EE7401 Homework 2

Instructions: Please answer all questions. Please write your final solutions in the boxes shown below, then scan (if needed) and upload via NTULearn/Assignments/Homework2 by 23:59, 7 November 2022. This is an open book assignment. Please do not copy from others or let others copy your solution. Penalties may apply for late submission.

Q1) *Dilemma in lion city:* Two students living in *Singa*pore, Dil Singh and Emma Lyon, can't decide whether to take 179 or 179A to JP (Jurong Point). They know the following facts.

- 179A takes 5 minutes less than 179 to reach JP.
- 179A arrival is a Poisson arrival with arrival rate λ. In other words, $\mathbf{x}(t) = \mathbf{n}(0, t)$ is a Poisson process, where $\mathbf{n}(0, t)$ is the number of arrivals of 179A between time 0 and t minute. $\mathbf{n}(0, t)$ is k with probability $P\{\mathbf{n}(0, t) = k\} = e^{-\lambda t}(\lambda t)^k / k!$ where t is in minutes.

1.1) Dil, Emma are waiting at the bus stop, and a 179 arrives at time $t = 0$ minute. Dil prefers to take the 179. Emma, however, prefers to wait for the 179A.

(1.1.a) Find $\hat{t}$, the time of arrival of the 179A, such that 179A (Emma's preference) reaches JP at the same time as the 179 (Dil's preference).

$\hat{t} =$ $\boxed{5}$

(2 marks)

(1.1.b) If the 179A arrives after $\hat{t}$, then Emma's preference reaches JP late. Find the probability P_{Emma}^{late} that Emma's preference reaches late, that is, the probability that no 179A arrives within time $\hat{t}$, or $P\{\mathbf{x}(\hat{t}) = 0\}$, in terms of λ.

$P_{Emma}^{late} =$ $\boxed{e^{-5\lambda}}$

(2 marks)

(1.1.c) Find the probability P_{Dil}^{late} that Dil's preference reaches late, in terms of λ.

$P_{Dil}^{late} =$ $\boxed{1 - e^{-5\lambda}}$

(3 marks)

(1.1.d) Find a condition on the arrival rate λ such that Emma's preference is better, that is, $P_{Emma}^{late} < P_{Dil}^{late}$:

$\boxed{\lambda > \frac{\ln 2}{5}}$

(3 marks)

1.2) Dil, Emma decides to meet at the bus stop and take a bus to JP. Dil comes to the bus stop at time $t = 0$ and waits for Emma. p 179A buses arrive between 0 and 10 minutes. Emma finally comes to the bus stop at time $t = 10$ and finds a visibly angry Dil waiting. Dil argues that the probability (say, P_1) of at least one 179A arriving in the next 5 minutes, given that he has already seen p 179A arriving in the past 10 minutes, is less than the usual probability of at least one 179A arriving in the next 5 minutes (say, P_2). Emma feels otherwise. Dil and Emma are comparing P_1 = the conditional probability of greater than p arrivals in 0 to 15 minutes given there were p arrivals in 0 to 10 minutes = $P\{\mathbf{x}(15) > p \mid \mathbf{x}(10) = p\}$, with P_2 = the probability of at least 1 arrival in 10 to 15 minutes = $P\{\mathbf{x}(15) - \mathbf{x}(10) \geq 1\}$.

(1.2.a) Compare $P\{\mathbf{x}(15) > p \mid \mathbf{x}(10) = p\}$ and $P\{\mathbf{x}(15) - \mathbf{x}(10) \geq 1\}$ to see if Dil is correct ($P_1 < P_2$), or if Emma is correct (P_1 not less than P_2). Show your steps.

This process is a homogeneous Poisson process, so it has independent increments in time interval $(0, 10)$ and $(10, 15)$.

