An Equation

Let x be a real number and $frac(x) = x - [|x|]$ (fractional part of x) where $[|x|]$ is the greatest integer function. For example: $frac(5.87) = 0.87$ or $frac(\frac{7}{5}) = \frac{2}{5}$ or $frac(-\frac{1}{3}) = \frac{2}{3}$.

Find a positive real number x such that $frac(x) + frac(\frac{1}{x}) = 1$.

SUBMIT your solution to

- Dr. Yuanhui Xiao @ matyxx@langate.gsu.edu **or**
- Dr. Erol Akbas @ matexa@langate.gsu.edu

before the **deadline**: **Friday, January 29, 2010, 5:00PM**.

You may get a copy of this problem from **the wall behind you**.

Problem of Last Month: Evaluation of Function Value

Let $f(x)$ be a function such that

$$f(x+y) + f(x-y) = 2(f(x) + f(y)), \quad \textit{and} \quad f(1) = 2.$$

Find $f(17/11)$.

Winner: Reimbay Reiimbayev.

Participant(s) with Correct Solution: Terresa Nguyen, Vivek Shah, Tsu-Way Tseng.

Solution. (Provided by Dr. Hossein Andikfar). We show that for $r \in \boldsymbol{Q}$, $f(r) = 2r^2$. By setting $x = y = 0$ we get $2f(0) = 4f(0)$ which implies $f(0) = 0$. Also

$$f(x-y) = 2[f(x) + f(y)] - f(x+y) = 2[f(y) + f(x)] - f(y+x) = f(y-x).$$

So, by setting $y = 0$ we get $f(x) = f(-x)$, $\forall x$. Now, by induction we prove that $f(nx) = n^2 f(x)$. The case of $n = 1$ is trivial, and for $n > 1$ we set $y = (n-1)x$. Then

$$\begin{aligned}
f(nx) + f((2-n)x) &= 2f(x) + 2f((n-1)x) \\
f(nx) + f((n-2)x) &= 2f(x) + 2f((n-1)x) \\
f(nx) + (n-2)^2 f(x) &= 2f(x) + 2(n-1)^2 f(x) \\
f(nx) &= [2(n-1)^2 + 2 - (n-2)^2] f(x) \\
f(nx) &= n^2 f(x).
\end{aligned}$$

Finally, in this equation we set $x = 1/n$ we get

$$f(1) = n^2 f(1/n) \Rightarrow 2 = n^2 f(1/n) \Rightarrow f(1/n) = 2/n^2.$$

So

$$f(m/n) = f(m \cdot 1/n) = m^2 f(1/n) = m^2 \cdot 2/n^2 = 2(m/n)^2.$$

Therefore,

$$f(17/11) = 2(17/11)^2 = 578/121.$$