Supplementary Material for Doubly Robust Estimation and Sensitivity Analysis for Marginal Structural Quantile Models by Chao Cheng, Liangyuan Hu, and Fan Li

This supplementary material provides proofs of main results and web tables and figures unshown in the paper.

Appendix A: Interpreting the causal parameter under a misspecified MSQM

Our MSQM parameter, as defined in equation (2) of the main manuscript, can be interpreted as a nonparametric causal effect under the nonparametric marginal structural model framework developed in Neugebauer & van der Laan (2007). To see why, we recall that our parameter of interest is the solution to equation (2), denoted by $\boldsymbol{\theta}_{q0}$ hereafter, which is treated as the true value of $\boldsymbol{\theta}_q$. We first note that $\boldsymbol{\theta}_{q0}$ can be equivalently written as the minimum of the following quantile loss function:

$$\begin{aligned}
\boldsymbol{\theta}_{q0} &= \underset{\boldsymbol{\theta}_q}{\operatorname{argmin}}\ E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} \rho(\overline{a}_K,\boldsymbol{Z})\cdot\delta_q(Y_{\overline{a}_K}-h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q))\right] \\
&=: \underset{\boldsymbol{\theta}_q}{\operatorname{argmin}}\ Q_{\text{loss}}(\boldsymbol{\theta}_q) \qquad \text{(s1)}
\end{aligned}$$

where $\delta_q(x)=x(q-\mathbb{I}(x\le 0))$ is the check function for the q-quantile (Koenker & Bassett 1978). The equivalence between the solution of (2) and the minimum of $Q_{\text{loss}}(\boldsymbol{\theta}_q)$ holds because

$$\begin{aligned}
&\frac{\partial Q_{\text{loss}}(\boldsymbol{\theta}_q)}{\partial\boldsymbol{\theta}_q} \\
=&\frac{\partial}{\partial\boldsymbol{\theta}_q}E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}\rho(\overline{a}_K,\boldsymbol{Z})\left\{q-\mathbb{I}(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q))\right\}\left\{Y_{\overline{a}_K}-h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\right\}\right] \\
=&\frac{\partial}{\partial\boldsymbol{\theta}_q}E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}\rho(\overline{a}_K,\boldsymbol{Z})\left\{h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\left(\mathbb{I}(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q))-q\right)-Y_{\overline{a}_K}\mathbb{I}(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q))\right\}\right] \\
=&\frac{\partial}{\partial\boldsymbol{\theta}_q}E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}\rho(\overline{a}_K,\boldsymbol{Z})\left\{h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\left(F_{Y_{\overline{a}_K}|\boldsymbol{Z}}\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)|\boldsymbol{Z}\right)-q\right)-\int_{-\infty}^{h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}uf_{Y_{\overline{a}_K}|\boldsymbol{Z}}(u|\boldsymbol{Z})du\right\}\right] \\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}\rho(\overline{a}_K,\boldsymbol{Z})\frac{\partial h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\partial\boldsymbol{\theta}_q}\left\{F_{Y_{\overline{a}_K}|\boldsymbol{Z}}\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)|\boldsymbol{Z}\right)-q\right\}\right] \\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}\rho(\overline{a}_K,\boldsymbol{Z})\frac{\partial h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\partial\boldsymbol{\theta}_q}\left\{\mathbb{I}(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q))-q\right\}\right]
\end{aligned}$$

=the left-hand side of equation (2) in the main manuscript.

We next show that the above minimization problem can be further transformed into minimizing the norm between the true causal quantile curve and the working MSQM, with the norm defined by the choice of $\rho(\overline{a}_K, \boldsymbol{Z})$ in a Hilbert space. To arrive at such a result, recall that when a linear quantile regression is misspecified, Angrist et al. (2006) showed that quantile regression coefficients still minimizes a weight sum of squared specification errors of the working linear model. Below, we generalize their result to the moment condition developed for MSQM. Specifically, we define the MSQM specification error as

$$\Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) = h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) - Q^{(q)}_{Y_{\overline{a}_K}|\boldsymbol{Z}},$$

which measures the difference between the MSQM and the true quantile causal curve. When the MSQM is misspecified, the following Proposition states that our parameter of interest, $\boldsymbol{\theta}_{q0}$, still minimizes a weighted sum of squared specification errors.

Proposition 1. *Suppose that (i) $f_{Y_{\overline{a}_K}|\boldsymbol{Z}}(y|\boldsymbol{Z})$ exists and is bounded almost surely, (ii) $E[Y^2_{\overline{a}_K}]$, $E[\rho(\overline{a}_K, \boldsymbol{Z})^2]$, $E[h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)^2]$, and $E\left[\frac{\partial h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\partial \boldsymbol{\theta}_q}\left\{\frac{\partial h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\partial \boldsymbol{\theta}_q}\right\}^T\right]$ are finite over all $\overline{a}_K \in \overline{\mathbb{A}}_K$ and $\boldsymbol{\theta}_q$ in its valid support; (iii) $\Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)^2$ is a convex function of $\boldsymbol{\theta}_q$; (iv) the solution of* (s1) *is unique. Then, $\boldsymbol{\theta}^*_{q0} = \boldsymbol{\theta}_{q0}$ uniquely solves the following equation*

$$\boldsymbol{\theta}^*_{q0} = \mathop{argmin}_{\boldsymbol{\theta}_q} E\left[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} \omega(\overline{a}_K, \boldsymbol{Z}, \boldsymbol{\theta}^*_{q0}) \Delta^2(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)\right], \tag{s2}$$

where

$$\omega(\overline{a}_K, \boldsymbol{Z}, \boldsymbol{\theta}^*_{q0}) = \frac{1}{2}\rho(\overline{a}_K, \boldsymbol{Z}) \int_0^1 f_{Y_{\overline{a}_K}|\boldsymbol{Z}}\left(u h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}^*_{q0}) + (1-u) Q^{(q)}_{Y_{\overline{a}_K}|\boldsymbol{Z}}\middle|\boldsymbol{Z}\right) du \geq 0$$

is a nonnegative weighting function.

Akin to Theorem 2 in Angrist et al. (2006), Proposition 1 indicates that the probability limit of the misspecified MSQM parameters, $\boldsymbol{\theta}_{q0}$, minimizes a weighted sum of squared specification errors—the squared difference between the true quantile of the potential outcome and an approximation given by the MSQM, with a weighting function $\omega(\overline{a}_K, \boldsymbol{Z}, \boldsymbol{\theta}_{q0})$ depending on the $\overline{a}_K$, $\boldsymbol{Z}$, and the true value of the MSQM parameters $\boldsymbol{\theta}_{q0}$ (rather than the MSQM parameters $\boldsymbol{\theta}_q$). In other words, $\boldsymbol{\theta}_{q0}$ is a unique fixed point to an iterated minimum distance approximation. Based on this representation, we can define $F_{\rho,\boldsymbol{\theta}_{q0}}$ such that $dF_{\rho,\boldsymbol{\theta}_{q0}}(\overline{a}_K, \boldsymbol{Z}) = \mathbb{I}(\overline{a}_K \in \overline{\mathbb{A}}_K)\omega(\overline{a}_K, \boldsymbol{Z}, \boldsymbol{\theta}_{q0}) dF_{\boldsymbol{Z}}(\boldsymbol{Z})$. Then the space of functions endowed with the inner product

$$\langle \lambda_1, \lambda_2 \rangle_{F_{\rho,\boldsymbol{\theta}_{q0}}} = \int_{\boldsymbol{Z}} \sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} \lambda_1(\overline{a}_K, \boldsymbol{Z}) \lambda_2(\overline{a}_K, \boldsymbol{Z}) dF_{\rho,\boldsymbol{\theta}_{q0}}(\overline{a}_K, \boldsymbol{Z})$$

is a Hilbert space with the norm $\|\lambda\|_{F_{\rho,\boldsymbol{\theta}_{q0}}} = \sqrt{\langle \lambda, \lambda \rangle_{F_{\rho,\boldsymbol{\theta}_{q0}}}}$. Then we can further rewrite (s2) as

$$\boldsymbol{\theta}_{q0} = \mathop{\mathrm{argmin}}_{\boldsymbol{\theta}_q} \int_{\boldsymbol{Z}} \sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} \left\{h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) - Q^{(q)}_{Y_{\overline{a}_K}|\boldsymbol{Z}}\right\}^2 \omega(\overline{a}_K, \boldsymbol{Z}, \boldsymbol{\theta}_{q0}) dF_{\boldsymbol{Z}}(\boldsymbol{Z}), \tag{s3}$$

$$= \underset{\boldsymbol{\theta}_q}{\operatorname{argmin}} \left\| h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) - Q^{(q)}_{Y_{\overline{a}_K}|\boldsymbol{Z}} \right\|_{F_{\rho,\boldsymbol{\theta}_{q0}}}. \tag{s4}$$

Equation (s4) is the final representation that sheds light on the nonparametric interpretation of the structural parameter $\boldsymbol{\theta}_{q0}$; that is, it minimizes the distance between the quantile causal curve and the working MSQM within the Hilbert space equipped with the norm $\|\lambda\|_{F_{\rho,\boldsymbol{\theta}_{q0}}}$. On a closer inspection, the norm of this space depends on $\rho(\overline{a}_K, \boldsymbol{Z})$, which is referred to as the causal kernel smoother in Neugebauer & van der Laan (2007). This causal kernel smoother determines the weighting scheme attached to each region of the causal curve defined by regimen $\overline{a}_K$ and covariate space $\boldsymbol{Z}$; the regions with higher weights will then be those we focus the investigation of the quantile causal curve. Different from Neugebauer & van der Laan (2007), the norm additionally depends on the conditional density $f_{Y_{\overline{a}_K}|\boldsymbol{Z}}$ and true value of the MSQM parameters $\boldsymbol{\theta}_{q0}$, owing to the unique definition of the loss function. But since these are true (albeit unknown) quantities that defines the norm, they do not affect the interpretation of the minimizer in equation (s4). In fact, one can view $\omega(\overline{a}_K, \boldsymbol{Z}, \boldsymbol{\theta}_{q0})$ as the generalized causal kernel smoother. Finally, when the specification error $\Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})$ is small, by Angrist et al. (2006), we have a more interpretable approximation to the generalized causal kernel smoother as

$$\omega(\overline{a}_K, \boldsymbol{Z}, \boldsymbol{\theta}_{q0}) \approx \frac{1}{4}\rho(\overline{a}_K, \boldsymbol{Z}) f_{Y_{\overline{a}_K}|\boldsymbol{Z}} \left(Q^{(q)}_{Y_{\overline{a}_K}|\boldsymbol{Z}} \middle| \boldsymbol{Z} \right),$$

which modifies the usual causal kernel smoother by the conditional outcome density evaluated at the true causal quantile. In summary, $\boldsymbol{\theta}_{q0}$ parameterizes the true quantile causal curve when the MSQM is correctly specified. When the the MSQM is misspecified, $\boldsymbol{\theta}_{q0}$ is the parameter for which $h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)$ best approximates the true quantile causal curve $Q^{(q)}_{Y_{\overline{a}_K}|\boldsymbol{Z}}$ in regions identified by the generalized causal kernel smoother. To this end, $\boldsymbol{\theta}_{q0}$ remains a localized and approximate summary measure of the quantile causal curve under a misspecified MSQM.

For completeness, we provide the proof for Proposition 1 below.

Proof of Proposition 1. The proof uses the same technique to that in Theorem 2 of Angrist et al. (2006). Define $\epsilon_{\overline{a}_K} = Y_{\overline{a}_K} - Q^{(q)}_{Y_{\overline{a}_K}|\boldsymbol{Z}}$. By the law of iterated expectation, the solution of (2) is equivalent to the solution to the following equation

$$E\left[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} E\left[d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) \left\{ \mathbb{I}(\epsilon_{\overline{a}_K} \le \Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)) - q \right\} | \boldsymbol{Z} \right] \right] = 0,$$

where conditions (i) and (ii) ensure the existence of expectation. Next, we show that

$$\begin{aligned}
&E\left[d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) \left\{ \mathbb{I}(\epsilon_{\overline{a}_K} \le \Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)) - q \right\} | \boldsymbol{Z} \right] \\
=& d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) \left\{ F_{\epsilon_{\overline{a}_K}|\boldsymbol{Z}} \left(\Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) | \boldsymbol{Z} \right) - F_{\epsilon_{\overline{a}_K}|\boldsymbol{Z}}(0|\boldsymbol{Z}) \right\} \\
=& d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) \int_0^{\Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)} f_{\epsilon_{\overline{a}_K}|\boldsymbol{Z}}(u|\boldsymbol{Z}) du \\
=& d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) \int_0^1 f_{\epsilon_{\overline{a}_K}|\boldsymbol{Z}} \left(u\Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) | \boldsymbol{Z} \right) \Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) du
\end{aligned}$$

$$
\begin{aligned}
=&d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)\Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) \int_0^1 f_{Y_{\overline{a}_K}|\boldsymbol{Z}} \left(uh(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}^*_{q0}) + (1-u)Q^{(q)}_{Y_{\overline{a}_K}|\boldsymbol{Z}} \middle| \boldsymbol{Z} \right) du \\
=&2\frac{\partial h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)}{\partial \boldsymbol{\theta}_q} \omega(\overline{a}_K, \boldsymbol{Z}, \boldsymbol{\theta}_q)\Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)
\end{aligned}
$$

where the first equality holds by law of iterated expectation and $F_{\epsilon_{\overline{a}_K}|\boldsymbol{Z}}(0|\boldsymbol{Z}) = F_{Y_{\overline{a}_K}|\boldsymbol{Z}} \left(Q^{(q)}_{Y_{\overline{a}_K}|\boldsymbol{Z}} | \boldsymbol{Z} \right) = q$, the second equality holds by the relationship between the cumulative distribution function and the density function, the third equality holds by fundamental theorem of calculus, the fourth equality holds by $f_{\epsilon_{\overline{a}_K}|\boldsymbol{Z}}(u|\boldsymbol{Z}) = f_{Y_{\overline{a}_K}|\boldsymbol{Z}} \left(u + Q^{(q)}_{Y_{\overline{a}_K}|\boldsymbol{Z}} | \boldsymbol{Z} \right)$, and the last equality holds by the definition of $\omega(\overline{a}_K, \boldsymbol{Z}, \boldsymbol{\theta}_q)$. It follows that $\boldsymbol{\theta}_{q0}$ is the unique solution of

$$
\mathcal{A}(\boldsymbol{\theta}_q) := 2 \times E \left[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} \frac{\partial h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)}{\partial \boldsymbol{\theta}_q} \omega(\overline{a}_K, \boldsymbol{Z}, \boldsymbol{\theta}_q)\Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) \right] = 0
$$

in terms of $\boldsymbol{\theta}_q$. Turning to the solution of (s2), the right-hand side of (s2) is a convex function of $\boldsymbol{\theta}_q$ because it is essentially an expectation of an weighted average of a convex function $\Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)^2$ (condition (iii)). Due to the convexity of (s2), $\boldsymbol{\theta}^*_{q0}$ uniquely solves the following first-order condition with respect to (s2):

$$
\mathcal{B}(\boldsymbol{\theta}_q) := 2 \times E \left[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} \frac{\partial h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)}{\partial \boldsymbol{\theta}_q} \omega(\overline{a}_K, \boldsymbol{Z}, \boldsymbol{\theta}_q)\Delta(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) \right] = 0.
$$

Thus, we obtain that the functions $\mathcal{A}(\boldsymbol{\theta}_q)$ and $\mathcal{B}(\boldsymbol{\theta}_q)$ are identical. Also notice that $\boldsymbol{\theta}_{q0}$ uniquely solves $\mathcal{A}(\boldsymbol{\theta}_q) = 0$ and $\boldsymbol{\theta}^*_{q0}$ uniquely solves $\mathcal{B}(\boldsymbol{\theta}_q) = 0$. It follows that $\boldsymbol{\theta}_{q0} = \boldsymbol{\theta}^*_{q0}$ and now we conclude our proof. □

Appendix B: the ICR estimator

Appendix B.1: Asymptotic properties of the ICR estimator

Define $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta})] = \mathbb{P}_n\left[\begin{pmatrix}\mathbb{U}_{\boldsymbol{\beta}_1}(\boldsymbol{O};\boldsymbol{\beta}_1,\boldsymbol{\beta}_2)\\ \mathbb{U}_{\boldsymbol{\beta}_2}(\boldsymbol{O};\boldsymbol{\beta}_2,\boldsymbol{\beta}_3)\\ \vdots \\ \mathbb{U}_{\boldsymbol{\beta}_K}(\boldsymbol{O};\boldsymbol{\beta}_K)\end{pmatrix}\right] = \mathbf{0}$ as the joint estimating equations for the unknown parameters in the outcome regression models, $\boldsymbol{\beta} = [\boldsymbol{\beta}_1^T,\ldots,\boldsymbol{\beta}_K^T]^T$, where $\mathbb{U}_{\boldsymbol{\beta}_k}(\boldsymbol{O};\boldsymbol{\beta}_k,\boldsymbol{\beta}_{k+1})$, $k = 1,\ldots,K-1$, are the estimating score given in equation (6) for $\boldsymbol{\beta}_k$ and $\mathbb{U}_{\boldsymbol{\beta}_K}(\boldsymbol{O};\boldsymbol{\beta}_K) = \mathbb{U}_{\boldsymbol{\beta}_K}(Y,\overline{A}_K,\overline{\boldsymbol{L}}_K)$ is the estimating score for $\boldsymbol{\beta}_K$. One can show the iterative procedure for estimating $\boldsymbol{\beta}$ provided in Section 3.2 is equivalent to solving the joint estimating equation, $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta})] = \mathbf{0}$. Also, abbreviate the ICR estimating equations for $\boldsymbol{\theta}_q$, equation (7), as $\mathbb{P}_n[\mathbb{U}^{\text{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\beta}_1)] = \mathbf{0}$. To proceed, we assume the following regularity conditions:

- 1. Assume that outcome regression models $\psi_k(Y_{\overline{A}_k,\underline{a}_{k+1}},\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_k)$, $k = 1,\ldots,K$, are correctly specified. Let $\boldsymbol{\beta}_0 = [\boldsymbol{\beta}_{1,0}^T,\ldots,\boldsymbol{\beta}_{K,0}^T]^T$ be the true parameter of $\boldsymbol{\beta}$ such that $\psi_k(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_{k,0}) = f_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|\overline{A}_k,\overline{\boldsymbol{L}}_k)$.

- 2. Let $\boldsymbol{\theta}_{q0}$ be the true parameter of $\boldsymbol{\theta}_q$ such that it is the solution of equation (2). Also, let $\boldsymbol{\Xi} = \boldsymbol{B}\times\boldsymbol{\Theta}$ be a bounded covex neighborhood of $(\boldsymbol{\beta}_0,\boldsymbol{\theta}_{q0})$. Assume that the estimating scores $\{\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta}),\mathbb{U}^{\text{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\beta}_1)\}$ and their first order derivatives with respect to $(\boldsymbol{\beta},\boldsymbol{\theta}_q)$ are continuous, dominated by certain square-integrable functions, and are also P-Glivenko-Cantelli for $(\boldsymbol{\beta},\boldsymbol{\theta}_q)\in\boldsymbol{\Xi}$.

- 3. Assume that $\boldsymbol{I}_{\boldsymbol{\beta}_0} = \mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\beta}}\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta})|_{\boldsymbol{\beta}=\boldsymbol{\beta}_0}\right]$ converges to a negative definite matrix $\boldsymbol{\mathcal{I}}_{\boldsymbol{\beta}_0} = \frac{\partial}{\partial\boldsymbol{\beta}}E\left[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta})\right]\Big|_{\boldsymbol{\beta}=\boldsymbol{\beta}_0}$. Also, $\boldsymbol{I}^{\text{ICR}}_{\boldsymbol{\theta}_{q0}} = \mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\mathbb{U}^{\text{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\beta}_{1,0})|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right]$ converges to a negative definite matrix $\boldsymbol{\mathcal{I}}^{\text{ICR}}_{\boldsymbol{\theta}_{q0}} = \frac{\partial}{\partial\boldsymbol{\theta}_q}E\left[\mathbb{U}^{\text{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\beta}_{1,0})\right]\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}$.

The asymptotic properties of the ICR estimator is summarized below:

Theorem S1. *Assuming $\mathcal{M}_{om}$ is correctly specified, under assumptions (A1)–(A4) and regularity conditions 1–3, $\widehat{\boldsymbol{\theta}}_q^{ICR}$ is CAN such that $\sqrt{n}(\widehat{\boldsymbol{\theta}}_q^{ICR}-\boldsymbol{\theta}_{q0})$ converge to $N(\mathbf{0},\boldsymbol{\Sigma}^{ICR})$, where $\boldsymbol{\Sigma}^{ICR}$ is defined in* (s9).

A consistent estimator of $\boldsymbol{\Sigma}^{\text{ICR}}$ is $\widehat{\boldsymbol{\Sigma}}^{\text{ICR}} = \left[\widehat{\boldsymbol{\mathcal{I}}}^{\text{ICR}}_{\boldsymbol{\theta}_q}\right]^{-1}\widehat{\boldsymbol{\mathcal{V}}}^{\text{ICR}}\left[\widehat{\boldsymbol{\mathcal{I}}}^{\text{ICR}}_{\boldsymbol{\theta}_q}\right]^{-T}$, where $\widehat{\boldsymbol{\mathcal{I}}}^{\text{ICR}}_{\boldsymbol{\theta}_q} = \mathbb{P}_n\frac{\partial}{\partial\boldsymbol{\theta}_q}\mathbb{U}^{\text{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}^{\text{ICR}}_q,\widehat{\boldsymbol{\beta}}_1)$ and

$$\widehat{\boldsymbol{\mathcal{V}}}^{\text{ICR}} = \mathbb{P}_n\left[\left(\mathbb{U}^{\text{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}^{\text{ICR}}_q,\widehat{\boldsymbol{\beta}}_1) - \widehat{\boldsymbol{\mathcal{S}}}^{\text{ICR}}_{\boldsymbol{\theta}_q,\boldsymbol{\beta}_1}\widehat{\boldsymbol{\mathcal{I}}}^{*}_{\boldsymbol{\beta}_1}\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\widehat{\boldsymbol{\beta}})\right)^{\otimes 2}\right].$$

Here, for vector $\boldsymbol{V}$, $\boldsymbol{V}^{\otimes 2} = \boldsymbol{V}\boldsymbol{V}^T$, $\widehat{\boldsymbol{\mathcal{S}}}^{\text{ICR}}_{\boldsymbol{\theta}_q,\boldsymbol{\beta}_1} = \mathbb{P}_n\frac{\partial}{\partial\boldsymbol{\beta}_1}\mathbb{U}^{\text{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}^{\text{ICR}}_q,\widehat{\boldsymbol{\beta}}_1)$, $\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta}) = [\mathbb{U}_{\boldsymbol{\beta}_1}(\boldsymbol{O};\boldsymbol{\beta}_1,\boldsymbol{\beta}_2)^T$, $\mathbb{U}_{\boldsymbol{\beta}_2}(\boldsymbol{O};\boldsymbol{\beta}_2,\boldsymbol{\beta}_3)^T,\ldots,\mathbb{U}_{\boldsymbol{\beta}_K}(\boldsymbol{O};\boldsymbol{\beta}_K)^T]^T$ is the joint estimating function for $\boldsymbol{\beta} = [\boldsymbol{\beta}_1^T,\boldsymbol{\beta}_2^T,\ldots,\boldsymbol{\beta}_K^T]^T$, and $\widehat{\boldsymbol{\mathcal{I}}}^{*}_{\boldsymbol{\beta}_1}$ is the first $d_{\boldsymbol{\beta}_1}$ rows of $\widehat{\boldsymbol{\mathcal{I}}}^{-1}_{\boldsymbol{\beta}}$ ($d_{\boldsymbol{\beta}_1}$ is the length of $\boldsymbol{\beta}_1$ and $\widehat{\boldsymbol{\mathcal{I}}}_{\boldsymbol{\beta}} = \mathbb{P}_n\frac{\partial}{\partial\boldsymbol{\beta}}\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O},\widehat{\boldsymbol{\beta}})$).

Below we prove Theorem S1. We first introduce several lemmas that will be used in the proof.

Lemma 1. *(g-formula representation of the potential outcome distributions) Under assumptions (A1)–(A3), we have that, for k in $\{1,\ldots,K-1\}$,*

$$f_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|\overline{a}_k,\overline{\boldsymbol{l}}_k)=\int_{\boldsymbol{l}_K}\cdots\int_{\boldsymbol{l}_{k+1}} f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\overline{\boldsymbol{l}}_K)\times\prod_{j=k+1}^{K} f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{k-1},\overline{\boldsymbol{l}}_{k-1})d\boldsymbol{l}_k, \tag{s5}$$

Similarly, we have

$$F_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|\overline{a}_k,\overline{\boldsymbol{l}}_k)=\int_{\boldsymbol{l}_K}\cdots\int_{\boldsymbol{l}_{k+1}} F_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\overline{\boldsymbol{l}}_K)\prod_{j=k+1}^{K} f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{k-1},\overline{\boldsymbol{l}}_{k-1})d\boldsymbol{l}_k.$$

Proof. By the law of total probability, we have that

$$\begin{aligned}
f_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|\overline{a}_k,\overline{\boldsymbol{l}}_k) &= \int_{\boldsymbol{l}_{k+1}} f_{Y_{\overline{A}_k,\underline{a}_{k+1}},\boldsymbol{L}_{k+1}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y,\boldsymbol{l}_{k+1}|\overline{a}_k,\overline{\boldsymbol{l}}_k)d\boldsymbol{l}_{k+1}\\
&= \int_{\boldsymbol{l}_{k+1}} f_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_{k+1}}(y|\overline{a}_k,\overline{\boldsymbol{l}}_{k+1})f_{\boldsymbol{L}_{k+1}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(\overline{a}_k,\overline{\boldsymbol{l}}_k)d\boldsymbol{l}_{k+1}\\
&= \int_{\boldsymbol{l}_{k+1}} f_{Y_{\overline{A}_{k+1},\underline{a}_{k+2}}|\overline{A}_{k+1},\overline{\boldsymbol{L}}_{k+1}}(y|\overline{a}_{k+1},\overline{\boldsymbol{l}}_{k+1})f_{\boldsymbol{L}_{k+1}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(\overline{a}_k,\overline{\boldsymbol{l}}_k)d\boldsymbol{l}_{k+1},
\end{aligned}$$

where the last equality is obtained by assumption (A2). Iteratively repeat the previous process from $f_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}$ to $f_{Y_{\overline{A}_{K-1},a_K}|\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1}}$, then one can obtain

$$f_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|\overline{a}_k,\overline{\boldsymbol{l}}_k)=\int_{\boldsymbol{l}_K}\cdots\int_{\boldsymbol{l}_{k+1}} f_{Y_{A_K}|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\overline{\boldsymbol{l}}_K)\times\prod_{j=k+1}^{K} f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{k-1},\overline{\boldsymbol{l}}_{k-1})\mathrm{d}\boldsymbol{l}_k,$$

which equals to (s5) by the consistency assumption (A1). □

Lemma 2. *If Assumptions (A1)–(A4) hold, $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta})]=\mathbf{0}$ and $\mathbb{P}_n[\mathbb{U}^{ICR}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\beta}_1)]=\mathbf{0}$ are unbiased estimating equations.*

Proof. We first show $E[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta})]=\mathbf{0}$ at $\boldsymbol{\beta}=\boldsymbol{\beta}_0$. First, we can show $E[\mathbb{U}_{\boldsymbol{\beta}_K}(\boldsymbol{O};\boldsymbol{\beta}_{K,0})]=E[\mathbb{U}_{\boldsymbol{\beta}_K}(Y,\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\beta}_{K,0})]=\mathbf{0}$ by definition of $\mathbb{U}_{\boldsymbol{\beta}_K}(Y,\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\beta}_K)$. Also, observing that

$$\begin{aligned}
&E[\mathbb{U}_{\boldsymbol{\beta}_k}(\boldsymbol{O};\boldsymbol{\beta}_{k,0},\boldsymbol{\beta}_{k+1,0})]\\
=&E\left[\sum_{\underline{a}_{k+1}\in\underline{\mathbb{A}}_{k+1}}\int_y \mathbb{U}_{\boldsymbol{\beta}_k}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_{k,0})\psi_{k+1}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\boldsymbol{\beta}_{k+1,0})\mathrm{d}y\right]\\
=&\sum_{\underline{a}_{k+1}\in\underline{\mathbb{A}}_{k+1}} E\left[\int_y \mathbb{U}_{\boldsymbol{\beta}_k}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_{k,0})f_{Y_{\overline{A}_{k+1},\underline{a}_{k+2}}|\overline{A}_{k+1},\overline{\boldsymbol{L}}_k}(y|\overline{A}_k,a_{k+1},\overline{\boldsymbol{L}}_{k+1})\mathrm{d}y\right]\\
&\qquad\text{(followed by point 1)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\underline{a}_{k+1}\in\underline{\mathbb{A}}_{k+1}} E\left[\int_y \mathbb{U}_{\boldsymbol{\beta}_k}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_{k,0}) f_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|\overline{A}_k,\overline{\boldsymbol{L}}_{k+1})\mathrm{d}y\right] \\
&\qquad\qquad \text{(followed by Assumption (A2), } Y_{\overline{A}_k,\underline{a}_{k+1}} \perp A_{k+1}|\overline{A}_k,\overline{\boldsymbol{L}}_{k+1}) \\
&= \sum_{\underline{a}_{k+1}\in\underline{\mathbb{A}}_{k+1}} E\left[E\left[\mathbb{U}_{\boldsymbol{\beta}_k}(Y_{\overline{A}_k,\underline{a}_{k+1}},\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_{k,0})\middle|\overline{A}_k,\overline{\boldsymbol{L}}_{k+1}\right]\right] \\
&= \sum_{\underline{a}_{k+1}\in\underline{\mathbb{A}}_{k+1}} E\left[\mathbb{U}_{\boldsymbol{\beta}_k}(Y_{\overline{A}_k,\underline{a}_{k+1}},\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_{k,0})\right] \\
&=\mathbf{0} \quad \text{(by definition of } \mathbb{U}_{\boldsymbol{\beta}_k}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_k)),
\end{aligned}
$$

we concludes $E[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta}_0)] = \mathbf{0}$. Similarly, one can easily verify that

$$
\begin{aligned}
&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K,\overline{\boldsymbol{L}}_K)\left\{\Psi_1\Big(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}),\overline{a}_K,\boldsymbol{L}_1;\boldsymbol{\beta}_{1,0}\Big)-q\right\}\right] \\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K,\overline{\boldsymbol{L}}_K)\left\{F_{Y_{A_1,\underline{a}_2}|A_1,\boldsymbol{L}_1}\Big(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})|a_1,\boldsymbol{L}_1\Big)-q\right\}\right] \quad \text{(by point 1)} \\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K,\overline{\boldsymbol{L}}_K)\left\{F_{Y_{\overline{a}_K}|\boldsymbol{L}_1}\Big(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})|\boldsymbol{L}_1\Big)-q\right\}\right] \quad \text{(by assumption (A2): } Y_{A_1,\underline{a}_2}\perp A_1|\boldsymbol{L}_1) \\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K,\overline{\boldsymbol{L}}_K)\left\{E\Big[\mathbb{I}(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}))|\boldsymbol{L}_1\Big]-q\right\}\right] \\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K,\overline{\boldsymbol{L}}_K)\left\{E\Big[\mathbb{I}(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}))|\boldsymbol{Z}\Big]-q\right\}\right] \\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K,\overline{\boldsymbol{L}}_K)\left\{\mathbb{I}(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}))-q\right\}\right] \\
=&\mathbf{0},
\end{aligned}
$$

which suggests $E[\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_{1,0})] = \mathbf{0}$ and completes the proof of Lemma 1. □

Lemma 3. *The estimator, $\widehat{\boldsymbol{\beta}}$, by solving $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta})] = \mathbf{0}$ is consistent and asymptotic normal (CAN). In addition, we have*

$$\sqrt{n}(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0) = -\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\beta}_0}\sqrt{n}\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta}_0)] + o_p(1). \tag{s6}$$

Specifically,

$$\sqrt{n}(\widehat{\boldsymbol{\beta}}_1-\boldsymbol{\beta}_{1,0}) = -\boldsymbol{\mathcal{I}}^{*}_{\boldsymbol{\beta}_{1,0}}\sqrt{n}\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta}_0)] + o_p(1), \tag{s7}$$

where $\boldsymbol{\mathcal{I}}^{*}_{\boldsymbol{\beta}_{1,0}}$ is the first $d_{\boldsymbol{\beta}_1}$ rows of $\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\beta}_0}$ and $d_{\boldsymbol{\beta}_1}$ is the length of $\boldsymbol{\beta}_1$.

Proof. By noting that $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta})] = \mathbf{0}$ is unbiased (Lemma 2), along with Point 2 and the negative definite property of $\boldsymbol{\mathcal{I}}_{\boldsymbol{\beta}_0}$ in point 3, one can easily concludes $\widehat{\boldsymbol{\beta}}_0$ is CAN following the standard proofs for M-estimators (e.g., van der Vaart 2000, Tsiatis 2006). Then, using a Taylor series for $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta})]$ around $\boldsymbol{\beta}=\boldsymbol{\beta}_0$ can deduce that (s6) holds. Finally, (s7) follows directly from (s6). $\square$

Now, we proceed to prove Theorem S1. The CAN property of $\widehat{\boldsymbol{\beta}}_1$ in Lemma 3 implies that, for all $\epsilon>0$, $\widehat{\boldsymbol{\beta}}_1=\boldsymbol{\beta}_1+o_p(n^{-1/2+\epsilon})$, a.s., which can be used to show that

$$\sup_{\boldsymbol{\theta}_q\in\boldsymbol{\Theta}}\left\|\mathbb{P}_n[\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\beta}}_1)]-\mathbb{P}_n[\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\beta}_{1,0})]\right\|=o(n^{-1/2+\epsilon}), \tag{s8}$$

where $\|\cdot\|$ denotes the L_2 vector norm. Because $E[\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_{1,0})]=\mathbf{0}$ (Lemma 2) and $\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\beta}_{1,0})|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q,0}}\right]$ converges to a negative definite matrix (point 3), it follows that $\mathbb{P}_n[\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\beta}_{1,0})]$ is bounded away from zero for $\boldsymbol{\theta}_q\neq\boldsymbol{\theta}_{q0}$. This, coupled with (s8) and the definition of $\widehat{\boldsymbol{\theta}}^{\mathrm{ICR}}_q$, implies that $\widehat{\boldsymbol{\theta}}^{\mathrm{ICR}}_q\to\boldsymbol{\theta}_{q0}$, a.s., as $n\to\infty$.

To prove asymptotic normality, we use a Taylor series, along with the Glivenko-Cantelli property in point 2 and negative definite property in point 3, to deduce that

$$\begin{aligned}\mathbf{0}=\mathbb{P}_n[\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}^{\mathrm{ICR}}_q,\widehat{\boldsymbol{\beta}}_1)]=&\mathbb{P}_n[\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_{1,0})]\\&+\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\beta}_1}\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_1)\Big|_{\boldsymbol{\beta}_1=\boldsymbol{\beta}_{1,0}}\right](\widehat{\boldsymbol{\beta}}_1-\boldsymbol{\beta}_{1,0})\\&+\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\beta}_{1,0})\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right](\widehat{\boldsymbol{\theta}}^{\mathrm{ICR}}_q-\boldsymbol{\theta}_{q0})+o_p(n^{-1/2}).\end{aligned}$$

This suggests

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\mathrm{ICR}}_q-\boldsymbol{\theta}_{q0})=-\left[\boldsymbol{\mathcal{I}}^{\mathrm{ICR}}_{\boldsymbol{\theta}_{q0}}\right]^{-1}\sqrt{n}\mathbb{P}_n\left[\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_{1,0})-\boldsymbol{\mathcal{S}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_{1,0}}\boldsymbol{\mathcal{I}}^*_{\boldsymbol{\beta}_{1,0}}\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta}_0)\right]+o_p(1),$$

where $\boldsymbol{\mathcal{S}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_{1,0}}=E\left[\frac{\partial}{\partial\boldsymbol{\beta}_1}\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_1)\Big|_{\boldsymbol{\beta}_1=\boldsymbol{\beta}_{1,0}}\right]$. Note that from Lemma 2 that the $\mathbb{P}_n$-term in the previous fromula has mean zero. By applying the Central Limit Theorem to the right-hand side of the previous formula, one can verify that $\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\mathrm{ICR}}_q-\boldsymbol{\theta}_{q0})$ converges to a multivariate normal distribution with mean zero and a finite variance-covariance matrix

$$\begin{aligned}\boldsymbol{\Sigma}^{\mathrm{ICR}}&=\lim_{n\to\infty}\left[\boldsymbol{\mathcal{I}}^{\mathrm{ICR}}_{\boldsymbol{\theta}_{q0}}\right]^{-1}\mathbb{P}_n\left[\left(\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_{1,0})-\boldsymbol{\mathcal{S}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_{1,0}}\boldsymbol{\mathcal{I}}^*_{\boldsymbol{\beta}_{1,0}}\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta}_0)\right)^{\otimes 2}\right]\left[\boldsymbol{\mathcal{I}}^{\mathrm{ICR}}_{\boldsymbol{\theta}_{q0}}\right]^{-T}\\&=\left[\boldsymbol{\mathcal{I}}^{\mathrm{ICR}}_{\boldsymbol{\theta}_{q0}}\right]^{-1}\boldsymbol{\mathcal{V}}^{\mathrm{ICR}}\left[\boldsymbol{\mathcal{I}}^{\mathrm{ICR}}_{\boldsymbol{\theta}_{q0}}\right]^{-T}\end{aligned} \tag{s9}$$

where $\boldsymbol{\mathcal{V}}^{\mathrm{ICR}}=E\left[\left(\mathbb{U}^{\mathrm{ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_{1,0})-\boldsymbol{\mathcal{S}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_{1,0}}\boldsymbol{\mathcal{I}}^*_{\boldsymbol{\beta}_{1,0}}\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta}_0)\right)^{\otimes 2}\right]$. We can show that $\widehat{\boldsymbol{\Sigma}}^{\mathrm{ICR}}$ defined below Theorem S1 is a consistent estimator of $\boldsymbol{\Sigma}^{\mathrm{ICR}}$ by applying the asymptotic normality results and points 2 and 3.

Appendix B.2: an example of ICR via heteroscedastic Gaussian linear regressions

We provide a concrete example of the ICR approach under Gaussian linear regression for the outcome distribution conditional on time-varying treatments and covariates, allowing for heteroscedasticity. This approach generalizes the approach considered in previous work for modelling the distribution of the potential outcome given a time-fixed treatment (Zhang et al. 2012, Xie et al. 2020, Yang & Zhang 2023) to time-varying treatments. Importantly, although the density of the potential outcome given treatment and covariate history is assumed Gaussian, the marginal density of the outcome is a complex mixture of Gaussian and can theoretically approximate a wide class of distributions, providing great flexibility in capturing local features of the marginal potential outcome distribution. Specifically, we consider $\psi_k(Y_{\overline{A}_k,\underline{a}_{k+1}}, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}_k)$ to follow a linear model with heteroscedasticity parameterized by $\boldsymbol{\beta}_k = [\boldsymbol{\delta}_k^T, \boldsymbol{\eta}_k^T]^T$:

$$Y_{\overline{A}_k,\underline{a}_{k+1}} = \boldsymbol{\delta}_k^T \boldsymbol{g}_k(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k) + \epsilon_k, \quad \epsilon_k \sim N\Big(0, \sigma_k^2(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\eta}_k)\Big). \tag{s10}$$

Here, $\boldsymbol{g}_k(\cdot)$ is a multi-dimensional function of $\{\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k\}$ to measure the conditional mean of $Y_{\overline{A}_k,\underline{a}_{k+1}}$ given $\overline{A}_k$ and $\overline{\boldsymbol{L}}_k$; for example, we can set $\boldsymbol{g}_k(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k) = [a_K, \ldots, a_{k+1}, A_k, \ldots, A_1, \boldsymbol{L}_k^T, \ldots, \boldsymbol{L}_1^T]^T$. Also, $\sigma_k^2(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\eta}_k)$ is a one-dimensional function with unknown parameter $\boldsymbol{\eta}_k$ to measure the conditional variance of $Y_{\overline{A}_k,\underline{a}_{k+1}}$ given $\overline{A}_k$ and $\overline{\boldsymbol{L}}_k$. For example, $\sigma_k^2(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\eta}_k) = \eta_0$ under the constant variance assumption. Given (s10), we can write the CDF corresponding to $\psi_k(y, \overline{a}_K, \bar{\boldsymbol{l}}_k; \boldsymbol{\beta}_k)$ as

$$\Psi_k\left(y, \overline{a}_K, \boldsymbol{l}_k; \boldsymbol{\beta}_k\right) = \Phi\left(y; \boldsymbol{\delta}_k^T \boldsymbol{g}_k(\overline{a}_K, \bar{\boldsymbol{l}}_k), \sigma_k^2(\overline{a}_K, \bar{\boldsymbol{l}}_k; \boldsymbol{\eta}_k)\right), \tag{s11}$$

where $\Phi(y; \mu, \sigma^2)$ is the CDF of a Gaussian random variable with mean μ and variance σ^2.

Given the linear regression setup, the following generalized estimating equation is a feasible estimating score for $\boldsymbol{\beta}_k = [\boldsymbol{\delta}_k^T, \boldsymbol{\eta}_k^T]^T$ if all $Y_{\overline{A}_k,\underline{a}_{k+1}}$ can be observed:

$$\begin{aligned} &\mathbb{U}_{\boldsymbol{\beta}_k}(Y_{\overline{A}_k,\underline{a}_{k+1}}, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k) \\ &= \left\{ \begin{array}{c} \boldsymbol{g}_k(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k) \left(Y_{\overline{A}_k,\underline{a}_{k+1}} - \boldsymbol{\delta}_k^T \boldsymbol{g}_k(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k)\right) \\ \frac{\partial \sigma_k^2(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\eta}_k)}{\partial \boldsymbol{\eta}_k} \left[\sigma_k^2(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\eta}_k) - \left(Y_{\overline{A}_k,\underline{a}_{k+1}} - \boldsymbol{\delta}_k^T \boldsymbol{g}_k(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k)\right)^2\right] \end{array} \right\}, \end{aligned}$$

where the first and second row correspond to the regression coefficients $\boldsymbol{\delta}_k$ and the variance parameters $\boldsymbol{\eta}_k$, respectively. After some algebra (Appendix B.3), the observed-data estimating equation (6) simplifies to

$$\mathbb{P}_n \left\{ \begin{array}{c} \sum_{\underline{a}_{k+1} \in \underline{\mathbb{A}}_{k+1}} \boldsymbol{g}_k(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k) \left(\widehat{Y}_{\overline{A}_k,\underline{a}_{k+1}} - \boldsymbol{\delta}_k^T \boldsymbol{g}_k(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k)\right) \\ \sum_{\underline{a}_{k+1} \in \underline{\mathbb{A}}_{k+1}} \frac{\partial \sigma_k^2(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\eta}_k)}{\partial \boldsymbol{\eta}_k} \left[\sigma_k^2(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\eta}_k) - \widehat{\epsilon}_k^2\right] \end{array} \right\} = \boldsymbol{0}, \tag{s12}$$

where $\widehat{Y}_{\overline{A}_k,\underline{a}_{k+1}} = \widehat{\boldsymbol{\delta}}_{k+1}^T \boldsymbol{g}_{k+1}(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_{k+1})$ and $\widehat{\epsilon}_k^2 = \widehat{Y}_{\overline{A}_k,\underline{a}_{k+1}}^2 + \sigma_{k+1}^2(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_{k+1}; \widehat{\boldsymbol{\eta}}_{k+1}) - 2\widehat{Y}_{\overline{A}_k,\underline{a}_{k+1}} \boldsymbol{\delta}_k^T \boldsymbol{g}_k(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k) + \left(\boldsymbol{\delta}_k^T \boldsymbol{g}_k(\overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k)\right)^2$. Therefore, the observed-data estimating equation in this example are explicit functions of $\boldsymbol{\beta}_k$ and we do not need to calculate the integrals in (6). Once we obtain $\widehat{\boldsymbol{\beta}}_1$, we can then estimate the MSQM parameter by solving (7), where the calculation of $\Psi_1\left(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q), \overline{a}_K, \boldsymbol{L}_1; \widehat{\boldsymbol{\beta}}_1\right)$ is based on (s11).

