13

Perturbation Theory, Regularization and Renormalization

13.1 The loop expansion

We will now return to perturbation theory. The perturbative expansion developed in Chapters 11 and 12 is limited to situations in which the coupling constant λ is small. This is a serious limitation. The perturbative expansion that we presented is also ambiguous even in situations in which the theory has many coupling constants and/or several fields that are coupled. In these cases, the naive expansion does not provide a criterion for how to organize the expansion. There are, however, other expansion schemes that, in principle, do not have these limitations. One such scheme is the familiar semiclassical WKB method in quantum mechanics. The problem is that WKB is really difficult to generalize to a field theory and it is not a feasible option. We will see in Chapter 19 that semiclassical methods also play an important role in quantum field theory. Another non-perturbative approach, that will be discussed in Chapter 17, involves taking the limit in which the rank of the symmetry group of the theory is large. These limits have proven very instructive to understand the behavior of many theories beyond perturbation theory.

It is implicit in the use of perturbation theory, and even more so in *defining* the theory by its perturbative expansion as it often done, is the assumption that this expansion is in some sense convergent in the weak coupling regime. It is obvious that this *cannot* be the case. Thus, in ϕ^4 theory changing the *sign* of the coupling constant λ from positive to negative, turns a theory with a stable classical vacuum state to one with a *metastable* vacuum state. This argument, originally formulated by Dyson in the context of QED, implies that all perturbative expansions have at most a vanishing radius of convergence and that the perturbation theory series is, at most, an asymptotic series. This fact is also true even in the simpler problem of the quantum anharmonic oscillator. Nevertheless it is often the case, and the anharmonic oscillator is a useful example in this sense, that the non-analyticities may be of the form of an essential singularity which cannot be detected to any finite order in perturbation theory. Another example is QED which, to date the most precise theory in physics. In that case changing the sign of the fine structure constant turns repulsion between like charges into an attraction leading to a massive instability of the ground state.

For these (and other) reasons it would be desirable to have other types of expansions. Within the framework of perturbation theory there is a way to organize the expansion in a way that is more general and effective: the loop expansion. We will see, however, that although the loop expansion does not escape from these problems, nevertheless it is a great formal tool. The loop expansion is essentially an expansion in powers of the number of internal momentum integrals ("loops") that appear in the perturbative expansion. The loop expansion involves introducing a formal expansion parameter, that we will call a, and organize perturbation theory as a series expansion in powers of a. Naturally, this is a formal procedure since in the theory we must have $a = 1$ and it is far from obvious that this can be set consistently. Here, for simplicity, we will work with ϕ^4 theory for a one-component real scalar field but this procedure generalizes to any theory we may be interested in. We will introduce the parameter a in the partition function much in the same way as $\hbar$ enters in the expression of the path-integral:

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}\phi e^{-\frac{1}{a}S[\phi]} \\
&= e^{-\frac{1}{a}\int d^dx \mathcal{L}_{\text{int}}[\frac{\delta}{\delta J}]} \times \int \mathcal{D}\phi\, e^{-\frac{1}{a}\int d^dx \mathcal{L}_0[\phi] + \int d^dx J(x)\phi(x)} &\quad (13.1)\\
&= e^{-\frac{1}{a}\int d^dx \mathcal{L}_{\text{int}}[\frac{\delta}{\delta J}]} \times e^{\frac{a}{2}\int d^dx \int d^dx' J(x) G_0(x,x') J(x')} &\quad (13.2)
\end{aligned}
$$

Thus the Feynman diagrammatic rules are the same as in Chapter 11, with the following changes: (a) every vertex now acquires a weight $\frac{1}{a}$, and (b) every propagator acquires a factor a. Thus, a graph with N external points and I internal lines will have a weight (to order n in perturbation theory) a^{I-n}. We now ask how many momentum integrations does a Feynman diagram have. In each diagram there are n δ-functions (that enforce momentum conservation). However each diagram must have one δ-function for overall momentum conservation. Since each internal propagator line carries momentum, the the number of independent momentum integrations L in each diagram is $L = I - (n - 1)$. Thus, the weight of a Feynman diagram is

$a^{I-n} = a^{L-1}$. Hence, the expansion in powers of a is really an expansion in powers of the number of independent integrals or *loops*.

An example is presented in Fig. 13.1 which is a Feynman diagram for $N = 4$ point function to $n - 3$ order in perturbation theory. This diagram has $I = 6$ internal propagator lines. Our rules imply the the number of loops is $L = 3$. In this case, the loop expansion coincides with an expansion in

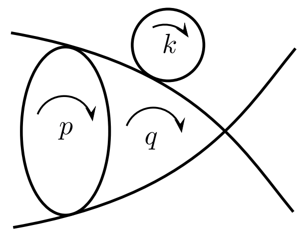


Figure 13.1 A three loop contribution, $L = 3$, to the $N = 4$ point function at order $n = 3$ in perturbation theory. The three internal momentum integrals run over the the momenta labeled by p, q and k, respectively.

powers of λ. However, if we had a theory with several coupling constants, the situation would be different.

13.1.1 The tree-level approximation

Let us compute the generating function $\Gamma[\bar{\phi}]$ to the tree-level order, $L = 0$ (no loops).

$\Gamma^{(2)}$: At the tree-level the 1-PI two-point function $\Gamma_0^{(2)}$ is just

$$\Gamma_0^{(2)}(k_1, k_2) = (2\pi)^D \delta^D(k_1 + k_2)(k_1^2 + m_0^2) \tag{13.3}$$

since at the tree level all 1PI graphs to the self-energy Σ contain at least one loop, and, hence, $\Sigma_0 = 0$.

$\Gamma^{(3)}$: The three-point function $\Gamma^{(3)}$ vanishes by symmetry at the tree level (and in fact to all orders in the loop expansion): $\Gamma^{(3)} = 0$

$\Gamma^{(4)}$: At the tree level, the four-point function $\Gamma_0^{(4)}$ is just given by (see Fig.13.2)

$$\Gamma_0^{(4)}(k_1, \ldots, k_4) = \lambda(2\pi)^D \delta^D(k_1 + \ldots + k_4) \tag{13.4}$$

$\Gamma^{(N)}$: In ϕ^4 theory all vertex functions with $N > 4$ vanish at the tree level.

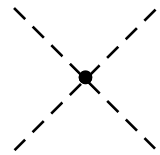


Figure 13.2 Tree level contribution to the four-point vertex function $\Gamma^{(4)}$.

So, at the tree level, the generating function $\Gamma[\phi]$ is

$$\Gamma[\bar{\phi}] = \sum_{N=1}^{\infty} \frac{1}{N!} \int \frac{d^D q_1}{(2\pi)^D} \dots \int \frac{d^D q_N}{(2\pi)^D} \Gamma^{(N)}(q_1, \dots, q_N) \bar{\phi}(-q_1) \dots \bar{\phi}(-q_N)$$
$$= \frac{1}{2!} \int \frac{d^D q_1}{(2\pi)^D} \int \frac{d^D q_2}{(2\pi)^D} (2\pi)^d \delta^d(q_1 + q_2)(k_1^2 + m_0^2) \bar{\phi}(-q_1) \bar{\phi}(-q_2)$$
$$+ \frac{1}{4!} \lambda \int \frac{d^D q_1}{(2\pi)^D} \dots \int \frac{d^D q_4}{(2\pi)^D} (2\pi)^d \delta^d(q_1 + \dots + q_4) \bar{\phi}(-q_1) \dots \bar{\phi}(-q_4) + O(\lambda^2)$$
$$(13.5)$$

Upon a Fourier transformation back to real space we find that the tree level effective action is just the classical action of ϕ^4 theory:

$$\Gamma_0(\bar{\phi}) = \int d^d x \left[\frac{1}{2} (\partial_\mu \bar{\phi})^2 + \frac{m_0^2}{2} \bar{\phi}^2 + \frac{\lambda}{4!} \bar{\phi}^4 \right] \qquad (13.6)$$

Then, at the tree level, we find two phases. If $m_0^2 > 0$, then $\Phi = 0$ and we are in the symmetric phase (with an unbroken symmetry), while for $m_0^2 < 0$, $\Phi = \pm \sqrt{\frac{6|m_0^2|}{\lambda}}$ we are in the phase in which the symmetry is spontaneously broken. Here, and below, Φ denotes the physical expectation value.

13.1.2 One Loop

We will now examine the role of quantum fluctuations by computing the effective potential at the one-loop level. At one loop $L = 1$, which implies that the Feynman diagrams that contribute must have the order n in perturbation theory equal to the number I of internal lines, $I = n$. In ϕ^4 theory, a graph with N external points, I internal lines and order n must satisfy the identity

$$4n = N + 2I \qquad (13.7)$$

since all the $4n$ lines that are emerging from the vertices must either be tied up together pairwise ('contracted'), or be attached to the N external points. From here we deduce that at the one loop diagrams must be such that $N = 2n$. The one-loop corrections are to $\Gamma^{(2)}$, $\Gamma^{(4)}$ and $\Gamma^{(6)}$ are shown in Fig. 13.3.

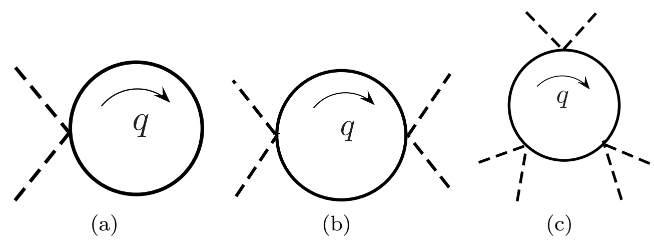


Figure 13.3 One loop contributions to (a) $\Gamma^{(2)}$, (b) $\Gamma^{(4)}$, and (c) $\Gamma^{(6)}$; q is the internal momentum of the loop.

In order to compute the one-loop contribution to effective potential, $U_1(\Phi)$, we need to compute the one-loop contribution to the N-point function, $\Gamma_1^{(N)}(0,\ldots,0)$, with $N = 2n$, with all the external momenta set to zero, $p_1 = p_2 = \ldots = p_{2n-1} = p_{2n} = 0$. A one-loop contribution to $\Gamma^{(N)}$ is shown in Fig.13.4. We find,

$$\Gamma_1^{(N)}(0,\ldots,0) = -\left(-\frac{\lambda}{4!}\right)^n \frac{1}{n!} S_n \int \frac{d^D q}{(2\pi)^D} \left(\frac{1}{q^2+m_0^2}\right)^n \times (N-1)! \quad (13.8)$$

where the symmetry factor is $S_n = (4\times 3)^n \times n!$, with $n!$ being the number of ways of reordering the vertices and $(N-1)!$ being the number of ways of attaching the external momenta, p_i $(i = 1,\ldots,2n)$, to the external vertices.

