Bayesian Hierarchical Network Autocorrelation Model and Its Application to the Study of Institutional Peer-Effects Involving Robotic Surgery

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Table 1.1. Bias and frequentist mean squared error of ρ.

Network Density		$\rho = -0.5$	$\rho = -0.2$	$\rho = 0$	$\rho = 0.2$	$\rho = 0.5$
0.2	Bias	0.034	0.026	0.017	0.007	-0.011
	MSE	0.026	0.028	0.029	0.029	0.025
0.4	Bias	0.020	-0.019	0.010	0.010	-0.053
	MSE	0.069	0.071	0.077	0.069	0.058
0.6	Bias	0.044	0.043	0.023	-0.057	-0.131
	MSE	0.145	0.151	0.152	0.120	0.108
0.8	Bias	0.025	-0.091	-0.133	-0.172	-0.298
	MSE	0.346	0.290	0.272	0.270	0.263

Note: For each value of ρ and network density, the results show the bias of the posterior median estimator of ρ and the posterior mean squared error (MSE) of ρ, respectively. The results are rounded to 3 decimal places.

Table 1.2. 95% coverage rates and average width of the credible interval of ρ.

Network Density		$\rho = -0.5$	$\rho = -0.2$	$\rho = 0$	$\rho = 0.2$	$\rho = 0.5$
0.2	Rate	0.944	0.94	0.946	0.956	0.946
	Width	0.636	0.664	0.670	0.659	0.603
0.4	Rate	0.96	0.948	0.954	0.952	0.958
	Width	1.089	1.105	1.079	1.017	0.896
0.6	Rate	0.972	0.966	0.958	0.972	0.96
	Width	1.572	1.519	1.455	1.392	1.232
0.8	Rate	0.962	0.97	0.97	0.97	0.962
	Width	2.290	2.199	2.069	1.959	1.777

Note: For each value of ρ and network density, the results show the 95% coverage rate (Rate) and average width of the credible interval (Width) of ρ, respectively. The results are rounded to 3 decimal places.

Table S2.1. Bias and frequentist mean squared error of ρ.

Network Density		$\rho = -0.5$	$\rho = -0.2$	$\rho = 0$	$\rho = 0.2$	$\rho = 0.5$
0.2	Bias	0.023	0.024	0.006	0.007	-0.011
	MSE	*0.025*	*0.029*	*0.027*	*0.029*	*0.025*
0.4	Bias	0.062	0.028	0.002	-0.019	-0.063
	MSE	*0.062*	*0.074*	*0.077*	*0.071*	*0.058*
0.6	Bias	0.151	0.062	0.007	-0.024	-0.152
	MSE	*0.127*	*0.115*	*0.126*	*0.104*	*0.120*
0.8	Bias	0.241	0.123	0.020	-0.102	-0.274
	MSE	*0.171*	*0.162*	*0.147*	*0.141*	*0.186*

Note: For each value of ρ and network density, the results show the bias of the posterior median estimator of ρ and the posterior mean squared error (MSE) of ρ, respectively. The results are rounded to 3 decimal places.

Table S2.2. 95% coverage rates and average width of the credible interval of ρ.

Network Density		$\rho = -0.5$	$\rho = -0.2$	$\rho = 0$	$\rho = 0.2$	$\rho = 0.5$
0.2	Rate	0.958	0.946	0.964	0.956	0.946
	Width	*0.614*	*0.669*	*0.672*	*0.659*	*0.603*
0.4	Rate	0.96	0.964	0.944	0.956	0.962
	Width	*0.963*	*1.053*	*1.047*	*1.025*	*0.913*
0.6	Rate	0.97	0.974	0.958	0.976	0.948
	Width	*1.251*	*1.336*	*1.343*	*1.306*	*1.198*
0.8	Rate	0.988	0.978	0.986	0.988	0.966
	Width	*1.531*	*1.550*	*1.560*	*1.553*	*1.517*

Note: For each value of ρ and network density, the results show the 95% coverage rate (Rate) and average width of the credible interval (Width) of ρ, respectively. The results are rounded to 3 decimal places.

Table S3.1. Bias and frequentist mean squared error of ρ.