Remark 1. *When the conditional distribution of the outcome remains skewed, regression model* (s10) *may likely be misspecified due to the potential non-Gaussian nature of the error term. In such cases, one could transform the observed outcome into a form that approximates a normal distribution, and then directly model this transformed outcome variable, denoted as $\widetilde{Y}$, using heteroscedastic Gaussian linear regression. For example, one can set $\widetilde{Y} = H(Y)$, where $H(.)$ is a monotone transformation function (e.g., $H(x) = \log(x)$ or $H(x)$ as the Box-Cox transformation). The MSQM can then be directly defined on $\widetilde{Y}_{\overline{a}_K} = H(Y_{\overline{a}_K})$. Once the MSQM for $Q^{(q)}_{\widetilde{Y}_{\overline{a}_K}|\boldsymbol{Z}}$ is obtained, then the q-th quantile of $Y_{\overline{a}_K}|\boldsymbol{Z}$ can be retrieved by applying the inverse transformation $H^{-1}\left(Q^{(q)}_{\widetilde{Y}_{\overline{a}_K}|\boldsymbol{Z}}\right)$. This is possible because quantiles exhibit the equivariance property to monotone transformations.*

Remark 2. *The ICR estimator is a natural extension of the iterative conditional expectation (ICE) estimator for MSMMs (Robins 1997), which specifies a sequence of regressions for $E[Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k]$, $k = K,\ldots,1$, to estimate the MSMM, $E[Y_{\overline{a}_K}|\boldsymbol{Z}]$. In the linear regression* (s10)*, if we ignore the parameter $\boldsymbol{\eta}_k$ in the error term ϵ_k and only iteratively use the first component of* (s12) *to obtain $\widehat{\boldsymbol{\delta}}_k$, $k = K,\ldots,1$, then we will have $\widehat{E}[Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k] = \widehat{\boldsymbol{\delta}}_k^T\boldsymbol{g}_k(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)$. This is exactly the procedure used in the ICE to estimate $E[Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k]$. The ICR approach generalizes the ICE approach from estimating $E[Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k]$ to the full distribution of $f_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}$, by adding unknown parameters $\boldsymbol{\eta}_k$ to characterize the distribution of the error term ϵ_k and proposing an generic algorithm inspired by the expected estimating equation to solve $[\boldsymbol{\delta}_k,\boldsymbol{\eta}_k]$.*

Remark 3. *Generally, statistical inference under the marginal structural median model (a special case of MSQM with $q = 0.5$) differs from that under the MSMM, even if the heteroscedastic Gaussian linear regression* (s10) *holds. Under conditional normality, the median and mean of potential outcome given baseline covariates coincide such that $Q^{0.5}_{Y_{\overline{a}_K}|\boldsymbol{L}_1} = E[Y_{\overline{a}_K}|\boldsymbol{L}_1]$. However, this does not guarantee that $Q^{0.5}_{Y_{\overline{a}_K}|\boldsymbol{Z}}$ equals $E[Y_{\overline{a}_K}|\boldsymbol{Z}]$ for $\boldsymbol{Z} \neq \boldsymbol{L}_1$ when we integrate over components of $\boldsymbol{L}_1$. For example, if $\boldsymbol{Z} = \emptyset$ and $\boldsymbol{L}_1$ only includes a scalar $L_1 \sim$ Bernoulli(0.8), then $Y_{\overline{a}_K}$ follows a mixture normal distribution with CDF, $F_{Y_{\overline{a}_K}}(y) = \sum_{l=0}^1 w_l\Phi(y;\boldsymbol{\delta}_1^T\boldsymbol{g}_1(\overline{a}_K,l),\sigma_1^2(\overline{a}_K,l;\boldsymbol{\eta}_1))$, where $w_0 = 0.2$ and $w_1 = 0.8$. It is evident that $F_{Y_{\overline{a}_K}}(y)$ does not represent a symmetrical distribution, implying that $Q^{0.5}_{Y_{\overline{a}_K}} \neq E[Y_{\overline{a}_K}]$. This observation underscores the necessity of employing MSQM over MSMM in causal inference when focusing on the center of the potential outcome distribution.*

Appendix B.3: Derivation of observed estimating equation (s12) used in the heterskedastic Gaussian linear regression

By definition given in (6), the observed-data estimating equation for $\boldsymbol{\beta}_k$ is

$$
\begin{aligned}
&\mathbb{P}_n\left[\sum_{\underline{a}_{k+1}\in\underline{\mathbb{A}}_{k+1}}\int_y \mathbb{U}_{\boldsymbol{\beta}_k}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\psi_{k+1}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\widehat{\boldsymbol{\beta}}_{k+1})\mathrm{d}y\right]\\
=&\mathbb{P}_n\left[\sum_{\underline{a}_{k+1}\in\underline{\mathbb{A}}_{k+1}}\int_y \begin{Bmatrix}\mathbb{U}_{\boldsymbol{\delta}_k}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\\ \mathbb{U}_{\boldsymbol{\eta}_k}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\end{Bmatrix}\psi_{k+1}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\widehat{\boldsymbol{\beta}}_{k+1})\mathrm{d}y\right], \qquad \text{(s13)}
\end{aligned}
$$

where

$$
\begin{Bmatrix} \mathbb{U}_{\boldsymbol{\delta}_k}(Y_{\overline{A}_k,\underline{a}_{k+1}},\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k) \\ \mathbb{U}_{\boldsymbol{\eta}_k}(Y_{\overline{A}_k,\underline{a}_{k+1}},\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k) \end{Bmatrix}
$$

$$
=\begin{Bmatrix} \boldsymbol{g}_k(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\left(Y_{\overline{A}_k,\underline{a}_{k+1}}-\boldsymbol{\delta}_k^T\boldsymbol{g}_k(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\right) \\ \frac{\partial\sigma_k^2(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\eta}_k)}{\partial\boldsymbol{\eta}_k}\left[\sigma_k^2(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\eta}_k)-\left(Y_{\overline{A}_k,\underline{a}_{k+1}}-\boldsymbol{\delta}_k^T\boldsymbol{g}_k(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\right)^2\right] \end{Bmatrix},
$$

Here,

$$
\begin{aligned}
&\int_y \mathbb{U}_{\boldsymbol{\delta}_k}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\psi_{k+1}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\boldsymbol{\beta}_{k+1})\mathrm{d}y \\
=&\boldsymbol{g}_k(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\left(\int_y y\psi_{k+1}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\boldsymbol{\beta}_{k+1})\mathrm{d}y-\boldsymbol{\delta}_k^T\boldsymbol{g}_k(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\right) \\
=&\boldsymbol{g}_k(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\left(E\left[Y_{\overline{A}_{k+1},\underline{a}_{k+2}}|\overline{A}_{k+1}=(\overline{A}_k,a_{k+1}),\boldsymbol{L}_{k+1}\right]-\boldsymbol{\delta}_k^T\boldsymbol{g}_k(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\right) \\
=&\boldsymbol{g}_k(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\left(\boldsymbol{\delta}_{k+1}^T\boldsymbol{g}_{k+1}(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1})-\boldsymbol{\delta}_k^T\boldsymbol{g}_k(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_y \mathbb{U}_{\boldsymbol{\eta}_k}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\psi_{k+1}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\boldsymbol{\beta}_{k+1})\mathrm{d}y \\
=&\frac{\partial\sigma_k^2(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\eta}_k)}{\partial\boldsymbol{\eta}_k}\left[\sigma_k^2(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\eta}_k)-\int_y y^2\psi_{k+1}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\boldsymbol{\beta}_{k+1})\mathrm{d}y\right. \\
&\left.+\boldsymbol{g}_k(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\int_y y\psi_{k+1}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\boldsymbol{\beta}_{k+1})\mathrm{d}y-\left(\boldsymbol{\delta}_k^T\boldsymbol{g}_k(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\right)^2\right],
\end{aligned}
$$

where $\int_y y\psi_{k+1}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\boldsymbol{\beta}_{k+1})\mathrm{d}y = \boldsymbol{\delta}_{k+1}^T\boldsymbol{g}_{k+1}(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1})$ as previously discussed and

$$
\begin{aligned}
&\int_y y^2\psi_{k+1}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\boldsymbol{\beta}_{k+1})\mathrm{d}y \\
=&E\left[Y^2_{\overline{A}_{k+1},\underline{a}_{k+2}}|\overline{A}_{k+1}=(\overline{A}_k,a_{k+1}),\boldsymbol{L}_{k+1}\right] \\
=&E^2\left[Y^2_{\overline{A}_{k+1},\underline{a}_{k+2}}|\overline{A}_{k+1}=(\overline{A}_k,a_{k+1}),\boldsymbol{L}_{k+1}\right]+\mathrm{Var}\left(Y_{\overline{A}_{k+1},\underline{a}_{k+2}}|\overline{A}_{k+1}=(\overline{A}_k,a_{k+1}),\boldsymbol{L}_{k+1}\right) \\
=&\left(\boldsymbol{\delta}_{k+1}^T\boldsymbol{g}_{k+1}(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1})\right)^2+\sigma_k^2(\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\boldsymbol{\eta}_k).
\end{aligned}
$$

Combining the previous discussions, one can easily conclude that (s13) equals to (s12).

Appendix C: Derivation of the Efficient Influence Function for $\boldsymbol{\theta}_q$ (Proof of Theorem 1)

Here, we shall use the semiparametric theory (Bickel et al. 1993) to derive the efficient influence function (EIF) for $\boldsymbol{\theta}_q$, $\mathbb{U}^{\text{eff}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)$. The likelihood function of the observed data $\boldsymbol{O}=\{\boldsymbol{L}_1, A_1, \boldsymbol{L}_2, A_2, \dots, \boldsymbol{L}_K, A_K, Y\}$ can be factorized as

$$f_{\boldsymbol{O}} = f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K} \times \prod_{k=2}^{K}\Big(f_{A_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}} \times f_{\boldsymbol{L}_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}}\Big) \times f_{A_1|\boldsymbol{L}_1} \times f_{\boldsymbol{L}_1^*|\boldsymbol{Z}} \times f_{\boldsymbol{Z}},$$

where $\boldsymbol{L}_1^* = \boldsymbol{L}_1 \backslash \boldsymbol{Z}$ is the subset of $\boldsymbol{L}_1$ excluding $\boldsymbol{Z}$. We consider a following parametric submodel $f_{\boldsymbol{O};\boldsymbol{\omega}}$ with unknown parameter $\boldsymbol{\omega}$ for $f_{\boldsymbol{O}}$, which contains the true model $f_{\boldsymbol{O}}$ at $\boldsymbol{\omega}=\boldsymbol{\omega}_0$ such that $f_{\boldsymbol{O};\boldsymbol{\omega}_0} = f_{\boldsymbol{O}}$. One can show that the score function of the parametric submodel is

$$\begin{aligned}
s_{\boldsymbol{\omega}}(\boldsymbol{O}) =& s_{\boldsymbol{\omega}}(Y|\overline{A}_K,\overline{\boldsymbol{L}}_K) + \sum_{k=2}^{K}\big(s_{\boldsymbol{\omega}}(A_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}) + s_{\boldsymbol{\omega}}(\boldsymbol{L}_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1})\big) + s_{\boldsymbol{\omega}}(A_1|\boldsymbol{L}_1) \\
&+ s_{\boldsymbol{\omega}}(\boldsymbol{L}_1^*|\boldsymbol{Z}) + s_{\boldsymbol{\omega}}(\boldsymbol{Z}). \qquad \text{(s14)}
\end{aligned}$$

Here, $s_{\boldsymbol{\omega}}(Y|X) = \partial f_{Y|X;\boldsymbol{\omega}}(Y|X)/\partial\boldsymbol{\omega}$ is the score function corresponding to the parametric model of the likelihood function $f_{Y|X;\boldsymbol{\omega}}$. We shall abbreviate $\partial f_{Y|X;\boldsymbol{\omega}}(Y|X)/\partial\boldsymbol{\omega}$ evaluated at $\boldsymbol{\omega}_0$ to $\dot{f}_{Y|X}(Y|X)$. To further simplify the notation, we also abbreviate the score function of the parametric submodel evaluated at the true parameter, $s_{\boldsymbol{\omega}_0}(\boldsymbol{O})$, as $s(\boldsymbol{O})$. We introduce the following lemma to simplify the proof.

Lemma 4. *We have the following identities connecting the score function to the parametric submodels for the observed outcome and time-varying covariates.*

$$\begin{aligned}
\dot{f}_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(Y|\overline{A}_K,\overline{\boldsymbol{L}}_K) &= \Big(s(\boldsymbol{O}) - E[s(\boldsymbol{O})|\overline{A}_K,\overline{\boldsymbol{L}}_K]\Big) f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(Y|\overline{A}_K,\overline{\boldsymbol{L}}_K), \\
\dot{f}_{\boldsymbol{L}_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}}(\boldsymbol{L}_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}) &= \Big(E[s(\boldsymbol{O})|\overline{A}_{k-1},\overline{\boldsymbol{L}}_k] - E[s(\boldsymbol{O})|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}]\Big)\times \\
&\qquad f_{\boldsymbol{L}_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}}(\boldsymbol{L}_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}), \quad \textit{for } k=2,\dots,K, \\
\dot{f}_{\boldsymbol{L}_1^*|\boldsymbol{Z}}(\boldsymbol{L}_1^*|\boldsymbol{Z}) &= \Big(E[s(\boldsymbol{O})|\boldsymbol{L}_1] - E[s(\boldsymbol{O})|\boldsymbol{Z}]\Big) f_{\boldsymbol{L}_1^*|\boldsymbol{Z}}(\boldsymbol{L}_1^*|\boldsymbol{Z}), \\
\dot{f}_{\boldsymbol{Z}}(\boldsymbol{Z}) &= \Big(E[s(\boldsymbol{O})|\boldsymbol{Z}] - E[s(\boldsymbol{O})]\Big) f_{\boldsymbol{Z}}(\boldsymbol{Z}).
\end{aligned}$$

Proof. We only prove the first identity, where the others can be proved similarly. Specifically, we have

$$\begin{aligned}
\dot{f}_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(Y|\overline{A}_K,\overline{\boldsymbol{L}}_K) =& \frac{\partial}{\partial\boldsymbol{\omega}} f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\omega}}(Y|\overline{A}_K,\overline{\boldsymbol{L}}_K)\Big|_{\boldsymbol{\omega}=\boldsymbol{\omega}_0} \\
=& \frac{\partial}{\partial\boldsymbol{\omega}} \log f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\omega}}(Y|\overline{A}_K,\overline{\boldsymbol{L}}_K)\Big|_{\boldsymbol{\omega}=\boldsymbol{\omega}_0} \times f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(Y|\overline{A}_K,\overline{\boldsymbol{L}}_K) \\
=& s(Y|\overline{A}_K,\overline{\boldsymbol{L}}_K) \times f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(Y|\overline{A}_K,\overline{\boldsymbol{L}}_K) \\
=& \Big(E[s(\boldsymbol{O})|Y,\overline{A}_K,\overline{\boldsymbol{L}}_K] - E[s(\boldsymbol{O})|\overline{A}_K,\overline{\boldsymbol{L}}_K]\Big) f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(Y|\overline{A}_K,\overline{\boldsymbol{L}}_K)
\end{aligned}$$

$$=\Big(s(\boldsymbol{O})-E[s(\boldsymbol{O})|\overline{A}_K,\overline{\boldsymbol{L}}_K]\Big)f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(Y|\overline{A}_K,\overline{\boldsymbol{L}}_K),$$

which concludes the first identity in Lemma 4. □

According to equation (4) and the g-formula expression of $f_{Y_{\overline{a}_K}|\boldsymbol{L}_1}(y|\boldsymbol{l}_1)$, we can rewrite $\boldsymbol{\theta}_q$ as the solution of the following equation that depends on the parametric submodel $f_{\boldsymbol{o};\boldsymbol{\omega}}$:

$$\boldsymbol{U}(\boldsymbol{\theta}_q,\boldsymbol{\omega})=\boldsymbol{0}, \tag{s15}$$

where

$$\boldsymbol{U}(\boldsymbol{\theta}_q,\boldsymbol{\omega})=\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}\int_{y,\bar{\boldsymbol{l}}_K}\xi(y,\overline{a}_K,\boldsymbol{z})f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\omega}}(y|\overline{a}_K,\bar{\boldsymbol{l}}_K)\left(\prod_{j=2}^K f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1};\boldsymbol{\omega}}(\boldsymbol{l}_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right)$$
$$f_{\boldsymbol{L}_1^*|\boldsymbol{Z};\boldsymbol{\omega}}(\boldsymbol{l}_1^*|\boldsymbol{z};\boldsymbol{\omega})f_{\boldsymbol{z};\boldsymbol{\omega}}(\boldsymbol{z})dyd\bar{\boldsymbol{l}}_K,$$

and $\xi(Y,\overline{a}_K,\boldsymbol{Z})=d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\left\{\mathbb{I}\Big(Y\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big)-q\right\}$. As shown in equation (s15), $\boldsymbol{\theta}_q$ is a function of $\boldsymbol{\omega}$ such that $\boldsymbol{\theta}_q=\boldsymbol{\theta}_q(\boldsymbol{\omega})$, where $\boldsymbol{\theta}_q(\boldsymbol{\omega})$ is solution of (s15) with $\boldsymbol{\omega}$ fixed. The true value of $\boldsymbol{\theta}_q$ (i.e., $\boldsymbol{\theta}_{q0}$) is $\boldsymbol{\theta}_q(\boldsymbol{\omega}_0)$. According to the semiparametric theory (Bickel et al. 1993), the EIF of $\boldsymbol{\theta}_q$, if it exists, must satisfy the following constraint,

$$\frac{\partial}{\partial\boldsymbol{\omega}}\boldsymbol{\theta}_q(\boldsymbol{\omega})\Big|_{\boldsymbol{\omega}=\boldsymbol{\omega}_0}=E\left[\mathbb{U}^{\text{eff}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)s(\boldsymbol{O})\right].$$

In what follows, we will derive $\mathbb{U}^{\text{eff}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)$ by calculating $\frac{\partial}{\partial\boldsymbol{\omega}}\boldsymbol{\theta}_q(\boldsymbol{\omega})\Big|_{\boldsymbol{\omega}=\boldsymbol{\omega}_0}$.

By the Implicit Function Theorem, we have that

$$\frac{\partial}{\partial\boldsymbol{\omega}}\boldsymbol{\theta}_q(\boldsymbol{\omega})\Big|_{\boldsymbol{\omega}=\boldsymbol{\omega}_0}=-\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\boldsymbol{U}(\boldsymbol{\theta}_q,\boldsymbol{\omega}_0)\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right]^{-1}\left[\frac{\partial}{\partial\boldsymbol{\omega}}\boldsymbol{U}(\boldsymbol{\theta}_{q0},\boldsymbol{\omega})\Big|_{\boldsymbol{\omega}=\boldsymbol{\omega}_0}\right],$$

where

$$-\frac{\partial}{\partial\boldsymbol{\theta}_q}\boldsymbol{U}(\boldsymbol{\theta}_q,\boldsymbol{\omega}_0)\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}=-\frac{\partial}{\partial\boldsymbol{\theta}_q}E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\left\{F_{Y_{\overline{a}_K}|\boldsymbol{Z}}\Big(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)|\boldsymbol{Z}\Big)-q\right\}\right]\Bigg|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}} \tag{s16}$$

is a constant matrix that does not depend on any of unknown parameters $\boldsymbol{\omega}$ and $\boldsymbol{\theta}_q$. Denote this constant matrix as $\boldsymbol{C}_q$ so that $\boldsymbol{C}_q=-\frac{\partial}{\partial\boldsymbol{\theta}_q}\boldsymbol{U}(\boldsymbol{\theta}_q,\boldsymbol{\omega}_0)\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}$. Next, we compute $\frac{\partial}{\partial\boldsymbol{\omega}}\boldsymbol{U}(\boldsymbol{\theta}_{q0},\boldsymbol{\omega})\Big|_{\boldsymbol{\omega}=\boldsymbol{\omega}_0}$. Specifically, we can decompose $\frac{\partial}{\partial\boldsymbol{\omega}}\boldsymbol{U}(\boldsymbol{\theta}_{q0},\boldsymbol{\omega})\Big|_{\boldsymbol{\omega}=\boldsymbol{\omega}_0}$ to the following formula

$$\frac{\partial}{\partial\boldsymbol{\omega}}\boldsymbol{U}(\boldsymbol{\theta}_{q0},\boldsymbol{\omega})\Big|_{\boldsymbol{\omega}=\boldsymbol{\omega}_0}=\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}\left\{E^{(K+1)}_{\overline{a}_K}+\left(\sum_{k=2}^K E^{(k)}_{\overline{a}_K}\right)+E^{(1)}_{\overline{a}_K}+E^{(0)}_{\overline{a}_K}\right\}, \tag{s17}$$

where

$$E^{(K+1)}_{\overline{a}_K}=\int_{y,\bar{\boldsymbol{l}}_K}\xi(y,\overline{a}_K,\boldsymbol{z})\dot{f}_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\bar{\boldsymbol{l}}_K)\left(\prod_{j=2}^K f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right)f_{\boldsymbol{L}_1^*|\boldsymbol{Z}}(\boldsymbol{l}_1^*|\boldsymbol{z})f_{\boldsymbol{Z}}(\boldsymbol{z})dyd\bar{\boldsymbol{l}}_K,$$
$$E^{(k)}_{\overline{a}_K}=\int_{y,\bar{\boldsymbol{l}}_K}\xi(y,\overline{a}_K,\boldsymbol{z})\dot{f}_{\boldsymbol{L}_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}}(\boldsymbol{l}_k|\overline{a}_{k-1},\bar{\boldsymbol{l}}_{k-1})f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\bar{\boldsymbol{l}}_K)\times$$

$$\left(\prod_{j\geq 2, j\neq k} f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right) f_{\boldsymbol{L}_1^*|\boldsymbol{Z}}(\boldsymbol{l}_1^*|\boldsymbol{z})f_{\boldsymbol{Z}}(\boldsymbol{z})dyd\bar{\boldsymbol{l}}_K, \quad \text{for } k=2,\ldots,K,$$

$$E_{\overline{a}_K}^{(1)} = \int_{y,\bar{\boldsymbol{l}}_K} \xi(y,\overline{a}_K,\boldsymbol{z})\dot{f}_{\boldsymbol{L}_1^*|\boldsymbol{Z}}(\boldsymbol{l}_1^*|\boldsymbol{z})f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\bar{\boldsymbol{l}}_K)\left(\prod_{j=2}^K f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right) f_{\boldsymbol{Z}}(\boldsymbol{z})dyd\bar{\boldsymbol{l}}_K,$$

$$E_{\overline{a}_K}^{(0)} = \int_{y,\bar{\boldsymbol{l}}_K} \xi(y,\overline{a}_K,\boldsymbol{z})\dot{f}_{\boldsymbol{Z}}(\boldsymbol{z})f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\bar{\boldsymbol{l}}_K)\left(\prod_{j=2}^K f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right) f_{\boldsymbol{L}_1^*|\boldsymbol{Z}}(\boldsymbol{l}_1^*|\boldsymbol{z})dyd\bar{\boldsymbol{l}}_K.$$

Then, we further simplify each $E_{\overline{a}_K}^{(k)}$, for $k=0,1,\ldots,K+1$. Specifically, we can rewrite $E_{\overline{a}_K}^{(K+1)}$ as

$$\begin{aligned}
E_{\overline{a}_K}^{(K+1)} =& \int_{\bar{\boldsymbol{l}}_K}\int_y \xi(y,\overline{a}_K,\boldsymbol{z})\dot{f}_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\bar{\boldsymbol{l}}_K)dy\left(\prod_{j=2}^K f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right) f_{\boldsymbol{L}_1^*|\boldsymbol{Z}}(\boldsymbol{l}_1^*|\boldsymbol{z})f_{\boldsymbol{Z}}(\boldsymbol{z})d\bar{\boldsymbol{l}}_K \\
=& \int_{\bar{\boldsymbol{l}}_K,\overline{a}_K}\int_y \xi(y,\overline{a}_K,\boldsymbol{z})\dot{f}_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\bar{\boldsymbol{l}}_K)dy \times \frac{\mathbb{I}(\overline{A}_K=\overline{a}_K)}{\pi_K(\overline{a}_K,\bar{\boldsymbol{l}}_K)} \times \left(\prod_{j=1}^K f_{A_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(a_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right) \times \\
& \left(\prod_{j=2}^K f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right) f_{\boldsymbol{L}_1^*|\boldsymbol{Z}}(\boldsymbol{l}_1^*|\boldsymbol{z})f_{\boldsymbol{Z}}(\boldsymbol{z})d\overline{a}_K d\bar{\boldsymbol{l}}_K \\
& \text{(by the first identity in Lemma 4, we then have)} \\
=& \int_{\bar{\boldsymbol{l}}_K,\overline{a}_K} \frac{\mathbb{I}(\overline{A}_K=\overline{a}_K)}{\pi_K(\overline{a}_K,\bar{\boldsymbol{l}}_K)} \times \int_y \xi(y,\overline{a}_K,\boldsymbol{z})\Big(s(\boldsymbol{o})-E[s(\boldsymbol{O})|\overline{a}_K,\bar{\boldsymbol{l}}_K]\Big) f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\bar{\boldsymbol{l}}_K)dy \times \\
& \left(\prod_{j=1}^K f_{A_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(a_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right)\left(\prod_{j=2}^K f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right) f_{\boldsymbol{L}_1^*|\boldsymbol{Z}}(\boldsymbol{l}_1^*|\boldsymbol{z})f_{\boldsymbol{Z}}(\boldsymbol{z})d\overline{a}_K d\bar{\boldsymbol{l}}_K \\
=& E\left[\frac{\mathbb{I}(\overline{A}_K=\overline{a}_K)}{\pi_K(\overline{a}_K,\bar{\boldsymbol{l}}_K)}\xi(Y,\overline{a}_K,\boldsymbol{Z})s(\boldsymbol{O})\right] - \int_{\bar{\boldsymbol{l}}_K,\overline{a}_K}\int_y \frac{\mathbb{I}(\overline{A}_K=\overline{a}_K)}{\pi_K(\overline{a}_K,\bar{\boldsymbol{l}}_K)} E[\xi(Y,\overline{a}_K,\boldsymbol{Z})|\overline{a}_K,\bar{\boldsymbol{l}}_K]s(\boldsymbol{o})f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\bar{\boldsymbol{l}}_K)dy \\
& \left(\prod_{j=1}^K f_{A_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(a_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right)\left(\prod_{j=2}^K f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right) f_{\boldsymbol{L}_1^*|\boldsymbol{Z}}(\boldsymbol{l}_1^*|\boldsymbol{z})f_{\boldsymbol{Z}}(\boldsymbol{z})d\overline{a}_K d\bar{\boldsymbol{l}}_K \\
=& E\left[\frac{\mathbb{I}(\overline{A}_K=\overline{a}_K)}{\pi_K(\overline{a}_K,\overline{\boldsymbol{L}}_K)}\xi(Y,\overline{a}_K,\boldsymbol{Z})s(\boldsymbol{O})\right] - E\left[\frac{\mathbb{I}(\overline{A}_K=\overline{a}_K)}{\pi_K(\overline{a}_K,\overline{\boldsymbol{L}}_K)}E[\xi(Y,\overline{a}_K,\boldsymbol{Z})|\overline{a}_K,\overline{\boldsymbol{L}}_K]s(\boldsymbol{O})\right] \\
=& E\left[\frac{\mathbb{I}(\overline{A}_K=\overline{a}_K)}{\pi_K(\overline{a}_K,\overline{\boldsymbol{L}}_K)}\left(\xi(Y,\overline{a}_K,\boldsymbol{Z})-E[\xi(Y,\overline{a}_K,\boldsymbol{Z})|\overline{a}_K,\overline{\boldsymbol{L}}_K]\right)s(\boldsymbol{O})\right] \\
=& E\left[\frac{\mathbb{I}(\overline{A}_K=\overline{a}_K)d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})}{\pi_K(\overline{a}_K,\overline{\boldsymbol{L}}_K)}\left(\mathbb{I}\Big(Y\leq h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big) - F_{Y_{\overline{a}_K}|\overline{A}_K,\overline{\boldsymbol{L}}_K}\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big|\overline{a}_K,\overline{\boldsymbol{L}}_K\right)\right)s(\boldsymbol{O})\right].
\end{aligned}$$

Also, we can rewrite $E_{\overline{a}_K}^{(k)}$, for $k=2,\ldots,K$, as

$$\begin{aligned}
E_{\overline{a}_K}^{(k)} =& \int_{\bar{\boldsymbol{l}}_{k-1},\overline{a}_{k-1}}\int_{y,\underline{\boldsymbol{l}}_k} \xi(y,\overline{a}_K,\boldsymbol{z})\dot{f}_{\boldsymbol{L}_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}}(\boldsymbol{l}_k|\overline{a}_{k-1},\bar{\boldsymbol{l}}_{k-1})f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\bar{\boldsymbol{l}}_K)\left(\prod_{j=k+1}^K f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right)dyd\underline{\boldsymbol{l}}_k \\
& \times \frac{\mathbb{I}(\overline{A}_{k-1}=\overline{a}_{k-1})}{\pi_{k-1}(\overline{a}_{k-1},\bar{\boldsymbol{l}}_{k-1})}\left(\prod_{j=1}^{k-1} f_{A_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(a_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right)\left(\prod_{j\geq 2}^{k-1} f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right) f_{\boldsymbol{L}_1^*|\boldsymbol{Z}}(\boldsymbol{l}_1^*|\boldsymbol{z})f_{\boldsymbol{Z}}(\boldsymbol{z})d\overline{a}_{k-1}d\bar{\boldsymbol{l}}_{k-1} \\
& \text{(by the second identity in Lemma 4, we then have)} \\
=& \int_{\bar{\boldsymbol{l}}_{k-1},\overline{a}_{k-1}} \frac{\mathbb{I}(\overline{A}_{k-1}=\overline{a}_{k-1})}{\pi_{k-1}(\overline{a}_{k-1},\bar{\boldsymbol{l}}_{k-1})}\int_{y,\underline{\boldsymbol{l}}_k} \xi(y,\overline{a}_K,\boldsymbol{z})\Big(E[s(\boldsymbol{O})|\overline{a}_{k-1},\bar{\boldsymbol{l}}_k]-E[s(\boldsymbol{O})|\overline{a}_{k-1},\bar{\boldsymbol{l}}_{k-1}]\Big)f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\bar{\boldsymbol{l}}_K) \\
& \times\left(\prod_{j=k}^K f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right)dyd\underline{\boldsymbol{l}}_k\left(\prod_{j=1}^{k-1} f_{A_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(a_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right)\left(\prod_{j\geq 2}^{k-1} f_{\boldsymbol{L}_j|\overline{A}_{j-1},\overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{j-1},\bar{\boldsymbol{l}}_{j-1})\right) \\
& \times f_{\boldsymbol{L}_1^*|\boldsymbol{Z}}(\boldsymbol{l}_1^*|\boldsymbol{z})f_{\boldsymbol{Z}}(\boldsymbol{z})d\overline{a}_{k-1}d\bar{\boldsymbol{l}}_{k-1}
\end{aligned}$$

$$
\begin{aligned}
=&E\left[\frac{\mathbb{I}(\overline{A}_{k-1}=\overline{a}_{k-1})}{\overline{\pi}_{k-1}(\overline{a}_{k-1},\overline{\boldsymbol{L}}_{k-1})}E[\xi(Y,\overline{a}_K,\boldsymbol{Z})|\overline{a}_{k-1},\overline{\boldsymbol{L}}_k]s(\boldsymbol{O})\right]-E\left[\frac{\mathbb{I}(\overline{A}_{k-1}=\overline{a}_{k-1})}{\overline{\pi}_{k-1}(\overline{a}_{k-1},\overline{\boldsymbol{L}}_{k-1})}E[\xi(Y,\overline{a}_K,\boldsymbol{Z})|\overline{a}_{k-1},\overline{\boldsymbol{L}}_{k-1}]s(\boldsymbol{O})\right]\\
=&E\left[\frac{\mathbb{I}(\overline{A}_{k-1}=\overline{a}_{k-1})d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})}{\overline{\pi}_{k-1}(\overline{a}_{k-1},\overline{\boldsymbol{L}}_{k-1})}\left(F_{Y_{\overline{A}_{k-1},\underline{a}_k}|\overline{A}_{k-1},\overline{\boldsymbol{L}}_k}\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big|\overline{a}_{k-1},\overline{\boldsymbol{L}}_k\right)\right.\right.\\
&\qquad\left.\left.-F_{Y_{\overline{A}_{k-1},\underline{a}_k}|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}}\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big|\overline{a}_{k-1},\overline{\boldsymbol{L}}_{k-1}\right)\right)s(\boldsymbol{O})\right]\\
&\qquad\text{(Noting assumption (A2) implies } F_{Y_{\overline{A}_{k-1},\underline{a}_k}|\overline{A}_{k-1},\overline{\boldsymbol{L}}_k}=F_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}\text{, we have)}\\
=&E\left[\frac{\mathbb{I}(\overline{A}_{k-1}=\overline{a}_{k-1})d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})}{\overline{\pi}_{k-1}(\overline{a}_{k-1},\overline{\boldsymbol{L}}_{k-1})}\left(F_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big|\overline{a}_k,\overline{\boldsymbol{L}}_k\right)\right.\right.\\
&\qquad\left.\left.-F_{Y_{\overline{A}_{k-1},\underline{a}_k}|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}}\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big|\overline{a}_{k-1},\overline{\boldsymbol{L}}_{k-1}\right)\right)s(\boldsymbol{O})\right].
\end{aligned}
$$

Using similar strategy, we can show

$$
\begin{aligned}
E_{\overline{a}_K}^{(1)}=&E\left[d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\left(F_{Y_{A_1,\underline{a}_2}|A_1,\boldsymbol{L}_1}\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big|a_1,\boldsymbol{L}_1\right)-F_{Y_{\overline{a}_K}|\boldsymbol{Z}}\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big|\boldsymbol{Z}\right)\right)s(\boldsymbol{O})\right],\\
E_{\overline{a}_K}^{(0)}=&E\left[d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\left(F_{Y_{\overline{a}_K}|\boldsymbol{Z}}\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big|\boldsymbol{Z}\right)-q\right)s(\boldsymbol{O})\right].
\end{aligned}
$$

Then, after some basic algebra, one can verify

$$
\begin{cases}
\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}E_{\overline{a}_K}^{(K+1)}=E\left[\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(K)}(\boldsymbol{O};\boldsymbol{\theta}_{q0})s(\boldsymbol{O})\right],\\
\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}E_{\overline{a}_K}^{(k)}=E\left[\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(k-1)}(\boldsymbol{O};\boldsymbol{\theta}_{q0})s(\boldsymbol{O})\right],\ \text{for } k=2,\ldots,K,\\
\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}E_{\overline{a}_K}^{(1)}+E_{\overline{a}_K}^{(0)}=E\left[\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(0)}(\boldsymbol{O};\boldsymbol{\theta}_{q0})s(\boldsymbol{O})\right],
\end{cases}
$$

where $\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(k)}(\boldsymbol{O};\boldsymbol{\theta}_q)$'s are defined in Theorem 1. It follows that

$$
\frac{\partial}{\partial\boldsymbol{\omega}}\boldsymbol{U}(\boldsymbol{\theta}_{q0},\boldsymbol{\omega})\Big|_{\boldsymbol{\omega}=\boldsymbol{\omega}_0}=E\left[\left(\boldsymbol{C}_q^{-1}\sum_{k=0}^{K}\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(k)}(\boldsymbol{O};\boldsymbol{\theta}_{q0})\right)s(\boldsymbol{O})\right].
$$

This concludes $\mathbb{U}_{\boldsymbol{\theta}_q}^{\text{eff}}(\boldsymbol{O};\boldsymbol{\theta}_q)=\boldsymbol{C}_q^{-1}\sum_{k=0}^{K}\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(k)}(\boldsymbol{O};\boldsymbol{\theta}_q)$ and we now complete the proof.

Appendix D: Asymptotic properties of the IPW estimators

Define $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha})] = \mathbb{P}_n\left[\begin{pmatrix}\mathbb{U}_{\boldsymbol{\alpha}_1}(\boldsymbol{O};\boldsymbol{\alpha}_1)\\ \mathbb{U}_{\boldsymbol{\alpha}_2}(\boldsymbol{O};\boldsymbol{\alpha}_2)\\ \vdots \\ \mathbb{U}_{\boldsymbol{\alpha}_K}(\boldsymbol{O};\boldsymbol{\alpha}_K)\end{pmatrix}\right] = \boldsymbol{0}$ as the joint estimating equations for the unknown parameters in the outcome regression models, $\boldsymbol{\alpha} = [\boldsymbol{\alpha}_1^T, \ldots, \boldsymbol{\alpha}_K^T]^T$, where $\mathbb{U}_{\boldsymbol{\alpha}_k}(\boldsymbol{O};\boldsymbol{\alpha}_k) = \frac{\partial}{\partial \boldsymbol{\alpha}_k}\log \pi_k(\overline{A}_k, \overline{\boldsymbol{L}}_k; \boldsymbol{\alpha}_k)$ is the log-likelihood score for $\boldsymbol{\alpha}_k$. Recall that the original IPW estimating equation (3) is

$$\mathbb{P}_n[\mathbb{U}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}})] = \mathbb{P}_n\left[\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K,\widehat{\boldsymbol{\alpha}})}\left\{\mathbb{I}\Big(Y \le h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big) - q\right\}\right]$$

and the smoothed IPW estimating equation (12) is

$$\mathbb{P}_n[\widetilde{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}})] = \mathbb{P}_n\left[\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K,\widehat{\boldsymbol{\alpha}})}\left\{\mathcal{K}\left(\frac{h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q) - Y_{\overline{a}_K}}{\tau_n}\right) - q\right\}\right]$$

where $\widehat{\boldsymbol{\alpha}}$ is obtained by solving $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha})] = \boldsymbol{0}$. Also, define

$$\mathbb{P}_n[\overline{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})]=\mathbb{P}_n\left[\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K,\boldsymbol{\alpha})}\left\{F_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}\Big(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)|\overline{A}_K,\overline{\boldsymbol{L}}_K\Big) - q\right\}\right]$$

where $F_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}$ is the true cumulative distribution function of Y given $\overline{A}_K$ and $\overline{\boldsymbol{L}}_K$. Besides Assumptions (A1)–(A4) and (B1)–(B2) shown in the paper, we require the following regularity conditions:

- $1'$. Assume that propensity score models $\pi_k(\overline{A}_k, \overline{\boldsymbol{L}}_k; \boldsymbol{\alpha}_k)$, $k = 1, \ldots, K$, are correctly specified. Let $\boldsymbol{\alpha}_0 = [\boldsymbol{\alpha}_{1,0}^T, \ldots, \boldsymbol{\alpha}_{K,0}^T]^T$ be the true parameter of $\boldsymbol{\alpha}$ such that $\pi_k(\overline{A}_k, \overline{\boldsymbol{L}}_k; \boldsymbol{\alpha}_{k,0}) = f_{A_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_k}(A_k|\overline{A}_{k-1}, \overline{\boldsymbol{L}}_k)$.

- $2'$. Let $\boldsymbol{\Xi}' = \boldsymbol{A} \times \boldsymbol{\Theta}$ be a bounded convex neighborhood of $(\boldsymbol{\alpha}_0, \boldsymbol{\theta}_{q0})$. Assume that original IPW estimating score, $\mathbb{U}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})$, is dominated by a square-integrable function and is also P-Glivenko-Cantelli for $(\boldsymbol{\alpha},\boldsymbol{\theta}_q) \in \boldsymbol{\Xi}'$. Also, suppose that the estimating scores $\{\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha}), \overline{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}), \widetilde{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\}$ and their first order derivatives with respect to $(\boldsymbol{\alpha},\boldsymbol{\theta}_q)$ are continuous, dominated by certain square-integrable functions, and are also P-Glivenko-Cantelli for $(\boldsymbol{\alpha},\boldsymbol{\theta}_q) \in \boldsymbol{\Xi}'$. Further assume that $\{\pi_k(\overline{A}_k, \overline{\boldsymbol{L}}_k; \boldsymbol{\alpha}_k), k = 1, \ldots, K\}$ is bounded away from 0 for any $\boldsymbol{\alpha} \in \boldsymbol{A}$.

- $3'$. Assume that $\boldsymbol{I}_{\boldsymbol{\alpha}_0} = \mathbb{P}_n\left[\frac{\partial}{\partial \boldsymbol{\alpha}}\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}\right]$ and $\overline{\boldsymbol{I}}^{\text{IPW}}_{\boldsymbol{\theta}_{q0}} = \mathbb{P}_n\left[\frac{\partial}{\partial \boldsymbol{\theta}_q}\overline{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right]$ converge to negative definite matrices, $\boldsymbol{\mathcal{I}}_{\boldsymbol{\alpha}_0} = \frac{\partial}{\partial \boldsymbol{\alpha}}E\left[\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha})\right]\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}$ and $\boldsymbol{\mathcal{I}}^{\text{IPW}}_{\boldsymbol{\theta}_{q0}} = \frac{\partial}{\partial \boldsymbol{\theta}_q}E\left[\overline{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)\right]\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}$, respectively.

- $4'$. Define $f'_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{A}_K,\overline{\boldsymbol{L}}_K)=\frac{\partial}{\partial y}f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\overline{\boldsymbol{L}}_K)$. Assume that $\{F_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{A}_K,\overline{\boldsymbol{L}}_K)$, $f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{A}_K,\overline{\boldsymbol{L}}_K)$, $f'_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{A}_K,\overline{\boldsymbol{L}}_K)\}$ are absolutely continuous with respect to $y\in[y_{\min},y_{\max}]$ and uniformly bounded.

The asymptotic properties of the IPW estimators are demostrated in the Theorem below:

Theorem S2. *Suppose that $\mathcal{M}_{ps}$ is correctly specified and Assumptions (A1)–(A4), (B1)–(B2), and regularity conditions $1'$–$4'$ hold. Then, $\widehat{\boldsymbol{\theta}}_q^{IPW}$ obtained from (12) is asymptotically equivalent to $\widehat{\boldsymbol{\theta}}_q^{IPW*}$ obtained by (3) such that $\sqrt{n}(\widehat{\boldsymbol{\theta}}_q^{IPW}-\widehat{\boldsymbol{\theta}}_q^{IPW*})=o_p(1)$. Moreover, both $\sqrt{n}(\widehat{\boldsymbol{\theta}}_q^{IPW*}-\boldsymbol{\theta}_{q0})$ and $\sqrt{n}(\widehat{\boldsymbol{\theta}}_q^{IPW}-\boldsymbol{\theta}_{q0})$ converge to $N(\mathbf{0},\boldsymbol{\Sigma}^{IPW})$, where $\boldsymbol{\Sigma}^{IPW}$ is defined in* (s32).