So, $P_1 = P\{\mathbf{x}(15) > p \mid \mathbf{x}(10) = p\} = P\{\mathbf{n}(10, 15) \geq 1\}$

And, $P_2 = P\{\mathbf{x}(15) - \mathbf{x}(10) \geq 1\} = P\{\mathbf{n}(10, 15) \geq 1\}$

Therefore, $P_1 = P_2$

(10 marks)

(1.2.b) Evaluate $P_2 = P\{\mathbf{x}(15) - \mathbf{x}(10) \geq 1\}$ for $\lambda = 0.2$ arrival/minute.

$$P\{\mathbf{x}(15) - \mathbf{x}(10) \geq 1\} = \boxed{1 - e^{-1}}$$

(5 marks)

Q2) *Fussy investor:* Mr. I. N. Vestor invests in the stock market. The unpredictable nature of the stock market means it is prudent to model the value of Mr. Vestor's investment at any given time by a wide sense stationary real process $\mathbf{x}(t)$. Let $R_{xx}(\tau)$ be the autocorrelation of $\mathbf{x}(t)$. Let $S_{xx}(\omega)$ be the power spectrum of $\mathbf{x}(t)$.

For obvious reasons, Vestor is always monitoring the rate of change of this value using an i-phone app. This rate of change $\mathbf{x}'(t)$ is essentially the output process when $\mathbf{x}(t)$ is passed through a differentiator. Recall that since differentiation in time domain is equivalent to multiplying by $j\omega$ in frequency domain, a differentiator is a linear system with frequency response $H(\omega) = j\omega$.

(2.a) Once a bull run (rate of change is positive, or the value increases) starts, Vestor asks you for how long will his good time continue. In effect, he is asking how the rate of change is correlated over time. Determine $R_{x'x'}(\tau)$, the autocorrelation of $\mathbf{x}'(t)$, in terms of $R_{xx}(\tau)$:

$$R_{x'x'}(\tau) = \boxed{-\frac{d^2 R_{xx}(\tau)}{d\tau^2}}$$

(8 marks)

(2.b) Find $S_{x'x'}(\omega)$, the power spectrum of $\mathbf{x}'(t)$, in terms of $S_{xx}(\omega)$:

$$S_{x'x'}(\omega) = \boxed{\omega^2 S_{xx}(\omega)}$$

(7 marks)

(2.c) After several years of bull run, I. N. Vestor accumulated a sizable wealth but was far from happy. When you asked him why, he said a larger value of $\mathbf{x}(t)$ would imply a negative rate of change. In order to convince him that this is not so, show that for a given t_1, the random variables $\mathbf{x}(t_1)$ and $\mathbf{x}'(t_1)$ are orthogonal.

Supposing that differentiator denoting system L:

$L[\mathbf{x}(t)] = \mathbf{x}'(t)$

So we have: $R_{xx'}(t_1, t_2) = L_2[R_{xx}(t_1, t_2)]$

Let $t_2 = t_1$

So we have: $\tau = t_2 - t_1 = 0$

Because $\mathbf{x}(t)$ is a wide-sense stationary process

We have: $R(t_1, t_2) = R(0) = constant$

So we have: $R_{xx'}(t_1, t_2) = L_2[R_{xx}(t_1, t_2)] = 0$

So $\mathbf{x}(t_1)$ and $\mathbf{x}'(t_1)$ are orthogonal.

(5 marks)

(2.d) Unfortunately Mr. Vestor never took EE7401 or any similar course, and failed to appreciate orthogonality. However, he understands what correlated means (recall that you already explained correlation to him in part 1.a). To convince him, show that for a given t_1, the random variables $\mathbf{x}(t_1)$ and $\mathbf{x}'(t_1)$ are uncorrelated.

Because $\mathbf{x}(t)$ is a wide-sense stationary process

We have: $E\{\mathbf{x}(t)\} = constant$

So we have: $E\{\mathbf{x}'(t)\} = L_2[E\{\mathbf{x}(t)\}] = 0$

$C_{xx'}(t_1, t_1) = R_{xx'}(t_1, t_1) - \eta_x(t_1)\eta_{x'}(t_1) = 0 - 0 = 0$

So $\mathbf{x}(t_1)$ and $\mathbf{x}'(t_1)$ are uncorrelated.