Network Density		$\rho = -0.5$	$\rho = -0.2$	$\rho = 0$	$\rho = 0.2$	$\rho = 0.5$
0.2	Bias	0.033	0.035	0.037	0.043	0.054
	MSE	*0.028*	*0.031*	*0.033*	*0.035*	*0.035*
0.4	Bias	0.048	0.068	0.086	0.101	0.074
	MSE	*0.088*	*0.103*	*0.109*	*0.107*	*0.068*
0.6	Bias	0.148	0.173	0.170	0.151	0.067
	MSE	*0.266*	*0.265*	*0.233*	*0.186*	*0.104*
0.8	Bias	0.312	0.283	0.233	0.165	0.026
	MSE	*0.591*	*0.475*	*0.377*	*0.278*	*0.161*

Note: For each value of ρ and network density, the results show the bias of the posterior median estimator of ρ and the posterior mean squared error (MSE) of ρ, respectively. The results are rounded to 3 decimal places.

Table S3.2. 95% coverage rates and average width of the credible interval of ρ.

Network Density		$\rho = -0.5$	$\rho = -0.2$	$\rho = 0$	$\rho = 0.2$	$\rho = 0.5$
0.2	Rate	0.942	0.934	0.930	0.924	0.916
	Width	*0.645*	*0.681*	*0.692*	*0.685*	*0.605*
0.4	Rate	0.946	0.934	0.924	0.906	0.906
	Width	*1.128*	*1.141*	*1.117*	*1.042*	*0.861*
0.6	Rate	0.920	0.912	0.916	0.920	0.944
	Width	*1.691*	*1.591*	*1.489*	*1.356*	*1.126*
0.8	Rate	0.912	0.918	0.926	0.934	0.944
	Width	*2.292*	*2.077*	*1.925*	*1.772*	*1.541*

Note: For each value of ρ and network density, the results show the 95% coverage rate (Rate) and average width of the credible interval (Width) of ρ, respectively. The results are rounded to 3 decimal places.

Table S4.1. Bias and frequentist mean squared error of ρ and α.

Network Density		$\rho=-0.5$		$\rho=-0.2$		$\rho=0$		$\rho=0.2$		$\rho=0.5$	
		α=2	α=5	α=2	α=5	α=2	α=5	α=2	α=5	α=2	α=5
d=0.2	Bias	0.070	0.021	0.024	0.013	0.010	-0.002	0.002	-0.025	-0.061	-0.040
	MSE	*0.068*	*0.028*	*0.059*	*0.030*	*0.053*	*0.035*	*0.049*	*0.035*	*0.046*	*0.027*
	Bias	-0.028	0.147	-0.046	0.147	0.005	-0.064	0.042	-0.103	0.091	0.175
	MSE	*1.537E-06*	*4.319E-05*	*4.211E-06*	*4.313E-05*	*5.278E-08*	*8.136E-06*	*3.484E-06*	*2.128E-05*	*1.642E-05*	*6.149E-05*
d=0.4	Bias	0.067	0.020	-0.017	0.014	-0.105	-0.026	-0.164	-0.046	-0.236	-0.100
	MSE	*0.306*	*0.097*	*0.212*	*0.083*	*0.232*	*0.092*	*0.204*	*0.075*	*0.178*	*0.063*
	Bias	-0.931	-0.205	0.103	0.180	-0.071	-0.198	0.189	0.248	0.206	0.195
	MSE	*1.735E-03*	*8.406E-05*	*2.112E-05*	*6.460E-05*	*1.016E-05*	*7.874E-05*	*7.146E-05*	*1.231E-04*	*8.464E-05*	*7.589E-05*

Note: The results in blue font represent the results of ρ while the results in black font represent the results of α. For each value of network density, the results show the bias of the posterior median estimator of ρ , the posterior mean squared error (MSE) of ρ, bias and the mean squared error (MSE) of the posterior median estimator and posterior distribution of α, respectively. The results are rounded to 3 decimal places.

Table S4.2. 95% coverage rates and average width of the credible interval of ρ and α.