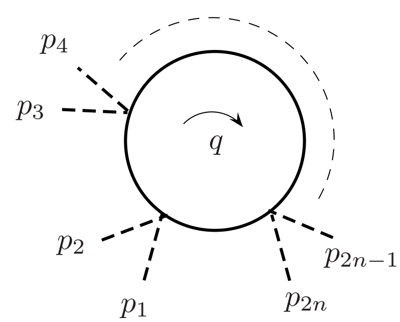


Figure 13.4 One loop contribution to $\Gamma^{(N)}$, with $N = 2n$; q is the internal momentum of the loop, and p_i (with $i = 1,\ldots 2n$) are the $2n$ external momenta.

Now, we may compute the corrections to the effective potential:

$$
\begin{aligned}
U_1[\Phi] &= \sum_{N=1}^{\infty} \frac{1}{N!} \Phi^N \Gamma_1^{(N)}(0,\ldots,0) \\
&= -\sum_{n=1}^{\infty} \frac{1}{(2n)!} \left(-\frac{\lambda \Phi^2}{4!} \right)^n \frac{(4\times 3)^n}{n!} n!(2n-1)! \int \frac{d^D q}{(2\pi)^D} \left(\frac{1}{q^2+m_0^2} \right)^n \\
&= -\sum_{n=1}^{\infty} \frac{1}{2n} \int \frac{d^D q}{(2\pi)^D} \left(-\frac{\lambda \Phi^2/2}{q^2+m_0^2} \right)^n
\end{aligned}
\tag{13.9}
$$

Using the power series expansion of the logarithm, $\ln(1+x) = -\sum_{N=1}^{\infty} \frac{1}{n}(-x)^n$, we obtain

$$
U_1[\Phi] = \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \ln \left(1 + \frac{\lambda \Phi^2/2}{q^2+m_0^2} \right) \tag{13.10}
$$

Therefore, the effective potential $U(\Phi)$, including the one loop corrections, is:

$$
U[\Phi] = \frac{m_0^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4 + \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \ln \left(q^2 + m_0^2 + \frac{\lambda \Phi^2}{2} \right) \tag{13.11}
$$

where we cancelled a constant, Φ-independent, term in Eq.(13.10) against the contribution of the functional determinant to the effective potential for the free massive scalar field theory.

Let us examine what we have done more carefully. In going from Eq.(13.9) to Eq.(13.10) to switched the order of the momentum integral with the sum over the index n of the order of perturbation theory and, in doing so, we used the Taylor series expansion of the logarithm. This is correct if the terms of the series are smaller than 1. The problem is that, for dimensionality D, the first $D/2$ terms of this series are divergent since the momentum integrals diverge. In particular, for $D = 4$ the first two terms diverge in the UV. These are, of course the quadratic UV divergence of the self-energy and the logarithmic UV divergence of the effective coupling constant, which we discussed in Sections 11.6 and 11.7.

The solution to this problem, as we saw, is to define a renormalized mass, μ^2, and a renormalized coupling constant g. The renormalized mass is defined by the condition

$$
\mu^2 = \Gamma^{(2)}(0) = m_0^2 + \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+m_0^2} + \ldots \tag{13.12}
$$

where we used the one-loop result. To one loop order we can invert the series as an expression of the bare mass m_0^2 in terms of the renormalized mass mu^2

$$m_0^2 = \mu^2 - \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + \mu^2} + \dots \tag{13.13}$$

We will see shortly that the error made in replacing the bare for the renormalized mass in the integral is cancelled by a two-loop correction. Furthermore, to one loop order, the renormalized two-point function, $\Gamma_R^{(2)}(p)$, is finite

$$\Gamma_R^{(2)}(p) = p^2 + \mu^2 \tag{13.14}$$

Similarly, we define a renormalized coupling constant g by the value of the the 4-point vertex function at zero external momenta, $p_i = 0$,

$$g = \Gamma^{(4)}(0) = \lambda - \frac{3}{2}\lambda^2 \left(\frac{1}{q^2 + m_0^2} \right)^2 + \dots \tag{13.15}$$

where we used the result at one-loop order. At this order in the loop expansion we can express the bare coupling λ as a power series of the renormalized coupling g

$$\lambda = g + \frac{3}{2} g^2 \int \frac{d^d}{(2\pi)^d} \left(\frac{1}{q^2 + \mu^2} \right)^2 + \dots \tag{13.16}$$

where we have replaced the bare mass with the renormalized mass, which si consistent to one loop order. Similarly, in the expression of the bare mass it is consistent to replace the bare coupling constant with the renormalized coupling constant. We will see that these replacements amount to taking into account two-loop corrections.

Thus the one-loop relation between the bare and the renormalized mass becomes

$$m_0^2 = \mu^2 - \frac{g}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + \mu^2} + \dots \tag{13.17}$$

To one-loop order, the renormalized 4-point function, $\Gamma_R^{(4)}(p_1, \dots, p_4)$, is

$$\begin{aligned} \Gamma_R^{(4)}(p_1, \dots, p_4) =& g - \frac{g^2}{2} \int \frac{d^D q}{(2\pi)^D} \left[\frac{1}{(q^2 + \mu^2)((p_1 + p_2 - q)^2 + \mu^2)} - \left(\frac{1}{q^2 + \mu^2} \right)^2 \right] \\ & +2 \text{ permutations} \end{aligned} \tag{13.18}$$

where the two permutations involve terms with the external momenta p_1 and p_3, and p_1 and p_4 respectively.

Finally, to one-loop order the (renormalized) effective potential $U_R[\Phi]$ is

$$U_R[\Phi] = \frac{\mu^2}{2}\Phi^2 + \frac{g}{4!}\Phi^4 + \frac{1}{2}\int \frac{d^D q}{(2\pi)^D} \ln\left(q^2 + \mu^2 + \frac{g\Phi^2}{2}\right)$$
$$- \frac{g}{2}\Phi^2 \int \frac{d^D q}{(2\pi)^D}\frac{1}{q^2+\mu^2} + \frac{g^2}{16}\Phi^4 \int \frac{d^D q}{(2\pi)^D}\left(\frac{1}{q^2+\mu^2}\right)^2 \qquad (13.19)$$

which is finite for $d < 6$ dimensions. The renormalized mass μ^2 and the renormalized coupling constant are defined in terms of the renormalized effective potential $U_R[\Phi]$ by the *renormalization conditions*

$$\mu^2 = \Gamma_R^{(2)}(0) = \left.\frac{\partial^2 U_R}{\partial \Phi^2}\right|_{\Phi=0}, \qquad g = \Gamma_R^{(4)}(0) = \left.\frac{\partial^4 U_R}{\partial \Phi^4}\right|_{\Phi=0} \qquad (13.20)$$

We end this one-loop discussion with several observations. One is that we have formally manipulated divergent integrals. In what follows we will see that we will need to go through a procedure of making them finite known *regularization.* This amounts to giving a definition of the theory at short distances and, as we will see, this can be done in several possible ways. The other question is that we have chosen to define the renormalized mass and coupling constant as the values of the two and four point vertex functions at zero external momenta. While this is an intuitive choice it is nevertheless arbitrary. One problem with this choice of renormalization conditions is that if we were to be interested in the renormalized massless theory, which as we will see defines a critical system, now the integrals contain infrared divergencies which will play an important role, and will require a change in the definition of the renormalized parameters. This is an important physical question which involves an operational definition of what the mass and the coupling constant are, and how are they measured.

Last, but not least, is the following observation. To one-loop order we defined two renormalized quantities, the mass and the coupling constant. Is this sufficient to all orders in perturbation theory? We will see next that already at two-loop order a new renormalization condition is needed: the wave function renormalization. But, how do we know if the number if renormalized (or effective) parameters does not grow (or even explode) with the order in perturbation theory? If that were to be the case, this theory would not have much of predictive power left! We will see that a theory with a finite number of renormalized parameters is what is called a renormalizable field theory, which play a key role in physics.

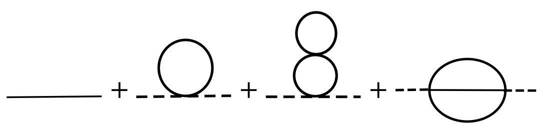
13.2 Perturbative renormalization to two loop order

We have just shown that, to one-loop order, the corrections to the two and four point functions amount to a redefinition of the bare mass m_0^2 and of the bare coupling constant λ in terms of a renormalized mass μ^2 and a renormalized coupling constant g. All the singular behavior (i.e. the sensitive dependence in the UV cutoff) is contained in the relation between these bare and renormalized quantities. We will now look at what happens to next order in perturbation theory and consider the two loop diagrams and examine if this program works at the two-loop level of if new physical effects appear.

13.2.1 Mass renormalization at two loops

The two loop contributions to the one-particle irreducible two point function $\Gamma^{(2)}$ are the last two terms of the following Feynman diagrams

$$\Gamma^{(2)}(p) =$$



(13.21)

At the two-loop level, renormalized mass μ^2 is then given by zero momentum limit of $\Gamma^{(2)}$ and has the formal expression

$$\mu^2 \equiv \Gamma^{(0)}(0) = m_0^2 + \frac{\lambda}{2} E_1(m_0^2) - \frac{\lambda^2}{4} E_2(m_0^2) E_1(m_0^2) - \frac{\lambda^2}{6} E_3(0, m_0^2) \tag{13.22}$$

where we used the notation

$$\begin{aligned}
E_1(m_0^2, \Lambda) &= \int^\Lambda \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m_0^2} \\
E_2(m_0^2, \Lambda) &= \int^\Lambda \frac{d^D q}{(2\pi)^D} \left(\frac{1}{q^2 + m_0^2} \right)^2 \\
E_3(p, m_0^2, \lambda) &= \int^\Lambda \frac{d^D q_1}{(2\pi)^D} \int^\Lambda \frac{d^D q_2}{(2\pi)^D} \frac{1}{(q_1^2 + m_0^2)(q_2^2 + m_0^2)((p - q_1 - q_2)^2 + m_0^2)}
\end{aligned} \tag{13.23}$$

where Λ is an UV cutoff scale (or regulator). Clearly, for a UV scale Λ, the degree of UV divergence of E_1 is Λ^{D-2}, of E_2 is Λ^{D-4} and of $E_3(0)$ is Λ^{2D-6}. In particular in $D = 4$ dimensions E_1 and E_3 are quadratically divergent while E_2 is logarithmically divergent. Hence, some UV regularization (or, rather, *definition*) must be supplied. We will do this shortly below.