(5 marks)

Q3) *When did you start your simulations:* Consider the simulation of a first order auto-regressive, or AR(1), process. A wide-sense stationary real white noise process $\mathbf{x}[n]$ with autocorrelation $R_{xx}[m] = 5\delta[m]$ is passed through the AR(1) filter with $a = 0.5$, such that the output is $\mathbf{y}[n] = \mathbf{x}[n] + 0.5\mathbf{y}[n-1]$.

$$\mathbf{x}[n] \longrightarrow \boxed{H(z) = \frac{1}{1 - 0.5z^{-1}}} \longrightarrow \mathbf{y}[n]$$

3.1) (3.1.a) Assume the above filter operates at all times. Then $\mathbf{y}[n]$ is jointly wide-sense stationary. Find the cross-correlation between $\mathbf{x}[n]$ and $\mathbf{y}[n]$, $R_{xy}[m] = E\{\mathbf{x}[n_1]\mathbf{y}[n_2]\}$, where $m = n_1 - n_2$. [Hint: There are many ways to solve this problem. You may use the frequency domain or the z domain approach, finding $S_{xy}(\omega)$ or $\mathbf{S}_{xy}(z)$ first, and then taking the inverse Fourier or z transform. You may express $\mathbf{y}[n]$ as a convolution of $\mathbf{x}[n]$ with $h[n]$ (impulse response), multiply the equation by $\mathbf{x}[n_1]$, and take the expectation.]

$$R_{xy}[m] = \begin{cases} \boxed{0} & \text{for } m > 0 \\ \boxed{5 \cdot 2^{m}} & \text{for } m \leq 0 \end{cases}$$

(4 marks)

(3.1.b) Find the autocorrelation of the AR(1) process, $R_{yy}[m]$.

$$R_{yy}[m] = \boxed{\frac{20}{3} \cdot 2^{-|m|}}$$

(2 marks)

$$\mathbf{x}[n] \xrightarrow[n=0]{} \boxed{H(z) = \frac{1}{1 - 0.5z^{-1}}} \longrightarrow \mathbf{y}[n]$$

3.2) Part (3.1) results are true if the simulation is started at $n = -\infty$. However, real-life simulations start at finite times. Consider a real-life simulation of the same AR(1) process as above, where the simulation starts at $n = 0$. $\mathbf{x}[n]$ is still the same wide-sense stationary with the same autocorrelation. However, the AR(1) filter starts at $n = 0$ (meaning, there was no filter before $n = 0$, or the output was zero). Therefore, the output becomes $\mathbf{y}[n] = \begin{cases} 0 & n < 0 \\ \mathbf{x}[n] + 0.5\mathbf{y}[n-1] & n \geq 0 \end{cases}$. Note that $\mathbf{y}[n]$ is no longer stationary. Therefore, the auto/cross-correlations involving $\mathbf{y}[n]$ no longer depend on the time difference m but depend on both times, like $R_{yy}[n_1, n_2]$. As a result, the power spectrums of $\mathbf{y}[n]$ do not exist, and the power spectrum based approaches can no longer be used to find the auto/cross-correlations. The time-domain approach may still be used.

(3.2.a) Express $\mathbf{y}[0]$ using only the input $\mathbf{x}[0]$. Express $\mathbf{y}[1]$ as a sum of only input terms of the form $\mathbf{x}[k]$. There should not be any past output term such as $\mathbf{y}[n-1]$. Continuing as above, express $\mathbf{y}[n_2]$ for any $n_2 \geq 0$ as a sum of only input terms of the form $\mathbf{x}[k]$. There should not be any past output term such as $\mathbf{y}[n_2 - 1]$.

$$\mathbf{y}[n_2] = \boxed{\sum_{k=0}^{n_2} 0.5^{n_2-k}\mathbf{x}[k]} \quad \text{for } n_2 \geq 0 \;\ldots\ldots\; \text{eq.(1)}$$

