining a set of renormalization constants such that the renormalized one-particle irreducible N-point vertex functions are related to the bare functions as

$$\Gamma_R^{(N)}(\{p_i\}; m_R, g_R, \kappa) = Z_\phi^{N/2} \Gamma^{(N)}(\{p_i\}, m_0, \lambda, \Lambda) \tag{16.1}$$

where Λ is the UV momentum regulator and κ is the renormalization scale. The relation between the bare and the renormalized theory is encoded in the renormalization constants (the wave function renormalization Z_ϕ, the renormalized mass m_R and the renormalized coupling constant g_R) such that as the UV cutoff is removed, $\Lambda \to \infty$, the renormalized vertex have a finite limit.

Here we will focus our attention on the massless theory, defined by the condition that the renormalized mass vanishes, $m_R = 0$. Thus, we will express the wave function renormalization Z_ϕ and the bare coupling constant λ as functions of the renormalized coupling constant and of the renormalization scale. Similarly the bare mass will be tuned to a value m_c such that

$m_R = 0$. Thus, we will write

$$Z_\phi = Z_\phi(g_R(\kappa), \kappa, \Lambda) \tag{16.2}$$
$$\lambda = \lambda(g_R(\kappa), \kappa, \Lambda) \tag{16.3}$$
$$m_c^2 = m_c^2(g_R(\kappa), \kappa, \Lambda) \tag{16.4}$$

where we made explicit the fact that the renormalized coupling constant g_R is actually not a constant but depends on the value we chose for the renormalization scale κ.

Since the value of the renormalization scale κ is arbitrary, the vertex functions defined at two different scales κ_1 and κ_2, must be related to each other since they correspond to the same bare theory,

$$\begin{aligned}\Gamma_R^{(N)}(\{p_i\}; g_R(\kappa_1), \kappa_1) &= Z_\phi^{N/2}(g_R(\kappa_1), \kappa_1)\Gamma^{(N)}(\{p_i\}, m_c, \lambda, \Lambda)\\ \Gamma_R^{(N)}(\{p_i\}; g_R(\kappa_2), \kappa_2) &= Z_\phi^{N/2}(; g_R(\kappa_2), \kappa_2)\Gamma^{(N)}(\{p_i\}, m_c, \lambda, \Lambda)\end{aligned} \tag{16.5}$$

From here we see that the renormalized vertex functions at the two scales are related by an expression of the form

$$\Gamma_R^{(N)}(\{p_i\}; g_R(\kappa_1), \kappa_1) = Z^{N/2}(g_R(\kappa_2), \kappa_2; g_R(\kappa_1), \kappa_1)\Gamma_R^{(N)}(\{p_i\}; g_R(\kappa_2), \kappa_2) \tag{16.6}$$

where we defined

$$Z(g_R(\kappa_2), \kappa_2; g_R(\kappa_1), \kappa_1) = \frac{Z_\phi(g_R(\kappa_1), \kappa_1, \Lambda)}{Z_\phi(g_R(\kappa_2), \kappa_2, \Lambda)} \tag{16.7}$$
$$= \frac{\partial}{dp^2}\Gamma_R^{(2)}(p, g_R(\kappa_1), \kappa_1)\Big|_{p^2=\kappa_2^2} \tag{16.8}$$

which has a finite limit as the UV cutoff is removed, $\Lambda \to \infty$.

Equation (16.6) is a relation between finite quantities at different scales and as such it is a finite quantity. It implies that a change in the renormalization scale κ is equivalent to a rescaling of the fields by $Z_\phi^{1/2}$ and a change of the renormalized coupling constant $g_1 = g_R(\kappa_1) \mapsto g_2 = g_R(\kappa_2)$, with

$$g_2 \equiv F(\kappa_2, \kappa_1, g_1) = Z_\phi^{-2}\Gamma_R^{(4)}(\{p_i\}g_1, \kappa_1)\Big|_{SP(\kappa_2)} \tag{16.9}$$

where $SP(\kappa)$ is the symmetric point of the four momenta $\{p_i\}$ with each momentum being at the scale κ, i.e. $p_i^2 = \kappa^2$. The mapping $g_2 = F(\kappa_2, \kappa_1, g_1)$, such that $g = F(\kappa, \kappa, g)$, defines a *flow* in the space of coupling constants. i.e. a renormalization group flow.

These relations apply to the full generating functional of renormalized vertex functions which then obeys

$$\Gamma_R\{\bar{\phi}, g_1, \kappa_1\} = \Gamma_R\{Z_\phi^{1/2}\bar{\phi}, F(\kappa_2, \kappa_1, g_1), \kappa_2\} \tag{16.10}$$

Since the bare theory is independent of our choice (and changes) of a renormalization scale, it is kept constant under these transformations. This can be expressed by stating that

$$\kappa\frac{\partial}{\partial\kappa}\Gamma^{(N)}(\{p_i\}, \lambda, m_c^2, \Lambda)\big|_{\lambda,\Lambda} = 0, \tag{16.11}$$

as the UV regulator $\Lambda \to \infty$. Consequently, we find

$$\kappa\frac{\partial}{\partial\kappa}\Big[Z_\phi^{-N/2}\Gamma_R^{(N)}(\{p_i\}, g_R(\kappa), \kappa)\Big]_{\lambda,\Lambda} = 0 \tag{16.12}$$

16.1.1 The Renormalization Group Equations

Therefore, the renormalized N-point vertex functions satisfy the partial differential equation

$$\left[\kappa\frac{\partial}{\partial\kappa} + \bar{\beta}(g_R, \kappa)\frac{\partial}{\partial g_R} - \frac{N}{2}\gamma_\phi(g_R, \kappa)\right]_{\lambda,\Lambda}\Gamma_R^{(N)}(\{p_i\}, g_R(\kappa), \kappa) = 0 \tag{16.13}$$

where we used the definitions

$$\bar{\beta}(g_R(\kappa), \kappa) = \kappa\frac{\partial g_R}{\partial\kappa}\Big|_{\lambda,\Lambda} \tag{16.14}$$

$$\gamma_\phi(g_R(\kappa), \kappa) = \kappa\frac{\partial \ln Z_\phi}{\partial\kappa}\Big|_{\lambda,\Lambda} \tag{16.15}$$

also with $\Lambda \to \infty$.

In general the coupling constant has dimensions. Let us define a dimensionless bare coupling constant u_0

$$\lambda = u_0\kappa^\epsilon \tag{16.16}$$

and a dimensionless renormalized coupling constant u such that

$$g_R = u\kappa^\epsilon \tag{16.17}$$

where $\epsilon = 4 - D = D - \Delta_4$, with $\Delta_4 = 4(D-2)/2$ being the scaling dimension of the operator ϕ^4 at the free massless fixed point.

In terms of the dimensionless renormalized coupling constant u, Eq.(16.13) becomes the *Callan-Symanzik Equation*

$$\left[\kappa\frac{\partial}{\partial\kappa} + \beta(u)\frac{\partial}{\partial u} - \frac{N}{2}\gamma_\phi(u)\right]_{\lambda,\Lambda}\Gamma_R^{(N)}(\{p_i\}, u(\kappa), \kappa) = 0 \tag{16.18}$$

(again, with $\Lambda \to \infty$) where $\beta(u)$ is the renormalization group beta function which is defined by

$$\beta(u) = \kappa \frac{\partial u}{\partial \kappa}\Big|_{\lambda,\Lambda} \tag{16.19}$$

and

$$\gamma_\phi(u) = \frac{\partial \ln Z_\phi}{\partial \ln \kappa}\Big|_{\lambda,\Lambda} \tag{16.20}$$

Notice that here we defined the sign of the beta function *opposite* to the sign we used in Chapter 15. Hence, a positive beta function means that the coupling constant increases as the momentum scale increases, and viceversa.

The formulas of Eqs. (16.19) and Eq.(16.20) as they stand are somewhat awkward to use since they involve he bare dimensionless coupling constant u_0 in terms of the dimensionless renormalized coupling constant u instead of the other way around. For this reason we use the chain rule to write

$$\kappa \left(\frac{\partial u}{\partial \kappa}\right)_\lambda = -\kappa \frac{\left(\frac{\partial \lambda}{\partial \kappa}\right)_u}{\left(\frac{\partial \lambda}{\partial u}\right)_\kappa} \tag{16.21}$$

where, by dimensional analysis,

$$\lambda = \kappa^\epsilon u_0(u, \kappa/\Lambda) \tag{16.22}$$

Using that

$$\kappa \left(\frac{\partial \lambda}{\partial \kappa}\right)_u = \epsilon \lambda \tag{16.23}$$

we find that the beta function is

$$\beta(u) = \kappa \left(\frac{\partial u}{\partial \kappa}\right)_\lambda = -\epsilon \left(\frac{\partial \ln u_0}{\partial u}\right)^{-1} \tag{16.24}$$

In this form the beta function $\beta(u)$ can be expressed as a power series expansion in the dimensionless renormalized coupling constant u. Each coefficient of this series is a function of $\epsilon = 4 - D$.

Similarly, we can rewrite the anomalous dimension $\gamma_\phi(u)$ as

$$\gamma_\phi(u) = \frac{\partial \ln Z_\phi}{\partial \ln \kappa}\Big|_\lambda = \kappa \left(\frac{\partial u}{\partial \kappa}\right)_\lambda \frac{\partial \ln Z_\phi}{\partial u} \tag{16.25}$$

We find

$$\gamma_\phi(u) = \beta(u) \frac{\partial \ln Z_\phi}{\partial u} \tag{16.26}$$

which also can be written as a power series expansion in the dimensionless renormalized coupling constant u.

16.1.2 General Solution of the Callan-Symanzik Equations

We will now solve the Callan-Symanzik equation, Eq.(16.18). Let $x = \ln \kappa$ and write the renormalized vertex function as follows

$$\Gamma_R^{(N)}(\{p_i\}, u, \kappa) = \exp\left(\frac{N}{2}\int_{u_1}^{u} du' \frac{\gamma_\phi(u')}{\beta(u')}\right)\Phi^{(N)}(\{p_i\}, u\kappa) \tag{16.27}$$

By requiring that this expression satisfies the Callan-Symanzik equation, Eq.(16.18), we find that the function $\Phi^{N)}$ must satisfy the simpler equation

$$\left[\frac{\partial}{\partial x} + \beta(u)\frac{\partial}{\partial u}\right]\Phi^{(N)}(\{p_i\}, u, \kappa) = 0 \tag{16.28}$$

It is straightforward to see that the solutions to this equation have the general form

$$\Phi^{(N)}(\{p_i\}, u, \kappa) = \mathcal{F}^{(N)}\left(\{p_i\}, x - \int_{u_2}^{u} \frac{du'}{\beta(u')}\right) \tag{16.29}$$

where $\mathcal{F}^{(N)}$ is an arbitrary (differentiable) function.

We conclude that Callan-Symanzik equation requires the renormalized vertex functions to have the following form

$$\Gamma^{(N)}(\{p_i\}, u\kappa) = \exp\left(\frac{N}{2}\int_{u_1}^{u} du' \frac{\gamma_\phi(u')}{\beta(u')}\right)\mathcal{F}^{(N)}\left(\{p_i\}, x - \int_{u_2}^{u} \frac{du'}{\beta(u')}\right) \tag{16.30}$$

where u_1 and u_2 are two integration constants. It should be apparent that the *scaling function* $\mathcal{F}^{(N)}$ cannot be obtained in perturbation theory.

Let us now rescale all the momenta $\{p_i\}$ by the same scale factor ρ, $p_i \to \rho p_i$. Dimensional analysis implies that as a result the vertex functions should be rescaled as follows

$$\Gamma_R^{(N)}(\{\rho p_i\}, u, \kappa) = \rho^{N+D-ND/2}\Gamma_R^{(N)}(\{p_i\}, u, \kappa) \tag{16.31}$$

Using the form of the general solution, Eq.(16.30), of the Callan-Symanzik equation, we find

$$\Gamma_R^{(N)}(\{\rho p_i\}, u, \kappa) = \rho^{N+D-ND/2} \exp\left(\frac{N}{2}\int_{u_1}^{u} du' \frac{\gamma_\phi(u')}{\beta(u')}\right) \times \mathcal{F}^{(N)}\left(\{p_i\}, x - \ln\rho - \int_{u_2}^{u} \frac{du'}{\beta(u')}\right) \tag{16.32}$$

where we used that, since $x = \ln\kappa$, the rescaling $\kappa \to \kappa/\rho$ is equivalent to the shift $x \to x - \ln\rho$.

We will now make explicit the notion that the renormalization group induces a flow in the space of coupling constants by introducing a *running* coupling constant $u(\rho)$ that is a function of the scale change ρ. To this effect, we define

$$s = \ln\rho = \int_{u}^{u(\rho)} \frac{du'}{\beta(u')} \tag{16.33}$$

such that the running coupling constant $u(s)$ obeys the differential equation

$$\frac{\partial u(s)}{\partial s} = \beta(u(s)) \tag{16.34}$$

with the initial condition for the flow $u(s = 0) = u$.

We can rewrite the solution of the Callan-Symanzik Equation, Eq.(16.32), in terms of the running coupling constant $u(\rho)$ defined in Eq.(16.33). The resulting expression is

$$\Gamma_R^{(N)}(\{\rho p_i\}, u, \kappa) = \rho^{N+D-ND/2}\Gamma_R^{(N)}(\{p_i\}, u, \kappa) \tag{16.35}$$

Using the form of the general solution, Eq.(16.30), of the Callan-Symanzik equation, we find

$$\Gamma_R^{(N)}(\{\rho p_i\}, u, \kappa) = \rho^{N+D-ND/2} \exp\left(\frac{N}{2}\left[\int_{u_1}^{u(s)} du' \frac{\gamma_\phi(u')}{\beta(u')} - \int_{u}^{u(s)} du' \frac{\gamma_\phi(u')}{\beta(u')}\right]\right) \times \mathcal{F}^{(N)}\left(\{p_i\}, x - \ln\rho - \int_{u_2}^{u(s)} \frac{du'}{\beta(u')}\right) \tag{16.36}$$

which can be written in the equivalent and more illuminating form

$$\Gamma_R^{(N)}(\{\rho p_i\}, u, \kappa) = \rho^{N+D-ND/2} \exp\left(\frac{N}{2}\int_{u}^{u(s)} du' \frac{\gamma_\phi(u')}{\beta(u')}\right)\Gamma_R^{(N)}(\{p_i\}, u(s), \kappa) \tag{16.37}$$

This result implies that a change in the momentum scale in the renormalized N-point vertex function is equivalent to

a) the rescaling of the vertex function by its canonical dimension,
b) the introduction a running coupling constant (i.e. a flow along a renormalization group trajectory),
c) an anomalous multiplicative factor associated with the anomalous dimension of the field

These changes embody the main effects of a renormalization group transformation. It should be self-evident that the construction used in this Chapter has the same physical content as the more intuitive and physically transparent approaches we discussed in Chapter 15.