The above results indicate that the IPW estimator obtained from the smoothed estimating equations are asymptotically equivalent to the original unsmoothed estimator. However, the smoothed estimating equation additionally allows for fast computation of the asymptotic variance matrices, $\boldsymbol{\Sigma}^{\text{IPW}}$ via the sandwich variance method. Specifically, a consistent estimator of $\boldsymbol{\Sigma}^{\text{IPW}}$ is $\widehat{\boldsymbol{\Sigma}}^{\text{IPW}}=\left[\widehat{\boldsymbol{\mathcal{I}}}_{\boldsymbol{\theta}_q}^{\text{IPW}}\right]^{-1}\widehat{\boldsymbol{\mathcal{V}}}^{\text{IPW}}\left[\widehat{\boldsymbol{\mathcal{I}}}_{\boldsymbol{\theta}_q}^{\text{IPW}}\right]^{-T}$, where $\widehat{\boldsymbol{\mathcal{I}}}_{\boldsymbol{\theta}_q}^{\text{IPW}}=\mathbb{P}_n\frac{\partial}{\partial\boldsymbol{\theta}_q}\widetilde{\mathbb{U}}_{\boldsymbol{\theta}_q}^{\text{IPW}}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}_q^{\text{IPW}},\widehat{\boldsymbol{\alpha}})$,

$$\widehat{\boldsymbol{\mathcal{V}}}^{\text{IPW}}=\mathbb{P}_n\left[\left(\widetilde{\mathbb{U}}_{\boldsymbol{\theta}_q}^{\text{IPW}}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}_q^{\text{IPW}},\widehat{\boldsymbol{\alpha}})-\widehat{\boldsymbol{\mathcal{S}}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}^{\text{IPW}}\widehat{\boldsymbol{\mathcal{I}}}_{\boldsymbol{\alpha}}^{-1}\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\widehat{\boldsymbol{\alpha}})\right)^{\otimes 2}\right],$$

$\widehat{\boldsymbol{\mathcal{S}}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}^{\text{IPW}}=\mathbb{P}_n\frac{\partial}{\partial\boldsymbol{\alpha}}\widetilde{\mathbb{U}}_{\boldsymbol{\theta}_q}^{\text{IPW}}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}_q^{\text{IPW}},\widehat{\boldsymbol{\alpha}})$, $\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha})=[\frac{\partial}{\partial\boldsymbol{\alpha}_1^T}\log\pi_1(A_1,\boldsymbol{L}_1;\boldsymbol{\alpha}_1),\ldots,\frac{\partial}{\partial\boldsymbol{\alpha}_K^T}\log\pi_K(\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\alpha}_K)]^T$ is the likelihood score for $\boldsymbol{\alpha}=[\boldsymbol{\alpha}_1^T,\ldots,\boldsymbol{\alpha}_K^T]^T$, and $\widehat{\boldsymbol{\mathcal{I}}}_{\boldsymbol{\alpha}}=\mathbb{P}_n\frac{\partial}{\partial\boldsymbol{\alpha}}\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O},\widehat{\boldsymbol{\alpha}})$.

Next, we prove Theorem S2. We first introduce several Lemmas that will be used in the proof.

Lemma 5. *The estimator, $\widehat{\boldsymbol{\alpha}}$, by solving $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha})]=\mathbf{0}$ is consistent and asymptotic normal (CAN). In addition, we have*

$$\sqrt{n}(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha})=-\boldsymbol{\mathcal{I}}_{\boldsymbol{\alpha}_0}^{-1}\sqrt{n}\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha}_0)]+o_p(1). \tag{s18}$$

Proof. By noting that $E[\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha})]=\mathbf{0}$, along with Point 2 and the negative definite property of $\boldsymbol{\mathcal{I}}_{\boldsymbol{\alpha}_0}$ in point 3, one can easily concludes $\widehat{\boldsymbol{\alpha}}_0$ is CAN following the standard proofs for M-estimators (e.g., van der Vaart 2000). Then, using a Taylor series for $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha})]$ around $\boldsymbol{\alpha}=\boldsymbol{\alpha}_0$ can deduce that (s18) holds. □

Lemma 6. *Let k_i, $i=1,2,\ldots,n$, be a sequence of non-random constants, then for every $c>0$ and $\epsilon>0$, we have that*

$$\sup_{\|\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0}\|<cn^{-1/3}}\Bigg|\sum_{i=1}^n k_i\mathbb{I}(Y_i\le h(\overline{A}_{iK},\boldsymbol{Z}_i;\boldsymbol{\theta}_q))-\sum_{i=1}^n k_i\mathbb{I}(Y_i\le h(\overline{A}_{iK},\boldsymbol{Z}_i;\boldsymbol{\theta}_{q0}))-$$
$$\sum_{i=1}^n k_iF_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(h(\overline{A}_{iK},\boldsymbol{Z}_i;\boldsymbol{\theta}_q)|\overline{A}_{iK},\overline{\boldsymbol{L}}_i)+\sum_{i=1}^n k_iF_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(h(\overline{A}_{iK},\boldsymbol{Z}_i;\boldsymbol{\theta}_{q0})|\overline{A}_{iK},\overline{\boldsymbol{L}}_i)\Bigg|=o_p(n^{1/2}). \tag{s19}$$

Proof. This lemma is a direct application of Lemma 1 in Lai & Ying (1988). □

Lemma 7. *Under point $1'$, we have that* *(i) The original IPW estimating equation,* $\mathbb{P}_n\left[\mathbb{U}^{IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\right]$*, is unbiased such that* $E\left[\mathbb{U}^{IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\right]=\mathbf{0}$.

(ii) $\mathbb{P}_n\left[\overline{\mathbb{U}}^{IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\right]$ *is also an unbiased estimating equation such that* $E\left[\overline{\mathbb{U}}^{IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\right]$ $=\mathbf{0}$.

Proof. We only prove point (i) and point (ii) can be proved by using a similar strategy. First, noticing that $Y=\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}Y_{\overline{a}_K}\mathbb{I}(\overline{A}_K=\overline{a}_K)$ by Assumption (A1), we have that

$$\mathbb{U}^{\mathrm{IPW}}_{\theta_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)=\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}\frac{d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\mathbb{I}(\overline{A}_K=\overline{a}_K)}{\overline{\pi}_K(\overline{a}_K,\overline{\boldsymbol{L}}_K,\overline{\boldsymbol{\alpha}}_{K,0})}\Big(\mathbb{I}\Big(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big)-q\Big)$$

Then, We can write $E\left[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\right]$ as an iterative expectation with the inner expectation conditional on $\overline{A}_{K-1}=\overline{a}_{K-1},\overline{\boldsymbol{L}}_K$:

$$\begin{aligned}
&E\left[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\right]\\
=&E\left[E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\left(\prod_{k=1}^{K}\frac{\mathbb{I}(A_k=a_k)}{\pi_k(a_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}\right)\{\mathbb{I}(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}))-q\}\Big|\overline{A}_{K-1}=\overline{a}_{K-1},\overline{\boldsymbol{L}}_K\right]\right]\\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\left(\prod_{k=1}^{K-1}\frac{\mathbb{I}(A_k=a_k)}{\pi_k(a_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}\right)E\left[\frac{\mathbb{I}(A_K=a_K)}{\pi_k(a_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\alpha}_{K,0})}\Big\{\mathbb{I}(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}))\right.\right.\\
&\left.\left.-q\Big\}\Big|\overline{A}_{K-1}=\overline{a}_{K-1},\overline{\boldsymbol{L}}_K\right]\right]\\
&\qquad\text{(by sequential ignorability, } Y_{\overline{a}_K}\perp A_K|\overline{A}_{K-1}=\overline{a}_{K-1},\overline{\boldsymbol{L}}_K\text{, we have below)}\\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\left(\prod_{k=1}^{K-1}\frac{\mathbb{I}(A_k=a_k)}{\pi_k(a_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}\right)\underbrace{E\left[\frac{\mathbb{I}(A_K=a_K)}{\pi_K(a_K,\overline{L}_K;\boldsymbol{\alpha}_{K,0})}\Big|\overline{A}_{K-1}=\overline{a}_{K-1},\overline{\boldsymbol{L}}_K\right]}_{=1}\right.\\
&\left.\quad\times E\left[\mathbb{I}(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}))-q\Big|\overline{A}_{K-1}=\overline{a}_{K-1},\overline{\boldsymbol{L}}_K\right]\right]\\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\left(\prod_{k=1}^{K-1}\frac{\mathbb{I}(A_k=a_k)}{\pi_k(a_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}\right)E\left[\mathbb{I}(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}))-q\Big|\overline{A}_{K-1}=\overline{a}_{K-1},\overline{\boldsymbol{L}}_K\right]\right].
\end{aligned}$$

Then, applying the iterative expectation strategy with the inner expectation conditional on $\overline{A}_{K-2}=\overline{a}_{K-2},\overline{\boldsymbol{L}}_{K-1}$, we can rewrite the last equation of the previous formula as

$$\begin{aligned}
&E\left[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\right]\\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\left(\prod_{k=1}^{K-2}\frac{\mathbb{I}(A_k=a_k)}{\pi_k(a_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}\right)E\Big[\mathbb{I}(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}))-q\Big|\overline{A}_{K-2}=\overline{a}_{K-2},\overline{\boldsymbol{L}}_{k-1}\Big]\right]
\end{aligned}$$

Repeating down to $k=1$ gives the following formula

$$E\left[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\right]=E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\times\left\{\mathbb{I}(Y_{\overline{a}_K}\leq h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}))-q\right\}\right]$$

which concludes that $E\left[\mathbb{U}^{\mathrm{IPW}}_{\theta_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\right]=\mathbf{0}$. This completes the proof.

□

Lemma 8. *(Asymptotic properties of the original unsmoothed IPW estimator) Under points 1′–4′, the estimator,* $\widehat{\boldsymbol{\theta}}_q^{IPW*}$*, by solving the original IPW estimating equation* $\mathbb{P}_n[\mathbb{U}^{IPW}_{\theta_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}})]=\mathbf{0}$ *is CAN.*

Proof. The CAN property of $\widehat{\boldsymbol{\alpha}}$ in Lemma 5 implies that, for all $\epsilon>0$, $\widehat{\boldsymbol{\alpha}}=\boldsymbol{\alpha}+o_p(n^{-1/2+\epsilon})$. This, along with point 2′, can be used to show that, for any $\epsilon>0$,

$$\sup_{\boldsymbol{\theta}_q\in\boldsymbol{\Theta}}\left\|\mathbb{P}_n[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}})]-\mathbb{P}_n[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)]\right\|=o_p(n^{-1/2+\epsilon}),\quad\text{a.s.}\tag{s20}$$

Moreover, observing that

$$\begin{aligned}
&E\left[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)-\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)\right]\\
=&E\left[\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K,\boldsymbol{\alpha}_0)}\times E\Big[\mathbb{I}(Y\leq h(\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\theta}_q))-F_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}\Big(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)|\overline{A}_K,\overline{\boldsymbol{L}}_K)\Big)\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\Big]\right]\\
=&E\left[\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K,\boldsymbol{\alpha}_0)}\times 0\right]=\mathbf{0}
\end{aligned}$$

and applying the uniformly strong law of large numbers, we can show

$$\sup_{(\boldsymbol{\alpha},\boldsymbol{\theta}_q)\in\boldsymbol{\Xi}'}\left\|\mathbb{P}_n[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)]-\mathbb{P}_n[\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)]\right\|=o_p(n^{-1/2+\epsilon})\quad\text{a.s.}$$

This, along with (s20), suggests that

$$\sup_{\boldsymbol{\theta}_q\in\boldsymbol{\Theta}}\left\|\mathbb{P}_n[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}})]-\mathbb{P}_n[\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)]\right\|=o_p(n^{-1/2+\epsilon}),\quad\text{a.s.}\tag{s21}$$

Because $E\left[\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\right]=\mathbf{0}$ (Lemma 7) and $\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right]$ converges to a negative definite matrix (point 3′), we have that there exists a positive constant M such that

$$\begin{aligned}
\|\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0}\|\leq& M\left\|E\left[\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)-\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\right]\right\|=M\left\|E\left[\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)\right]\right\|\\
\leq& M\left\|\mathbb{P}_n\left[\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)\right]\right\|+o_p(n^{-1/2+\epsilon})
\end{aligned}$$

for $\boldsymbol{\theta}_q$ in a small neighborhood of $\boldsymbol{\theta}_{q0}$. This, coupled with (s21), suggests that

$$\|\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0}\|\leq M\left\|\mathbb{P}_n\left[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}})\right]\right\|+o_p(n^{-1/2+\epsilon}).$$

Now, replacing $\widehat{\boldsymbol{\theta}}_q$ in the previous equation by $\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}*}$ and also noticing that $\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}*}$ solves $\mathbb{P}_n\left[\mathbb{U}_{\boldsymbol{\theta}_q}^{\mathrm{IPW}}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}})\right]=\mathbf{0}$, we have obtained

$$\|\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}*}-\boldsymbol{\theta}_{q0}\|=o_p(n^{-1/2+\epsilon}) \tag{s22}$$

such that $\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}*}$ is $\sqrt{n}$-consistent.

Next, we prove the asymptotic normality. Let $D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})=\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K,\overline{\boldsymbol{\alpha}}_K)}$, $H(\boldsymbol{\theta}_q)=\mathbb{I}\Big(Y\le h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big)-q$, and $S(\boldsymbol{\theta}_q)=F_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)|\overline{A}_K,\overline{\boldsymbol{L}}_K)-q$. For any $(\boldsymbol{\alpha},\boldsymbol{\theta}_q)$ in the $O(n^{-1/3})$ neighborhood of $(\boldsymbol{\alpha}_0,\boldsymbol{\theta}_{q0})$ (i.e., $\|\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0}\|<cn^{-1/3}$ and $\|\boldsymbol{\alpha}-\boldsymbol{\alpha}_0\|<cn^{-1/3}$ for any fixed constant c), we can show

$$\begin{aligned}
&\mathbb{P}_n\Big[\mathbb{U}_{\boldsymbol{\theta}_q}^{\mathrm{IPW}}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})-\mathbb{U}_{\boldsymbol{\theta}_q}^{\mathrm{IPW}}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\Big]\\
=&\mathbb{P}_n\Big[\Big(D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})-D(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\Big)S(\boldsymbol{\theta}_{q0})\Big]+\mathbb{P}_n\Big[D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})\Big(H(\boldsymbol{\theta}_q)-H(\boldsymbol{\theta}_{q0})\Big)\Big]\\
&+\mathbb{P}_n\Big[\Big(D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})-D(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\Big)\Big(H(\boldsymbol{\theta}_{q0})-S(\boldsymbol{\theta}_{q0})\Big)\Big],
\end{aligned}$$

where,

$$\begin{aligned}
&\mathbb{P}_n\Big[D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})\Big(H(\boldsymbol{\theta}_q)-H(\boldsymbol{\theta}_{q0})\Big)\Big]\\
=&\mathbb{P}_n\Big[D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})\Big(S(\boldsymbol{\theta}_q)-S(\boldsymbol{\theta}_{q0})\Big)\Big]+\underbrace{\mathbb{P}_n\Big[D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})\Big(H(\boldsymbol{\theta}_q)-H(\boldsymbol{\theta}_{q0})-S(\boldsymbol{\theta}_q)+S(\boldsymbol{\theta}_{q0})\Big)\Big]}_{=o_p(n^{-1/2})\text{ by (s19) in Lemma 6}}\\
=&\mathbb{P}_n\Big[D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})\Big(S(\boldsymbol{\theta}_q)-S(\boldsymbol{\theta}_{q0})\Big)\Big]+o_p(n^{-1/2}),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}_n\Big[\Big(D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})-D(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\Big)\Big(H(\boldsymbol{\theta}_{q0})-S(\boldsymbol{\theta}_{q0})\Big)\Big]\\
&\quad\Big(\text{by a Taylor expansion of }(\boldsymbol{\theta}_q,\boldsymbol{\alpha})\text{ around }(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\text{, we have that}\Big)\\
=&\mathbb{P}_n\Big[\frac{\partial D(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}}\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}\Big(H(\boldsymbol{\theta}_{q0})-S(\boldsymbol{\theta}_{q0})\Big)\Big](\boldsymbol{\alpha}-\boldsymbol{\alpha}_0)\\
&+\mathbb{P}_n\Big[\frac{\partial D(\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)}{\partial\boldsymbol{\theta}_q}\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\Big(H(\boldsymbol{\theta}_{q0})-S(\boldsymbol{\theta}_{q0})\Big)\Big](\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0})+o_p(n^{-1/2})\\
=&\frac{\partial E[D(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha})(H(\boldsymbol{\theta}_{q0})-S(\boldsymbol{\theta}_{q0}))]}{\partial\boldsymbol{\alpha}}\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}(\boldsymbol{\alpha}-\boldsymbol{\alpha}_0)\\
&+\frac{\partial E[D(\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)(H(\boldsymbol{\theta}_{q0})-S(\boldsymbol{\theta}_{q0}))]}{\partial\boldsymbol{\theta}_q}\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}(\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0})+o_p(n^{-1/2})\\
=&o_p(n^{-1/2}),
\end{aligned} \tag{s23}$$

where the last equality of the previous formula holds because $E[D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})(H(\boldsymbol{\theta}_{q0})-S(\boldsymbol{\theta}_{q0}))]=\mathbf{0}$. It follows that

$$\mathbb{P}_n\Big[\mathbb{U}_{\boldsymbol{\theta}_q}^{\mathrm{IPW}}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})-\mathbb{U}_{\boldsymbol{\theta}_q}^{\mathrm{IPW}}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\Big]$$

$$=\mathbb{P}_n\Big[D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})\Big(S(\boldsymbol{\theta}_q)-S(\boldsymbol{\theta}_{q0})\Big)\Big]+\mathbb{P}_n\Big[\Big(D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})-D(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\Big)S(\boldsymbol{\theta}_{q0})\Big]+o_p(n^{-1/2})$$
$$=\mathbb{P}_n\Big[\overline{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})-\overline{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\Big]+o_p(n^{-1/2}).$$

In the previous formula, by applying a first-order Taylor's expansion of $\overline{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})$ around $(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)$, one can obtain

$$\begin{aligned}\mathbb{P}_n[\mathbb{U}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})]=&\mathbb{P}_n[\mathbb{U}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)]+\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\alpha}}\overline{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}\right](\boldsymbol{\alpha}-\boldsymbol{\alpha}_0)\\&+\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\overline{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right](\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0})+o_p(n^{-1/2}).\end{aligned}\tag{s24}$$

Because $(\widehat{\boldsymbol{\theta}}^{\text{IPW}*}_q,\widehat{\boldsymbol{\alpha}})$ is in the $O(n^{-1/3})$ neighborhood of $(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)$ with probability tending to 1, $\mathbb{P}_n\Big[\mathbb{U}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}^{\text{IPW}}_q,\widehat{\boldsymbol{\alpha}})\Big]=\boldsymbol{0}$, and $\mathbb{P}_n\Big[\overline{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\Big]=\boldsymbol{0}$, we have that

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\text{IPW}*}_q-\boldsymbol{\theta}_{q0})=-\Big[\boldsymbol{\mathcal{I}}^{\text{IPW}}_{\boldsymbol{\theta}_{q0}}\Big]^{-1}\sqrt{n}\mathbb{P}_n\Big[\mathbb{U}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)-\boldsymbol{\mathcal{S}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0}\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}_0}\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha}_0)\Big]+o_p(1),\tag{s25}$$

where $\boldsymbol{\mathcal{S}}^{\text{IPW}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0}=E\left[\frac{\partial}{\partial\boldsymbol{\alpha}}\overline{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}\right]$. Note that the $\mathbb{P}_n$-term in the previous formula has mean zero. Then, by applying the central limit theorem to the right-hand side of the previous formula, one can easily verify that $\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\text{IPW}*}_q-\boldsymbol{\theta}_{q0})$ converges to a multivariate normal distribution with mean zero and finite variance-covariance matrix. This completes the proof. □

Lemma 9. *Under points 1′–4′ and conditions (B1)–(B2) regarding the local distribution function $\mathcal{K}(.)$ and bandwidth τ_n, the original IPW estimating equation, $\mathbb{P}_n[\mathbb{U}^{IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})]$, and the smoothed IPW estimating equation, $\mathbb{P}_n[\widetilde{\mathbb{U}}^{IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})]$, are asymptotically equivalent such that*

$$\sqrt{n}\mathbb{P}_n\Big[\widetilde{\mathbb{U}}^{IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})-\mathbb{U}^{IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\Big]=o_p(1),\tag{s26}$$

uniformly in $(\boldsymbol{\theta}_q,\boldsymbol{\alpha})\in\boldsymbol{\Xi}'$.

Proof. Define $\epsilon=Y-h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)$ and $d=\epsilon/\tau_n$. The cumulative distribution function of ϵ given $\overline{A}_K,\overline{\boldsymbol{L}}_K$ is $F_{\epsilon|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{A}_K,\overline{\boldsymbol{L}}_K)=F_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}\Big(y+h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)|\overline{A}_K,\overline{\boldsymbol{L}}_K\Big)$. Also, the density function $f_{\epsilon|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{A}_K,\overline{\boldsymbol{L}}_K)=f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}\Big(y+h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)|\overline{A}_K,\overline{\boldsymbol{L}}_K\Big)$. In this proof, we shall abbreviate $F_{\epsilon|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{A}_K,\overline{\boldsymbol{L}}_K)$ and $f_{\epsilon|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{A}_K,\overline{\boldsymbol{L}}_K)$ to $F_\epsilon(y)$ and $f_\epsilon(y)$, respectively.

Without loss of generality, let $\widetilde{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q,j}(\boldsymbol{\theta}_q,\boldsymbol{\alpha})$ and $\mathbb{U}^{\text{IPW}}_{\boldsymbol{\theta}_q,j}(\boldsymbol{\theta}_q,\boldsymbol{\alpha})$ be the j^{th} element of $\widetilde{\mathbb{U}}^{\text{IPW}}_n(\boldsymbol{\theta}_q,\boldsymbol{\alpha})$ and $\mathbb{U}^{\text{IPW}}_n(\boldsymbol{\theta}_q,\boldsymbol{\alpha})$, respectively. If one can show that, for every j,

$$\left|E\left[\sqrt{n}\mathbb{P}_n\Big[\widetilde{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})-\mathbb{U}^{\text{IPW}}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\Big]\right]\right|=o(1),\tag{s27}$$
$$\text{Var}\left(\sqrt{n}\mathbb{P}_n\Big[\widetilde{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})-\mathbb{U}^{\text{IPW}}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\Big]\right)=o(1),\tag{s28}$$

uniformly for $(\boldsymbol{\theta}_q, \boldsymbol{\alpha}) \in \boldsymbol{\Xi}'$, we will complete the proof.

Observing $\mathcal{K}(-d) - \mathbb{I}(d \leq 0) = \mathcal{K}(-|d|)(1 - 2\mathbb{I}(d \leq 0))$, we have that

$$
\begin{aligned}
&E\Big[\mathcal{K}(-d) - \mathbb{I}(d \leq 0)|\overline{A}_K, \overline{\boldsymbol{L}}_K\Big] \\
=&\int_\epsilon \mathcal{K}(-|d|)(1 - 2\mathbb{I}(d \leq 0))f_\epsilon(\epsilon)d\epsilon \\
=&\int_\epsilon \mathcal{K}(-|\epsilon|/\tau_n)(1 - 2\mathbb{I}(\epsilon \leq 0))f_\epsilon(\epsilon)d\epsilon \\
=&\tau_n \int_t \mathcal{K}(-|t|)(1 - 2\mathbb{I}(t \leq 0))f_\epsilon(\tau_n t)dt \quad (\text{let } t = \epsilon/\tau_n) \\
=&\tau_n \int_t \mathcal{K}(-|t|)(1 - 2\mathbb{I}(t \leq 0))\left(f_\epsilon(0) + f_\epsilon'(\psi(t))\tau_n t\right)dt,
\end{aligned}
$$

where $\psi(t)$ is between 0 and $\tau_n t$. Because $\int \mathcal{K}(-|t|)(1 - 2\mathbb{I}(t \leq 0))dt = 0$, we have $\int \mathcal{K}(-|t|)(1 - 2\mathbb{I}(t \leq 0))f_\epsilon(0)dt = 0$. Also, $f_\epsilon'(\cdot)$ is bounded (point $4'$), and therefore

$$
\left|\tau_n \int \mathcal{K}(-|t|)(1 - 2\mathbb{I}(t \leq 0))f_\epsilon'(\psi(t))\tau_n t dt\right| \leq M\tau_n^2,
$$

for some fixed $M > 0$. Therefore, we obtain

$$
\left|E\Big[\mathcal{K}(-d) - \mathbb{I}(d \leq 0)|\overline{A}_K, \overline{\boldsymbol{L}}_K\Big]\right| \leq M\tau_n^2. \tag{s29}
$$

It follows that, as $n \to \infty$,

$$
\begin{aligned}
&\left|E\left[\sqrt{n}\mathbb{P}_n\Big[\widetilde{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}) - \mathbb{U}^{\text{IPW}}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\Big]\right]\right| \\
=&\left|\sqrt{n}E\left[\frac{d_j(\overline{A}_K, \boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K, \overline{\boldsymbol{L}}_K, \boldsymbol{\alpha})}\Big(K(-d) - \mathbb{I}(d \leq 0)\Big)\right]\right| \\
&\qquad (\text{In above, } d_j(\overline{A}_K, \boldsymbol{Z};\boldsymbol{\theta}_q) \text{ is the } j^{th} \text{ element of } d(\overline{A}_K, \boldsymbol{Z};\boldsymbol{\theta}_q)) \\
=&\left|\sqrt{n}E\left[\frac{d_j(\overline{A}_K, \boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K, \overline{\boldsymbol{L}}_K, \boldsymbol{\alpha})}E\Big[K(-d) - \mathbb{I}(d \leq 0)\Big|\overline{A}_K, \overline{\boldsymbol{L}}_K\Big]\right]\right| \\
\leq&\sqrt{n} \times \sup_{(\boldsymbol{\alpha},\boldsymbol{\theta}_q)\in\boldsymbol{\Xi}'} E\left|\frac{d_j(\overline{A}_K, \boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K, \overline{\boldsymbol{L}}_K, \boldsymbol{\alpha})}\right| \times M\tau_n^2 \\
=&O(n^{1/2}\tau_n^2) = o(1).
\end{aligned}
$$

Now, we conclude (s27). Next, we shall prove (s28). Specifically,

$$
\begin{aligned}
&\text{Var}\left(\sqrt{n}\mathbb{P}_n\Big[\widetilde{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}) - \mathbb{U}^{\text{IPW}}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\Big]\right) \\
=&\text{Var}\left[\frac{d_j(\overline{A}_K, \boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K, \overline{\boldsymbol{L}}_K, \boldsymbol{\alpha})}\mathcal{K}(-|d|)(1 - 2\mathbb{I}(d \leq 0))\right] \\
\leq&E\left[\left(\frac{d_j(\overline{A}_K, \boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K, \overline{\boldsymbol{L}}_K, \boldsymbol{\alpha})}\mathcal{K}(-|d|)(1 - 2\mathbb{I}(d \leq 0))\right)^2\right]
\end{aligned}
$$

$$=E\left[\frac{d_j^2(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K^2(\overline{A}_K,\overline{\boldsymbol{L}}_K,\boldsymbol{\alpha})}\mathcal{K}^2(-|d|)\right]$$
$$=E\left[\frac{d_j^2(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K^2(\overline{A}_K,\overline{\boldsymbol{L}}_K,\boldsymbol{\alpha})}E\left[\mathcal{K}^2(-|d|)|\overline{A}_K,\overline{\boldsymbol{L}}_K\right]\right],$$

where

$$\begin{aligned}
&E[\mathcal{K}^2(-|d|)|\overline{A}_K,\overline{\boldsymbol{L}}_K]\\
=&\int_\epsilon \mathcal{K}^2(-|d|)f_\epsilon(\epsilon)d\epsilon=\tau_n\int_t \mathcal{K}^2(-|t|)f_\epsilon(\tau_n t)dt\\
=&\tau_n\int_{t>n^{1/4}}\mathcal{K}^2(-|t|)f_\epsilon(\tau_n t)dt+\tau_n\int_{t\le n^{1/4}}\mathcal{K}^2(-|t|)f_\epsilon(\tau_n t)dt\\
\le&\mathcal{K}^2(-|n^{1/4}|)\int_{t>n^{1/4}}f_\epsilon(\tau_n t)d(\tau_n t)+\tau_n n^{1/4}f_\epsilon(\psi)\quad(\psi \text{ is a value between } -\tau_n n^{1/4} \text{ and } \tau_n n^{1/4})\\
\le&\mathcal{K}^2(-n^{1/4})+\tau_n n^{1/4}f_\epsilon(\psi)\\
&\quad(\text{since } f_\epsilon(\psi)\le M \text{ for a constant } M \text{ as it is uniformly bounded (point } 4'), \text{ we have})\\
\le&\mathcal{K}^2(-n^{1/4})+M\tau_n n^{1/4}.
\end{aligned}$$

It suggests that

$$\begin{aligned}
\mathrm{Var}\Big(\sqrt{n}\mathbb{P}_n\Big[\widetilde{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})-\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\Big]\Big)\le&E\left[\frac{d_j^2(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K^2(\overline{A}_K,\overline{\boldsymbol{L}}_K,\boldsymbol{\alpha})}\right]\times(\mathcal{K}^2(-n^{1/4})+M\tau_n n^{1/4})\\
=&o(1).
\end{aligned}$$

This concludes that (s28) holds. We now complete the proof.

□

Lemma 10. *Define* $\widetilde{\boldsymbol{S}}^{IPW}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}=\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\alpha}}\widetilde{\mathbb{U}}^{IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\right]$, $\boldsymbol{\mathcal{S}}^{IPW}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}=E\left[\frac{\partial}{\partial\boldsymbol{\alpha}}\overline{\mathbb{U}}^{IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\right]$, $\widetilde{\boldsymbol{I}}^{IPW}_{\boldsymbol{\theta}_q}=\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\widetilde{\mathbb{U}}^{IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\right]$, *and* $\boldsymbol{\mathcal{I}}^{IPW}_{\boldsymbol{\theta}_q}=E\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\overline{\mathbb{U}}^{IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\right]$, *then we can show*

$$\widetilde{\boldsymbol{S}}^{IPW}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}=\boldsymbol{\mathcal{S}}^{IPW}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}+o_p(1)\quad \text{and}\quad \widetilde{\boldsymbol{I}}^{IPW}_{\boldsymbol{\theta}_q}=\boldsymbol{\mathcal{I}}^{IPW}_{\boldsymbol{\theta}_q}+o_p(1),$$

uniformly in $(\boldsymbol{\theta}_q,\boldsymbol{\alpha})\in\boldsymbol{\Xi}'$.

Proof. Here, we shall follow the notations used in the proof of Lemma 9. Let $\overline{\boldsymbol{S}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}=\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\alpha}}\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\right]$. We can show that

$$\begin{aligned}
\widetilde{\boldsymbol{S}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}-\overline{\boldsymbol{S}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}=&\mathbb{P}_n\left[\frac{\partial D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}}\Big(\mathcal{K}(-d)-F_\epsilon(0)\Big)\right]\\
=&E\left[\frac{\partial D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}}\Big(E[\mathcal{K}(-d)|\overline{A}_K,\overline{\boldsymbol{L}}_K]-F_\epsilon(0)\Big)\right]+o_p(1),
\end{aligned}$$

where

$$\Big|E[\mathcal{K}(d)|\overline{A}_K,\overline{\boldsymbol{L}}_K]-F_\epsilon(0)\Big|=\Big|E[\mathcal{K}(-d)|\overline{A}_k,\overline{\boldsymbol{L}}_K]-E[\mathbb{I}(d\le 0)|\overline{A}_k,\overline{\boldsymbol{L}}_K]\Big|$$

$$
\begin{aligned}
=&\left|E[\mathcal{K}(-d)-\mathbb{I}(d\leq 0)|\overline{A}_k,\overline{\boldsymbol{L}}_K]\right|\\
\leq& M\tau_n^2 \quad \text{(see equation (s29))}\\
=&o(n^{-1/2}).
\end{aligned}
$$

This suggests $\widetilde{\boldsymbol{S}}^{\text{IPW}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}=\overline{\boldsymbol{S}}^{\text{IPW}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}+o_p(1)$. Then, noticing $E\left[\overline{\boldsymbol{S}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}\right]=\boldsymbol{\mathcal{S}}^{\text{IPW}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}$, we have $\overline{\boldsymbol{S}}^{\text{IPW}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}=\boldsymbol{\mathcal{S}}^{\text{IPW}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}+o_p(1)$ by uniform law of large numbers. This concludes $\widetilde{\boldsymbol{S}}^{\text{IPW}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}=\boldsymbol{\mathcal{S}}^{\text{IPW}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}+o_p(1)$.

Define $h'(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)=\frac{\partial}{\partial\boldsymbol{\theta}_q}h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)$ and $\overline{\boldsymbol{I}}^{\text{IPW}}_{\boldsymbol{\theta}_q}=\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\overline{\mathbb{U}}^{\text{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\right]$. We can show that

$$
\begin{aligned}
\widetilde{\boldsymbol{I}}^{\text{IPW}}_{\boldsymbol{\theta}_q}-\overline{\boldsymbol{I}}^{\text{IPW}}_{\boldsymbol{\theta}_q}=&\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})\left(\mathcal{K}(d)-F_\epsilon(0)\right)\right]+\mathbb{P}_n\left[h'(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})\left(\frac{1}{\tau_n}\mathcal{K}'(d)-f_\epsilon(0)\right)\right]\\
=&E\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})\left(E\left[\mathcal{K}(d)|\overline{A}_K,\overline{\boldsymbol{L}}_K\right]-F_\epsilon(0)\right)\right]\\
&+E\left[h'(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)D(\boldsymbol{\theta}_q,\boldsymbol{\alpha})\left(E\left[\frac{1}{\tau_n}\mathcal{K}'(d)|\overline{A}_K,\overline{\boldsymbol{L}}_K\right]-f_\epsilon(0)\right)\right]+o_p(1) \quad \text{(s30)}
\end{aligned}
$$

Since $\left|E[\mathcal{K}(d)|\overline{A}_K,\overline{\boldsymbol{L}}_K]-F_\epsilon(0)\right|=o(1)$, the first term in the previous formula is $o(1)$. Moreover,

$$
\begin{aligned}
\left|\frac{1}{\tau_n}E[\mathcal{K}'(d)|\overline{A}_K,\overline{\boldsymbol{L}}_K]-f_\epsilon(0)\right|=&\left|\frac{1}{\tau_n}\int_\epsilon\mathcal{K}'\left(\frac{\epsilon}{\tau_n}\right)f_\epsilon(\epsilon)d\epsilon-f_\epsilon(0)\right|\\
=&\left|\int_t\mathcal{K}'(t)\,f_\epsilon(t\tau_n)dt-f_\epsilon(0)\right|\\
=&\left|\int_t\mathcal{K}'(t)\left(f_\epsilon(0)+t\tau_nf_\epsilon'(\psi(t))\right)dt-f_\epsilon(0)\right|\\
=&\left|\int_t\mathcal{K}'(t)\,t\tau_nf_\epsilon'(\psi(t))dt\right|\\
\leq&\tau_n\int_t\left|\mathcal{K}'(t)\,tf_\epsilon'(\psi(t))\right|dt,
\end{aligned}
$$

where $\psi(t)$ lies between 0 and τ_nt. Because $f_\epsilon'(\cdot)$ is uniformly bounded (point $4'$), there exists a positive constant M such that $f_\epsilon'(\psi(t))<M$, which implies $\tau_n\int_t\left|\mathcal{K}'(t)\,tf_\epsilon'(\psi(t))\right|dt\leq M\tau_n\int_t\left|\mathcal{K}'(t)\,t\right|dt=o(1)$. Now we conclude

$$
\left|\frac{1}{\tau_n}E[\mathcal{K}'(d)|\overline{A}_K,\overline{\boldsymbol{L}}_K]-f_\epsilon(0)\right|=o(1),
$$

which suggests the second term of (s30) is $o(1)$. Therefore, we conclude

$$
\widetilde{\boldsymbol{I}}^{\text{IPW}}_{\boldsymbol{\theta}_q}=\overline{\boldsymbol{I}}^{\text{IPW}}_{\boldsymbol{\theta}_q}+o_p(1).
$$

This and observing $\overline{\boldsymbol{I}}^{\text{IPW}}_{\boldsymbol{\theta}_q}=\boldsymbol{\mathcal{I}}^{\text{IPW}}_{\boldsymbol{\theta}_q}+o(1)$ by uniform law of large numbers, we conclude

$$
\widetilde{\boldsymbol{I}}^{\text{IPW}}_{\boldsymbol{\theta}_q}=\boldsymbol{\mathcal{I}}^{\text{IPW}}_{\boldsymbol{\theta}_q}+o_p(1).
$$

□

Now, we proceed the proof of Theorem S2. Combining (s21) and (s26), one can show

$$\sup_{\boldsymbol{\theta}_q\in\boldsymbol{\Theta}}\left\|\mathbb{P}_n[\widetilde{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}})]-\mathbb{P}_n[\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)]\right\|=o_p(n^{-1/2+\epsilon}),\quad \text{a.s.}$$

Then, following similar arguments for proving the $\sqrt{n}$-consistency of $\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}*}$ in (s22) of Lemma 8, one can also deduce that $\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}}$ is $\sqrt{n}$-consistent.

Next, we prove the asymptotic normality. According to Lemma 9, we have

$$\mathbb{P}_n\left[\widetilde{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\right]=\mathbb{P}_n\left[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\right]+o_p(n^{-1/2})$$

Based on the expansion of $\mathbb{P}_n\left[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\right]$ (see (s24)) in the proof of Lemma 8, we have

$$\begin{aligned}\mathbb{P}_n\left[\widetilde{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\right]=&\mathbb{P}_n[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)]\\&+\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\alpha}}\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}\right](\boldsymbol{\alpha}-\boldsymbol{\alpha}_0)\\&+\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}_0)\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right](\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0})+o_p(n^{-1/2}),\end{aligned}$$

for any $(\boldsymbol{\theta}_q,\boldsymbol{\alpha})$ in the $O(n^{-1/3})$ neighborhood of $(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)$. Because $(\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}},\widehat{\boldsymbol{\alpha}})$ is in the $n^{-1/3}$ neighborhood of $(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)$ and $\mathbb{P}_n\left[\widetilde{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}},\widehat{\boldsymbol{\alpha}})\right]=\mathbf{0}$, we have

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}}-\boldsymbol{\theta}_{q0})=-\left[\boldsymbol{\mathcal{I}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_{q0}}\right]^{-1}\sqrt{n}\mathbb{P}_n\left[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)-\boldsymbol{\mathcal{S}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0}\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}_0}\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha}_0)\right]+o_p(1).\tag{s31}$$

This concludes that $\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}}$ and $\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}*}$ share the same influence function and therefore they are asymptotically equivalent such that $\sqrt{n}(\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}}-\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}*})=o_p(1)$.

By (s31), we can deduce that $\sqrt{n}(\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}}-\boldsymbol{\theta}_{q0})$ converges to a multivariate normal distribution with mean zero and a finite variance-covariance matrix

$$\boldsymbol{\Sigma}^{\mathrm{IPW}}=\left[\boldsymbol{\mathcal{I}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_{q0}}\right]^{-1}\boldsymbol{\mathcal{V}}^{\mathrm{IPW}}\left[\boldsymbol{\mathcal{I}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_{q0}}\right]^{-T}\tag{s32}$$

where $\boldsymbol{\mathcal{V}}^{\mathrm{IPW}}=E\left[\left(\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)-\boldsymbol{\mathcal{S}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0}\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}_0}\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha}_0)\right)^{\otimes 2}\right]$. To show $\widehat{\boldsymbol{\Sigma}}^{\mathrm{IPW}}$ defined below Theorem S2 is a consistent estimator of $\boldsymbol{\Sigma}^{\mathrm{IPW}}$, one may use the asymptotic normality results for $(\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\theta}}_q^{\mathrm{IPW}})$ along with $\mathbb{P}_n[\widetilde{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)]=\mathbb{P}_n[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)]+o_p(n^{-1/2})$, $\widetilde{\boldsymbol{S}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}=\boldsymbol{\mathcal{S}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}+o_p(1)$ and $\widetilde{\boldsymbol{I}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}=\boldsymbol{\mathcal{I}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}+o_p(1)$ (Lemma 10).

Appendix E: Asymptotic properties of the doubly robust estimators (Proof of Theorem 2)

Recall that the original doubly robust estimator $\widehat{\boldsymbol{\theta}}_q^{\mathrm{DR}*}$ is obtained by solving the doubly robust estimating equation, $\mathbb{P}_n\left[\mathbb{U}_{\boldsymbol{\theta}_q}^{\mathrm{DR}}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})\right]=\mathbf{0}$. Because this estimating equation is discontinuous with respect to $\boldsymbol{\theta}_q$, we propose to solve the smoothed doubly robust estimating equation, $\mathbb{P}_n\left[\widetilde{\mathbb{U}}_{\boldsymbol{\theta}_q}^{\mathrm{DR}}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})\right]=\mathbf{0}$, to obtain $\widehat{\boldsymbol{\theta}}_q^{\mathrm{DR}}$. Also, we define

$$
\begin{aligned}
&\mathbb{P}_n\left\{\overline{\mathbb{U}}_{\boldsymbol{\theta}_q}^{\mathrm{DR}}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\right\}\\
=&\mathbb{P}_n\left\{\overline{\mathbb{U}}_{\boldsymbol{\theta}_q}^{(K)}(\boldsymbol{O};\boldsymbol{\theta}_q,\overline{\boldsymbol{\alpha}}_K,\boldsymbol{\beta}_K)+\sum_{k=1}^{K-1}\mathbb{U}_{\boldsymbol{\theta}_q}^{(k)}(\boldsymbol{O};\boldsymbol{\theta}_q,\overline{\boldsymbol{\alpha}}_k,\boldsymbol{\beta}_k,\boldsymbol{\beta}_{k+1})+\mathbb{U}_{\boldsymbol{\theta}_q}^{(0)}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\beta}_1)\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
&\overline{\mathbb{U}}_{\boldsymbol{\theta}_q}^{(K)}(\boldsymbol{O};\boldsymbol{\theta}_q,\overline{\boldsymbol{\alpha}}_K,\boldsymbol{\beta}_K)\\
=&\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\alpha})}\left\{F_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}\Big(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)|\overline{A}_K,\overline{\boldsymbol{L}}_K\Big)-\Psi_K\Big(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\beta}_K\Big)\right\}
\end{aligned}
$$

and $F_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}$ is the true cumulative density function of Y given $\overline{A}_K$ and $\overline{\boldsymbol{L}}_K$.

In this section, we investigate the asymptotic properties of $\widehat{\boldsymbol{\theta}}_q^{\mathrm{DR}*}$ and $\widehat{\boldsymbol{\theta}}_q^{\mathrm{DR}}$. Assume the following:

- $1''$. Assume that either the sequence of propensity score models $\pi_k(\overline{A}_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_k)$, $k=1,\dots,K$, or the sequence of outcome regression models, $\psi_k(Y_{\overline{A}_k,\underline{a}_{k+1}},\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_k)$, $k=1,\dots,K$, are correctly specified. Let $\boldsymbol{\alpha}_0=[\boldsymbol{\alpha}_{1,0}^T,\dots,\boldsymbol{\alpha}_{K,0}^T]^T$ and $\boldsymbol{\beta}_0=[\boldsymbol{\beta}_{1,0}^T,\dots,\boldsymbol{\beta}_{K,0}^T]^T$ be the true parameter of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively.

- $2''$. There exists a finite vector $\boldsymbol{\alpha}^*$ such that $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha})]=\mathbf{0}$, where $\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha})$ is the estimating score for $\boldsymbol{\alpha}$ (also see Appendix D for the definition). And there exists a finite vector $\boldsymbol{\beta}^*$ such that $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta})]=\mathbf{0}$, where $\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta})$ is the estimating score for $\boldsymbol{\beta}$ (also see Appendix B for the definition). If the propensity score models are correctly specified, then $\boldsymbol{\alpha}^*=\boldsymbol{\alpha}_0$; if the outcome regression models are correctly specified, then $\boldsymbol{\beta}^*=\boldsymbol{\beta}_0$.