(3 marks)

(3.2.b) Find the cross-correlation between $\mathbf{x}[n]$ and $\mathbf{y}[n]$, $R_{xy}[n_1, n_2] = E\{\mathbf{x}[n_1]\mathbf{y}[n_2]\}$, for all $n_1, n_2 \geq 0$ by multiplying eq.(1) by $\mathbf{x}[n_1]$, and taking the expectation of both sides. [Hint: You should find 2 cases: write answers in 2 left boxes, and write the cases in 2 right boxes.]

$$R_{xy}[n_1, n_2] = \begin{cases} \boxed{5 \cdot 2^{-n_2}} & \text{for } n_1 \geq 0, n_2 \geq 0, \boxed{n_1 < n_2} \\ \boxed{} & \text{for } n_1 \geq 0, n_2 \geq 0, \boxed{} \end{cases}$$

(4 marks)

(3.2.c) Extend your result of (3.2.b) to all possible n_1, n_2 values. [Hint: You should find 2 cases: write answers in the 2 left boxes, and write the condition of the primary case in the right box.]

$$R_{xy}[n_1, n_2] = \begin{cases} \boxed{5 \cdot 2^{-n_2}} & \text{for } n_1 \geq 0, n_2 \geq 0, \boxed{n_1 < n_2} \\ \boxed{0} & \text{otherwise} \end{cases}$$

(2 marks)

(3.2.d) Find the autocorrelation of the AR(1) process, $R_{yy}[n_1, n_2]$ for all possible n_1, n_2 values. [Hint: Replace both $\mathbf{y}[n_1]$ and $\mathbf{y}[n_2]$ in $E\{\mathbf{y}[n_1]\mathbf{y}[n_2]\}$ by eq.(1) twice, then take the expectation of this double summation. You should find 3 cases (it is possible to combine 2 primary cases into a single case): write answers in the 3 left boxes, and write the conditions of 2 primary cases in the 2 right boxes.]

$$R_{yy}[n_1, n_2] = \begin{cases} \boxed{\frac{5}{3} \cdot 2^{n_1-n_2}(4 - 2^{-2n_1})} & \text{for } n_1 \geq 0, n_2 \geq 0, \boxed{n_1 < n_2} \\ \boxed{} & \text{for } n_1 \geq 0, n_2 \geq 0, \boxed{} \\ \boxed{0} & \text{otherwise} \end{cases}$$

(6 marks)

(3.2.e) For all non-negative n_1 and n_2, is $R_{xy}[n_1, n_2]$ of (3.2.c) equal to $R_{xy}[m]$ of (3.1.a)? If not, when are they equal?
For all non-negative n_1 and n_2, is $R_{yy}[n_1, n_2]$ of (3.2.d) equal to $R_{yy}[m]$ of (3.1.b)? If not, when are they equal?

For all non-negative n_1 and n_2, $R_{xy}[n_1, n_2]$ of (3.2.c) is not equal to $R_{xy}[m]$ of (3.1.a).

For all non-negative n_1 and n_2, $R_{xy}[n_1, n_2]$ of (3.2.d) is not equal to $R_{xy}[m]$ of (3.1.b).

(4 marks)

Q4) *Bandlimited process does change with time:* In lecture we have upper bounded the change in value of a bandlimited process over a small time τ. Here we obtain a lower bound (and a new upper bound). First, two intermediate results are obtained as below.