By inspection, we see that the first two-loop contribution to $\Gamma^{(2)}(p)$ (the third term in Eq. (13.21), is just an insertion of the one-loop digram (the 2nd term) inside the propagator. Indeed, if we carry the definition of the bare mass m_0^2 in terms of the renormalized mass μ^2 of Eq. (13.17) beyond the leading term (i.e. the terms denoted by the ellipsis) we find

$$\begin{aligned} E_1(m_0^2) &= \int^{\Lambda} \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m_0^2} = \int^{\Lambda} \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + \mu^2 - \frac{\lambda}{2} E_1(\mu^2) + O(\lambda^2)} \\ &= E_1(\mu^2) + \frac{\lambda}{2} E_1(\mu^2) E_2(\mu^2) + O(\lambda^2) \end{aligned} \tag{13.24}$$

Thus, the expression of the renormalized mass at two-loop order becomes

$$\begin{aligned} \mu^2 &= m_0^2 + \frac{\lambda}{2}\left(E_1(\mu^2) + \frac{\lambda}{2} E_1(\mu^2) E_2(\mu^2)\right) - \frac{\lambda^2}{4} E_2(m_0^2) E_1(m_0^2) - \frac{\lambda^2}{6} E_3(0, m_0^2) \\ &= m_0^2 - \frac{\lambda}{2} E_1(\mu^2) - \frac{\lambda^2}{6} E_3(\mu^2) + O(\lambda^3) \end{aligned} \tag{13.25}$$

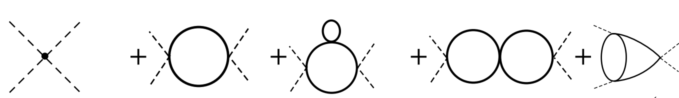
Equivalently, at two-loop order, the expression of the bare mass m_0^2 in terms of the renormalized mass μ^2 is

$$m_0^2 = \mu^2 - \frac{\lambda}{2} E_1(\mu^2) + \frac{\lambda^2}{6} E_3(0, \mu^2) + O(\lambda^3) \tag{13.26}$$

Notice that the one-loop renormalization has partially cancelled the two-loop contribution.

13.2.2 *Coupling constant renormalization at two loops*

Let us now turn to the renormalization of the coupling coupling constant at two loops. At the two-loop level the bare one-particle irreducible four point function is given by the following Feynman diagrams

$\Gamma^{(4)}(p_1, \ldots, p_4) =$  (13.27)

The actual expression is

$$
\begin{aligned}
\Gamma^{(4)}(p_1,\ldots,p_4) =&\lambda - \frac{\lambda^2}{2}\left[I(p_1+p_2,m_0^2) + 2 \text{ permutations}\right]\\
&+\frac{\lambda^3}{4}\left[I(p_1+p_2;m_0^2)^2 + 2 \text{ permutations}\right]\\
&+\frac{\lambda^3}{2}\left[I_3(p_1+p_2;m_0^2)E_1(m_0^2) + 2 \text{ permutations}\right]\\
&+\frac{\lambda^3}{2}\left[I_4(p_1,\ldots,p_4;m_0^2) + 5 \text{ permutations}\right] + O(\lambda^4)
\end{aligned}
\tag{13.28}
$$

where we denoted the (singular) integrals by

$$
I(p;m_0^2) = \int_q \frac{1}{(q^2+m_0^2)((p-q)^2+m_0^2)} \tag{13.29}
$$

$$
I_3(p;m_0^2) = \int_q \frac{1}{(q^2+m_0^2)^2((p-q)^2+m_0^2)} \tag{13.30}
$$

$$
I_4(\{p_i\};m_0^2) = \int\int_{q_1,q_2} \frac{1}{(q_1^2+m_0^2)(q_2^2+m_0^2)((p_1+p_2-q_1)^2+m_0^2)((p_3+q_1-q_2)^2+m_0^2)} \tag{13.31}
$$

where we used the notation

$$
\int_q \equiv \int^\Lambda \frac{d^Dq}{(2\pi)^D} \tag{13.32}
$$

By counting powers we see $I(p;m_0^2)$ scales with the UV cutoff Λ as Λ^{D-4} (and it has a $\ln\Lambda$ divergence in $D=4$), whereas I_3 scales as Λ^{D-6} (and it is finite in $D=4$ dimensions), and I_4 scales as $\Lambda^{2(D-4)}$ (and has a $\ln^2\Lambda$ divergence in $D=4$).

Let us first carry out the mass renormalization the expressions involved in Eq.(13.28) which lead to a partial cancellation. After that is done Eq.(13.28) becomes

$$
\begin{aligned}
\Gamma^{(4)}(p_1,\ldots,p_4) =&\lambda - \frac{\lambda^2}{2}\left[I(p_1+p_2,\mu^2) + 2 \text{ permutations}\right]\\
&+\frac{\lambda^3}{4}\left[I(p_1+p_2;\mu^2)^2 + 2 \text{ permutations}\right]\\
&+\frac{\lambda^3}{2}\left[I_4(p_1,\ldots,p_4;\mu^2) + 5 \text{ permutations}\right] + O(\lambda^4)
\end{aligned}
\tag{13.33}
$$

Next we define the renormalized coupling constant at the two loop level:

$$g = \Gamma^{(4)}([p_i = 0]) = \lambda - \frac{3}{2}\lambda^2 E_2(\mu^2) + \frac{3}{4}\lambda^3 E_2^2(\mu^2) + 3\lambda^3 I_4([p_i = 0]; \mu^2) \tag{13.34}$$

which is inverted as

$$\lambda = g + \frac{3}{2}g^2 E_2(\mu^2) + \frac{15}{4}g^3 E_2^2(\mu^2) - 3g^3 I_4([p_i = 0]; \mu^2) + O(g^4) \tag{13.35}$$

After this renormalizations the four point function becomes

$$\begin{aligned}\Gamma^{(4)}(p_1, \ldots, p_4) =& g - \frac{g^2}{2}\left\{[I(p_1 + p_2, \mu^2) - E_2(\mu^2)] + 2 \text{ permutations}\right\}\\ &+\frac{\lambda^3}{4}\left\{[I(p_1 + p_2; \mu^2) - E_2(\mu^2)]^2 + 2 \text{ permutations}\right\}\\ &+\frac{\lambda^3}{2}\{[I_4(p_1, \ldots, p_4; \mu^2) - I_4([p_i = 0]; \mu^2)]\\ &- E_2(\mu^2)[I(p_1 + p_2; \mu^2) - E_2(\mu^2)] + 5 \text{ permutations}\} + O(g^4)\end{aligned} \tag{13.36}$$

The coupling constant renormalization changes the two point function to

$$\Gamma^{(2)}(p; \mu^2) = p^2 + \mu^2 - \frac{g^2}{6}\left[E_3(p; \mu^2) - E_3(0; \mu^2)\right] + O(g^3) \tag{13.37}$$

and the relation between the bare and the renormalized mass:

$$m_0^2 = \mu^2 - \frac{g}{2}E_1(\mu^2) - \frac{3}{4}g^2 E_1(\mu^2)E_2(\mu^2) + \frac{g^2}{6}E_3(0; \mu^2) + O(g^3) \tag{13.38}$$

13.2.3 Wave function renormalization

The preceding results now tell us that, after mass renormalization at two-loop level, the one-particle irreducible two point function $\Gamma^{(2)}(p)$ becomes

$$\Gamma^{(2)}(p) = p^2 + \mu^2 - \frac{g^2}{6}\left[E_3(p, \mu^2) - E_3(0, \mu^2)\right] + O(g^3) \tag{13.39}$$

There is still, however, the momentum-dependent contribution of $\Gamma^{(2)}(p)$, the last term of the right hand side of Eq.(13.39). The subtracted expression, $E_3(p, \mu^2) - E_3(0, \mu^2)$, is logarithmically divergent in $D = 4$ dimensions. Clearly, this is unaffected by the mass renormalization. It is also obvious that it cannot be taken care of by the coupling constant renormalization. Hence, we will need a new renormalization which, for historical reasons, is known as the *wave function renormalization*, which we will discuss now.

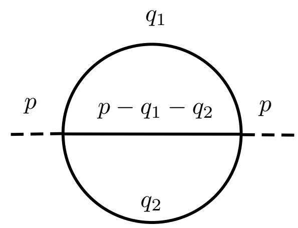


Figure 13.5 Feynman diagram with the leading contribution to the wave function renormalization

Since the remaining singular behavior in $\Gamma^{(2)}(p)$ comes from its momentum dependence, we have to interpret it as a factor in front of p^2. Equivalently, this new renormalization is a change in the prefactor of the gradient term of the action (more on this below). This suggests that we define a renormalized one-particle irreducible two point function by a rescaling of the form

$$\Gamma_R^{(2)}(p, \mu_R^2) = Z_\phi(g, \mu^2, \Lambda)\Gamma^{(2)}(p, \mu^2, \Lambda) \tag{13.40}$$

such that

$$\Gamma_R^{(2)}(p, \mu_R^2) = p^2 + \mu_R^2 \tag{13.41}$$

Notice that the rescaling of the two point function is forcing us to change the definition of the renormalized mass from μ^2 to μ_R^2. The introduction of the new renormalization "constant" $Z_\phi(g, \mu^2, \Lambda)$ is equivalent to a rescaling of the field ϕ by $Z_\phi^{1/2}$. For this reason this procedure is known as the "wave function renormalization". This name, invented in the context of QED in the late 1940s, is still retained even though it is a highly misleading term.

The wave function renormalization constant Z_ϕ has the expansion

$$Z_\phi = 1 + z_1 g + z_2 g^2 + \ldots \tag{13.42}$$

Since the wave function renormalization only appears (in ϕ^4 theory) at the two-loop level, we see that we must set $z_1 = 0$. Furthermore, demanding that $\Gamma_R^{(2)}$ has the form of Eq.(13.41) or, what is the same, that

$$\Gamma_R^{(2)}(0) = \mu_R^2, \qquad \text{and} \qquad \left.\frac{\partial \Gamma_R^{(2)}(p)}{\partial p^2}\right|_{p=0} = 1 \tag{13.43}$$

we find

$$1 = Z_\phi \left(1 - \frac{g^2}{6} \left.\frac{\partial E_3(p, \mu^2)}{\partial p^2}\right|_{p=0} \right) \tag{13.44}$$

from which we conclude that the (logarithmically divergent) quantity z_2 is

$$z_2 = \frac{1}{6}\frac{\partial E_3(p,\mu^2)}{\partial p^2}\Big|_{p=0} \tag{13.45}$$

On the other hand, this rescaling changes the new renormalized mass to μ_R^2

$$\mu_R^2 = (1 + z_2 g^2)\mu^2 \tag{13.46}$$

The introduction of the wave function renormalization, in turn, affects the definition of the renormalized coupling. The renormalized one-particle irreducible four-point function is defined as

$$\Gamma_R^{(4)}(\{p_i\}; g_R, \mu_R^2) = Z_\phi^2 \Gamma^{(4)}(\{p_i\}; g, \mu^2) \tag{13.47}$$

