Probability and Random Process Exams

CHEN SHANG

Nov 2022

Personal answers.

May not always be right.

2018-2019

1.

(a)

$p_{Y|X}(y|x) = f_Z(\frac{1}{2}y - 3x|x)$

$p_{Y|X}(y|1) = f_Z(\frac{1}{2}y - 3) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{8}y^2+\frac{3}{2}y-\frac{9}{2}}$

$p_{Y|X}(y|0) = f_Z(\frac{1}{2}y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{8}y^2}$

(b)

$p_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)}{\sum_{x'} p_X(x')f_{Y|X}(y|x')}p_X(x)$

$p_X(0) = 1 - p,\ p_X(1) = p$

$p_{X|Y}(1|y) = \frac{f_{Y|X}(y|1)}{p_X(0)f_{Y|X}(y|0)+p_X(1)f_{Y|X}(y|1)}p_X(1) = \frac{pe^{\frac{3}{2}y-\frac{9}{2}}}{1-p+pe^{\frac{3}{2}y-\frac{9}{2}}}$

$p_{X|Y}(0|y) = \frac{f_{Y|X}(y|0)}{p_X(0)f_{Y|X}(y|0)+p_X(1)f_{Y|X}(y|1)}p_X(0) = \frac{1-p}{1-p+pe^{\frac{3}{2}y-\frac{9}{2}}}$

$$X = \begin{cases} 0 & y < \frac{2}{3}\ln(1-p) - \frac{2}{3}\ln p + 3 \\ 1 & \text{otherwise} \end{cases}$$

2.

(a)

$f_Y(y) = \int_0^3 f_{Y|X}(y|x) f_X(x) dx$

$= \frac{c}{3}\int_0^3 xe^{-xy} dx$

$= -\frac{c}{3y}\int_0^3 xd(e^{-xy})$

$= -\frac{c}{3y}(xe^{-xy}\mid_0^3 - \int_0^3 e^{-xy} dx)$

$= \begin{cases} \frac{c}{3y^2}[1 - e^{-3y}(3y+1)] & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$\int_0^{+\infty} f_Y(y) dy = c = 1$

(b)

$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)}{f_Y(y)} f_X(x)$

$f_{X|Y}(x|\frac{1}{3}) = \frac{f_{Y|X}(\frac{1}{3}|x)}{f_Y(\frac{1}{3})} f_X(x) = \begin{cases} \frac{xe^{-\frac{1}{3}x}}{9(1-2e^{-1})} & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$

$\hat{x} = E(X|Y = \frac{1}{3}) = \int_0^3 x f_{X|Y}(x|\frac{1}{3}) dx = \frac{6-15e^{-1}}{1-2e^{-1}}$

3.

(a)

(i)

$40 = 20 + 20$

$p = \frac{1}{C_{10}^2} = \frac{1}{45}$

(ii)

$p = \frac{1+C_3^2+C_4^2}{C_{10}^2} = \frac{2}{9}$

(b)

(i)

$\eta_x(t) = \frac{1}{2}g(t) + \frac{1}{2}(-g(t)) = 0$

(ii)

$R(t_1, t_2) = E(x(t_1)x^*(t_2)) = \frac{1}{2}g(t_1)g^*(t_2) + \frac{1}{2}(-g(t_1))(-g^*(t_2))$

$R(t_1, t_2) = g(t_1)g^*(t_2)$

No.

(c)

$R_{ss}(t_1, t_2) = E(s(t_1)s^*(t_2)) = \alpha^2 E(e^{jp_1(t_1-t_2)})E(e^{jp_2(t_1^2-t_2^2)})$

4.

(a)

(b)

5.

(a)

(i)

State 0: at the home

State 1: at the office

$$P = \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{bmatrix}$$

(ii)

$$\begin{cases} Y = YP \\ p_{home} + p_{office} = 1 \end{cases}$$

$p_{office} = \frac{2}{3}$

(iii)

$$P = \begin{bmatrix} 0.2 & 0.8 \\ 0.4 + a & 0.6 - a \end{bmatrix}$$

$$\begin{cases} Y = YP \\ p_{home} + p_{office} = 1 \end{cases}$$

$p_{office} = \frac{0.8}{1.2+a}$

No.

(b)

2019-2020

1.