16.1.3 Fixed Points and Scaling Behavior

We will now discuss the consequences of Eq.(16.37). This is a rather complex expression. We will split the analysis in two steps by a) first considering its scaling predictions at a fixed point of the renormalization group ,and b) looking at the corrections to the scaling behavior as the fixed point is approached (or departed).

Scaling behavior at a Fixed Point

We begin by looking at the predictions for the behavior of the renormalized N-point vertex functions at an RG fixed point. As in Chapter 15, a fixed point of the RG is the value u^* of the dimensionless renormalized coupling constant at which its beta function has a zero: $\beta(u^*) = 0$.

At a fixed point u^*, the integral in the second factor of Eq.(16.37) can be computed exactly. The result s

$$\int_u^{u(s)} du' \frac{\gamma_\phi(u')}{\beta(u')} = \int_{s_0}^{s_0+s} \gamma_\phi(u(s'))ds' = \gamma_\phi(u^*)s = \gamma_\phi(u^*)\ln\rho \qquad (16.38)$$

Hence, the integral is just the value of the anomalous dimension γ_ϕ *at the fixed point* u^*, multiplied by the logarithm of the scale change. Therefore, at the fixed point, the N-point vertex function obeys the scaling relation

$$\Gamma_R^{(N)}(\{\rho p_i\}, u^*, \kappa) = \rho^{N+D-ND/2-N\gamma_\phi(u^*)/2}\Gamma_R^{(N)}(\{p_i\}, u^*, \kappa) \qquad (16.39)$$

In the case of the $N = 2$ point function we obtain

$$\Gamma_R^{(2)}(\rho p, u^*, \kappa) = \rho^{2-\gamma_\phi(u^*)}\Gamma_R^{(2)}(p, u^*, \kappa). \qquad (16.40)$$

If we choose the scale change to be $\rho = \kappa/p$, Eq. (16.40) implies that

$$\Gamma_R^{(2)}(p, u^*, \kappa) = \left(\frac{p}{\kappa}\right)^{2-\gamma_\phi(u^*)} \Gamma_R^{(2)}(\kappa, u^*, \kappa) \equiv p^2 \left(\frac{p}{\kappa}\right)^{-\gamma_\phi(u^*)}, \qquad (16.41)$$

since, by definition, the renormalized two-point vertex function of a massless theory is $\Gamma_R^{(2)}(\kappa, u^*, \kappa) = \kappa^2$. Furthermore, this result implies that the renormalized two-point function in real space has the scaling behavior

$$\langle \phi(x)\phi(y)\rangle_R^* = \frac{\text{const.}}{|x-y|^{D-2+\eta}} \tag{16.42}$$

Hence, the scaling dimension of the field ϕ, Δ_ϕ, at the fixed point is

$$\Delta_\phi = \frac{D-2}{2} + \frac{\gamma_\phi^*}{2}. \tag{16.43}$$

and

$$\eta = \gamma_\phi^*, \tag{16.44}$$

which explains why $\gamma_\phi(u^*)$ is called the *anomalous* (or fractal) dimension of the field since it measures the deviation of Δ_ϕ, the scaling dimension of the field ϕ, from its free-field value, $(D-2)/2$.

The main conclusion is that at a general fixed point the scaling behavior of the observables are generally different from those of a free field theory. This result is essentially nonperturbative.

Corrections to Scaling

Most physical systems are not at a fixed point. Therefore it will be important to quantify what corrections, if any, there will be to the predicted scaling behavior away from the fixed point. as we saw in Chapter 15 the fixed points can either be stable or unstable and, hence, the flow may be attractive or repulsive. We will consider both cases. These effects are called corrections to scaling behavior. The types of corrections to scaling depends on the behavior of the beta function near the fixed point. Since, by construction, the beta function is a regular function of the coupling constant, this question depends then on the order of the zero associated with the fixed point.

Let su consider first the simplest case of a linear zero and write

$$\beta(u) = \beta'(u^*)(u-u^*) + O((u-u^*)^2) \tag{16.45}$$

Likewise, we can expand the anomalous dimension function $\gamma_\phi(u)$ near the fixed point u^*,

$$\gamma_\phi(u) = \gamma_\phi(u^*) + \gamma_\phi'(u^*)(u-u^*) + O((u-u^*)^2) \tag{16.46}$$

We will have to consider two cases:

a) If $\beta'(u^*) > 0$, then as $\rho \to 0$ (i.e. $s \to -\infty$), the coupling constant u flows *into* the fixed point, $u \to u^*$, in the IR. This is the case of an IR stable

(UV unstable) fixed point. In other words, this fixed point is stable at long distances (low energies). In this case, the associated operator is *irrelevant*. This is called an IR fixed point.

b) If $\beta'(u^*) < 0$, then as $\rho \to \infty$ (i.e. $s \to +\infty$), the coupling constant u flows *into* of the fixed point, $u \to u^*$, in the UV. This is the UV stable (IR unstable) fixed point. In this case, this fixed point is stable at short distances (high energies). In this case, the associated operator is *relevant*. This is called a UV fixed point.

We see that the theory will generally have several fixed points. as the energy scale is lowered, the RG flow goes from the UV fixed point to the IR fixed point.

Near the fixed point, the integral of Eq.(16.38) becomes

$$\begin{aligned}\int_u^{u(s)} du' \frac{\gamma_\phi(u')}{\beta(u')} =& \int_u^{u(s)} \left[\frac{\gamma_\phi(u^*) + \gamma_\phi'(u^*)(u-u^*) + \dots}{\beta'(u^*)(u-u^*) + \dots} \right] du' \\ =& \frac{\gamma_\phi(u^*)}{\beta'(u^*)} \ln\left[\frac{u(s)-u^*}{u-u^*}\right] + \frac{\gamma_\phi'(u^*)}{\beta'(u^*)}(u(s)-u) + \dots\end{aligned} \tag{16.47}$$

Then,

$$\exp\left(\frac{N}{2}\int_u^{u(s)} du' \frac{\gamma_\phi(u')}{\beta(u')}\right) = \exp\left(-\frac{N}{2}\gamma_\phi(u^*)s - \frac{N}{2}\frac{\gamma_\phi(u^*)}{\beta'(u^*)}(u(s)-u)\right) \tag{16.48}$$

In the case of case a), the IR fixed point, the coupling u flows into the fixed point at u^* in the IR (as $\rho \to 0$). Hence in this case the scaling form o the N point vertex function becomes

$$\begin{aligned}\Gamma_R^{(N)}(\{\rho p_i\}, u, \kappa) = \rho^{N+D-\frac{ND}{2}-\frac{N}{2}\gamma_\phi(u^*)} \exp\left(-\frac{N}{2}\frac{\gamma_\phi'(u^*)}{\beta'(u^*)}(u^*-u)\right) \\ \times \Gamma_R^{(N)}(\{p_i\}, u^*, \kappa)\end{aligned} \tag{16.49}$$

For the two-point vertex function we now find

$$\Gamma_R^{(2)}(p, u, \kappa) = p^2 \left(\frac{p}{\kappa}\right)^{-\gamma_\phi(u^*)} \exp\left(-\frac{\gamma_\phi'(u^*)}{\beta'(u^*)}(u^*-u)\right) \tag{16.50}$$

Hence, the correction to scaling at long distances (low energies) near the IR fixed point amounts to a non-universal multiplicative factor correction to the two point function (and for the other N-point functions).

It is obvious to see that in case b), the UV fixed point, we obtain the same result but at short distances (or low energies).

Marginality and Renormalizability

More interesting is the case of a *marginal* operator. In this case, the beta function has a double zero, i.e.

$$\beta(u) = A(u - u^*)^2 + \dots \tag{16.51}$$

This is the case in all theories at their renormalizable dimension, i.e. the critical dimension D_c. In the marginal case, the scaling dimension of the operator associated with the coupling constant u is equal to the dimension of spacetime, $\Delta = D$. This is what happens for ϕ^4 theory in $D = 4$ dimensions, and also in the case of gauge theories. In the case of the non-linear sigma model the critical dimension is $D_c = 2$.

Here too we have two cases: a) $A > 0$, a marginally stable IR fixed point, and b) $A < 0$, a marginally stable UV fixed point. The case $A < 0$ is specially important as it describes asymptotically free theories.

We first need to solve the equation of the beta function

$$\frac{\partial u}{\partial s} = A(u - u^*)^2 \tag{16.52}$$

The solution is

$$u(s) = u^* - \frac{u - u^*}{A(u^* - u)s + 1} \tag{16.53}$$

and

$$s = -\frac{1}{A}\left[\frac{1}{u(s) - u^*} - \frac{1}{u - u^*}\right] \tag{16.54}$$

As $u(s) \to u^*$ we find

$$s = -\frac{1}{A(u(s) - u^*)} \tag{16.55}$$

Using these results, we get

$$\int_u^{u(s)} du' \frac{\gamma_\phi(u')}{\beta(u')} = \gamma_\phi(u^*)s + \frac{\gamma'_\phi(u^*)}{A} \ln\left(\frac{u(s) - u^*}{u - u^*}\right) \tag{16.56}$$

and, again as $u(s) \to u^*$,

$$\exp\left(-\frac{N}{2}\int_u^{u(s)} du' \frac{\gamma_\phi(u')}{\beta(u')}\right) = \text{const. } \rho^{-\frac{N}{2}\gamma_\phi(u^*)} (\ln \rho)^{\frac{N}{2}\frac{\gamma'_\phi(u^*)}{A}} \tag{16.57}$$

In particular, we find the two point function near the fixed point now is

$$\Gamma_R^{(2)}(p, u, \kappa) = \text{const } p^2 \left(\frac{p}{\kappa}\right)^{-\gamma_\phi(u^*)} \left(\ln \frac{p}{\kappa}\right)^{\gamma'_\phi(u^*)/A} \tag{16.58}$$

Thus, in the marginal cases, we find a logarithmic correction to the fixed point behavior. This is a consequence of the slow change of the coupling constant near the fixed point. This result applies for the IR fixed point at low energies ($p \to 0$) and the UV fixed point at high energies ($p \to \infty$).

16.2 Perturbative Renormalization Group for Massless ϕ^4 Theory

Our next task is to compute the renormalization group functions for ϕ^4 theory. We will only consider the massless theory and we will work to two-loop order.

16.2.1 Renormalization constants to two-loop order

We will now use dimensional regularization to compute the coefficients of the renormalization constants for the massless theory of Eq.(13.65) and Eq.(13.67).

We begin with the wave function renormalization. As we saw, in ϕ^4 theory, first non-trivial contribution appears at two loop order. The coefficient of this contribution, called z_2, is given in Eq.(13.65). It can be written as

$$z_2 = \frac{1}{6}\kappa^{2\epsilon} E'_3(\kappa) \tag{16.59}$$

where $E'_3(\kappa)$ is the integral

$$\left.\frac{\partial E_3(p)}{\partial p^2}\right|_{p^2=\kappa^2} \tag{16.60}$$

where

$$E_3(p) = \int \frac{d^D q_1}{(2\pi)^D} \int \frac{d^D q_2}{(2\pi)^D} \frac{1}{q_1^2 q_2^2 (p - q_1 - q_2)^2} \tag{16.61}$$

Using dimensional regularization for the integral in Eq.(16.60), one finds

$$z_2 = -\frac{1}{48}\left(\frac{1}{\epsilon} + \frac{5}{4}\right) \tag{16.62}$$

where, as before, $\epsilon = 4 - D$. Notice that z_2 has a simple pole in ϵ. Notice that we are keeping only the singular part as $\epsilon \to 0$.

Next we look at the coefficients λ_2 and λ_3, of the expansion for the coupling constant to two-loop order, Eq.(13.67). The computation of the coefficient λ_2 involves only the the one loop integral of Eq.(13.29) of the massless theory, evaluated at the symmetric point of the four external momenta with a momentum transfer scale $P^2 = \kappa^2$.

$$I_{SP} = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 (P-q)^2} \tag{16.63}$$

We have already done this calculation in Sec.13.8.1 using dimensional regularization. Using these results on finds that at finite but small $\epsilon = 4 - D$, the coefficient is

$$\lambda_2 = \frac{3}{2}\left(\frac{1}{\epsilon} + \frac{1}{2}\right) \tag{16.64}$$

As shown in Eq.(13.67), the coefficient λ_2 involved the one loop integral I_{SP} of Eq.(13.29), the two loop integrals I_{4SP} of Eq. (13.31),

$$I_{4SP} = \int \frac{d^D q_1}{(2\pi)^D} \int \frac{d^D q_2}{(2\pi)^D} \frac{1}{q_1^2 (p_1 + p_2 - q_1)^2 q_2^2 (p_3 - (q_1 + q_2))^2} \tag{16.65}$$

(again, with the external momenta at the symmetry point), and the coefficient z_2 computed above. Using these results and the computation of I_{4SP} at the symmetric point (c.f. Amit's book) we obtain

$$\lambda_3 = \frac{9}{4\epsilon^2} + \frac{37}{24\epsilon} \tag{16.66}$$

Notice the double-pole in ϵ of this two-loop coefficient.

16.2.2 Renormalization Group Functions and Fixed Points

Our next task is to compute the beta function $\beta(u)$ and the anomalous dimension γ_ϕ in terms of the expressions of Eq.(16.4) that relate the bare and renormalized coupling constant and mass, and the wave function renormalization Z_ϕ. We will use the two-loop results for ϕ^4 theory shown in the preceding section, Sec.16.2.1. There we wrote an expansion of the dimensionless bare coupling constant u_0 as a power series in the dimensionless renormalized coupling constant u of the form

$$u_0 = u\left(1 + \lambda_2 u + \lambda_3 u^2 + \ldots\right) \tag{16.67}$$

and a similar expansion for the wave function renormalization Z_ϕ,

$$Z_\phi = 1 + z_2 u^2 + \ldots \tag{16.68}$$

These coefficients can be computed in terms of Feynman diagrams either a) by using renormalization conditions, or b) by using the minimal subtraction scheme in dimensionally regularized diagrams. Here we will do it both ways.

The beta function $\beta(u)$, using the result of Eq.(16.24), can be written as a series expansion in powers of u:

$$\beta(u) = -\epsilon\left(\frac{\partial \ln u_0}{\partial u}\right)^{-1} = -\epsilon u\left(1 - \lambda_2 u + 2(\lambda_2^2 - \lambda_3)u^2\right) + \dots \tag{16.69}$$

where λ_2 and λ_3 are given explicitly in Eqs.(16.64) and (16.66).