This definition accounts for the contributions of two-loop corrections in the internal propagators. Similarly, the renormalized one-particle irreducible N-point functions are defined to be

$$\Gamma_R^{(N)}(\{p_i\}; g_R, \mu_R^2) = Z_\phi^{N/2} \Gamma^{(N)}(\{p_i\}; g, \mu^2) \tag{13.48}$$

13.2.4 Renormalization Conditions

Let us summarize the procedure of perturbative renormalization that we outlined and that we carried out to two loop level. This is what we have done:

a: We computed the bare two and four one-particle irreducible functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$ as a function of the bare mass m_0^2 and the bare coupling constant λ.

b: We replaced the bare mass m_0^2 first by μ^2 and later by $\mu_R^2 = \mu^2 Z_\phi$. The renormalized mass was *defined* by the renormalization condition

$$\lim_{p\to 0} \Gamma_R^{(2)}(p) = \mu_R^2 \tag{13.49}$$

c: We replaced the bare coupling constant λ first by g and later by the renormalized coupling constant $g_R = g Z_\phi^2$. The renormalized coupling constant was *defined* by the renormalization condition

$$\lim_{\{p_i\}\to 0} \Gamma_R^{(4)}(p_1, \ldots, p_4) = g_R \tag{13.50}$$

d: The wave function renormalization Z_ϕ was obtained by demanding the renormalization condition

$$\frac{\partial \Gamma_R^{(2)}(p)}{\partial p^2}\Big|_{p=0} = 1 \tag{13.51}$$

e: We *defined* the renormalized functions

$$\Gamma_R^{(N)} = Z_\phi^{N/2}\Gamma^{(N)}, \tag{13.52}$$

which are UV-finite functions of the renormalized mass μ_R^2 and of the renormalized coupling constant g_R.

Notice that these are definitions that we chose to make and that there are many possible such definitions. In particular, all the singular (divergent) behavior is "hidden" in the relation between the bare and the renormalized quantities. By this procedure, at least up to two loops, we succeeded in removing the strong dependence on the UV definition of the theory. However, how do we know if this procedure will suffice *to all orders* in perturbation theory? In other words, how do we know if the number of renormalized parameters does not grow with the order of perturbation theory? Clearly, a theory with an infinite number of arbitrary parameters would not be a theory at all! We will address this problem shortly.

Formally, the renormalized mass μ_R^2, the renormalized coupling constant g_R and the wave function renormalization Z_ϕ are functions of the bare mass m_0^2, the bare coupling constant λ and the UV regulator (or cutoff scale) Λ, of the form

$$\begin{aligned}
\mu_R^2 =& Z_\phi \mu^2(m_0^2, \lambda, \Lambda)\\
g_R =& Z_\phi^2 g(m_0^2, \lambda, \Lambda)\\
Z_\phi =& Z_\phi(m_0^2, \lambda, \Lambda)
\end{aligned} \tag{13.53}$$

which are given by their expressions in the perturbation theory expansion in powers of the coupling constant λ. Alternatively, we can invert these relations and write expressions for the bare parameters in terms of the renormalized ones,

$$\begin{aligned}
m_0^2 =& Z_\phi m_0^2(\mu_R^2, g_R, \Lambda)\\
\lambda =& Z_\phi^2 g(\mu_R^2, g_R, \Lambda)\\
Z_\phi =& Z_\phi(\mu_R^2, g_R, \Lambda)
\end{aligned} \tag{13.54}$$

Then, the renormalized vertex functions

$$\Gamma_R^{(N)}(\{p_i\}, \mu_R^2, g_R, \Lambda) = Z_\phi^{N/2}\Gamma^{(N)}(m_0^2, \lambda, \Lambda) \tag{13.55}$$

have a finite limit as the regulator $\Lambda \to \infty$ to every order in an expansion in powers of the *renormalized* coupling constant g_R.

We will now sketch the computation of the renormalization constants to two-loop order in ϕ^4 theory in $D = 4$ dimensions. For simplicity we will discuss only the massless case, and hence we will require that $\mu_R^2 = 0$. However, since in the massless theory has infrared divergencies, we will need to redefine our renormalization conditions. In general, in ϕ^4 theory in $D = 4$ dimensions, we have three renormalization conditions

$$\Gamma_R^{(2)}(0, \mu_R^2, g_R) = \mu_R^2, \qquad \text{fixes the renormalized mass} \tag{13.56}$$

$$\frac{\partial}{\partial p^2}\Gamma_R^{(2)}(p, \mu_R^2, g_R)\Big|_{p=0} = 1, \qquad \text{fixes the wavefunction renormalization} \tag{13.57}$$

$$\Gamma_R^{(4)}(\{p_i = 0\}, \mu_R^2, g_R) = g_R \qquad \text{fixes the coupling constant renormalization} \tag{13.58}$$

These definitions are fine for the massive case but not for the massless case, $\mu_R^2 = 0$ due to IR divergencies in the expressions. In the massless case, $\mu_R^2 = 0$, we will instead impose renormalization conditions at a fixed momentum scale κ:

$$\begin{aligned}
\Gamma_R^{(2)}(0, g_R) =& 0 \\
\frac{\partial}{\partial p^2}\Gamma_R^{(2)}(p, g_R)\Big|_{p^2=\kappa^2} =& 1 \\
\Gamma_R^{(4)}(\{p_i\}, g_R)\Big|_{SP} =& g_R
\end{aligned} \tag{13.59}$$

where SP denotes the symmetric arrangement of the four external momenta $\{p_i\}$ (with $i = 1, \dots, 4$), such that $p_i \cdot p_j = \frac{\kappa^2}{2}(4\delta_{ij} - 1)$. With this choice we have $P^2 = (p_i + p_j)^2 = \kappa^2$. Since the renormalized quantities are defined a a fixed momentum scale κ, the renormalization constants will also be functions of that scale.

To proceed we first need to find the value of the bare mass m_0^2 for which the renormalized mass vanishes, $\mu_R^2 = 0$. We will call this value of the bare mass $m_c^2(\lambda, \Lambda)$, which is a function of the bare coupling constant and of the UV regulator. We already done something like this at the one-loop level in Section 11.6, where we observed that this was the equivalent to find the correction due to fluctuations to the critical temperature for the phase transition in the Landau theory of phase transitions where we identified $m_c^2 = T_c - T_0$, with T_0 being the bare (or mean field) value of the critical temperature and T_c its value corrected by fluctuations.

At two loop order we find that $m_c^2(\lambda, \Lambda)$ is the solution of the equation

$$0 = m_c^2 + \frac{\lambda}{2} D_1(m_c^2, \Lambda) - \frac{\lambda^2}{4} D_1(m_c^2, \Lambda) E_2(m_c^2, \Lambda) - \frac{\lambda^2}{6} E_3(0, m_c^2, \Lambda) + O(\lambda^3) \tag{13.60}$$

The solution to this equation as a power series expansion in powers of the bare coupling constant λ is

$$m_c^2(\lambda, \Lambda) = -\frac{\lambda}{2} E_1(0, \Lambda) + \frac{\lambda^2}{6} E_3(0, 0, \Lambda) + O(\lambda^3) \tag{13.61}$$

where, as before, we used the notation

$$\begin{aligned} E_1(0, \Lambda) &= \int^{\Lambda} \frac{d^D q}{(2\pi)^D} \frac{1}{q^2} \\ E_3(0, 0, \Lambda) &= \int^{\Lambda} \frac{d^D q_1}{(2\pi)^D} \int^{\Lambda} \frac{d^D q_2}{(2\pi)^D} \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2} \end{aligned} \tag{13.62}$$

where Λ is an unspecified UV regulator (or cutoff). Notice that both integrals diverge like Λ^2 in the UV in in $D = 4$ dimensions and are finite in the IR.

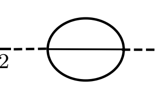
Next we need to do the wave function and the coupling constant renormalizations. In each case, we write the bare coupling constant λ and the the wave function renormalization Z_ϕ as a power series in the renormalized coupling constant g_R, of the form

$$\begin{aligned} \lambda &= g_R + \lambda_2 g_R^2 + \lambda_3 g_R^3 + O(g_R^4) \\ Z_\phi &= 1 + z_2 g_R^2 + O(g_R^3) \end{aligned} \tag{13.63}$$

where we used the fact that there is no wave function renormalization and one-loop level and hence we already set $z_1 = 0$.

The renormalization condition of Eq.(13.57), dictates that Z_ϕ must obey the condition

$$1 = Z_\phi \left[1 - \frac{\lambda^2}{6} \frac{\partial}{\partial p^2} \quad \cdots \quad \right]_{p^2 = \kappa^2} \tag{13.64}$$

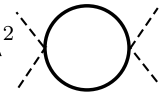
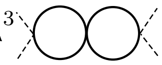
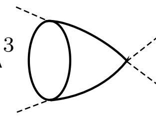


From which it follows that z_2 is given by the expression

$$z_2 = \frac{1}{6} \frac{\partial}{\partial p^2} E_3(p, 0, \Lambda) \Big|_{p^2 = \kappa^2} \tag{13.65}$$

Note that, although the integral $E_3(p, 0, \Lambda)$ diverges quadratically in the UV, as Λ^2 (in $D = 4$ dimensions), the derivative $\frac{\partial E_3}{\partial p^2}$ diverges logarithmically with the UV cutoff Λ.

Likewise, Eq.(13.58) leads to the requirement that the renormalized coupling constant g_R should obey

$g_R = Z_\phi^2 \Big[\lambda - \frac{3}{2}\lambda^2$  $+ \frac{3}{4}\lambda^3$  $+ 3\lambda^3$  $\Big]_{SP}$ (13.66)

Collecting terms and expanding order by order we find that the constants λ_2 and λ_3 (defined in Eq.(13.63)) are given by

$$\begin{aligned} \lambda_2 &= \frac{3}{2} I_{SP} \\ \lambda_3 &= \frac{15}{4} I_{SP}^2 - 3I_{4SP} - 2z_2 \end{aligned} \tag{13.67}$$

where I_{SP} and I_{4SP} are the integrals for the bubble diagrams defined in Eqs.(13.29), (13.30) and (13.31) computed in the massless theory at the symmetric point of the external momenta with momentum scale κ.

Thus, we have reduced the problem of computing the renormalization constants to the evaluation of singular integrals which, themselves, are ill-defined unless we supply a definition of the theory of short distances. This we will do in Section 13.5 where we will discuss different regularization schemes.

