MATH2923: Advance Analysis

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Abstract

Thank you for stopping by to read this. These are notes collated from lectures and tutorials as I took this course.

Contents

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1. Axioms and Bounds

1.1 Foundations and Axioms

1.1.1 Axioms

Axiom 1.1 *(Peano Axioms for $\mathbb{N}$). The set of natural numbers is a set $\mathbb{N}$ with a distinguished element 0 and a successor function S that is*

1. *S: $\mathbb{N} \to \mathbb{N}/0$ is injective.*
2. *If $0 \in N$ and $S(n) \in N\ \forall n \in \mathbb{N}$, then $N = \mathbb{N}$.*

Definition 1.2 *(Rational Numbers). The set of rational numbers $\mathbb{Q}$ is the set $\{\frac{p}{q}\text{: } p,\ q \in \mathbb{Z}\}$.*

Definition 1.3 *(Group). A group (G, .) is a set with a binary operator . (such as the addition operator) defined on it. It satisfies the 4 group axioms:*

1. *Closure: For all a, b in G, a . b $\in$ G.*
2. *Associative: For all a, b in G, (a . b) . c = a . (b . c).*
3. *Identity Element: There is an identity element e such that e . a = a.*
4. *Inverse Element: For all a in G, there exist an inverse element a' such that a.a' = e.*

Remark 1.4 *A group is referred to as abelian if a . b = b . a*

Remark 1.5 *A monoid is a group without the property of inverse elements.*

Definition 1.6 *(Integers $\mathbb{Z}$). The set of integers $\mathbb{Z}$ with the addition (+) and multiplication (.) binary operator defined on it ($\mathbb{Z}$, +, .) is an abelian group under addition ($\mathbb{Z}$, +) and a monoid under multiplication ($\mathbb{Z}$, .). The multiplication operator is distributive with respect to addition.*

Definition 1.7 *(Field). A field is a set with 2 operations addition and multiplication such that the it is an abelian group under addition, an abelian group under multiplication for nonzero elements, and the multiplication is distributive over addition.*

Remark 1.8 *The set of rational numbers $\mathbb{Q}$ and real numbers $\mathbb{R}$ forms a field.*

We now seek to derive the properties of $\mathbb{R}$ from the 3 axioms: field, order, and completeness.

Axiom 1.9 *(Field Axioms). The field axioms are the properties of a field as defined earlier.*

We now establish the notion of *order* in a set.

Axiom 1.10 *(Order Axioms). There exists a relation $<$ on $\mathbb{R}$ with the following 4 properties.*

1. $x < y$ *iff* $0 < y - x$;
2. *If* $0 < x, y$ *then* $0 < x + y$;
3. *If* $0 < x, y$ *then* $0 < xy$;
4. *For every* $x \in \mathbb{R}$, *either* $0 < -x$, $0 < x$, *or* $x = 0$.

Remark 1.11 *Note that we only used the $<$ operator since that is the only thing that has been defined so far.*

Remark 1.12 *What we have now constructed are **partially ordered sets**. Note that this is different to **total order sets**, which are sets that require every element of the set to be comparable to one another.*

1.1.2 Boundedness

Before we define the *completeness* axiom, we need to define some other things first.

Definition 1.13 *(Upper and Lower Bounds). The set $A \subseteq \mathbb{R}$ is bounded from above (below) if there exists $m \in \mathbb{R}$ such that $x \leq m$ ($x \geq m$) for all $x \in A$. Every m that satisfies this property is known as an upper bound (lower bound) of A.*

Definition 1.14 *(Bounded). A set is bounded if it is bounded from above **and** bounded from below.*

Definition 1.15 *(Supremum). Let $A \subseteq \mathbb{R}$ be a subset bounded from above and is **non-empty**. We say that $M \in \mathbb{R}$ is a supremum of A if it satisfies the **two** conditions that*

(i) *M is an upper bound of A;*

(ii) *$M \leq m$ for every upper bound m of A.*

Definition 1.16 *(Infimum). Let $A \subseteq \mathbb{R}$ be a subset bounded from below and is **non-empty**. We say that $N \in \mathbb{R}$ is a infimum of A if it satisfies the **two** conditions that*

(i) *N is an lower bound of A;*

(ii) *$N \geq n$ for every lower bound n of A.*

Remark 1.17 *(Least Upper Bound and Greatest Lower Bound). We refer to the supremum (infimum) as the least upper bound (greatest lower bound). An unbounded set has a supremum of ∞ and an infimum of $-\infty$.*

Remark 1.18 *The supremum (infimum) of a set is **unique**.*

Definition 1.19 *(Maximum and Minimum). A supremum (infimum) that is also* ***in*** *the set A is known as a maximum (minimum).*

Remark 1.20 *A maximum is* ***always*** *a supremum but a supremum is not always a maximum.*

Collecting all this, we can now state the third axiom.

Axiom 1.21 *(Least Upper Bound Axiom). Every non-empty set $A \subseteq \mathbb{R}$ which is bounded from above* ***admits*** *a supremum.*

Additionally, we could phrase it in terms of the **infimum**.

Axiom 1.22 *(Greatest Lower Bound Axiom). Every non-empty set $A \subseteq \mathbb{R}$ which is bounded from below* ***admits*** *an infimum.*

Remark 1.23 *The only field that satisfies all 3 axioms (field, order, and completeness) is the field of real numbers.*

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2. Basic Properties of Real Numbers

2.1.3 More things on Supremum and Infimum

Lemma 2.24 *Let $A \neq \emptyset$ be a subset of $\mathbb{R}$. The following assertions are true.*

(i) If $t <$ sup A, then there exists $a \in A$ such that $t < a$;

(ii) If $t >$ inf A, then there exists $a \in A$ such that $t > a$;

Proof: i) Suppose there did not exist an element a ∈ A. Then that means t ≤ a for all a ∈ A. Therefore t is a smaller upper bound than the supremum, which is a contradiction. Hence, there exists an element a ∈ A that is greater than t.

2) Suppose there did not exist an element a ∈ A. Then that means t ≥ a for all a ∈ A. Therefore t is a larger lower bound that the infimum, which is a contradiction. Hence, there exists an element a ∈ A that is lower than t. ∎

From the greatest lower bound axiom, it guarantees the existence of supremums for sets bounded above. We can use this axiom to show that this is equivalent to the existence of infimums for sets bounded by below.

Proposition 2.25 *The following 3 statements are equivalent:*

(i) sup A exists for every non-empty $A \subseteq \mathbb{R}$ bounded from above.

(ii) inf A exists for every non-empty $A \subseteq \mathbb{R}$ bounded from below.

(iii) If A, $B \subseteq \mathbb{R}$ are sets such that $a \leq b$ for all $a \in A$ and $b \in B$. Then there exists $c \in \mathbb{R}$ with $a \leq c \leq b$ for all $a \in A$ and $b \in B$.

Proof:(Sketch).

$(1 \rightarrow 2)$

Construct a set of lower bounds for $B \subseteq \mathbb{R}$. Denote this set as set A. Then show that the supremum of set A is the infimum of set B.

$(2 \rightarrow 3)$

Let $a \leq b$ for all $a \in A, b \in B$. Then define $c := infB$. We then need to show c is $a \leq c \leq b$ for all $a \in A$ and $b \in B$.