(a)

$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dydx = 1$
$\int_0^1\int_0^{1-x} cxy\ dydx = 1$
$c = 24$
$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \int_0^{1-y} 24xy\ dx = \begin{cases} 12y(1-y)^2 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

(b)

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{2x}{(1-y)^2} & x \ge 0,\ y \ge 0,\ x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$

(c)

$\hat{x} = \mathrm{E}(X|Y=y) = \int_0^{1-y} x f_{X|Y}(x|y)dx = \frac{2}{3}(1-y)$

(d)

$\mathrm{E}(X^2|Y=y) = \int_0^{1-y} x^2 f_{X|Y}(x|y)dx = \frac{1}{2}(1-y)^2$
$\mathrm{Var}(X|Y=y) = \mathrm{E}(X^2|Y=y) - \mathrm{E}^2(X|Y=y) = \frac{1}{18}(1-y)^2$
$\mathrm{MSE} = \mathrm{E}(\mathrm{Var}(X|Y=y)) = \int_0^1 \frac{1}{18} y(1-y)^2 dy = \frac{1}{216}$

2.

(a)

$p_{X,Y}(x,y)$

(b)

$p_{X,Y}(x,3)$

(c)

$0.08 + 0.08 + 0.24 = p_Y(3)$

$\frac{p_{X,Y}(x,3)}{p_Y(3)} = p_{X|Y}(x|3)$

(d)

$p_{Y|X}(3|4),\ p_{Y|X}(3|5),\ p_{Y|X}(3|6)$

3.

(a)

(i)

$P(A) = \frac{9}{90} = \frac{1}{10}$

$P(B) = \frac{10}{90} = \frac{1}{9}$

(ii)

$P(C) = P(A) + P(B) - P(AB) = \frac{1}{5}$

(b)

(i)

$\eta_y[n] = E(x[n]) - E(x[n-3]) = 0$

(ii)

$E(y^2[n]) = E(x^2[n] - 2x[n]x[n-3] + x^2[n-3]) = R_{xx}[0] - 2R_{xx}[3] + R_{xx}[0]$

$E(y^2[n]) = 2 - 2e^{-1.5} + 2 = 4 - 2e^{-1.5}$

(iii)

$$R_{yy}[m] = E(y[n]y[n+m]) = \begin{cases} 2R_{xx}[0] - R_{xx}[m-3] - R_{xx}[m+3] \\ todo\ Week11\ 2:05:47 \end{cases}$$

$c = 2 - e^{1.5} - e^{-1.5}$

4.

(a)

(b)

5.

(a)

(i)

$\eta_Z(t) = \sum_{i=1}^{n} b_i y_i(t) = \sum_{i=1}^{n} 0 = 0$

$R_{yy}(t_1, t_2) = C_{yy}(t_1, t_2) + \eta(t_1)\eta^*(t_2) = 0,\ t_1 \neq t_2$

$R_{zz}(\tau) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j E(y_i(t) y_j(t+\tau)) = \sum_{i=1}^{n} b_i^2 e^{-a_i|\tau|}$

WSS.

(ii)

$C_{zz}(\tau) = R_{zz}(\tau) = \sum_{i=1}^{n} b_i^2 e^{-a_i|\tau|}$

$\frac{1}{T}\int_0^T C_{zz}(\tau) d\tau = \frac{1}{T}\sum_{i=1}^{n} b_i^2 \frac{1-e^{-a_i T}}{a_i} \to 0$

(iii)

???

(b)

2020-2021

1.

(a)

$\int_0^{10} a\,(1-0.1x)\,dx = 1$

$a = 0.2$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 0.2x - 0.01x^2 & 0 \le x \le 10 \\ 1 & x > 10 \end{cases}$$

$y = F_X(x)$

$y_1 = 0.0591,\ y_2 = 0.19,\ y_3 = 0.4375,\ y_4 = 0.6975,\ y_5 = 0.91$

(b)

$\int_0^{10} bzdz = 1$

$b = 0.02$

$$F_Z(z) = \begin{cases} 0 & z < 0 \\ 0.01z^2 & 0 \le z \le 10 \\ 1 & z > 10 \end{cases}$$

$z = 10\sqrt{F_X(x)}$

2.