13.3 Subtractions, counterterms and renormalized Lagrangians

The procedure of renormalized perturbation theory relates the bare connected N-point functions, $G_c^{(N)}(\{p_i\}; m_0^2, \lambda, \Lambda)$, which depend on N external momenta $\{p_i\}$ (with $i = 1, \ldots, N$), the bare mass m_0^2 and the bare coupling constant λ (and of some as yet unspecified UV regulator, or cutoff, Λ), to a renormalized connected N-point function, $G_{cR}^{(N)}(\{p_i\}; \mu_R^2, g_R)$, that depends on the N external momenta, the renormalized mass μ_R^2 and the renormalized coupling constant g_R:

$$G_{cR}^{(N)}(\{p_i\}; \mu_R^2, g_R) = Z_\phi^{-N/2} G_c^{(N)}(\{p_i\}; m_0^2, \lambda, \Lambda) \tag{13.68}$$

In this way, the renormalized N point functions formally do not depend on the regulator scale Λ, although, as we will see later on, they do depend on the renormalization procedure. At least at a formal level, the renormalized N point functions describe a continuum quantum field theory (i.e. without a cutoff). All the strong dependence on the UV definition of the theory (i.e. the regulator) is encoded in the relation between the bare and renormalized quantities.

The bare connected N point functions are determined by the generating functional $F[J] = \ln Z[J]$, where $Z[J]$ is the partition function for a theory with the bare Lagrangian (density) $\mathcal{L}_B$

$$\mathcal{L}_B = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2} m_0^2 \phi^2 + \frac{\lambda}{4!}\phi^4 - J\phi \tag{13.69}$$

Formally we should expect that the renormalized connected N point functions should be determined from a generating functional $F_R[J]$ for a renormalized Lagrangian $\mathcal{L}_R$ of the same form, i.e.

$$\mathcal{L}_R = \frac{1}{2}\left(\partial_\mu \phi_R\right)^2 + \frac{1}{2}\mu_R^2 \phi^2 + \frac{g_R}{4!}\phi_R^4 - J\phi_R \tag{13.70}$$

which depends on the renormalized mass μ_R^2 and the renormalized coupling constant g_R. Here ϕ_R is the "renormalized field",

$$\phi_R \equiv Z_\phi^{-1/2}\phi \tag{13.71}$$

that differs from the field ϕ by a multiplicative rescaling factor $Z_\phi^{-1/2}$.

Upon rescaling the field as in Eq.(13.71), in the bare Lagrangian, Eq.(13.69), we can write to the bare Lagrangian as

$$\mathcal{L}_B = \frac{1}{2} Z_\phi \left(\partial_\mu \phi_R\right)^2 + \frac{1}{2} Z_\phi m_0^2 \phi^2 + \frac{1}{4!}\lambda Z_\phi^2 \phi_R^4 - J\phi_R \tag{13.72}$$

Then, the difference $\Delta\mathcal{L} = \mathcal{L}_B - \mathcal{L}_R$ between the bare and the renormalized Lagrangians is

$$\Delta\mathcal{L} = \frac{1}{2}(Z_\phi - 1)\left(\partial_\mu \phi_R\right)^2 + \frac{1}{2}(Z_\phi m_0^2 - \mu_R^2)\phi^2 + \frac{1}{4!}(\lambda Z_\phi^2 - g_R)\phi_R^4 - J\phi_R \tag{13.73}$$

The procedure of renormalized theory can then be regarded as one in which the bare Lagrangian $\mathcal{L}_B$ is *subtracted* by a set of *counterterms* shown in Eq.(13.73) which have the *same form* as the bare Lagrangian.

Thus, the program of renormalizing a quantum field theory (defined by its perturbation theory) can be recast as a systematic classification of the possible counterterms needed to make the perturbation theory UV finite. Here we have assumed that the bare and the renormalized Lagrangians have the same structure (and not only the same symmetries) and hence so do the counterterms. If additional operators were to arise at some order in perturbation theory, these operators, and their counterterms, must be added to the Lagrangian to insure consistency, or *renormalizability.*

This program traces back its origins to the work by Feynman, Schwinger and Tomonaga (later expanded by Bogoliubov, Symanzik and many others) in Quantum Electrodynamics which, to this date, is the most precise and the most successful quantum field theory. On the other hand, in spite of its many notable successes, this program has several drawbacks. In the first place it relies entirely in the perturbative definition of the theory. The perturbation series is *at best* an asymptotic series with a vanishing radius of convergence, which cannot be an analytic function of the bare coupling (even once a UV regulator is defined) since changing the sign of the coupling constant λ from positive to negative turns a theory with a stable vacuum state to another one without a stable vacuum state (unless operators with higher powers of the field are added explicitly to the action to insure stability).

The other and more serious conceptual problem is that the procedure that we followed is physically obscure. There is a lot of important physics that is hidden away in the relation between bare and renormalized quantities. We will clarify these questions when we discuss the Renormalization Group in a later chapter. There we will see that, at a price, we can formulate a non-perturbative definition of the theory beyond its definition in terms of its perturbation series. One important concept that we will encounter is that the coupling constants, and for that matter all the parameters of the Lagrangian, are not really fixed quantities but depend on the energy (and momentum) scale at which they are defined (or measured).

13.4 Dimensional Analysis and Perturbative Renormalizability

In the previous sections we saw that, at least to the second order in the loop expansion, it is possible to recast the effects of fluctuations in a renormalization of a set of parameters, i.e. the coupling constant, the mass, and the wavefunction renormalization. It is implicit in what we did the assumption that there is a *finite* number of such renormalized parameters. When this is the case, we will say that a theory, such as ϕ^4 theory in $D = 4$ dimensions, is renormalizable. Or, rather, to be more precise, we should say perturbatively renormalizable since it is a statement on the perturbative expansion. In a later chapter we will discuss the renormalization group and there we will see that there are several possible renormalized theories defined by a *fixed point* of the renormalization group.

For now we will discuss perturbative renormalization (i.e. in the weak coupling regime) in several theories focusing specifically in the case of ϕ^4 theory.

13.4.1 Dimensional Analysis

We begin by doing dimensional analysis in several theories of physical interest that we have discussed in other chapters. When we say dimensional analysis it will always be assumed that this is done at the level of the *free* *field theory*. We will see in a later chapter that interactions can (and do) change this analysis in profound ways. Here we will introduce two key concepts: scaling dimensions of operators and critical dimensions of spacetime. We will see that there is an intimate relation between critical dimensions and perturbative renormalizability.

Scalar Fields

Let us begin with a theory of a scalar field ϕ which for simplicity we take it to have only one real component. The extension to many components (real or complex) will not change the essence of our analysis. The Euclidean action has the general form

$$S = \int d^D x \, \mathcal{L} = \int d^D x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{m_0^2}{2} \phi^2 + \sum_n \frac{\lambda_n}{n!} \phi^n \right] \tag{13.74}$$

Since the action S should be *dimensionless*, the Euclidean Lagrangian (density) must scale as the inverse volume, $[\mathcal{L}] = L^{-D}$ where L is a length scale. It then follows that the field ϕ must have units of $[\phi] = L^{-(D-2)/2}$. Or, in terms of a momentum scale Λ, the field scales as $[\phi] = \Lambda^{(D-2)/2}$. By consistency, we find that the mass has unites of $[m_0] = L^{-1} = \Lambda$. Similarly, the operators ϕ^n must have the units $[\phi^n] = L^{-n(D-2)/2}$and the coupling constants λ_n, defined in the action of Eq.(13.74), must scale as $[\lambda_n] = L^{\frac{nD}{2}-D-n} = \Lambda^{D+n=nD/2}$.

We will say that an operator $\mathcal{O}$ has *scaling dimension* $\Delta_{\mathcal{O}}$ if its units are $[\mathcal{O} = L^{-\Delta_{\mathcal{O}}}$. Clearly, the free scalar field ϕ has scaling dimension $\Delta_\phi = (D-2)/2$ and ϕ^n has (free field) scaling dimension $\Delta_n = n\Delta_\phi$. Now we ask when can the coupling constant λ_n be *dimensionless*. For this to happen the scaling dimension of the operator ϕ^n must be equal to the dimensionality D of Euclidean space. This can only happen at the certain *critical dimension* $D_n^c = 2n/(n-2)$. Hence, the critical dimension for ϕ^3 is 6, for ϕ^4 is 4, for ϕ^6 is 3, etc. Also we see that, as $n \to \infty$, $D_n^c \to 2$. Hence in $D = 2$ dimensions the field ϕ and all its powers are dimensionless and their scaling dimensions are zero.

This dimensional analysis also tells us the the connected N-point functions of the scalar field, $G_N(x_1, \ldots, x_N) = \langle \phi(x_1) \ldots \phi(x_n) \rangle$, have units $[G_N] = [\phi]^N = L^{-N(D-2)/2} = \Lambda^{N(D-2)/2}$. Their Fourier transforms $\tilde{G}_N(p_1, \ldots, p_N)$ have units $[\tilde{G}_N]\Lambda^{-ND} = \Lambda^{-N(D+2)/2}$. It is easy to see that the Fourier transforms of the one-particle irreducible N-point vertex functions $\bar{\Gamma}^N$ (where we factored out the delta function for momentum conservation) have units $[\bar{\Gamma}^N] = \Lambda^{N+D-ND/2}$, which are the same units of the coupling constants λ_N.

Non-linear sigma models

Non-linear sigma models are a class of scalar field theories in which the field obeys certain local constraints and that in such theories the global symmetry is realized nonlinearly. The prototype non-linear sigma model is an N-component real scalar field $n^a(x)$ (with $a = 1, \ldots, N$) that satisfies the local constraint

$$\boldsymbol{n}^2(x) = 1 \tag{13.75}$$

The global symmetry in this theory is $O(N)$. The Euclidean action of the non-linear sigma model is

$$S = \frac{1}{2g} \int d^D x \, \left(\partial_\mu \boldsymbol{n}(x)\right)^2 \tag{13.76}$$

where g is a coupling constant.

The constraint of the non-linear sigma model, Eq.(13.75), requires that the field $\boldsymbol{n}$ be *dimensionless*. Hence, the coupling constant g must have units $[g] = L^{D-2}$. It follows that the critical dimension of the non-linear sigma model is $D_c = 2$. We will come back to non-linear sigma models in later chapters where we will find that this analysis holds for all such models.

Fermi Fields

Let us now discuss the theory of an N-component relativistic Fermi field ψ_a, with $a = 1, \ldots, N$ (here we drop the Dirac indices). The action has a free field Dirac term plus some local interaction terms

$$S = \int d^D x \, \left[\bar{\psi}_a \left(i\gamma^\mu \partial_\mu - m_0\right) \psi_a + g(\bar{\psi}_a \psi_a)^2\right] \tag{13.77}$$

This is the Gross-Neveu model with a global $SU(N)$ symmetry. Here g is the coupling constant.

Once again, by requiring that action S be dimensionless we find the scaling dimension of the field, which now is $\Delta_\psi = (D-1)/2$, and the mass has units $[m_0] = L^{-1} = \Lambda$ (as it should be!). The scaling dimension of the interaction operator $(\bar{\psi}\psi)^2$ is $\Delta = 4\Delta_\psi = 2(D-1)$. It follows that the coupling constant g must have units $[g] = -D + 2(D-1) = D-2$. Hence, g is dimensionless is $D_c = 2$ and this is the critical dimension of the Gross-Neveu model. Notice that this is the same as the critical dimension of the non-linear sigma model.