Network Density		$\rho = -0.5$		$\rho = -0.2$		$\rho = 0$		$\rho = 0.2$		$\rho = 0.5$	
		α=2	α=5	α=2	α=5	α=2	α=5	α=2	α=5	α=2	α=5
d=0.2	Rate	0.950	0.962	0.958	0.962	0.962	0.936	0.952	0.938	0.926	0.940
	Width	*0.945*	*0.673*	*0.941*	*0.700*	*0.897*	*0.707*	*0.865*	*0.683*	*0.739*	*0.594*
	Rate	0.938	0.944	0.942	0.946	0.958	0.938	0.946	0.946	0.948	0.950
	Width	*0.323*	*0.599*	*0.333*	*0.614*	*0.328*	*0.624*	*0.346*	*0.633*	*0.343*	*0.640*
d=0.4	Rate	0.970	0.954	0.980	0.966	0.972	0.940	0.952	0.934	0.928	0.918
	Width	*2.161*	*1.193*	*1.956*	*1.177*	*1.831*	*1.126*	*1.649*	*1.044*	*1.407*	*0.886*
	Rate	0.966	0.948	0.972	0.952	0.974	0.934	0.958	0.952	0.934	0.946
	Width	*1.590*	*0.503*	*1.352*	*0.526*	*1.257*	*0.528*	*1.126*	*0.533*	*0.980*	*0.556*

Note: The results in blue font represent the results of ρ while the results in black font represent the results of α. For each value of network density, the results show the 95% coverage rate (Rate) and average width of the credible interval (Width) of ρ, the 95% coverage rate (Rate) and average width of the credible interval (Width) of α, respectively. The results are rounded to 3 decimal places.

Table S5.1. Bias and frequentist mean squared error of ρ and α.

Network Density		$\rho=-0.5$		$\rho=-0.2$		$\rho=0$		$\rho=0.2$		$\rho=0.5$	
		α=2	α=5	α=2	α=5	α=2	α=5	α=2	α=5	α=2	α=5
d=0.2	Bias	0.062	0.033	0.049	0.016	0.009	-0.004	0.022	-0.017	-0.029	-0.040
	MSE	*0.050*	*0.029*	*0.057*	*0.033*	*0.050*	*0.033*	*0.055*	*0.034*	*0.037*	*0.027*
	Bias	0.018	0.022	0.035	-0.081	-0.022	0.085	0.018	-0.103	0.011	0.175
	MSE	*6.212E-07*	*9.724E-07*	*2.421E-06*	*1.302E-05*	*9.363E-07*	*1.433E-05*	*6.321E-07*	*2.128E-05*	*2.508E-07*	*6.149E-05*
d=0.4	Bias	0.364	0.069	0.134	0.019	0.021	0.007	-0.123	-0.059	-0.243	-0.094
	MSE	*0.239*	*0.073*	*0.145*	*0.080*	*0.125*	*0.081*	*0.135*	*0.073*	*0.172*	*0.056*
	Bias	-0.133	0.188	0.119	-0.017	0.204	0.050	0.516	-0.066	-0.113	-0.064
	MSE	*3.519E-05*	*7.054E-05*	*2.837E-05*	*6.120E-07*	*8.332E-05*	*5.000E-06*	*5.329E-04*	*8.820E-06*	*2.553E-05*	*8.135E-06*

Note: The results in blue font represent the results of ρ while the results in black font represent the results of α. For each value of network density, the results show the bias of the posterior median estimator of ρ , the posterior mean squared error (MSE) of ρ, bias and the mean squared error (MSE) of the posterior median estimator and posterior distribution of α, respectively. The results are rounded to 3 decimal places.

Table S5.2. 95% coverage rates and average width of the credible interval of ρ and α.

Network Density		$\rho=-0.5$		$\rho=-0.2$		$\rho=0$		$\rho=0.2$		$\rho=0.5$	
		α=2	α=5	α=2	α=5	α=2	α=5	α=2	α=5	α=2	α=5
d=0.2	Rate	0.964	0.968	0.966	0.954	0.960	0.950	0.932	0.936	0.958	0.940
	Width	*0.894*	*0.667*	*0.955*	*0.710*	*0.910*	*0.714*	*0.860*	*0.681*	*0.730*	*0.594*
	Rate	0.964	0.960	0.956	0.946	0.956	0.946	0.946	0.946	0.962	0.950
	Width	*0.315*	*0.609*	*0.343*	*0.613*	*0.341*	*0.626*	*0.358*	*0.633*	*0.356*	*0.640*
d=0.4	Rate	0.984	0.952	0.984	0.954	0.984	0.942	0.980	0.956	0.914	0.944
	Width	*1.579*	*1.000*	*1.529*	*1.083*	*1.496*	*1.078*	*1.440*	*1.040*	*1.316*	*0.877*
	Rate	0.968	0.950	0.978	0.956	0.978	0.966	0.968	0.960	0.956	0.968
	Width	*1.453*	*0.499*	*1.281*	*0.514*	*1.201*	*0.531*	*1.086*	*0.544*	*0.976*	*0.555*

Note: The results in blue font represent the results of ρ while the results in black font represent the results of α. For each value of network density, the results show the 95% coverage rate (Rate) and average width of the credible interval (Width) of ρ, the 95% coverage rate (Rate) and average width of the credible interval (Width) of α, respectively. The results are rounded to 3 decimal places.

