EE7401 Probability and Random Processes

Homework 1 Solutions

1. (35 marks) (Estimation vs. Detection) Let

$$X = \begin{cases} 1 & \text{with probability } \frac{1}{2}, \\ -1 & \text{with probability } \frac{1}{2}, \end{cases}$$

and the noise $Z \sim \mathrm{Unif}\,(-2, 2)$ be independent random variables. Their sum $Y = X + Z$ is observed.

(a) Find the conditional pmf $p_{X|Y}(x \mid y)$. Find the MMSE of X given Y and its MSE.

(b) Suppose we use a decoder to decide whether $X = 1$ or -1. Using the pmf $p_{X|Y}(x \mid y)$ found in part (a), find the MAP decoder and its probability of error. Compare the MAP decoder's MSE to the minimum MSE.

Solution.

(a) Since X and Z are independent,

$$\begin{aligned} f_Y(y) &= p_X(y) * f_Z(y) \\ &= \begin{cases} \frac{1}{4} & -1 \le y \le 1 \\ \frac{1}{8} & -3 \le y < -1,\ 1 < y \le 3 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

and

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{4} & -2 \le y - x \le 2 \\ 0 & \text{elsewhere.} \end{cases}$$

Now the aposteriori pmf of X

$$\begin{aligned} p_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \frac{f_{Y|X}(y|x)p_X(x)}{f_Y(y)} \end{aligned}$$

is non-zero for $x = \pm 1$ only:

$$p_{X|Y}(1|y) = \begin{cases} 0 & -3 \leq y < -1 \\ \frac{1}{2} & -1 \leq y \leq 1 \\ 1 & 1 < y \leq 3 \end{cases}$$

$$p_{X|Y}(-1|y) = \begin{cases} 1 & -3 \leq y < -1 \\ \frac{1}{2} & -1 \leq y \leq 1 \\ 0 & 1 < y \leq 3. \end{cases}$$

Thus the best MSE estimate

$$\begin{aligned} g(Y) &= E[X|Y] \\ &= \sum_{x \in \mathcal{X}} x \cdot p_{X|Y}(x|Y) \\ &= \begin{cases} -1 & -3 \leq Y < -1 \\ 0 & -1 \leq Y \leq 1 \\ 1 & 1 < Y \leq 3 \end{cases} \end{aligned}$$

and its MSE is

$$\begin{aligned} E_Y \mathrm{Var}(X|Y) &= E_Y \left(E_X[X^2|Y] - (E_X[X|Y])^2 \right) \\ &= E(1 - (g(Y))^2) \\ &= 1 - E(g(Y))^2 \\ &= 1 - \int_{-\infty}^{\infty} (g(y))^2 f_Y(y)\, dy \\ &= 1 - \left(\int_{-3}^{-1} 1.\frac{1}{8}\, dy + \int_{-1}^{1} 0.\frac{1}{4}\, dy + \int_{1}^{3} 1.\frac{1}{8}\, dy \right) \\ &= 1 - \int_{-3}^{-1} \frac{1}{8}\, dy - \int_{1}^{3} \frac{1}{8}\, dy \\ &= 1 - \frac{1}{4} - \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

(b) The optimal decoder is given by the MAP rule. Using $p_{X|Y}(x|y)$ found above, the MAP rule reduces to

$$D(y) = \begin{cases} -1 & -3 \le y < -1 \\ -1 \text{ or } 1 & -1 \le y \le 1 \\ 1 & 1 < y \le 3. \end{cases}$$

Since the output can be either value for the center range of Y, a symmetrical decoder is sufficient:

$$D(y) = \begin{cases} 1 & y \ge 0 \\ -1 & y < 0. \end{cases}$$

The probability of decoding error

$$\begin{aligned} P\{D(Y) \ne X\} &= P\{X = 1, Y < 0\} + P\{X = -1, Y \ge 0\} \\ &= P\{Y < 0|X = 1\} \cdot p_X(1) + P\{Y \ge 0|X = -1\} \cdot p_X(-1) \\ &= \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} \\ &= \frac{1}{4}. \end{aligned}$$

Considering the the decoder as an estimator, its MSE

$$\begin{aligned} E(D(Y) - X)^2 &= \frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 2^2 \\ &= 1. \end{aligned}$$

Thus the MSE of the optimal decoder is twice that of the minimum mean square error estimator.

2. (45 marks) Given a Gaussian random vector $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

(a) What are the pdfs of

i. X_1,

ii. $X_2 + X_3$,

iii. $2X_1 + X_2 + X_3$, and

iv. X_3 given (X_1, X_2)?

(b) What is $\mathbb{P}(2X_1 + X_2 + X_3 < 0)$?

(c) What is the joint pdf of $\mathbf{Y} = \mathbf{A}\mathbf{X}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}?$$

Solution.

(a) i. The marginal pdfs of a jointly gaussian pdf are gaussian. Therefore

$$X_1 \sim \mathcal{N}(1, 1).$$

ii. Since

$$\sigma_{23} = 0$$

and X_2 and X_3 are jointly gaussian, they are also independent. Thus the means and variances sum:

$$\begin{aligned} \mu &= \mu_2 + \mu_3 \\ &= 5 + 2 \\ &= 7 \end{aligned}$$

$$\begin{aligned} \sigma^2 &= \sigma_2^2 + \sigma_3^2 \\ &= 4 + 9 \\ &= 13. \end{aligned}$$

The sum of two jointly gaussian random variables is also gaussian, and therefore

$$X_2 + X_3 \sim \mathcal{N}(7, 13).$$

iii. Since this is a linear combination of GRV's, i.e.

$$2X_1 + X_2 + X_3 = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

it is also a GRV with mean

$$\begin{aligned} \mu &= \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \\ &= 9 \end{aligned}$$

and variance

$$\begin{aligned} \sigma^2 &= \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \\ &= 21. \end{aligned}$$

Thus

$$2X_1 + X_2 + X_3 \sim \mathcal{N}(9, 21).$$

iv. Since X_3 and X_1 are jointly gaussian and uncorrelated, they are therefore independent, and similarly X_3 and X_2 are independent also. Thus the conditional pdf of $X_3|X_1, X_2$ is the same as the pdf of X_3:

$$X_3|(X_1, X_2) \sim \mathcal{N}(2, 9).$$

(b) Let

$$Y = 2X_1 + X_2 + X_3.$$

Now

$$Y \sim \mathcal{N}(9, 21)$$

so

$$\begin{aligned} P\{2X_1 + X_2 + X_3 < 0\} &= P\{Y < 0\} \\ &= \Phi\left(\frac{0 - \mu_Y}{\sigma_y}\right) \\ &= \Phi\left(\frac{-9}{\sqrt{21}}\right) \\ &= \Phi(-1.96) \\ &= Q(1.96) \\ &= 2.48 \times 10^{-2}. \end{aligned}$$

(c)

$$\begin{aligned}
\mu_{\mathbf{Y}} &= A\mu_{\mathbf{X}} \\
&= \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 9 \\ -2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\Sigma_{\mathbf{Y}} &= A\Sigma_{\mathbf{X}}A^T \\
&= \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 21 & 6 \\ 6 & 12 \end{bmatrix}.
\end{aligned}$$

Thus

$$\mathbf{Y} \sim \mathcal{N}\left(\begin{bmatrix} 9 \\ -2 \end{bmatrix}, \begin{bmatrix} 21 & 6 \\ 6 & 12 \end{bmatrix} \right).$$