$(3 \rightarrow 1)$

Construct a set of upper bounds for A and denote it as B. Then we claim that c is the supremum of A. Hence, show that c is an upper bound of A and it is the smallest element out of the set of upper bounds in B. ∎

Proposition 2.26 *Let $A, B \subseteq \mathbb{R}$ such that $A \subseteq B$. This implies that $supA \leq supB$.*

Proof: An upperbound of B is also an upperbound of A. This is true for all upperbounds of B. Hence, by definition of a supremum, $supA \leq supB$. ∎

Proposition 2.27 *Let $A \subseteq \mathbb{R}$ and non-empty. Then*

$$supA = -inf(-A).$$

Proof: Look at the infimum of -A. We have that $M \coloneqq inf(-A)$. Therefore, $-M \leq -x$ for all x in A. Hence, we have that $M \geq x$ for all x in A, therefore M is an upperbound of A. Now for any lower bounds $-m \leq -x$ for all $x \in A$, we have that $m \geq x$ for all $x \in A$ but $m \geq M$ and therefore M is the supremum. ∎

Proposition 2.28 *Let $A + B = \{x + y : x \in A, y \in B\}$. Then*

$$sup(A + B) = supA + supB.$$

Proposition 2.29 *Let $s \coloneqq supA \notin A$, then there exists a strictly increasing $x_n \in A$ with $sup_{n \in \mathbb{N}} x_n = s$.*

2.1.4 Basic Properties of the Real Numbers

Proposition 2.30 *(Archimedian Property of $\mathbb{N}$). For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$.*

Proof: (Sketch).

To prove this, use a proof by contradiction. ∎

Corollary 2.31 *For every $\epsilon > 0$, there exist a $n \in \mathbb{N}$ such that:*

$$0 < \frac{1}{n} < \epsilon$$

Proof: Using the Archimedian property by setting x as $\frac{1}{\epsilon}$,

$$\frac{1}{\epsilon} < n$$

We then rearrange and the proof is complete. ∎

We can look at the distribution of the rational and irrational numbers in $\mathbb{R}$.

Proposition 2.32 *(Density of rational and irrational numbers). Suppose that a, $b \in \mathbb{R}$ with $a < b$. Then:*

(i) There exists $q \in \mathbb{Q}$ such that $a < q < b$ (The rationals are dense in $\mathbb{R}$);

(ii) There exists $i \in \mathbb{I}$ with $a < i < b$ (The irrationals $\mathbb{I}$ are dense in $\mathbb{R}$).

We can now show that $\mathbb{Q}$ does not satisfy the least upper bound property.

Proposition 2.33 *$x^2 = 2$ has no solution in $\mathbb{Q}$.*

Proof: (By contradiction). Assume $x^2 = 2$ has a solution, say $\frac{p}{q}$ where p, q $\in \mathbb{Z}$ and p, q have no common factors. So: $(\frac{p}{q})^2 = 2$, hence $p^2 = 2q^2$. Then p^2 is **even** since it is divisible by 2. Then p is even because if p was odd then p^2 would be odd. So p = 2m for some $m \in \mathbb{Z}$, $\rightarrow p^2 = 4m^2 = 2q^2$. Then $2m^2 = q^2$. Hence, q is even. Then this contradicts our assumption that p and q have no common factors. Therefore, $x^2 = 2$ must not have a solution. ∎

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3. Complex Numbers and Inner Product

3.1.5 Introduction to Complex Numbers

For every $z \in \mathbb{C}$, there exists $a, b \in \mathbb{R}$ such that:

$$z = a + bi$$

where we define:

(i) a is known as the **real** part of z;

(ii) b is the **imaginary** part of z;

(iii) $\bar{z} = a - bi$ is the **complex conjugate** of z;

(iv) $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2} \geq 0$ is the **modulus** of z.

The modulus of z is also known as the distance of the point (a,b) from the origin (0, 0) whilst the complex conjugate is the reflection of z on the real axis.

Proposition 3.34 *The compex numbers has the following properties:*

(i) $Re\ z = \frac{1}{2}(z + \bar{z})$ *and* $Im\ z = \frac{1}{2i}(z - \bar{z})$*;*

(ii) $\overline{z+w} = \bar{z}+\bar{w}$, $\overline{zw} = \bar{z}\bar{w}$, $\overline{\frac{1}{z}} = \frac{1}{\bar{z}}$*;*

(iii) $|zw| = |z||w|$ *and* $|z + w| \leq |z| + |w|$.

3.1.6 Higher Dimensional Spaces

We define $\mathbb{K}$ to be either $\mathbb{R}$ or $\mathbb{C}$. For $n \in \mathbb{N}$, we denote by $\mathbb{K}^n$ the set:

$$\mathbb{K}^n = \{\mathbf{x} = (x_1, x_2, ..., x_n) : x_j \in \mathbb{K}, \forall j = 1, ..., n\}$$

$(\mathbb{K}^n, +, .)$ forms a vector space over $\mathbb{K}$ for every $N \in \mathbb{N}$. Elements in $\mathbb{K}$ are known as *vectors* whilst elements in $\mathbb{K}$ are known as *scalars*. We can define the 2 operations of *addition by vectors* and *multiplication by scalars*.

Definition 3.35 *(Vector Space). A set V with two operations vector addition (+) and scalar multiplication (.) is called a vector space over* $\mathbb{K}$ *if the following properties hold:*

(i) (V, +) is an ***abelian group****;*

(ii) Scalar multiplication is compatible with field multiplication. $\alpha(\beta v) = \alpha\beta(v)$ *for all* $\beta, \alpha \in \mathbb{K}$ *and every* $v \in V$*;*

(iii) *Multiplicative identity for the field is also the multiplicative identity for scalar multiplication.* $1.v = v$ *for all* $v \in V$;

(iv) *Scalar multiplication is* ***distributive*** *with respect to both vector addition* $[\alpha(v + w) = \alpha v + \alpha w]$ *and field addition* $[(\alpha + \beta)v = \alpha v + \beta v]$.

3.1.7 Inner Product and Euclidean Norm

Definition 3.36 *(Inner Product). Given a vector space* V *over the field* F*, the* ***inner product*** *is a map* $<.,.>: V \times V \to \mathbb{R}$. *Given vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{K}^N$*, we define their inner product by:*

$$x.y = \sum_{i=1}^{N} x_i \bar{y}_i.$$

Remark 3.37 *The inner product associates every pair of vectors with a scalar known as the inner product of the vectors. This scalar helps us understand the notion of similarity between two objects in a vector space. The complex conjugate is necessary for* y *or else this leads to inconsistencies with the inner product axioms, in particular, the positive definiteness property would not be satisfied.*

Remark 3.38 *If* $\mathbb{K} = \mathbb{R}$*, then this definition coincides with the definition of the* ***dot product*** *as the complex conjugate of a real number is itself.*

We now look at the properties of the inner product in $\mathbb{K}^N$.

Proposition 3.39 *Let* x, y, $z \in \mathbb{K}^N$ *and* $\alpha, \beta \in \mathbb{K}$. *Then*

(i) *(Positive Definiteness).* $x \cdot x \geq 0$ *with equality iff* $x = 0$;

(ii) *(Conjugate Symmetry).* $x \cdot y = \overline{y \cdot x}$;

(iii) *(Linearity in the first argument).* $(\alpha x + \beta y) \cdot z = \alpha(x \cdot z) + \beta(y \cdot z)$.

Remark 3.40 *When we take an inner product with a vector and itself, the inner product is always a non-negative real number. Additionally, the order of inner product matters if we aren't operating on* $\mathbb{R}$.

Furthermore, we note that the dot product in $\mathbb{K}^N$ is conjugate linear in the second argument. That is

$$< x, \alpha y + \beta z >= \bar{\alpha} < x, y > + \bar{\beta} < x, z >$$

for all $\alpha, \beta \in \mathbb{K}$ and for all $x, y, z \in \mathbb{K}^N$.

When we augment a vector space with an inner product, we will then have an **inner product space**.

Definition 3.41 *(Euclidean Norm). For every* $x \in \mathbb{K}^N$*, we can define:*

$$||x|| = \sqrt{< x, x >} = \sqrt{\sum_{j=1}^{N} |x_j|^2} \geq 0.$$

Remark 3.42 $||x||= 0$ *iff* $< x, x > = 0$ *iff* $x = \mathbf{x}$, *where* x *is a vector in* $\mathbb{K}^N$.