- $3''$ Let $\boldsymbol{\Xi}''=\boldsymbol{A}\times\boldsymbol{B}\times\boldsymbol{\Theta}$ be a bounded convex neighborhood of $(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*,\boldsymbol{\theta}_{q0})$. Suppose that $\mathbb{U}_{\boldsymbol{\theta}_q}^{(K)}(\boldsymbol{O};\boldsymbol{\theta}_q,\overline{\boldsymbol{\alpha}}_K,\boldsymbol{\beta}_K)$ is dominated by certain square-integrable function and is also P-Glivenko-Cantelli for $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}_q)\in\boldsymbol{\Xi}''$. Assume that the functions $\{\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha}),\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta}),\widetilde{\mathbb{U}}_{\boldsymbol{\theta}_q}^{(K)}(\boldsymbol{O};\boldsymbol{\theta}_q,\overline{\boldsymbol{\alpha}}_K,\boldsymbol{\beta}),\overline{\mathbb{U}}_{\boldsymbol{\theta}_q}^{(K)}(\boldsymbol{O};\boldsymbol{\theta}_q,\overline{\boldsymbol{\alpha}}_K,\boldsymbol{\beta}),\mathbb{U}_{\boldsymbol{\theta}_q}^{(0)}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\beta}_1),\mathbb{U}_{\boldsymbol{\theta}_q}^{(k)}(\boldsymbol{O};\boldsymbol{\theta}_q,\overline{\boldsymbol{\alpha}}_k,\boldsymbol{\beta}_k,\boldsymbol{\beta}_{k+1}),k=1,\dots,K-1\}$ and their first order derivatives with respect to $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}_q)$ are continuous, dominated by certain square-integrable functions, and are also P-Glivenko-Cantelli for $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}_q)\in\boldsymbol{\Xi}''$. Further assume that $\{\pi_k(\overline{A}_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_k),k=1,\dots,K\}$ is bounded away from 0 for any $\boldsymbol{\alpha}\in\boldsymbol{A}$.

- $4''$. Assume that $\boldsymbol{\mathcal{I}}_{\boldsymbol{\alpha}^*}=\frac{\partial}{\partial\boldsymbol{\alpha}}E\left[\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha})\right]\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^*}$ and $\boldsymbol{\mathcal{I}}_{\boldsymbol{\beta}^*}=\frac{\partial}{\partial\boldsymbol{\beta}}E\left[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta})\right]\Big|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}$ are nagative definite. Also, $\boldsymbol{\mathcal{I}}_{\boldsymbol{\theta}_{q0}}^{\mathrm{DR}}=\frac{\partial}{\partial\boldsymbol{\theta}_q}E\left[\overline{\mathbb{U}}_{\boldsymbol{\theta}_q}^{\mathrm{DR}}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\right]\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}$ is negative definite.

- $5''$. Define $f'_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{A}_K,\overline{\boldsymbol{L}}_K)=\frac{\partial}{\partial y}f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\overline{\boldsymbol{L}}_K)$. Assume that $\{F_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{A}_K,\overline{\boldsymbol{L}}_K)$, $f_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{A}_K,\overline{\boldsymbol{L}}_K), f'_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{A}_K,\overline{\boldsymbol{L}}_K)\}$ are absolutely continuous with respect to $y\in[y_{\min},y_{\max}]$ and uniformly bounded.

Lemma 11. *Under points 2″–4″, we have*

$$\sqrt{n}(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)=-\boldsymbol{\mathcal{I}}_{\boldsymbol{\alpha}^*}^{-1}\sqrt{n}\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha}^*)]+o_p(1),$$
$$\sqrt{n}(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}^*)=-\boldsymbol{\mathcal{I}}_{\boldsymbol{\beta}^*}^{-1}\sqrt{n}\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta}^*)]+o_p(1).$$

Proof. The proof is similar to that of Lemma 3 and 5 and is omitted here. □

Lemma 12. *Under points 1″ and 2′, we have*

(i) The original doubly robust estimating equation, $\mathbb{P}_n\left[\mathbb{U}^{DR}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\right]$, *is unbiased such that* $E\left[\mathbb{U}^{DR}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\right]=\mathbf{0}$.

(ii) Moreover, $\mathbb{P}_n\left[\overline{\mathbb{U}}^{DR}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\right]$ *is also an unbiased estimating equation such that* $E\left[\overline{\mathbb{U}}^{DR}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\right]=\mathbf{0}$.

Proof. Here we only prove item (i), where item (ii) can be proved by the same strategy. Item (i) is a conclusion of the following two facts.

(1) $E\left[\mathbb{U}^{\mathbf{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\right]=\mathbf{0}$ **if the outcome models are correct** $(\boldsymbol{\beta}^*=\boldsymbol{\beta}_0)$.

Notice that, if the outcome models are correct ($\boldsymbol{\beta}^*=\boldsymbol{\beta}_0$), we have $\Psi_k(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_{k,0})=F_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|\overline{A}_k,\overline{\boldsymbol{L}}_k)$, $k=1,\ldots,K$. First, we can show

$$\begin{aligned}
&E\left[\mathbb{U}^{(K)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}_0)|\overline{A}_K,\overline{\boldsymbol{L}}_K\right]\\
=&E\left[\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\alpha}^*)}\left\{\mathbb{I}\Big(Y\le h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big)-\Psi_K\Big(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}),\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\beta}_{K,0}\Big)\right\}\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right]\\
=&\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\alpha}^*)}\left\{E\left[\mathbb{I}\Big(Y\le h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big)\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right]-F_{Y_{\overline{A}_K}|\overline{A}_K,\overline{\boldsymbol{L}}_K}\Big(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})|\overline{A}_K,\overline{\boldsymbol{L}}_K\Big)\right\}\\
=&\mathbf{0},
\end{aligned}$$

which suggests $E\left[\mathbb{U}^{(K)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}_0)\right]=\mathbf{0}$. Similarly, we can show that, for $k=1,\ldots,K-1$,

$$\begin{aligned}
&E\left[\mathbb{U}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}_0)|\overline{A}_k,\overline{\boldsymbol{L}}_k\right]\\
=&\sum_{\underline{a}_{k+1}}\frac{d(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_{q0})}{\overline{\pi}_k(\overline{A}_k,\overline{\boldsymbol{L}}_k;\overline{\boldsymbol{\alpha}}_k^*)}E\Bigg[\Big\{\Psi_{k+1}\Big(h(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_{q0}),\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\boldsymbol{\beta}_{k+1,0}\Big)-\\
&\qquad\Psi_k\Big(h(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_{q0}),\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_{k,0}\Big)\Big\}\Big|\overline{A}_k,\overline{\boldsymbol{L}}_k\Bigg]\\
=&\sum_{\underline{a}_{k+1}}\frac{d(\overline{A}_k,\underline{a}_{k+1},,\boldsymbol{Z};\boldsymbol{\theta}_{q0})}{\overline{\pi}_k(\overline{A}_k,\overline{\boldsymbol{L}}_k;\overline{\boldsymbol{\alpha}}_k^*)}\Bigg\{E\left[F_{Y_{\overline{A}_{k+1},\underline{a}_{k+2}}|\overline{A}_{k+1},\overline{\boldsymbol{L}}_{k+1}}\Big(h(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_{q0})|\overline{A}_k,a_{k+1},\overline{\boldsymbol{L}}_{k+1}\Big)\Big|\overline{A}_k,\overline{\boldsymbol{L}}_k\right]
\end{aligned}$$

$$- F_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}\Big(h(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_{q0})|\overline{A}_k,\overline{\boldsymbol{L}}_k\Big)\Big\}$$

where

$$\begin{aligned}
& E\left[F_{Y_{\overline{A}_{k+1},\underline{a}_{k+2}}|\overline{A}_{k+1},\overline{\boldsymbol{L}}_{k+1}}\Big(h(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_{q0})|\overline{A}_k,a_{k+1},\overline{\boldsymbol{L}}_{k+1}\Big)\Big|\overline{A}_k,\overline{\boldsymbol{L}}_k\right] \\
=& E\left[E\left[\mathbb{I}(Y_{\overline{A}_{k+1},\underline{a}_{k+2}} \le h(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_{q0}))|\overline{A}_k,a_{k+1},\overline{\boldsymbol{L}}_{k+1}\right]\Big|\overline{A}_k,\overline{\boldsymbol{L}}_k\right] \\
=& E\left[E\left[\mathbb{I}(Y_{\overline{A}_k,\underline{a}_{k+1}} \le h(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_{q0}))|\overline{A}_k,\overline{\boldsymbol{L}}_{k+1}\right]\Big|\overline{A}_k,\overline{\boldsymbol{L}}_k\right] \quad (\text{by A2 } Y_{\overline{a}_K} \perp A_{k+1}|\overline{A}_k=\overline{a}_k,\overline{\boldsymbol{L}}_{k+1}) \\
=& E\left[\mathbb{I}(Y_{\overline{A}_k,\underline{a}_{k+1}} \le h(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_{q0}))\Big|\overline{A}_k,\overline{\boldsymbol{L}}_k\right] = F_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}\Big(h(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_{q0})|\overline{A}_k,\overline{\boldsymbol{L}}_k\Big).
\end{aligned}$$

It follows that $E\left[\mathbb{U}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}_0)|\overline{A}_k,\overline{\boldsymbol{L}}_k\right]=\mathbf{0}$ and thus $E\left[\mathbb{U}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}_0)\right]=\mathbf{0}$, for $k=1,\ldots,K-1$. Finally, from the arguments in the proof of Lemma 2, we know $E\left[\mathbb{U}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}_0)\right]=\mathbf{0}$.

Combining the previous discussions, we conclude

$$E\left[\mathbb{U}^{\text{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\right]=\sum_{k=0}^{K}E\left[\mathbb{U}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}_0)\right]=\mathbf{0}.$$

(2) $E\left[\mathbb{U}^{\mathbf{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\right]=\mathbf{0}$ if the propensity score models are correct ($\boldsymbol{\alpha}^*=\boldsymbol{\alpha}_0$)

We can rewrite the doubly robust estimating score as

$$\mathbb{U}^{\text{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})=\sum_{k=0}^{K}\mathbb{T}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta}),$$

where

$$\begin{aligned}
\mathbb{T}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\alpha},\boldsymbol{\beta}) &= \frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\alpha})}\times\mathbb{I}\left(Y\le h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\right)-\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\times q, \\
\mathbb{T}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta}) &= \sum_{\underline{a}_k\in\underline{\mathbb{A}}_k}\frac{d(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_{k-1}(\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1};\overline{\boldsymbol{\alpha}}_{k-1})}\times\Psi_k\left(h(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{A}_{k-1},\underline{a}_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_k\right) \\
&\quad -\sum_{\underline{a}_{k+1}\in\underline{\mathbb{A}}_{k+1}}\frac{d(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_k(\overline{A}_k,\overline{\boldsymbol{L}}_k;\overline{\boldsymbol{\alpha}}_k)}\times\Psi_k(h(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_k),
\end{aligned}$$

for $k=1,\ldots,K$, and $\overline{\pi}_0(\overline{A}_{-1},\overline{\boldsymbol{L}}_{-1})$ is defined as 1 and $\sum_{\underline{a}_{K+1}\in\underline{\mathbb{A}}_{K+1}}[V]$ used in $\mathbb{T}^{(K)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})$ is defined as V.

We can show

$$E[\mathbb{T}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0,\boldsymbol{\beta}^*)] = E\left[\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K;\overline{\boldsymbol{\alpha}}_K)}\times\mathbb{I}\left(Y\le h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\right)-\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\times q\right]$$

$$
\begin{aligned}
&= E\left[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} \frac{d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})}{\overline{\pi}_K(\overline{a}_K, \overline{\boldsymbol{L}}_K; \overline{\boldsymbol{\alpha}}_{K,0})} \Big\{ \mathbb{I}\left(Y_{\overline{a}_K} \le h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})\right) - q \Big\} \right] \\
&= E\left[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0) \right].
\end{aligned}
$$

Because the IPW estimating score is unbiased when $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0$(see Lemma 7), we conclude $E[\mathbb{T}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*)] = E[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0)] = \boldsymbol{0}$.

Next, we can show that the expectation of the second term in $\mathbb{T}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*)$ ($k = 1, \ldots, K$) is

$$
\begin{aligned}
& E\left[\sum_{\underline{a}_{k+1} \in \underline{\mathbb{A}}_{k+1}} \frac{d(\overline{A}_k, \underline{a}_{k+1}, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})}{\overline{\pi}_k(\overline{A}_k, \overline{\boldsymbol{L}}_k; \overline{\boldsymbol{\alpha}}_{k,0})} \times \Psi_k(h(\overline{A}_k, \underline{a}_{k+1}, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}), \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}^*_k)\right] \\
&= E\left[\sum_{\underline{a}_k \in \underline{\mathbb{A}}_k} \frac{d(\overline{A}_{k-1}, \underline{a}_k, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}) \times \mathbb{I}(A_k = a_k)}{\overline{\pi}_{k-1}(\overline{A}_{k-1}, \overline{\boldsymbol{L}}_{k-1}; \overline{\boldsymbol{\alpha}}_{k-1,0}) \times \pi_k(a_k, \overline{A}_{k-1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\alpha}_{k,0})} \times \Psi_k\Big(h(\overline{A}_{k-1}, \underline{a}_k, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}), \overline{A}_{k-1}, \underline{a}_k, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}^*_k\Big)\right] \\
&= E\left\{\sum_{\underline{a}_k \in \underline{\mathbb{A}}_k} E\left[\frac{d(\overline{A}_{k-1}, \underline{a}_k, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}) \times \mathbb{I}(A_k = a_k)}{\overline{\pi}_{k-1}(\overline{A}_{k-1}, \overline{\boldsymbol{L}}_{k-1}; \overline{\boldsymbol{\alpha}}_{k-1,0}) \times \pi_k(a_k, \overline{A}_{k-1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\alpha}_{k,0})}\right.\right. \\
&\qquad \left.\left. \times \Psi_k\Big(h(\overline{A}_{k-1}, \underline{a}_k, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}), \overline{A}_{k-1}, \underline{a}_k, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}^*_k\Big) \Big| \overline{A}_{k-1}, a_k, \overline{\boldsymbol{L}}_k \right]\right\} \\
&= E\left\{\sum_{\underline{a}_k \in \underline{\mathbb{A}}_k} \frac{d(\overline{A}_{k-1}, \underline{a}_k, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})}{\overline{\pi}_{k-1}(\overline{A}_{k-1}, \overline{\boldsymbol{L}}_{k-1}; \overline{\boldsymbol{\alpha}}_{k-1,0})} \times \Psi_k\Big(h(\overline{A}_{k-1}, \underline{a}_k, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}), \overline{A}_{k-1}, \underline{a}_k, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}^*_k\Big) \times \right. \\
&\qquad \left. \underbrace{E\left[\frac{\mathbb{I}(A_k = a_k)}{\pi_k(a_k, \overline{A}_{k-1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\alpha}_{k,0})} \Big| \overline{A}_{k-1}, a_k, \overline{\boldsymbol{L}}_k\right]}_{=1, \text{ by } E[\mathbb{I}(A_k = a_k) | \overline{A}_{k-1}, a_k, \overline{\boldsymbol{L}}_k] = \pi_k(a_k, \overline{A}_{k-1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\alpha}_{k,0})} \right\} \\
&= E\left[\sum_{\underline{a}_k \in \underline{\mathbb{A}}_k} \frac{d(\overline{A}_{k-1}, \underline{a}_k, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})}{\overline{\pi}_{k-1}(\overline{A}_{k-1}, \overline{\boldsymbol{L}}_{k-1}; \overline{\boldsymbol{\alpha}}_{k-1,0})} \times \Psi_k\Big(h(\overline{A}_{k-1}, \underline{a}_k, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}), \overline{A}_{k-1}, \underline{a}_k, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}^*_k\Big)\right],
\end{aligned}
$$

which is actually the expectation of the first term in $\mathbb{T}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*)$. This suggests that $E[\mathbb{T}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*)] = 0$. This completes the proof. □

Lemma 13. *Under points 1″–5″, the doubly robust estimator, $\widehat{\boldsymbol{\theta}}_q^{DR*}$, by solving the original doubly robust estimating equation $\mathbb{P}_n[\mathbb{U}^{DR}_{\theta_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})] = \boldsymbol{0}$ is CAN.*

Proof. Lemma 11 suggests that $\widehat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}^* + o_p(n^{-1/2+\epsilon})$ and $\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta}^* + o_p(n^{-1/2+\epsilon})$. This, along with point 3″, can be used to show that, for any $\epsilon > 0$,

$$
\sup_{\boldsymbol{\theta}_q \in \boldsymbol{\Theta}} \left\| \mathbb{P}_n[\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})] - \mathbb{P}_n[\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)] \right\| = o_p(n^{-1/2+\epsilon}), \quad \text{a.s.} \tag{s33}
$$

Moreover, oberving that $E\Big[\mathbb{U}^{\mathrm{DR}}_{\theta_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) - \overline{\mathbb{U}}^{\mathrm{DR}}_{\theta_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)\Big] = \boldsymbol{0}$, we can show

$$
\sup_{(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta}_q) \in \boldsymbol{\Xi}''} \left\| \mathbb{P}_n[\mathbb{U}^{\mathrm{DR}}_{\theta_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)] - \mathbb{P}_n[\overline{\mathbb{U}}^{\mathrm{DR}}_{\theta_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)] \right\| = o(n^{-1/2+\epsilon}) \quad a.s.
$$

by uniform law of large numbers. This and (s33) suggest that

$$\sup_{\boldsymbol{\theta}_q\in\boldsymbol{\Theta}}\left\|\mathbb{P}_n[\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})]-\mathbb{P}_n[\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)]\right\|=o(n^{-1/2+\epsilon}),\quad \text{a.s.} \tag{s34}$$

Notice that $E[\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)]=\mathbf{0}$ (Lemma 12) and $\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right]$ converges to a negative definite matrix, $\boldsymbol{\mathcal{I}}^{\mathrm{DR}}_{\boldsymbol{\theta}_{q0}}$ (point 4″), which implies existence of a positive constant M such that

$$\begin{aligned}\|\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0}\| &\le M\left\|E\left[\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)-\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\right]\right\|\\ &= M\left\|E\left[\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\right]\right\|\\ &\le M\left\|\mathbb{P}_n\left[\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\right]\right\|+o_p(n^{-1/2+\epsilon})\end{aligned}$$

for $\boldsymbol{\theta}_q$ in a small neighborhood of $\boldsymbol{\theta}_{q0}$. This, combined with (s34), suggests that

$$\|\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0}\|\le M\left\|\mathbb{P}_n\left[\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})\right]\right\|+o_p(n^{-1/2+\epsilon}).$$

We can then replace $\boldsymbol{\theta}_q$ in the previous equation by $\widehat{\boldsymbol{\theta}}_q^{\mathrm{DR}*}$ and also recall that $\widehat{\boldsymbol{\theta}}_q^{\mathrm{DR}*}$ solves $\mathbb{P}_n\left[\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\theta}})\right]=\mathbf{0}$, we have obtained

$$\|\widehat{\boldsymbol{\theta}}_q^{\mathrm{DR}*}-\boldsymbol{\theta}_{q0}\|=o_p(n^{-1/2+\epsilon}) \tag{s35}$$

such that $\widehat{\boldsymbol{\theta}}_q^{\mathrm{DR}*}$ is $\sqrt{n}$-consistent.

Next, we prove the asymptotic normality. Specifically, we can show

$$\begin{aligned}&\mathbb{P}_n\Big[\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})-\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\Big]\\ =&\mathbb{P}_n\Big[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})-\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*)\Big]+\mathbb{P}_n\Big[\mathbb{H}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})-\mathbb{H}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\Big],\end{aligned}$$

where $\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})$ is the unsmoothed IPW estimating score and

$$\begin{aligned}\mathbb{H}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})=&-\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\alpha})}\times\Psi_K\Big(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\beta}_K\Big)\\ &+\sum_{k=1}^{K-1}\mathbb{U}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\overline{\boldsymbol{\alpha}}_k,\boldsymbol{\beta}_k,\boldsymbol{\beta}_{k+1})+\mathbb{U}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\beta}_1).\end{aligned}$$

From the arguments in the proof of Lemma 8, one can easily verify that

$$\mathbb{P}_n\Big[\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})-\mathbb{U}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*)\Big]=\mathbb{P}_n\Big[\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})-\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*)\Big]+o_p(n^{-1/2}),$$

for any $(\boldsymbol{\theta}_q,\boldsymbol{\alpha})$ in the $O(n^{-1/3})$ neighborhood of $(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*)$. This suggests that

$$\mathbb{P}_n\Big[\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})-\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\Big]$$

$$=\mathbb{P}_n\Big[\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})-\overline{\mathbb{U}}^{\mathrm{IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*)\Big]+\mathbb{P}_n\Big[\mathbb{H}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})-\mathbb{H}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\Big]+o_p(n^{-1/2})$$
$$=\mathbb{P}_n\Big[\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})-\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\Big]+o_p(n^{-1/2})$$

By a Taylor's expansion of $\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})$ around $(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0,\boldsymbol{\beta}_0)$, one can obtain

$$\begin{aligned}&\mathbb{P}_n\Big[\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})-\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\Big]\\
=&\mathbb{P}_n\Big[\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\Big]+\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\alpha}}\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha},\boldsymbol{\beta}^*)\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^*}\right](\boldsymbol{\alpha}-\boldsymbol{\alpha}^*)\\
&+\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\alpha}}\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta})\Big|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}\right](\boldsymbol{\beta}-\boldsymbol{\beta}^*)+\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right](\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0})\\
&+o_p(n^{-1/2})\end{aligned}\tag{s36}$$

Because $(\widehat{\boldsymbol{\theta}}^{\mathrm{DR}*}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})$ is in the $O(n^{-1/3})$ neighborhood of $(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$ with probability tending to 1, $\mathbb{P}_n\Big[\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}^{\mathrm{DR}}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})\Big]=\mathbf{0}$, and $\mathbb{P}_n\Big[\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\Big]=\mathbf{0}$, we have that

$$\begin{aligned}\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\mathrm{DR}*}_q-\boldsymbol{\theta}_{q0})=-\Big[\boldsymbol{\mathcal{I}}^{\mathrm{DR}}_{\boldsymbol{\theta}_{q0}}\Big]^{-1}\sqrt{n}\mathbb{P}_n\Big[&\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)-\boldsymbol{\mathcal{S}}^{\mathrm{DR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*}\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}^*}\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha}^*)\\
&-\boldsymbol{\mathcal{S}}^{\mathrm{DR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\beta}^*}\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\beta}^*}\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta}^*)\Big]+o_p(1),\end{aligned}\tag{s37}$$

where $\boldsymbol{\mathcal{S}}^{\mathrm{DR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*}=E\left[\frac{\partial}{\partial\boldsymbol{\alpha}}\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha},\boldsymbol{\beta}^*)\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^*}\right]$ and $\boldsymbol{\mathcal{S}}^{\mathrm{DR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\beta}^*}=\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\beta}}\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta})\Big|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}\right]$. Note that the $\mathbb{P}_n$-term in the previous formula has mean zero. Then, by applying the central limit theorem to the right-hand side of the previous formula, one can easily verify that $\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\mathrm{DR}*}_q-\boldsymbol{\theta}_{q0})$ converges to a zero-mean multivariate normal distribution with mean zero and a finite variance-covariance matrix. This completes the proof. □

Lemma 14. *Under points 1″–5″ and conditions (B1)–(B2) regarding the local distribution function $\mathcal{K}(.)$ and bandwidth τ_n, the original doubly robust estimating equation, $\mathbb{P}_n[\mathbb{U}^{DR}_n(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})]$, and the smoothed doubly robust estimating equation, $\mathbb{P}_n[\widetilde{\mathbb{U}}^{DR}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})]$, are asymptotically equivalent such that*

$$\sqrt{n}\mathbb{P}_n\Big[\widetilde{\mathbb{U}}^{DR}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})-\mathbb{U}^{DR}_n(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\Big]=o_p(1),\tag{s38}$$

uniformly in $(\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\in\boldsymbol{\Xi}''$.

Proof. The proof is similar to that for Lemma 9 and is omitted for brevity. □

Lemma 15. *Define $\widetilde{\boldsymbol{S}}^{DR}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}=\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\alpha}}\widetilde{\mathbb{U}}^{DR}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha})\right]$, $\boldsymbol{\mathcal{S}}^{IPW}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta}}=E\left[\frac{\partial}{\partial\boldsymbol{\alpha}}\overline{\mathbb{U}}^{DR}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\right]$, $\widetilde{\boldsymbol{I}}^{DR}_{\boldsymbol{\theta}_q}=\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\widetilde{\mathbb{U}}^{IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\right]$, and $\boldsymbol{\mathcal{I}}^{DR}_{\boldsymbol{\theta}_q}=E\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\overline{\mathbb{U}}^{DR}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\right]$, then we can show*

$$\widetilde{\boldsymbol{S}}^{DR}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}=\boldsymbol{\mathcal{S}}^{DR}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}+o_p(1)\quad\textit{and}\quad\widetilde{\boldsymbol{I}}^{DR}_{\boldsymbol{\theta}_q}=\boldsymbol{\mathcal{I}}^{DR}_{\boldsymbol{\theta}_q}+o_p(1),$$

uniformly in $(\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\in\boldsymbol{\Xi}''$.

Proof. The proof is similar to that for Lemma 10 and is omitted for brevity. □

Proof of Theorem 2

Combining (s34) and (s38), one can show

$$\sup_{\boldsymbol{\theta}_q\in\boldsymbol{\Theta}}\left\|\mathbb{P}_n[\widetilde{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})]-\mathbb{P}_n[\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)]\right\|=o(n^{-1/2+\epsilon}),\quad \text{a.s.}$$

Then, following arguments for proving the $\sqrt{n}$-consistency of $\widehat{\boldsymbol{\theta}}^{\mathrm{DR}*}_q$ in (s35) of Lemma 13, one can also deduce that $\widehat{\boldsymbol{\theta}}^{\mathrm{DR}}_q$ is $\sqrt{n}$-consistent.

Next, we prove the asymptotic normality. According to Lemma 14, we have

$$\mathbb{P}_n\Big[\widetilde{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\Big]=\mathbb{P}_n\Big[\mathbb{U}^{\mathrm{DR}}_{n}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\Big]+o_p(n^{-1/2})$$

By the expansion of $\mathbb{P}_n\Big[\mathbb{U}^{\mathrm{DR}}_{n}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\Big]$ (see (s36)) in the proof of Lemma 13, we have

$$\begin{aligned}\mathbb{P}_n\Big[\widetilde{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\Big]=&\mathbb{P}_n[\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)]\\&+\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\alpha}}\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha},\boldsymbol{\beta}^*)\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^*}\right](\boldsymbol{\alpha}-\boldsymbol{\alpha}^*)\\&+\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\beta}}\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta})\Big|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}\right](\boldsymbol{\beta}-\boldsymbol{\beta}^*)\\&+\mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\overline{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right](\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0})+o_p(n^{-1/2}),\end{aligned}$$

for any $(\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})$ in the $O(n^{-1/3})$ neighborhood of $(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$. Because $(\widehat{\boldsymbol{\theta}}^{\mathrm{DR}}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})$ is in the $O(n^{-1/3})$ neighborhood of $(\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)$ with probability tending to 1 and $\mathbb{P}_n\Big[\widetilde{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}^{\mathrm{DR}}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})\Big]=\boldsymbol{0}$, we have

$$\begin{aligned}\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\mathrm{DR}}_q-\boldsymbol{\theta}_{q0})=-&\Big[\boldsymbol{\mathcal{I}}^{\mathrm{DR}}_{\boldsymbol{\theta}_{q0}}\Big]^{-1}\sqrt{n}\mathbb{P}_n\Big[\mathbb{U}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)-\boldsymbol{\mathcal{S}}^{\mathrm{DR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*}\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}^*}\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha}^*)\\&-\boldsymbol{\mathcal{S}}^{\mathrm{DR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\beta}^*}\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\beta}^*}\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta}^*)\Big]+o_p(1).\end{aligned}$$

This concludes that $\widehat{\boldsymbol{\theta}}^{\mathrm{DR}}_q$ and $\widehat{\boldsymbol{\theta}}^{\mathrm{DR}*}_q$ share the same influence function and therefore they are asymptotically equivalent such that $\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\mathrm{DR}}_q-\widehat{\boldsymbol{\theta}}^{\mathrm{DR}*}_q)=o_p(1)$. In addition, we can deduce that $\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\mathrm{DR}}_q-\boldsymbol{\theta}_{q0})$ converges to a multivariate normal distribution with mean zero and a finite variance-covariance matrix

$$\boldsymbol{\Sigma}^{\mathrm{DR}}=\Big[\boldsymbol{\mathcal{I}}^{\mathrm{DR}}_{\boldsymbol{\theta}_{q0}}\Big]^{-1}\boldsymbol{\mathcal{V}}^{\mathrm{DR}}\Big[\boldsymbol{\mathcal{I}}^{\mathrm{DR}}_{\boldsymbol{\theta}_{q0}}\Big]^{-T}$$

where $\boldsymbol{\mathcal{V}}^{\mathrm{DR}}=E\left[\Big(\widetilde{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*)-\boldsymbol{\mathcal{S}}^{\mathrm{DR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*}\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}^*}\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\boldsymbol{\alpha}^*)-\boldsymbol{\mathcal{S}}^{\mathrm{DR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\beta}^*}\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\beta}^*}\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\boldsymbol{\beta}^*)\Big)^{\otimes 2}\right]$.

We can show that $\boldsymbol{\Sigma}^{\mathrm{DR}}$ can be consistently estimated by $\widehat{\boldsymbol{\Sigma}}^{\mathrm{DR}}=\Big[\widehat{\boldsymbol{\mathcal{I}}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}\Big]^{-1}\widehat{\boldsymbol{\mathcal{V}}}^{\mathrm{DR}}\Big[\widehat{\boldsymbol{\mathcal{I}}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}\Big]^{-T}$, where $\widehat{\boldsymbol{\mathcal{I}}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}=\mathbb{P}_n\frac{\partial}{\partial\boldsymbol{\theta}_q}\widetilde{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}^{\mathrm{DR}}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})$,

$$\widehat{\boldsymbol{\mathcal{V}}}^{\mathrm{DR}}=\mathbb{P}_n\left[\Big(\widetilde{\mathbb{U}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}^{\mathrm{DR}}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})-\widehat{\boldsymbol{\mathcal{S}}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}}\widehat{\boldsymbol{\mathcal{I}}}^{-1}_{\boldsymbol{\alpha}}\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O};\widehat{\boldsymbol{\alpha}})-\widehat{\boldsymbol{\mathcal{S}}}^{\mathrm{DR}}_{\boldsymbol{\theta}_q,\boldsymbol{\beta}}\widehat{\boldsymbol{\mathcal{I}}}^{-1}_{\boldsymbol{\beta}}\mathbb{U}_{\boldsymbol{\beta}}(\boldsymbol{O};\widehat{\boldsymbol{\beta}})\Big)^{\otimes 2}\right],$$

and $\widehat{\boldsymbol{\mathcal{S}}}^{\text{DR}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}} = \mathbb{P}_n \frac{\partial}{\partial \boldsymbol{\alpha}} \widetilde{\mathbb{U}}^{\text{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \widehat{\boldsymbol{\theta}}^{\text{DR}}_q, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})$, $\widehat{\boldsymbol{\mathcal{S}}}^{\text{DR}}_{\boldsymbol{\theta}_q,\boldsymbol{\beta}} = \mathbb{P}_n \frac{\partial}{\partial \boldsymbol{\beta}} \widetilde{\mathbb{U}}^{\text{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \widehat{\boldsymbol{\theta}}^{\text{DR}}_q, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})$. To show $\widehat{\boldsymbol{\Sigma}}^{\text{DR}}$ is a consistent estimator of $\boldsymbol{\Sigma}^{\text{DR}}$, one may use the asymptotic normality results for the point estimates $\left\{\widehat{\boldsymbol{\theta}}^{\text{DR}}_q, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}\right\}$ along with $\mathbb{P}_n[\widetilde{\mathbb{U}}^{\text{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)] = \mathbb{P}_n[\mathbb{U}^{\text{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)] + o_p(n^{-1/2})$ (Lemma 14), $\widetilde{\boldsymbol{S}}^{\text{DR}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}} = \boldsymbol{\mathcal{S}}^{\text{DR}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}} + o_p(1)$ (Lemma 15), $\widetilde{\boldsymbol{I}}^{\text{DR}}_{\boldsymbol{\theta}_q} = \boldsymbol{\mathcal{I}}^{\text{DR}}_{\boldsymbol{\theta}_q} + o_p(1)$ (Lemma 15), and $\widetilde{\boldsymbol{S}}^{\text{DR}}_{\boldsymbol{\theta}_q,\boldsymbol{\beta}} = \boldsymbol{\mathcal{S}}^{\text{DR}}_{\boldsymbol{\theta}_q,\boldsymbol{\beta}}$.

If both $\mathcal{M}_{ps}$ and $\mathcal{M}_{om}$ are correctly specified (i.e., $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}_0$ and $\boldsymbol{\beta}^* = \boldsymbol{\beta}_0$), $\widehat{\boldsymbol{\theta}}^{\text{DR}}_q$ will achieve the semiparametric efficiency bound such that $\boldsymbol{\Sigma}^{\text{DR}} = E[\mathbb{U}^{\text{eff}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0})^{\otimes 2}]$. One can easily verify this by observing that following 3 points hold when $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}_0$ and $\boldsymbol{\beta}^* = \boldsymbol{\beta}_0$:

- First, $\boldsymbol{\mathcal{I}}^{\text{DR}}_{\boldsymbol{\theta}_{q0}} = -\boldsymbol{C}_q$, where $\boldsymbol{C}_q = -\frac{\partial}{\partial \boldsymbol{\theta}_q} \boldsymbol{U}(\boldsymbol{\theta}_q, \boldsymbol{\omega}_0)\Big|_{\boldsymbol{\theta}_q = \boldsymbol{\theta}_{q0}}$ is defined in (s16).
- Second, $\mathbb{U}^{\text{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*) = \mathbb{U}^{\text{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0) = \sum_{k=0}^{K} \boldsymbol{\psi}^{(k)}(\boldsymbol{O}; \boldsymbol{\theta}_{q0})$, where $\boldsymbol{\psi}^{(k)}(\boldsymbol{O}; \boldsymbol{\theta}_{q0})$ is defined in Theorem 1.
- Third, $\boldsymbol{\mathcal{S}}^{\text{DR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}^*} = \boldsymbol{\mathcal{S}}^{\text{DR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0} = \boldsymbol{0}$ and $\boldsymbol{\mathcal{S}}^{\text{DR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\beta}^*} = \boldsymbol{\mathcal{S}}^{\text{DR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_0} = \boldsymbol{0}$ because Theorem 1 suggests that $\mathbb{U}^{\text{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ is orthogonal to the likelihood score of the nuisance parametric models $\mathcal{M}_{ps}$ and $\mathcal{M}_{om}$.

Combining the above three points, one can conclude that

$$\begin{aligned}
\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\text{DR}}_q - \boldsymbol{\theta}_{q0}) =& -\left[\boldsymbol{\mathcal{I}}^{\text{DR}}_{\boldsymbol{\theta}_{q0}}\right]^{-1} \sqrt{n}\mathbb{P}_n\left[\mathbb{U}^{\text{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)\right] + o_p(1) \\
=& \boldsymbol{C}_q^{-1}\sqrt{n}\mathbb{P}_n\left[\sum_{k=0}^{K} \boldsymbol{\psi}^{(k)}(\boldsymbol{O}; \boldsymbol{\theta}_{q0})\right] + o_p(1) \\
=& \sqrt{n}\mathbb{P}_n\left[\mathbb{U}^{\text{eff}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0})^{\otimes 2}\right] + o_p(1)
\end{aligned}$$

and therefore $\boldsymbol{\Sigma}^{\text{DR}} = E[\mathbb{U}^{\text{eff}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0})^{\otimes 2}]$ as $n \to \infty$. □

Appendix F: Technical results for a double machine learning MSQM estimator

In this section, we establish a nonparametric efficiency theory for a double machine learning MSQM when nonparametric methods or machine learners are employed to estimate the nuisance functions $h_{nuisance} = \left\{\pi_k(\overline{A}_k, \overline{\boldsymbol{L}}_k), F_{Y_{\overline{A}_k, \underline{a}_{k+1}}|\overline{A}_k, \overline{\boldsymbol{L}}_k}, \text{ for } k = 1, \ldots, K\right\}$ based on a cross-fitting procedure (Chernozhukov et al. 2018). We shall first assume there exists certain nonparametric or data-adaptive machine learners for $h_{nuisance}$ and a discussion on how to nonparametrically estimate $h_{nuisance}$ is provided later. For cross-fitting, we randomly split the data into V (e.g., 5) non-overlapping groups with approximately equal size (group size difference ≤ 1). For $v \in \{1, \ldots, V\}$, let $\mathcal{O}_v$ be the data in the v-th group and $\mathcal{O}_{-v}$ be the data in the other $V-1$ groups excluding $\mathcal{O}_v$. For each v, the nuisance function estimates on group $\mathcal{O}_v$, $\widehat{h}^{\text{np},v}_{nuisance} = \left\{\widehat{\pi}^{\text{np},v}_k(\overline{A}_k, \overline{\boldsymbol{L}}_k), \widehat{F}^{\text{np},v}_{Y_{\overline{A}_k, \underline{a}_{k+1}}|\overline{A}_k, \overline{\boldsymbol{L}}_k}, \text{ for } k = 1, \ldots, K\right\}$, are computed based on nonparametric or machine learning methods trained on data $\mathcal{O}_{-v}$. Therefore, the overall nuisance function estimates under the entire data set is the concatenation, $\widehat{h}^{\text{np}}_{nuisance} = \{\widehat{h}^{\text{np},1}_{nuisance}, \ldots, \widehat{h}^{\text{np},V}_{nuisance}\}$. The cross-fitting based nonparametric estimator $\widehat{\boldsymbol{\theta}}^{\text{np}}_q$ is therefore obtained by solving the doubly robust estimating equation $\mathbb{P}_n\left[\sum_{k=0}^{K} \widehat{\boldsymbol{\psi}}^{(k),\text{np}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q)\right] = \boldsymbol{0}$, where $\widehat{\boldsymbol{\psi}}^{(k),\text{np}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q)$ is $\boldsymbol{\psi}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q)$ evaluated under $\widehat{h}^{\text{np}}_{nuisance}$ and $\boldsymbol{\psi}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q)$ is defined in Theorem 1. The following theorem entails the required rate conditions for consistency, efficiency, and asymptotic normality of $\widehat{\boldsymbol{\theta}}^{\text{np}}_q$.

Theorem S3. *Suppose that Assumptions (A1)–(A4) hold. Then, if all elements in $\widehat{h}^{np}_{nuisance}$ are consistent with the rates of convergence*

$$\|\widehat{\pi}^{np}_k(\overline{A}_k, \overline{\boldsymbol{L}}_k) - \pi_k(\overline{A}_k, \overline{\boldsymbol{L}}_k)\|_p \times \|\widehat{F}^{np}_{Y_{\overline{A}_k, \underline{a}_{k+1}}|\overline{A}_k, \overline{\boldsymbol{L}}_k}(y|\overline{A}_k, \overline{\boldsymbol{L}}_k) - F_{Y_{\overline{A}_k, \underline{a}_{k+1}}|\overline{A}_k, \overline{\boldsymbol{L}}_k}(y|\overline{A}_k, \overline{\boldsymbol{L}}_k)\|_p = o_p(n^{-1/2})$$

for all $k \in \{1, \ldots, K\}$ and uniformly for $y \in [y_{\min}, y_{\max}]$, then $\widehat{\boldsymbol{\theta}}^{np}_q$ is CAN and its asymptotic variance achieves the semiparametric efficiency bound. Here, $\|\cdot\|_p$ defines the $L_2(P)$-norm such that $\|g(V)\|^2_p = \int_v g^2(v) dF_V(v)$.

Proof of Theorem S3 is provided at the end of this section. Theorem S3 suggests that $\widehat{\boldsymbol{\theta}}^{\text{np}}_q$ is CAN and semiparametrically efficient if all nuisance functions in $h_{nusiance}$ are consistently estimated (say by machine learning methods) and the product between the convergence rate of the propensity score estimator and that of the outcome distribution estimator at the k-th time period, for all $k = 1, \ldots, K$, are at least $o_p(n^{-1/2})$. Therefore, when all nuisance functions are consistent and converge faster than $o_p(n^{-1/4})$, $\widehat{\boldsymbol{\theta}}^{\text{np}}_q$ is CAN and semiparametrically efficient. Moreover, $\widehat{\boldsymbol{\theta}}^{\text{np}}_q$ also attains semiparametric efficiency under some alternative conditions. For example, if the propensity score estimator converges to the truth at a rate of $o_p(n^{-2/5})$, then a $o_p(n^{-1/10})$ rate of convergence of the outcome distribution estimator would ensure the product convergence rate to be $o_p(n^{-1/2})$, and thus $\widehat{\boldsymbol{\theta}}^{\text{np}}_q$ would remain CAN and semiparametrically efficient. As another example, if the propensity score estimator converges to truth at a parametric rate at $O_p(n^{-1/2})$, then we only require the estimate of outcome distributions to be consistent, and then $\widehat{\boldsymbol{\theta}}^{\text{np}}_q$ is semiparametric efficiency without requirement on the convergence rate of the outcome distribution estimator.

In practice, the set of propensity scores $\{\pi_k(\overline{A}_k, \overline{\boldsymbol{L}}_k), k = 1, \ldots, K\}$ can be estimated through many state-of-art machine learners designed for binary classification given a set of predictors (e.g., the neural networks by Chen & White (1999), L_2-boosting by Luo et al. (2016), random forests by Wager & Walther (2015), the series estimator by Newey (1997)), which usually can attain a rate of convergence with $o_p(n^{-1/4})$ or faster. However, using nonparametric methods or machine learners to estimate the set of outcome distributions $\{F_{Y_{\bar{A}_k, \underline{a}_{k+1}}|\bar{A}_k, \bar{\boldsymbol{L}}_k}, k = 1, \ldots, K\}$ may be not trivial. This is because the sequence of conditional distributions $\left\{F_{Y_{\bar{A}_k, \underline{a}_{k+1}}|\bar{A}_k, \bar{\boldsymbol{L}}_k}, k = 1, \ldots, K\right\}$ depends on the potential outcomes rather than the observed outcome (except $k = K$ so that $Y_{\bar{A}_K} = Y$ is observed), which precludes direct using existing machining learning methods for estimating conditional CDFs. Moreover, the iterative algorithm used in Section 3.2 is specifically designed for parametric specification of the outcome distributions and may not easily incorporate data-adaptive machine learning methods with general data structure. However, under the special case when all baseline and time-dependent confounders, $\{\boldsymbol{L}_1, \ldots, \boldsymbol{L}_K\}$, are categorical, we can obtain a plug-in estimate of $F_{Y_{\overline{A}_k, \underline{a}_{k+1}}|\overline{A}_k, \overline{\boldsymbol{L}}_k}$ based on the g-formula representation of potential outcome distribution (Lemma 1):

$$
\begin{aligned}
& F_{Y_{\overline{A}_k, \underline{a}_{k+1}}|\overline{A}_k, \overline{\boldsymbol{L}}_k}(y|\overline{a}_k, \bar{\boldsymbol{l}}_k) \\
= & \begin{cases} F_{Y|\overline{A}_K, \overline{\boldsymbol{L}}_K}(y|\overline{a}_K, \bar{\boldsymbol{l}}_K) & \text{if } k = K, \\ \int_{\boldsymbol{l}_K} \cdots \int_{\boldsymbol{l}_{k+1}} F_{Y|\overline{A}_K, \overline{\boldsymbol{L}}_K}(y|\overline{a}_K, \bar{\boldsymbol{l}}_K) \times \prod_{j=k+1}^{K} f_{\boldsymbol{L}_j|\overline{A}_{j-1}, \overline{\boldsymbol{L}}_{j-1}}(\boldsymbol{l}_j|\overline{a}_{k-1}, \bar{\boldsymbol{l}}_{k-1}) \mathrm{d}\boldsymbol{l}_k & \text{if } k < K, \end{cases}
\end{aligned}
$$

which depends on a conditional outcome CDF $F_{Y|\overline{A}_K, \overline{\boldsymbol{L}}_K}$ and conditional densities of all post-baseline covariates $f_{\boldsymbol{L}_k|\overline{A}_{k-1}, \overline{\boldsymbol{L}}_{k-1}}$. In practice, the conditional outcome CDF, $F_{Y|\overline{A}_K, \overline{\boldsymbol{L}}_K}$, can be estimated based on existing nonparametric or machine learning methods for conditional distribution functions; for example, the Kernel smoothing method (Jiang 2017), the highly adaptive lasso method (Hejazi et al. 2022), the neural networks (Hu & Nan 2023), or the multi-stage adaptable lasso method (Chiang & Huang 2012). In principal, the Kernel smoothing method (Jiang 2017) can guarantee a rate of convergence faster than $o_p(n^{-1/4})$ when $\overline{\boldsymbol{L}}_K$ are categorical, and therefore the resultant plug-in estimator of $F_{Y_{\overline{A}_k, \underline{a}_{k+1}}|\overline{A}_k, \overline{\boldsymbol{L}}_k}(y|\overline{a}_k, \bar{\boldsymbol{l}}_k)$ based on the g-formula also converges with a rate faster than $o_p(n^{-1/4})$ if appropriate machine learners are chosen for estimating $f_{\boldsymbol{L}_k|\overline{A}_{k-1}, \overline{\boldsymbol{L}}_{k-1}}$ (e.g., the methods in Tu (2019)). However, if the time-varying covariates are high-dimensional with many continues components, this approach may be practically infeasible and its rate of convergence would also be slower due to curse of dimensionality. Future investigations are needed to identify feasible algorithms to alleviate such challenges.