(a)

$\int_0^4 b_1 ydy = 1,\ b_1 = \frac{1}{8}$

$\int_0^4 b_2(4-y)dy = 1,\ b_2 = \frac{1}{8}$

$$f_Y(y) = f_{Y|X}(y|0)f_X(0) + f_{Y|X}(y|2)f_X(2) = \begin{cases} \frac{3}{10} - \frac{1}{40}y & 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$f_{X|Y}(0|2.5) = \frac{f_{Y|X}(2.5|0)f_X(0)}{f_Y(2.5)} = 0.5263$

$f_{X|Y}(2|2.5) = \frac{f_{Y|X}(2.5|2)f_X(2)}{f_Y(2.5)} = 0.4737$

$\hat{x} = 0$

(b)

$\mathrm{E}(x|y) = \sum 0 \cdot f_{X|Y}(0|2.5) + 2 \cdot f_{X|Y}(2|2.5) = 0.9474$

3.

(a)

$\eta_y(t) = \int_0^{10} e^{\mathbf{x}t} \frac{1}{10} dx = \frac{e^{10\,t}-1}{10\,t}$

(b)

$\mathbf{y} = e^{\mathbf{x}t},\ \mathbf{x} = \frac{\ln y}{t}$

$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{x}} = te^{\mathbf{x}t} = t\mathbf{y}$

$f_Y(y,t) = \frac{1}{yt} f_X(\frac{\ln y}{t})U(y) = \begin{cases} \frac{1}{10yt} & 1 \leq y \leq e^{10t} \\ 0 & \text{otherwise} \end{cases}$

(c)

$R_{yy}(t_1,t_2) = E(e^{xt_1}e^{xt_2}) = E(e^{x(t_1+t_2)}) = E(y(t_1+t_2)) = \eta_y(t_1+t_2)$

(d)

$R_{yy}(t_1,t_2) = \eta_y(t_1+t_2) = \frac{e^{10(t_1+t_2)}-1}{10(t_1+t_2)}$

4.

(a)

(i)

$S_{xx}(\omega) = \sum_{m=-1}^{1} R_{xx}[m]e^{-j\omega m} = e^{j\omega} + 2 + e^{-j\omega} = 2 + 2cos\omega$

$\mathbf{S}_{xx}(z) = \sum_{m=-1}^{1} R_{xx}[m]z^{-m} = z + 2 + z^{-1}$

(ii)

$R_{yy}[m] = E(x[n]x[n+m]) = 5R_{xx}[m] - 2R_{xx}[m+1] - 2R_{xx}[m-1]$

$$R_{yy}[m] = \begin{cases} 6 & m = 0 \\ 1 & m = 1, -1 \\ -2 & m = 2, -2 \\ 0 & \text{otherwise} \end{cases}$$

(iii)

$S_{yy}(\omega) = -2e^{j2\omega} + e^{j\omega} + 6 + e^{-j\omega} - 2e^{-j2\omega} = 6 + 2cos\omega - 4cos2\omega$

$\mathbf{S}_{yy}(z) = -2z^2 + z + 6 + z^{-1} - 2z^{-2}$

(b)

(i)

State 0:negative

State 1:asymptomatic

State 2:symptomatic

$$\Pi = \begin{bmatrix} 1-p-q & p & q \\ 3p & 1-3p & 0 \\ 5q & 1-5q & 0 \end{bmatrix}$$

(ii)

$P(\Pi - I) = 0,\ p_0 + p_1 + p_2 = 1$

$p_0 = \frac{15}{23},\ p_1 = \frac{5}{23},\ p_2 = \frac{3}{23}$

$35000 \times \frac{3}{23} = 4562$

(iii)

$N \geq 6.25$

2021-2022

1.

(a)

$p_{Y|X}(y|x):$

$a + 0.2 + 0.3 = 1,\ a = 0.5$

$b + 0.1 + 0.2 = 1,\ b = 0.7$

$c + 0.5 + 0.4 = 1,\ c = 0.1$

$p_{X|Y}(x|y):$

$a + 0.1 + 0.5 = 1,\ a = 0.4$

$b + 0.2 + 0.4 = 1,\ b = 0.4$

$c + 0.3 + 0.2 = 1,\ c = 0.5$

$p_{X,Y}(x, y):$

???

(b)

$d + 0.3 + 0.4 = 1,\ d = 0.3$

Table 1 can't be PMF of $p_{X,Y}(x, y)$, as sum of all values are great than 1.