Remark 3.43 *The norm is a function that assigns a strictly positive length to each vector in a vector space except for the zero vector. Hence, we take the inner product of a vector x with itself. The norm motivates the idea of the* ***length*** *of vectors. Here, we see the norm being expressed as the square root of the inner product of a vector and itself.*

Remark 3.44 *For complex number z, the norm of z, which we denote as* $||z||$, *is simply* $\sqrt{z\bar{z}}$. *If we had a vector of complex numbers* $||\boldsymbol{v}||= ||(v_1, ..., v_n)||= \sqrt{|v_1|^2+|v_2|^2+ ... + |v_n|^2} = \sqrt{\sum_{k=1}^{n}|v_k|^2}$.

Definition 3.45 *(Cauchy-Schwarz Inequality). For every* x, $y \in \mathbb{K}^N$, *we have that:*

$$|< x, y > | \leq \; ||x||.||y||$$

with equality if and only if x *and* y *are linearly dependent, that is, there exists* $t \in \mathbb{K}$ *such that* $x = ty$ *or* $y = tx$.

Theorem 3.46 *(Euclidean Norms). Let* x, $y \in \mathbb{K}^N$ *and* $\alpha \in \mathbb{K}$. *Then*

(i) (Positive Definiteness) $||x||\geq 0$ *with equality iff* $x = 0$;

(ii) (Positive Homogeneity) $||\alpha x||= |\alpha|||x||$;

(iii) (Triangle Inequality) $||x + y||\leq ||x||+||y||$

Definition 3.47 *(Reversed Triangle Inequality). If* x, $y \in \mathbb{K}^N$ *then*

$$\Big|\Big|\Big| ||x||-||y|| \Big|\Big|\Big| \leq ||x - y||.$$

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4. Sequences and Limits

4.2 Sequences and Limits

4.2.1 Sequences

Definition 4.48 *(Sequence). Let X be a nonempty set. A sequence is an enumerated collection of objects in which repitition is allowed. A sequence in a set X is a function x: $\mathbb{N} \to X$. We write $x_n := x(n)\ \forall n \in \mathbb{N}$.*

Remark 4.49 *The domain of the sequence is either the first n natural numbers for a sequence of finite length n or the set of natural numbers for infinite sequences.*

Remark 4.50 *An example of a sequence $x_n = n^2$, where it maps a number from natural number and applies the function n^2 to it.*

We denote sequences at $\{x_n\}_{n\geq 0}$.

Recall that $\mathbb{R}$ has order whilst $\mathbb{C}$ does not. Hence, we have different definitions of convergence for the 2 fields.

Definition 4.51 *(Bounded Sequence in $\mathbb{R}$). The sequence $\{x_n\}_{n\geq 0}$ is* ***bounded from above*** *if there exists $M \in \mathbb{R}$ such that*

$$x_n \leq M \quad \forall\, n \in \mathbb{N}.$$

The sequence $\{x_n\}_{n\geq 0}$ is ***bounded from below*** *if there exists $m \in \mathbb{R}$ such that*

$$x_n \geq M \quad \forall\, n \in \mathbb{N}.$$

A sequence is ***bounded*** *if it is bounded from above and below. In other words, there exists a positive constant C such that*

$$|x_n| \leq C, \quad \forall\, n \in \mathbb{N}.$$

In other words, there exists $C > 0$ such that

$$-C \leq x_n \leq C, \quad \forall\, n \in \mathbb{N}.$$

Remark 4.52 *The reason that it is the same value C for the upper and lower bound is that we are just simply picking an arbitrary C whereby it bounds the modulus of the sequence. Hence, we make C large enough to bound such a sequence and we choose this as it lets us prove things in a more simpler manner.*

Definition 4.53 *(Bounded Sequence in $\mathbb{K}$). A sequence $\{x_n\}_{n\geq 0}$ in $\mathbb{K}^N$ is called* ***bounded*** *if $\{||x_n||\}_{n\geq 0}$ is bounded as a sequence in $\mathbb{R}$, that is, there exists $C > 0$ such that*

$$||x_n|| \leq C, \quad \forall n \in \mathbb{N}.$$

Remark 4.54 *A sequence in $\mathbb{K}^N$ means that every element x in $\{x_n\}_{n\geq 0}$ is a N-dimensional vector. We then take the norm of each vector to get a real number scalar. We now have a sequence of norms. We then can apply the same definition as a bounded sequence in $\mathbb{C}$.*

Definition 4.55 *(Sequence Convergence for $\mathbb{R}$). A sequence $\{x_n\}_{n\geq 0}$ in $\mathbb{R}$ is said to* ***converge*** *to $x \in \mathbb{R}$ if for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ suhc that*

$$|x_n - x| < \epsilon, \text{ for all } n \geq n_\epsilon.$$

This is equivalent to

$$x - \epsilon < x_n < x + \epsilon.$$

Definition 4.56 *(Sequence Convergence for $\mathbb{K}^N$). A sequence $\{x_n\}_{n\geq 0}$ in $\mathbb{K}^N$ is said to* ***converge*** *to $x \in \mathbb{K}^N$ if for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that*

$$||x_n - x|| < \epsilon, \text{ for all } n \geq n_\epsilon.$$

In other words, $\{||x_n - x||\}_{n\geq 0}$ converges to 0, where x is the limit. The sequence of norms converges to 0.

If $||x_n|| \to 0$, then $x_n \to 0$. When $\{x_n\} \in \mathbb{K}^N$ converges to $x \in \mathbb{K}^N$, we write $x_n \to x$ as $n \to \infty$ or that $\lim_{n\to\infty} x_n = x$.

Proposition 4.57 *(Relating types of convergence). A sequence of $\{x_n\}$ in $\mathbb{K}^N$ that converges to $x \in \mathbb{K}^N$ is said to converge in its sequence of norms $||x_n|| \to ||x||$ as $n \to \infty$ AND the sequence $\{x_n\}$ is a* ***bounded sequence*** *in $\mathbb{K}^N$.*

Proof: To prove the convergence of norms, use the reverse triangle inequality combined with the definition of convergence. We use reverse triangle inequality to get that

$$|\ ||x_n|| - ||x||\ | \leq ||x_n - x||.$$

As $x_n \to x$ by assumption, $||x_n - x|| \to 0$ as $n \to \infty$. Hence, we have that

$$|\ ||x_n|| - ||x||\ | \leq ||x_n - x|| \leq \epsilon$$

for all $n \geq n_\epsilon$. Therefore, we have the desired term of

$$|\ ||x_n|| - ||x||\ | \leq \epsilon.$$

To prove boundedness, we use the fact that the sequence of the norms is bounded for some rank M by a constant. We can then bound the sequence by the max of the constant M or the any of the first M-1 elements.

$$||x_n|| \leq C\ \forall n \in \mathbb{N}$$

is what we aim for. We use the convergence of sequence of norms and $\epsilon = 1$ so that

$$|\ ||x_n|| - ||x||\ | \leq ||x_n - x|| \leq 1$$

Hence we have that

$$||x_n|| \leq ||x|| + 1\ \forall n \geq n_1.$$

Hence, we define our constant to be

$$C = max\{1 + ||x||, ||x_0||, ||x_1||, ..., ||x_{n-1}||\}.$$

■

Remark 4.58 *If the sequence of norms converges to 0, then the sequence converges to 0. This does not hold for any other element other than 0. Hence, the convergence of the sequence of norms does not imply the convergence of the sequence.*

Remark 4.59 *If a norm of sequences is convergent, the sequence is also bounded. In general, to show a sequence is bounded, we show that is bounded from a number onwards and hence there will only be a finite number of elements that we have not bound yet which we can then easily bound.*

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5. Limit Laws

5.2.2 Consequences of limits

Proposition 5.60 *Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that $x_n \to x$ as $n \to \infty$ for some $x \in \mathbb{R}$.*

Let $a, b \in \mathbb{R}$ satisfy $a < x < b$. Then there exists $m \in \mathbb{N}$ such that $a < x_n < b$ for all $n \geq m$.