Similarly, we can expand the anomalous dimensions γ_ϕ:

$$\gamma_\phi = \beta(u)\frac{\partial \ln Z_\phi}{\partial u} = -2\epsilon\, z_2 u^2 + \dots \tag{16.70}$$

where z_2 was given in Eq.(16.62).

In Chapter 13 we used renormalization conditions to calculate the coefficients λ_2, λ_3 and z_2, Eqs. (16.64), (16.66) and (16.62) respectively. using these results we find the following result for the beta function $\beta(u)$ and the anomalous dimension γ_ϕ to two loop order

$$\beta(u) = \kappa\frac{\partial u}{\partial \kappa} = -\,\epsilon u + \frac{3}{2}\left(1 + \frac{\epsilon}{2}\right)u^2 - \frac{17}{12}u^3 + O(u^4) \tag{16.71}$$

$$\gamma_\phi = \frac{1}{24}u^2 + O(u^3) \tag{16.72}$$

Notice that although the coefficients λ_2, λ_3 and z_2 have poles in ϵ, the beta function and the anomalous dimension are regular functions of ϵ.

The fixed points are the zeros of the beta function which is a polynomial in the renormalized dimensionless coupling constant u. By construction it always has a zero at $u = 0$, representing the free field theory. For $D < 4$ (or, $\epsilon > 0$), for small enough ϵ it also has a zero at a value u^* which is a power series expansion in ϵ. At two loop order, the non-trivial (Wilson-Fisher) fixed point is at

$$u^* = \frac{2}{3}\epsilon + \frac{34}{9}\epsilon^2 + O(\epsilon^3) \tag{16.73}$$

Since the beta function is a polynomial function of u to any finite order in perturbation theory one may wonder if there are other possible fixed points. However, only the trivial fixed point at $u = 0$ and the Wilson-Fisher fixed point $u^* = f(\epsilon)$, where $f(\epsilon)$ is a regular function of epsilon that vanishes as $\epsilon \to 0$, should be considered, since all other fixed points will be zeros at a finite distance from $u = 0$ that is not tuned by ϵ.

The anomalous dimension γ_ϕ of Eq.(16.72) at this fixed point is

$$\gamma_\phi(u^*) = \frac{\epsilon^2}{54} + O(\epsilon^3) \tag{16.74}$$

Thus, as promised, we see that at two-loop order the field ϕ at the Wilson-Fisher fixed point has an anomalous dimension.

It is now easy to see that for $D < 4$, the free field fixed point is unstable in the IR since the dimension of the operator ϕ^4 is $\Delta_4(FF) = 2(D-2) > 0$. In contrast, the two-loop beta function, Eq.(16.71), tells us that the dimension of ϕ^4 at the Wilson-Fisher fixed point is $\Delta_4(WF) = 4 - \frac{17}{27}\epsilon^2 > D$. Thus, the same operator is relevant at one fixed point (free field) and irrelevant at the other (Wilson-Fisher).

16.3 Dimensional Regularization with Minimal Subtraction

We will now use dimensional regularization within the Minimal Subtraction (MS) scheme to compute the renormalization functions. Once again we will consider the massless theory in D dimensions. As we saw, the Feynman diagrams are expressed in terms of integrals develop singularities in the form of poles in $\epsilon = 4 - D$ (as $D \to 4$). In fact, the only vertex functions that have primitive divergencies that develop poles in ϵ are $\frac{\partial \Gamma^{(2)}}{\partial p^2}$ and $\Gamma^{(4)}$. The poles in ϵ correspond to logarithmic divergencies in the momentum scale at $D = 4$.

Once again we will write the bare dimensionless coupling constant u_0 and the wave function renormalization Z_ϕ as a series expansion in powers of the dimensionless renormalized coupling constant u,

$$u_0 = u\big(1 + \sum_{n=0}^{\infty} \lambda_n(\epsilon) u^n\big) \tag{16.75}$$

$$Z_\phi = 1 + \sum_{n=1}^{\infty} z_n(\epsilon) u^n \tag{16.76}$$

The coefficients $\{\lambda_n(\epsilon)\}$ and $\{z_n(\epsilon)\}$, are singular functions of ϵ chosen in such a way that

$$\Gamma_R^{(2)}(p; u, \kappa) = Z_\phi \Gamma^{(2)}(p; u_0, \kappa) \tag{16.77}$$

$$\Gamma_R^{(4)}(\{p_i\}; u, \kappa) = Z_\phi^2 \Gamma^{(4)}(\{p_i\}; u_0, \kappa) \tag{16.78}$$

are finite as $D \to 4$. In the Minimal Subtraction scheme the coefficients are determined by imposing the condition that the poles in ϵ found in $\Gamma^{(2)}$ and $\Gamma^{(4)}$ are minimally subtracted.

This scheme works as follows. We first write the expressions for the bare two-point function to two loop order:

$$\Gamma^{(p)}(p;u_0,\kappa) = p^2(1 - B_2 u_0^2) \tag{16.79}$$

(where we imposed the condition that the renormalized theory is massless), and similarly for the bare four-point function

$$\Gamma^{(4)}(\{p_i\};u_0,\kappa) = \kappa^\epsilon u_0 \left(1 - A_1 u_0 + (A_2^{(1)} + A_2^{(2)}) u_0^2\right) \tag{16.80}$$

where

$$A_1 = \frac{1}{2}\left[I\left(\frac{p_1+p_2}{\kappa}\right) + \text{two permutations}\right] \tag{16.81}$$

$$A_2^{(2)} = \frac{1}{2}\left[I_4\left(\frac{p_1}{\kappa},\ldots,\frac{p_4}{\kappa}\right) + \text{five permutations}\right] \tag{16.82}$$

The integrals were given in Sec.13.2 (c.f. Eq.(13.36)).

Consider first $\Gamma^{(2)}$. To order u^2 we find

$$\Gamma_R^{(2)}(p,\kappa) = p^2(1 + z_1 u + z_2 u^2)(1 - B_2 u^2) \tag{16.83}$$

$$= p^2\left[1 + z_1 u + (z_2 - B_2)u^2 + O(u^3)\right] \tag{16.84}$$

Thus, we set

$$z_1 = 0, \quad z_2 = [B_2]_{\text{sing}} \tag{16.85}$$

where $[B_2]_{\text{sing}}$ is the singular part of B_2. Hence,

$$z_2 = [B_2]_{\text{sing}} = -\frac{1}{48\epsilon} \tag{16.86}$$

Therefore, in the MS scheme the wave function renormalization is

$$Z_\phi = 1 - \frac{u^2}{48\epsilon} + O(u^3) \tag{16.87}$$

Next we use the same approach for the four-point function. To order u^3 we can write the renormalized four-point vertex function as

$$\begin{aligned}\Gamma_R^{(4)}(p_1,\ldots,p_4;u,\kappa) =& \kappa^\epsilon(1 + 2z_2u^2)(u + \lambda_2 u^2 + \lambda_3 u^3)\\ &-(u^2 + 2\lambda_2 u^3)\frac{1}{2}\left[I\left(\frac{p_1+p_2}{\kappa}\right) + \text{two permutations}\right]\\ &+u^3\left\{\frac{1}{4}\left[I^2\left(\frac{p_1+p_2}{\kappa}\right) + \text{two permutations}\right]\right.\\ &\left.+\frac{1}{2}\left[I_4\left(\frac{p_1}{\kappa},\ldots,\frac{p_4}{\kappa}\right) + \text{five permutations}\right]\right\}\end{aligned} \tag{16.88}$$

Collecting terms we get

$$\begin{aligned}\Gamma_R^{(4)}(p_1,\ldots,p_4;u,\kappa) =&\kappa^{\epsilon}\Big\{u + u^2\Big[\lambda_2 - \frac{1}{2}\Big[I\Big(\frac{p_1+p_2}{\kappa}\Big) + \text{two \ permutations}\Big]\Big] \\ &+u^3\Big[\lambda_3 + 2z_2 - \frac{3}{2}\lambda_2\Big[I\Big(\frac{p_1+p_2}{\kappa}\Big) + \text{two permutations}\Big] \\ &+ \frac{1}{4}\Big[I^2\Big(\frac{p_1+p_2}{\kappa}\Big) + \text{two permutations}\Big] \qquad (16.89) \\ &\frac{1}{2}\Big[I_4\Big(\frac{p_1}{\kappa},\ldots,\frac{p_4}{\kappa}\Big) + \text{five permutations}\Big]\Big]\Big\} \qquad (16.90)\end{aligned}$$

We then cancel the singularities by setting

$$\lambda_2 = \frac{1}{2}\Big[\Big[I\Big(\frac{p_1+p_2}{\kappa}\Big)\Big]_{\text{sing}} + \text{two permutations}\Big] = \frac{3}{2\epsilon} \tag{16.91}$$

Similarly

$$\begin{aligned}\lambda_3 =& -2z_2 + \lambda_2\Big[\Big[I\Big(\frac{p_1+p_2}{\kappa}\Big)\Big]_{\text{sing}} + \text{two permutations}\Big] \\ &- \frac{1}{4}\Big[\Big[I^2\Big(\frac{p_1+p_2}{\kappa}\Big)\Big]_{\text{sing}} + \text{two permutations}\Big] \\ &- \frac{1}{2}\Big[\Big[I_4\Big(\frac{p_1}{\kappa},\ldots,\frac{p_4}{\kappa}\Big)\Big]_{\text{sing}} + \text{five permutations}\Big]\end{aligned} \tag{16.92}$$

from where we find

$$\lambda_3 = \frac{9}{4\epsilon^2} - \frac{17}{24\epsilon} \tag{16.93}$$

We find that to two-loop order, the bare and renormalized dimensionless coupling constants are related by

$$u_0 = u + \frac{3}{2\epsilon}u^2 + \Big(\frac{9}{4\epsilon^2} - \frac{17}{24\epsilon}\Big)u^3 + O(u^4) \tag{16.94}$$

We can now use these results to obtain the beta function and the anomalous dimension to two-loop order in the MS scheme:

$$\beta(u) = -\epsilon u + \frac{3}{2}u^2 - \frac{17}{12}u^3 + O(u^4) \tag{16.95}$$

$$\gamma_\phi(u) = \frac{u^2}{24} + O(u^3) \tag{16.96}$$

Notice that the expression for the two-loop beta functions of Eq.(16.71) and Eq.(16.95) are slightly different. These differences arise from the different renormalization schemes used. Nevertheless, these differences only amount to a redefinition of the location of the fixed point which do not change the value of the exponents.

16.4 The non-linear sigma model in $D = 2$ dimensions

In Sec.12.5 we derived the low-energy effective action of a ϕ^4 theory with an $O(N)$ global symmetry spontaneously broken to its $O(N-1)$ subgroup. The action of the non-linear sigma model is

$$S = \frac{1}{2g} \int d^D g \left(\partial_\mu \boldsymbol{n}(x)\right)^2, \qquad \text{with } \boldsymbol{n}^2(x) = 1 \tag{16.97}$$

where the constraint is imposed locally. Here g is the coupling constant. In this section we will discuss the renormalizability and the renormalization group for the non-linear sigma model in $D = 2$ and $D = 2 + \epsilon$ dimensions. As we saw in Section 13.4, by dimensional analysis we expect the non-linear sigma model to be renormalizable in $D = 2$ space-time dimensions. Although this theory is non-renormalizable as an expansion around the trivial fixed point at $g = 0$, it is renormalizable around its non-trivial fixed point in $D = 2 + \epsilon$, which we expect to have the same universal properties as the Wilson-Fisher fixed point in the same dimension.

The non-linear sigma model is interesting for several reasons. It arises in high-energy physics as the low-energy limit of theories of spontaneously broken chiral symmetry and, hence, as a model to describe pions. in that context, the coupling constant g is the inverse of the pion decay constant. It also arises in classical statistical mechanics as the long-wavelength description of the Heisenberg model of a classical ferromagnet, described in terms of an N-component spin degree of freedom with unit length. Here, the coupling constant is T/J, where T is the temperature and J is the exchange constant. This model also arises as the effective action of quantum antiferromagnets. Generalizations of the non-linear sigma model also play a key role in perturbative string theory, where (supersymmetric) non-linear sigma models describe contributions to the string amplitudes, and even in the theory of localization of electrons in random potentials.

However, in addition to its relevance to wide areas of physics, this model of particular interest since it has many analogies with non-abelian four-dimensional gauge theories, while being much simpler.

16.5 Generalizations of the Non-Linear Sigma Model

The $O(N)$ non-linear sigma model, whose action is given in Eq.(16.97), has a simple geometrical interpretation which leads to many generalizations. The field $\boldsymbol{n}(x)$ of the non-linear sigma model takes values on the N-dimensional sphere S_N which is the coset of the broken symmetry group $O(N)$ with the unbroken global symmetry $O(N-1)$, i.e. $S_{N-1} \cong O(N)/O(N-1)$. In this sense, the field can be regarded as a mapping of the Euclidean space-time $\mathbb{R}^D$ onto the sphere S_{N-1}. We will call the sphere S_N the *target manifold*. Hence, the target space of the non-linear sigma model is a manifold that is a coset. We will see in Chapter 19 that, with natural boundary conditions, it can also be regarded as a map from the space-time compactified to the sphere S_D onto the sphere S_{N-1}.

16.5.1 The $\mathbb{CP}^{N-1}$ non-linear sigma model

The first generalization that we will discuss is the $\mathbb{CP}^N$ non-linear sigma model. Here $\mathbb{CP}^{N-1}$ is the complex projective space of N dimensions. Let us define a real field with $M = N^2 - 1$ components, $n^a(x)$ (with $a = 1, \ldots, M$) and a field $z_\alpha(x)$, with N *complex* components, such that

$$n^a(x) = \sum_{\alpha,\beta=1}^{N} z^*_\alpha(x)\, \tau^a_{\alpha\beta}\, z_\beta(x), \tag{16.98}$$

with the constraint

$$\sum_{\alpha=1}^{N} |z_\alpha(x)|^2 = 1 \tag{16.99}$$

In Eq.(16.98) $\tau^a_{\alpha\beta}$ are the $N^2 - 1$ generators of $SU(N)$, and satisfy

$$\sum_{a=1}^{N^2-1} \tau^a_{\alpha\beta}\tau^a_{\gamma\delta} = N\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta} \tag{16.100}$$

It follows that the constraint of Eq.(16.99) of the complex field z_α implies that the n^a field is also constrained such that

$$\boldsymbol{n}^2 = N - 1 \tag{16.101}$$

In the special case $N = 2$, we have a mapping of a constrained two-component complex field z_α to the *three-component* constrained real field n^a. In this special case, the symmetry is $SU(2)$ and the mapping of Eq.(16.98) relates $SU(2)$ of the field z_α with the group $O(3)$ of the field n^a. For $N = 2$, this mapping, known as the Hopf map, relates the fundamental (spinor) representation to $SU(2)$ to the adjoint (vector) representation of $SU(2)$.

