Finite and infinite basis GPs

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From infinite linear models to Gaussian processes

Consider the class of functions (sums of squared exponentials):

$$\begin{aligned} f(x) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=-N/2}^{N/2} \gamma_n \exp(-(x - \tfrac{n}{\sqrt{N}})^2), \;\text{ where }\; \gamma_n \sim \mathcal{N}(0,1),\ \forall n \\ &= \int_{-\infty}^{\infty} \gamma(u) \exp(-(x-u)^2)\,du, \;\text{ where }\; \gamma(u) \sim \mathcal{N}(0,1),\ \forall u. \end{aligned}$$

The mean function is:

$$\mu(x) \;=\; \mathrm{E}[f(x)] \;=\; \int_{-\infty}^{\infty} \exp(-(x-u)^2) \int_{-\infty}^{\infty} \gamma(u) p(\gamma(u)) d\gamma(u) \;\; du \;=\; 0,$$

and the covariance function:

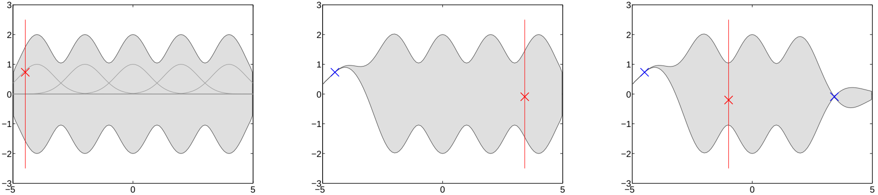
$$\mathrm{E}[f(x)f(x')] \;=\; \int \exp\big(-(x-u)^2 - (x'-u)^2\big)du$$

$$= \int \exp\big(-2(u - \frac{x+x'}{2})^2 + \frac{(x+x')^2}{2} - x^2 - x'^2\big)du \;\propto\; \exp\big(-\frac{(x-x')^2}{2}\big).$$

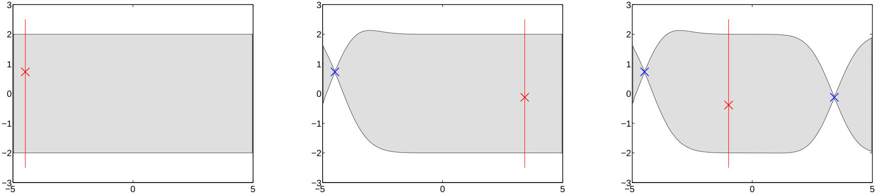
Thus, the squared exponential covariance function is equivalent to regression using infinitely many Gaussian shaped basis functions placed everywhere, not just at your training points!

Using finitely many basis functions may be dangerous!(1)

Finite linear model with 5 localized basis functions)

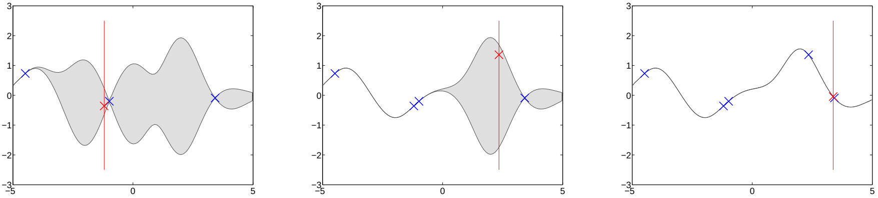


Gaussian process with infinitely many localized basis functions

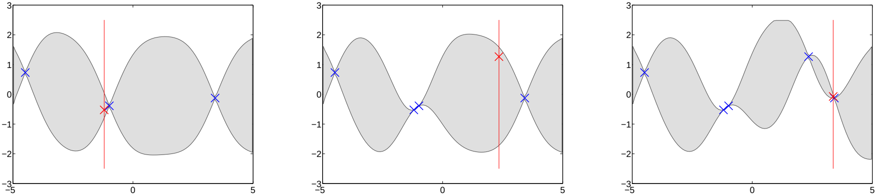


Using finitely many basis functions may be dangerous!(2)

Finite linear model with 5 localized basis functions)

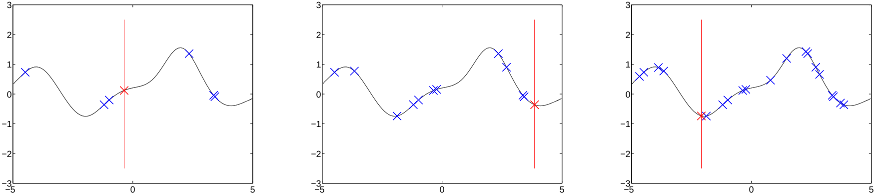


Gaussian process with infinitely many localized basis functions



Using finitely many basis functions may be dangerous!(3)

Finite linear model with 5 localized basis functions)



Gaussian process with infinitely many localized basis functions