Problem of the Month — November 2009

Let $f(x)$ be a function such that

$$f(x+y) + f(x-y) = 2(f(x) + f(y)), \quad \text{and} \quad f(1) = 2.$$

Find $f(17/11)$.

Deadline: November 30, 2009, 5:00pm.

- You may get a copy of this from the wall behind you.
- Submit your solution to
  - Dr. Yuanhui Xiao @ *matyxx@langate.gsu.edu* or
  - Dr. Erol Akabs @ *matexa@langate.gsu.edu*,
  - or drop a hard copy in their mailbox before the deadline.

Problem of October:

Duel: Three men, Fermat, Galois, and Hilbert, decide to fight a pistol duel. Theyll stand at the corners of an equilateral triangle, and each man, in order, will aim and shoot wherever he pleases. They choose randomly who will be shooting first, second, and third, and will continue in that order until two of them are dead. All three know that Fermat always hits his target 90% accurate, Galois is 70% accurate, and Hilbert hits his mark half the time. Assuming that all three adopt the best strategy and that nobody is killed by a wild shot not intended for him, who has the best chance to survive, and why? Also, find the survival probabilities for each man.

Solution: To solve this problem. We need introduce several notations. The letters F, G, H will denote the shooters. $P(A)$ denotes the probability that shooter A hits his target, and $P(A^c) = 1 - P(A)$. $D(A|B)$ denotes the probability that shooter A will die if B shoots first, $D_A(B|C)$ denotes the probability that A shooter will die if C shoots just after A is killed. To have higher survival rate, any shooter will shoot the better shooter first once he has the chance. With this in mind, we can build the following three systems of linear equations:

SL1:

$$\begin{aligned}
D(F|F) &= P(F) \times D_G(F|H) + P(F^c) \times (D(F|G) + D(F|H))/2, \\
D(F|G) &= P(G) + P(G^c) \times (D(F|F) + D(F|H))/2, \\
D(F|H) &= P(H) + P(H^c) \times (D(F|F) + D(F|G))/2;
\end{aligned}$$

SL2:

$$\begin{aligned}
D(G|F) &= P(F) + P(F^c) \times (D(G|G) + D(G|H))/2, \\
D(G|G) &= P(G) \times D_F(G|H) + P(G^c) \times (D(G|F) + D(G|H))/2, \\
D(G|H) &= P(H) \times D_F(G|G) + P(H^c) \times (D(G|F) + D(G|G))/2;
\end{aligned}$$

SL3:

$$\begin{aligned}
D(H|F) &= P(F) \times D_G(H|H) + P(F^c) \times (D(H|G) + D(H|H))/2, \\
D(H|G) &= P(G) \times D_F(H|H) + P(G^c) \times (D(H|F) + D(H|H))/2, \\
D(H|H) &= P(H) \times D_F(H|G) + P(H^c) \times (D(H|F) + D(H|G))/2;
\end{aligned}$$

The three systems are independent of each other, and they can be solved separately. The three systems of equations have the same form:

$$\begin{bmatrix} 1 & -P(F^c) & -P(F^c) \\ -P(G^c) & 1 & -P(G^c) \\ -P(H^c) & -P(H^c) & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

where $x_1,\ x_2,\ x_3$ are the probabilities that the same shooter will die for three cases: 1). F shoots first, 2). G shoots first, 3) H shoots first, respectively. (Thus, the probability that the shooter will survive is $1-(x_1+x_2+x_3)/3$.) The right side is different for the three systems.

For SL1

$$\begin{bmatrix} b_1 = & P(F) \times D_G(F|H) \\ b_2 = & P(G) \\ b_3 = & P(H) \end{bmatrix}.$$

For SL2,

$$\begin{bmatrix} b_1 = & P(F) \\ b_2 = & P(G) \times D_F(G|H) \\ b_3 = & P(H) \times D_F(G|G) \end{bmatrix}.$$

For SL3,

$$\begin{bmatrix} b_1 = & P(F) \times D_G(H|H) \\ b_2 = & P(G) \times D_F(H|H) \\ b_3 = & P(H) \times D_F(H|G) \end{bmatrix}.$$

To compute b_i's, we need know the probabilities $D_G(F|H)$, $D_F(G|H)$, etc. Obviously, this is the situation when there are two shooters. These probabilities are computed as follows.

Let $D(A|AB)$ denotes the probability that A will die when A shoots first and $D(A|BA)$ the probability that A will die when B shoots first, then we have

$$\begin{aligned} D(A|AB) &= D(A|BA)P(A^c) \\ D(A|BA) &= D(A|AB)P(B^c) + P(B). \end{aligned}$$

So,

$$\begin{aligned} D(A|AB) &= \frac{P(A^c)P(B)}{1 - P(A^c)P(B^c)} \\ D(A|BA) &= \frac{P(B)}{1 - P(A^c)P(B^c)}. \end{aligned}$$

The computation is better to be done in a computer. After a tedious calculation, we know the chance for F to survive is 0.2208, that for G is 0.3131, and that for H is .4731. (Please note, the three probabilities sum to one!) So, the best shooter has least chance to survive. This is understandable since he is the target of both other shooters. The least accurate shooter has the best chance to survive. This result may be surprising. However, this is reasonable since the other two shooters will not choose him as the target before one shooter is killed.

Winner: No winner this time. Only Mr. Robert Xu submitted his answer. Unfortunately, his answer is incorrect.