Proof: (Sketch). Recall that $x_n - \epsilon \leq x_n \leq x_n + \epsilon$, We then choose an ϵ such that $(x - \epsilon, x + \epsilon) \subset (a, b)$ for all $m \geq n_\epsilon$. ■

What this proposition states is that we can always find an open interval with the sequence in the the open interval after a certain rank.

Lemma 5.61 *(Special Case of Squeeze Law). Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be sequence in $\mathbb{R}$ such that $b_n \to 0$ as $n \to \infty$.*

1. *Assume that there exists $M \in \mathbb{N}$ such that*

$$0 \leq a_n \leq b_n, \ \forall n \geq M$$

   Then $a_n \to 0$ as $n \to \infty$.

2. *Assume that $\{a_n\}_{n\geq 0}$ is **at least** bounded. Then*

$$a_n b_n \to 0 \text{ as } n \to \infty$$

Proof:(Sketch).

(i)

Use the fact that $b_n < \epsilon$ for all $\epsilon > 0$. As b_n bounds a_n, then $a_n < \epsilon$ and hence $a_n \to 0$.

(ii) As $a_n \to C$, we have that

$$|a_n b_n| \leq C|b_n|.$$

Hence, we know that $C|b_n| \to 0$ and hence $|a_n b_n| \to 0$. ■

5.2.3 Limit Laws

Theorem 5.62 *(Limit Laws). Let (x_n), (y_n) be a sequence in $\mathbb{K}^N$ such that $x_n \to x$ and $y_n \to y$ in $\mathbb{K}^N$. Let (α_n) be a sequence in $\mathbb{K}$ such that $\alpha_n \to \alpha$. Then*

1. *$x_n + y_n \to x + y$ as $n \to \infty$;*

2. *$\alpha_n x_n \to \alpha x$ as $n \to \infty$;*

3. *$< x_n, y_n > \to < x, y >$ as $n \to \infty$;*

4. *If $\alpha \neq 0$, there exists $M \in \mathbb{N}$ such that $|\alpha_n| > \frac{|\alpha|}{2}$, for all $n \geq M$ and $\frac{1}{\alpha_n} \to \frac{1}{\alpha}$ as $n \to \infty$.*

We finally note that sequences in $\mathbb{K}^N$ converges if and only if they converge component wise. Furthermore, a sequence in $\mathbb{C}$ converges if and only if their real and imaginary parts converge. Note that we can split up the series of a complex number into the series of its real and imaginary parts.

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6. Monotone Sequences

6.2.4 Squeeze Law

Theorem 6.63 *(Squeeze Law). Let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, and $\{x_n\}_{n\geq 0}$ be sequences in $\mathbb{R}$. Suppose that there exists $M \in \mathbb{N}$ such that*

$$a_n \leq x_n \leq b_n, \ \forall n \geq M$$

If $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ converge to the same number $x \in \mathbb{R}$, then $x_n \to x$ as $n \to \infty$.

Proof: (Sketch). Look at $0 = a_n - a_n \leq x_n - a_n \leq b_n - a_n$. We arrive at that $x_n - a_n \to 0$ as $n \to \infty$. Then

$$x_n = (x_n - a_n) + a_n \to 0 + x = x.$$

■

Proposition 6.64 *(Preservation of inequalities). Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be sequences in $\mathbb{R}$. Assume that there exists $m \in \mathbb{N}$ such that*

$$a_n \leq b_n \ \ \forall \, n \geq m.$$

If $a_n \to a$ as $n \to \infty$ and $b_n \to b$ as $n \to \infty$, with $a, b \in \mathbb{R}$, then we have that

$$a \leq b.$$

Proof: (Sketch). Prove by contradiction. Define a new sequence $c_n \coloneqq b_n - a_n$, which is positive for all $n \geq m$. Now assume that $a > b$. You should arrive at $c_n < 0$ for all $n \geq m$, which is a contradiction. ■

Remark 6.65 *Note that preservation of **equalities** does not necessarily hold. In other words, if $a_n < b_n$ for all $n \in \mathbb{N}$, we **cannot** conclude that $a < b$.*

Definition 6.66 *(Monotone Sequences). Let (a_n) be a sequence in $\mathbb{R}$. We say that (a_n) is*

(i) Increasing (increasing eventually) if $a_n \leq a_{n+1}$ for all $n \geq 0$ (if there exists $m \in \mathbb{N}$ such that $a_n \leq a_{n+1}$).

(ii) Strictly increasing (strictly increasing eventually) if $a_n \leq a_{n+1}$ for all $n \geq 0$ (if there exists $m \in \mathbb{N}$ such that $a_n \leq a_{n+1}$ for all $n \geq m$).

We define decreasing (decreasing eventually) and strictly decreasing (strictly decreasing eventually) by reversing the inequalities.

Definition 6.67 *(Monotone). A sequence $(a_n) \in \mathbb{R}$ is called **monotone** if it is either increasing or decreasing.*

Remark 6.68 *Note that all these definitions do not apply to vectors as they do not have order.*

Theorem 6.69 *(Monotone Convergence Theorem). Every sequence in $\mathbb{R}$ which is monotone and bounded must converge.*

(i) If $\{a_n\}_{n\geq 0}$ is increasing and bounded (from above), then

$$\lim_{n\to\infty} a_n = sup_{n\in\mathbb{N}} a_n;$$

(ii) If $\{a_n\}_{n\geq 0}$ is decreasing and bounded (from below), then

$$\lim_{n\to\infty} b_n = inf_{n\in\mathbb{N}} b_n.$$

So if we have a sequence that is monotone and bounded, then it's limit is its supremum/infimum.

Proof: We prove the case for increasing sequences.

Construct a set $A = \{a_n : n \in \mathbb{N}\}$. We know that $A \subseteq \mathbb{R}$ and $A \neq \emptyset$. By least upper bound axiom, this set has a supremum as it is bounded from above as the sequence is bounded. Then, using the definition of the supremum, there exists $n_\epsilon \in \mathbb{N}$ such that

$$a - \epsilon < a_{n_\epsilon}.$$

Furthermore, we can expand this to

$$a - \epsilon < a_{n_\epsilon} < a < a + \epsilon.$$

Then, for all $n \geq n_\epsilon$, and the fact that a_n is increasing sequence

$$a - \epsilon < a_{n_\epsilon} < a_n < a < a + \epsilon.$$

Hence, we have that

$$a - \epsilon < a_n < a + \epsilon,$$

for all $n \geq n_\epsilon$.

$$|a_n - a| < \epsilon,$$

for all $n \geq n_\epsilon$. Hence, the sequence $\{a_n\}_{n\geq 0} \to a$. ■

Remark 6.70 *Any sequence in $\mathbb{R}$ which is* ***bounded*** *and* ***eventually*** *monotone must converge. Here, $\lim_{n\to\infty} a_n = sup_{n\geq m} a_n$ where $m \in \mathbb{N}$ such that $a_n \leq a_{n+1}$ where for all $n \geq m$.*

Note that we take the rank from where the sequence is eventually increasing here. Furthermore, a sequence that is convergent is definitely bounded but not necessarily monotonic.

Now we look at sequences that are not necessarily monotonic.

Definition 6.71 *(Divergence). Let $\{a_n\}_{n\geq 0}$ be a sequence in $\mathbb{R}$.*

(i) Then $\lim_{n\to\infty} a_n = +\infty$ if for all $M \in \mathbb{R}$, there exists $n_m \in \mathbb{N}$ such that $a_n > M$ for all $n \geq n_M$.

(ii) Then $\lim_{n\to\infty} b_n = -\infty$ if for all $M \in \mathbb{R}$, there exists $n_m \in \mathbb{N}$ such that $b_n < M$ for all $n \geq n_M$.