In the general case, the N component complex field is in the fundamental representation of $SU(N)$. With the constraint of Eq.(16.99), it describes the breaking of $SU(N)$ down to its $SU(N-1)$ subgroup. However, the real field n^a of Eq.(16.98) is invariant under the *local* $U(1)$ (gauge) symmetry

$$z_\alpha(x) \mapsto e^{i\phi(x)} z_\alpha(x) \tag{16.102}$$

Configurations of the field z_α differing by this $U(1)$ gauge transformations are physically equivalent. Hence, in this case the target manifold is not the coset $SU(N)/SU(N-1)$ but, instead, it is the complex projective space

$$\mathbb{CP}^{N-1} \cong \frac{SU(N)}{(SU(N-1) \otimes U(1))} \tag{16.103}$$

The simplest local Lagrangian that has these symmetries is

$$\mathcal{L}[z_\alpha, z_\alpha^*, \mathcal{A}_\mu] = \frac{1}{g^2} \int d^D x \left| \left(\partial_\mu - i\mathcal{A}_\mu(x)\right) z_\alpha(x) \right|^2, \tag{16.104}$$

with the local constraint $||z||^2 = 1$, and where $A_\mu(x)$ is a $U(1)$ gauge field. This action is invariant under the global symmetry

$$z_\alpha(x) \mapsto U_{\alpha\beta} z_\beta(x), \tag{16.105}$$

where $U \in SU(N)$, and the local $U(1)$ (gauge) symmetry,

$$z_\alpha(x) \mapsto e^{i\phi(x)} z_\alpha(x), \qquad \mathcal{A}_\mu(x) \mapsto \mathcal{A}_\mu(x) + \partial_\mu \phi(x) \tag{16.106}$$

Notice that the action of the $\mathbb{CP}^{N-1}$ non-linear sigma model, Eq.(16.104) depends on the $U(1)$ gauge field $\mathcal{A}_\mu$ only through the covariant derivative of the complex field z_α. The (Euclidean) partition function for the $\mathbb{CP}^{N-1}$ non-linear sigma model is

$$Z_{\mathbb{CP}^{N-1}} = \int \mathcal{D}z_\alpha \mathcal{D}z_\alpha^* \mathcal{D}\mathcal{A}_\mu \exp\left(-\int d^D x \, \mathcal{L}[z_\alpha, z_\alpha^*, \mathcal{A}_\mu] \right) \prod_x \delta(||z(x)||^2 - 1) \tag{16.107}$$

It is apparent from this partition function that the gauge field $\mathcal{A}_\mu$ does not have any dynamics of its own and, furthermore, that the action is a quadratic form in the gauge field. Thus we can integrate out this field explicitly . Indeed, the equation of motion of the gauge field is

$$\frac{\delta S}{\delta \mathcal{A}_\mu} = \frac{\delta}{\delta \mathcal{A}_\mu} \left(|\partial_\mu z_\alpha|^2 + i \left(z_\alpha^* \partial_\mu z_\alpha - (\partial_\mu z_\alpha)^* z_\alpha \right) \mathcal{A}_\mu + \mathcal{A}_\mu^2 \right) = 0 \tag{16.108}$$

which is equivalent to the identification

$$\mathcal{A}_\mu(x) \equiv -\frac{i}{2} \left(z_\alpha^* \partial_\mu z_\alpha - (\partial_\mu z_\alpha)^* z_\alpha \right) \tag{16.109}$$

If we now substitute back this expression for the gauge field A_μ back in the Lagrangian of Eq.(16.104) we obtain a non-linear action for the complex field z_α. In the special case of $N = 2$ one obtains the identity

$$\frac{1}{4} \left(\partial_\mu \boldsymbol{n} \right)^2 = \left| \left(\partial_\mu - i\mathcal{A}_\mu \right) z_\alpha \right|^2 \tag{16.110}$$

and the resulting action is equal to the action pf the $O(3)$ non-linear sigma model. This defines the Hopf mapping of the sphere S_3 of the z field onto the sphere S_2 of the $\boldsymbol{n}$ space.

By inspecting the Lagrangian of Eq.(16.104), we see that the coupling constant of the $\mathbb{CP}^{N-1}$ non-linear sigma model has the same units as the coupling constant of the $O(N)$ non-linear sigma model. Thus, these models too are expected to be renormalizable in $D = 2$ dimensions.

16.5.2 The Principal chiral Field

One generalization is to the principal chiral field. Let G be a compact Lie group and let the principal chiral field be $g(x)$ that takes values on G, i.e. $g(x) \in G$. the Lagrangian is

$$\mathcal{L} = \frac{1}{2u^2} \mathrm{tr} \left(\partial_\mu g^{-1}(x) \partial^\mu g(x) \right) \tag{16.111}$$

where u is the coupling constant. This Lagrangian is invariant under global transformations

$$g(x) \mapsto h^{-1} g(x) v \tag{16.112}$$

where $h \in G$ and $v \in G$. The global symmetry is thus $G_R \otimes G_L$. The non-linear nature of the field is hidden in the condition that $g(x)$ is a group element. For instance, if $G = U(N)$, then $g(x)$ satisfies $g(x)^{-1} = g(x)^\dagger$. Similar constraints apply more generally.

16.5.3 General non-Linear Sigma Models

This construction can be made more general. Consider a field $\phi(x)$ whose target space is a differentiable manifold M. The Euclidean Lagrangian is

$$\mathcal{L}[\phi] = \frac{1}{2u} a^{2-D} g_{ij}[\phi(x)] \partial_\mu \phi^i(x) \partial_\mu \phi^j(x) \tag{16.113}$$

where u is the dimensionless coupling constant and $g^{ij}[\phi(x)]$ is a Riemannian metric on the manifold M, with a being a short-distance cutoff scale. A homogeneous space of the form of a coset $M = G/H$ is a special case. We already discussed the cases in which M is a sphere and a complex projective space. An example, arising in the theory of Anderson localization, is the case of a coset of the form $O(n+m)/(O(n) \otimes O(m))$ (in the limit $n, m \to 0$!). In string theory, the field is the coordinate of the bosonic string in a target space, such as the Calabi-Yau manifolds.

16.6 The $O(N)$ non-linear sigma model in perturbation theory

All non-linear sigma models are renormalizable in $D = 2$ space-time dimensions. Here we will focus on the simpler case of the $O(N)$ model, following the work of Polyakov, and Brezin and Zinn-Justin. The general case was proven by Friedan and will comment on his main results at the end.

In order to discuss the structure of perturbation theory for the $O(N)$ non-linear sigma model we will need to make a choice of coordinates on its target space, the sphere. To this end we will decompose the field into a longitudinal field $\sigma(x)$, representing the component along the direction of symmetry breaking, and $N-1$ fields $\boldsymbol{\pi}(x)$, representing the Goldstone bosons, transverse to the direction of symmetry breaking.

Hence, we write $\boldsymbol{n}(x) = (\sigma(x), \boldsymbol{\pi}(x))$, subject to the local constraint $\boldsymbol{n}^2(x) = \sigma^2(x) + \boldsymbol{\pi}^2(x) = 1$. The partition function of the $O(N)$ non-linear sigma model is the functional integral

$$Z[\boldsymbol{J}(x)] = \int \mathcal{D}\sigma\mathcal{D}\boldsymbol{\pi} \prod_x \delta(\sigma(x)^2 + \boldsymbol{\pi}(x)^2 - 1) \; \exp\left(-\int d^D x \; \mathcal{L}[\sigma, \boldsymbol{\pi}; \boldsymbol{J}]\right) \tag{16.114}$$

where $\mathcal{L}[\sigma, \boldsymbol{\pi}; \boldsymbol{J}]$ is the Euclidean Lagrangian of the non-linear sigma model

$$\mathcal{L}[\sigma, \boldsymbol{\pi}; \boldsymbol{J}] = \frac{1}{2g}\left((\partial_\mu \sigma)^2 + (\partial_\mu \boldsymbol{\pi})^2\right) - (J_\sigma(x)\sigma(x) + \boldsymbol{J}_{\boldsymbol{\pi}}(x) \cdot \boldsymbol{\pi}(x)) \tag{16.115}$$

and where $\boldsymbol{J}(x) = (J_\sigma(x), \boldsymbol{J}_{\boldsymbol{\pi}}(x))$ is a symmetry breaking field.

Except for the local constraint $\boldsymbol{n}^2 = 1$, this theory looks like a free field. We can deal with the constraint in one of two ays. One option is to replace the delta function that enforces the constraint by a path integral over a Lagrange multiplier field $\lambda(x)$,

$$\prod_x \delta(\sigma(x)^2 + \boldsymbol{\pi}(x)^2 - 1) = \int \mathcal{D}\lambda \exp(-\int d^D x \lambda(x)(\sigma(x)^2 + \boldsymbol{\pi}(x)^2 - 1)) \tag{16.116}$$

This leads to a path integral over the fields σ, $\boldsymbol{\pi}$, and λ. The other option is to solve the constraint and work with fewer degrees of freedom. in this section we will use the second option which, in the end, is a matter of choice. Notice that this issue is very similar to the problem in gauge theory and the role of gauge fixing. there we had the choice of fixing the gauge first and quantizing later, or to quantize first and impose the gauge condition later.

Thus, we will first integrate out the longitudinal component $\sigma(x)$ using that

$$\int d\sigma \; \delta(\sigma^2 + \boldsymbol{\pi}^2 - 1) \; F(\sigma, \boldsymbol{\pi}) = \frac{1}{2\sqrt{1-\boldsymbol{\pi}^2}} \; F(\sqrt{1-\boldsymbol{\pi}^2}, \boldsymbol{\pi}) \tag{16.117}$$

In other words, the quantity $\mathcal{J}[\boldsymbol{\pi}(x)]$,

$$\mathcal{J}[\boldsymbol{\pi}(x)] = \prod_x \left(2\sqrt{1-\boldsymbol{\pi}^2(x)}\right)^{-1} \tag{16.118}$$

is the Jacobian of the change of variables. In fact, in our choice coordinates,

$$\mathcal{D}\boldsymbol{\pi}\ \mathcal{J}[\boldsymbol{\pi}] \equiv \frac{\mathcal{D}\boldsymbol{\pi}}{2\sqrt{1-\boldsymbol{\pi}^2}} \tag{16.119}$$

(where we used a short-hand notation for the Jacobian) is the $O(N)$-invariant Haar measure for the sphere S_{N-1}.

Then, we can write the partition function as

$$Z[\boldsymbol{J}] = \int \frac{\mathcal{D}\boldsymbol{\pi}}{2\sqrt{1-\boldsymbol{\pi}^2}} \exp(-S_{\text{eff}}[\boldsymbol{\pi};\boldsymbol{J}]) \tag{16.120}$$

where the effective action is

$$\begin{aligned} S_{\text{eff}}[\boldsymbol{\pi};\boldsymbol{J}] =& \frac{1}{2g}\int d^Dx \left[\left(\partial_\mu\sqrt{1-\boldsymbol{\pi}^2(x)}\right)^2 + (\partial_\mu\boldsymbol{\pi}(x))^2\right] \\ & - \int d^Dx \left(J_\sigma(x)\sqrt{1-\boldsymbol{\pi}^2(x)} + \boldsymbol{J}_{\boldsymbol{\pi}}(x)\cdot\boldsymbol{\pi}(x)\right) \end{aligned} \tag{16.121}$$

It is important to note that, in spite of the fact that effective action $S_{\text{eff}}[\boldsymbol{\pi};\boldsymbol{J}]$ seems to have only a global $O(N-1)$ symmetry, the inclusion of the Jacobian factor in the functional integral makes renders the partition function globally $O(N)$ invariant. The inclusion of the Jacobian factor will play a key role in what follows. Notice that we could have alternatively incorporate the Jacobian factor in the effective action with a contact term of the form

$$S_{\text{contact}} = -\frac{1}{2a^D}\int d^Dx\ \ln(1-\boldsymbol{\pi}^2(x)) \tag{16.122}$$

where a is a short-distance cutoff (where we used a lattice regulator).

In what follows we will denote $J_\sigma = H$ and $\boldsymbol{J}_{\boldsymbol{\pi}} = \boldsymbol{J}$ (which is an $N-1$-component vector). We will work with the partition function in the form

$$Z[H,\boldsymbol{J}] = \int \frac{\mathcal{D}\boldsymbol{\pi}}{2\sqrt{1-\boldsymbol{\pi}^2}}\ \exp\left[-\frac{1}{g}S[\boldsymbol{\pi},H] + \int d^Dx\ \boldsymbol{J}(x)\cdot\boldsymbol{\pi}(x)\right] \tag{16.123}$$

where

$$S[\boldsymbol{\pi},H] = \int d^Dx \left[\frac{1}{2}(\partial_\mu\boldsymbol{\pi}(x))^2 + \frac{1}{2}\frac{(\boldsymbol{\pi}(x)\cdot\partial_\mu\boldsymbol{\pi}(x))^2}{(1-\boldsymbol{\pi}^2(x))} - H(x)\sqrt{1-\boldsymbol{\pi}^2(x)}\right] \tag{16.124}$$

With the form of the action of Eq.(16.124) we see that an expansion of the partition function in powers of the coupling constant g is just the loop expansion for a theory with the action of Eq.(16.124).

16.6.1 Ward Identities

Ward identities play a central role in the renormalization of the non-linear sigma model and in the proof of renormalizability. To see how this works we should recall that the actual global symmetry is not $O(N)$ but the coset $O(N)/O(N-1)$.