Proof of Theorem S3. We prove the asymptotic properties of $\widehat{\boldsymbol{\theta}}_q^{\mathrm{np}}$ given in Theorem S3. Recall that $\widehat{\boldsymbol{\theta}}_q^{\mathrm{np}}$ is given as the solution of $\mathbb{P}_n[\widehat{\mathbb{U}}^{\mathrm{np}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q)] = \boldsymbol{0}$ with $\widehat{\mathbb{U}}^{\mathrm{np}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q) = \sum_{k=0}^{K} \widehat{\boldsymbol{\psi}}^{(k),\mathrm{np}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q)$. We write $\mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q) = \sum_{k=0}^{K} \boldsymbol{\psi}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q)$ as the (non-normalized) EIF of $\boldsymbol{\theta}_q$. The following regularity conditions are required to proceed with the proof:

- 1^*. Assume that all elements in the nuisance functions $h_{nuisance}$ are consistently estimated with rate of convergence

$$\|\widehat{\pi}_k^{\mathrm{np}}(\overline{A}_k, \overline{\boldsymbol{L}}_k) - \pi_k(\overline{A}_k, \overline{\boldsymbol{L}}_k)\|_p \times \|\widehat{F}^{\mathrm{np}}_{Y_{\overline{A}_k, \underline{a}_{k+1}}|\overline{A}_k, \overline{\boldsymbol{L}}_k}(y|\overline{A}_k, \overline{\boldsymbol{L}}_k) - F_{Y_{\overline{A}_k, \underline{a}_{k+1}}|\overline{A}_k, \overline{\boldsymbol{L}}_k}(y|\overline{A}_k, \overline{\boldsymbol{L}}_k)\|_p = o_p(n^{-1/2})$$

for all $k \in \{1,\ldots,K\}$ and uniformly for $y \in [y_{\min}, y_{\max}]$, where $\|\cdot\|_p$ denotes the $L_2(P)$-norm such that $\|g(V)\|_p^2 = \int_v g^2(v)dF_V(v)$.

- 2^*. Let $\boldsymbol{\Theta}$ be a bounded convex neighborhood of $\boldsymbol{\theta}_{q0}$. Assume that $\boldsymbol{\psi}^{(K)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)$ is dominated by a square-integral function and is P-Glivenko-Cantelli for $\boldsymbol{\theta}_q \in \boldsymbol{\Theta}$. Suppose that the class of functions $\{\boldsymbol{\psi}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)$, for $k = 0,\ldots,K-1\}$ and their first order derivative with respect to $\boldsymbol{\theta}_q$ are continuous, dominated by certain square-integrable functions, and is P-Glivenko-Cantelli for $\boldsymbol{\theta}_q \in \boldsymbol{\Theta}$. Assume that $\{\pi_k(\overline{A}_k, \overline{\boldsymbol{L}}_k), k = 1,\ldots,K\}$ is bounded away from 0.

- 3^*. Assume that $\boldsymbol{C}_q$ defined in (s16) is positive definite. Also notice that $\boldsymbol{C}_q = -\frac{\partial}{\partial\boldsymbol{\theta}_q} E\left[\boldsymbol{\psi}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}$ by the definition of $\boldsymbol{\psi}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)$.

Based on the cross-fitting procedure, we can rewrite

$$\mathbb{P}_n\left[\widehat{\mathbb{U}}^{\mathrm{np}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right] = \frac{1}{n}\sum_{v=1}^{V} n_v \mathbb{P}_{n_v}\left[\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right],$$

where n_v is the size of the v-the group $\mathcal{O}_v$, $\mathbb{P}_{n_v}[\cdot]$ is the empirical mean operator on $\mathcal{O}_v$, and $\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q) = \sum_{k=0}^{K}\widehat{\boldsymbol{\psi}}^{(k),\mathrm{np},v}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)$ is $\mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)$ evaluated under $\widehat{h}^{\mathrm{np},v}_{nuisance}$, which is the nonparametric estimator of $h_{nuisance}$ based on the leave-one-out sample $\mathcal{O}_{-v}$. We can further decompose $\mathbb{P}_{n_v}\left[\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]$ as

$$\begin{aligned}\mathbb{P}_{n_v}\left[\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right] =& \mathbb{P}_{n_v}\left[\mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right] + \underbrace{(\mathbb{P}_{n_v}-E)\left[\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q) - \mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]}_{=:R_1}\\ &+ \underbrace{E\left[\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q) - \mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]}_{=:R_2}. \end{aligned} \tag{s39}$$

We can show that $R_1 = o_p(n^{-1/2})$ due to the cross-fitting procedure and also $R_2 = o_p(n^{-1/2})$ by the regularity condition 1^*. For brevity, we only prove the j-th dimension of R_1 and R_2, denoted by $R_{1,j}$ and $R_{2,j}$, are $o_p(n^{-1/2})$.

For $j \in \{1,\ldots,p\}$, let $\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)$ and $\mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)$ as the j-th dimension of $\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)$ and $\mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)$, respectively. By the independence suggested by cross-fitting and the fact that

$$\begin{aligned}\mathrm{Var}\left\{(\mathbb{P}_{n_v}-E)\left[\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q) - \mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\Big|\mathcal{O}_{-v}\right\} =& \mathrm{Var}\left\{\mathbb{P}_{n_v}\left[\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q) - \mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\Big|\mathcal{O}_{-v}\right\}\\ =& \frac{1}{n_v}\mathrm{Var}\left\{\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q) - \mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\Big|\mathcal{O}_{-v}\right\}\\ =& \frac{1}{n_v}\|\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q) - \mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\|^2,\end{aligned}$$

we have, for any $\epsilon > 0$,

$$\mathbb{P}\left\{\frac{\sqrt{n_v}\left|(\mathbb{P}_{n_v}-E)\left[\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q) - \mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\right|}{\|\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q) - \mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\|} \geq \epsilon\right\}$$

$$
\begin{aligned}
=&E\left[\mathbb{P}\left\{\frac{\sqrt{n_v}\left|(\mathbb{P}_{n_v}-E)\left[\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)-\mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\right|}{\|\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)-\mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\|}\geq\epsilon\Big|\mathcal{O}_{-v}\right\}\right]\\
\leq&\frac{1}{\epsilon^2}E\left[\mathrm{Var}\left\{\frac{\sqrt{n_v}\left|(\mathbb{P}_{n_v}-E)\left[\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)-\mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\right|}{\|\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)-\mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\|}\Big|\mathcal{O}_{-v}\right\}\right]\\
=&\epsilon^{-2},
\end{aligned}
$$

where the second row to the third row hold as a result of the Markov's inequality. Therefore, we have obtained $R_{1,j}=(\mathbb{P}_{n_v}-E)\left[\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)-\mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]=O_p(n_v^{-1/2}\|\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)-\mathbb{U}^{\mathrm{eff}*}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\|)=o_p(n_v^{-1/2})$. Because V is a finte number and we partition the data as evenly as possible, $n_v/n=O_p(1)$ and therefore $R_{1,j}=o_p(n^{-1/2})$.

Next we show $R_{2,j}=o_p(n^{-1/2})$. We shall abbreviate $F_k=F_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}$ and $\widehat{F}^{\mathrm{np},v}_k=\widehat{F}^{\mathrm{np},v}_{Y_{\overline{A}_k,\underline{a}_{k+1}}|\overline{A}_k,\overline{\boldsymbol{L}}_k}$ for brevity. First, noting that $E\left[\widehat{\mathbb{U}}^{\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]=\sum_{k=1}^K E\left[\widehat{\boldsymbol{\psi}}^{(k),\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]$ and and the expectation for the K-th component is

$$
\begin{aligned}
&E\left[\widehat{\boldsymbol{\psi}}^{(K),\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\\
=&E\left[\frac{d_j(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\widehat{\overline{\pi}}^{\mathrm{np},v}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K)}\left\{\mathbb{I}\left(Y\leq h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\right)-\widehat{F}^{\mathrm{np},v}_K\left(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right)\right\}\right]\\
=&E\left[\frac{d_j(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\widehat{\overline{\pi}}^{\mathrm{np},v}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K)}\left\{F_K\left(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right)-\widehat{F}^{\mathrm{np},v}_K\left(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right)\right\}\right]\\
=&E\left[\left(\frac{1}{\widehat{\pi}^{\mathrm{np},v}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K)}-\frac{1}{\pi_K(\overline{A}_K,\overline{\boldsymbol{L}}_K)}\right)\frac{d_j(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\widehat{\overline{\pi}}^{\mathrm{np},v}_{K-1}(\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1})}\left\{F_K\left(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right)-\widehat{F}^{\mathrm{np},v}_K\left(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right)\right\}\right]\\
&+E\left[\frac{d_j(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\pi_K(\overline{A}_K,\overline{\boldsymbol{L}}_K)\widehat{\overline{\pi}}^{\mathrm{np},v}_{K-1}(\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1})}\left\{F_K\left(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right)-\widehat{F}^{\mathrm{np},v}_K\left(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right)\right\}\right]\\
=&\underbrace{E\left[\frac{d_j(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\pi_K(\overline{A}_K,\overline{\boldsymbol{L}}_K)\widehat{\overline{\pi}}^{\mathrm{np},v}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K)}\left\{\pi_K(\overline{A}_K,\overline{\boldsymbol{L}}_K)-\widehat{\pi}^{\mathrm{np},v}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K)\right\}\left\{F_K\left(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right)-\widehat{F}^{\mathrm{np},v}_K\left(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right)\right\}\right]}_{=o_p(n^{-1/2})\text{ by point }1^*\text{ with Cauchy–Schwarz inequality}}\\
&+E\left[\sum_{a_K\in\mathbb{A}_K}\frac{d_j(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\widehat{\overline{\pi}}^{\mathrm{np},v}_{K-1}(\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1})}\left\{F_K\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},a_K,\overline{\boldsymbol{L}}_K\right)-\widehat{F}^{\mathrm{np},v}_K\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},a_K,\overline{\boldsymbol{L}}_K\right)\right\}\right]\\
=&E\left[\sum_{a_K\in\mathbb{A}_K}\frac{d_j(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\widehat{\overline{\pi}}^{\mathrm{np},v}_{K-1}(\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1})}\left\{F_K\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},a_K,\overline{\boldsymbol{L}}_K\right)-\widehat{F}^{\mathrm{np},v}_K\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},a_K,\overline{\boldsymbol{L}}_K\right)\right\}\right]+o_p(n^{-1/2}),
\end{aligned}
$$

which then can be used to obtain

$$
\begin{aligned}
&\sum_{k=K-1}^{K}E\left[\widehat{\boldsymbol{\psi}}^{(k),\mathrm{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\\
=&E\left[\sum_{a_K\in\mathbb{A}_K}\frac{d_j(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\widehat{\overline{\pi}}^{\mathrm{np},v}_{K-1}(\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1})}\left\{F_K\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},a_K,\overline{\boldsymbol{L}}_K\right)-\widehat{F}^{\mathrm{np},v}_K\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},a_K,\overline{\boldsymbol{L}}_K\right)\right\}\right]+o_p(n^{-1/2})\\
&+E\left[\sum_{a_K\in\mathbb{A}_K}\frac{d_j(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\widehat{\overline{\pi}}^{\mathrm{np},v}_{K-1}(\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1})}\left\{\widehat{F}^{\mathrm{np},v}_K\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},a_K,\overline{\boldsymbol{L}}_K\right)-\widehat{F}^{\mathrm{np},v}_{K-1}\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)\right\}\right]\\
=&E\left[\sum_{a_K\in\mathbb{A}_K}\frac{d_j(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\widehat{\overline{\pi}}^{\mathrm{np},v}_{K-1}(\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1})}\left\{F_K\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},a_K,\overline{\boldsymbol{L}}_K\right)-\widehat{F}^{\mathrm{np},v}_{K-1}\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)\right\}\right]+o_p(n^{-1/2})\\
=&E\left[\sum_{a_K\in\mathbb{A}_K}\frac{d_j(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\widehat{\overline{\pi}}^{\mathrm{np},v}_{K-1}(\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1})}\left\{F_{K-1}\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)-\widehat{F}^{\mathrm{np},v}_{K-1}\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)\right\}\right]+o_p(n^{-1/2})\\
&\text{(the previous step follows by Assumption (A2) and law of iterated expectation)}\\
=&E\left[\sum_{a_K\in\mathbb{A}_K}\frac{d_j(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\pi_{K-1}(\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1})\widehat{\overline{\pi}}^{\mathrm{np},v}_{K-2}(\overline{A}_{K-2},\overline{\boldsymbol{L}}_{K-2})}\left\{F_{K-1}\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)-\widehat{F}^{\mathrm{np},v}_{K-1}\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)\right\}\right]
\end{aligned}
$$

$$+E\Bigg[\sum_{\underline{a}_K\in\mathbb{A}_K}\left(\frac{1}{\widehat{\pi}^{\text{np},v}_{K-1}(\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1})}-\frac{1}{\pi_{K-1}(\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1})}\right)\frac{d_j(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\widehat{\overline{\pi}}^{\text{np},v}_{K-2}(\overline{A}_{K-2},\overline{\boldsymbol{L}}_{K-2})}\Big\{F_{K-1}\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)-$$

$$\widehat{F}^{\text{np},v}_{K-1}\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)\Big\}\Bigg]+o_p(n^{-1/2})$$

$$=E\Bigg[\sum_{\underline{a}_{K-1}\in\underline{\mathbb{A}}_{K-1}}\frac{d_j(\overline{A}_{K-2},\underline{a}_{K-1},\boldsymbol{Z};\boldsymbol{\theta}_q)}{\widehat{\overline{\pi}}^{\text{np},v}_{K-2}(\overline{A}_{K-2},\overline{\boldsymbol{L}}_{K-2})}\Big\{F_{K-1}\left(h(\overline{A}_{K-2},\underline{a}_{K-1},\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-2},a_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)-\widehat{F}^{\text{np},v}_{K-1}\left(h(\overline{A}_{K-2},\underline{a}_{K-1},\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-2},a_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)\Big\}\Bigg]$$

$$+\underbrace{\begin{array}{c}E\Bigg[\sum_{\underline{a}_K\in\mathbb{A}_K}\frac{d_j(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\pi_{K-1}(\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1})\widehat{\overline{\pi}}^{\text{np},v}_{K-1}(\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1})}\Big\{\pi_{K-1}(\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1})-\widehat{\pi}^{\text{np},v}_{K-1}(\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1})\Big\}\times\\ \Big\{F_{K-1}\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)-\widehat{F}^{\text{np},v}_{K-1}\left(h(\overline{A}_{K-1},a_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)\Big\}\Bigg]\end{array}}_{=o_p(n^{-1/2})\text{by point }1^*\text{ and Cauchy–Schwarz inequality}}+o_p(n^{-1/2})$$

$$=E\Bigg[\sum_{\underline{a}_{K-1}\in\underline{\mathbb{A}}_{K-1}}\frac{d_j(\overline{A}_{K-2},\underline{a}_{K-1},\boldsymbol{Z};\boldsymbol{\theta}_q)}{\widehat{\overline{\pi}}^{\text{np},v}_{K-2}(\overline{A}_{K-2},\overline{\boldsymbol{L}}_{K-2})}\Big\{F_{K-1}\left(h(\overline{A}_{K-2},\underline{a}_{K-1},\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-2},a_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)-\widehat{F}^{\text{np},v}_{K-1}\left(h(\overline{A}_{K-2},\underline{a}_{K-1},\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_{K-2},a_{K-1},\overline{\boldsymbol{L}}_{K-1}\right)\Big\}\Bigg]$$

$$+o_p(n^{-1/2}).$$

Applying the previous strategy from $k=K-1$ to 0 iteratively, one can obtain

$$\begin{aligned}\sum_{k=0}^{K}E\left[\widehat{\boldsymbol{\psi}}^{(k),\text{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]&=E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\left\{F_{Y_{\overline{a}_K}|\boldsymbol{L}_1}\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\boldsymbol{L}_1\right)-q\right\}\right]+o_p(n^{-1/2})\\&=E\left[\mathbb{U}^{\text{eff*}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]+o_p(n^{-1/2}).\end{aligned}$$

Notice that the left-hand side of the previous equation is exactly $E\left[\widehat{\mathbb{U}}^{\text{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]$, we deduce that $R_{2,j}=E\left[\widehat{\mathbb{U}}^{\text{np},v}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)-\mathbb{U}^{\text{eff*}}_{\boldsymbol{\theta}_q,j}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]=o_p(n^{-1/2})$.

Now, we have obtained that $\mathbb{P}_{n_v}[\widehat{\mathbb{U}}^{\text{np},v}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)]=\mathbb{P}_{n_v}\left[\mathbb{U}^{\text{eff*}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]+o_p(n^{-1/2})$, thus

$$\begin{aligned}\mathbb{P}_n\left[\widehat{\mathbb{U}}^{\text{np}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]&=\frac{1}{n}\sum_{v=1}^{V}n_v\mathbb{P}_{n_v}\left[\widehat{\mathbb{U}}^{\text{np},v}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\\&=\sum_{v=1}^{V}\left\{\frac{n_v}{n}\mathbb{P}_{n_v}\left[\mathbb{U}^{\text{eff*}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]+o_p\left(\frac{n_v}{n^{3/2}}\right)\right\}\\&=\mathbb{P}_n\left[\mathbb{U}^{\text{eff*}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]+o_p(n^{-1/2})\end{aligned}\tag{s40}$$

uniformly for $\boldsymbol{\theta}_q\in\boldsymbol{\Theta}$. In addition, noticing that $E[\boldsymbol{\psi}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)]=\boldsymbol{0}$ for all $\boldsymbol{\theta}_q\in\boldsymbol{\Theta}$ across $k=1,\ldots,K$, we have $E[\mathbb{U}^{\text{eff*}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)-\boldsymbol{\psi}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)]=\boldsymbol{0}$ and therefore

$$\sup_{\boldsymbol{\theta}_q\in\boldsymbol{\Theta}}\left\|\mathbb{P}_n[\mathbb{U}^{\text{eff*}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)]-\mathbb{P}_n[\boldsymbol{\psi}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)]\right\|=o_p(n^{-1/2+\epsilon})\quad\text{a.s.}$$

by uniform law of large numbers. This and (s40) suggest that

$$\sup_{\boldsymbol{\theta}_q\in\boldsymbol{\Theta}}\left\|\mathbb{P}_n[\widehat{\mathbb{U}}^{\text{np}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)]-\mathbb{P}_n[\boldsymbol{\psi}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)]\right\|=o_p(n^{-1/2+\epsilon}),\quad\text{a.s.}\tag{s41}$$

Moreover, point 3^* implies the existence of a positive constant M such that

$$\|\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0}\|\le M\left\|E\left[\boldsymbol{\psi}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)-\boldsymbol{\psi}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0})\right]\right\|$$

$$
\begin{aligned}
&= M \left\| E\left[\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(0)}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\right\| \\
&\leq M \left\|\mathbb{P}_n\left[\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(0)}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\right\| + o_p(n^{-1/2+\epsilon})
\end{aligned}
$$

for $\boldsymbol{\theta}_q$ in a small neighborhood of $\boldsymbol{\theta}_{q0}$, where the equality from the first to the second row follows by $E[\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(0)}(\boldsymbol{O};\boldsymbol{\theta}_{q0})] = \mathbf{0}$. This, combined with (s41), suggests that

$$
\|\boldsymbol{\theta}_q - \boldsymbol{\theta}_{q0}\| \leq M \left\|\mathbb{P}_n\left[\widehat{\mathbb{U}}_{\boldsymbol{\theta}_q}^{\mathrm{np}}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\right\| + o_p(n^{-1/2+\epsilon}).
$$

We can then replace $\boldsymbol{\theta}_q$ in the previous equation by $\widehat{\boldsymbol{\theta}}_q^{\mathrm{np}}$ and also recall that $\widehat{\boldsymbol{\theta}}_q^{\mathrm{np}}$ solves $\mathbb{P}_n\left[\widehat{\mathbb{U}}_{\boldsymbol{\theta}_q}^{\mathrm{np}}(\boldsymbol{O};\boldsymbol{\theta}_q)\right] = \mathbf{0}$, we have obtained $\|\widehat{\boldsymbol{\theta}}_q^{\mathrm{DR}*} - \boldsymbol{\theta}_{q0}\| = o_p(n^{-1/2+\epsilon})$ such that $\widehat{\boldsymbol{\theta}}_q^{\mathrm{np}}$ is $\sqrt{n}$-consistent.

To prove the asymptotic normality and semiparametric efficiency of $\widehat{\boldsymbol{\theta}}_q^{\mathrm{np}}$, notice that $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\theta}_q}^{\mathrm{eff}*}(\boldsymbol{O};\boldsymbol{\theta}_q) - \mathbb{U}_{\boldsymbol{\theta}_q}^{\mathrm{eff}*}(\boldsymbol{O};\boldsymbol{\theta}_{q0})]$ admits the following decomposition for any $\boldsymbol{\theta}_q$ in $O(n^{-1/3})$ neighborhood of $\boldsymbol{\theta}_{q0}$:

$$
\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\theta}_q}^{\mathrm{eff}*}(\boldsymbol{O};\boldsymbol{\theta}_q) - \mathbb{U}_{\boldsymbol{\theta}_q}^{\mathrm{eff}*}(\boldsymbol{O};\boldsymbol{\theta}_{q0})] = \sum_{k=0}^{K} \mathbb{P}_n[\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(k)}(\boldsymbol{O};\boldsymbol{\theta}_q) - \boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(k)}(\boldsymbol{O};\boldsymbol{\theta}_{q0})],
$$

where

$$
\begin{aligned}
&\mathbb{P}_n[\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(K)}(\boldsymbol{O};\boldsymbol{\theta}_q) - \boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(K)}(\boldsymbol{O};\boldsymbol{\theta}_{q0})] \\
=&\underbrace{\begin{aligned}\mathbb{P}_n\Big[\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K)}\Big\{&\mathbb{I}\left(Y \leq h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\right) - F_{Y_{\overline{A}_K}|\overline{A}_K,\overline{\boldsymbol{L}}_K}\left(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right) \\
&-\mathbb{I}\left(Y \leq h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\right) + F_{Y_{\overline{A}_K}|\overline{A}_K,\overline{\boldsymbol{L}}_K}\left(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right)\Big\}\Big]\end{aligned}}_{=o_p(n^{-1/2}) \text{ by (s19) in Lemma 6}} \\
&+\underbrace{\mathbb{P}_n\left[\left\{\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K)} - \frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K)}\right\}\left\{\mathbb{I}\left(Y \leq h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\right) - F_{Y_{\overline{A}_K}|\overline{A}_K,\overline{\boldsymbol{L}}_K}\left(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\Big|\overline{A}_K,\overline{\boldsymbol{L}}_K\right)\right\}\right]}_{=o_p(n^{-1/2}) \text{ by similar arguments in (s23)}} \\
=&o_p(n^{-1/2}).
\end{aligned}
$$

Moreover, observing point 2* and using a first-order Taylor expansion for $\boldsymbol{\theta}_q$ around $\boldsymbol{\theta}_{q0}$, we can show that for $k \in \{1,\ldots,K-1\}$,

$$
\begin{aligned}
\mathbb{P}_n[\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(k)}(\boldsymbol{O};\boldsymbol{\theta}_q) - \boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(k)}(\boldsymbol{O};\boldsymbol{\theta}_{q0})] &= \mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(k)}(\boldsymbol{O};\boldsymbol{\theta}_q)\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right](\boldsymbol{\theta}_q - \boldsymbol{\theta}_{q0}) + o_p(n^{-1/2}) \\
&= E\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(k)}(\boldsymbol{O};\boldsymbol{\theta}_q)\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right](\boldsymbol{\theta}_q - \boldsymbol{\theta}_{q0}) + o_p(n^{-1/2}) \\
&= \frac{\partial}{\partial\boldsymbol{\theta}_q}E\left[\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(k)}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}} \times (\boldsymbol{\theta}_q - \boldsymbol{\theta}_{q0}) + o_p(n^{-1/2}) \\
&= o_p(n^{-1/2}),
\end{aligned}
$$

where the last equality follows by $E\left[\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(k)}(\boldsymbol{O};\boldsymbol{\theta}_q)\right] = \mathbf{0}$ for all $\boldsymbol{\theta}_q \in \boldsymbol{\Theta}$ and $k \in \{1,\ldots,K-1\}$. In addition, for $k = 0$, we have

$$
\mathbb{P}_n[\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(0)}(\boldsymbol{O};\boldsymbol{\theta}_q) - \boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(0)}(\boldsymbol{O};\boldsymbol{\theta}_{q0})] = \mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\theta}_q}\boldsymbol{\psi}_{\boldsymbol{\theta}_q}^{(0)}(\boldsymbol{O};\boldsymbol{\theta}_q)\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right](\boldsymbol{\theta}_q - \boldsymbol{\theta}_{q0}) + o_p(n^{-1/2})
$$

$$
\begin{aligned}
&= E\left[\frac{\partial}{\partial \boldsymbol{\theta}_q}\boldsymbol{\psi}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\right](\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0})+o_p(n^{-1/2})\\
&= \underbrace{\left\{\frac{\partial}{\partial \boldsymbol{\theta}_q}E\left[\boldsymbol{\psi}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)\right]\right\}\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}}_{=-\boldsymbol{C}_q \text{ as defined in (s16)}}(\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0})+o_p(n^{-1/2})\\
&= -\boldsymbol{C}_q(\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0})+o_p(n^{-1/2}).
\end{aligned}
$$

Combining the previous discussions, we have

$$
\mathbb{P}_n[\mathbb{U}^{\text{eff}*}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)-\mathbb{U}^{\text{eff}*}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0})] = -\boldsymbol{C}_q(\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0})+o_p(n^{-1/2}).
$$

This, coupled with (s40), suggests that

$$
\mathbb{P}_n[\widehat{\mathbb{U}}^{\text{np}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q)] = \mathbb{P}_n[\mathbb{U}^{\text{eff}*}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0})] - \boldsymbol{C}_q(\boldsymbol{\theta}_q-\boldsymbol{\theta}_{q0})+o_p(n^{-1/2}).
$$

Noting that $\widehat{\boldsymbol{\theta}}^{\text{np}}_q$ is in the $O(n^{-1/3})$ neighborhood of $\boldsymbol{\theta}_{q0}$ and also the solution of the left-hand side of the previous equation, we then conclude

$$
\begin{aligned}
\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\text{np}}_q-\boldsymbol{\theta}_{q0}) &= \boldsymbol{C}_q^{-1}\times\sqrt{n}\mathbb{P}_n[\mathbb{U}^{\text{eff}*}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0})]+o_p(1)\\
&= \sqrt{n}\mathbb{P}_n[\mathbb{U}^{\text{eff}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0})]+o_p(1)
\end{aligned}
$$

This confirms that $\widehat{\boldsymbol{\theta}}^{\text{np}}_q$ is asymptotically normal and its asymptotic variance achieves the semiparametric efficiency lower bound. □

Appendix G: A technical interpretation of the working confounding function

A technical interpretation of the working confounding function is provided in the following Proposition.

Proposition 2. *Suppose that (i) the first term in the true confounding function (13), $F_{Y_{\bar{a}_K}|\bar{A}_k,\bar{\boldsymbol{L}}_k}(y|\bar{a}_k,\bar{\boldsymbol{l}}_k)$, is a Gaussian CDF with mean $\mu_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)$ and variance $\sigma_k^2(\bar{a}_K,\bar{\boldsymbol{l}}_k)$, and (ii) the two distributions involved in the true confounding function only have a mean shift such that $E[Y_{\bar{a}_K}|\bar{A}_k=\bar{a}_k,\bar{\boldsymbol{L}}_k=\bar{\boldsymbol{l}}_k]-E[Y_{\bar{a}_K}|\bar{A}_k=(1-a_k,\bar{a}_{k-1}),\bar{\boldsymbol{L}}_k=\bar{\boldsymbol{l}}_k]=s_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)$. Then, if the mean shift is not large, the working confounding function defined in (14) hold approximately, with $r_k(\bar{a}_K,\bar{\boldsymbol{l}}_k,\boldsymbol{\gamma}_{k1})=-\frac{s_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)}{\sqrt{2\pi\sigma_k^2(\bar{a}_K;\bar{\boldsymbol{l}}_k)}}$, $b_k(\bar{a}_K,\bar{\boldsymbol{l}}_k,\boldsymbol{\gamma}_{k2})=|s_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)|$, and $m_k(\bar{a}_K,\bar{\boldsymbol{l}}_k;\boldsymbol{\gamma}_{k3})=\mu_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)$.*

Proof of Proposition 2 is given at the end of this section. Proposition 2 suggests that the working confounding function (14) is a good approximation to the true confounding function (13) when the potential outcome $Y_{\bar{a}_K}$ given $\bar{A}_k=\bar{a}_k$ and $\bar{\boldsymbol{L}}_k=\bar{\boldsymbol{l}}_k$ is normally distributed and the two distributions used in the confounding function only have a mean shift. Moreover, we have that the *area* component approximately equals to the absolute mean shift value and the *location* component approximately equals to the conditional mean of $F_{Y_{\bar{a}_K}|\bar{A}_k,\bar{\boldsymbol{L}}_k}(y|\bar{a}_k,\bar{\boldsymbol{l}}_k)$, which is $\boldsymbol{\delta}_k^T\boldsymbol{g}_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)$ in (s10) if the Gaussian linear regression in Appendix B.2 are used to model the potential outcome distribution.

Proof of Proposition 2. Let $\phi(y;\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$ be the population density function for $N(\mu,\sigma^2)$. Also, let $\phi'(y;\mu,\sigma^2)=-\frac{y-\mu}{\sigma^2}\times\phi(y;\mu,\sigma^2)$ denote the first derivative of $\phi(y;\mu,\sigma^2)$ in terms of y. Then, the sensitivity function has the follow form

$$
\begin{aligned}
c_k(y,\bar{a}_K,\bar{\boldsymbol{l}}_k)=&F_{Y_{\bar{a}_K}|\bar{A}_k,\bar{\boldsymbol{L}}_k}(y|\bar{a}_k,\bar{\boldsymbol{l}}_k)-F_{Y_{\bar{a}_K}|\bar{A}_k,\bar{\boldsymbol{L}}_k}(y|1-a_k,\bar{a}_{k-1},\bar{\boldsymbol{l}}_k)\\
=&\int_{-\infty}^{y}\phi\Big(u;\mu_k(\bar{a}_K,\bar{\boldsymbol{l}}_k),\sigma_k^2(\bar{a}_K,\bar{\boldsymbol{l}}_k)\Big)-\phi\Big(u;\mu_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)-s_k(\bar{a}_K,\bar{\boldsymbol{l}}_k),\sigma_k^2(\bar{a}_K,\bar{\boldsymbol{l}}_k)\Big)du.
\end{aligned}
$$

Since the mean shift $s_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)$ is assumed small, we can apply a first-order Taylor expansion on $\phi\Big(u;\mu_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)-s_k(\bar{a}_K,\bar{\boldsymbol{l}}_k),\sigma_k^2(\bar{a}_K,\bar{\boldsymbol{l}}_k)\Big)$ with respect to $s_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)$ around 0. This suggests that

$$
\begin{aligned}
c_k(y,\bar{a}_K,\bar{\boldsymbol{l}}_k)\approx&\int_{-\infty}^{y}\phi\Big(u;\mu_k(\bar{a}_K,\bar{\boldsymbol{l}}_k),\sigma_k^2(\bar{a}_K,\bar{\boldsymbol{l}}_k)\Big)-\phi\Big(u;\mu_k(\bar{a}_K,\bar{\boldsymbol{l}}_k),\sigma_k^2(\bar{a}_K,\bar{\boldsymbol{l}}_k)\Big)\\
&\qquad+\phi\Big(u;\mu_k(\bar{a}_K,\bar{\boldsymbol{l}}_k),\sigma_k^2(\bar{a}_K,\bar{\boldsymbol{l}}_k)\Big)\times\frac{(u-\mu_k(\bar{a}_K,\bar{\boldsymbol{l}}_k))\times s_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)}{\sigma_k^2(\bar{a}_K,\bar{\boldsymbol{l}}_k)}du\\
=&\int_{-\infty}^{y}\phi\Big(u;\mu_k(\bar{a}_K,\bar{\boldsymbol{l}}_k),\sigma_k^2(\bar{a}_K,\bar{\boldsymbol{l}}_k)\Big)\times\frac{(u-\mu_k(\bar{a}_K,\bar{\boldsymbol{l}}_k))\times s_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)}{\sigma_k^2(\bar{a}_K,\bar{\boldsymbol{l}}_k)}du\\
=&\int_{-\infty}^{y}s_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)\times\phi'\Big(y;\mu_k(\bar{a}_K,\bar{\boldsymbol{l}}_k),\sigma_k^2(\bar{a}_K,\bar{\boldsymbol{l}}_k)\Big)du\\
=&-s_k(\bar{a}_K,\bar{\boldsymbol{l}}_k)\times\phi\Big(u;\mu_k(\bar{a}_K,\bar{\boldsymbol{l}}_k),\sigma_k^2(\bar{a}_K,\bar{\boldsymbol{l}}_k)\Big)
\end{aligned}
$$

$$
= -\frac{s_k(\overline{a}_K, \bar{\boldsymbol{l}}_k)}{\sqrt{2\pi\sigma_k^2(\overline{a}_K, \bar{\boldsymbol{l}}_k)}} \times \exp\left\{-\frac{\Big(y - \mu_k(\overline{a}_K, \bar{\boldsymbol{l}}_k)\Big)^2}{2\sigma_k^2(\overline{a}_K, \bar{\boldsymbol{l}}_k)}\right\}.
$$

It is then immediate that this is identical to (14) with $r_k(\overline{a}_K, \bar{\boldsymbol{l}}_k, \boldsymbol{\gamma}_{1k}) = -\frac{s_k(\overline{a}_K, \bar{\boldsymbol{l}}_k)}{\sqrt{2\pi\sigma_k^2(\overline{a}_K; \bar{\boldsymbol{l}}_k)}}$, $b_k(\overline{a}_K, \bar{\boldsymbol{l}}_k, \boldsymbol{\gamma}_{2k}) = |s_k(\overline{a}_K, \bar{\boldsymbol{l}}_k)|$, and $m_k(\overline{a}_K, \bar{\boldsymbol{l}}_k; \boldsymbol{\gamma}_{3k}) = \mu_k(\overline{a}_K, \bar{\boldsymbol{l}}_k)$. $\square$

Appendix H: Supporting information for the bias-corrected estimators with unmeasured confounding

Define $c_k^*(y,\overline{a}_K,\overline{\boldsymbol{l}}_k;\boldsymbol{\gamma}_k,\boldsymbol{\alpha}_k)=c_k(y,\overline{a}_K,\overline{\boldsymbol{l}}_k;\boldsymbol{\gamma}_k)\times\pi_k(1-a_k,\overline{a}_{k-1},\overline{\boldsymbol{l}}_k;\boldsymbol{\alpha}_k)$. The bias-corrected IPW estimator, $\widehat{\boldsymbol{\theta}}_q^{\text{BC-IPW}}$, solves the following bias-corrected IPW estimating equation

$$\mathbb{P}_n\left[\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K,\widehat{\overline{\boldsymbol{\alpha}}}_K)}\left\{\mathcal{K}\left(\frac{h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)-Y}{\tau_n}\right)-\sum_{k=1}^K c_k^*\Big(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{A}_K,\overline{\boldsymbol{L}}_k;\boldsymbol{\gamma}_k,\widehat{\boldsymbol{\alpha}}_k\Big)-q\right\}\right]=\mathbf{0}. \tag{s42}$$

Compared to the estimating equation for $\widehat{\boldsymbol{\theta}}_q^{\text{IPW}}$ (12), the bias-corrected estimating equation (s42) includes an extra term, $\sum_{k=1}^K c_k^*\Big(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{A}_K,\overline{\boldsymbol{L}}_k;\boldsymbol{\gamma}_k,\boldsymbol{\alpha}_k\Big)$, to remove the hidden bias due to unmeasured confounding.

With unmeasure confounding, the ICR estimator is inconsistent and requires the following modifications to restore consistency. First, we compute $\widehat{\boldsymbol{\beta}}_K$ by solving $\mathbb{P}_n\left[\mathbb{U}_{\boldsymbol{\beta}_K}(Y,\overline{A}_K,\overline{\boldsymbol{L}}_K)\right]=\mathbf{0}$. Then, for $k=K-1,\dots,1$, we iteratively compute $\widehat{\boldsymbol{\beta}}_k$ by solving

$$\mathbb{P}_n\Bigg[\sum_{\underline{a}_{k+1}\in\underline{\mathbb{A}}_{k+1}}\int_y \mathbb{U}_{\boldsymbol{\beta}_k}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k)\Big\{\psi_{k+1}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\boldsymbol{\beta}_{k+1}) - c_{k+1}^{*'}(y,\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_{k+1};\boldsymbol{\gamma}_{k+1},\widehat{\boldsymbol{\alpha}}_{k+1})\Big\}\mathrm{d}y\Bigg]=\mathbf{0},$$

where $\widehat{\boldsymbol{\alpha}}_k$ is obtained by maximum likelihood as in Section 3.1 and $c_k^{*'}(y,\overline{a}_K,\overline{\boldsymbol{l}}_k;\boldsymbol{\gamma}_k,\boldsymbol{\alpha}_k)=c_k'(y,\overline{a}_K,\overline{\boldsymbol{l}}_k;\boldsymbol{\gamma}_k)\pi_k(1-a_k,\overline{a}_{k-1},\overline{\boldsymbol{l}}_k;\boldsymbol{\alpha}_k)$ is the partial derivative of $c_k^*(y,\overline{a}_K,\overline{\boldsymbol{l}}_k;\boldsymbol{\gamma}_k,\boldsymbol{\alpha}_k)$ in terms of y. Finally, the bias-corrected ICR estimator, denoted by $\widehat{\boldsymbol{\theta}}_q^{\text{BC-ICR}}$, is obtained by solving

$$\mathbb{P}_n\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\left\{\Psi_1\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{a}_K,\boldsymbol{L}_1;\widehat{\boldsymbol{\beta}}_1\right)-c_1^*(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{a}_K,\boldsymbol{L}_1;\gamma_1,\widehat{\boldsymbol{\alpha}}_1)-q\right\}\right]=\mathbf{0}.$$

We can further combine the bias-corrected IPW and bias-corrected ICR estimators to construct a bias-corrected doubly robust estimator $\widehat{\boldsymbol{\theta}}_q^{\text{BC-DR}}$ for the causal parameter $\boldsymbol{\theta}_q$, where the expression is given in Section 5.2. The following Theorem summarizes the asymptotic properties of $\widehat{\boldsymbol{\theta}}_q^{\text{BC-IPW}}$, $\widehat{\boldsymbol{\theta}}_q^{\text{BC-ICR}}$, and $\widehat{\boldsymbol{\theta}}_q^{\text{BC-DR}}$.

Theorem S4. *Suppose that the assumptions (A1) and (A3)–(A4) hold and the working confounding functions are correctly specified; i.e.,* $c_k(y,\overline{a}_K,\overline{\boldsymbol{l}}_k;\boldsymbol{\gamma}_k)\equiv c_k(y,\overline{a}_K,\overline{\boldsymbol{l}}_k)$ *for all* $y\in[y_{\min},y_{\max}]$, $k=1,\dots,K$.

(i) If $\mathcal{M}_{ps}$ is correctly specified and conditions (B1)–(B2) regarding the smoothness function hold, $\widehat{\boldsymbol{\theta}}_q^{\text{BC-IPW}}$ is CAN such that $\sqrt{n}(\widehat{\boldsymbol{\theta}}_q^{\text{BC-IPW}}-\boldsymbol{\theta}_{q0})$ converges to $N(\mathbf{0},\boldsymbol{\Sigma}^{\text{BC-IPW}})$, where $\boldsymbol{\Sigma}^{\text{BC-IPW}}$ is defined in Appendix H.1.

(ii) If both $\mathcal{M}_{ps}$ and $\mathcal{M}_{om}$ are correctly specified, $\widehat{\boldsymbol{\theta}}_q^{\text{BC-ICR}}$ is CAN such that $\sqrt{n}(\widehat{\boldsymbol{\theta}}_q^{\text{BC-ICR}}-\boldsymbol{\theta}_{q0})$ converges to $N(\mathbf{0},\boldsymbol{\Sigma}^{\text{BC-ICR}})$, where $\boldsymbol{\Sigma}^{\text{BC-ICR}}$ is defined in Appendix H.2.

(iii) If $\mathcal{M}_{ps}$ is correctly specified and conditions (B1)–(B2) regrading the smoothness function hold, $\widehat{\boldsymbol{\theta}}_q^{BC\text{-}DR}$ is CAN such that $\sqrt{n}(\widehat{\boldsymbol{\theta}}_q^{BC\text{-}DR} - \boldsymbol{\theta}_{q0})$ converges to $N(\mathbf{0}, \boldsymbol{\Sigma}^{BC\text{-}DR})$, where $\boldsymbol{\Sigma}^{BC\text{-}DR}$ is defined in Appendix H.3.

The proofs of Theorem S4(i)–(iii) are given in Appendix H.1–H.3. We also provide consistent estimators of the asymptotic variances, $\boldsymbol{\Sigma}^{\text{BC-IPW}}$, $\boldsymbol{\Sigma}^{\text{BC-ICR}}$, and $\boldsymbol{\Sigma}^{\text{BC-DR}}$, in Appendix H.1, H.2, and H.3, respectively.

Appendix H.1: Proof of Theorem S4(i)

Rewrite the bias-corrected IPW estimating equation (s42) as $\mathbb{P}_n\left[\widetilde{\mathbb{U}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}})\right]=\mathbf{0}$. Define

$$
\begin{aligned}
&\mathbb{P}_n\left[\mathbb{U}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}})\right]\\
=&\mathbb{P}_n\left[\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K,\widehat{\boldsymbol{\alpha}})}\left\{\mathbb{I}(Y\le h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q))-\sum_{k=1}^{K}c_k^*\left(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{A}_K,\overline{\boldsymbol{L}}_k;\boldsymbol{\gamma}_k,\widehat{\boldsymbol{\alpha}}_k\right)-q\right\}\right]=\mathbf{0}
\end{aligned}
\tag{s43}
$$

as the unsmoothed counterpart of $\mathbb{P}_n\left[\widetilde{\mathbb{U}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\widehat{\boldsymbol{\alpha}},\boldsymbol{\theta}_q)\right]=\mathbf{0}$. Also, define $\mathbb{P}_n\left[\overline{\mathbb{U}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}})\right]$ as $\mathbb{P}_n\left[\mathbb{U}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}})\right]$ with $\mathbb{I}(Y\le h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q))$ in replacement by $F_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)|\overline{A}_K,\overline{\boldsymbol{L}}_K)$. The following lemma will be used repeatedly in the proof of the unbiasedness of the bias-corrected estimating equations.

