(Multivariate) Gaussian (Normal) Probability Densities

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Gaussian Density

The probability density of a D-dimensional Gaussian with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ is given by

$$p(\mathbf{x}|\boldsymbol{\mu},\Sigma) \;=\; \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\Sigma) \;=\; \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\big(-\tfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\big),$$

and we also write

$$\mathbf{x}|\boldsymbol{\mu},\Sigma \;\sim\; \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\Sigma).$$

The covariance matrix Σ must be symmetric and positive definite.

In the special (scalar) case where $D = 1$ we have

$$p(x|\mu,\sigma^2) \;=\; \frac{1}{\sqrt{2\pi\sigma^2}} \exp\big(-\tfrac{1}{2}(x-\mu)^2/\sigma^2\big),$$

where σ^2 is the variance and σ is the standard deviation.

The *standard* Gaussian has $\boldsymbol{\mu} = \mathbf{0}$ and $\Sigma = I$ (the unit matrix).

Parametrisation

There are two commonly used parametrisations of Gaussians

- *standard* parametrisation:
  - *mean* $\boldsymbol{\mu}$ and
  - *covariance* Σ
- *natural* parametrisation:
  - *natural mean* $\boldsymbol{\nu} = \Sigma^{-1}\boldsymbol{\mu}$ and
  - *precision* matrix $\mathsf{R} = \Sigma^{-1}$.

Different operations are more convenient in either parametrisation.

Gaussian Pictures

Gaussian Properties

Gaussians are closed both under marginalisation and conditioning.
If $\mathbf{x}$ and $\mathbf{y}$ are jointly Gaussian

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

Kullback-Leibler Divergence (Relative Entropy)

The Kullback-Leibler (KL) divergence between continuous distributions is

$$\mathcal{KL}(q(x)\|p(x)) \;=\; \int q(x)\log\frac{q(x)}{p(x)}dx.$$

The KL divergence is an asymmetric measure of distance between distributions.
The KL divergence between two Gaussians is

$$\mathcal{KL}(\mathcal{N}_0\|\mathcal{N}_1) \;=\; \tfrac{1}{2}\log|\Sigma_1\Sigma_0^{-1}| + \tfrac{1}{2}\operatorname{tr}\big(\Sigma_1^{-1}\big((\mu_0-\mu_1)(\mu_0-\mu_1)^\top+\Sigma_0-\Sigma_1\big)\big).$$

KL matching constrained Gaussians

It is often convenient to approximate one distribution with another, simpler one, by finding the *closest match* within a constrained family.

Minimizing KL divergence between a general Gaussian $\mathcal{N}_g$ and a factorized Gaussian $\mathcal{N}_f$ will match the means $\boldsymbol{\mu}_f = \boldsymbol{\mu}_g$ and for the covariances either:

$$\frac{\partial \mathcal{KL}(\mathcal{N}_f \| \mathcal{N}_g)}{\partial \Sigma_f} = -\tfrac{1}{2}\Sigma_f^{-1} + \tfrac{1}{2}\Sigma_g^{-1} = 0 \;\Rightarrow\; (\Sigma_f)_{ii} = 1/(\Sigma_g^{-1})_{ii},$$

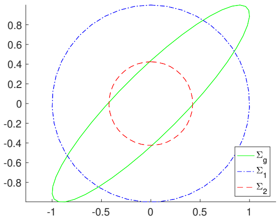
or

$$\frac{\partial \mathcal{KL}(\mathcal{N}_g \| \mathcal{N}_f)}{\partial \Sigma_f} = \tfrac{1}{2}\Sigma_f^{-1} - \tfrac{1}{2}\Sigma_f^{-1}\Sigma_g\Sigma_f^{-1} = 0 \;\Rightarrow\; (\Sigma_f)_{ii} = (\Sigma_g)_{ii}.$$

Interpretation:

- averaging wrt the *factorized* Gaussian, the fitted variance equals the *conditional* variance of Σ_g,
- averaging wrt the *general* Gaussian, the fitted variance equals the *marginal* variance of Σ_g,

with straight forward generalization to block diagonal Gaussians.



Appendix: Some useful Gaussian identities

If x is multivariate Gaussian with mean μ and covariance matrix Σ

$$p(\mathbf{x}; \mu, \Sigma) \;=\; (2\pi|\Sigma|)^{-D/2} \exp\big(-(\mathbf{x}-\mu)^\top \Sigma^{-1} (\mathbf{x}-\mu)/2\big),$$

then

$$\begin{aligned} \mathbb{E}[\mathbf{x}] &= \mu, \\ \mathbb{V}[\mathbf{x}] &= \mathbb{E}[(\mathbf{x}-\mathbb{E}[\mathbf{x}])^2] \;=\; \Sigma. \end{aligned}$$

For any matrix A, if $\mathbf{z} = A\mathbf{x}$ then $\mathbf{z}$ is Gaussian and

$$\begin{aligned} \mathbb{E}[\mathbf{z}] &= A\mu, \\ \mathbb{V}[\mathbf{z}] &= A\Sigma A^\top. \end{aligned}$$

Matrix and Gaussian identities cheat sheet

Matrix identities

- Matrix inversion lemma (Woodbury, Sherman & Morrison formula)

$$(Z + UWV^\top)^{-1} = Z^{-1} - Z^{-1}U(W^{-1} + V^\top Z^{-1}U)^{-1}V^\top Z^{-1}$$

- A similar equation exists for determinants

$$|Z + UWV^\top| = |Z|\,|W|\,|W^{-1} + V^\top Z^{-1}U|$$

The product of two Gaussian density functions

$$\mathcal{N}(\mathbf{x}|\mathbf{a}, A)\,\mathcal{N}(P^\top \mathbf{x}|\mathbf{b}, B) = z_c\,\mathcal{N}(\mathbf{x}|\mathbf{c}, C)$$

- is proportional to a Gaussian density function with covariance and mean

$$C = \left(A^{-1} + P\,B^{-1}P^\top\right)^{-1} \qquad\qquad \mathbf{c} = C\,\left(A^{-1}\mathbf{a} + P\,B^{-1}\,\mathbf{b}\right)$$

- and has a normalizing constant z_c that is Gaussian both in **a** and in **b**

$$z_c = (2\,\pi)^{-\frac{m}{2}}|B + P^\top A\,P|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\mathbf{b} - P^\top\,\mathbf{a})^\top \left(B + P^\top A\,P\right)^{-1}(\mathbf{b} - P^\top\,\mathbf{a})\right)$$