Python Package for Octonions (pyoctonion)

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Content

- Algebra of complex numbers

Definition

The complex numbers are the pair of real numbers, which is usually written as

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R},\ i^2 = -1\}$$

Basic Arithmetics

$$Let \quad z_1 = a + bi \quad and \quad z_2 = c + di \quad then$$

$$z_1 + z_2 = (a + c) + (b + d)i$$
$$z_1 - z_2 = (a - c) + (b - d)i$$
$$z_1 . z_2 = (a + bi)(c + di) = (ac - bd) + (bc + ad)i$$
$$\frac{z_1}{z_2} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

Definition

- We define the complex conjugate $\bar{z}$ of a complex number
$$z = a + bi \quad by \quad \bar{z} = a - bi.$$
- The norm $|z|$ of a complex number z is defined by
$$|z|^2 = z\bar{z} = a^2 + b^2$$
- Any nonzero complex number has a unique inverse
$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

Complex numbers are commutative ring with every non-zero element has multiplicative inverse. Thus, $(\mathbb{C}, +, .)$ is a field.

- History of quaternions
  The quaternions were discovered by Sir William Rowan Hamilton in 1843, after struggling unsuccessfully to construct an algebra in three dimensions. On 16 October 1843, as Hamilton was walking along a canal in Dublin, he realized how to construct an algebra in four dimensions instead. Hamiton carved the equation onto the base of the Brougham Bridge.

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1}$$

Figure: Brougham Bridge in Dublin

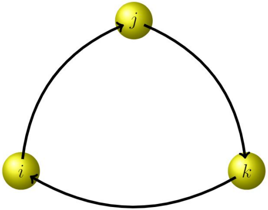

Figure: Quaternion multiplication circle

$$ij = k = -ji \qquad jk = i = -kj \qquad ki = j = -ik$$

A quaternion q can be represented as four real numbers (q_0, q_1, q_2, q_3), usually written

$$q = q_0 + q_1 i + q_2 j + q_3 k = (q_0 + q_1 i) + (q_2 + q_3 i)j$$

- The quaternion conjugate defined by $\bar{q} = q_1 - q_2 i - q_3 j - q_4 k$
- The norm of quaternion $||q||$, defined by

$$||q||^2 = q\bar{q} = q_1^2 + q_2^2 + q_3^2 + q_4^2.$$

Octonions were first introduced by John T. Graves and Arthur Cayley.

$$x = x_0 + x_1 i + x_2 j + x_3 k + x_4 l + x_5 il + x_6 jl + x_7 kl$$

$$i^2 = j^2 = k^2 = (kl)^2 = (jl)^2 = (il)^2 = l^2 = -1$$

Definition

We define the octonionic conjugate $\bar{x}$ and the norm $||x||$ of an octonion x as

$$\bar{x} = x_0 - x_1 i - x_2 j - x_3 k - x_4 l - x_5 il - x_6 jl - x_7 kl$$

$$||x||^2 = x\bar{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2$$

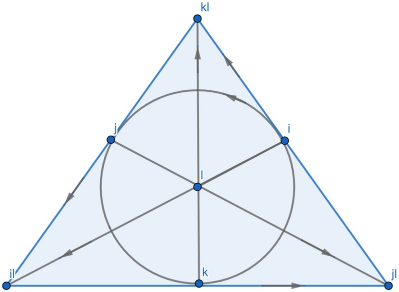

Figure: The octonionic multiplication table

- In general, Octonions are not associative

note that

$$(ij)l = kl \qquad but \qquad i(jl) = -kl$$

Why python package is useful?

- Lack of commutativity and associativity of octonions make it difficult to do calculations in octonions.