Lemma 16. *If the propensity score models are correct (i.e., $\pi_k(\overline{A}_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})=f_{A_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}}$, $k=1,\ldots,K$), and the working confounding functions are correct (i.e., $c_k(y,\overline{a}_K,\overline{\boldsymbol{l}}_k;\boldsymbol{\gamma}_k)=c_k(y,\overline{a}_K,\overline{\boldsymbol{l}}_k)$, $k=1,\ldots,K$), then we have*

$$
c_k^*\left(y,\overline{a}_K,\overline{\boldsymbol{l}}_k;\boldsymbol{\gamma}_k,\boldsymbol{\alpha}_{k,0}\right)=F_{Y_{\overline{a}_K}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|\overline{a}_k,\overline{\boldsymbol{l}}_k)-F_{Y_{\overline{a}_K}|\overline{A}_{k-1},\overline{\boldsymbol{L}}_k}(y|\overline{a}_{k-1},\overline{\boldsymbol{l}}_k),
$$

which measures the difference of the probability of the potential outcome $Y_{\overline{a}_K}<y$ given $\overline{A}_k=\overline{a}_K$ and $\overline{\boldsymbol{L}}_k=\overline{\boldsymbol{l}}_k$ as compared to it given $\overline{A}_{k-1}=\overline{a}_{k-1}$ and $\overline{\boldsymbol{L}}_k=\overline{\boldsymbol{l}}_k$.

Proof. This is immediate after observing the following relationship:

$$
\begin{aligned}
&c_k^*\left(y,\overline{a}_K,\overline{\boldsymbol{l}}_k;\boldsymbol{\gamma}_k,\boldsymbol{\alpha}_{k,0}\right)\\
=&c_k\left(y,\overline{a}_K,\overline{\boldsymbol{l}}_k;\boldsymbol{\gamma}_k\right)\times\pi_k(1-a_k,\overline{a}_{k-1},\overline{\boldsymbol{l}}_k;\boldsymbol{\alpha}_{k,0})\\
=&c_k\left(y,\overline{a}_K,\overline{\boldsymbol{l}}_k\right)\times f_{A_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_k}(1-a_k|\overline{a}_{k-1},\overline{\boldsymbol{l}}_k)\\
=&\Big(F_{Y_{\overline{a}_K}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|\overline{a}_k,\overline{\boldsymbol{l}}_k)-F_{Y_{\overline{a}_K}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|1-a_k,\overline{a}_{k-1},\overline{\boldsymbol{l}}_k)\Big)\times f_{A_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_k}(1-a_k|\overline{a}_{k-1},\overline{\boldsymbol{l}}_k)\\
=&F_{Y_{\overline{a}_K}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|\overline{a}_k,\overline{\boldsymbol{l}}_k)f_{A_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_k}-F_{Y_{\overline{a}_K}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|\overline{a}_k,\overline{\boldsymbol{l}}_k)f_{A_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_k}(a_k|\overline{a}_{k-1},\overline{\boldsymbol{l}}_k)\\
&\quad-F_{Y_{\overline{a}_K}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|1-a_k,\overline{a}_{k-1},\overline{\boldsymbol{l}}_k)f_{A_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_k}(1-a_k|\overline{a}_{k-1},\overline{\boldsymbol{l}}_k)\\
=&F_{Y_{\overline{a}_K}|\overline{A}_k,\overline{\boldsymbol{L}}_k}(y|\overline{a}_k,\overline{\boldsymbol{l}}_k)-F_{Y_{\overline{a}_K}|\overline{A}_{k-1},\overline{\boldsymbol{L}}_k}(y|\overline{a}_{k-1},\overline{\boldsymbol{l}}_k).
\end{aligned}
$$

$\square$

When the propensity score models are correct (i.e., $\pi_k(\overline{A}_k, \overline{\boldsymbol{L}}_k; \boldsymbol{\alpha}_{k,0}) = f_{A_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}}$, $k = 1,\dots,K$), the following Lemma shows that (s43) is an unbiased estimating equation such that $E[\mathbb{U}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)] = \mathbf{0}$ if Assumptions (A1), (A3) and (A4) hold and the working confounding functions are correct (i.e., $c_k(y,\overline{A}_K,\overline{\boldsymbol{L}}_k;\boldsymbol{\gamma}_k) = c_k(y,\overline{A}_K,\overline{\boldsymbol{L}}_k)$, $k = 1,\dots,K$).

Lemma 17. *We have that $E[\mathbb{U}^{BC\text{-}IPW}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)] = \mathbf{0}$ if identifiability assumptions (A1), (A3), and (A4) hold and the working confounding functions are correct (i.e., $c_k(y,\overline{A}_K,\overline{\boldsymbol{L}}_k;\boldsymbol{\gamma}_k) = c_k(y,\overline{A}_K,\overline{\boldsymbol{L}}_k)$, $k = 1,\dots,K$).*

Proof. First, we can show

$$
\begin{aligned}
&E\left[\mathbb{U}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)\right]\\
=&E\left[E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\left(\prod_{k=1}^{K}\frac{\mathbb{I}(A_k=a_k)}{\pi_k(a_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}\right)\{\mathbb{I}(Y_{\overline{a}_K}\leq h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}))\right.\right.\\
&\left.\left.\left.-\sum_{k=1}^{K}c_k^*\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{a}_K,\overline{\boldsymbol{L}}_k;\boldsymbol{\gamma}_k,\boldsymbol{\alpha}_{k,0}\right)-q\right)\right\}\middle|\overline{A}_K=\overline{a}_K,\overline{\boldsymbol{L}}_K\right]\right]\\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\left(\prod_{k=1}^{K}\frac{\mathbb{I}(A_k=a_k)}{\pi_k(a_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}\right)\left\{F_{Y_{\overline{A}_K}|\overline{A}_K,\overline{\boldsymbol{L}}_K}(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})|\overline{a}_K,\overline{\boldsymbol{L}}_K)\right.\right.\\
&\left.\left.-\sum_{k=1}^{K}c_k^*\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}),\overline{a}_K,\overline{\boldsymbol{L}}_k;\boldsymbol{\gamma}_k,\boldsymbol{\alpha}_{k,0}\right)-q\right)\right\}\right] \qquad \text{(s44)}
\end{aligned}
$$

Then, we can rewrite the previous expectation as an iterative expectation with the internal expectation conditional on $\overline{A}_{K-1} = \overline{a}_{K-1}, \overline{\boldsymbol{L}}_K$:

$$
\begin{aligned}
\text{(s44)} =&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\left(\prod_{k=1}^{K-1}\frac{\mathbb{I}(A_k=a_k)}{\pi_k(a_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}\right)E\left[\frac{\mathbb{I}(A_K=a_K)}{\pi_k(a_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\alpha}_{K,0})}\times\right.\right.\\
&\left.\left.\left\{F_{Y_{\overline{A}_K}|\overline{A}_K,\overline{\boldsymbol{L}}_K}(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})|\overline{a}_K,\overline{\boldsymbol{L}}_K)-\sum_{k=1}^{K}c_k\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}),\overline{a}_K,\overline{\boldsymbol{L}}_k;\boldsymbol{\gamma}_k,\boldsymbol{\alpha}_{k,0}\right)-q\right\}\middle|\overline{A}_{K-1}=\overline{a}_{K-1},\overline{\boldsymbol{L}}_K\right]\right]
\end{aligned}
$$

(by Lemma 16, we have $F_{Y_{\overline{A}_K}|\overline{A}_K,\overline{\boldsymbol{L}}_K}(y|\overline{a}_K,\overline{\boldsymbol{L}}_K) =$

$F_{Y_{\overline{A}_{K-1},a_K}|\overline{A}_{K-1},\overline{\boldsymbol{L}}_K}(y|\overline{a}_{K-1},\overline{\boldsymbol{L}}_K) + c_K^*(y,\overline{a}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\gamma}_K,\boldsymbol{\alpha}_{K,0})$, and therefore)

$$
\begin{aligned}
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\left(\prod_{k=1}^{K-1}\frac{\mathbb{I}(A_k=a_k)}{\pi_k(a_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}\right)\underbrace{E\left[\frac{\mathbb{I}(A_K=a_K)}{\pi_K(a_K,\overline{L}_K;\boldsymbol{\alpha}_{K,0})}\middle|\overline{A}_{K-1}=\overline{a}_{K-1},\overline{\boldsymbol{L}}_K\right]}_{=1}\times\right.\\
&\left.E\left[F_{Y_{\overline{A}_{K-1},a_K}|\overline{A}_{K-1},\overline{\boldsymbol{L}}_K}(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})|\overline{a}_{K-1},\overline{\boldsymbol{L}}_K)-\sum_{k=1}^{K-1}c_k\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}),\overline{a}_K,\overline{\boldsymbol{L}}_k;\boldsymbol{\gamma}_k,\boldsymbol{\alpha}_{k,0}\right)-q\middle|\overline{A}_{K-1}=\overline{a}_{K-1},\overline{\boldsymbol{L}}_K\right]\right]\\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\left(\prod_{k=1}^{K-1}\frac{\mathbb{I}(A_k=a_k)}{\pi_k(a_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}\right)\times\right.\\
&\left.\left\{F_{Y_{\overline{A}_{K-1},a_K}|\overline{A}_{K-1},\overline{\boldsymbol{L}}_K}(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})|\overline{a}_{K-1},\overline{\boldsymbol{L}}_K)-\sum_{k=1}^{K-1}c_k\left(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0}),\overline{a}_K,\overline{\boldsymbol{L}}_k;\boldsymbol{\gamma}_k,\boldsymbol{\alpha}_{k,0}\right)-q\right\}\right]
\end{aligned}
$$

Then, applying the iterative expectation strategy with the inner expectation conditional on $\overline{A}_{K-2} = \overline{a}_{K-2}, \overline{\boldsymbol{L}}_{K-1}$, we can rewrite the last equation of the previous formula as

$$(\text{s44}) = E\left[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}) \left(\prod_{k=1}^{K-2} \frac{\mathbb{I}(A_k = a_k)}{\pi_k(a_k, \overline{\boldsymbol{L}}_k; \boldsymbol{\alpha}_{k,0})}\right) \times \right.$$
$$\left.\left\{F_{Y_{\overline{A}_{K-2}, \underline{a}_{k-1}} | \overline{A}_{K-2}, \overline{\boldsymbol{L}}_{K-1}}(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}) | \overline{a}_{K-2}, \overline{\boldsymbol{L}}_{K-1}) - \sum_{k=1}^{K-2} c_k\left(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}), \overline{a}_K, \overline{\boldsymbol{L}}_k; \boldsymbol{\gamma}_k, \boldsymbol{\alpha}_{k,0}\right) - q\right\}\right]$$

Repeating down to $k = 1$ gives the following formula

$$E\left[\mathbb{U}^{\text{BC-IPW}*}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0)\right] = E\left[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}) \times \left\{\mathbb{I}(Y_{\overline{a}_K} \leq h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})) - q\right\}\right]$$

This completes the proof. $\square$

Now we ensure that unsmoothed bias-corrected IPW estimating equation, $\mathbb{P}_n\left[\mathbb{U}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \widehat{\boldsymbol{\alpha}})\right] = \boldsymbol{0}$, is an unbiased estimating equation. Then, applying similar strategy used in Lemma 8, we can show that the bias-corrected IPW estimator by solving the unsmoothed bias-corrected IPW estimating equation, denoted by $\widehat{\boldsymbol{\theta}}^{\text{BC-IPW}*}_q$, is consistent and asymptotically normal if the propensity score models are correctly specified and the working confounding functions are true. Finally, we can also show that $\widehat{\boldsymbol{\theta}}^{\text{BC-IPW}}_q$ by solving the smoothed bias-corrected IPW estimating equation, $\mathbb{P}_n\left[\widetilde{\mathbb{U}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \widehat{\boldsymbol{\alpha}})\right] = \boldsymbol{0}$, is consistent and asymptotically normal by applying the similar strategy used in the proof for Theorem S2. Specifically, we can show

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\text{BC-IPW}}_q - \boldsymbol{\theta}_{q0}) = -\left[\boldsymbol{\mathcal{I}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_{q0}}\right]^{-1} \sqrt{n}\mathbb{P}_n\left[\mathbb{U}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0) - \boldsymbol{\mathcal{S}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0} \boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}_0} \mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \boldsymbol{\alpha}_0)\right] + o_p(1),$$

where $\boldsymbol{\mathcal{I}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_{q0}} = \frac{\partial}{\partial \boldsymbol{\theta}_q} E\left[\overline{\mathbb{U}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}_0)\right]\Big|_{\boldsymbol{\theta}_q = \boldsymbol{\theta}_{q0}}$ and $\boldsymbol{\mathcal{S}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0} = \frac{\partial}{\partial \boldsymbol{\theta}_q} E\left[\overline{\mathbb{U}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha})\right]\Big|_{\boldsymbol{\alpha} = \boldsymbol{\alpha}_0}$, and definitions of $\boldsymbol{\mathcal{I}}_{\boldsymbol{\alpha}_0}$ and $\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \boldsymbol{\alpha}_0)$ are given in Appendix C. Therefore, one can conclude that $\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\text{BC-IPW}}_q - \boldsymbol{\theta}_{q0})$ converges to a multivariate normal distribution with mean zero and a finite variance-covariance matrix $\boldsymbol{\Sigma}^{\text{BC-IPW}} = \left[\boldsymbol{\mathcal{I}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_{q0}}\right]^{-1} \boldsymbol{\mathcal{V}}^{\text{BC-IPW}} \left[\boldsymbol{\mathcal{I}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_{q0}}\right]^{-T}$, where $\boldsymbol{\mathcal{V}}^{\text{BC-IPW}} = E\left[\left(\mathbb{U}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0) - \boldsymbol{\mathcal{S}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0} \boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}_0} \mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \boldsymbol{\alpha}_0)\right)^{\otimes 2}\right]$.

A consistent estimator of $\boldsymbol{\Sigma}^{\text{BC-IPW}}$ can be $\widehat{\boldsymbol{\Sigma}}^{\text{BC-IPW}} = \left[\widehat{\boldsymbol{\mathcal{I}}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}\right]^{-1} \widehat{\boldsymbol{\mathcal{V}}}^{\text{BC-IPW}} \left[\widehat{\boldsymbol{\mathcal{I}}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}\right]^{-T}$, where $\widehat{\boldsymbol{\mathcal{I}}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q} = \mathbb{P}_n\left[\frac{\partial}{\partial \boldsymbol{\theta}_q} \widetilde{\mathbb{U}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \widehat{\boldsymbol{\alpha}})\Big|_{\boldsymbol{\theta}_q = \widehat{\boldsymbol{\theta}}^{\text{BC-IPW}}_q}\right]$, $\widehat{\boldsymbol{\mathcal{V}}}^{\text{BC-IPW}} = \mathbb{P}_n\left[\left(\widetilde{\mathbb{U}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \widehat{\boldsymbol{\theta}}^{\text{BC-IPW}}_q, \widehat{\boldsymbol{\alpha}}) - \widehat{\boldsymbol{\mathcal{S}}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q, \boldsymbol{\alpha}} \widehat{\boldsymbol{\mathcal{I}}}^{-1}_{\boldsymbol{\alpha}} \mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \widehat{\boldsymbol{\alpha}})\right)^{\otimes 2}\right]$, $\widehat{\boldsymbol{\mathcal{S}}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q, \boldsymbol{\alpha}} = \mathbb{P}_n\left[\frac{\partial}{\partial \boldsymbol{\alpha}} \widetilde{\mathbb{U}}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \widehat{\boldsymbol{\theta}}^{\text{BC-IPW}}_q, \boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha} = \widehat{\boldsymbol{\alpha}}}\right]$, and $\widehat{\boldsymbol{\mathcal{I}}}_{\boldsymbol{\alpha}} = \mathbb{P}_n\left[\frac{\partial}{\partial \boldsymbol{\alpha}} \mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}, \boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha} = \widehat{\boldsymbol{\alpha}}}\right]$.

Appendix H.2: Proof of Theorem S4(ii)

According to the bias-corrected ICR estimator, the estimating equation for solving $\boldsymbol{\beta} = [\boldsymbol{\beta}_1^T, \boldsymbol{\beta}_2^T, \ldots, \boldsymbol{\beta}_K^T]$ is $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\beta}}^{\text{BC-ICR}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_2, \boldsymbol{\beta})] = \mathbb{P}_n \left[\begin{pmatrix} \mathbb{U}_{\boldsymbol{\beta}_1}^{\text{BC-ICR}}(\boldsymbol{O}; \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\alpha}_2) \\ \mathbb{U}_{\boldsymbol{\beta}_2}^{\text{BC-ICR}}(\boldsymbol{O}; \boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\alpha}_3) \\ \vdots \\ \mathbb{U}_{\boldsymbol{\beta}_K}^{\text{BC-ICR}}(\boldsymbol{O}; \boldsymbol{\beta}_K) \end{pmatrix} \right] = \mathbf{0}$, where $\underline{\boldsymbol{\alpha}}_2 = [\boldsymbol{\alpha}_2^T, \ldots, \boldsymbol{\alpha}_K^T]^T$, $\mathbb{U}_{\boldsymbol{\beta}_K}^{\text{BC-ICR}}(\boldsymbol{O}; \boldsymbol{\beta}_K) = \mathbb{P}_n\left[\mathbb{U}_{\boldsymbol{\beta}_K}(Y, \overline{A}_K, \overline{\boldsymbol{L}}_K; \boldsymbol{\beta}_K)\right]$ and

$$
\begin{aligned}
&\mathbb{U}_{\boldsymbol{\beta}_k}^{\text{BC-ICR}}(\boldsymbol{O}; \boldsymbol{\beta}_k, \boldsymbol{\beta}_{k+1}, \boldsymbol{\alpha}_{k+1}) \\
&= \sum_{\underline{a}_{k+1} \in \underline{\mathbb{A}}_{k+1}} \int_y \mathbb{U}_{\boldsymbol{\beta}_k}(y, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k) \Big\{ \psi_{k+1}(y, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_{k+1}; \boldsymbol{\beta}_{k+1}) - c_{k+1}^{*'}(y, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_{k+1}; \boldsymbol{\gamma}_{k+1}, \boldsymbol{\alpha}_{k+1}) \Big\} \mathrm{d}y.
\end{aligned}
$$

Then, the bias-corrected ICR estimator $\widehat{\boldsymbol{\theta}}_q^{\text{BC-ICR}}$ is obtained by solving $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\theta}_q}^{\text{BC-ICR}}(\boldsymbol{O}; \boldsymbol{\theta}_q, \widehat{\boldsymbol{\alpha}}_1, \widehat{\boldsymbol{\beta}}_1)] = \mathbf{0}$, where

$$
\begin{aligned}
&\mathbb{U}_{\boldsymbol{\theta}_q}^{\text{BC-ICR}}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1) \\
&= \sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} d(\overline{a}_K, \boldsymbol{Z}) \left\{ \Psi_1\left(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q), \overline{a}_K, \boldsymbol{L}_1; \boldsymbol{\beta}_1\right) - c_1^*(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q), \overline{a}_K, \boldsymbol{L}_1; \boldsymbol{\gamma}_1, \boldsymbol{\alpha}_1) - q \right\}
\end{aligned}
$$

In the following lemma, we show that $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\beta}}^{\text{BC-ICR}}(\boldsymbol{O}; \boldsymbol{\theta}_q, \underline{\boldsymbol{\alpha}}_2, \boldsymbol{\beta})] = \mathbf{0}$ and $\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\theta}_q}^{\text{BC-ICR}}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)] = \mathbf{0}$ are unbiased estimating equations, if identifiability assumptions (A1), (A3), and (A4) hold, the propensity score models are correct (i.e., $\pi_k(\overline{A}_k, \overline{\boldsymbol{L}}_k; \boldsymbol{\alpha}_{k,0}) = f_{A_k|\overline{A}_{k-1}, \overline{\boldsymbol{L}}_{k-1}}$, $k = 1, \ldots, K$), the outcome regression models are true (i.e., $\psi_k(y, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}_{k,0}) = f_{Y_{\overline{A}_k, \underline{a}_{k+1}}|\overline{A}_k, \overline{\boldsymbol{L}}_k}(y|\overline{A}_k, \overline{\boldsymbol{L}}_k)$, $k = 1, \ldots, K$), and the working confounding functions are correct (i.e., $c_k(y, \overline{A}_K, \overline{\boldsymbol{L}}_k; \boldsymbol{\gamma}_k) = c_k(y, \overline{A}_K, \overline{\boldsymbol{L}}_k)$, $k = 1, \ldots, K$).

Lemma 18. *If Assumptions (A1), (A3), and (A4) hold, the propensity score models are correct, the outcome regression models are true, and the working confounding functions are correct, we have that*

(i) $E[\mathbb{U}_{\boldsymbol{\beta}}^{BC\text{-}ICR}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_{2,0}, \boldsymbol{\beta}_0)] = \mathbf{0}$,

(ii) $E[\mathbb{U}_{\boldsymbol{\theta}_q}^{BC\text{-}ICR}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_{1,0}, \boldsymbol{\beta}_{1,0})] = \mathbf{0}$.

Proof. We first show $E[\mathbb{U}_{\boldsymbol{\beta}}^{\text{BC-ICR}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_{2,0}, \boldsymbol{\beta}_0)] = \mathbf{0}$. Specifically, $E[\mathbb{U}_{\boldsymbol{\beta}_K}^{\text{BC-ICR}}(\boldsymbol{O}; \boldsymbol{\beta}_{K,0})] = E[\mathbb{U}_{\boldsymbol{\beta}_K}(Y, \overline{A}_K, \overline{\boldsymbol{L}}_K;$
$\boldsymbol{\beta}_{K,0})] = \mathbf{0}$ by definition of $\mathbb{U}_{\boldsymbol{\beta}_K}(Y, \overline{A}_K, \overline{\boldsymbol{L}}_K; \boldsymbol{\beta}_K)$. Also, observing that

$$
\begin{aligned}
&E[\mathbb{U}_{\boldsymbol{\beta}_k}(\boldsymbol{O}; \boldsymbol{\beta}_{k,0}, \boldsymbol{\beta}_{k+1,0}, \boldsymbol{\alpha}_{k+1,0})] \\
&= \sum_{\underline{a}_{k+1} \in \underline{\mathbb{A}}_{k+1}} E\left[\int_y \mathbb{U}_{\boldsymbol{\beta}_k}(y, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}_k) \Big\{ \psi_{k+1}(y, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_{k+1}; \boldsymbol{\beta}_{k+1,0}) - c_{k+1}^{*'}(y, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_{k+1}; \boldsymbol{\gamma}_{k+1}, \boldsymbol{\alpha}_{k+1,0}) \Big\} \mathrm{d}y \right] \\
&= \sum_{\underline{a}_{k+1} \in \underline{\mathbb{A}}_{k+1}} E\left[\int_y \mathbb{U}_{\boldsymbol{\beta}_k}(y, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}_k) \Big\{ f_{Y_{\overline{A}_{k+1}, \underline{a}_{k+2}}|\overline{A}_{k+1}, \overline{\boldsymbol{L}}_{k+1}}(y|\overline{A}_k, a_{k+1}, \overline{\boldsymbol{L}}_{k+1}) - c_{k+1}^{*'}(y, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_{k+1}; \boldsymbol{\gamma}_{k+1}, \boldsymbol{\alpha}_{k+1,0}) \Big\} \mathrm{d}y \right] \\
&\qquad \text{(by Lemma 16, we have that)} \\
&= \sum_{\underline{a}_{k+1} \in \underline{\mathbb{A}}_{k+1}} E\left[\int_y \mathbb{U}_{\boldsymbol{\beta}_k}(y, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}_k) f_{Y_{\overline{A}_k, \underline{a}_{k+1}}|\overline{A}_k, \overline{\boldsymbol{L}}_{k+1}}(y|\overline{A}_k, \overline{\boldsymbol{L}}_{k+1}) \mathrm{d}y \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\underline{a}_{k+1}\in\underline{\mathbb{A}}_{k+1}} E\left[E\left[\mathbb{U}_{\boldsymbol{\beta}_k}(Y_{\overline{A}_k,\underline{a}_{k+1}}, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}_{k,0})\Big|\overline{A}_k, \overline{\boldsymbol{L}}_{k+1}\right]\right] \\
&= \sum_{\underline{a}_{k+1}\in\underline{\mathbb{A}}_{k+1}} E\left[\mathbb{U}_{\boldsymbol{\beta}_k}(Y_{\overline{A}_k,\underline{a}_{k+1}}, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}_{k,0})\right] \\
&=\mathbf{0} \quad \text{(by definition of } \mathbb{U}_{\boldsymbol{\beta}_k}(y, \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}_k)\text{)},
\end{aligned}
$$

we concludes $E[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_{2,0}, \boldsymbol{\beta}_0)] = \mathbf{0}$. Similarly, one can easily verifies that

$$
\begin{aligned}
&E[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_{1,0}, \boldsymbol{\beta}_{1,0})] \\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})\left\{\Psi_1\left(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q), \overline{a}_K, \boldsymbol{L}_1; \boldsymbol{\beta}_{1,0}\right) - c_1^*(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q), \overline{a}_K, \boldsymbol{L}_1; \boldsymbol{\gamma}_1, \boldsymbol{\alpha}_{1,0}) - q\right\}\right] \\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})\left\{F_{Y_{A_1,\underline{a}_2}|A_1,\boldsymbol{L}_1}\Big(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})|a_1, \boldsymbol{L}_1\Big) - c_1^*(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q), \overline{a}_K, \boldsymbol{L}_1; \boldsymbol{\gamma}_1, \boldsymbol{\alpha}_{1,0}) - q\right\}\right] \\
&\quad\text{(by Lemma 16, we have that)} \\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})\left\{F_{Y_{\overline{a}_K}|\boldsymbol{L}_1}\Big(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})|\boldsymbol{L}_1\Big) - q\right\}\right] \\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})\left\{E\left[\mathbb{I}\Big(Y_{\overline{a}_K} \le h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})\Big)\Big|\boldsymbol{L}_1\right] - q\right\}\right] \\
=&E\left[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K} d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})\left\{\mathbb{I}\Big(Y_{\overline{a}_K} \le h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})\Big) - q\right\}\right] \\
=&\mathbf{0}.
\end{aligned}
$$

Now we ensure that $\mathbb{P}_n[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_2, \boldsymbol{\beta})] = \mathbf{0}$ and $\mathbb{P}_n[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)] = \mathbf{0}$ are unbiased estimating equations for $\boldsymbol{\beta}$ and $\boldsymbol{\theta}_q$. $\square$

Noting that $\mathbb{P}_n[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_2, \boldsymbol{\beta})] = \mathbf{0}$ is an unbiased estimating equation (Lemma 18), one can show that $\widehat{\boldsymbol{\beta}}$ is consistent under similar regularity conditions to those used in Appendix B. Then, using a Taylor series for $\mathbb{P}_n[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_2, \boldsymbol{\beta})] = \mathbf{0}$ around $\underline{\boldsymbol{\alpha}}_2 = \underline{\boldsymbol{\alpha}}_{2,0}$, and $\boldsymbol{\beta} = \boldsymbol{\beta}_0$, we have that

$$
\begin{aligned}
\mathbb{P}_n[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \widehat{\underline{\boldsymbol{\alpha}}}_2, \widehat{\boldsymbol{\beta}})] =&\mathbb{P}_n[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_{2,0}, \boldsymbol{\beta}_0)] \\
&+ \mathbb{P}_n\left[\frac{\partial}{\partial\boldsymbol{\beta}}\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_{2,0}, \boldsymbol{\beta}_0)\Big|_{\boldsymbol{\beta}=\boldsymbol{\beta}_0}\right](\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) \\
&+ \mathbb{P}_n\left[\frac{\partial}{\partial\underline{\boldsymbol{\alpha}}_2}\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_{2,0}, \boldsymbol{\beta}_0)\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}\right](\widehat{\underline{\boldsymbol{\alpha}}}_2 - \underline{\boldsymbol{\alpha}}_{2,0}) + o_p(n^{-1/2})
\end{aligned}
$$

This suggests

$$
\sqrt{n}(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0) = -\left[\boldsymbol{\mathcal{I}}^{\text{BC-ICR}}_{\boldsymbol{\beta}_0}\right]^{-1}\sqrt{n}\mathbb{P}_n\left[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_{2,0}, \boldsymbol{\beta}_0) - \boldsymbol{\mathcal{S}}_{\boldsymbol{\beta}_0,\underline{\boldsymbol{\alpha}}_{2,0}}\boldsymbol{\mathcal{I}}^*_{\underline{\boldsymbol{\alpha}}_{2,0}}\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \boldsymbol{\alpha}_0)\right]+o_p(1),
$$

where $\boldsymbol{\mathcal{I}}^{\text{BC-ICR}}_{\boldsymbol{\beta}_0} = \frac{\partial}{\partial \boldsymbol{\beta}} E[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_{2,0}, \boldsymbol{\beta})]\Big|_{\boldsymbol{\beta}=\boldsymbol{\beta}_0}$, $\boldsymbol{\mathcal{S}}_{\boldsymbol{\beta}_0,\underline{\boldsymbol{\alpha}}_{2,0}} = \frac{\partial}{\partial \underline{\boldsymbol{\alpha}}_2} E[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_2, \boldsymbol{\beta}_0)]\Big|_{\underline{\boldsymbol{\alpha}}_2=\underline{\boldsymbol{\alpha}}_{2,0}}$, and $\boldsymbol{\mathcal{I}}^*_{\underline{\boldsymbol{\alpha}}_{2,0}}$ is the last $d_{\underline{\boldsymbol{\alpha}}_2}$ rows of $\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}_0}$, and $d_{\underline{\boldsymbol{\alpha}}_2}$ is the length of $\underline{\boldsymbol{\alpha}}_2$. This concludes that $\widehat{\boldsymbol{\beta}}$ is CAN.

Finally, applying similar strategy used in the proof for Theorem S1, one can show that the augment ICR estimator, $\widehat{\boldsymbol{\theta}}^{\text{BC-ICR}}_q$, is CAN. Using a Tayler series for $\mathbb{P}_n[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)] = \mathbf{0}$, we have that

$$\begin{aligned}
\mathbb{P}_n[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \widehat{\boldsymbol{\theta}}^{\text{BC-ICR}}_q, \widehat{\boldsymbol{\alpha}}_1, \widehat{\boldsymbol{\beta}}_1)] =& \mathbb{P}_n[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_{1,0}, \boldsymbol{\beta}_{1,0})] \\
&+ \mathbb{P}_n\Big[\frac{\partial}{\partial \boldsymbol{\theta}_q} \mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}_{1,0}, \boldsymbol{\beta}_{1,0})\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}\Big](\widehat{\boldsymbol{\theta}}^{\text{BC-ICR}}_q - \boldsymbol{\theta}_{q0}) \\
&+ \mathbb{P}_n\Big[\frac{\partial}{\partial \boldsymbol{\alpha}_1} \mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_{1,0})\Big|_{\boldsymbol{\alpha}_1=\boldsymbol{\alpha}_{1,0}}\Big](\widehat{\boldsymbol{\alpha}}_1 - \boldsymbol{\alpha}_{1,0}) \\
&+ \mathbb{P}_n\Big[\frac{\partial}{\partial \boldsymbol{\beta}_1} \mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_{1,0}, \boldsymbol{\beta}_1)\Big|_{\boldsymbol{\beta}_1=\boldsymbol{\beta}_{1,0}}\Big](\widehat{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_{1,0}) + o_p(n^{-1/2}),
\end{aligned}$$

which implies that $\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\text{BC-ICR}}_q - \boldsymbol{\theta}_{q0})$ converges to a multivariate normal distribution with mean zero and variance-covariance matrix

$$\boldsymbol{\Sigma}^{\text{BC-ICR}} = \Big[\boldsymbol{\mathcal{I}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_{q0}}\Big]^{-1} \boldsymbol{\mathcal{V}}^{\text{BC-ICR}} \Big[\boldsymbol{\mathcal{I}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_{q0}}\Big]^{-T},$$

where $\boldsymbol{\mathcal{I}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_{q0}} = \frac{\partial}{\partial \boldsymbol{\theta}_q} E[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}_{1,0}, \boldsymbol{\beta}_{1,0})]\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}$, $\boldsymbol{\mathcal{V}}^{\text{BC-ICR}} = E\Big[\Big(\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_{1,0}, \boldsymbol{\beta}_{1,0}) - \boldsymbol{\mathcal{S}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_{1,0}} \boldsymbol{\mathcal{I}}^*_{\boldsymbol{\alpha}_{1,0}} \mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \boldsymbol{\alpha}_0) - \boldsymbol{\mathcal{S}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_{1,0}} \boldsymbol{\mathcal{I}}^*_{\boldsymbol{\beta}_{1,0}} \mathbb{U}^{\text{BC-ICR}*}_{\boldsymbol{\beta}}(\boldsymbol{O}; \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)\Big)^{\otimes 2}\Big]$, $\boldsymbol{\mathcal{S}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_{1,0}} = \frac{\partial}{\partial \boldsymbol{\alpha}_1} E[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_{1,0})]\Big|_{\boldsymbol{\alpha}_1=\boldsymbol{\alpha}_{1,0}}$, $\boldsymbol{\mathcal{S}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\beta}_{1,0}} = \frac{\partial}{\partial \boldsymbol{\beta}_1} E[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_{1,0}, \boldsymbol{\beta}_1)]\Big|_{\boldsymbol{\beta}_1=\boldsymbol{\beta}_{1,0}}$, $\boldsymbol{\mathcal{I}}^*_{\boldsymbol{\alpha}_{1,0}}$ is the first $d_{\boldsymbol{\alpha}_1}$ rows of $\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}_0}$ ($d_{\boldsymbol{\alpha}_1}$ is the length of $\boldsymbol{\alpha}_1$), $\boldsymbol{\mathcal{I}}^*_{\boldsymbol{\beta}_{1,0}}$ is the first $d_{\boldsymbol{\beta}_1}$ rows of $\Big[\boldsymbol{\mathcal{I}}^{\text{BC-ICR}}_{\boldsymbol{\beta}_0}\Big]^{-1}$ ($d_{\boldsymbol{\beta}_1}$ is the length of $\boldsymbol{\beta}_1$), and $\mathbb{U}^{\text{BC-ICR}*}_{\boldsymbol{\beta}}(\boldsymbol{O}; \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0) = \mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_{2,0}, \boldsymbol{\beta}_0) - \boldsymbol{\mathcal{S}}_{\boldsymbol{\beta}_0,\underline{\boldsymbol{\alpha}}_{2,0}} \boldsymbol{\mathcal{I}}^*_{\underline{\boldsymbol{\alpha}}_{2,0}} \mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \boldsymbol{\alpha}_0)$.

Furthermore, $\widehat{\boldsymbol{\Sigma}}^{\text{BC-ICR}} = \Big[\widehat{\boldsymbol{\mathcal{I}}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}\Big]^{-1} \widehat{\boldsymbol{\mathcal{V}}}^{\text{BC-ICR}} \Big[\widehat{\boldsymbol{\mathcal{I}}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}\Big]^{-T}$ is a consistent estimator of $\boldsymbol{\Sigma}^{\text{BC-ICR}}$, where $\widehat{\boldsymbol{\mathcal{I}}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q} = \mathbb{P}_n\Big[\frac{\partial}{\partial \boldsymbol{\theta}_q} \mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \widehat{\boldsymbol{\alpha}}_{1,0}, \widehat{\boldsymbol{\beta}}_{1,0})\Big|_{\boldsymbol{\theta}_q=\widehat{\boldsymbol{\theta}}^{\text{BC-ICR}}_q}\Big]$, and $\widehat{\boldsymbol{\mathcal{V}}}^{\text{BC-ICR}} = \mathbb{P}_n\Big[\Big(\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \widehat{\boldsymbol{\theta}}^{\text{BC-ICR}}_q, \widehat{\boldsymbol{\alpha}}_{1,0}, \widehat{\boldsymbol{\beta}}_{1,0}) - \widehat{\boldsymbol{\mathcal{S}}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q,\boldsymbol{\alpha}_1} \widehat{\boldsymbol{\mathcal{I}}}^*_{\boldsymbol{\alpha}_1} \mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \widehat{\boldsymbol{\alpha}}) - \widehat{\boldsymbol{\mathcal{S}}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q,\boldsymbol{\beta}_1} \widehat{\boldsymbol{\mathcal{I}}}^*_{\boldsymbol{\beta}_1} \mathbb{U}^{\text{BC-ICR}*}_{\boldsymbol{\beta}}(\boldsymbol{O}; \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})\Big)^{\otimes 2}\Big]$, $\boldsymbol{\mathcal{S}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_{1,0}} = E\Big[\frac{\partial}{\partial \boldsymbol{\alpha}_1} \mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_{1,0})\Big|_{\boldsymbol{\alpha}_1=\boldsymbol{\alpha}_{1,0}}\Big]$, $\widehat{\boldsymbol{\mathcal{S}}}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q,\boldsymbol{\beta}_1} = \mathbb{P}_n\Big[\frac{\partial}{\partial \boldsymbol{\beta}_1} \mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \widehat{\boldsymbol{\theta}}^{\text{BC-ICR}}_q, \widehat{\boldsymbol{\alpha}}_1, \boldsymbol{\beta}_1)\Big|_{\boldsymbol{\beta}_1=\widehat{\boldsymbol{\beta}}_1}\Big]$, $\widehat{\boldsymbol{\mathcal{I}}}^*_{\boldsymbol{\alpha}_1}$ is the first $d_{\boldsymbol{\alpha}_1}$ rows of $\widehat{\boldsymbol{\mathcal{I}}}^{-1}_{\boldsymbol{\alpha}}$ ($d_{\boldsymbol{\alpha}_1}$ is the length of $\boldsymbol{\alpha}_1$ and $\widehat{\boldsymbol{\mathcal{I}}}_{\boldsymbol{\alpha}} = \mathbb{P}_n[\frac{\partial}{\partial \boldsymbol{\alpha}} \mathbb{U}(\boldsymbol{O}; \boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\widehat{\boldsymbol{\alpha}}}]$), $\widehat{\boldsymbol{\mathcal{I}}}^*_{\boldsymbol{\beta}_1}$ is the first $d_{\boldsymbol{\beta}_1}$ rows of $\Big[\widehat{\boldsymbol{\mathcal{I}}}^{\text{BC-ICR}}_{\boldsymbol{\beta}}\Big]^{-1}$ ($d_{\boldsymbol{\beta}_1}$ is the length of $\boldsymbol{\beta}_1$ and $\widehat{\boldsymbol{\mathcal{I}}}^{\text{BC-ICR}}_{\boldsymbol{\beta}} = \mathbb{P}_n\Big[\frac{\partial}{\partial \boldsymbol{\beta}} \mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \widehat{\underline{\boldsymbol{\alpha}}}_2, \boldsymbol{\beta})\Big|_{\boldsymbol{\beta}=\widehat{\boldsymbol{\beta}}}\Big]$), and $\mathbb{U}^{\text{BC-ICR}*}_{\boldsymbol{\beta}}(\boldsymbol{O}; \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}) = \mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \widehat{\underline{\boldsymbol{\alpha}}}_2, \widehat{\boldsymbol{\beta}}) - \widehat{\boldsymbol{\mathcal{S}}}_{\boldsymbol{\beta},\underline{\boldsymbol{\alpha}}_2} \widehat{\boldsymbol{\mathcal{I}}}^*_{\underline{\boldsymbol{\alpha}}_2} \mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \widehat{\boldsymbol{\alpha}})$ (here, $\widehat{\boldsymbol{\mathcal{S}}}_{\boldsymbol{\beta},\underline{\boldsymbol{\alpha}}_2} = \mathbb{P}_n\Big[\frac{\partial}{\partial \underline{\boldsymbol{\alpha}}_2} \mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_2, \widehat{\boldsymbol{\beta}})\Big|_{\underline{\boldsymbol{\alpha}}_2=\widehat{\underline{\boldsymbol{\alpha}}}_2}\Big]$, $\widehat{\boldsymbol{\mathcal{I}}}^*_{\underline{\boldsymbol{\alpha}}_2}$ is the last $d_{\underline{\boldsymbol{\alpha}}_2}$ rows of $\widehat{\boldsymbol{\mathcal{I}}}^{-1}_{\boldsymbol{\alpha}}$, and $d_{\underline{\boldsymbol{\alpha}}_2}$ is the length of $\underline{\boldsymbol{\alpha}}_2$).

Appendix H.3: Proof of Theorem S4(iii)

Define $\mathbb{P}_n\left[\mathbb{U}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\right]$ as the unsmoothed bias-corrected doubly robust estimating equation, which replaces $\mathcal{K}\left(\frac{h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)-Y}{\tau_n}\right)$ used in the bias-corrected doubly robust estimating equation, $\mathbb{P}_n\left[\widetilde{\mathbb{U}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\right]$, with $\mathbb{I}\left(Y\le h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\right)$. Specifically,

$$\mathbb{P}_n\left\{\mathbb{U}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\right\}=\mathbb{P}_n\left\{\mathbb{U}^{(K)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})+\sum_{k=0}^{K-1}\widetilde{\mathbb{U}}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\right\}=\boldsymbol{0},$$

where $\mathbb{U}^{(K)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})$ is defined in Section 4.1 and $\widetilde{\mathbb{U}}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})$, $k=1,\ldots,K-1$, and $\widetilde{\mathbb{U}}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})$ are defined in Section 6.3. Also, define $\mathbb{P}_n\left\{\overline{\mathbb{U}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\right\}$ as $\mathbb{P}_n\left\{\mathbb{U}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\right\}$ with $\mathbb{U}^{(K)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})$ in replacement by

$$\begin{aligned}&\overline{\mathbb{U}}^{(K)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\\ =&\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\alpha})}\left\{F_{Y|\overline{A}_K,\overline{\boldsymbol{L}}_K}\Big(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)|\overline{A}_K,\overline{\boldsymbol{L}}_K\Big)-\Psi_K\Big(h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{A}_K,\overline{\boldsymbol{L}}_K;\boldsymbol{\beta}_K\Big)\right\}\end{aligned}$$

The following Lemma shows that $\mathbb{P}_n\left[\mathbb{U}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})\right]$ is an unbiased estimating equation if Assumptions (A1), (A3), and (A4) hold, the confounding functions are true, the propensity score models are correct (i.e., $\pi_k(\overline{A}_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})=f_{A_k|\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1}}$, $k=1,\ldots,K$, $k=1,\ldots,K$), but regardless of whether the outcome models are correct or not.