Table S6.1. Bias and frequentist mean squared error of ρ and α.

Network Density		$\rho = -0.5$		$\rho = -0.2$		$\rho = 0$		$\rho = 0.2$		$\rho = 0.5$	
		α=2	α=5	α=2	α=5	α=2	α=5	α=2	α=5	α=2	α=5
d=0.2	Bias	0.065	0.020	0.069	0.022	0.071	0.024	0.071	0.023	0.042	0.010
	MSE	*0.068*	*0.030*	*0.074*	*0.034*	*0.073*	*0.036*	*0.068*	*0.035*	*0.044*	*0.027*
	Bias	-0.036	0.148	-0.043	0.150	-0.048	0.151	-0.053	0.150	-0.061	0.149
	MSE	*2.632E-06*	*4.356E-05*	*3.761E-06*	*4.526E-05*	*4.663E-06*	*4.556E-05*	*5.711E-06*	*4.497E-05*	*7.491E-06*	*4.451E-05*
d=0.4	Bias	0.347	0.045	0.230	0.047	0.155	0.041	0.074	0.025	-0.050	-0.017
	MSE	*0.581*	*0.110*	*0.394*	*0.113*	*0.298*	*0.107*	*0.217*	*0.094*	*0.133*	*0.065*
	Bias	0.158	-0.112	0.203	-0.114	0.236	-0.115	0.267	-0.116	0.311	-0.115
	MSE	*5.001E-05*	*2.500E-05*	*8.276E-05*	*2.582E-05*	*1.112E-04*	*2.633E-05*	*1.429E-04*	*2.691E-05*	*1.933E-04*	*2.643E-05*

Note: The results in blue font represent the results of ρ while the results in black font represent the results of α. For each value of network density, the results show the bias of the posterior median estimator of ρ , the posterior mean squared error (MSE) of ρ, bias and the mean squared error (MSE) of the posterior median estimator and posterior distribution of α, respectively. The results are rounded to 3 decimal places.

Table S6.2. 95% coverage rates and average width of the credible interval of ρ and α.

Network Density		$\rho = -0.5$		$\rho = -0.2$		$\rho = 0$		$\rho = 0.2$		$\rho = 0.5$	
		α=2	α=5	α=2	α=5	α=2	α=5	α=2	α=5	α=2	α=5
d=0.2	Rate	0.938	0.962	0.934	0.958	0.926	0.948	0.920	0.944	0.910	0.922
	Width	*0.967*	*0.686*	*0.977*	*0.717*	*0.949*	*0.716*	*0.888*	*0.694*	*0.722*	*0.599*
	Rate	0.944	0.942	0.934	0.946	0.928	0.948	0.934	0.948	0.944	0.940
	Width	*0.321*	*0.600*	*0.341*	*0.616*	*0.352*	*0.626*	*0.358*	*0.632*	*0.356*	*0.638*
d=0.4	Rate	0.932	0.948	0.932	0.944	0.938	0.936	0.930	0.930	0.932	0.928
	Width	*2.226*	*1.242*	*1.975*	*1.224*	*1.804*	*1.177*	*1.632*	*1.094*	*1.349*	*0.897*
	Rate	0.918	0.966	0.920	0.964	0.922	0.958	0.934	0.960	0.948	0.964
	Width	*1.705*	*0.512*	*1.468*	*0.527*	*1.327*	*0.536*	*1.201*	*0.545*	*1.033*	*0.558*

Note: The results in blue font represent the results of ρ while the results in black font represent the results of α. For each value of network density, the results show the 95% coverage rate (Rate) and average width of the credible interval (Width) of ρ, the 95% coverage rate (Rate) and average width of the credible interval (Width) of α, respectively. The results are rounded to 3 decimal places.