Another case of interest is a theory of a Dirac field ψ and a scalar field ϕ coupled through a Yukawa coupling of the form $g_Y \bar{\psi}\psi\phi$. Since the scaling dimension of the Dirac field is $\Delta_\psi = (D-1)/2$ and the scaling dimension of the scalar field is $\Delta_\phi = (D-2)/2$, the units of the Yukawa coupling are

$[g_Y] = (D-4)/2$. Hence, the critical dimension for the Yukawa coupling is $D_c = 4$.

Gauge Theories

We now turn to the case of gauge theories. Let us consider the general case of a non-abelian Yang-Mills theory. This analysis also holds fir the special case of the abelian (Maxwell) gauge theory. Here the gauge field A_μ is a connection and takes values in the algebra of a gauge group G. The covariant derivative is $D_\mu = \partial_\mu - iA_\mu$. Dimensional analysis in this case is useful in the asymptotically weak coupling regime, $g \to 0$, where the gauge fields might be expected to essentially behave classically. We will see in the sequel that, here too, this analysis can also serve to assess the importance of low-order quantum fluctuations (i.e. one loop). There we will see that our expectations are correct in some cases (e.g. QED) but incorrect in other cases (e.g. Yang-Mills).

From this definition, it follows that the gauge field A_μ has units of inverse length and hence its scaling dimension is $\Delta_A = 1$, regardless of the dimension of spacetime. Notice we we have not included the coupling constant in the definition of the covariant derivative. Since the field tensor is the commutator two covariant derivatives, $F_{\mu\nu} = i[D_\mu, D_\nu]$, it has scaling dimension $\Delta_F = 2$.

On the other hand, the Yang-Mills action is

$$S = \frac{1}{4g^2} \int d^D x \, \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \tag{13.78}$$

where introduced the Yang-Mills coupling constant g. Again, since S is dimensionless, it follows that the Yang-Mills coupling constant g has units $[g] = L^{-(D-4)/2}$.

Hence, the critical dimension of all gauge theories (with a continuous gauge group) is $D_c = 4$. This analysis also holds for gauge theories (minimally) coupled to matter fields, be the case of Dirac fermions (as in QED and QCD) or complex scalar fields (as in scalar electrodynamics and in Higgs models). In all cases gauge invariance requires that there should be only one coupling constant. In a later chapter we will discuss gauge theories with a discrete gauge group and we will see that their critical dimension is $D_c = 2$.

13.4.2 Criterion for Perturbative Renormalizability

Earlier in this chapter we discussed in detail the program of perturbative renormalization in ϕ^4 field theory. There we worked laboriously to repackage the results, up to two loop order, in *the same form* as the classical theory by defining a finite number of renormalized parameters, e.g. the renormalized mass and the renormalized coupling constant. We also saw that at the two-loop level we needed to introduce a new concept, the wave function (or field) renormalization. Although we have not yet discussed it here, products of fields at short distances become composite operators which have their own renormalizations.

With some variants, this program has been carried out for all the theories mentioned in the previous subsection. For instance, in the case of non-linear sigma-models and Yang-Mills gauge theories the expansion in powers of the coupling constant is an expansion about the classical vacuum state. But in all cases one makes the (explicit or implicit) assumption that the theory has a small parameter that, at least qualitatively, might control the use of a perturbative expansion. this is the case in QED, whose expansion parameter is the fin structure constant $\alpha = 1/137$, but it is nit the case in most other theories, particularly in Yang-Mills.

Theories that require a *finite number* of renormalizations to account for their UV divergencies are said to be (perturbatively) renormalizable field theories. This criterion implicitly always uses free field (or classical) theory as a reference theory. In later chapters, when we discuss the renormalization group, we will see that one can define theories withe respect to other *fixed points* where the theory is not free (or classical). From now on, when we use the term “renormalizable” it will be meant “perturbatively renormalizable”. Examples of renormalizable field theories are ϕ^4 theory in $D = 4$ dimensions, non-linear sigma models in $D = 2$ dimensions, Gross-Neveu models in $D = 2$ dimensions, gauge theories in $D = 4$ dimensions, QED and QCD in $D = 4$ dimensions and Higgs models in $D = 4$ dimensions.

The reader will note that the theories we mentioned are renormalizable at their critical dimensions where their coupling constants are dimensionless. We will now see why this is the case. To make the argument concrete we will examine perturbative ϕ^4 field theory and its divergent Feynman diagrams. There are two types of divergent diagrams, such as the ones are shown in Fig.13.6.

On the other hand, at higher orders in perturbation theory, we can find more complex divergent diagrams, such as those shown in Fig.13.7. However, as we can see, this class of diagrams result from insertions of lower order diagrams into themselves.

Divergent diagrams that do not arise from lower order insertions, i.e. those of Fig.13.6, are said to be *primitively divergent*. In ϕ^4 theory in $D = 4$ dimensions these are the only primitively divergent diagrams. Clearly, the

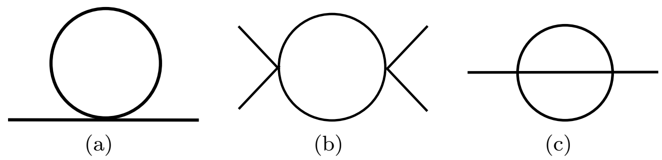


Figure 13.6 Primitively UV divergent diagrams in ϕ^4 theory: a) the tadpole diagram, b) the bubble diagram and c) the watermelon diagram.

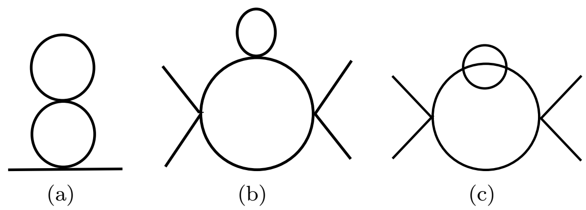


Figure 13.7 Examples of non-primitively UV divergent diagrams in ϕ^4 theory: a) the tadpole insertion in a tadpole diagram, b) tadpole insertion in the bubble diagram and c) watermelon insertion in the bubble diagram.

number and type of primitively divergent diagrams depends on the theory and on the dimension D.

Here when we say that a diagram is divergent we use implicitly its superficial degree of divergence which follows from a simple power counting argument. Consider for instance a scalar field theory with a ϕ^r interaction. Let us consider the perturbative contributions to the N-point vertex function to n-th order in perturbation theory. As a function of an UV regulator Λ, these contributions scale as $\Lambda^{\delta(r,D,N,n)}$. For a diagram with L internal loops, the quantity $\delta(r,D,N,n)$ must be just the difference between the phase space contribution and the number I of propagators in the integrand of the diagram. Thus, we must have

$$\delta(r,D,N,n) = LD - 2I, \quad \text{where} \quad L = I - (n-1) \tag{13.79}$$

On the other hand, in a diagram for the N point function the propagator lines must either connect the internal vertices with themselves or with the external points. Hence,

$$nr = N + 2I \tag{13.80}$$

We then find that the superficial degree of divergence $\delta(r,D,N,n)$ is

$$\delta(r,D,N,n) = \left(r + D - \frac{ND}{2}\right) - n\delta_r \tag{13.81}$$

where

$$\delta_r = r + D - \frac{rD}{2} \equiv D - \Delta_r \tag{13.82}$$

where we introduced the scaling dimension $\Delta_r = r\Delta_\phi = r(D-2)/2$ of the operator ϕ^r in dimension D, defined in Section 13.4.1. We now recognize the quantity δ_r as giving the units of the coupling constant λ_r of the operator ϕ^r.

This dimensional analysis tells us that the superficial degree of UV divergence of the diagrams that contribute to a vertex function depends on the canonical scaling dimension Δ_r of the operator ϕ^r. Eq.(13.81) depends linearly with the order n in perturbation theory. We will have three cases:

i) If $\delta(r, D, N, n) < 0$, the superficial degree of UV divergence of the N point function will increase as the order n of perturbation theory increases. Clearly, if the degree of UV divergence increases with the order of perturbation theory, we will have to introduce an infinite number of parameters (couplings) to account for this singular behavior. For this reason, a theory with $\delta_r < 0$ is said to be "non-renormalizable".
ii) Conversely, if $\delta_r > 0$, the superficial degree of UV divergence decreases as the order of perturbation theory increases. In that case the theory is said to be "super-renormalizable".
iii) However, if $\delta_r = 0$ the superficial degree of divergence does not depend on the order n in perturbation theory. However, from Eq.(13.82) we see that $\delta_r = 0$ only if the scaling dimension satisfies $\Delta_r = D$. Hence, the dimension D must be equal to the critical dimension for ϕ^r, where the coupling constant λ_r is dimensionless.

We can now change the question somewhat ask: what vertex functions $\Gamma^{(N)}$ have primitive divergencies in ϕ^r theory at its critical dimension $D_c(r) = 2r/(r-2)$? The answer is those vertex functions such that $\delta(D_c(r)) \geq 0$. This yields the condition $N + D_c(r) - ND_c(r)/2 \geq 0$. Hence, vertex functions $\Gamma^{(N)}$ with $N \leq 2D_c(r)/(D_c(r)-2) = r$ have primitively divergent diagrams. In particular, in ϕ^4 theory in $D = 4$ dimensions, $\Gamma^{(2)}$ and $\Gamma^{(4)}$ vertex functions have, respectively, quadratic and logarithmically primitively divergent diagrams, while in ϕ^6 theory in $D = 3$ dimensions $\Gamma^{(2)}$, $\Gamma^{(4)}$ and $\Gamma^{(6)}$ vertex functions have quadratic, linear and logarithmically divergent primitive diagrams, etc.

The analysis that we have just made only sets the stage for the proof of renormalizability at the critical dimension. In addition a more sophisticated analysis is needed to prove renormalizability, which we will not do here. In a later chapter we will prove the renormalizability of the $O(N)$ non-linear sigma model using a Ward identity.

In this section we have focused exclusively on the UV behavior of a theory of the type of ϕ^4. However, the infrared (IR) is just as interesting and in some sense physically more important. The reason being that IR divergencies signal a breakdown of perturbation theory indicating that the perturbative ground state may be unstable. IR divergencies are suppressed (or, rather, controlled) by a finite renormalized mass, μ_R. However, IR divergencies come back with a vengeance in the massless limit, $\mu_R \to 0$. So, it is natural to inquire how does perturbation theory behave in the IR. It is easy to see that if a theory is non-renormalizable, $\delta_r < 0$, then the vertex functions are finite in the IR. Conversely, super-renormalizable theories, with $\delta_r > 0$, are non-trivial in the IR and their degree of IR divergence increases with the order of perturbation theory. Only in the renormalizable case, for which $\delta_r = 0$, the degree of UV and IR divergence is independent of the order of perturbation theory. In particular, at the critical dimension the theory has logarithmic divergencies which blow up at both the IR and the UV.