Lemma 19. *If Assumptions (A1), (A3), and (A4) hold, the confounding functions are true, then* $E\left[\mathbb{U}^{BC\text{-}DR}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0,\boldsymbol{\beta}^*)\right]=\boldsymbol{0}$, *regardless of whether* $\boldsymbol{\beta}^*=\boldsymbol{\beta}_0$ *or not.*

Proof. We can rewrite the bias-corrected doubly robust estimating score as

$$\mathbb{U}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})=\sum_{k=1}^{K}\widetilde{\mathbb{T}}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta}),$$

where

$$\begin{aligned}\widetilde{\mathbb{T}}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})=&\frac{d(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_K(\overline{A}_K,\overline{\boldsymbol{L}}_K;\overline{\boldsymbol{\alpha}}_K)}\times\mathbb{I}\left(Y\le h(\overline{A}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\right)\\ &-\sum_{k=1}^{K}\sum_{\underline{a}_k\in\underline{\mathbb{A}}_k}\frac{d(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z})}{\overline{\pi}_{k-1}(\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1};\overline{\boldsymbol{\alpha}}_{k-1})}c^*_k\Big(h(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{A}_{k-1},\underline{a}_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\gamma}_k,\boldsymbol{\alpha}_k\Big)\\ &-\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\times q,\\ \widetilde{\mathbb{T}}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\boldsymbol{\alpha},\boldsymbol{\beta})=&\sum_{\underline{a}_k\in\underline{\mathbb{A}}_k}\frac{d(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z};\boldsymbol{\theta}_q)}{\overline{\pi}_{k-1}(\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1};\overline{\boldsymbol{\alpha}}_{k-1})}\times\Psi_k\left(h(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{A}_{k-1},\underline{a}_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_k\right)\end{aligned}$$

$$- \sum_{\underline{a}_{k+1} \in \underline{\mathbb{A}}_{k+1}} \frac{d(\overline{A}_k, \underline{a}_{k+1}, \boldsymbol{Z}; \boldsymbol{\theta}_q)}{\overline{\pi}_k(\overline{A}_k, \overline{\boldsymbol{L}}_k; \overline{\boldsymbol{\alpha}}_k)} \times \Psi_k(h(\overline{A}_k, \underline{a}_{k+1}, \boldsymbol{Z}; \boldsymbol{\theta}_q), \overline{A}_k, \underline{a}_{k+1}, \overline{\boldsymbol{L}}_k; \boldsymbol{\beta}_k),$$

for $k = 1, \ldots, K$.

We first show $E[\widetilde{\mathbb{T}}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*)] = \boldsymbol{0}$. By the consistency assumption (A1), we have

$$\begin{aligned}
&\widetilde{\mathbb{T}}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*) \\
=& \sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} \frac{d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}) \mathbb{I}(\overline{A}_K = \overline{a}_K)}{\overline{\pi}_K(\overline{a}_K, \overline{\boldsymbol{L}}_K; \overline{\boldsymbol{\alpha}}_{K,0})} \times \mathbb{I}\left(Y_{\overline{a}_K} \leq h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0})\right) - \\
&\sum_{k=1}^{K} \sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} \frac{d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}) \mathbb{I}(\overline{A}_{k-1} = \overline{a}_{k-1})}{\overline{\pi}_{k-1}(\overline{a}_{k-1}, \overline{\boldsymbol{L}}_{k-1}; \overline{\boldsymbol{\alpha}}_{k-1})} c^*_k\Big(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}), \overline{a}_K, \overline{\boldsymbol{L}}_k; \boldsymbol{\gamma}_k, \boldsymbol{\alpha}_{k,0}\Big) - \sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) \times q,
\end{aligned}$$

where the expectation of the second term in the right hand side of the equation is

$$\begin{aligned}
&E\left[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} \frac{d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}) \mathbb{I}(\overline{A}_{k-1} = \overline{a}_{k-1})}{\overline{\pi}_{k-1}(\overline{a}_{k-1}, \overline{\boldsymbol{L}}_{k-1}; \overline{\boldsymbol{\alpha}}_{k-1,0})} c^*_k\Big(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q), \overline{a}_K, \overline{\boldsymbol{L}}_k; \boldsymbol{\gamma}_k, \boldsymbol{\alpha}_{k,0}\Big)\right] \\
=&E\Bigg[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} \frac{d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}) \mathbb{I}(\overline{A}_{k-1} = \overline{a}_{k-1})}{\overline{\pi}_{k-1}(\overline{a}_{k-1}, \overline{\boldsymbol{L}}_{k-1}; \overline{\boldsymbol{\alpha}}_{k-1,0})} \times \prod_{j=k}^{K} \underbrace{E\left[\frac{\mathbb{I}(A_j = a_j)}{\pi_j(\overline{a}_j, \overline{\boldsymbol{L}}_j; \boldsymbol{\alpha}_{j,0})} \Big| A_j = a_j, \overline{\boldsymbol{L}}_j\right]}_{=1} \times \\
&c^*_k\Big(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q), \overline{a}_K, \overline{\boldsymbol{L}}_k; \boldsymbol{\gamma}_k, \boldsymbol{\alpha}_{k,0}\Big)\Bigg] \\
&\quad \text{(by law of iterated expectation, we have)} \\
=&E\left[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} \frac{d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}) \mathbb{I}(\overline{A}_K = \overline{a}_K)}{\overline{\pi}_K(\overline{a}_K, \overline{\boldsymbol{L}}_K; \overline{\boldsymbol{\alpha}}_{K,0})} c_k\Big(h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q), \overline{a}_K, \overline{\boldsymbol{L}}_k; \boldsymbol{\gamma}_k, \boldsymbol{\alpha}_{k,0}\Big)\right]
\end{aligned}$$

and the expectation of the third term in the right hand side of the equation is

$$\begin{aligned}
E\left[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) \times q\right] =& E\left[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) \times \prod_{j=1}^{K} \underbrace{E\left[\frac{\mathbb{I}(A_j = a_j)}{\pi_j(\overline{a}_j, \overline{\boldsymbol{L}}_j; \boldsymbol{\alpha}_{j,0})} \Big| A_j = a_j, \overline{\boldsymbol{L}}_j\right]}_{=1} \times q\right] \\
&\text{(by law of iterated expectation, we have)} \\
=&E\left[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} \frac{d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_{q0}) \mathbb{I}(\overline{A}_K = \overline{a}_K)}{\overline{\pi}_K(\overline{a}_K, \overline{\boldsymbol{L}}_K; \overline{\boldsymbol{\alpha}}_{K,0})} \times q\right].
\end{aligned}$$

This suggests that

$$\begin{aligned}
&E\left[\widetilde{\mathbb{T}}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*)\right] \\
=&E\Bigg[\sum_{\overline{a}_K \in \overline{\mathbb{A}}_K} \frac{d(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q) \mathbb{I}(\overline{A}_K = \overline{a}_K)}{\overline{\pi}_K(\overline{a}_K, \overline{\boldsymbol{L}}_K; \overline{\boldsymbol{\alpha}}_{K,0})} \times \mathbb{I}\left(Y_{\overline{a}_K} \leq h(\overline{a}_K, \boldsymbol{Z}; \boldsymbol{\theta}_q)\right) -
\end{aligned}$$

$$\sum_{k=1}^{K}\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}\frac{d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\mathbb{I}(\overline{A}_K=\overline{a}_K)}{\pi_K(\overline{a}_K,\overline{\boldsymbol{L}}_K;\overline{\boldsymbol{\alpha}}_{K,0})}c_k^*\Big(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{a}_K,\overline{\boldsymbol{L}}_k;\boldsymbol{\gamma}_k,\boldsymbol{\alpha}_{k,0}\Big)-\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}\frac{d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\mathbb{I}(\overline{A}_K=\overline{a}_K)}{\pi_K(\overline{a}_K,\overline{\boldsymbol{L}}_K;\overline{\boldsymbol{\alpha}}_{K,0})}\times q\Bigg]$$

$$=E\Bigg[\sum_{\overline{a}_K\in\overline{\mathbb{A}}_K}\frac{d(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\mathbb{I}(\overline{A}_K=\overline{a}_K)}{\pi_K(\overline{a}_K,\overline{\boldsymbol{L}}_K;\overline{\boldsymbol{\alpha}}_{K,0})}\Big\{\mathbb{I}\left(Y_{\overline{a}_K}\le h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q)\right)-c_k\Big(h(\overline{a}_K,\boldsymbol{Z};\boldsymbol{\theta}_q),\overline{a}_K,\overline{\boldsymbol{L}}_k\Big)-q\Big\}\Bigg]$$

$$=E[\mathbb{U}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)]$$

$$=\boldsymbol{0}\quad(\text{by Lemma 17, we have } E[\mathbb{U}^{\text{BC-IPW}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0)]=\boldsymbol{0}).$$

Now we conclude $E[\widetilde{\mathbb{T}}^{(0)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\overline{\boldsymbol{\alpha}}_{K,0})]=\boldsymbol{0}$.

Next, we can show that the expectation of the second term in $\widetilde{\mathbb{T}}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0,\boldsymbol{\beta}^*)$ is

$$E\left[\sum_{\underline{a}_{k+1}\in\underline{\mathbb{A}}_{k+1}}\frac{d(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_{q0})}{\overline{\pi}_k(\overline{A}_k,\overline{\boldsymbol{L}}_k;\overline{\boldsymbol{\alpha}}_{k,0})}\times\Psi_k(h(\overline{A}_k,\underline{a}_{k+1},\boldsymbol{Z};\boldsymbol{\theta}_{q0}),\overline{A}_k,\underline{a}_{k+1},\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_k^*)\right]$$

$$=E\left[\sum_{\underline{a}_k\in\underline{\mathbb{A}}_k}\frac{d(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\times\mathbb{I}(A_k=a_k)}{\overline{\pi}_{k-1}(\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1};\overline{\boldsymbol{\alpha}}_{k-1,0})\times\pi_k(a_k,\overline{A}_{k-1},\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}\times\Psi_k\Big(h(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z};\boldsymbol{\theta}_{q0}),\overline{A}_{k-1},\underline{a}_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_k^*\Big)\right]$$

$$=E\Bigg\{\sum_{\underline{a}_k\in\underline{\mathbb{A}}_k}E\Bigg[\frac{d(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z};\boldsymbol{\theta}_{q0})\times\mathbb{I}(A_k=a_k)}{\overline{\pi}_{k-1}(\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1};\overline{\boldsymbol{\alpha}}_{k-1,0})\times\pi_k(a_k,\overline{A}_{k-1},\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}$$

$$\times\Psi_k\Big(h(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z};\boldsymbol{\theta}_{q0}),\overline{A}_{k-1},\underline{a}_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_k^*\Big)\Big|\overline{A}_{k-1},a_k,\overline{\boldsymbol{L}}_k\Bigg]\Bigg\}$$

$$=E\Bigg\{\sum_{\underline{a}_k\in\underline{\mathbb{A}}_k}\frac{d(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z};\boldsymbol{\theta}_{q0})}{\overline{\pi}_{k-1}(\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1};\overline{\boldsymbol{\alpha}}_{k-1,0})}\times\Psi_k\Big(h(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z};\boldsymbol{\theta}_{q0}),\overline{A}_{k-1},\underline{a}_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_k^*\Big)\times$$

$$\underbrace{E\left[\frac{\mathbb{I}(A_k=a_k)}{\pi_k(a_k,\overline{A}_{k-1},\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}\Big|\overline{A}_{k-1},a_k,\overline{\boldsymbol{L}}_k\right]}_{=1,\text{ by } E[\mathbb{I}(A_k=a_k)|\overline{A}_{k-1},a_k,\overline{\boldsymbol{L}}_k]=\pi_k(a_k,\overline{A}_{k-1},\overline{\boldsymbol{L}}_k;\boldsymbol{\alpha}_{k,0})}\Bigg\}$$

$$=E\left[\sum_{\underline{a}_k\in\underline{\mathbb{A}}_k}\frac{d(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z};\boldsymbol{\theta}_{q0})}{\overline{\pi}_{k-1}(\overline{A}_{k-1},\overline{\boldsymbol{L}}_{k-1};\overline{\boldsymbol{\alpha}}_{k-1,0})}\times\Psi_k\Big(h(\overline{A}_{k-1},\underline{a}_k,\boldsymbol{Z};\boldsymbol{\theta}_{q0}),\overline{A}_{k-1},\underline{a}_k,\overline{\boldsymbol{L}}_k;\boldsymbol{\beta}_k^*\Big)\right],$$

which is actually the expectation of the first term in $\widetilde{\mathbb{T}}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0,\boldsymbol{\beta}^*)$. This suggests that $E[\widetilde{\mathbb{T}}^{(k)}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0,\boldsymbol{\beta}^*)]=\boldsymbol{0}$, for $k=1,\ldots,K$. This completes the proof. □

By Lemma 19, we ensure that $E\left[\mathbb{U}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0,\boldsymbol{\beta}^*)\right]=\boldsymbol{0}$, whether $\boldsymbol{\beta}^*=\boldsymbol{\beta}_0$ or not. Then, applying similar strategy used in Lemma 13, we can show that the bias-corrected doubly robust estimator by solving the unsmoothed bias-corrected doubly robust estimating equation $\mathbb{P}_n\left[\mathbb{U}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})\right]=\boldsymbol{0}$, denoted by $\widehat{\boldsymbol{\theta}}_q^{\text{BC-DR}*}$, is consistent and asymptotically normal if the propensity score models are correctly specified and the sensitivity functions are true. Finally, we can also show that $\widehat{\boldsymbol{\theta}}_q^{\text{BC-DR}}$ by solving the smoothed bias-corrected IPW estimating equation, $\mathbb{P}_n\left[\widetilde{\mathbb{U}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_q,\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})\right]=\boldsymbol{0}$, is consistent by applying the similar strategy used in the proof for Theorem 2. Also, one can show

$$\mathbb{P}_n\Big[\widetilde{\mathbb{U}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\widehat{\boldsymbol{\theta}}_q^{\text{BC-DR}},\widehat{\boldsymbol{\alpha}},\widehat{\boldsymbol{\beta}})\Big]=\mathbb{P}_n[\mathbb{U}^{\text{DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O};\boldsymbol{\theta}_{q0},\boldsymbol{\alpha}_0,\boldsymbol{\beta}^*)]$$

$$+ \mathbb{P}_n \left[\frac{\partial}{\partial \boldsymbol{\alpha}} \overline{\mathbb{U}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}, \boldsymbol{\beta}^*)\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \right] (\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0)$$
$$+ \mathbb{P}_n \left[\frac{\partial}{\partial \boldsymbol{\beta}} \overline{\mathbb{U}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta})\Big|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*} \right] (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*)$$
$$+ \mathbb{P}_n \left[\frac{\partial}{\partial \boldsymbol{\theta}_q} \overline{\mathbb{U}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*)\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}} \right] (\widehat{\boldsymbol{\theta}}^{\text{BC-DR}}_q - \boldsymbol{\theta}_{q0}) + o_p(n^{-1/2}),$$

where $\boldsymbol{\beta}^*$ is the convergent value of $\widehat{\boldsymbol{\beta}}$, which may not equal $\boldsymbol{\beta}_0$ if the outcome models are not correctly specified. Recall that

$$\sqrt{n}(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0) = -\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}_0} \sqrt{n}\mathbb{P}_n[\mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \boldsymbol{\alpha}_0)] + o_p(1),$$
$$\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*) = -\left[\boldsymbol{\mathcal{I}}^{\text{BC-ICR}}_{\boldsymbol{\beta}^*}\right]^{-1} \sqrt{n}\mathbb{P}_n \left[\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_{2,0}, \boldsymbol{\beta}^*) - \boldsymbol{\mathcal{S}}_{\boldsymbol{\beta}^*, \underline{\boldsymbol{\alpha}}_{2,0}} \boldsymbol{\mathcal{I}}^*_{\underline{\boldsymbol{\alpha}}_{2,0}} \mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \boldsymbol{\alpha}_0)\right] + o_p(1),$$

where $\mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_2, \boldsymbol{\beta})$ is given in Appendix H.2, $\boldsymbol{\mathcal{I}}^{\text{BC-ICR}}_{\boldsymbol{\beta}^*} = E\left[\frac{\partial}{\partial \boldsymbol{\beta}} \mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_{2,0}, \boldsymbol{\beta})\Big|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}\right]$, $\boldsymbol{\mathcal{S}}_{\boldsymbol{\beta}^*, \underline{\boldsymbol{\alpha}}_{2,0}} = E\left[\frac{\partial}{\partial \underline{\boldsymbol{\alpha}}_2} \mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_2, \boldsymbol{\beta}^*)\Big|_{\underline{\boldsymbol{\alpha}}_2=\underline{\boldsymbol{\alpha}}_{2,0}}\right]$, and $\boldsymbol{\mathcal{I}}^*_{\underline{\boldsymbol{\alpha}}_{2,0}}$ is the last $d_{\underline{\boldsymbol{\alpha}}_2}$ rows of $\boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}_0}$, and $d_{\underline{\boldsymbol{\alpha}}_2}$ is the length of $\underline{\boldsymbol{\alpha}}_2$. Therefore, we have

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\text{BC-DR}}_q - \boldsymbol{\theta}_{q0}) = -\left[\boldsymbol{\mathcal{I}}^{\text{BC-DR}}_{\boldsymbol{\theta}_{q0}}\right]^{-1} \sqrt{n}\mathbb{P}_n \Big[\mathbb{U}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*) - \boldsymbol{\mathcal{S}}^{\text{BC-DR}}_{\boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0} \boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}_0} \mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \boldsymbol{\alpha}_0)$$
$$- \boldsymbol{\mathcal{S}}^{\text{BC-DR}}_{\boldsymbol{\theta}_{q0}, \boldsymbol{\beta}^*} [\boldsymbol{\mathcal{I}}^{\text{BC-ICR}}_{\boldsymbol{\beta}^*}]^{-1} \mathbb{U}^{\text{BC-ICR}*}_{\boldsymbol{\beta}}(\boldsymbol{O}; \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*)\Big] + o_p(1),$$

where $\boldsymbol{\mathcal{I}}^{\text{BC-DR}}_{\boldsymbol{\theta}_{q0}} = \frac{\partial}{\partial \boldsymbol{\theta}_q} E\left[\mathbb{U}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*)\right]\Big|_{\boldsymbol{\theta}_q=\boldsymbol{\theta}_{q0}}$, $\boldsymbol{\mathcal{S}}^{\text{BC-DR}}_{\boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0} = \frac{\partial}{\partial \boldsymbol{\alpha}} E\left[\mathbb{U}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}, \boldsymbol{\beta}^*)\right]\Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}$, $\boldsymbol{\mathcal{S}}^{\text{BC-DR}}_{\boldsymbol{\theta}_{q0}, \boldsymbol{\beta}^*} = \frac{\partial}{\partial \boldsymbol{\beta}} E\left[\mathbb{U}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta})\right]\Big|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}$, and $\mathbb{U}^{\text{BC-ICR}*}_{\boldsymbol{\beta}}(\boldsymbol{O}; \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*) = \mathbb{U}^{\text{BC-ICR}}_{\boldsymbol{\beta}}(\boldsymbol{O}; \underline{\boldsymbol{\alpha}}_{2,0}, \boldsymbol{\beta}^*) - \boldsymbol{\mathcal{S}}_{\boldsymbol{\beta}^*, \underline{\boldsymbol{\alpha}}_{2,0}} \boldsymbol{\mathcal{I}}^*_{\underline{\boldsymbol{\alpha}}_{2,0}} \mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \boldsymbol{\alpha}_0)$. This deduce that $\sqrt{n}(\widehat{\boldsymbol{\theta}}^{\text{BC-DR}}_q - \boldsymbol{\theta}_{q0})$ converges to a multivariate normal distribution with mean zero and a finite variance-covariance matrix

$$\boldsymbol{\Sigma}^{\text{BC-DR}} = \left[\boldsymbol{\mathcal{I}}^{\text{BC-DR}}_{\boldsymbol{\theta}_{q0}}\right]^{-1} \boldsymbol{\mathcal{V}}^{\text{BC-DR}} \left[\boldsymbol{\mathcal{I}}^{\text{BC-DR}}_{\boldsymbol{\theta}_{q0}}\right]^{-T},$$

where

$$\boldsymbol{\mathcal{V}}^{\text{BC-DR}} = E\left[\left(\mathbb{U}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*) - \boldsymbol{\mathcal{S}}^{\text{BC-DR}}_{\boldsymbol{\theta}_{q0}, \boldsymbol{\alpha}_0} \boldsymbol{\mathcal{I}}^{-1}_{\boldsymbol{\alpha}_0} \mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \boldsymbol{\alpha}_0) - \boldsymbol{\mathcal{S}}^{\text{BC-DR}}_{\boldsymbol{\theta}_{q0}, \boldsymbol{\beta}^*} [\boldsymbol{\mathcal{I}}^{\text{BC-ICR}}_{\boldsymbol{\beta}^*}]^{-1} \mathbb{U}^{\text{BC-ICR}*}_{\boldsymbol{\beta}}(\boldsymbol{O}; \boldsymbol{\alpha}_0, \boldsymbol{\beta}^*)\right)^{\otimes 2}\right].$$

The variance-covariance matrix, $\boldsymbol{\Sigma}^{\text{BC-DR}}$ can be consistently estimated by

$$\widehat{\boldsymbol{\Sigma}}^{\text{BC-DR}} = \left[\widehat{\boldsymbol{\mathcal{I}}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}\right]^{-1} \widehat{\boldsymbol{\mathcal{V}}}^{\text{BC-DR}} \left[\widehat{\boldsymbol{\mathcal{I}}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}\right]^{-T},$$

where $\widehat{\boldsymbol{\mathcal{I}}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q} = \mathbb{P}_n\left[\frac{\partial}{\partial \boldsymbol{\theta}_q} \widetilde{\mathbb{U}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \boldsymbol{\theta}_q, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})\Big|_{\boldsymbol{\theta}_q=\widehat{\boldsymbol{\theta}}^{\text{BC-DR}}_q}\right]$,

$$\widehat{\boldsymbol{\mathcal{V}}}^{\text{BC-DR}} = \mathbb{P}_n\left[\left(\widetilde{\mathbb{U}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}, \widehat{\boldsymbol{\theta}}^{\text{BC-DR}}_q, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}) - \widehat{\boldsymbol{\mathcal{S}}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q, \boldsymbol{\alpha}} \widehat{\boldsymbol{\mathcal{I}}}^{-1}_{\boldsymbol{\alpha}} \mathbb{U}_{\boldsymbol{\alpha}}(\boldsymbol{O}; \widehat{\boldsymbol{\alpha}}) - \widehat{\boldsymbol{\mathcal{S}}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q, \boldsymbol{\beta}} \left[\widehat{\boldsymbol{\mathcal{I}}}^{\text{BC-ICR}}_{\boldsymbol{\beta}}\right]^{-1} \mathbb{U}^{\text{BC-ICR}*}_{\boldsymbol{\beta}}(\boldsymbol{O}; \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})\right)^{\otimes 2}\right],$$

$$\widehat{\boldsymbol{\mathcal{S}}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q, \boldsymbol{\alpha}} = \mathbb{P}_n\left[\frac{\partial}{\partial \boldsymbol{\alpha}} \widetilde{\mathbb{U}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \widehat{\boldsymbol{\theta}}^{\text{BC-DR}}_q, \boldsymbol{\alpha}, \widehat{\boldsymbol{\beta}})\Big|_{\boldsymbol{\alpha}=\widehat{\boldsymbol{\alpha}}}\right], \widehat{\boldsymbol{\mathcal{S}}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q, \boldsymbol{\beta}} = \mathbb{P}_n\left[\frac{\partial}{\partial \boldsymbol{\beta}} \widetilde{\mathbb{U}}^{\text{BC-DR}}_{\boldsymbol{\theta}_q}(\boldsymbol{O}; \widehat{\boldsymbol{\theta}}^{\text{DR}}_q, \widehat{\boldsymbol{\alpha}}, \boldsymbol{\beta})\Big|_{\boldsymbol{\beta}=\widehat{\boldsymbol{\beta}}}\right],$$

and definitions of $\widehat{\boldsymbol{\mathcal{I}}}^{\text{BC-ICR}}_{\boldsymbol{\beta}}$ and $\mathbb{U}^{\text{BC-ICR}*}_{\boldsymbol{\beta}}(\boldsymbol{O}; \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})$ are given at the end of Appendix H.2.

Appendix I: Supporting information for simulation study

We elucidate the data generation process for the simulation study as below. Recall that we consider $K = 3$ time periods and therefore the potential outcome is $Y_{\overline{a}_3}$. Also, we define the potential values of post-baseline covariates as $\boldsymbol{L}_{2,a_1} = [L_{21,a_1}, L_{22,a_1}]^T$ and $\boldsymbol{L}_{3,\overline{a}_2} = [L_{31,\overline{a}_2}, L_{32,\overline{a}_2}]^T$, where $\boldsymbol{L}_{2,a_1}$ is the potential value of $\boldsymbol{L}_2 = [L_{21}, L_{22}]^T$ that a subject would have had is he/she followed $A_1 = a_1$ in the first period, and $\boldsymbol{L}_{3,\overline{a}_2}$ is the potential value of $\boldsymbol{L}_3 = [L_{31}, L_{32}]^T$ that a subject would have had is he/she followed $\overline{A}_2 = \overline{a}_2$ in the first two periods. We shall first generate the baseline covariates and the potential values of the post-baseline covariates and outcome. Specifically, we first simulate the baseline covariates $\boldsymbol{L}_1 = [L_{11}, L_{12}]^T$ from two independent standard normal distributions $N(0, 1)$. Then, we generate $\boldsymbol{L}_{2,a_1} = [L_{21,a_1}, L_{22,a_1}]$ for every $a_1 \in \{0, 1\}$ by $L_{21,a_1} = X_{21} + a_1$ and $L_{22,a_1} = X_{22} + 2a_1$, where X_{21} and X_{22} are generated as two independent $N(0, 1)$ random variables. Next, we generate $\boldsymbol{L}_{3,\overline{a}_2} = [L_{31,\overline{a}_2}, L_{32,\overline{a}_2}]^T$ for every $\overline{a}_2 \in \{0, 1\}^2$ by $L_{31,\overline{a}_2} = X_{31} + a_2$ and $L_{32,\overline{a}_2} = X_{32} + 2a_2$, where X_{31} and X_{32} are simulated by two independent $N(0, 1)$. Finally, the potential outcome $Y_{\overline{a}_3}$, for every $\overline{a}_3 \in \{0, 1\}^3$, was generated by the following normal distribution

$$Y_{\overline{a}_3} \sim N\Big(10 - 10\sum_{k=1}^{3} a_k + 2L_{11} + 2L_{12} + 2L_{21,a_1} + 2L_{22,a_1} + 2L_{31,\overline{a}_2} + 2L_{32,\overline{a}_2}, 4 + 12a_3\Big).$$

Based on the previous data generate process on time-varying covariates and outcome, one can easily deduce that the marginal distribution of the potential outcome is $Y_{\overline{a}_3} \sim N(10 - 4a_1 - 4a_2 - 10a_3, 28 + 12a_3)$. Therefore, the correct quantile model for the q-th quantile of $Y_{\overline{a}_3}$ is

$$\begin{aligned} Q^{(q)}_{Y_{\overline{a}_3}} &= (10 + \sqrt{28}z_q) - 4a_1 - 4a_2 + \Big((\sqrt{40} - \sqrt{28})z_q - 10\Big)a_3 \\ &\approx (10 + 5.292z_q) - 4a_1 - 4a_2 + (-10 + 1.033z_q)a_3 \end{aligned}$$

where z_q is the q-th quantile for a standard normal distribution such that $z_{0.5} = 0$, $z_{0.75} = 0.675$, and $z_{0.25} = -0.675$.

Until now, we have obtained the baseline covariates $\boldsymbol{L}_1$, the potential values of the post-baseline covariates, $\{\boldsymbol{L}_{2,a_1}, \boldsymbol{L}_{3,\overline{a}_2}\}$, and the potential outcome $Y_{\overline{a}_3}$. Next, we generate the treatment regimen and the observed values of the post-baseline covariates and outcome. Specifically, we generate the first treatment A_1 based on the probability mass function $f_{A_1|\boldsymbol{L}_1}(1|\boldsymbol{L}_1) = \text{logit}\big(\alpha_{10} + \phi L_{11} + 2\phi\mathbb{I}(L_{12} > 0)\big)$, where $\text{logit}(x) = \log(x/(1-x))$, choice of ϕ is given at the end of this paragraph, and α_{10} is chosen such that half of the subjects receive treatment at period 1. Then, $\boldsymbol{L}_2$ is generated by selecting $\boldsymbol{L}_{2,a_1}$ with $A_1 = a_1$. Next, the second treatment A_2 is generated by $f_{A_2|A_1,\overline{\boldsymbol{L}}_2}(1|A_1, \overline{\boldsymbol{L}}_2) = \text{logit}\big(\alpha_{20} - 1.5A_1 + \phi L_{21} + 2\phi\mathbb{I}(L_{22} > 0)\big)$, where α_{20} is chosen such that half of the subjects receive treatment at period 2. Then, the covariates after the second treatment, $\boldsymbol{L}_3$, are generated by selecting $\boldsymbol{L}_{3,\overline{a}_2}$ with $\overline{A}_2 = \overline{a}_2$. Next, we generate the third treatment by $f_{A_3|\overline{A}_2,\overline{\boldsymbol{L}}_3}(1|\overline{A}_2, \overline{\boldsymbol{L}}_3) = \text{logit}\big(\alpha_{30} - 1.5A_2 + \phi L_{31} + 2\phi\mathbb{I}(L_{32} > 0)\big)$, where α_{30} is chosen such that half of the subjects receive treatment at period 3. Finally, we choose $Y_{\overline{a}_3}$ with $\overline{A}_3 = \overline{a}_3$ as Y. In the data generation process, the parameter ϕ controls the propensity scores overlap between the treatment and control group at each time period. In Scenarios I and II, we choose $\phi = 1$ and 1.5 to represent a moderate and weak overlap of the treatment-specific propensity score distribution. The distribution of the true propensity scores is shown in Figure S1 in the Supplementary Material.

In Section 6.2, we need to calculate the true confounding function (13). Similar to Hu et al. (2022), we used a sufficient large simulated data set with 2 million observations, along with the working confounding function (14) to approximate true confounding functions (13) in each Case. We briefly explain the procedure for obtaining the true confounding functions in Case 3, and similar procedures apply for Cases 1 and 2. In Case 3, we observe $\overline{\boldsymbol{L}}_3 = [L_{11}, L_{12}, L_{21}, L_{22}, L_{32}]^T$, thus $c_1(y, \overline{a}_3, \boldsymbol{l}_1) \equiv c_2(y, \overline{a}_3, \bar{\boldsymbol{l}}_2) \equiv 0$ by construction. Based on Proposition 2 in Supplementary Material, $c_3(y, \overline{a}_3, \bar{\boldsymbol{l}}_3)$ can be approximated by (14) based on the the following three parameters: $\mu_3(\overline{a}_3, \bar{\boldsymbol{l}}_3) = E[Y_{\overline{a}_3} | \overline{A}_3 = \overline{a}_3, \overline{\boldsymbol{L}}_3 = \bar{\boldsymbol{l}}_3]$, $\sigma_3^2(\overline{a}_3, \bar{\boldsymbol{l}}_3) = \text{Var}(Y_{\overline{a}_3} | \overline{A}_3 = \overline{a}_3, \overline{\boldsymbol{L}}_3 = \bar{\boldsymbol{l}}_3)$, and $s_3(\overline{a}_3, \bar{\boldsymbol{l}}_3) = E[Y_{\overline{a}_3} | \overline{A}_3 = \overline{a}_3, \overline{\boldsymbol{L}}_3 = \bar{\boldsymbol{l}}_3] - E[Y_{\overline{a}_3} | \overline{A}_3 = (1 - a_3, \overline{a}_2), \overline{\boldsymbol{L}}_3 = \bar{\boldsymbol{l}}_3]$. As the potential outcome $Y_{\overline{a}_3}$ for all $\overline{a}_3 \in \mathbb{A}_3$ are known based on the data generating process, we fit working regression models on the potential outcomes to approximate the three sensitivity parameters. Specifically, to calculate $\mu_3(\overline{a}_3, \bar{\boldsymbol{l}}_3)$, we fit a linear regression of $Y_{\overline{a}_3}$ on $(A_1 = a_1, A_2 = a_2, A_3 = a_3, L_{11}, L_{12}, L_{21}, L_{22}, L_{32})$ to calculate $\widehat{E}[Y_{\overline{a}_3} | \overline{A}_3 = \overline{a}_3, \overline{\boldsymbol{L}}_3 = \bar{\boldsymbol{l}}_3]$ and approximate $\mu_3(\overline{a}_3, \bar{\boldsymbol{l}}_3)$. Similarly, $\sigma_3^2(\overline{a}_3, \bar{\boldsymbol{l}}_3)$ is approximated by fitting a linear regression of the squared residuals, $(Y_{\overline{a}_3} - \widehat{E}[Y_{\overline{a}_3} | \overline{A}_3 = \overline{a}_3, \overline{\boldsymbol{L}}_3])^2$ on $(A_1 = a_1, A_2 = a_2, A_3 = a_3, L_{11}, L_{12}, L_{21}, L_{22}, L_{32})$. Lastly, $s_3(\overline{a}_3, \bar{\boldsymbol{l}}_3)$ is approximated by $\widehat{E}[Y_{\overline{a}_3} | \overline{A}_3 = \overline{a}_3, \overline{\boldsymbol{L}}_3 = \bar{\boldsymbol{l}}_3] - \widehat{E}[Y_{\overline{a}_3} | \overline{A}_3 = (1 - a_3, \overline{a}_2), \overline{\boldsymbol{L}}_3 = \bar{\boldsymbol{l}}_3]$, where $\widehat{E}[Y_{\overline{a}_3} | \overline{A}_3 = (1 - a_3, \overline{a}_2), \overline{\boldsymbol{L}}_3 = \bar{\boldsymbol{l}}_3]$ is obtained by fitting a linear regression of $Y_{\overline{a}_3}$ on $(A_1 = a_1, A_2 = a_2, A_3 = 1 - a_3, L_{11}, L_{12}, L_{21}, L_{22}, L_{32})$. Finally, the confounding function $c_3(y, \overline{a}_3, \bar{\boldsymbol{l}}_3)$ is obtained by evaluating (14) at the approximated parameter values.

References

Angrist, J., Chernozhukov, V. & Fernández-Val, I. (2006), 'Quantile regression under misspecification, with an application to the us wage structure', *Econometrica* **74**(2), 539–563.

Bickel, P. J., Klaassen, C. A., Bickel, P. J., Ritov, Y., Klaassen, J., Wellner, J. A. & Ritov, Y. (1993), *Efficient and adaptive estimation for semiparametric models*, Springer.

Chen, X. & White, H. (1999), 'Improved rates and asymptotic normality for nonparametric neural network estimators', *IEEE Transactions on Information Theory* **45**(2), 682–691.

Chernozhukov, V., Chetverikov, D., Demirer, M., Duflo, E., Hansen, C., Newey, W. & Robins, J. (2018), 'Double/debiased machine learning for treatment and structural parameters', *The Econometrics Journal* **21**(1), C1–C68.

Chiang, C.-T. & Huang, M.-Y. (2012), 'New estimation and inference procedures for a single-index conditional distribution model', *Journal of Multivariate Analysis* **111**, 271–285.

Hejazi, N. S., van der Laan, M. J. & Benkeser, D. (2022), 'haldensify: Highly adaptive lasso conditional density estimation inr', *Journal of Open Source Software* **7**(77), 4522.

Hu, B. & Nan, B. (2023), 'Conditional distribution function estimation using neural networks for censored and uncensored data', *Journal of Machine Learning Research* **24**(223), 1–26.

Hu, L., Zou, J., Gu, C., Ji, J., Lopez, M. & Kale, M. (2022), 'A flexible sensitivity analysis approach for unmeasured confounding with multiple treatments and a binary outcome with application to seer-medicare lung cancer data', *The Annals of Applied Statistics* **16**(2), 1014–1037.

Jiang, H. (2017), Uniform convergence rates for kernel density estimation, *in* 'International Conference on Machine Learning', PMLR, pp. 1694–1703.

Koenker, R. & Bassett, G. (1978), 'Regression quantiles', *Econometrica* pp. 33–50.

Lai, T. L. & Ying, Z. (1988), 'Stochastic integrals of empirical-type processes with applications to censored regression', *Journal of Multivariate Analysis* **27**(2), 334–358.

Luo, Y., Spindler, M. & Kück, J. (2016), 'High-dimensional l_2 boosting: Rate of convergence', *arXiv preprint arXiv:1602.08927* .

Neugebauer, R. & van der Laan, M. (2007), 'Nonparametric causal effects based on marginal structural models', *Journal of Statistical Planning and Inference* **137**(2), 419–434.

Newey, W. K. (1997), 'Convergence rates and asymptotic normality for series estimators', *Journal of Econometrics* **79**(1), 147–168.

Robins, J. M. (1997), Causal inference from complex longitudinal data, *in* 'Latent variable modeling and applications to causality', Springer, pp. 69–117.

Tsiatis, A. A. (2006), ‘Semiparametric theory and missing data’.

Tu, C. (2019), ‘Comparison of various machine learning algorithms for estimating generalized propensity score’, *Journal of Statistical Computation and Simulation* **89**(4), 708–719.

van der Vaart, A. W. (2000), *Asymptotic statistics*, Vol. 3, Cambridge university press.

Wager, S. & Walther, G. (2015), ‘Adaptive concentration of regression trees, with application to random forests’, *arXiv preprint arXiv:1503.06388* .

Xie, Y., Cotton, C. & Zhu, Y. (2020), ‘Multiply robust estimation of causal quantile treatment effects’, *Statistics in Medicine* **39**(28), 4238–4251.

Yang, S. & Zhang, Y. (2023), ‘Multiply robust matching estimators of average and quantile treatment effects’, *Scandinavian Journal of Statistics* **50**(1), 235–265.

Zhang, Z., Chen, Z., Troendle, J. F. & Zhang, J. (2012), ‘Causal inference on quantiles with an obstetric application’, *Biometrics* **68**(3), 697–706.

Table S1: Correct and incorrect specifications of the propensity score models and outcome regression models in the Simulation Study (Section 5)

	A. Propensity score models ($\mathcal{M}_{ps}$)
True	$\pi_1(1,\boldsymbol{L}_1;\boldsymbol{\alpha}_1)=\text{logit}\Big([1,L_{11},\mathbb{I}(L_{12}>0)]\boldsymbol{\alpha}_1\Big)$
	$\pi_2(1,A_1,\overline{\boldsymbol{L}}_2;\boldsymbol{\alpha}_2)=\text{logit}\Big([1,A_1,L_{21},\mathbb{I}(L_{22}>0)]\boldsymbol{\alpha}_2\Big)$
	$\pi_3(1,\overline{A}_2,\overline{\boldsymbol{L}}_3;\boldsymbol{\alpha}_3)=\text{logit}\Big([1,A_2,L_{31},\mathbb{I}(L_{32}>0)]\boldsymbol{\alpha}_3\Big)$
False	$\pi_1(1,\boldsymbol{L}_1;\boldsymbol{\alpha}_1)=\text{logit}\Big([1,\mathbb{I}(L_{11}\times L_{12}>0)]\boldsymbol{\alpha}_1\Big)$
	$\pi_2(1,A_1,\overline{\boldsymbol{L}}_2;\boldsymbol{\alpha}_2)=\text{logit}\Big([1,A_1,\mathbb{I}(L_{21}\times L_{22}>0)]\boldsymbol{\alpha}_2\Big)$
	$\pi_3(1,\overline{A}_2,\overline{\boldsymbol{L}}_3;\boldsymbol{\alpha}_3)=\text{logit}\Big([1,A_2,\mathbb{I}(L_{31}\times L_{32}>0)]\boldsymbol{\alpha}_3\Big)$
	B. Outcome regression models ($\mathcal{M}_{om}$)*
True	$\psi_1(Y_{A_1,\underline{a}_2},A_1,\underline{a}_2,\boldsymbol{L}_1;\boldsymbol{\beta}_1):Y_{A_1,\underline{a}_2}=[A_1,a_2,a_3,L_{11},L_{12}]\boldsymbol{\delta}_1+\epsilon_1,\ \text{where } \epsilon_1\sim N(0,[1,a_3]\boldsymbol{\eta}_1)$
	$\psi_2(Y_{\overline{A}_2,a_3},\overline{A}_2,a_3,\overline{\boldsymbol{L}}_2;\boldsymbol{\beta}_2):Y_{\overline{A}_2,a_3}=[A_1,A_2,a_3,L_{11},L_{12},L_{21},L_{22}]\boldsymbol{\delta}_2+\epsilon_2,\ \text{where } \epsilon_2\sim N(0,[1,a_3]\boldsymbol{\eta}_2)$
	$\psi_3(Y_{\overline{A}_3},\overline{A}_3,\overline{\boldsymbol{L}}_3;\boldsymbol{\beta}_3):Y_{\overline{A}_3}=[A_1,A_2,A_3,L_{11},L_{12},L_{21},L_{22},L_{31},L_{32}]\boldsymbol{\delta}_3+\epsilon_3,\ \text{where } \epsilon_3\sim N(0,[1,A_3]\boldsymbol{\eta}_3)$
False	$\psi_1(Y_{A_1,\underline{a}_2},A_1,\underline{a}_2,\boldsymbol{L}_1;\boldsymbol{\beta}_1):Y_{A_1,\underline{a}_2}=[A_1,a_2,a_3,L_{11},L_{12}^2]\boldsymbol{\delta}_1+\epsilon_1,\ \text{where } \epsilon_1\sim N(0,[1,a_3]\boldsymbol{\eta}_1)$
	$\psi_2(Y_{\overline{A}_2,a_3},\overline{A}_2,a_3,\overline{\boldsymbol{L}}_2;\boldsymbol{\beta}_2):Y_{\overline{A}_2,a_3}=[A_1,A_2,a_3,L_{11},L_{12}^2,L_{21},L_{22}^2]\boldsymbol{\delta}_2+\epsilon_2,\ \text{where } \epsilon_2\sim N(0,[1,a_3]\boldsymbol{\eta}_2)$
	$\psi_3(Y_{\overline{A}_3},\overline{A}_3,\overline{\boldsymbol{L}}_3;\boldsymbol{\beta}_3):Y_{\overline{A}_3}=[A_1,A_2,A_3,L_{11},L_{12}^2,L_{21},L_{22}^2,L_{31},L_{32}^2]\boldsymbol{\delta}_3+\epsilon_3,\ \text{where } \epsilon_3\sim N(0,[1,A_3]\boldsymbol{\eta}_3)$

[*] Both the correct and incorrect outcome regression models can fit into the linear regression frameworks in Appendix A.2.

Table S2: Simulation results for estimating $\boldsymbol{\theta}_{0.75} = [\theta_{0.75,0}, \theta_{0.75,1}, \theta_{0.75,2}, \theta_{0.75,3}]$ in Scenario I (the treatment-specific propensity scores are moderately overlapped). We considered both correct (denoted by 'T') or incorrect (denoted by 'F') specifications of the propensity score models ($\mathcal{M}_{ps}$) and the outcome regression models ($\mathcal{M}_{om}$). The results are summarized as percent bias (%), standard error, and 95% confidence interval coverage rate (%).

		Specification		Percent bias (%)				Standard error			
Model	Method	$\mathcal{M}_{ps}$	$\mathcal{M}_{om}$	$\hat{\theta}_{0.75,0}$	$\hat{\theta}_{0.75,1}$	$\hat{\theta}_{0.75,2}$	$\hat{\theta}_{0.75,3}$	$\hat{\theta}_{0.75,0}$	$\hat{\theta}_{0.75,1}$	$\hat{\theta}_{0.75,2}$	$\hat{\theta}_{0.75,3}$
QR	Unadjusted			-27.74	-63.61	-53.47	-24.34	0.29	0.33	0.32	0.33
	Adjusted			-16.29	149.86	149.92	-6.84	0.23	0.38	0.38	0.31
MSQM	IPW	T		0.54	-1.15	-0.49	0.16	0.91	0.72	0.73	0.73
	IPW	F		-27.81	-69.37	-54.73	-22.44	0.27	0.29	0.29	0.30
	ICR		T	-0.12	-0.11	-0.26	0.05	0.25	0.26	0.21	0.16
	ICR		F	-11.80	-34.18	-18.19	-8.13	0.23	0.26	0.24	0.22
	DR	T	T	0.82	0.19	-0.14	0.39	0.45	0.49	0.47	0.41
	DR	T	F	0.72	-0.25	0.11	0.78	0.76	0.64	0.60	0.60
	DR	F	T	1.40	1.02	0.18	0.55	0.28	0.31	0.27	0.22
	DR	F	F	-11.52	-36.29	-19.30	-7.12	0.29	0.30	0.30	0.26
		Specification		Coverage rate[1] (%)				Coverage rate[2] (%)			
Model	Method	$\mathcal{M}_{ps}$	$\mathcal{M}_{om}$	$\hat{\theta}_{0.75,0}$	$\hat{\theta}_{0.75,1}$	$\hat{\theta}_{0.75,2}$	$\hat{\theta}_{0.75,3}$	$\hat{\theta}_{0.75,0}$	$\hat{\theta}_{0.75,1}$	$\hat{\theta}_{0.75,2}$	$\hat{\theta}_{0.75,3}$
QR	Unadjusted			0.0	0.0	0.0	0.0	0.0	0.1	1.1	0.8
	Adjusted			0.0	0.0	0.0	45.1	0.0	0.0	0.0	50.0
MSQM	IPW	T		91.4	93.9	94.0	93.9	93.9	95.7	95.7	94.6
	IPW	F		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.4
	ICR		T	94.3	94.3	95.6	95.8	95.1	94.8	95.4	95.4
	ICR		F	0.0	0.1	13.2	6.6	1.2	6.3	47.5	38.1
	DR	T	T	93.0	94.1	94.4	94.1	93.8	96.7	94.7	94.3
	DR	T	F	94.2	93.3	95.6	95.2	94.0	95.6	95.5	96.2
	DR	F	T	91.9	96.0	95.4	94.6	92.9	94.9	95.8	95.5
	DR	F	F	0.0	0.2	27.4	29.8	7.5	14.0	64.4	60.8

* The percent bias was calculated as the mean of the ratio of bias to the true value over 1,000 replications, i.e., Bias(%) $= mean(\frac{\widehat{p}-p}{p}) \times 100\%$, where p denotes the true value of the causal parameter and $\widehat{p}$ is its point estimate. The standard error is defined as the squared root of the empirical variance of causal parameter from the 1,000 replications. The coverage rate[1] is by a Wald-type 95% confidence interval using the derived asymptotic variance formula. The coverage rate[2] is by a non-parametric percentile bootstrapping confidence interval using the 2.5% and 97.5% percentiles of the bootstrap distribution.