Let us consider a global infinitesimal transformation mixing the fields σ and $\boldsymbol{\pi}$:

$$\delta\boldsymbol{\pi}(x) = \sqrt{1-\boldsymbol{\pi}^2(x)}\,\boldsymbol{\omega}, \qquad \delta\sqrt{1-\boldsymbol{\pi}^2(x)} = -\boldsymbol{\omega}\cdot\boldsymbol{\pi}(x) \tag{16.125}$$

where $\boldsymbol{\omega}$ is an $N-1$-component constant infinitesimal vector. We will assume that the UV regulator of the theory is consistent with the global symmetry. This is the case with dimensional regularization and with lattice regularization, and other schemes. Hence, the the action, the integration measure of the functional integral and the regularization preserve the full global symmetry. hence the partition function is invariant. If we define

$$F[\boldsymbol{J},H] = g\ln Z[\boldsymbol{J},H] \tag{16.126}$$

which is the generating function of the connected correlators of the non-linear sigma model. Then, the invariance of the partition function under the infinitesimal transformation of the fields, Eq.(16.125), implies the Ward Identity

$$\int d^Dx\,\left[J_i(x)\frac{\delta F}{\delta H(x)} - H(x)\frac{\delta F}{\delta J_i(x)}\right] = 0 \tag{16.127}$$

As in Chapter 12, we define $\Gamma[\boldsymbol{\pi},H]$ the generating functional of the one-particle irreducible vertex functions of the $\boldsymbol{\pi}$ fields as the Legendre transform of F:

$$\Gamma[\boldsymbol{\pi},H] = \int d^Dx\,\langle\boldsymbol{\pi}(x)\rangle\cdot\boldsymbol{J}(x) - F[\boldsymbol{J},H] \tag{16.128}$$

As in Chapter 12, the following identities hold,

$$\begin{aligned}\langle\boldsymbol{\pi}(x)\rangle &= \frac{\delta F}{\delta\boldsymbol{J}(x)}, \qquad \frac{\delta F}{\delta H(x)} = \langle\sigma(x)\rangle\\ \frac{\delta\Gamma}{\delta\boldsymbol{\pi}(x)} &= \boldsymbol{J}(x), \qquad \frac{\delta\Gamma}{\delta H(x)} = \langle\sigma(x)\rangle\end{aligned} \tag{16.129}$$

where

$$\frac{\delta\Gamma}{\delta H} = -\frac{\delta F}{\delta H} \tag{16.130}$$

The Ward Identity for Γ is

$$\int d^D x \left[\frac{\delta \Gamma}{\delta \boldsymbol{\pi}(x)} \frac{\delta \Gamma}{\delta H(x)} + H(x) \boldsymbol{\pi}(x) \right] = 0 \tag{16.131}$$

where we used the notation $\langle \boldsymbol{\pi}(x) \rangle \equiv \boldsymbol{\pi}(x)$. We will see below that this ward Identity can be used to show that the non-linear sigma model is renormalizable in $D = 2$ dimensions.

16.6.2 Primitive Divergencies

The first step is to analyze the primitive divergencies in the expansion in powers in g. We will see that $D = 2$ is special. Iin two dimensions the coupling constant g is dimensionless, and so are the field $\boldsymbol{\pi}$, and the field $\sigma = \sqrt{1 - \boldsymbol{\pi}^2}$. The operator $\left(\partial_\mu \boldsymbol{\pi}\right)^2$ has dimension 2 as well as the operator $\left(\boldsymbol{\pi} \cdot \partial_\mu \boldsymbol{\pi}\right)^2 / \left(1 - \boldsymbol{\pi}^2\right)$ and the symmetry breaking field H. In particular, if we were to expand the action of Eq.(16.124) in powers of the field $\boldsymbol{\pi}$, the operators that are obtained in each term of the expansion all have dimension 2 since all of them have just two derivatives and powers of the field $\boldsymbol{\pi}$. That is to say, all the operators of the expansion are *equally* relevant (marginal, actually). Thus, we cannot truncate the expansion at any order. Moreover, a truncation of the expansion would break the symmetry since they are related by symmetry. A general term in this expansion is a vertex of the form $\left(\boldsymbol{\pi} \cdot \partial_\mu \boldsymbol{\pi}\right)^2 \boldsymbol{\pi}^{2n}$, see Figs. 16.1 a and b.

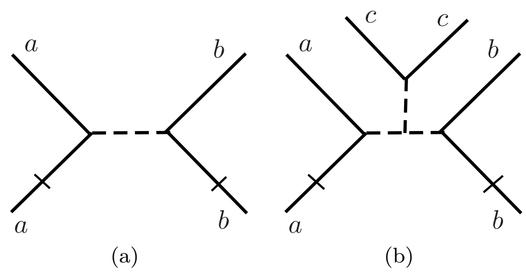


Figure 16.1 Feynman rules for the $O(N)$ non-linear sigma model: a) $n = 0$ vertex, and b) $n = 1$ vertex. Here $a, b, c = 1, \dots, N-1$. Vertices with $n \geq 2$ have more insertions. The dashes represent derivatives on these external legs. The broken line is shown for clarity.

Let us first determine the Feynman rules for the NLSM. This requires that we formally expand the action of Eq.(16.124) in powers of the coupling constant g. To make this process more explicit we will define the rescaled field $\boldsymbol{\varphi}(x) = \boldsymbol{\pi}(x)/\sqrt{g}$ and expand the resulting action. The result of the Lagrangian

$$
\begin{aligned}
\mathcal{L} =& \frac{1}{2}\left(\partial_\mu \boldsymbol{\varphi}\right)^2 + \sum_{n=0}^{\infty} \frac{g^{n+1}}{2}\left(\boldsymbol{\varphi}^2\right)^n \left(\boldsymbol{\varphi}\cdot\partial_\mu\boldsymbol{\varphi}\right)^2 \\
& - H(x)\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2(n-1))!}{2^{2n-1}(n-1)!n!} g^n \left(\boldsymbol{\varphi}^2\right)^n\right) + \sqrt{g}\boldsymbol{J}_{\boldsymbol{\pi}}\cdot\boldsymbol{\varphi}
\end{aligned}
\tag{16.132}
$$

The lowest order vertex, with $n = 0$, is shown in Fig.16.1a. In momentum space it carries a weight of $\frac{g}{2}\boldsymbol{q}^{(1)}\cdot\boldsymbol{q}^{(2)}$, where $\boldsymbol{q}^{(1)}$ and $\boldsymbol{q}^{(2)}$ are the momenta on the two external legs.

Let us consider the Feynman diagrams obtained at one-loop order with $H = 0$ and $\boldsymbol{J} = 0$. The leading contributions to the propagator of the $\boldsymbol{\pi}$ field are shown in Fig.16.2 a and b. The diagram of Fig.16.2a has one derivative on each external leg. The internal loop yields a contribution proportional to the integral

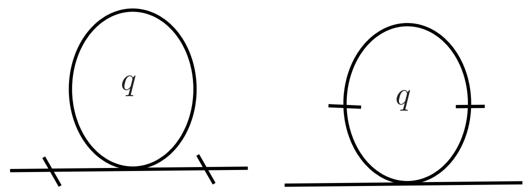
$$
\int \frac{d^2q}{(2\pi)^2}\frac{1}{q^2} \propto \ln\Lambda \tag{16.133}
$$

and is logarithmically divergent in the UV momentum cutoff λ. This graph contributes to a term in the effective action which in momentum space of the form $p^2|\boldsymbol{\pi}(p)|^2$, and hence to the wave function renormalization. In contrast, the contribution of the graph of Fig. 16.2b has the form

$$
\int \frac{d^2q}{(2\pi)^2}\frac{1}{q^2} q^\mu q^\nu \propto g^{\mu\nu}\Lambda^2 \tag{16.134}
$$

is quadratically divergent in the UV cutoff Λ and looks like a mass renormalization.

However, the field $\boldsymbol{\pi}$ is a Goldstone boson and the Ward Identity guarantees that it should be massless at every order in perturbation theory. So, what has gone wrong? The remedy to this problem is readily found by noting that the action of the NLSM has a contact term, Eq. (16.122), arising from the integration measure. The contact term yields a contribution at quadratic order in the field $\boldsymbol{\pi}$ which also looks like a quadratically divergent mass term and which exactly cancels the offending term. Below we will prove this statement explicitly. On the other hand, dimensional regularization is often used (as we will in the sequel). We already saw that this regularization replaces a logarithmic divergence with a pole in ϵ. Furthermore, in dimensional regularization quadratic divergencies are regularized to zero and one does not have to be concerned with this problem. However, in other schemes, such as lattice regularization, these cancellations must be checked at every order in perturbation theory.



(a) (b)

Figure 16.2 One loop contributions to the two-point function of $\boldsymbol{\pi}$. The dashes denote derivatives in the external lines (a), and in the internal loop (b).

16.7 Renormalizability of the 2D Non-Linear Sigma Model

We will now examine the restrictions that the Ward Identities on the possible structure of the singularities. By power counting we only need to worry of operators of scaling dimensions two or less since (a) any operator with higher dimension would make the theory non-renormalizable, and (b) such operators will be irrelevant at distance scales large compared to the UV cutoff $a \sim \Lambda^{-1}$.

To this end let us expand the effective one-particle irreducible action Γ in powers of the coupling constant g,

$$\Gamma = \sum_{n=0}^{\infty} g^n \Gamma^{(n)} \tag{16.135}$$

This is done by organizing the Feynman graphs in powers of g. At lowest order in g, the tree level, we recover, as expected, the action of the NLSM, i.e.

$$\begin{aligned}\Gamma^{(0)} =&S(\boldsymbol{\pi}, H)\\ =&\int d^2x \left[\frac{1}{2}\left(\partial_\mu \boldsymbol{\varphi}\right)^2 + \frac{1}{2}\frac{\left(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}\right)^2}{1-\boldsymbol{\pi}^2} - H(x)\sqrt{1-\boldsymbol{\pi}^2}\right] + \text{measure terms}\end{aligned} \tag{16.136}$$

We will now expand the effective action to order one loop, $\Gamma = \Gamma^{(0)} + g\Gamma^{(1)}$, use the Ward Identity of eq.(16.131), and demand that it must hold at every order in g, resulting in the requirement

$$\int d^2x \left[\frac{\delta\Gamma^{(0)}}{\delta\boldsymbol{\pi}} \frac{\delta\Gamma^{(1)}}{\delta H} + \frac{\delta\Gamma^{(1)}}{\delta\boldsymbol{\pi}} \frac{\delta\Gamma^{(0)}}{\delta H} \right] = 0 \tag{16.137}$$

We now define the operator

$$\Gamma^{(0)}\star \equiv \int d^2x \left[\frac{\delta\Gamma^{(0)}}{\delta\boldsymbol{\pi}} \frac{\delta}{\delta H} + \frac{\delta\Gamma^{(0)}}{\delta H} \frac{\delta}{\delta\boldsymbol{\pi}} \right] \tag{16.138}$$

in terms of which Eq.(16.137) becomes

$$\Gamma^{(0)} \star \Gamma^{(1)} = 0 \tag{16.139}$$

As the cutoff is removed, $\Lambda \to \infty$, the quantity $\Gamma^{(1)}$ will develop a singular part, that we will denote by $\Gamma^{(1)}_{\rm div}$. However, since the divergent and finite parts (the latter one will be denoted by $\Gamma^{(1)}_{\rm reg}$) have different dependence in the regulator, they must obey Eq.(16.139) separately. Hence, $\Gamma^{(1)}_{\rm div}$ must satisfy an equation of the form of Eq.(16.139). The divergence contained in $\Gamma^{(1)}$ can be canceled by adding a counterterm to the action $S(\boldsymbol{\pi}, H)$ of the form $gS_1(\boldsymbol{\pi}, H)$ such that

$$S_1(\boldsymbol{\pi}, H) = -\Gamma^{(1)}_{\rm div} + O(g) \tag{16.140}$$

In this way, the new, renormalized, action $S + gS_1$ satisfies the Ward Identity to all orders in g.

By power counting we know that $\Gamma^{(1)}_{\rm div}$ is a local function with scaling dimension 2 of the field $\boldsymbol{\pi}$. Since the scaling dimension of $H(x)$ is also 2, it follows that $\Gamma^{(1)}_{\rm div} = O(H)$. Thus, we can write $\Gamma^{(1)}_{\rm div}$ as an expression of the form

$$\Gamma^{(1)}_{\rm div} = \int d^2x \left[B(\boldsymbol{\pi}) + H(x) C(\boldsymbol{\pi}) \right] \tag{16.141}$$

where $B(\boldsymbol{\pi})$ contains at most two derivatives and $C(\boldsymbol{\pi})$ has no derivatives. Now, the requirement that $\Gamma^{(1)}_{|rmdiv}$ obeys Eq.(16.139), yields the condition on B and C

$$0 = \int d^2x \left[\frac{\delta\Gamma^{(0)}}{\delta H(x)} \frac{\delta C}{\delta\boldsymbol{\pi}} H(x) + \frac{\delta\Gamma^{(0)}}{\delta\boldsymbol{\pi}} C\boldsymbol{\pi}) + \frac{\delta\Gamma^{(0)}}{\delta\boldsymbol{H}(x)} \frac{\delta B}{\delta\boldsymbol{\pi}} \right] \tag{16.142}$$

From the expression of the tree level action, $\Gamma^{(0)}$ we find the explicit expressions

$$\begin{aligned}\frac{\delta\Gamma^{(0)}}{\delta H(x)} &= -\sqrt{1-\boldsymbol{\pi}^2}\\ \frac{\delta\Gamma^{(0)}}{\delta\boldsymbol{\pi}} &= -\partial^2\boldsymbol{\pi} + \frac{(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})}{1-\boldsymbol{\pi}^2}\partial_\mu\boldsymbol{\pi} + \frac{(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2}{(1-\boldsymbol{\pi}^2)^2}\boldsymbol{\pi} + \frac{H(x)}{\sqrt{1-\boldsymbol{\pi}^2}}\boldsymbol{\pi}\end{aligned} \tag{16.143}$$

Upon collecting terms we can write

$$\begin{aligned}&\int d^2x\left[-\sqrt{1-\boldsymbol{\pi}^2}\frac{\delta C}{\delta\boldsymbol{\pi}} + \frac{\boldsymbol{\pi}}{\sqrt{1-\boldsymbol{\pi}^2}}C(\boldsymbol{\pi})\right]H(x)\\ &+\int d^2x\left\{\left[-\partial^2\boldsymbol{\pi} + \boldsymbol{\pi}\frac{\partial^2(1-\boldsymbol{\pi}^2)^{1/2}}{\sqrt{1-\boldsymbol{\pi}^2}}\right]C - \sqrt{1-\boldsymbol{\pi}^2}\frac{\delta B}{\delta\boldsymbol{\pi}}\right\} = 0\end{aligned} \tag{16.144}$$

Since $H(x)$ and $\boldsymbol{\pi}(x)$ are arbitrary functions of the coordinates, this integral implies that $C(\boldsymbol{\pi})$ must be the solution of

$$\sqrt{1-\boldsymbol{\pi}^2}\frac{\delta C}{\delta\boldsymbol{\pi}} = \frac{\boldsymbol{\pi}}{\sqrt{1-\boldsymbol{\pi}^2}}C(\boldsymbol{\pi}) \tag{16.145}$$

and that

$$\int d^2x\left\{\left[-\partial^2\boldsymbol{\pi} + \boldsymbol{\pi}\frac{\partial^2\sqrt{1-\boldsymbol{\pi}^2}}{\sqrt{1-\boldsymbol{\pi}^2}}\right]C - \sqrt{1-\boldsymbol{\pi}^2}\frac{\delta B}{\delta\boldsymbol{\pi}} = 0 \tag{16.146}$$

The most general solution is

$$\Gamma^{(1)}_{\text{div}} = \lambda S^{(0)} + \mu\int d^2x\left[\frac{(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2}{(1-\boldsymbol{\pi}^2)^2} + \frac{H(x)}{\sqrt{1-\boldsymbol{\pi}^2}}\right] \tag{16.147}$$

where $S^{(0)}$ is the classical action and λ and μ are two singular functions of the regulator.