4.1) Assume that the time is bounded by $|\tau| < (\pi/\sigma)$ and that the frequency is bounded by $|\omega| \leq \sigma$. Then, using the fact that if $0 < \varphi < (\pi/2)$, then $(2\varphi/\pi) < \sin\varphi < \varphi$, find a lower bound and an upper bound on $sin^2(\omega\tau/2)$:

$$\boxed{\frac{\omega^2\tau^2}{\pi^2}} \leq sin^2(\omega\tau/2) \leq \boxed{\frac{\omega^2\tau^2}{4}}$$

(4 marks)

4.2) Let the power spectrum of $\mathbf{x}(t)$ be $S_{xx}(\omega)$. If $\mathbf{x}(t)$ is passed through a differentiator (frequency response $H(\omega) = j\omega$), then you have already found $S_{x'x'}(\omega)$, the power spectrum of the output $\mathbf{x}'(t)$, in terms of $S_{xx}(\omega)$, in Q(2.b). Copy this $S_{x'x'}(\omega)$ below to obtain the autocorrelation of the output $\mathbf{x}'(t)$ as

$$R_{x'x'}(\tau) = \int_{-\infty}^{\infty} \boxed{\omega^2} S_{xx}(\omega)e^{j\omega\tau}d\omega/2\pi$$

Now, putting $\tau = 0$, the average power of the output $\mathbf{x}'(t)$ is:

$$E\{|\mathbf{x}'(t)|^2\} = \int_{-\infty}^{\infty} \boxed{\omega^2} S_{xx}(\omega)d\omega/2\pi$$

(2 marks)

4.3) (4.3.a) Express the expectation of the square of the change in $\mathbf{x}(t)$ over time τ, $E\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\}$, using its autocorrelation:

$$E\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\} = 2R_{xx}(\boxed{0}) - R_{xx}(\boxed{\tau}) - R_{xx}(\boxed{\tau})$$

(5 marks)

(4.3.b) Use (4.3.a), $R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}\frac{d\omega}{2\pi}$, and $1 - cos\theta = 2sin^2(\theta/2)$, to get:

$$E\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\} = \int_{-\infty}^{\infty} S_{xx}(\omega) \boxed{4\sin^2\frac{\omega\tau}{2}} d\omega/2\pi$$

(4 marks)

(4.3.c) Let us say the integral you obtained in part (4.3.b) is $\int_{-\infty}^{\infty} S_{xx}(\omega)g(\omega)d\omega/2\pi$ for some function $g(\omega)$ that you wrote inside the box. Now, since $\mathbf{x}(t)$ is bandlimited, its power spectrum $S_{xx}(\omega) = 0$ for $|\omega| > \sigma$. Therefore, this integral's limits may be changed, $E\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\} = \int_{-\infty}^{\infty} S_{xx}(\omega)g(\omega)d\omega/2\pi = \int_{-\sigma}^{\sigma} S_{xx}(\omega)g(\omega)d\omega/2\pi$. Apply the lower and upper bounds on $sin^2(\omega\tau/2)$ from part (4.1) to obtain the lower and upper bounds on the expectation:

$$\int_{-\sigma}^{\sigma} S_{xx}(\omega)\,\omega^2 \boxed{4\frac{\tau^2}{\pi^2}} \frac{d\omega}{2\pi} \le E\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\} \le \int_{-\sigma}^{\sigma} S_{xx}(\omega)\,\omega^2 \boxed{\tau^2} \frac{d\omega}{2\pi}$$

(4 marks)

(4.3.d) The difficulty is that, unlike in the lecture, we can't replace ω by σ in the lower bound. Therefore, we need to evaluate $\int_{-\sigma}^{\sigma} S_{xx}(\omega)\omega^2 d\omega/2\pi$. This has already been done in part (4.2) using the differentiated process $\mathbf{x}'(t)$. Use the result of (4.2) on the lower and upper bounds of (4.3.c) to obtain the final result:

$$\boxed{4\sigma^2\frac{\tau^2}{\pi^2}} E\{|\mathbf{x}'(t)|^2\} \le E\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)|^2\} \le \boxed{\sigma^2\tau^2} E\{|\mathbf{x}'(t)|^2\}$$

(2 marks)

(4.3.e) Is the upper bound of (4.3.d) smaller, or larger, than the upper bound found in the class?

As $E|\mathbf{x}'(t)|^2 = R(0)$, the upper bound of (4.3.d) is equal to upper bound found in class.

(4 marks)