Hierarchical network autocorrelation model 1

$$\begin{gathered} Y = Z\theta + B\delta + \varepsilon \\ \delta = \rho W\delta + X\beta + \tau \\ \varepsilon \sim N(0, \sigma^2 I_N) \\ \tau \sim N(0, \omega^2 I_g) \end{gathered} \tag{1}$$

Following the notations in the main text, Y indicates a vector of length N containing the values of a response variable for N observations. Z is a $(N \times k)$ matrix for k observation level covariates whose first column is a vector of 1_s for the intercept. δ is a vector of length g representing the random effect of network actors and B is a $(N \times g)$ allocating matrix linking the random effect δ back to the response variable. X is a $(g \times l)$ matrix for l actor level covariates. ε and τ represent the errors at each level. W is a $(g \times g)$ matrix consisting of the ties between the actors in a network. The ij^{th} entry of W, W_{ij}, represents the influence of actor j on actor i. In particular, W is constrained to be a non-negative row-stochastic matrix whose diagonal element is zero reflecting the absence of negative influences, the specification of relative exposures as being the conduit to social influence, and the absence of self-ties in the network. The focal parameter ρ measures the magnitude of the peer effect in the network and thus indirectly the effect of alters (actors linked through the network to the focal actor to which the individual whose outcome is being analyzed is associated) on the outcomes of the individuals of other actors.

The likelihood function of the model is:

$$f(Y|\theta, \delta, \sigma^2) = (2\pi\sigma^2)^{-N/2} \exp\left(-\frac{(Y - Z\theta - B\delta)^T (Y - Z\theta - B\delta)}{2\sigma^2}\right).$$

We assign the uniform prior distributions on $\theta, \beta, \sigma, and\ \omega$, that is $p(\theta, \beta, \sigma^2, \omega^2) \propto \frac{1}{\sigma} * \frac{1}{\omega}$. As discussed in the main text, we use a flat prior for ρ and the parameter space of ρ in the prior is restricted to $(1/\lambda_{min}, 1/\lambda_{max})$, that is $p(\rho) \propto 1$, $1/\lambda_{min} < \rho < 1/\lambda_{max}$, where λ_{max} and λ_{min} is the max and min eigenvalues of the row-normalized W.
In addition, the prior distribution of δ is:

$$p(\delta|\beta, \omega^2, \rho) = |A|(2\pi\omega^2)^{-\frac{g}{2}} \exp\left(-\frac{(A\delta - X\beta)^T (A\delta - X\beta)}{2\omega^2}\right).$$

Where,

$$A = I_g - \rho W.$$

Then, the conditional posteriors of $\sigma^2, \omega^2, \beta, \theta$ can be written as:

$$P(\sigma^2|\delta, \theta, \beta, \omega^2, \rho, Y) \sim IG\left(\frac{N-1}{2}, \frac{(Y - Z\theta - B\delta)^T (Y - Z\theta - B\delta)}{2}\right)$$

$$P(\omega^2|\delta, \theta, \beta, \sigma^2, \rho, Y) \sim IG\left(\frac{g-1}{2}, \frac{(A\delta - X\beta)^T (A\delta - X\beta)}{2}\right)$$

$$P(\beta|\delta, \sigma^2, \rho, \omega^2, \theta, Y) \sim N((X^T X)^{-1} X^T A\delta, (X^T X)^{-1}\omega^2)$$

$$P(\theta|\delta, \sigma^2, \rho, \omega^2, \beta, Y) \sim N((Z^T Z)^{-1} Z^T (Y - B\delta), (Z^T Z)^{-1}\sigma^2)$$

For ρ,

$$p(\rho|\beta,\omega^2,\delta,\sigma^2,\theta,Y) \propto |A|\exp\,(-\frac{(A\delta-X\beta)^T(A\delta-X\beta)}{2\omega^2})$$

The conditional posterior of $\boldsymbol{\rho}$ does not have a well-known form for direct sampling. Therefore, we use the Metropolis-Hastings algorithm with an independent candidate generating function. To approximate the target distribution, we firstly approximate $\boldsymbol{ln|A|}$ using the quadratic polynomial approximation by Taylor series at $\boldsymbol{\rho}=\mathbf{0}$ (Dittrich *et al.*, 2017):

$|A| = \prod(1-\rho\lambda_i)$, where λ_i $(i=1,\dots,g)$ are eigenvalues of W (Ord, 1975) and $ln|A| = \sum \ln(1-\rho\lambda_i)$.

By a second-order Taylor series approximation,

$$ln|A| \approx \sum -\rho\lambda_i - \rho^2{\lambda_i}^2/2 = \sum -\rho^2{\lambda_i}^2/2 \text{ as } \sum\lambda_i = tr(W) = 0.$$

That is, $|A| \approx \exp\left(\sum -\rho\lambda_i - \frac{\rho^2{\lambda_i}^2}{2}\right) = \exp\left(\sum -\frac{\rho^2{\lambda_i}^2}{2}\right)$.