If we now define $S^{(1)} = -\Gamma^{(1)}_{\text{div}}$, and write the field $\boldsymbol{\pi}$ in terms of a rescaled field $Z^{1/2}\boldsymbol{\pi}$ (where we recognize this as the wave function renormalization), we can write the renormalized action as the expression

$$S = \int d^2x\left\{\frac{Z}{Z_1}\left[\frac{1}{2}(\partial_\mu\boldsymbol{\pi})^2 + \frac{1}{2}\frac{(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2}{\left(\frac{1}{Z}-\boldsymbol{\pi}^2\right)}\right] - H\sqrt{\frac{1}{Z}-\boldsymbol{\pi}^2}\right\} \tag{16.148}$$

where we defined

$$\begin{aligned}\frac{Z}{Z_1} &= 1-\lambda g + O(g^2)\\ Z &= 1-2\mu g + O(g^2)\end{aligned} \tag{16.149}$$

We now define the renormalized dimensionless coupling constant t_R and the renormalized field H_R by the conditions

$$
\begin{aligned}
g =& t_R Z_1 \kappa^{2-D} \\
H =& H_R \frac{Z_1}{\sqrt{Z}}
\end{aligned}
\tag{16.150}
$$

where κ is an arbitrary renormalization scale. Thus, to one loop order, it is sufficient to do a renormalization of the coupling constant and a wave function renormalization.

However, since we rescaled the field we must modify the transformation laws to read

$$
\delta\boldsymbol{\pi} = \sqrt{\frac{1}{Z} - \boldsymbol{\pi}^2}\boldsymbol{\omega} \tag{16.151}
$$

and the measure $d\boldsymbol{\pi}/\sqrt{1-\boldsymbol{\pi}^2}$ is no longer invariant. To solve this problem we must replace the measure by $d\boldsymbol{\pi}/\sqrt{1/Z-\boldsymbol{\pi}^2}$ which is an effect that will become apparent at two-loop level. Hence we have succeeded in making the renormalized one-particle irreducible function Γ finite and satisfies the ward Identity.

We will now show that the one-loop result implies that that the NLSM is renormalizable to all orders in the loop expansion. We will now see that at every order the key is the Ward Identity. We will use the following induction argument. We will assume that to order $n-1$ we succeeded in renormalizing the theory. Thus, to order n we must satisfy

$$
\Gamma^{(0)} \star \Gamma^{(n)} = -\left(\Gamma^{(1)} \star \Gamma^{(n-1)} + \Gamma^{(2)} \star \Gamma^{(n-2)} + \dots\right) \tag{16.152}
$$

However, by hypothesis, the right hand side contain only renormalized terms. Hence the singular part $\Gamma^{(n)}_{\text{div}}$ must also satisfy the same equation that we solved at one loop order, i.e.

$$
\Gamma^{(0)} \star \Gamma^{(n)}_{\text{div}} = 0 \tag{16.153}
$$

Hence, $\Gamma^{(n)}_{\text{div}}$ has the same form as $\Gamma^{(1)}_{\text{div}}$. Therefore, to all orders in a expansion in the coupling constant, we can renormalize the NLSM by renormalizing the coupling constant and with a wave function renormalization. This completes the proof of renormalizability.

In summary, in two dimensions, the renormalized action is

$$
\frac{S}{g} = \int d^2x \left\{ \frac{Z}{2Z_1 g}\left[\left(\partial_\mu \boldsymbol{\pi}\right)^2 + \left(\partial_\mu \sqrt{1 - Z\boldsymbol{\pi}^2}\right)^2 \right] - \frac{H}{g}\sqrt{\frac{1}{Z} - \boldsymbol{\pi}^2} \right\} \tag{16.154}
$$

In general dimension $D > 2$ the result is

$$\frac{S}{g} = \frac{\kappa^{D-2}}{2Z_1 t} \int d^2x \left\{ Z\left[(\partial_\mu \boldsymbol{\pi})^2 + \frac{(\boldsymbol{\pi} \cdot \partial_\mu \boldsymbol{\pi})^2}{(\frac{1}{Z} - \boldsymbol{\pi}^2)^2} \right] - \frac{HZ_1}{\sqrt{Z}} \sqrt{1 - Z\boldsymbol{\pi}^2} \right\} \quad (16.155)$$

where

$$g = t\kappa^{2-D} \quad (16.156)$$

16.7.1 Renormalization to One Loop Order

Let us carry out this program to one loop order. It will suffice to study the renormalization of the two-point function of the $\boldsymbol{\pi}$ fields. We first observe that the bare propagator of the $\boldsymbol{\pi}$ field in momentum space is

$$G_0^{ij}(p) = \delta_{ij} \frac{g}{p^2 + \frac{H}{g}} \quad (16.157)$$

where $i, j = 1, \ldots, N-1$, and that (except terms coming from the measure) every vertex contributes with a weight of $1/g$.The coupling constant g is thus the parameter that organizes the loop expansion.

To zeroth (tree-level) order the one-particle irreducible two-point function then is

$$\Gamma_{ij}^{(2)}(p) = \frac{1}{g}\left(p^2 + \frac{H}{g}\right) \quad (16.158)$$

The Feynman diagrams for the one-loop contributions to $\Gamma^{(2)}(p)$ are shown in Fig.16.3. The explicit expression for $\Gamma^{(2)}$ at one-loop order is

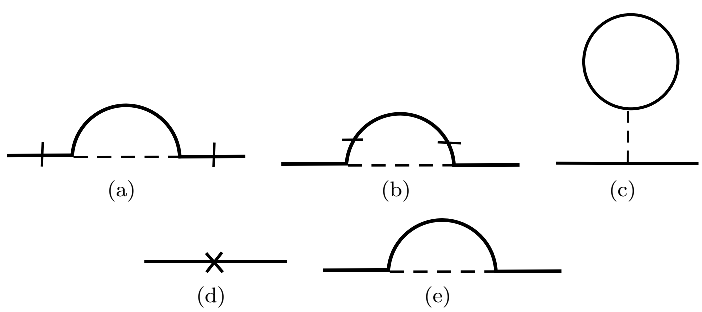


Figure 16.3 One loop contributions to $\Gamma^{(2)}$. The dashes denote derivatives in the external lines (a), and in the internal loop (b); the tadpole diagram (c) originates from the symmetry breaking field H; here we included the lowest order (quadratic) term coming from the integration measure (d).

$$
\begin{aligned}
\Gamma^{(2)}(p) =& \frac{1}{g}\left(p^2 + \frac{H}{g}\right) \\
&+p^2 \int \frac{d^D q}{(2\pi)^D} \frac{1}{p^2+\frac{H}{g}} + \int \frac{d^D q}{(2\pi)^D} \frac{q^2}{q^2+\frac{H}{g}} + \frac{H}{2g}(N-1) \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+\frac{H}{g}} \\
&-\frac{\Lambda^D}{(2\pi)^D} + \frac{H}{g} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+\frac{H}{g}}
\end{aligned}
\tag{16.159}
$$

Here the terms on the second line are the expressions for the Feynman diagrams of Fig.16.3a-c, and those in the third line correspond to the diagrams of Fig.16.3 d and e. Notice the *negative* contribution of the first term of the third line which arises from the measure.

It is straightforward to see that the following identity holds

$$
\int \frac{d^D q}{(2\pi)^D} \frac{q^2}{q^2+\frac{H}{g}} - \frac{\Lambda^D}{(2\pi)^D} + \frac{H}{g} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+\frac{H}{g}} = 0 \tag{16.160}
$$

In $D = 2$ this identity tells us that the quadratically divergent contributions, that could have induced a mass term (as well as the N-independent logarithmic divergent contributions), cancel exactly. This is a consequence of the Ward Identity.

As a result the final form of the 1PI two-point function to one loop order is

$$
\Gamma^{(2)}(p) = \frac{1}{g}\left(q^2 + \frac{H}{g}\right) + \left(p^2 + \frac{(N-1)}{2g} H\right) \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+\frac{H}{g}} \tag{16.161}
$$

The divergencies in $\Gamma^{(2)}$ can be taken care of by means of a renormalized dimensionless coupling constant t and a wave function renormalization Z,

$$
\Gamma^{(2)}_R(p, t, H_R, \kappa) = Z\Gamma^{(2)}(p, g, H, \Lambda) \tag{16.162}
$$

with

$$
g = t\kappa^{-\epsilon} Z_1 \tag{16.163}
$$

where we set $\epsilon = D - 2$, and

$$
H = H_R \frac{Z_1}{\sqrt{Z}} \tag{16.164}
$$

The renormalization constants Z_1 and Z can be expanded in a power series of the dimensionless renormalized coupling constant t_R:

$$\begin{aligned} Z =&1 + at + O(t^2) \\ Z_1 =&1 + bt + O(t^2) \end{aligned} \tag{16.165}$$

The renormalized 1PI two point function becomes

$$\begin{aligned} \Gamma_R^{(2)} =&\frac{p^2}{t}\kappa^{D-2}\Big[1 + t\Big(a - b + \kappa^{2-D}\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2 + \frac{H_R}{t}}\Big) + \dots\Big] \\ &+\frac{H_R}{t}\kappa^{D-2}\Big[1 + t\Big(\frac{a}{2} + \kappa^{2-D}\frac{(N-1)}{2}\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2 + \frac{H_R}{t}}\Big) + \dots\Big] \end{aligned} \tag{16.166}$$

The coefficients a and b will be chosen so that $\Gamma_R^{(2)}(p)$ is finite. There are many ways of doing this. For instance we can choose these coefficients to cancel the singular part of the integral in the expressions of Eq.(16.166). If we use dimensional regularization this procedure is equivalent to the minimal subtraction procedure of t'Hooft and Veltman. Here we will use the following somewhat different choice that subtracts the the expressions at a renormalization scale κ,

$$\begin{aligned} a =& -\kappa^{2-D}(N-1)\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2 + \kappa^2} \\ b =& -\kappa^{2-D}\frac{(N-1)}{2}\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2 + \kappa^2} \end{aligned} \tag{16.167}$$

Hence,

$$a - b = -\kappa^{2-D}\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2 + \kappa^2} \tag{16.168}$$

With this choice the renormalization constants Z and Z_1 become

$$\begin{aligned} Z =&1 - t\kappa^{2-D}\ (N-1)\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2 + \kappa^2} \\ Z_1 =&1 - t\kappa^{2-D}\ (N-2)\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2 + \kappa^2} \end{aligned} \tag{16.169}$$

We will next study the behavior of the integral for dimensions $D = 2+\epsilon$ and expand it in powers of ϵ,

$$\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2 + \kappa^2} = \frac{1}{(4\pi)^{D/2}}\Gamma\left(1 - \frac{D}{2}\right)\kappa^{D-2} \tag{16.170}$$

Using the asymptotic expression for the Euler Gamma function

$$\Gamma(z) = \frac{1}{z} - \gamma + O(z) \tag{16.171}$$

where γ is the Euler-Mascheroni constant, the integral becomes

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + \kappa^2} = -\frac{1}{2\pi\epsilon} + O(1) \tag{16.172}$$

With these prescriptions the renormalization constants take the simple form

$$\begin{aligned} Z &= 1 + \frac{(N-1)}{\epsilon} t + O(t^2) \\ Z_1 &= 1 + \frac{(N-2)}{\epsilon} t + O(t^2) \end{aligned} \tag{16.173}$$

Hence this procedure is equivalent to Minimal Subtraction.

Hence, we find that, to lowest order in ϵ, the renormalized 1PI two point function is

$$\Gamma_R^{(2)}(p, t, H_R, \kappa = 1) = \frac{1}{t}\left(p^2 + \frac{H_R}{t}\right) - \frac{1}{2}\left(p^2 + \frac{(N-2)}{2}\frac{H_R}{t}\right) \ln\left(\frac{H_R}{t\kappa^2}\right) + O(t^2) \tag{16.174}$$

Hence, it is not possible to set $H_R \to 0$. In other terms, the true IR behavior is not accessible to perturbation theory.

16.7.2 Renormalization Group Analysis of the Non-Linear Sigma Model

We next compute the beta function of the non-linear sigma model,

$$\beta(t) = \kappa \frac{\partial t}{\partial \kappa}\bigg|_{\text{bare}} \tag{16.175}$$

where we hold the bare theory fixed. Since the bare coupling constant g and the renormalized coupling constant t are related by $g = \kappa^{-\epsilon} Z_1 t$, varying the renormalization scale κ at fixed bare coupling constant,

$$\kappa \frac{\partial g}{\partial \kappa} = 0 \tag{16.176}$$

leads to the result

$$0 = -\epsilon t + \beta(t)\left(1 + t\frac{\partial \ln Z_1}{\partial t}\right) \tag{16.177}$$

From the result of Eq.(16.173), we find

$$\frac{\partial \ln Z_1}{\partial t} = \frac{N-2}{\epsilon} \tag{16.178}$$

Hence, the beta function is given by

$$\beta(t) = \epsilon t - (N-2)t^2 + O(t^3) \tag{16.179}$$

Similarly, we find that the anomalous dimension is

$$\begin{aligned}\gamma(t) =& \kappa \frac{\partial \ln Z}{\partial \kappa}\Big|_{\text{bare}} \\ =& \beta(t) \frac{\partial \ln Z}{\partial t} \\ =& (N-1)t + O(t^2)\end{aligned} \tag{16.180}$$

Notice that, contrary to what we found in ϕ^4 theory, the non-linear sigma model has an anomalous dimension already at one loop level.

We can similarly derive the Callan-Symanzik equations to the N-point 1PI vertex functions of the $\boldsymbol{\pi}$ fields by requiring that the bare functions, $\Gamma_B^{(N)}$ remain constant as the renormalization scale changes,

$$\kappa \frac{\partial \Gamma_B^{(N)}}{\partial \kappa}(p, g, H, \Lambda) = 0 \tag{16.181}$$

resulting in the Callan-Symanzik equation

$$\Big[\kappa \frac{\partial}{\partial \kappa} + \beta(t)\frac{\partial}{\partial t} - \frac{N}{2}\gamma(t) + \Big(\frac{\gamma(t)}{2} + \frac{\beta(t)}{t} - (D-2)\Big) H_R \frac{\partial}{\partial H_R}\Big]\Gamma_R^{(N)}(p, t, H_R, \kappa) = 0 \tag{16.182}$$

We will shortly solve this equation along the RG flow near a fixed point.