In a later chapter, when we discuss the theory of the renormalization group, operators with $\delta < 0$ will be called *irrelevant* operators (in that their effect is negligible in the IR), those with $\delta > 0$ will be called *relevant* operators (they dominate in the IR) and those with $\delta = 0$ will be called *marginal* operators. In the framework of the renormalization group a renormalizable theory is one whose Lagrangian has only marginal and relevant operators.

13.5 Regularization

We now must face the UV divergencies of the Feynman diagrams and devise some procedure to be used to define the theory in the UV. These procedures are known as regularizations. There are in principle many ways to regularize the theory and we will consider a few. As we will see, most regularization procedures break one symmetry or other, such as Lorentz/Euclidean invariance or some internal symmetry, and hence the effects of the regularization must be considered with care.

13.5.1 Momentum Cutoffs

The simplest and most intuitive approach is to define some sort of cutoff procedure which suppresses the contributions from very large momenta to each Feynman diagram. This can be done in (a) momentum space, invoking a UV momentum space cutoff Λ, or (b) in position space, by invoking a short-distance cutoff a. For the sake of definiteness we will consider the one-loop contribution to the one-particle irreducible 4-point function in ϕ^4 theory, shown in Fig. 13.8.

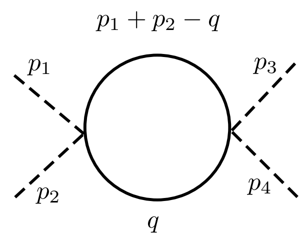


Figure 13.8 Feynman diagram with the leading contribution to the four-point vertex function

We can regularize the diagram of Fig. 13.8 using a sharp momentum cutoff Λ to make the following expression UV finite

$$I_{\text{reg}}(p) = \frac{\lambda^2}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m_0^2)((q-p)^2 + m_0^2)} f_\Lambda(p) + 2 \text{ permutations} \tag{13.83}$$

where $p = p_1 + p_2$ is the momentum transfer, $f_\Lambda(p) = \theta(\Lambda - |p|)$, and Λ is the UV momentum cutoff. Here $\theta(x)$ is the Heaviside (step) function: $\theta(x) = 1$ if $x > 0$, and zero otherwise.

A sharp momentum cutoff is a simple procedure which is useful only at one loop order. However, multi-loop integrals with momentum cutoff are cumbersome analytically, to say the least, and are not compatible with full Lorentz (and Euclidean) invariance. More importantly, the cutoff procedure is not compatible with gauge invariance which makes it ineffective outside the boundaries of theories with only global symmetries.

To improve the analytic tractability other momentum cutoff regularizations are used. Thus, instead of cutting off the momentum integrals at a definite momentum scale Λ, as in Eq.(13.83), smooth cutoff procedures have been introduced. They amount to multiply the integrand of the Feynman diagrams by a smooth cutoff function $f_\Lambda(p)$, such a gaussian regulator, $f_\Lambda(p) = \exp(-p^2/\Lambda^2)$. More common is the use of rational functions, e.g.

$$f_\Lambda(p) = \left(\frac{\Lambda^2}{p^2 + \Lambda^2} \right)^n \tag{13.84}$$

where the integer n is chosen to make the diagram UV finite.

13.5.2 Lattice regularization

Another possible regularization is a lattice cutoff. What this means is to define the (Euclidean) theory on a lattice, normally a hypercubic lattice of dimension D with lattice spacing a. In other words, the Euclidean field theory then becomes identical to a system in classical statistical mechanics. In the case of ϕ^4 theory, the degrees of freedom are real fields defined on the sites of the lattice. Once this is done, the momenta of the fields is the restricted to the first Brillouin zone of a hypercubic lattice which, in the thermodynamic limit where the linear size of the lattice is infinite, defined on the D-dimensional torus $|q_\mu| \leq \pi/a$ (with $\mu = 1, \ldots, D$).

In this approach the bare propagators become

$$G(p, m_0^2) = \frac{1}{\frac{2}{a^2}\sum_{\mu=1}^{D}[1-\cos(q_\mu a)] + m_0^2} \tag{13.85}$$

Much as in the case of a momentum cutoff, to compute the loop integrals (even one-loop!) is a challenging task. In practice, theories regularized on a lattice can be studied numerically using classical Monte Carlo techniques, outside the framework of perturbation theory. On the other hand, the lattice regularization breaks the continuous rotational and translational Euclidean invariance down to the point group symmetry and discrete translation symmetries of the lattice. The recovery of these continuous symmetries can only be achieved by tuning the theory of a critical point where the lattice effects become "irrelevant operators" close enough to a fixed point of the renormalization group, as will be discussed in a later chapter. We should note here that, in contrast with naive cuttoff procedures, lattice regularization is (or can be) compatible with local gauge invariance, as shown by Wilson, Kogut and Susskind. We will discuss this question in a later chapter.

13.5.3 Pauli-Villars regularization

Given the drawbacks of sharp momentum cutoff procedures, other approaches have been devised. A procedure widely applied to the regularization of Feynman diagrams in QED is known as Pauli-Villars. In the Pauli-Villars approach the bare propagator $G_0(p; m_0)$, is replaced by a regularized propagator, $G_0^{\text{reg}}(p; m_0)$, which differs from the bare propagator by enough number of subtractions to render the Feynman diagrams UV finite. This is done by introducing a set of (unobservable) very massive fields, with masses M_i. The regularized propagator is defined to me

$$G_0^{\text{reg}}(p;m_0) = G_0(p;m_0) + \sum_i c_i G_0(p;M_i) \tag{13.86}$$

where $G_0(p;M_i)$ are the propagators of the regulating heavy fields. The coefficients shown in this sum, c_i, are chosen in such a way that the Feynman diagrams are finite in the UV and the regularized propagator be smooth. Let us consider the case of a single heavy field with mass $M \gg m_0$. In order to suppress the strong divergent behavior in the UV one needs to subtract the behavior at large momenta. In this case the regularized propagator is

$$\begin{aligned} G_0^{\text{reg}}(p;m_0) &= G_0(p;m_0) - G_0(p;M) \\ &= \frac{1}{p^2+m_0^2} - \frac{1}{p^2+M^2} \\ &= \frac{M^2-m_0^2}{(p^2+m_0^2)(p^2+M^2)} \end{aligned} \tag{13.87}$$

Hence, the regularized propagator behaves at large momentum as $1/p^4$. With this prescription clearly the one loop diagram of Fig.13.8 is now finite for to $D < 8$ dimensions. However, Eq.(13.87) shows that the propagator of the heavy regulator fields is *negative*. In the Minkowski signature this implies a violation of unitarity. The result is that in a theory regulated *á la* Pauli-Villars it is necessary to prove that unitarity is preserved. Nevertheless, there are theories, such as those with anomalies, Pauli-Villars regularization is the the only practical regularization method left.

13.5.4 Dimensional regularization

The complex analytic structure of multi-loop Feynman diagrams regularized with Pauli-Villars regulators (and, for that matter, with any soft cutoff as well) motivated the introduction of so-called analytic regularization methods. In section 8.7.2 we already encountered one such method, the ζ-function regularization approach for the calculation of functional determinants.

In the case of Feynman diagrams, an early analytic regularization approach consisted in replacing the Euclidean propagators as follows

$$\frac{1}{p^2+m_0^2} \mapsto \lim_{\eta\to 1} \frac{1}{(p^2+m_0^2)^\eta} \tag{13.88}$$

Thus, since the Feynman diagram shown Fig. 13.8 is logarithmically divergent in $D = 4$ dimensions, it becomes finite for any $\eta > 1$. However, this regularization has the serious drawback that it changes the analytic structure of the Feynman diagrams since for any value of η the poles of the propagators are replaced by branch cuts. In fact, the free field theory of such propagators is non-local.

The most powerful and widely used regularization procedure is *dimensional regularization*. In this regularization, introduced in 1972 by Gerard 't Hooft and Martinus Veltman and, independently and simultaneously, by Carlos Bollini and Juan José Giambiagi, the Feynman diagrams are computed in a general dimension $D > D_c$ (where D_c is typically 4) where they are convergent. The resulting expressions are the analytically continued to the complex D plane. In these expressions the UV divergencies are replaced by poles as functions of $D - D_c$. The regularized expressions of Feynman diagrams are obtained by subtracting these poles, a procedure known as *minimal subtraction*.

One key advantage of dimensional regularization is that it is manifestly compatible with gauge invariance. For this reason, dimensional regularization was the key tool in the proof of renormalizability of non-abelian Yang-Mills gauge theories and had, and still does, a huge impact in our understanding of modern day particle physics. Dimensional regularization is also the key analytic tool in high precision computation of critical exponents in the theory of phase transitions.

However, for all its great successes and its great power, dimensional regularization has limitations. By construction it relies on the assumption the the quantities of interest depend smoothly on the dimensionality D and cannot be used for quantities that cannot be continued in dimension. This is a problem in relativistic quantum field theories that involve fermions (Dirac or Majorana). Indeed, while many properties of spinors can be continued in dimension, some cannot. One such problem is chiral symmetry and associated the chiral anomaly which exists only in even space-time dimensions D and whose expression involves the Levi-Civita totally antisymmetric tensor. Similarly, in odd space-time dimensions fermionic theories have parity (or time-reversal) anomalies which also involve the Levi-Civita tensor. The expressions of for these anomalies are specific for a given dimension cannot be unambiguously analytically continued in dimensionality. Consequently, dimensional regularization does not work for such problems.

13.6 Computation of Feynman diagrams in Pauli-Villars regularization

We will now compute the Feynman diagram shown in Fig. 13.8 in different regularization schemes and compare the results. More specifically we will do the computation using a) a Pauli-Villars regularization and b) dimensional regularization.

Let us regularize the one of the terms for the Feynman diagram given in Eq.(13.83) with a cutoff function roughly of the form of Eq.(13.84) with $n = 2$, resulting in the expression

$$I_D^{\text{reg}}(p^2) = \frac{\lambda^2}{2}\int \frac{d^D q}{(2\pi)^D}\frac{1}{(q^2+m_0^2)((q-p)^2+m_0^2)}\left(\frac{\Lambda^2}{q^2+\Lambda^2}\right)\left(\frac{\Lambda^2}{(q-p)^2+\Lambda^2}\right) \tag{13.89}$$

By power counting we see that this regularized expression is finite for dimension $D < 8$. Using partial fractions in pairs of factors, we obtain

$$\begin{aligned} I_D^{\text{reg}}(p^2) =& \frac{\lambda^2}{2} \\ &\times \int \frac{d^D q}{(2\pi)^D}\left[\frac{1}{(q^2+m_0^2)}-\frac{1}{q^2+\Lambda^2}\right]\left[\frac{1}{((q-p)^2+m_0^2)}-\frac{1}{(q-p)^2+\Lambda^2}\right] \end{aligned} \tag{13.90}$$

where we have omitted a prefactor $[\Lambda^2/(\Lambda^2-m_0^2)]^2$ which approaches 1 for $\Lambda \gg m_0$. Hence, we obtained essentially the same expression we would have found with Pauli-Villars regularization with a large regulator mass $M = \Lambda$. Notice that we could have chosen to use instead a single smooth cutoff function $f_\Lambda(p)$ with $n = 1$ which would have rendered the diagram fin ite for $D < 6$. Clearly there is some degree of leeway in how one chooses to regularize the Feynman diagram.

