Solutions to 2D-DCT Exercise

(a)

$$S_{uv} = \alpha(u)\alpha(v)\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} s_{ij}\cos\frac{(2i+1)u\pi}{2N}\cos\frac{(2j+1)v\pi}{2N}$$

$$= \alpha(u)\sum_{i=0}^{N-1}\cos\frac{(2i+1)u\pi}{2N}\underbrace{\left\{\alpha(v)\sum_{j=0}^{N-1} s_{ij}\cos\frac{(2j+1)v\pi}{2N}\right\}}_{F_{iv}}$$

$$= \alpha(u)\sum_{i=0}^{N-1} F_{iv}\cos\frac{(2i+1)u\pi}{2N}$$

Hence, 2D-DCT (S_{uv}) can be computed using a two-stage 1D-DCT:

First Stage: $\quad F_{iv} = \alpha(v)\sum_{j=0}^{N-1} s_{ij}\cos\frac{(2j+1)v\pi}{2N} \qquad i, v = 0, 1, \cdots, N-1$

Second Stage: $\quad S_{uv} = \alpha(u)\sum_{i=0}^{N-1} F_{iv}\cos\frac{(2i+1)u\pi}{2N} \qquad u, v = 0, 1, \cdots, N-1$

<u>For 4×4 2D-DCT:</u>

First Stage: $\quad F_{iv} = \alpha(v)\sum_{j=0}^{3} s_{ij}\cos\frac{(2j+1)v\pi}{8} \qquad i, v = 0, 1, 2, 3$

Second Stage: $\quad S_{uv} = \alpha(u)\sum_{i=0}^{3} F_{iv}\cos\frac{(2i+1)u\pi}{8} \qquad u, v = 0, 1, 2, 3$

Given image:

$$\mathbf{A} = \begin{bmatrix} 10 & 0 & 0 & 10 \\ 10 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>First Stage</u>

First row:

$$s_{0j} = \begin{bmatrix} 10 & 0 & 0 & 10 \end{bmatrix}$$

$$F_{0v} = \alpha(v)\left\{10\cos\frac{v\pi}{8} + 10\cos\frac{7v\pi}{8}\right\} = \alpha(v)\left\{10\times[1+(-1)^v]\cos\frac{v\pi}{8}\right\}$$

$$F_{00} = \frac{1}{2}\{20\} = 10$$

$$F_{01} = 0$$

$$F_{02} = \frac{1}{\sqrt{2}}\left\{10\times 2\times\cos\frac{2\pi}{8}\right\} = 10$$

$$F_{03} = 0$$

$$F_{0v} = \begin{bmatrix} 10 & 0 & 10 & 0 \end{bmatrix}$$

Second row same as first row

$F_{1v} = \begin{bmatrix} 10 & 0 & 10 & 0 \end{bmatrix}$

Third and fourth row are equal to zero.

$F_{2v} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

$F_{3v} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

Hence, after the first stage 1D-DCT, we obtain:

$$F_{iv} = \begin{bmatrix} 10 & 0 & 10 & 0 \\ 10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Second Stage

First column:

$$S_{u0} = \alpha(u)\left\{10\cos\frac{u\pi}{8} + 10\cos\frac{3u\pi}{8}\right\}$$

$$F_{i0} = \begin{bmatrix} 10 \\ 10 \\ 0 \\ 0 \end{bmatrix},$$

$$S_{00} = \frac{1}{2}\{10 + 10\} = 10$$

$$S_{10} = \frac{1}{\sqrt{2}}\left\{10\cos\frac{\pi}{8} + 10\cos\frac{3\pi}{8}\right\} = 9.2388 \quad , \qquad \therefore S_{u0} = \begin{bmatrix} 10 \\ 9.2388 \\ 0 \\ -3.8268 \end{bmatrix}$$

$$S_{20} = \frac{1}{\sqrt{2}}\left\{10\cos\frac{2\pi}{8} + 10\cos\frac{6\pi}{8}\right\} = 0$$

$$S_{30} = \frac{1}{\sqrt{2}}\left\{10\cos\frac{3\pi}{8} + 10\cos\frac{9\pi}{8}\right\} = -3.8268$$

The third column is the same as the first column:

$$S_{u1} = S_{u3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad S_{u2} = \begin{bmatrix} 10 \\ 9.2388 \\ 0 \\ -3.8268 \end{bmatrix}$$

Therefore, the 2D-DCT is

$$S_{uv} = \begin{bmatrix} 10 & 0 & 10 & 0 \\ 9.2388 & 0 & 9.2388 & 0 \\ 0 & 0 & 0 & 0 \\ -3.8268 & 0 & -3.8268 & 0 \end{bmatrix}$$

Alternatively, we can use 2D DCT matrix computation.

$$\mathbf{T}[i, j] = \begin{cases} \frac{1}{\sqrt{N}}, & \text{if } i = 0 \\ \sqrt{\frac{2}{N}} \cdot \cos\frac{(2j+1)\cdot i\pi}{2N}, & \text{if } i > 0 \end{cases}$$

DCT matrix is computed using the formula above.

T =

0.5000	0.5000	0.5000	0.5000
0.6533	0.2706	-0.2706	-0.6533
0.5000	-0.5000	-0.5000	0.5000
0.2706	-0.6533	0.6533	-0.2706

The 2D DCT is computed using:

$$F(u, v) = \mathbf{T} \cdot f(i, j) \cdot \mathbf{T}^T$$

$$F = \begin{bmatrix} 10 & 0 & 10 & 0 \\ 9.2388 & 0 & 9.2388 & 0 \\ 0 & 0 & 0 & 0 \\ -3.8268 & 0 & -3.8268 & 0 \end{bmatrix}$$