We next need to find the fixed points of this theory, i.e. the values t^* where $\beta(t^*) = 0$. We have two cases.

a) For $D \leq 2$ the only finite fixed point is at $t^* = 0$. This fixed point is IR unstable. The case of $D = 2$ is special, and we will discuss in detail below. In this case the fixed point is marginally IR unstable.

b) For $D > 2$ the fixed point at $t^* = 0$ is IR stable and has a finite basin of attraction in the IR. This fixed point represents the spontaneously broken symmetry state and has $N-1$ exactly massless Goldstone bosons. At this fixed point the theory is no longer renormalizable in the sense that it has a large number irrelevant operators. On the other hand, for $D > 2$ a new finite fixed point emerges at

$$t_c = \frac{\epsilon}{N-2} + O(\epsilon^2) \tag{16.183}$$

This fixed point is IR unstable. We will see that this is the UV fixed point of the theory. Thus, for $t < t_c$ the theory flows in the IR towards the fixed point at $t^* = 0$, and for $t > t_c$ the theory flows in the IR towards $t \to \infty$. While the behavior in this phase is not accessible to perturbation theory. Nevertheless we will be able to infer that in this phase the correlation length is finite and that the symmetry is unbroken. To justify this inference requires a non-perturbative definition of the theory, such as a lattice regularization where it becomes identical to the O(N) Heisenberg model of classical Statistical Mechanics, where this phase is simply the high temperature phase. Another option, that we will discuss in the next chapter, is to define this theory in terms of its $1/N$ expansion.

We will now inquire on the structure of the correlators. Dimensionally, in momentum space the 1PI vertex function has units $[\Gamma_R^{(N)}] = \kappa^D$. Hence, under a change of momentum scale ρ, the renormalized vertex functions must obey

$$\Gamma_R^{(N)}(\{p_i\}, t, H_R, \kappa) = \rho^D \Gamma_R^{(N)}\left(\left\{\frac{p_i}{\rho}\right\}, t, \frac{H_R}{\rho^2}, \frac{\kappa}{\rho}\right) \tag{16.184}$$

As we saw earlier in this chapter, the correlation length ξ (i.e. the inverse of the mass), satisfies the Callan-Symanzik-type equation

$$\left(\kappa\frac{\partial}{\partial\kappa} + \beta(t)\frac{\partial}{\partial t}\right)\xi(t,\kappa) = 0 \tag{16.185}$$

For $t < t_c$, the solution is

$$\xi(t,\kappa) = \kappa^{-1}\exp\left(\int_0^t \frac{dt'}{\beta(t')}\right) \tag{16.186}$$

Since the slope of the beta function at t_c is $\beta'(t_c) = -\epsilon$, we find that the correlation length has the universal form

$$\xi(t,\kappa) = \kappa^{-1}\left|\frac{t-t_c}{t_c}\right|^{-\nu} \tag{16.187}$$

with a correlation length exponent

$$\nu = -\frac{1}{\beta(t_c)} = \frac{1}{\epsilon} + O(1) \tag{16.188}$$

Hence, as $t \to t_c$ from below, the correlation length diverges in the IR. In a later chapter we will see that at t_c the theory exhibits conformal invariance. Notice that, at this one-loop order of the ϵ expansion, the value of th exponent ν is the same for all N. This is only correct at one-loop order.

Now we will examine the behavior of the vacuum expectation value of the field σ, i.e. magnitude of the broken symmetry. At the classical level $\sigma = \sqrt{1-\boldsymbol{\pi}^2}$. Here we will set the symmetry breaking field to be zero,

$H = 0$. Under the renormalization procedure that we use the field σ, just as the field $\boldsymbol{\pi}$, must be rescaled by the wave function renormalization

$$\sigma_R(t, \kappa) = Z^{-1/2} \sigma_B(g, \Lambda) \tag{16.189}$$

Consequently, the renormalized field σ_R obeys the Callan-Symanzik equation

$$\left(\beta(t) \frac{\partial}{\partial t} + \frac{\gamma(t)}{2} \right) \sigma_R(t, \kappa) = 0 \tag{16.190}$$

which has the solution

$$\sigma_R(t, \kappa) = \text{const.} \exp\left(-\frac{1}{2} \int_0^t \frac{\gamma(t')}{\beta(t')} dt' \right) \tag{16.191}$$

For $t \to t_c$ from below we obtain that $\sigma_R(t)$ obeys the scaling law

$$\sigma_R(t, \kappa) \sim |t - t_c|^\beta \tag{16.192}$$

where the exponent β is

$$\beta = -\frac{\gamma(t_c)}{2\beta'(t_c)} = \frac{N-1}{2(N-2)} + O(\epsilon) \tag{16.193}$$

Notice that as $n \to \infty$, $\beta \to 1/2$. We will revisit this result in the next chapter when we discuss the large N regime of field theories.

Finally let us examine the renormalization of the two-point function. Here too we will set $H = 0$. For a change of scale ρ, we define the running dimensionless coupling constant $t(\rho)$,

$$t(\rho) = \exp\left(\int_t^{t(\rho)} \frac{dt'}{\beta(t')} \right) \tag{16.194}$$

and find

$$\begin{aligned}
\Gamma_R^{(2)}(p, t, \kappa) =& \rho^D \Gamma_R^{(2)} \left(\frac{p}{\rho}, t, \frac{\kappa}{\rho} \right) \\
=& \rho^2 \exp\left(- \int_t^{t(\rho)} \frac{(\gamma(t') - \epsilon)}{\beta(t')} dt' \right) \Gamma_R^{(2)} \left(\frac{p}{\rho}, t(\rho), \kappa \right)
\end{aligned} \tag{16.195}$$

where we have set $D = 2 + \epsilon$.

At the fixed point, $t(\rho) = t_c$, we get

$$\begin{aligned}
\Gamma_R^{(2)}(p, t_c, \kappa) =& \rho^2 e^{-(\gamma(t_c) - \epsilon) \ln \rho} \, \Gamma_R^{(2)} \left(\frac{p}{\rho}, t_c, \kappa \right) \\
\equiv& \rho^{2-\eta} \Gamma_R^{(2)} \left(\frac{p}{\rho}, t_c, \kappa \right)
\end{aligned} \tag{16.196}$$

From the condition that

$$\Gamma_R^{(2)}(\kappa, t_c, \kappa) = \kappa^2 \tag{16.197}$$

we find the standard result

$$\Gamma_R^{(2)}(p, t_c, \kappa) = \left(\frac{p}{\kappa}\right)^{2-\eta} \kappa^2 \tag{16.198}$$

where the exponent η is

$$\eta = \gamma(t_c) - \epsilon = \frac{\epsilon}{N-2} + O(\epsilon^2) \tag{16.199}$$

Thus, we find a non-vanishing anomalous dimension already at one loop order. In contrast, in ϕ^4 theory the anomalous dimension only appears at two-loop order.

We now turn to the important case of $D = 2$ dimensions. In $D = 2$ dimensions $t_c = 0$ and the one-loop beta function reduces to

$$\beta(t) = -(N-2)t^2 + O(t^3) \tag{16.200}$$

and

$$\gamma(t) = (N-1)t + O(t^2) \tag{16.201}$$

From the integral

$$\int_{t_0}^{t} \frac{dt'}{\beta(t')} = \frac{1}{N-2}\left(\frac{1}{t} - \frac{1}{t_0}\right) \tag{16.202}$$

The correlation length $\xi(t, \kappa)$ now behaves as

$$\xi(t, \kappa) = \left[\kappa^{-1} \exp\left(-\frac{1}{(N-2)t_0}\right)\right] \exp\left(\frac{1}{(N-2)t}\right) \tag{16.203}$$

Hence, at $D = 2$ the correlation length diverges as $t \to 0$ with an essential singularity. We will shortly find the same behavior in $D = 4$ dimensional Yang-Mills gauge theory.

Furthermore, the running coupling constant $t(\rho)$ now obeys

$$\ln \rho = \frac{1}{N-2}\left(\frac{1}{t(\rho)} - \frac{1}{t}\right) \tag{16.204}$$

Hence, as the momentum scale ρ increases, the running coupling constant $t(\rho)$ flows to zero, albeit logarithmically slowly,

$$t(\rho) = \frac{t}{1 + (N-2)t \ln \rho} \approx \frac{1}{(N-2) \ln \rho} \to 0, \text{ as } \rho \to \infty \tag{16.205}$$

In other terms the effective (running) coupling constant becomes very weak at large momenta (or short distances). This behavior is known as *asymptotic* *freedom*. In this regime, the renormalized coupling constant is weak and renormalized perturbation theory works.

The flip side of this result is the behavior at long distances. By inspection of Eq.(16.205) we see that there is a scale ρ^* where $t(\rho^*) \to \infty$. It easy to see that this scale is determined by the correlation length, $\rho^* \sim \xi^{-1}$. To be more precise, this momentum scale sets a lower bound to the applicability of renormalized perturbation theory. The physics at length scales longer than ξ (even the existence of ξ itself!) is beyond the reach of perturbation theory.

This result is most remarkable. In $D = 2$ dimensions the classical action is dimensionless and so is the coupling constant. However, as we see, at the quantum level the theory flows to strong coupling in the IR and a non-trivial length (and hence mass) scale appears. This phenomenon is often called dimensional transmutation and it is characteristic of asymptotically free theories.

In this context, it is instructive to compute the explicit form of the two-point 1PI function $\Gamma_R^{(2)}$. This can be done by evaluating the integrals of Eq.(16.195). Here we will need to extract the changes in the two-point function due to the effects of the renormalization group flow. We have already discussed this problem in Sec.16.1.3 where we discussed the effects of corrections to scaling. Using the beta function $\beta(t)$ and the anomalous dimensions $\gamma(t)$ in $D = 2$ we find

$$\begin{aligned}\exp\left(-\int_t^{t(\rho)} \frac{\gamma(t')}{\beta(t')} dt'\right) &= \left(\frac{t(\rho)}{t}\right)^{(N-1)/(N-2)} \\ &= \left(1 + t(N-2)\ln\rho\right)^{-(N-1)/(N-2)} \end{aligned} \tag{16.206}$$

Setting $\rho = p/\kappa$ and plugging this result into the general expression of Eq.(16.195) we find that the two-point function has a logarithmic correction to scaling of the form

$$\Gamma_R^{(2)}(p, t, \kappa) = p^2 \, \left(t(N-2)\ln(p/\kappa)\right)^{-(N-1)/(N-2)} \tag{16.207}$$

This is the behavior of the two-point function at short distances (or high energies), $\Lambda \gg p \gg \kappa$. The same behavior is found in four-dimensional non-abelian gauge theories.

16.8 Renormalization of Yang-Mills Gauge Theories in $D = 4$ Dimensions

We will now discuss briefly the renormalization of four dimensional gauge theories. This material is discussed extensively in several classic textbooks, particularly in the book by Peskin and Schroeder. Here we will highlight the main points and compare with what we did in ϕ^4 theory and the non-liner sigma model.

For the sake of definiteness consider a theory of quarks and gluons. This is a Yang-Mills gauge theory with (color) gauge group $G = SU(N_c)$. The Lagrangian of this theory contains gauge fields (gluons) A_μ^a in the adjoint representation of $SU(N_c)$ (and, hence, $a = 1, \ldots, N_c^2 - 1$) and Dirac fermions (quarks) ψ_i that carry the quantum numbers of the fundamental, N_c-dimensional, representation of $SU(N_c)$. Here the color index $i = 1, \ldots, N_c$. We omit for now the Dirac indices.

16.8.1 Perturbation Theory

Here we will use the path-integral quantization of Yang-Mills theory discussed in Sec.9.8, in the Feynman-'t Hooft gauges with parameter λ. The Lagrangian of this theory is

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}^a F_a^{\mu\nu} + \bar{\psi} \left(i\not{D}[A] - M \right) \psi + \frac{\lambda}{2g^2} \left(\partial_\mu A_a^\mu \right)^2 - \bar{\eta}^a \partial_\mu D_{ab}^\mu [A] \eta^b \tag{16.208}$$

where $D_{ij}^\mu[A] = \delta_{ij}\partial_\mu - igA_\mu^a t_{ij}^a$ is the covariant derivative in the fundamental representation of $SU(N_c)$ (with t_{ij}^a being the generators of $SU(N_c)$ in the fundamental representation), and $D_\mu^{ab}[A] = \delta_{ab}\partial_\mu - gf_{abc}A_\mu^c$ is the covariant derivative in the adjoint representation. Here η_a are the ghost fields.

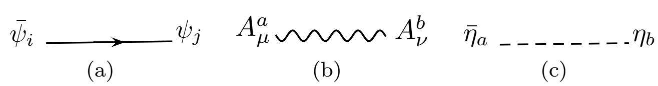


Figure 16.4 QCD propagators: a) quark propagator, b) gluon propagator, and c) ghost propagator.

The Feynman rules in the 't Hooft - Feynam gauges are as follows. The propagator of the fermions (quarks), represented by a full oriented line (shown in Fig.16.4a), is

$$S_{ij}(p) = \frac{i}{\not{p} - M + i\epsilon} \delta_{ij}, \tag{16.209}$$

with $i, j = 1, \ldots, N_c$. The propagator of the gauge field (in the Feynman gauge, with $\lambda = 1$) is represented by a wavy line (shown in Fig.16.4b) and is given by

$$\mathcal{D}^{ab}_{\mu\nu}(p) = -\frac{i}{p^2 + i\epsilon}\delta_{ab}g_{\mu\nu} \tag{16.210}$$

with $a, b = 1, \ldots, N_c^2 - 1$, and $g_{\mu\nu}$ is the metric tensor of Minkowski space-time. The propagator of the ghost fields (shown as a broken line in Fig.16.4c) is

$$\mathcal{C}_{ab}(p) = \frac{i}{p^2 + i\epsilon}\delta_{ab} \tag{16.211}$$

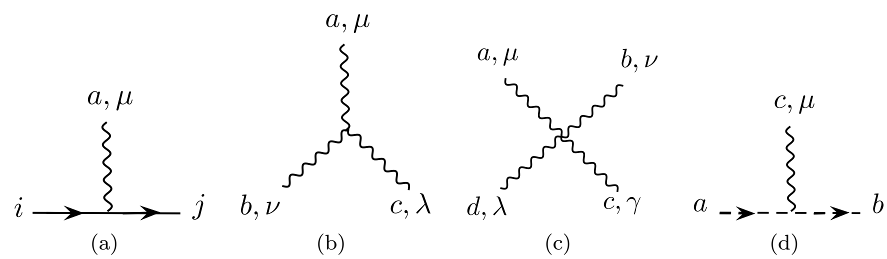


Figure 16.5 QCD vertices: a) quark-gluon vertex, b) trilinear gluon vertex, c) quadrilinear gluon vertex, and d) ghost-gluon vertex.