We will now proceed to compute the regularized expression $I_D^{\text{reg}}(p^2)$, shown in Eq.(13.90). To this end, let $A.0$ be a positive real number. Then we introduce the Feynman-Schwinger parameter x through the integral

$$\frac{1}{A} = \frac{1}{2}\int_0^\infty dx\, e^{-Ax/2} \tag{13.91}$$

to raise all the denominator factors in Eq.(13.90) to the argument of exponentials. Since in Eq.(13.90) we have two factors inside the momentum integral we will need to introduce an expression of the form of Eq.(13.91) for each factor, resulting in the expression

$$I_D^{\text{reg}}(p^2) = \frac{\lambda^2}{8}\int_0^\infty dx \int_0^\infty dy \int \frac{d^D q}{(2\pi)^D}\left(e^{-x(q^2+m_0^2)/2} - e^{-x(q^2+\Lambda^2)/2}\right) \times \left(e^{-x((q-p)^2+m_0^2)/2} - e^{-x((q-p)^2+\Lambda^2)/2}\right) \tag{13.92}$$

Using the gaussian integral identity (with $A > 0$)

$$\int \frac{d^D q}{(2\pi)^D}\, e^{-\frac{A}{2}q^2 - B\boldsymbol{q}\cdot\boldsymbol{p}} = \frac{1}{(2\pi A)^{D/2}} e^{+\frac{B}{2A}p^2} \tag{13.93}$$

Eq.(13.92) becomes

$$I_D^{\text{reg}}(p^2) = \frac{\lambda^2}{8}\int_0^\infty dx \int_0^\infty dy \frac{e^{-\frac{xy}{2(x+y)}p^2}}{(2\pi(x+y))^{D/2}}\left(e^{-xm_0^2/2} - e^{-x\Lambda^2/2}\right)\left(e^{-ym_0^2/2} - e^{-y\Lambda^2/2}\right) \tag{13.94}$$

Next, after the change of variables

$$x = uv, \qquad y = (1-u)v, \tag{13.95}$$

with $0 \le v < \infty$ and $0 \le u \le 1$, Eq.(13.92) becomes

$$I_D^{\text{reg}}(p^2) = \frac{\lambda^2}{8(2\pi)^{D/2}}\int_0^1 du \int_0^\infty dv v^{(2-D)/2} e^{-uv(1-u)p^2/2} \times \left(e^{-vm_0^2/2} + e^{-v\Lambda^2/2} - e^{-uvm_0^2/2-(1-u)v\Lambda^2/2} - e^{-uv\Lambda^2/2-(1-u)vm_0^2/2}\right) \tag{13.96}$$

We can now carry out explicitly the integral over the variable v using the result

$$\int_0^\infty dv\; v^{(2-D)/2} e^{-\gamma v} = \gamma^{(D-4)/2}\Gamma(2-D/2) \tag{13.97}$$

where $\Gamma(z)$ is the Euler Gamma-function

$$\Gamma(z) = \int_0^\infty dt\; t^{z-1}e^{-t} \tag{13.98}$$

which is well defined in the domain Re $z > 0$. We obtain

$$I_D^{\rm reg}(p^2) = \frac{\lambda^2}{2}\frac{\Gamma(2-D/2)}{4(2\pi)^{D/2}}\int_0^1 du\,\Bigg\{\Big[\frac{2}{u(1-u)p^2+m_0^2}\Big]^{2-D/2}+\Big[\frac{2}{u(1-u)p^2+\Lambda^2}\Big]^{2-D/2}$$
$$-\Big[\frac{2}{u(1-u)p^2+um_0^2+(1-u)\Lambda^2}\Big]^{2-D/2}-\Big[\frac{2}{u(1-u)p^2+u\Lambda^2+(1-u)m_0^2}\Big]^{2-D/2}\Bigg\} \tag{13.99}$$

In $D = 4$ dimensions the regularized expression for $I_4^{\rm reg}(p^2)$ becomes (for $\Lambda \gg m_0$)

$$I_4^{\rm reg}(p^2) = \frac{\lambda^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{m_0^2}\right)-\frac{\lambda^2}{16\pi^2}-\frac{\lambda^2}{32\pi^2}\int_0^1 du\ \ln\left[1+u(1-u)\frac{p^2}{m_0^2}\right] \tag{13.100}$$

where we exhibited a logarithmically divergent part separate from a finite part. This regularized result is manifestly rotationally invariant in Euclidean space and Lorentz invariant when continued to Minkowski spacetime.

A similar expression is found for the massless case, $m_0 \to 0$,

$$I_4^{\rm reg}(p^2) = \frac{\lambda^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)-\frac{\lambda^2}{16\pi^2}-\frac{\lambda^2}{32\pi^2}\int_0^1 du\ \ln\left[u(1-u)\frac{p^2}{\mu^2}\right] \tag{13.101}$$

where μ is an *arbitrary* mass scale needed for dimensional reasons. Upon doing the integral left in Eq.(13.101) we find the simpler result for the massless $m_0 = 0$ case

$$I_4^{\rm reg}(p^2) = \frac{\lambda^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{p^2}\right) \tag{13.102}$$

This result shows that in the massless case, there is an infrared logarithmic divergence as $p \to 0$. In both cases, massive and massless, the UV regulator Λ enters in an essential and singular way.

13.7 Computation of Feynman diagrams with dimensional regularization

We will now compute the same Feynman diagram, of Eq.(13.83), but using dimensional regularization. Thus, as we explained above, we will compute the Feynman diagram without any explicit UV regularization as a function of the dimension of the Euclidean spacetime D. We will seek the domain of dimensions D for which the diagram is finite and define its value at the dimension of physical interest, say $D = 4$, by means of an analytic continuation. As we will see, the UV singular behavior will appear in the form of poles in the dependence on the dimension D, regarded as a complex variable.

13.7.1 A one-loop example

The expression for the Feynman diagram of Eq.(13.83) in general dimension D is obtained by setting the regulator $\Lambda \to \infty$ in Eq.(13.99). The result is

$$I_D(p^2) = \frac{\lambda^2}{2} \frac{\Gamma(2 - D/2)}{4(2\pi)^{D/2}} \int_0^1 du \left[\frac{2}{u(1-u)p^2 + m_0^2} \right]^{2-D/2} \tag{13.103}$$

Except for the singularities of the Gamma function, this expression is finite.

The Euler Gamma function, $\Gamma(z)$, is a function which is analytic in the domain $\text{Re } z > 0$. In the complex plane the Gamma function has simple poles at the negative integers, $z = -n$, with $n \in \mathbb{N}$, and $z = 0$. More explicitly, we can use the Weierstrass representation of the Gamma function

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t} = \int_0^1 dt\, t^{z-1} e^{-t} + \int_1^\infty t^{z-1} e^{-t} \tag{13.104}$$

The second integral is clearly finite since the integration range does not reach down to $t = 0$. On the other hand, the first integral has a finite integration range, the interval $(0, 1)$ and hence we can expand the exponential in its power series expansion and integrate term by term. The result is the Weierstrass representation

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \Gamma_{\text{reg}}(z) \tag{13.105}$$

where $\Gamma_{\text{reg}}(z)$, the regularized Gamma function, is

$$\Gamma_{\text{reg}}(z) = \int_1^\infty t^{z-1} e^{-t} \tag{13.106}$$

Thus, the Weierstrass representation of Eq.(13.105), expresses the Gamma function as a sum of a regularized function $\Gamma_{\text{reg}}(z)$, and a series of simple poles on the negative real axis at the negative integers and zero. It also tells us how to analytically continue the Gamma function from the domain $\text{Re } z > 0$ to the complex plane $\mathbb{C}$.

Furthermore, in the vicinity of its leading pole, at $z = 0$, the Gamma function has the asymptotic behavior

$$\Gamma(z) = \frac{1}{z} - \gamma + O(z), \qquad \text{as } z \to 0 \tag{13.107}$$

where

$$\gamma = \lim_{n \to \infty} \left(-\ln n + \sum_{k=1}^{n} \frac{1}{k} \right) = 0.5772 \ldots \tag{13.108}$$

is the Euler-Mascheroni constant.

Armed with these results, we can write the expression of Eq.(13.103), and after setting $\epsilon = 4 - D$, in the form

$$I_D(p) = \frac{\lambda^2}{2} \frac{\mu^{-\epsilon}}{4(2\pi)^{D/2}} 2^{\epsilon/2} \Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 du \left[\frac{\mu^2}{u(1u)p^2 + m_0^2} \right]^{\epsilon/2} \tag{13.109}$$

where, once again, μ is an arbitrary mass scale.

In the limit $D \to 4$ dimensions, Eq.(13.109) becomes

$$I_D(p) = \frac{\lambda^2}{32\pi^2} \left[\frac{2}{\epsilon} + \ln(4\pi) - \gamma + \int_0^1 du \, \ln\left(\frac{\mu^2}{u(1-u)p^2 + m_0^2} \right) \right] \tag{13.110}$$

Thus, the bare expression of this Feynman diagram is split into a singular term, with a pole in ϵ, and a finite regular term in $D = 4$ dimensions. Dimensional regularization defines the value of this Feynman diagram as this expression with the pole term subtracted. Clearly, one could have also subtracted also some piece of the finite term and that prescription would have been equally correct. The procedure of subtracting just the contribution of the pole is known as Dimensional Regularization with Minimal Subtraction (or MS).

In the massless limit, $m_0 \to 0$, Eq.(13.110) becomes

$$I_D(p) = \frac{\lambda^2}{32\pi^2} \left[\frac{2}{\epsilon} + \ln(4\pi) - \gamma + 2 + \ln\left(\frac{\mu^2}{p^2} \right) \right] \tag{13.111}$$

which should be compared with the same result using Pauli-Villars regularization, Eq.(13.102). Clearly, the factor $\ln(\Lambda^2/\mu^2)$ in the Pauli-Villars result corresponds to the pole $2/\epsilon$ in dimensional regularization. The same comparison holds for the massive case. Notice, however, that the *finite parts* are *different* in different regularizations.