Therefore,

$$p(\rho|\beta,\omega^2,\delta,\sigma^2,\theta,Y) \propto \exp\left(\sum -\frac{\rho^2{\lambda_i}^2}{2}\right)\exp\left(-\frac{(A\delta-X\beta)^T(A\delta-X\beta)}{2\omega^2}\right) \sim TN(\mu^*,V^*), 1/\lambda_{min} < \rho < 1/\lambda_{max}, (A1)$$

where, $TN(\mu^*,V^*)$ is the truncated normal distribution candidate generating distribution with mean μ^* and variance V^*:

$$\mu^* = \frac{\delta^T W^T(\delta - X\beta)}{\omega^2\sum{\lambda_i}^2 + \delta^T W^T W\delta}$$

$$V^* = \frac{\omega^2}{\omega^2\sum{\lambda_i}^2 + \delta^T W^T W\delta}$$

The acceptance ratio γ for a Metropolis-Hastings algorithm with an Independence candidate generating function is:

$$\gamma = \min\,(1, \frac{\pi(\rho_p)N^T(\rho_c,\mu^*,V^*)}{\pi(\rho_c)N^T(\rho_p,\mu^*,V^*)})$$

where ρ_c is the current state of ρ and ρ_p is the proposed state of ρ. $\pi(\cdot)$ is the desired distribution with $\pi(\rho) = |A|\exp\,(-\frac{(A\delta-X\beta)^T(A\delta-X\beta)}{2\omega^2})$.

To perform a Metropolis-Hastings step we randomly draw a random variable u from $U(0,1)$ and accept the proposed state if $u \le \gamma$, otherwise rejecting the proposed state and staying at current state if $u > \gamma$.

Similarly, for $p(\,\rho) \propto 1, -1 < \rho < 1$, the resulting candidate generating distribution of ρ is $TN(\mu^*,V^*)$, $-1 < \rho < 1$. For $p(\rho) \propto \frac{1}{(1/\lambda_{max}-\rho)(\rho-1/\lambda_{min})}$, the conditional posterior of ρ is:

$$p(\rho|\beta,\omega^2,\delta,\sigma^2,\theta,Y) \propto |A|\exp\,(-\frac{(A\delta-X\beta)^T(A\delta-X\beta)}{2\omega^2})\frac{1}{(1/\lambda_{max}-\rho)(\rho-1/\lambda_{min})}$$

The above distribution does not have a well-known closed-form that can be used for directly sampling. Therefore, we use the same candidate generating distribution introduced in Equation (A1) above to sample ρ.

The conditional posterior of δ is:

$$p(\delta|\beta, \omega^2, \rho, \sigma^2, \theta, Y) \propto \exp\left(-\frac{(A\delta - X\beta)^T(A\delta - X\beta)}{2\omega^2} - \frac{(Y - Z\theta - B\delta)^T(Y - Z\theta - B\delta)}{2\sigma^2}\right)$$

$$= \exp\left(\frac{\frac{\sigma^2}{w^2}(A\delta - X\beta)^T(A\delta - X\beta) + (Y - Z\theta - B\delta)^T(Y - Z\theta - B\delta)}{-2\sigma^2}\right).$$

Let $H = \frac{\sigma}{\omega}A$, so that

$$= \exp\left(\frac{\left(H\delta - \frac{\sigma}{\omega}X\beta\right)^T\left(H\delta - \frac{\sigma}{\omega}X\beta\right) + (Y - Z\theta - B\delta)^T(Y - Z\theta - B\delta)}{-2\sigma^2}\right).$$

Let $\Delta = \frac{\sigma}{\omega}X\beta$, so that

$$\left(H\delta - \frac{\sigma}{\omega}X\beta\right)^T\left(H\delta - \frac{\sigma}{\omega}X\beta\right) = (\delta^T H^T - \Delta^T)(H\delta - \Delta).$$

Letting $M = Y - Z\theta$ we obtain

$$(Y - Z\theta - B\delta)^T(Y - Z\theta - B\delta) = (M - B\delta)^T(M - B\delta) = (M^T - \delta^T B^T)(M - B\delta)$$

and

$$(H\delta - \Delta)^T(H\delta - \Delta) + (M - B\delta)^T(M - B\delta) \propto \delta^T(H^T H + B^T B)\delta - 2(\Delta^T H + M^T B)\delta$$