Table S3: Simulation results for estimating $\boldsymbol{\theta}_{0.25} = [\theta_{0.25,0}, \theta_{0.25,1}, \theta_{0.25,2}, \theta_{0.25,3}]$ in Scenario I (the treatment-specific propensity scores are moderately overlapped). We considered both correct (denoted by 'T') or incorrect (denoted by 'F') specifications of the propensity score models ($\mathcal{M}_{ps}$) and the outcome regression models ($\mathcal{M}_{om}$). The results are summarized as percent bias (%), standard error, and 95% confidence interval coverage rate (%).

		Specification		Percent bias (%)				Standard error			
Model	Method	$\mathcal{M}_{ps}$	$\mathcal{M}_{om}$	$\hat{\theta}_{0.25,0}$	$\hat{\theta}_{0.25,1}$	$\hat{\theta}_{0.25,2}$	$\hat{\theta}_{0.25,3}$	$\hat{\theta}_{0.25,0}$	$\hat{\theta}_{0.25,1}$	$\hat{\theta}_{0.25,2}$	$\hat{\theta}_{0.25,3}$
QR	Unadjusted			-47.73	-60.61	-50.92	-20.75	0.29	0.32	0.33	0.34
	Adjusted			34.88	150.43	150.46	6.07	0.21	0.38	0.38	0.29
MSQM	IPW	T		-2.11	-0.76	0.56	-0.61	0.52	0.70	0.67	0.67
	IPW	F		-53.38	-66.22	-53.10	-19.42	0.26	0.29	0.31	0.31
	ICR		T	-0.11	-0.11	-0.26	-0.09	0.20	0.26	0.21	0.15
	ICR		F	-20.17	-34.18	-18.19	-7.81	0.21	0.26	0.24	0.23
	DR	T	T	-1.56	0.95	-0.24	-0.23	0.34	0.54	0.49	0.44
	DR	T	F	-1.82	0.31	-0.63	-0.49	0.41	0.60	0.57	0.55
	DR	F	T	-1.03	1.18	0.92	0.29	0.24	0.29	0.27	0.23
	DR	F	F	-25.11	-36.34	-22.35	-7.52	0.26	0.30	0.29	0.27
		Specification		Coverage rate[1] (%)				Coverage rate[2] (%)			
Model	Method	$\mathcal{M}_{ps}$	$\mathcal{M}_{om}$	$\hat{\theta}_{0.25,0}$	$\hat{\theta}_{0.25,1}$	$\hat{\theta}_{0.25,2}$	$\hat{\theta}_{0.25,3}$	$\hat{\theta}_{0.25,0}$	$\hat{\theta}_{0.25,1}$	$\hat{\theta}_{0.25,2}$	$\hat{\theta}_{0.25,3}$
QR	Unadjusted			0.0	0.0	0.0	0.0	0.0	0.1	2.7	0.7
	Adjusted			0.0	0.0	0.0	43.6	0.0	0.0	0.0	48.2
MSQM	IPW	T		92.6	92.5	93.9	94.9	91.3	94.5	94.9	95.5
	IPW	F		0.0	0.0	0.0	0.0	0.0	0.0	0.7	0.9
	ICR		T	94.5	94.3	95.6	95.5	94.4	94.8	95.4	95.4
	ICR		F	0.0	0.1	13.2	4.0	2.5	6.3	47.5	30.9
	DR	T	T	93.2	93.5	94.5	95.1	93.8	94.5	96.3	95.1
	DR	T	F	93.3	93.9	94.9	94.4	94.1	95.7	95.4	94.0
	DR	F	T	95.6	95.5	94.6	94.8	95.7	94.8	94.1	95.0
	DR	F	F	0.0	0.6	13.7	17.9	1.0	13.4	53.3	53.0

[*] The percent bias was calculated as the mean of the ratio of bias to the true value over 1,000 replications, i.e., Bias(%) $= mean(\frac{\widehat{p}-p}{p}) \times 100\%$, where p denotes the true value of the causal parameter and $\widehat{p}$ is its point estimate. The standard error is defined as the squared root of the empirical variance of causal parameter from the 1,000 replications. The coverage rate[1] is by a Wald-type 95% confidence interval using the derived asymptotic variance formula. The coverage rate[2] is by a non-parametric percentile bootstrapping confidence interval using the 2.5% and 97.5% percentiles of the bootstrap distribution.

Table S4: Simulation results for estimating $\boldsymbol{\theta}_{0.5} = [\theta_{0.5,0}, \theta_{0.5,1}, \theta_{0.5,2}, \theta_{0.5,3}]$ in Scenario II (the treatment-specific propensity scores are weakly overlapped). We considered both correct (denoted by 'T') or incorrect (denoted by 'F') specifications of the propensity score models ($\mathcal{M}_{ps}$) and the outcome regression models ($\mathcal{M}_{om}$). The results are summarized as percent bias (%), standard error, and 95% confidence interval coverage rate (%).

		Specification		Percent bias (%)				Standard error			
Model	Method	$\mathcal{M}_{ps}$	$\mathcal{M}_{om}$	$\widehat{\theta}_{0.5,0}$	$\widehat{\theta}_{0.5,1}$	$\widehat{\theta}_{0.5,2}$	$\widehat{\theta}_{0.5,3}$	$\widehat{\theta}_{0.5,0}$	$\widehat{\theta}_{0.5,1}$	$\widehat{\theta}_{0.5,2}$	$\widehat{\theta}_{0.5,3}$
QR	Unadjusted			-40.98	-73.56	-62.30	-27.64	0.25	0.28	0.30	0.30
	Adjusted			0.07	150.40	150.58	0.24	0.21	0.36	0.38	0.29
MSQM	IPW	T		-2.12	-2.94	0.82	-0.52	1.21	1.21	1.19	1.28
	IPW	F		-44.49	-82.57	-66.23	-25.91	0.23	0.26	0.28	0.28
	ICR		T	0.05	0.07	0.44	-0.04	0.21	0.27	0.22	0.16
	ICR		F	-19.07	-44.75	-23.46	-11.02	0.22	0.25	0.25	0.22
	DR	T	T	0.20	0.06	1.71	0.26	0.55	0.84	0.79	0.71
	DR	T	F	-0.72	-1.73	2.46	-0.35	0.94	1.09	1.10	1.08
	DR	F	T	1.56	2.78	2.52	0.75	0.23	0.30	0.26	0.21
	DR	F	F	-20.85	-47.99	-26.63	-9.91	0.25	0.28	0.28	0.27
		Specification		Coverage rate[1] (%)				Coverage rate[2] (%)			
Model	Method	$\mathcal{M}_{ps}$	$\mathcal{M}_{om}$	$\widehat{\theta}_{0.5,0}$	$\widehat{\theta}_{0.5,1}$	$\widehat{\theta}_{0.5,2}$	$\widehat{\theta}_{0.5,3}$	$\widehat{\theta}_{0.5,0}$	$\widehat{\theta}_{0.5,1}$	$\widehat{\theta}_{0.5,2}$	$\widehat{\theta}_{0.5,3}$
QR	Unadjusted			0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	Adjusted			97.6	0.0	0.0	96.1	96.4	0.0	0.0	96.3
MSQM	IPW	T		87.5	89.8	91.2	90.5	88.8	93.1	95.2	92.6
	IPW	F		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	ICR		T	96.1	95.6	93.6	96.0	95.5	94.2	94.4	95.3
	ICR		F	0.0	0.0	3.3	0.1	0.0	0.2	24.8	8.8
	DR	T	T	95.3	93.4	92.7	92.9	95.7	94.4	95.5	96.1
	DR	T	F	91.0	92.4	92.3	93.6	93.5	95.6	96.0	95.5
	DR	F	T	92.3	95.2	92.6	93.6	91.9	92.5	94.4	94.3
	DR	F	F	0.0	0.0	3.4	3.9	0.0	0.6	30.7	32.6

[*] The percent bias was calculated as the mean of the ratio of bias to the true value over 1,000 replications, i.e., Bias(%) $= mean(\frac{\widehat{p}-p}{p}) \times 100\%$, where p denotes the true value of the causal parameter and $\widehat{p}$ is its point estimate. The standard error is defined as the squared root of the empirical variance of causal parameter from the 1,000 replications. The coverage rate[1] is by a Wald-type 95% confidence interval using the derived asymptotic variance formula. The coverage rate[2] is by a non-parametric percentile bootstrapping confidence interval using the 2.5% and 97.5% percentiles of the bootstrap distribution.

Table S5: Simulation results for estimating $\boldsymbol{\theta}_{0.75} = [\theta_{0.75,0}, \theta_{0.75,1}, \theta_{0.75,2}, \theta_{0.75,3}]$ in Scenario II (the treatment-specific propensity scores are weakly overlapped). We considered both correct (denoted by 'T') or incorrect (denoted by 'F') specifications of the propensity score models ($\mathcal{M}_{ps}$) and the outcome regression models ($\mathcal{M}_{om}$). The results are summarized as percent bias (%), standard error, and 95% confidence interval coverage rate (%).

		Specification		Percent bias (%)				Standard error			
Model	Method	$\mathcal{M}_{ps}$	$\mathcal{M}_{om}$	$\hat{\theta}_{0.75,0}$	$\hat{\theta}_{0.75,1}$	$\hat{\theta}_{0.75,2}$	$\hat{\theta}_{0.75,3}$	$\hat{\theta}_{0.75,0}$	$\hat{\theta}_{0.75,1}$	$\hat{\theta}_{0.75,2}$	$\hat{\theta}_{0.75,3}$
QR	Unadjusted			-34.02	-75.90	-64.17	-30.48	0.28	0.31	0.31	0.32
	Adjusted			-16.42	149.59	150.20	-6.74	0.22	0.39	0.38	0.31
MSQM	IPW	T		-2.20	-6.56	-2.11	-0.82	1.54	1.25	1.25	1.36
	IPW	F		-35.39	-84.90	-67.85	-28.25	0.25	0.29	0.29	0.30
	ICR		T	0.01	0.07	0.44	0.01	0.25	0.27	0.22	0.17
	ICR		F	-15.97	-44.81	-23.47	-11.39	0.24	0.25	0.25	0.22
	DR	T	T	0.92	-1.06	1.56	0.59	0.80	0.92	0.85	0.72
	DR	T	F	-1.55	-5.03	-0.22	0.38	1.41	1.18	1.18	1.18
	DR	F	T	1.31	0.65	1.00	0.48	0.27	0.33	0.29	0.22
	DR	F	F	-16.31	-48.63	-24.78	-10.13	0.28	0.30	0.30	0.27
		Specification		Coverage rate[1] (%)				Coverage rate[2] (%)			
Model	Method	$\mathcal{M}_{ps}$	$\mathcal{M}_{om}$	$\hat{\theta}_{0.75,0}$	$\hat{\theta}_{0.75,1}$	$\hat{\theta}_{0.75,2}$	$\hat{\theta}_{0.75,3}$	$\hat{\theta}_{0.75,0}$	$\hat{\theta}_{0.75,1}$	$\hat{\theta}_{0.75,2}$	$\hat{\theta}_{0.75,3}$
QR	Unadjusted			0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0
	Adjusted			0.0	0.0	0.0	50.5	0.0	0.0	0.0	54.6
MSQM	IPW	T		80.2	87.5	90.4	88.9	84.3	92.4	93.9	92.6
	IPW	F		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	ICR		T	96.4	95.6	93.6	94.9	95.9	94.2	94.4	95.2
	ICR		F	0.0	0.0	3.3	0.1	0.0	0.2	24.8	13.7
	DR	T	T	93.6	93.6	93.5	93.2	95.2	95.2	95.5	94.9
	DR	T	F	82.9	88.6	92.0	91.4	88.3	93.2	94.8	94.6
	DR	F	T	91.1	95.8	93.9	94.6	92.0	93.9	94.6	94.7
	DR	F	F	0.0	0.0	8.9	7.4	0.1	1.8	37.1	42.6

* The percent bias was calculated as the mean of the ratio of bias to the true value over 1,000 replications, i.e., Bias(%) $= mean(\frac{\widehat{p}-p}{p}) \times 100\%$, where p denotes the true value of the causal parameter and $\widehat{p}$ is its point estimate. The standard error is defined as the squared root of the empirical variance of causal parameter from the 1,000 replications. The coverage rate[1] is by a Wald-type 95% confidence interval using the derived asymptotic variance formula. The coverage rate[2] is by a non-parametric percentile bootstrapping confidence interval using the 2.5% and 97.5% percentiles of the bootstrap distribution.

Table S6: Simulation results for estimating $\boldsymbol{\theta}_{0.25} = [\theta_{0.25,0}, \theta_{0.25,1}, \theta_{0.25,2}, \theta_{0.25,3}]$ in Scenario II (the treatment-specific propensity scores are weakly overlapped). We considered both correct (denoted by 'T') or incorrect (denoted by 'F') specifications of the propensity score models ($\mathcal{M}_{ps}$) and the outcome regression models ($\mathcal{M}_{om}$). The results are summarized as percent bias (%), standard error, and 95% confidence interval coverage rate (%).

		Specification		Percent bias (%)				Standard error			
Model	Method	$\mathcal{M}_{ps}$	$\mathcal{M}_{om}$	$\hat{\theta}_{0.25,0}$	$\hat{\theta}_{0.25,1}$	$\hat{\theta}_{0.25,2}$	$\hat{\theta}_{0.25,3}$	$\hat{\theta}_{0.25,0}$	$\hat{\theta}_{0.25,1}$	$\hat{\theta}_{0.25,2}$	$\hat{\theta}_{0.25,3}$
QR	Unadjusted			-55.83	-70.92	-60.52	-25.43	0.26	0.31	0.33	0.33
	Adjusted			34.63	149.86	150.43	6.10	0.24	0.41	0.42	0.30
MSQM	IPW	T		-4.42	-3.56	-0.74	-2.12	0.97	1.21	1.26	1.25
	IPW	F		-63.83	-80.17	-64.91	-23.96	0.25	0.30	0.30	0.31
	ICR		T	0.10	0.07	0.44	-0.12	0.19	0.27	0.22	0.17
	ICR		F	-25.63	-44.76	-23.46	-10.72	0.21	0.25	0.25	0.23
	DR	T	T	-1.81	1.89	1.36	-0.38	0.58	0.93	0.98	0.83
	DR	T	F	-2.23	-1.57	1.04	-1.19	1.05	1.17	1.25	1.21
	DR	F	T	-0.23	1.90	2.00	0.31	0.25	0.33	0.29	0.24
	DR	F	F	-32.53	-48.34	-29.97	-10.35	0.26	0.31	0.32	0.30
		Specification		Coverage rate[1] (%)				Coverage rate[2] (%)			
Model	Method	$\mathcal{M}_{ps}$	$\mathcal{M}_{om}$	$\hat{\theta}_{0.25,0}$	$\hat{\theta}_{0.25,1}$	$\hat{\theta}_{0.25,2}$	$\hat{\theta}_{0.25,3}$	$\hat{\theta}_{0.25,0}$	$\hat{\theta}_{0.25,1}$	$\hat{\theta}_{0.25,2}$	$\hat{\theta}_{0.25,3}$
QR	Unadjusted			0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1
	Adjusted			0.0	0.0	0.0	46.7	0.0	0.0	0.0	51.8
MSQM	IPW	T		88.2	89.4	89.6	89.2	89.5	93.9	94.1	91.2
	IPW	F		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1
	ICR		T	95.5	95.6	93.6	96.2	94.7	94.2	94.4	95.4
	ICR		F	0.0	0.0	3.3	0.2	0.1	0.2	24.8	9.8
	DR	T	T	94.1	92.8	91.7	91.8	94.5	96.0	95.8	95.3
	DR	T	F	91.3	90.7	91.2	90.9	94.1	97.0	97.2	94.0
	DR	F	T	95.1	95.5	93.4	94.8	94.8	94.9	94.6	94.8
	DR	F	F	0.0	0.0	3.9	4.2	0.0	1.7	27.6	28.8

* The percent bias was calculated as the mean of the ratio of bias to the true value over 1,000 replications, i.e., Bias(%) $= mean(\frac{\widehat{p}-p}{p}) \times 100\%$, where p denotes the true value of the causal parameter and $\widehat{p}$ is its point estimate. The standard error is defined as the squared root of the empirical variance of causal parameter from the 1,000 replications. The coverage rate[1] is by a Wald-type 95% confidence interval using the derived asymptotic variance formula. The coverage rate[2] is by a non-parametric percentile bootstrapping confidence interval using the 2.5% and 97.5% percentiles of the bootstrap distribution.

Table S7: Sensitivity analysis simulation results on estimating $\boldsymbol{\theta}_{0.5} = [\theta_{0.5,0}, \theta_{0.5,1}, \theta_{0.5,2}, \theta_{0.5,3}]$ to unmeasured baseline confounding, based on the uncorrected estimators (upper panel) and bias-corrected estimators (lower panel). We considered both correct (denoted by 'T') or incorrect (denoted by 'F') specifications of the propensity score models ($\mathcal{M}_{ps}$) and the outcome regression models ($\mathcal{M}_{om}$).

	Specification		Percent bias (Monte Carlo S.E)				Coverage rate			
Method	$\mathcal{M}_{ps}$	$\mathcal{M}_{om}$	$\hat{\theta}_{0.5,0}$	$\hat{\theta}_{0.5,1}$	$\hat{\theta}_{0.5,2}$	$\hat{\theta}_{0.5,3}$	$\hat{\theta}_{0.5,0}$	$\hat{\theta}_{0.5,1}$	$\hat{\theta}_{0.5,2}$	$\hat{\theta}_{0.5,3}$
Panel A: uncorrected estimates ignoring unmeasured confounding effect										
IPW	T		-8.77 (0.62)	-41.49 (0.66)	-0.76 (0.68)	0.11 (0.68)	64.5	26.8	93.9	93.8
IPW	F		-35.84 (0.34)	-67.29 (0.39)	-53.81 (0.39)	-20.73 (0.39)	0.0	0.0	0.0	0.0
ICR		T	-7.81 (0.30)	-38.63 (0.37)	0.18 (0.32)	0.00 (0.26)	25.9	0.8	96.0	94.7
ICR		F	-20.46 (0.31)	-64.55 (0.35)	-17.67 (0.34)	-7.86 (0.32)	0.0	0.0	47.6	29.6
DR	T	T	-8.30 (0.45)	-41.82 (0.54)	0.10 (0.53)	0.32 (0.48)	51.4	13.4	94.0	94.0
DR	T	F	-8.47 (0.53)	-41.6 (0.59)	-0.49 (0.61)	0.24 (0.58)	61.5	20.8	94.1	94.6
DR	F	T	-6.53 (0.34)	-37.31 (0.42)	2.35 (0.38)	0.81 (0.33)	51.5	4.8	94.0	93.6
DR	F	F	-21.41 (0.35)	-66.59 (0.40)	-19.54 (0.40)	-6.90 (0.37)	0.0	0.0	50.8	53.9
Panel B: bias-corrected estimates adjusting for unmeasured confounding										
BC-IPW	T		-0.60 (0.65)	-1.04 (0.66)	-1.09 (0.69)	0.16 (0.69)	94.2	95.0	93.3	93.4
BC-IPW	F		-27.87 (0.35)	-29.36 (0.39)	-52.65 (0.38)	-20.41 (0.38)	0.0	14.4	0.0	0.0
BC-ICR	T	T	0.11 (0.30)	0.78 (0.37)	0.18 (0.32)	0.00 (0.26)	95.1	94.2	96.0	94.7
BC-ICR	T	F	-12.75 (0.32)	-26.33 (0.36)	-17.86 (0.34)	-7.91 (0.32)	2.4	16.2	47.5	29.6
BC-ICR	F	T	-0.04 (0.30)	0.36 (0.37)	0.17 (0.31)	0.00 (0.26)	95.0	94.2	96.0	94.6
BC-ICR	F	F	-12.67 (0.32)	-26.50 (0.36)	-17.61 (0.34)	-7.80 (0.31)	2.1	14.1	47.5	29.7
BC-DR	T	T	-0.06 (0.47)	-0.67 (0.54)	-0.18 (0.53)	0.14 (0.48)	94.4	94.9	95.1	93.8
BC-DR	T	F	-0.25 (0.56)	-1.00 (0.59)	-0.12 (0.61)	0.18 (0.59)	94.9	95.6	94.1	93.6
BC-DR	F	T	1.50 (0.34)	3.51 (0.41)	2.19 (0.37)	0.74 (0.32)	91.3	93.6	94.2	92.8
BC-DR	F	F	-13.23 (0.36)	-26.64 (0.40)	-19.03 (0.40)	-6.81 (0.37)	5.2	22.3	50.4	54.0

* The percent bias was calculated as the mean of the ratio of bias to the true value over 1,000 replications. The standard error is defined as the squared root of the empirical variance of causal parameter from the 1,000 replications. The coverage rate is by a Wald-type 95% confidence interval using the derived asymptotic variance formula. Note that, different from the ICR approach, the bias-corrected ICR estimator requires specifications of both $\mathcal{M}_{ps}$ and $\mathcal{M}_{om}$, therefore four scenarios with correct/incorrect specification of $\mathcal{M}_{ps}$ and $\mathcal{M}_{om}$ are considered for the bias-corrected ICR estimator in Panel B.

Table S8: Sensitivity analysis simulation results on estimating $\boldsymbol{\theta}_{0.5} = [\theta_{0.5,0}, \theta_{0.5,1}, \theta_{0.5,2}, \theta_{0.5,3}]$ to unmeasured confounding at the second period, based on the uncorrected estimators (upper panel) and bias-corrected estimators (lower panel).We considered both correct (denoted by 'T') or incorrect (denoted by 'F') specifications of the propensity score models ($\mathcal{M}_{ps}$) and the outcome regression models ($\mathcal{M}_{om}$).

	Specification		Percent bias (Monte Carlo S.E)				Coverage rate			
Method	$\mathcal{M}_{ps}$	$\mathcal{M}_{om}$	$\hat{\theta}_{0.5,0}$	$\hat{\theta}_{0.5,1}$	$\hat{\theta}_{0.5,2}$	$\hat{\theta}_{0.5,3}$	$\hat{\theta}_{0.5,0}$	$\hat{\theta}_{0.5,1}$	$\hat{\theta}_{0.5,2}$	$\hat{\theta}_{0.5,3}$
Panel A: uncorrected estimates ignoring unmeasured confounding effect										
IPW	T		-7.76 (0.66)	3.13 (0.68)	-41.62 (0.75)	0.22 (0.70)	69.3	93.7	32.3	94.7
IPW	F		-34.91 (0.34)	-61.78 (0.39)	-57.78 (0.38)	-20.84 (0.38)	0.0	0.0	0.0	0.0
ICR		T	-7.09 (0.28)	4.20 (0.36)	-39.34 (0.30)	0.04 (0.26)	30.5	92.4	0.0	94.5
ICR		F	-20.76 (0.30)	-30.59 (0.36)	-53.58 (0.32)	-7.87 (0.32)	0.0	6.2	0.0	27.5
DR	T	T	-7.67 (0.44)	4.41 (0.55)	-41.87 (0.54)	0.25 (0.50)	58.3	93.4	11.5	93.3
DR	T	F	-7.54 (0.58)	4.42 (0.63)	-41.86 (0.65)	0.33 (0.63)	66.5	94.1	21.4	94.8
DR	F	T	-5.91 (0.33)	7.37 (0.40)	-38.62 (0.35)	0.75 (0.33)	56.7	88.8	0.4	92.9
DR	F	F	-21.28 (0.35)	-29.27 (0.40)	-56.99 (0.37)	-6.99 (0.37)	0.0	16.8	0.0	51.5
Panel B: bias-corrected estimates adjusting for unmeasured confounding										
BC-IPW	T		-0.28 (0.70)	-0.39 (0.69)	-0.44 (0.75)	0.14 (0.71)	94.2	94.7	93.2	94.2
BC-IPW	F		-27.37 (0.34)	-64.74 (0.39)	-18.28 (0.38)	-20.36 (0.38)	0.0	0.0	52.4	0.0
BC-ICR	T	T	0.09 (0.29)	0.53 (0.37)	0.22 (0.30)	0.04 (0.26)	95.3	94.2	95.2	94.5
BC-ICR	T	F	-13.54 (0.30)	-34.23 (0.36)	-14.01 (0.32)	-7.87 (0.32)	0.5	3.3	61.0	27.5
BC-ICR	F	T	0.07 (0.29)	0.57 (0.37)	0.20 (0.30)	0.04 (0.26)	95.4	93.9	95.2	94.5
BC-ICR	F	F	-13.54 (0.30)	-34.19 (0.36)	-14.01 (0.32)	-7.87 (0.32)	0.5	3.3	61.0	27.5
BC-DR	T	T	-0.02 (0.46)	0.46 (0.55)	-1.05 (0.53)	0.16 (0.49)	94.8	95.3	95.0	93.3
BC-DR	T	F	-0.03 (0.62)	0.55 (0.64)	-0.73 (0.65)	0.31 (0.63)	94.5	94.5	94.3	94.8
BC-DR	F	T	1.46 (0.32)	3.42 (0.40)	2.20 (0.34)	0.74 (0.32)	93.1	93.5	93.8	92.9
BC-DR	F	F	-13.63 (0.35)	-32.75 (0.41)	-16.48 (0.37)	-6.74 (0.37)	2.7	8.8	57.2	53.5

* The percent bias was calculated as the mean of the ratio of bias to the true value over 1,000 replications, i.e., Bias(%) $= mean(\frac{\widehat{p}-p}{p}) \times 100\%$, where p denotes the true value of the causal parameter and $\widehat{p}$ is its point estimate. The standard error is defined as the squared root of the empirical variance of causal parameter from the 1,000 replications. The coverage rate is by a Wald-type 95% confidence interval using the derived asymptotic variance formula. Note that, different from the ICR approach, the bias-corrected ICR estimator requires specifications of both $\mathcal{M}_{ps}$ and $\mathcal{M}_{om}$, therefore four scenarios with correct/incorrect specification of $\mathcal{M}_{ps}$ and $\mathcal{M}_{om}$ are considered for the bias-corrected ICR estimator in Panel B.

Table S9: Sensitivity analysis simulation results on estimating $\boldsymbol{\theta}_{0.5} = [\theta_{0.5,0}, \theta_{0.5,1}, \theta_{0.5,2}, \theta_{0.5,3}]$ to unmeasured confounding at the third period, based on the uncorrected estimators (upper panel) and bias-corrected estimators (lower panel). We considered both correct (denoted by 'T') or incorrect (denoted by 'F') specifications of the propensity score models ($\mathcal{M}_{ps}$) and the outcome regression models ($\mathcal{M}_{om}$).

	Specification		Percent bias (Monte Carlo S.E)				Coverage rate			
Method	$\mathcal{M}_{ps}$	$\mathcal{M}_{om}$	$\hat{\theta}_{0.5,0}$	$\hat{\theta}_{0.5,1}$	$\hat{\theta}_{0.5,2}$	$\hat{\theta}_{0.5,3}$	$\hat{\theta}_{0.5,0}$	$\hat{\theta}_{0.5,1}$	$\hat{\theta}_{0.5,2}$	$\hat{\theta}_{0.5,3}$
Panel A: uncorrected estimates ignoring unmeasured confounding effect										
IPW	T		-7.76 (0.68)	-0.44 (0.65)	4.04 (0.70)	-16.94 (0.72)	67.5	95.0	93.1	30.3
IPW	F		-34.85 (0.34)	-67.05 (0.38)	-48.36 (0.39)	-22.42 (0.39)	0.0	0.0	0.0	0.0
ICR		T	-7.18 (0.28)	0.34 (0.36)	3.72 (0.28)	-15.87 (0.24)	29.5	94.8	92.1	0.0
ICR		F	-20.80 (0.30)	-33.72 (0.36)	-14.56 (0.32)	-22.09 (0.29)	0.0	2.8	56.2	0.0
DR	T	T	-7.54 (0.43)	0.34 (0.54)	3.56 (0.5)	-16.81 (0.47)	55.7	94.7	91.9	5.5
DR	T	F	-7.49 (0.54)	0.51 (0.58)	3.82 (0.6)	-17.06 (0.58)	65.6	95.8	93.2	17.0
DR	F	T	-5.92 (0.33)	3.74 (0.40)	6.53 (0.34)	-15.79 (0.30)	54.6	93.3	89.2	0.0
DR	F	F	-21.26 (0.34)	-34.8 (0.40)	-13.69 (0.38)	-22.14 (0.34)	0.0	5.9	68.4	0.0
Panel B: bias-corrected estimates adjusting for unmeasured confounding										
BC-IPW	T		-0.23 (0.69)	-0.65 (0.65)	0.33 (0.71)	-0.34 (0.72)	93.2	94.7	94.2	94.4
BC-IPW	F		-27.24 (0.34)	-66.82 (0.38)	-52.09 (0.39)	-6.02 (0.38)	0.0	0.0	0.0	67.4
BC-ICR	T	T	0.02 (0.29)	0.46 (0.37)	0.22 (0.30)	-0.04 (0.24)	94.2	94.7	94.8	95.0
BC-ICR	T	F	-13.6 (0.31)	-33.73 (0.36)	-18.03 (0.32)	-6.28 (0.29)	0.8	3.0	41.1	41.8
BC-ICR	F	T	0.04 (0.29)	0.29 (0.36)	0.34 (0.29)	-0.06 (0.24)	94.5	94.8	94.8	94.8
BC-ICR	F	F	-13.59 (0.31)	-33.74 (0.36)	-17.97 (0.32)	-6.27 (0.29)	0.7	2.8	40.7	41.8
BC-DR	T	T	0.17 (0.44)	0.27 (0.54)	0.05 (0.51)	-0.39 (0.46)	95.5	95.3	94.3	94.4
BC-DR	T	F	0.12 (0.57)	0.17 (0.59)	0.83 (0.61)	-0.65 (0.58)	94.9	96	95.0	94.1
BC-DR	F	T	1.55 (0.33)	3.32 (0.40)	2.48 (0.34)	0.69 (0.30)	91.2	93.0	93.5	93.6
BC-DR	F	F	-13.65 (0.35)	-34.86 (0.40)	-17.74 (0.38)	-5.66 (0.34)	2.8	6.3	53.2	62.9

* The percent bias was calculated as the mean of the ratio of bias to the true value over 1,000 replications, i.e., Bias(%) $= mean(\frac{\widehat{p}-p}{p}) \times 100\%$, where p denotes the true value of the causal parameter and $\widehat{p}$ is its point estimate. The standard error is defined as the squared root of the empirical variance of causal parameter from the 1,000 replications. The coverage rate is by a Wald-type 95% confidence interval using the derived asymptotic variance formula. Note that, different from the ICR approach, the bias-corrected estimator requires specifications of both $\mathcal{M}_{ps}$ and $\mathcal{M}_{om}$, therefore four scenarios with correct/incorrect specification of $\mathcal{M}_{ps}$ and $\mathcal{M}_{om}$ are considered for the bias-corrected estimator in Panel B.

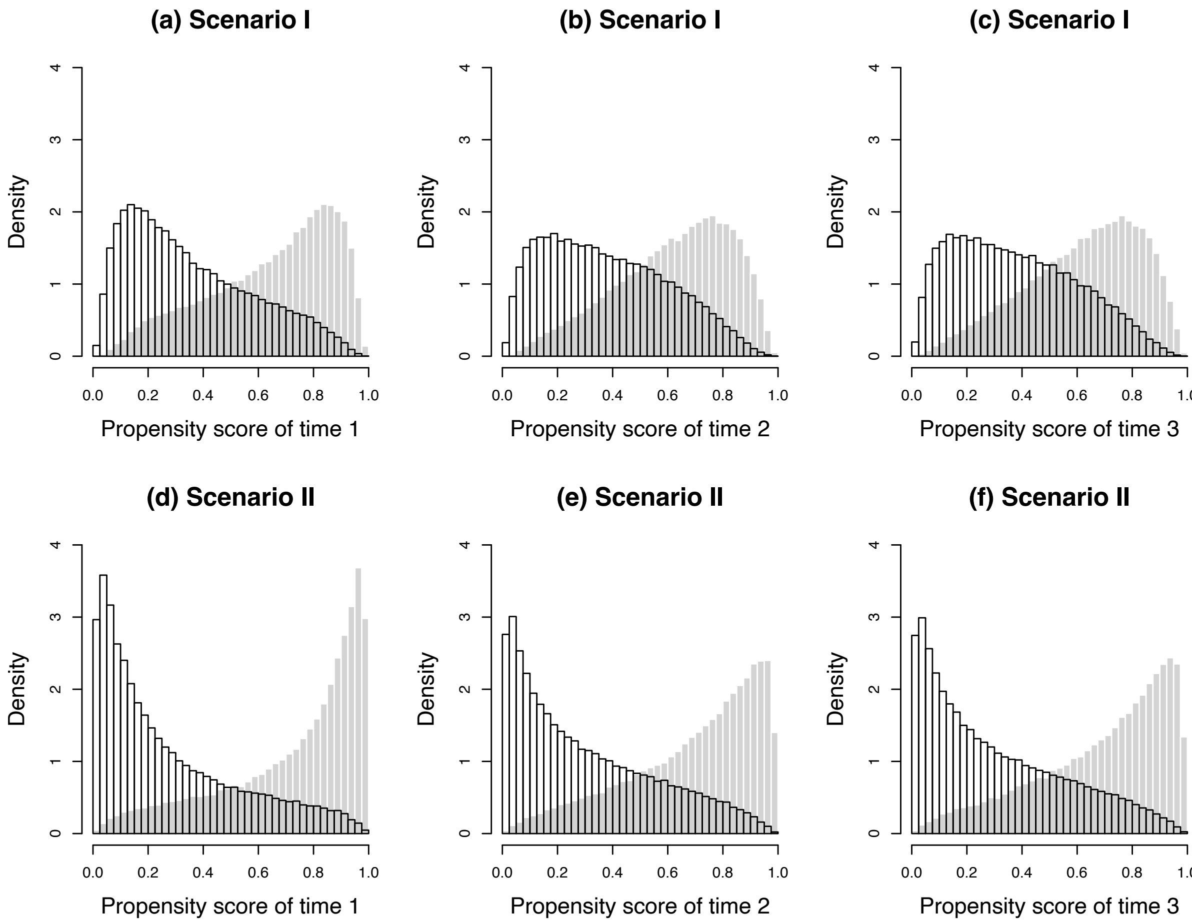

Figure S1: Distributions of true propensity scores in Scenario I (upper row) and Scenario II (lower row). The gray shaded bars indicate the treated group; the unshaded bars indicate the control group.

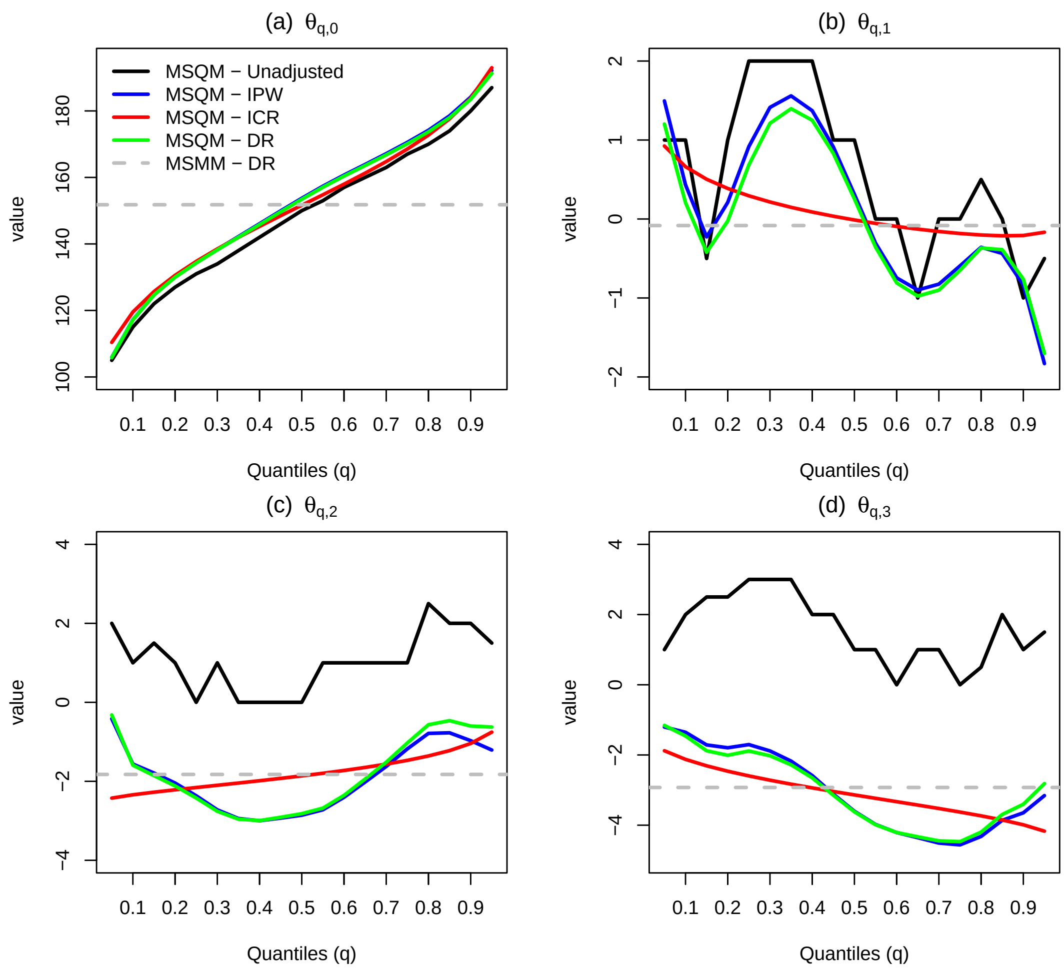

Figure S2: Analysis of the YNHHS Severe Inpatient Hypertension Dataset: point estimates of $\theta_{q,k}$ in the MSQM. The gray-dotted horizontal line in panels (a)–(d) is the doubly robust estimation of ξ_0, ξ_1, ξ_2, and ξ_3 for the MSMM.

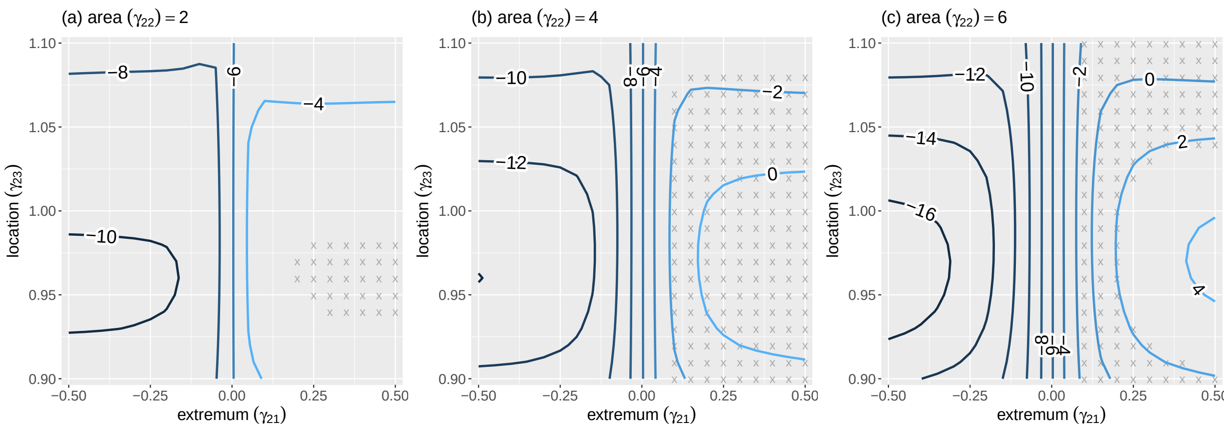

Figure S3: A sensitivity analysis for $\mathrm{TE}_{0.5}$ in relation to unmeasured confounding at the second period. The bias-corrected doubly robust estimates are illustrated through contours for fixed values of $(\gamma_{21}, \gamma_{23})$, with γ_{22} set to 2 (Panel a), 4 (Panel b), 6 (Panel c). The regions marked with '×' denote the values of $(\gamma_{21}, \gamma_{23})$ where the corresponding 95% interval estimate encompasses 0.

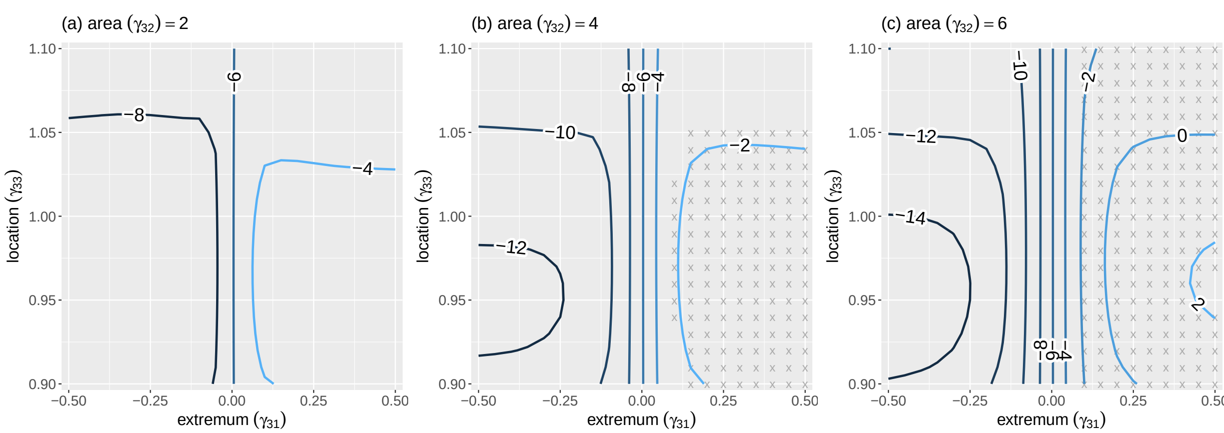

Figure S4: A sensitivity analysis for $\mathrm{TE}_{0.5}$ in relation to unmeasured confounding at the third period. The bias-corrected doubly robust estimates are illustrated through contours for fixed values of $(\gamma_{31}, \gamma_{33})$, with γ_{32} set to 2 (Panel a), 4 (Panel b), 6 (Panel c). The regions marked with '×' denote the values of $(\gamma_{31}, \gamma_{33})$ where the corresponding 95% interval estimate encompasses 0.