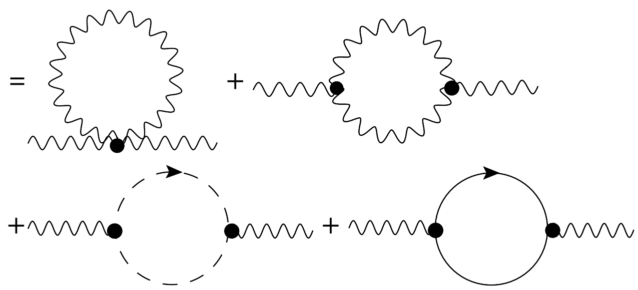
This theory has several vertices: a quark-gluon vertex (shown in Fig.16.5a) with weight $-ig\gamma^\mu t^a_{ij}$, a trilinear gluon vertex (shown in Fig.16.5b) with weight $-g((q_\mu - k_\mu)g_{\nu\lambda} + \text{two permutations})f^{abc}$, a quadrilinear gluon vertex (shown in Fig.16.5c) with weight $-ig^2 f^{abe} f^{dce}(g^{\mu\gamma}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\gamma})+$ five permutations, and a ghost-gluon vertex (shown in Fig.16.5d) with weight $-gf^{abc}q^\mu$. Notice the important fact that both the weights of both the trilinear gluon vertex and the ghost-gluon vertex carry factors that are linear in momentum.

At the tree level, the 1PI gluon propagator in the Feynman gauge ($\lambda = 1$) is

$$\Gamma_0^{(2)ab}{}_{\mu\nu}(p) = -ip^2\delta_{ab}g_{\mu\nu} \tag{16.212}$$

We will now consider its one-loop corrections, $\Gamma^{(2)}_{\text{one loop}}$. These gluon self-energy corrections are represented by the following sum of Feynman diagrams

$\Gamma^{(2)}_{\text{one loop}} =$

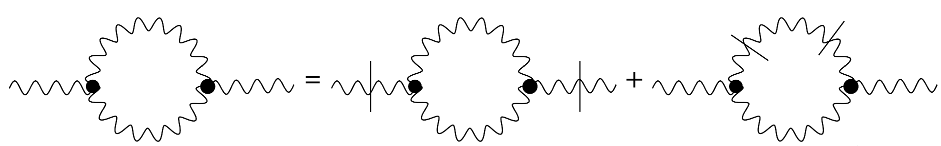


(16.213)

The pure gluon contributions, the first three terms of the right hand side of this equation, are characteristic of the non-linearities of Yang-Mills theory and are absent in Quantum Electrodynamics (QED). The last term, the quark loop, on the other hand, is similar to the electron loop in QED. If the theory has a flavor symmetry $SU(N_f)$, the quark loop has a coefficient of N_f. Here we will spare the reader the explicit expressions of these Feynman diagrams which can be found in many textbooks. Notice that, due to Fermi statistics, both the ghost loop and the quark loop each have a minus sign in their weight.

A superficial inspection of these one-loop contributions to the gluon self-energy seem to suggest that the gluons acquire a mass due to these radiative corrections. If this were to be the case it would be disastrous since mass terms for the gauge fields are forbidden by local gauge invariance. A closer examination reveals that this is not the case. The reason lies in the momentum dependence of the trilinear gluon vertex and of the ghost-gluon vertex.

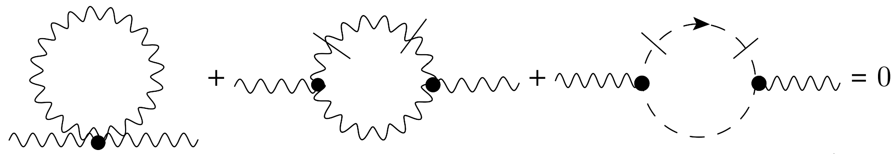
Indeed, due to the momentum dependence of these vertices there will be an explicit factor of the momentum either on the external legs of the gluon bubble diagram (the second term of the right hand side of Eq.(16.213)) or two momenta on the internal gluon loop. Thus, the gluon bubble diagram is actually a sum of two terms, one in which the there is a factor of the momentum on each internal gluon leg and one in which there are two factors of momentum in the internal gluon loop.



(16.214)

where the dashes indicate a momentum factor. The same considerations apply to the ghost loop diagram (the third term of the right hand side of Eq.(16.213)). Thus, we have three possible contributions to a potential gluon mass term: one coming from the gluon tadpole diagram (this has the quadrilinear vertex), one from the gluon bubble (with two factors of momentum on the internal gluon loop) and one from the ghost loop (again, with two factors of momentum in the internal ghost loop).

Fortunately these three contributions to a potential gluon mass term exactly cancel out!

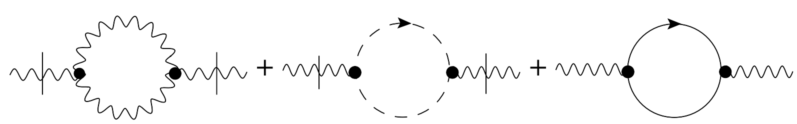


(16.215)

Of course, this cancellation is not an accident. It is a consequence of gauge invariance and it is a manifestation of the Ward Identities of Yang-Mills theory. However, this argument is formal in the sense that the integrals that enter in these Feynman diagrams are divergent. In particular, in $D = 4$ dimensions, the contributions of the Feynman diagrams that could produce a gluon mass term are quadratically divergent. Hence, for this cancellation to be obeyed it is crucial that the regularization used be consistent with gauge invariance. In a non-abelian gauge theory this is a non-trivial requirement. It is here where the use of dimensional regularization plays a key role for two reasons. One is that dimensional regularization is naturally compatible with gauge invariance. The other (and related) reason is that in dimensional regularization quadratic divergencies are regularized to zero.

16.8.2 Renormalization Group

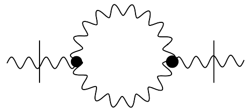
Once the cancellation of the quadratic divergencies is taken care of we find that the one-loop contribution to the gluon 1PI two-point function is the sum of three Feynman diagrams

$\Gamma^{(2)}_{\text{one loop}} =$ 

(16.216)

These diagrams are logarithmically divergent in $D = 4$ dimensions.

These three contributions have the same form:



$$= Cg^2\Big[\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2(p+q)^2}\Big]p^2 i(p^2 g^{\mu\nu} - p^\mu p^\nu)\delta_{ab} \tag{16.217}$$

where C is a constant that is different for the three diagrams. What is important is that this constant depends on the gauge group and on the representation of the fermions. The integral in Eq.(16.217) is manifestly logarithmically divergent as $D \to 4$ and, hence, has a pole in $\epsilon = D - 4$. The ghost loop term yields a contribution of the same form but with a minus sign to account for the fermionic nature of ghosts. Finally, the fermion-loop diagram yields an expression of the same form (and also negative). The fermion loop is the only contribution to the photon self-energy since Maxwell's theory is free.

Putting it all together we find that the one loop correction to the 1PI gluon two point function in the Feynman gauges ($\lambda = 1$) is (no quarks)

$$\Gamma^{\mu\nu}_{\text{one loop}\, ab}(p) = -C(N)\delta_{ab}\frac{g^2}{16\pi^2}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\frac{5}{3}\left[-\frac{2}{\epsilon} + \ln\left(\frac{p^2}{\mu^2}\right)\right] \tag{16.218}$$

where μ is a renormalization scale. Here we used that for the color group $SU(N)$ the constant is the quadratic Casimir operator of the adjoint representation and it is simply equal to $C(N) = N$. Here we used that if T^a are the generators of $SU(N)$ in the adjoint representation they obey

$$\text{tr}(T_a T_b) = -C(N)\delta_{ab}, \quad (T_a)_{bc} = i f_{abc} \tag{16.219}$$

where f_{abc} are the structure constants of $SU(N)$ and $C(N)$ is the quadratic Casimir..

Following the renormalization prescription of dimensional regularization with minimal subtraction, the pole in ϵ can be cancelled by a wave function renormalization of the non-abelian gauge field A^a_μ,

$$\Gamma^{(2)\mu\nu}_{R\;\;ab}(p,\mu) = Z_3^{-1}\Gamma^{(2)^{\mu\nu}}ab \tag{16.220}$$

where

$$Z_3 = g^2\frac{C}{16\pi^2}\left[\frac{5}{3} + \frac{1}{2}\left(1 - \frac{1}{\lambda}\right)\right]\frac{2}{\epsilon} \tag{16.221}$$

The renormalizability of Yang-Mills theory implies that the counterterms have the same form as the terms of the bare Lagrangian to all orders in perturbation theory. The renormalized Yang-Mills Lagrangian has then the form

$$\begin{aligned}\mathcal{L}_{YM} + \delta\mathcal{L}_{YM} =& \text{tr}\frac{1}{2}Z_3\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 + \frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2 \\ & - gZ_1\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)[A^\mu, A^\nu] + \frac{g^2}{2}Z_4[A_\mu, A_\nu] \\ & - \tilde{Z}_3\partial_\mu\bar{\eta}\partial^\mu\eta + g\tilde{Z}_1\partial_\mu\bar{\eta}_b A^\mu_a \eta_c f_{abc}\end{aligned} \tag{16.222}$$

where we denoted the matrix-valued gauge field $A_\mu = t_a A_\mu^a$. The relation between the renormalized and bare fields is

$$\begin{aligned} A^\mu =& Z_3^{1/2} A_R^\mu, \quad \eta = \tilde{Z}_3^{1/2} \eta_R, \quad \bar{\eta} = Z_3^{1/2} \eta_R \\ g =& Z_1 Z_3^{-3/2} g_R, \quad \lambda = Z_3^{-1} \lambda_R \end{aligned} \tag{16.223}$$

When coupled to fermions (quarks) the renormalized Lagrangian has the following additional terms

$$\mathcal{L}_{\text{fermions}} = Z_2 \bar{\psi} i \partial\!\!\!/ \psi - Z_2 m \bar{\psi} \psi - i g Z_{1F} \bar{\psi} A\!\!\!/_a t^a \psi \tag{16.224}$$

Clearly these renormalization constants cannot be independent since otherwise the renormalized Lagrangian would not be gauge-invariant. This condition is enforced by the Ward identities of this theory, known as the Slavnov-Taylor identities. A consequence of these identities is that the renormalization constants satisfy the relations

$$\frac{Z_4}{Z_1} = \frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_{1F}}{Z_2} \tag{16.225}$$

Hence, this theory, just as in the case of the non-linear sigma model, only has a coupling constant renormalization and a wave function renormalization. The coupling constant renormalization is

$$g_R = Z_g g \tag{16.226}$$

where, one loop order,

$$Z_g \equiv Z_1^{-1} Z_3^{3/2} = 1 + \frac{g^2}{16\pi^2} \left(\frac{11}{6} C - \frac{2}{3} T_f \right) \tag{16.227}$$

where

$$\operatorname{tr}(t_a t_b) = -T_f \delta_{ab} \tag{16.228}$$

Here T_f depends on the representation. For the fundamental representation $T_f = 1/2$.

16.8.3 QCD: Asymptotic Freedom

The one-loop beta function is

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} = -\frac{g^3}{8\pi^2} \left(\frac{11}{6} C - \frac{2}{3} N_f T_f \right) \tag{16.229}$$

where N_f is the number of fermion flavors. For $SU(N)$, $C = N$ and $T_f = 1/2$. The resulting beta function is

$$\beta(g) = -ag^3 \tag{16.230}$$

where

$$a = \frac{1}{16\pi^2}\left(\frac{11}{3}N - \frac{2}{3}N_f\right) \tag{16.231}$$

For the color gauge group $SU(3)$, the quantity in parenthesis is $(33-2N_f)/3$.

Thus, provided the quantity in parenthesis in the beta function of Eq.(16.229) is positive, the theory is asymptotically free. The solution of this equation is the same than what we found in the non-linear sigma model in $D = 2$. Thus, if κ is an arbitrary renormalization scale, then the running coupling constant is

$$g^2(\kappa) = \frac{1}{\text{const.} + 2a\ln\kappa} \tag{16.232}$$

For large κ (high energies), the running coupling constant becomes weak

$$g^2(\kappa) \approx \frac{1}{2a\ln\kappa} \to 0 \text{ as } \kappa \to \infty \tag{16.233}$$

Also,

$$\frac{1}{g^2(\kappa)} - \frac{1}{g^2(\kappa^*)} = 2a\ln\left(\frac{\kappa}{\kappa^*}\right) \tag{16.234}$$

Conversely, in the IR the running coupling constant flows to large values. Let the renormalization scale $\kappa \sim \Lambda$ (the UV scale), then we can as at what scale $\kappa^* = 1/\xi$ does the running coupling constant become strong? (i.e. diverge). Here too this scale has the form

$$\xi = \frac{1}{\Lambda}\exp\left(\frac{1}{2ag^2}\right) \tag{16.235}$$

This means that, a) perturbation theory works on scales shorter than ξ, and b) the IR, long distance, behavior is not accessible to perturbation theory. This behavior will persist so long as the coefficient a of the beta function of Eq.(16.231) remains positive. Already at the one-loop level one can see that if the number of flavors does not become greater than a critical value $N_f > N_{f,c}$. For $SU(3)$ this number is $N_{f,c} \sim 16$.

Notice that in the case of quantum electrodynamics, whose gauge theory is free, the coefficient is always negative. In this case the coupling constant flows to zero in the IR but becomes strong in the UV, just as in the case of ϕ^4 theory.

As the reader can see, there is a close analogy in the behaviors of the

2D non-linear sigma model and $D = 4$ non-abelian gauge theories. Both theories are asymptotically free in the UV and flow to strong coupling in the IR. Hence, in both theories the IR behavior is the regime in which the field fluctuations become wild and strongly non-classical. In Chapter 18 we will see that in this regime the non-linear sigma model is in its symmetric (unbroken) phase and that the gauge theory is in its confinement regime.

16.8.4 QED: Asymptotic "Triviality"

In the case of quantum electrodynamics, the gauge sector of the theory is free and the RG flow is entirely due to the coupling between fermions (electrons and positrons) with the electromagnetic field. Although the renormalization of the theory has the same structure it is substantially simpler. Indeed, in this case only the fermion loop contributes to the photon self-energy. The resulting beta function in $D = 4$ dimensions for QED is

$$\beta(g) = +\frac{N_f}{24\pi^2} g^3 \tag{16.236}$$

Thus, contrary to the case of QCD, the coupling constant becomes weak in the IR (and strong the the UV). This is analogous to what we found for the case of ϕ^4 theory. This behavior is usually referred to as the triviality of QED in the sense that the effective coupling constant vanishes in the IR and hence the photon-mediated scattering between electrons becomes very weak at very low energies.

There is a dynamical scale also in this theory, and has the same form as in Eq.(16.235). However, since the coupling constant runs to weak coupling in the IR this scale doe not represent here a breakdown of perturbation theory or of the vacuum state. The difference is that this scale now represents a *short-distance* (or high-energy) regime that is inaccessible to perturbation theory.