Note that when talking about divergence of a sequence, we discuss that we can always find a rank where every sequence is greater/lesser than any element in $\mathbb{R}$, rather than the other way around! Furthermore, note that talking about converging to ∞ or having a limit of ∞ doesn't quite make much sense as they are not proper limits.

Proposition 6.72 *(Unbounded sequences).*

*(i) Let $\{a_n\}_{n\geq 0}$ be a sequence in $\mathbb{R}$. If $\{a_n\}$ is **increasing** and unbounded from above, then $\lim_{n\to\infty} a_n = +\infty$.*

*(ii) Let $\{b_n\}_{n\geq 0}$ be a sequence in $\mathbb{R}$. If $\{a_n\}$ is **decreasing** and unbounded from below, then $\lim_{n\to\infty} b_n = -\infty$.*

Proof: The proof follows easily from the definition of unboundedness.

(i)

Since $\{a_n\}_{n\geq 0}$ is unbounded from above, for all $M \in \mathbb{R}$, there exists $n_m \in \mathbb{N}$ such that $a_{n_m} > M$. Therefore, from the fact that $\{a_n\}$ is increasing

$$M < a_{n_m} \leq a_n$$

for all $n \geq n_m$. Hence,

$$\lim_{n\to\infty} = +\infty.$$

(ii)

Consider $\{-a_n\}_{n\geq 0}$ and apply proof from part (i). ■

We can now look at our first convergence criterion for sequences, which can be used to derive basic limits.

Proposition 6.73 *(Cauchy Ratio Test for Sequences). Suppose that $a_n \geq 0$ for all $n \in \mathbb{N}$ and that $a_{n+1}/a_n \to L$ as $n \to \infty$. Then 3 cases can occur*

(i) If $L \in [0, 1)$, then the sequence converges and $a_n \to 0$;

(ii) If $L \geq 1$, then the sequence diverges and $a_n \to \infty$;

(iii) If $L = 1$ or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent sequence that satisfy this case.

Proof: First we prove part (i).

(i) We assume that the limit L $\in [0, 1)$. Then, $\{a_n\}$ is eventually decreasing since $0 \leq L < 1$, then there exists a rank $M \in \mathbb{N}$ such that $0 \leq \frac{a_{n+1}}{a_n} < 1$ for all $n \geq M$. This sequence is obviously bounded from below by 0 and hence, $a_n \to a \in [0, \infty)$ as $n \to \infty$. We show that $a = 0$ by contradiction. Assume that $a \neq 0$, then we know that

$$\lim_{n\to\infty} a_n = \lim_{n+1\to\infty} a_{n+1} = a.$$

Therefore, looking at our limit L

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{a}{a} = 1.$$

However, we assumed that $L < 1$ and hence we have a contradiction. Therefore a $= 1$.

(ii) We now assume that the limit $L \in (1, \infty]$. Since $1 < L \leq \infty$, then we have that from a rank M, $1 < \frac{a_{n+1}}{a_n} \leq \infty$ for all $n > M$. Hence, $a_n < a_{n+1}$, so the sequence is eventually increasing. If the sequence is unbounded from above, we know that the $\lim_{n\to\infty} a_n = +\infty$, which is what we want. We prove that the sequence is unbounded from above by contradiction. Let us assume that the sequence $\{a_n\}$ is bounded from above and hence, by the least upper bound axiom, $\{a_n\}$ converges. Therefore, $\lim_{n\to\infty} a_n \to a \in (1, \infty]$. However, for the limit $L := \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{a}{a} = 1$. This is a contradiction as we assumed that $L > 1$. Hence, the sequence $\{a_n\}$ is unbounded from above and $\lim_{n\to\infty} a_n = \infty$.

(iii) We see that both case (i) and case (ii) can arrive if L = 1. Hence, the test is inconclusive. ■

This test provides sufficient conditions for a sequence of positive numbers to converge to 0 or diverge to $+\infty$. These are the only 2 responses you should have.

We can use this to show some examples of growth.

Proposition 6.74 *(Exponential Decay Dominates Polynomial Growth). Let $k \in \mathbb{N}$ be fixed and $a \geq 0$. Then*

$$\lim_{n\to\infty} n^k a^n \begin{cases} 0 & a \in [0,1) \\ \infty & a \geq 1 \end{cases}$$

Proposition 6.75 *(Exponential Growth). Let $k \in \mathbb{N}$ be fixed and $a \in \mathbb{C}$. We have*

(i) $n^k a^n \to 0$ if and only if $|a|< 1$;

(ii) $a^n \to 0$ if $|a|< 1$, $a^n \to 1$ if $a = 1$, and a^n diverges if $|a_n|\geq 1$ and $a \neq 1$.

Proposition 6.76 *(Factorial Growth Dominates Exponential Growth). Let $a \in \mathbb{C}$. Then*

$$\lim_{n\to\infty} \frac{a_n}{n!} = 0.$$

Remark 6.77 *Note that the Cauchy ratio test, can also be applied to **series**.*

(i) If $L \in [0, 1)$, then $\sum_{k=0}^{\infty} a_n$ is absolutely convergent;

(ii) If $L \geq 1$, then $\sum_{k=0}^{\infty} a_n$ is divergent;

(iii) If $L = 1$ or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.

Basic limits to remember.

1. $a^n \to 0$ iff $|a|< 1$;
2. a^n diverges if $|a|\geq 1$ and $a \neq 1$;
3. $n^k a^n \to 0$ if $k \geq 1$ and $|a|< 1$;
4. $\frac{a^n}{n!} \to 0$ for all $a \in \mathbb{C}$;
5. $\sqrt[n]{n} \to 1$;
6. $\sqrt[n]{a} \to 1$ for all $a > 0$;
7. $(1 + \frac{1}{n})^n \to e$.

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7. Existence of the nth root.

7.2.5 Arithmetic-Geometric Mean Inequality

Definition 7.78 *(Arithmetic Mean). Given a sequence of positive numbers $x_1, ..., x_n$, we have the arithmetic mean being*

$$\frac{x_1 + x_2 + ... + x_n}{n}.$$

Definition 7.79 *(Geometric Mean). Given a sequence of positive numbers $x_1, ..., x_n$, we have the geometric mean being*

$$\sqrt{x_1 x_2 ... x_n}.$$

The geometric mean is always smaller than the arithmetic mean.

Lemma 7.80 *(Bernoulli's Inequality). For $x \geq -1$ and $n \geq 1$, we have*

$$(1 + x)^n \geq 1 + nx$$

with strict inequality unless n = 1 or x = 0.

Proof: (Sketch). Prove by induction to show this. ■

Theorem 7.81 *(Arithmetic-Geometric Mean Inequality). Suppose that $n \in \mathbb{N}$ and that $x_1, ..., x_n \geq 0$. Then*

$$x_1 x_2 ... x_n \leq (\frac{x_1 + x_2 + ... + x_n}{n})^n$$

with strict inequality unless $x_1 = x_2 = ... = x_n$.

7.2.6 Existence of nth root

Theorem 7.82 *For every $a > 0$, and every $n \in \mathbb{N}$ 0 there exists an unique* ***positive*** *number α such that $\alpha^n = a$. We denote $\alpha = a^{\frac{1}{n}}$. In other words, every positive number has an unique nth root.*

Proof:(Sketch).

(Proof of uniqueness). Look at a strictly increasing function $f(x) = x^n - a$. Any strictly monotone function is injective. Hence f(x) = 0 has **at most** one root.

(Proof of existence). We need to construct a sequence of positive numbers which is monotonically decreasing s.t. the limit is α. We use Newton's method which gives us

$$\begin{cases} x_0 & a \\ x_{k+1} & \frac{1}{n}[(n-1)x_k + \frac{a}{x_k^{n-1}}] \text{ for all k } \geq 0 \end{cases}$$

This sequence is strictly positive and monotone decreasing. Hence, the sequence must converge and resultantly, its limit is α, where $\alpha^n = a$. ■

Proposition 7.83 *(Fundamental Limits).*

1. $\lim_{n\to\infty} \sqrt[n]{n} = 1$;
2. *For all* $a > 0$, $\lim_{n\to\infty} \sqrt[n]{a} = 1$.