If Table 1 is PMF of $p_{X|Y}(x|y)$:

$p_X(X = 4) = 0.4 \times 0.3 + 0.1 \times 0.4 + 0.5 \times 0.3 = 0.31$

$p_X(X = 5) = 0.2 \times 0.3 + 0.4 \times 0.4 + 0.4 \times 0.3 = 0.34$

$p_X(X = 6) = 0.3 \times 0.3 + 0.2 \times 0.4 + 0.5 \times 0.3 = 0.32$

$p_X(X = 4) + p_X(X = 5) + p_X(X = 6) \neq 1$

So Table 1 is not PMF of $p_{X|Y}(x|y)$.

Therefore, Table 1 is PMF of $p_{Y|X}(y|x)$.

(c)

$$\begin{cases} 0.5p_X(4) + 0.1p_X(5) + 0.5p_X(6) = 0.3 \\ 0.2p_X(4) + 0.7p_X(5) + 0.4p_X(6) = 0.4 \\ 0.3p_X(4) + 0.2p_X(5) + 0.1p_X(6) = 0.3 \end{cases}$$

$$\begin{cases} p_X(4) = 0.75 \\ p_X(5) = 0.5 \\ p_X(6) = -0.25 \end{cases}$$

???

2.

(a)

$f_Y(y) = \int_0^{+\infty} f_{X,Y}(x,y)dx = \lambda y$

$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} ye^{-xy} & x \geq 0,\ 1 \leq y \leq 5 \\ 0 & \text{otherwise} \end{cases}$

$\mathrm{E}(X|Y) = \int_0^{+\infty} x f_{X|Y}(x|y)dx = \frac{1}{y}$

$\mathrm{E}(X^2|Y) = \int_0^{+\infty} x^2 f_{X|Y}(x|y)dx = \frac{2}{y^2}$

$\mathrm{MMSE} = \mathrm{Var}(X|Y) = \mathrm{E}(X^2|Y) - \mathrm{E}^2(X|Y) = \frac{1}{y^2}$

$\hat{x} = \mathrm{E}(X|Y = 2) = \frac{1}{2}$

(b)

$\int_1^5 f_Y(y) = 12\lambda = 1,\ \lambda = \frac{1}{12}$

$\hat{y} = E(Y) = \int_1^5 y \cdot \frac{1}{12} y dy = \frac{31}{9}$

$E(Y^2) = \int_1^5 y^2 \cdot \frac{1}{12} y dy = 13$

$\mathrm{MSE} = \mathrm{Var}(Y) = E(Y^2) - E^2(Y) = 1.1358$

3.

(a)

(i)

$p = (1-p)^{m-1}p$

(ii)

$p = \sum_{m=1}^{n-1}[(1-p)^{m-1}p][(1-p)^{n-m-1}p] = (n-1)(1-p)^{n-2}p^2$

(b)

(i)

$E(\mathbf{y}(t)) = \mu + 2$

(ii)

$R_{yy}(\tau) = E(\mathbf{y}(t)\mathbf{y}(t+\tau)) = E[(\mathbf{x}(t)+2)(\mathbf{x}(t+\tau)+2)]$

$R_{yy}(\tau) = E[\mathbf{x}(t)\mathbf{x}(t+\tau) + 2\mathbf{x}(t) + 2\mathbf{x}(t+\tau) + 4] = e^{-|\tau|} + 4\mu + 4$

(iii)

$\mu = -1$

4.

(a)

(i)

$$E(\mathbf{x}(t)) = \int_0^3 \frac{1}{3}\mathbf{x}(\tau)d\tau = \begin{cases} \frac{7000000000}{3}t & 0 < t < 3 \\ 0 & t \leq 0 \\ 7000000000 & t \geq 3 \end{cases}$$

(ii)

$$E(a[\mathbf{x}(t)]^b) = a\int_0^3 \frac{1}{3}\mathbf{x}^b(\tau)d\tau = \begin{cases} a\frac{7000000000^b}{3}t & 0 < t < 3 \\ 0 & t \leq 0 \\ a7000000000^b & t \geq 3 \end{cases}$$

(b)

5.

(a)

(i)

$$\Pi = \begin{bmatrix} 0.6 + 0.4e^{-\tau} & 0.4 - 0.4e^{-\tau} \\ 0.4 - 0.6e^{-\tau} & 0.6 + 0.6e^{-\tau} \end{bmatrix}$$

(ii)

$$\Pi'(0^+) = \begin{bmatrix} -0.4 & 0.4 \\ 0.6 & -0.6 \end{bmatrix}$$

(iii)

$$P(0.693) = P(0)\Pi(0.693) = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}$$

(b)