- pyoctonion package allow you to do calculations in octonions such as
  - addition, substraction, multiplication and division
  - calculate norm
  - conjugate
  - inverse
  - power
  - find associator of three octonions
  - check alternative property
  - check certain identities (Ex: Moufang identity)
  - write program in octonions (Ex: Find solutions to octonionic left qudratic equation of the form : $x^2 + bx + c = 0, \ b, c \in \mathbb{O}$)
  - solve characteristic equation and find real eigenvalues of 3×3 octonionic Hermitian matrices

Main usage of pyoctonion package in my research

Left spectrum of 2×2 octonionic Hermitian matrix

Let A be 2×2 octonionic Hermitian matrix.

$$A = \begin{bmatrix} p & \bar{a} \\ a & m \end{bmatrix}$$

where $p, m \in \mathbb{R}, \ a \in \mathbb{O}$.
the left spectrum of A is

$$\sigma_l(A) = \left\{ p + \bar{a}x : x^2 + \frac{a(p-m)x}{|a|^2} - \frac{a^2}{|a|^2} = 0 \right\}$$

Example

Let

$$A = \begin{bmatrix} 5 & 2+\mathbf{k}+\mathbf{l} \\ 2-\mathbf{k}-\mathbf{l} & 3 \end{bmatrix}$$

Then left eigenvalus of A is of the form

$$\sigma_l(A) = \{p + \bar{a}x : x^2 + \frac{a(p-m)x}{|a|^2} - \frac{a^2}{|a|^2} = 0\}.$$

since, $p = 5, \ m = 3, \bar{a} = 2 + \mathbf{k} + \mathbf{l}$

$$\sigma_l(A) = \left\{5 + (2+\mathbf{k}+\mathbf{l})x : x^2 + \frac{(2-\mathbf{k}-\mathbf{l})x}{3} - \frac{(1-2\mathbf{k}-2\mathbf{l})}{3} = 0\right\}$$

Note that we need to solve the quadratic quation

$$x^2 + \frac{(2-\mathbf{k}-\mathbf{l})x}{3} - \frac{(1-2\mathbf{k}-2\mathbf{l})}{3} = 0$$

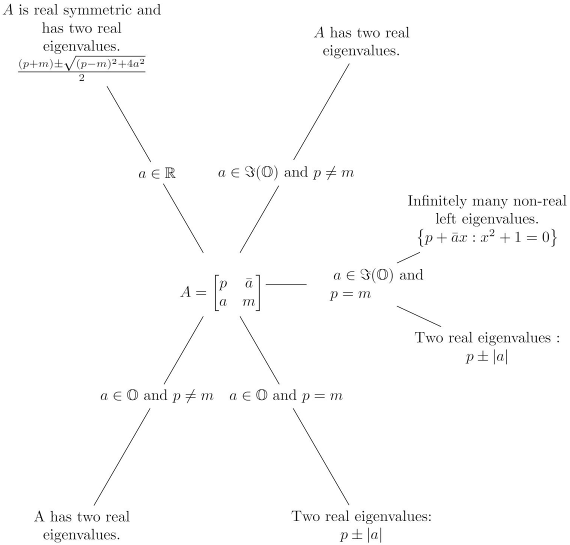

Figure: Classification of left spectrum of 2×2 octonionic Hermitian matrices.

Consider the 3×3 octonionic Hermitian matrix.

$$A = \begin{pmatrix} p & a & \bar{b} \\ \bar{a} & m & c \\ b & \bar{c} & n \end{pmatrix}$$

with $p, m, n \in \mathbb{R}$ and $a, b, c \in \mathbb{O}$

$$trA = p + m + n,$$

$$\mu(A) = pm + pn + mn - |a|^2 - |b|^2 - |c|^2,$$
$$\det A = pmn + b(ac) + \overline{b(ac)} - n|a|^2 - m|b|^2 - p|c|^2$$

Theorem (Tevian Dray in 1998)

The real eigenvalues of the 3×3 octonionic Hermitian matrix A satisfy the modified characteristic equation

$$det(\lambda I - A) = \lambda^3 - (trA)\lambda^2 + \mu(A)\lambda - detA = r$$

where r is either of the two roots of

$$r^2 + 4\Phi(a, b, c)r - |[a, b, c]|^2 = 0$$

and where

$$\Phi(a, b, c) = \frac{1}{2} Re([a, \bar{b}]c).$$

Example (we can find real eigenvalues of A using pyoctonion package)

Let

$$A = \begin{bmatrix} 1 & 1+i & l+jl \\ 1-i & 2 & l \\ -l-jl & -l & 3 \end{bmatrix}$$