7.2.7 The Euler Number

Recall that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and that for x = 1, we get the **Euler number**

$$S_n = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} \text{ for all n } \in \mathbb{N};$$

We also note that e_n can be expressed as

$$e_n = \lim_{n\to\infty} (1 + \frac{1}{n})^n \text{ for all n } \in \mathbb{N}/0.$$

Hence, we can see we can express the Euler number as the sum of the infinite series or the limit of a sequence. From this, we can gather several statements regarding the Euler number.

Theorem 7.84 *(Euler Number).*

(i) The sequence e_n is strictly increasing;

(ii) The sequence $e_n \in (2, 4]$ for all $n > 2$;

(iii) There exists a limit $e := \lim_{n\to\infty} e_n \in (2, 4]$;

Proof: (Sketch).
Let $e_n = (1 + \frac{1}{n})^n$.

(i) Use AM-GM inequality;

(ii) Let $\frac{e_n}{4}$ and use AM-GM inequality;

(iii) Use (i) and (ii).

■

Theorem 7.85 *(Relationship between e_n and s_n).*
Let $e_n = (1 + \frac{1}{n})^n$ and $s_n = \sum_{k=0}^{\infty} \frac{1}{k!}$. Then

(i) $e_n \leq s_n$ for all $n \geq 1$;

(ii) $s_m \leq e$ *for all* $m \geq 1$;

(iii) (By Squeeze Law) $\lim_{n\to\infty} S_n = e = \sum_{k=0}^{\infty} \frac{1}{k!}$.

Proof: (Sketch). First recall the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k}.$$

(iv) Re-express

$$(1+\frac{1}{n})^n = \sum_{k=0}^{n} \binom{n}{k} (\frac{1}{n})^k.$$

Do some algebra manipulation by taking a 1 out and then notice that the sum is less than s_n.

(v)

(vi) We have that $e_n \leq s_n \leq e$ and by squeeze law, $\lim_{n\to\infty} s_n = e$. ■

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8. Limit Superior and Inferior

8.2.8 Limit Superior and Inferior

We don't always require sequences to converge, we in fact can look at subsequences. In this section we look at arbitrary sequences in $\mathbb{R}$ and construct 2 monotone sequences associated with it. These 2 sequences are then guaranteed to converge.

Definition 8.86 *(Limit Inferior). The limit inferior of a sequence $\{x_n\}$ is*

$$\lim_{n\to\infty} inf x_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} (\inf_{k\geq n} x_k).$$

Definition 8.87 *(Limit Superior). The limit superior of a sequence $\{x_n\}$ is*

$$\lim_{n\to\infty} sup x_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} (\sup_{k\geq n} x_k).$$

Remark 8.88 *The sequence constructed in the limit inferior a_n is an* ***increasing*** *sequence whilst the sequence constructed in the limit superior is a* ***decreasing sequence****. Recall that whenever we talk about infimums/supremums of a sequence, we need to first turn it into a set, check if the set is non-empty and it is a subset of $\mathbb{R}$.*

When we look at the constructed nested sequence, we see whether is new sequence of supremums/infimum is bounded or not. If it is bounded, then it converges to a limit or else it diverges.

If the sequence is unbounded, we define the limit inferior (superior) to be $-\infty$ (∞). Intuitively, we can look at the list of limits of the subsequences and find the smallest (largest) limit.

Note that if the limit is ∞, it means there exist at LEAST one subsequence that goes to ∞, it does not mean every subsequence goes to ∞. Recall that we only use the word converge if there is a limit and if not, we use the phrase tends to/diverges.

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9. Further Properties of the Limit Superior and Inferior

9.2.9 Further Properties of the Limit Superior and Inferior

Theorem 9.89 *Let $\{x_n\}_{n\geq 0}$ be a sequence in $\mathbb{R}$. We have the following*

(i) $\lim_{n\to\infty} inf x_n \leq \lim_{n\to\infty} sup x_n$;

(ii) $\{x_n\}_{n\geq 0}$ *converges to* $x \in \mathbb{R}$ *if and only if* $\{x_n\}$ *is bounded* ***and*** $\lim_{n\to\infty} inf x_n = \lim_{n\to\infty} sup x_n = x$.

(iii) If $\{x_n\}_{n\geq 0}$ *converges, then*

$$\lim_{n\to\infty} inf x_n = \lim_{n\to\infty} sup x_n = \lim_{n\to\infty} x_n.$$

Proof: (Sketch).

(i) Use the fact that $a_n \leq x_n \leq b_n$ for all $n \geq \mathbb{N}$. If either $\lim_{n\to\infty} inf x_n = -\infty$ or $\lim_{n\to\infty} sup x_n = \infty$, then the statement is trivial. Assume they are both finite. This implies that $\{x_n\}$ is bounded. Now using the preservation of inequalities in the limit, $\lim_{n\to\infty} inf x_n \leq \lim_{n\to\infty} sup x_n$.

(ii) $\Leftarrow$

Use squeeze law so that $x_n \to x$ as $n \to \infty$.

$\Rightarrow$

Use the definition that for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $x - \epsilon < x_n < x + \epsilon$ for all $n \geq n_\epsilon$. Use the fact that

$$x - \epsilon \leq \ inf_{k\geq n} x_k \leq x_n \leq \ sup_{k\geq n} x_k \ \leq x + \epsilon$$

for all n $\geq n_\epsilon$ to conclude the proof by using the definition of limits for the limit inferior and superior.

■

Remark 9.90 *Recall for the second point, we need the extra condition that the limit inferior and superior to coincide as bounded sequences does not imply convergence.*

Theorem 9.91 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be arbitrary sequences in* $\mathbb{R}$. *Then*

(i)

$$\lim_{n\to\infty} sup(x_n + y_n) \leq \lim_{n\to\infty} sup x_n + \lim_{n\to\infty} sup y_n$$

provided that the RHS is neither $\infty - \infty$ *or* $-\infty + \infty$. *Moreover, if either sequence converges, then we get equality.*

(ii) If $\{x_n\}$ *and* $\{y_n\}$ *are* ***non-negative numbers*** *then*

$$\lim_{n\to\infty} sup(x_n y_n) \leq (\lim_{n\to\infty} sup\ x_n)(\lim_{n\to\infty} sup\ y_n)$$

provided that the RHS is neither $0.\infty$ *or* $\infty.0$. *If either* $\{x_n\}$ *or* $\{y_n\}$ *converges to a positive number, then we have equality.*

Don't forget the useful property that $\lim_{n\to\infty} inf(-x_n) = -\lim_{n\to\infty} sup(x_n)$.

Proof: (Sketch) For i and ii, we arrive at the inequality by looking at the monotonic sequence constructed by taking the sup or inf of subsequences. We then analyse the differences between our original sequence and new subsequence. We then add or multiply them together to arrive at our inequality. To show equality, look at the sequence that does **not** converge and manipulate it by adding the one that does converge to it. This should get us the inequality to prove equality. ■

Hence, note that the definition of a limit does not actually help us compute limits, we can only use it to verify limits.

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10. Subsequences and Cauchy Sequences

10.2.10 Subsequences

Definition 10.92 *(Subsequence). Let $\{x_n\}_{n\geq 0}$ be a sequence in X. Let $\{n_k\}_{k\geq 0}$ be a strictly increasing sequence in $\mathbb{N}$ $n_0 < n_1 < ...$ for $n_k \in \mathbb{N}$ for all $k \geq 0$. Then $\{x_{n_k}\}_{k\geq 0}$ is called a **subsequence** of $\{x_n\}$.*

Notice that we now have an index on our index for subsequences. Furthermore, for subsequences, we have to pick elements from the sequence in a strictly increasing order.