Finally, let $D = H^T H + B^T B$, $C = \Delta^T H + M^T B$ so that

$$\delta^T D\delta - 2C\delta \propto (\delta - D^{-1}C^T)^T D(\delta - D^{-1}C^T).$$

Hence,

$$p(\delta|\beta, \omega^2, \rho, \sigma^2, \theta, Y) \sim \mathrm{N}(D^{-1}C^T, \sigma^2 D^{-1}).$$

Hierarchical network autocorrelation model 2

$$\begin{gathered} Y = Z\theta + B[\delta + \alpha W_1\delta] + \varepsilon \\ \delta = \rho W_2\delta + X\beta + \tau \\ \varepsilon \sim N(0, \sigma^2 I_N) \\ \tau \sim N(0, \omega^2 I_g) \end{gathered} \tag{2}$$

α is used to measure the direct effect of alters on the outcome of individuals from the ego through the network while we maintain the same notation for the other parameters as for model 1. With only a single source of network relationships, we set $W_1 = W_2 = W$.

The likelihood function of the model is:

$$f(Y|\theta,\delta,\sigma^2,\alpha) = (2\pi\sigma^2)^{-N/2}\exp\left(-\frac{\left(Y-Z\theta-B(\delta+\alpha W\delta)\right)^T\left(Y-Z\theta-B(\delta+\alpha W\delta)\right)}{2\sigma^2}\right).$$

We assign a flat prior for α, that is $p(\alpha) \propto 1$ and the prior distributions for the other parameters align with these in the model (1).

Let $K = Y - Z\theta - B\delta$, then the conditional posterior of α can be expressed as:

$$p(\alpha|\beta,\omega^2,\rho,\sigma^2,\theta,Y,\delta) \sim N\left(\frac{\delta^T W^T B^T K}{\delta^T W^T B^T B W\delta}, \frac{\sigma^2}{\delta^T W^T B^T B W\delta}\right)$$

Moreover, the conditional posterior of δ is:

$$\begin{aligned} p(\delta|\beta,\omega^2,\rho,\sigma^2,\theta,Y,\alpha) & \\ &\propto \exp\left(-\frac{(A\delta-X\beta)^T(A\delta-X\beta)}{2\omega^2}\right. \\ &\left.-\frac{(Y-Z\theta-B(\delta+\alpha W\delta))^T(Y-Z\theta-B(\delta+\alpha W\delta))}{2\sigma^2}\right) \end{aligned}$$

Let $G = B[I_g + \alpha W]$, $D = H^T H + G^T G$, $C = \Delta^T H + M^T G$.

Therefore, $p(\delta|\beta,\omega^2,\rho,\sigma^2,\theta,Y,\alpha) \propto N(D^{-1}C^T, \sigma^2 D^{-1})$.

Similar to the model in (1), the conditional posterior of θ, σ^2, ω^2, β and ρ follow that

$$P(\theta|\delta,\sigma^2,\rho,\omega^2,\beta,Y,\alpha) \sim N((Z^T Z)^{-1} Z^T (Y - B(\delta+\alpha W\delta)), (Z^T Z)^{-1}\sigma^2)$$

$$P(\sigma^2|\delta,\theta,\beta,\omega^2,\rho,Y,\alpha) \sim IG\left(\frac{N-1}{2}, \frac{\left(Y-Z\theta-B(\delta+\alpha W\delta)\right)^T(Y-Z\theta-B(\delta+\alpha W\delta))}{2}\right)$$

$$P(\omega^2|\delta,\theta,\beta,\sigma^2,\rho,Y,\alpha) \sim IG\left(\frac{g-1}{2}, \frac{(A\delta-X\beta)^T(A\delta-X\beta)}{2}\right)$$

$$P(\beta|\delta,\sigma^2,\rho,\omega^2,\theta,Y,\alpha) \sim N((X^T X)^{-1} X^T A\delta, (X^T X)^{-1}\omega^2)$$

$$p(\rho|\beta,\omega^2,\delta,\sigma^2,\theta,Y,\alpha) \propto |A| \exp\left(-\frac{(A\delta-X\beta)^T(A\delta-X\beta)}{2\omega^2}\right) p(\rho)$$

To sample ρ, we use the same approach used for the model in (1).