Definition 10.93 *(Accumulation Point). If $\{x_{n_k}\}_k$ is **convergent**, then its limit is called an **accumulation point** of $\{x_n\}_{n\geq 0}$.*

So in other words, accumulation points are simply the limits of subsequences.

Proposition 10.94 *Let $\{x_n\}$ be a sequence in $\mathbb{R}$.*

*If $\lim_{n\to\infty} inf(x_n)$ exists in $\mathbb{R}$, then $\lim_{n\to\infty} inf(x_n)$ is the **smallest accumulation point** of $\{x_n\}$.*

*If $\lim_{n\to\infty} sup(x_n)$ exists in $\mathbb{R}$, then $\lim_{n\to\infty} sup(x_n)$ is the **largest accumulation point** of $\{x_n\}$.*

This means that there exists a subsequence $\{x_{n_k}\}$ that converges to the limit superior or inferior. In other words, if we have a bounded sequence, then its limit superior/inferior exists and serves as an accumulation point, which is the limit of a convergent subsequence.

Definition 10.95 *(Cauchy Sequence). A sequence $\{x_n\}$ in $\mathbb{K}^N$ is called a Cauchy sequence if for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that*

$$||x_m - x_n|| < \epsilon$$

for all $m, n \geq n_\epsilon$.

That is, there exists a rank where the norm difference between **any** 2 elements beyond the rank is smaller than ϵ.

Proposition 10.96 *Every convergent sequence in $\mathbb{K}^N$ is a Cauchy sequence.*

Proof: (Sketch). Use the definition of convergence in addition to adding and subtracting the limit term x and then use triangle inequality. ■

We now seek to show the **completeness** of $\mathbb{K}^N$. We split the proof up into 3 parts

1. Show that every Cauchy sequence in bounded in $\mathbb{K}^N$;
2. Show that every bounded sequence has a convergent subsequence (Bolzano Weierstrass theorem);

3. A Cauchy sequence in $\mathbb{K}^N$ with a convergent subsequence converges.

Proposition 10.97 *Every Cauchy Sequence in $\mathbb{K}^N$ is bounded.*

Proof: Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{K}^N$. Take $\epsilon = 1$. Then there exists a rank n_1 such that $||x_m - x_n||< 1$ for all $m, n \geq n_1$.

Let $n = n_1$, hence we have that

$$||x_m||= ||(x_m - x_{n_1}) + x_{n_1}||.$$

By triange inequality

$$\leq ||(x_m - x_{n_1})||+||x_{n_1}||$$
$$< 1 + ||x_{n_1}||.$$

Then we define the constant C as

$$C := max\{||x_0||, ||x_1||, ||x_2||, ..., ||x_{n_1-1}, 1 + ||x_{n_1}||\}$$

Then $||x_m||\leq C$ for all $m \geq 0$. Thus, $\{x_n\}$ is bounded. ■

Theorem 10.98 *(Bolzano-Weiestrass Theorem). Every bounded sequence in $\mathbb{K}^N$ contains a convergent subsequence.*

Proof: We can prove this by induction. We assume that a bounded sequence in $\mathbb{R}$ contains a convergent subsequence. Using induction, we show that any bounded sequence in $\mathbb{R}^{N+1}$ contains a convergent subsequence.

In particular, for $\mathbb{R}^{N+1}$, we have that

$$\{x_n\} = \{y_n\} + \{z_n\}$$

where $\{y_n\} \in \mathbb{R}^N$ and $\{z_n\} \in \mathbb{R}$. Due to property of norms

$$||x_n||= ||y_n||+||z_n||$$

so then $\{x_n\}$ converges iff $\{y_n\}$ and $\{z_n\}$ converges. By induction hypothesis $\{y_{n_k}\}$ is bounded and hence converges to y and by the base case, $\{z_{n_{k_j}}\}$ converges to z. Note that z has an extra index as it needs to be along same subsequence as y first. Resultantly, $\{x_{n_{k_j}}\}$ is a subsequence that converges to $(y, z) \in \mathbb{R}^{N+1}$. Hence $\{x_n\}$ has a convergent subsequence. ■

Proposition 10.99 *A Cauchy sequence is convergent if and only if it has a convergent subsequence. Then, they converge to the same limit.*

Theorem 10.100 *(Completeness of $\mathbb{K}^N$). A sequence $\{x_n\}_{n\geq 0}$ in $\mathbb{K}^N$ converges if and only if it is a Cauchy sequence.*

Proof: (Sketch). Use the definition of Cauchy sequence and the definition of a convergent subsequence. Pick a rank large enough for the subsequence and use the triangle inequality to show that

$$\lim_{m\to\infty} x_m = x,$$

where x is the limit of the subsequence $\{x_{n_k}\}$. ■

Here, the least upper bound axiom in $\mathbb{R}$ is equivalent to the fact that $\mathbb{R}$ has the Archimedian property and that every Cauchy sequence in $\mathbb{R}$ converges. The advantage of our new definition is that it does not depend on the order properties which is found in $\mathbb{R}$ and also generalises.

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11. Introduction to Series

11.3 Series

11.3.1 Series

Informally, a series is simply an infinite sum $\sum_{k=0}^{\infty} a_k$.

Definition 11.101 *(Series). We set*

$$s_n \coloneqq \sum_{k=0}^{n} a_k = a_0 + a_1 + ... + a_n.$$

We call s_n the n-th partial sum of $\sum_{k=0}^{\infty} a_k$ and (s_n) the sequence of partial sums.

If (s_n) converges we say that the series converges. In particular,

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n.$$

In this case, the series converges or else it diverges.

So we look at the limit of partial sums when looking at the sum of the infinite series. So a sequence of partial sums is literally a sequence whereby element in index k is the sum of the first k terms in the sequence.

We now look at important series that we need to know.

Definition 11.102 *(Geometric Series). The geometric series is defined as*

$$\sum_{k=0}^{\infty} a^k$$

where $a \in \mathbb{C}$.

Note that if $a \neq 1$, we have that

$$\sum_{k=0}^{\infty} a^k \text{ converges } \leftrightarrow |a| < 1$$

In particular for this case, we have that

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$$

Definition 11.103 *(Alternating Series). Let $a_k = (-1)^k$, so then the series is $\sum_{k=0}^{\infty}(-1)^k$.*

Note that $s_n = 0$ for n is odd and $s_n = 1$ for n is even. Clearly, s_n diverges.

We look at our first necessary condition for a series to converge.

Lemma 11.104 *If $\sum_{k=0}^{\infty} a_k$ converges, then necessarily $\lim_{k\to\infty} a_k = 0$.*

Proof: We denote the limit of the partial sums s_n as

$$L := \lim_{n\to\infty} s_n$$

as the series converges. Now,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0.$$

■

The contrapositive of this is one of the most useful tests for us.

Proposition 11.105 *(Divergence Test / Term Test). If the sequence $\{a_k\} \nrightarrow 0$, then the series $\sum_{k=0}^{\infty} a_k$ diverges.*

Remark 11.106 *This test is something you should always use at the start. If the sequence does not converge to 0, then we can say that the series diverges. However, be careful because if the sequence converges to 0, it is non-conclusive so we don't know whether has the sequence converge. In particular, if the limit is 0, the series may **still** diverge.*

Definition 11.107 *(Harmonic Series). The Harmonic series is defined as*

$$\sum_{k=1}^{\infty} \frac{1}{k}.$$

The Harmonic series diverges to ∞ despite the limit of a_k converging to 0.

The convergence properties of a series are determined by the convergence properties of sequences.

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12. Tests for convergence of series

12.3.2 Tests for convergence of series

Theorem 12.108 *(Cauchy Criterion for Series). Let $\{x_n\}_{n\geq 0}$ be a sequence in $\mathbb{K}^N$.*

Then $\sum_{k=0}^{\infty} a_k$ converges in $\mathbb{K}^N$ $\leftrightarrow$ its sequence $\{s_n\}_{n\geq 0}$ of partial sums is a Cauchy sequence in $\mathbb{K}^N$ $\leftrightarrow$ for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ s.t. $||s_m - s_n||< \epsilon$ for all $m > n \geq n_\epsilon$ which can be expressed as

$$||\sum_{k=0}^{m} a_k - \sum_{k=0}^{n} a_k||< \epsilon$$

$$||\sum_{k=n+1}^{m} a_k||< \epsilon.$$

Remark 12.109 *This is useful for theoretical purposes and rarely used on actual series. However, it does apply to all series.*

Definition 12.110 *(Non-negative Series). A series $\sum_{k=0}^{\infty}$ is called a non-negative series if $a_k \geq 0$, for all $k \geq 0$.*

Resultantly, the sequence of partial sums is a monotonic increasing sequence since it is the summation of positive terms. Hence, we now only need to show it is bounded in order for us to show it has a limit.

Proposition 12.111 *Let $\sum_{k=0}^{\infty} a_k$ be a series with* ***non-negative*** *terms. Define s_n to be the n-th partial sum of the series $s_n \coloneqq \sum_{k=0}^{n} a_k$. The sequence s_n is increasing. There, we have that*

$$\{s_n\} \text{ converges if and only if it is bounded.}$$

This proposition is the basis for the comparison tests.

Theorem 12.112 *(The Comparison Test). Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be non-negative series.*

Assume that there exists a positive constant C and $m \in \mathbb{N}$ such that

$$a_k \leq Cb_k, \ \forall k \geq m.$$

The following holds

(i) If $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ converges;

(ii) If $\sum_{k=0}^{\infty} a_k$ diverges, then $\sum_{k=0}^{\infty} b_k$ diverges.

Proof: (Sketch).

To show (i), it suffices to show that the sequence of partial sums for a_k is bounded from above and hence establishes that $\sum_{k=0}^{\infty} a_k$ is bounded. Use the relationship between the sequences of partial sums to show this.

(ii) is simply the contrapositive of (i). ■

Hence, we have a series that bounds another series from a certain rank onwards and we can use that to determine whether is the series bounded or not.

Corollary 12.113 *(Limit Comparison Test). Suppose that $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are non-negative series. Then if*

$$\lim_{n\to\infty} sup\frac{a_n}{b_n} = \lim_{n\to\infty} (sup_{k\geq n}\frac{a_k}{b_k}) < \infty$$

Then the following assertions are true.

(i) If $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ converges;

(ii) If $\sum_{k=0}^{\infty} a_k$ diverges, then $\sum_{k=0}^{\infty} b_k$ diverges too.

Remark 12.114 *If the limit comparison test yields ∞, then b_k is growing slower than a_k, hence if b_k diverges, then a_k diverges. If the test yields 0, then b_k is growing faster than a_k, so if b_k converges, then a_k converges.*

We construct another test from the comparison test.

Proposition 12.115 *(Maclaurin–Cauchy Integral Test). Let $N \in \mathbb{N}$ and let f be* ***non-negative*** *and defined on the unbounded interval $[N, \infty)$ on which the function f is* ***monotone decreasing****. Then, the series $\sum_{n=N}^{\infty} f(n)$ converges to a real number if and only if*

$$\int_N^{\infty} f(x)dx = \lim_{n\to\infty} \int_N^n f(x)dx < \infty.$$

Proof:(Sketch). Recall that by definition of Riemann sums, we have

$$\sum_{k=N}^{n} a_k - a_1 \leq \int_N^n f(x)dx \leq \sum_{k=N}^{n} a_k - a_n.$$

Both the partial sums and integral are monotone increasing functions. They converge if and only if they are bounded. Make an assumption whether is the integral bounded or not and from that, use the comparison test to determine whether does the original series converge. ■

We can look at the relationships between sequences.

Proposition 12.116 *(Equivalence relation between sequences). Two sequences $\{a_n\}$ and $\{b_n\}$ in $\mathbb{K}$ are equivalent if*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

From this, we say that the equivalence of sequences are an equivalence relation $a_n \sim b_n$.

Proposition 12.117 *If $a_n \sim b_n$, then $\{a_n\}$ converges if and only if $\{b_n\}$ converges. Furthermore, in the case of convergence, the limit of $\{a_n\}$ and $\{b_n\}$ are the same.*

Proposition 12.118 *(Convergence of equivalence series). If $a_n \sim b_n$, then the series $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=0}^{\infty} b_k$ converges.*

Proof:(Sketch). Use the limit comparison test to show this for both $\frac{a_n}{b_n}$ and $\frac{b_n}{a_n}$. ■

Definition 12.119 *(Contraction). Let $f : \mathbb{K}^N \to \mathbb{K}^N$ and suppose that there exists $L \in (0, 1)$ such that*

$$||f(x) - f(y)|| \leq L||x - y||$$

for all $x, y \in \mathbb{K}^N$. A function with this property is known as a ***contraction*** *as any pair of image points is closer together than the original points.*

Proposition 12.120 *For a function with the contraction property, there exists a unique point in $\mathbb{K}^N$ such that $x = f(x)$.*

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13. More Tests and Alternating Series.

13.3.3 Cauchy Condensation Test

Proposition 13.121 *(Cauchy Condensation Test). Suppose that (a_k) is a decreasing sequence with $a_k \geq 0$ for all $k \in \mathbb{N}$. Then $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=0}^{\infty} 2^n a_{2^n}$ converges.*

Remark 13.122 *Note that after you apply the Cauchy condensation test, you can apply another test to see if $\sum_{k=0}^{\infty} 2^n a_{2^n}$ converges or not.*

Proposition 13.123 *(The P-Series). The P-series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$. The series diverges for $p \leq 1$, where it is the Harmonic series when $p = 1$.

Definition 13.124 *(Riemann Zeta Function). One of the most famous series defined to be*

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

for $p > 1$.

13.3.4 Alternating Series and Conditional Convergence

Definition 13.125 *(Alternating Harmonic Series). The Alternating Harmonic series is defined to be*

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}.$$

Note that unlike the Harmonic series, the alternating harmonic series **converges**.

Proof: (Sketch). Break up the alternating harmonic series into its odd and even subsequence. The odd subsequence will be decreasing whilst the even subsequence will be increasing. Find the bounds of both subsequences and combine them together. ■

Theorem 13.126 *(Lebniz Test for Alternating Series). Let (a_k) be a non-negative decreasing sequence with $a_k \to 0$. Then the alternating series*

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

converges. Moreover, $a_0 - a_1 \leq \sum_{k=0}^{\infty} a_k \leq a_0$.

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14. Tests for convergence of series

14.3.5 Tests for convergence of series

Definition 14.127 *(Unconditional Convergence). A series $\sum_{k=0}^{\infty} a_k \in \mathbb{K}^N$ is called unconditionaly convergent to $\alpha \in \mathbb{K}^N$ if for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum_{k=0}^{\infty} a_{\sigma(k)}$ converges to α.*

What this means is that any permutation of the series will still lead to the same limit. A series that does not satisfy this is known to be **conditionally convergent** to some $\alpha \in \mathbb{K}^N$.

Notice this is the idea required for expectations in probability theory. If something converges conditionally (therefore not absolute convergence), then expectation can be a different value depending on the order of summation we use!

Definition 14.128 *(Absolute Convergence). A series $\sum_{k=0}^{\infty} a_k \in \mathbb{K}^N$ is called absolutely convergent if $\sum_{k=0}^{\infty} ||a_k||$ is a convergent series in $\mathbb{R}$.*

Hence, we only need to look at the sequence of norms to check for absolute convergence. Hence, if the sequence of norms converges, then $\sum_{k=0}^{\infty} a_k$ also converges.

Definition 14.129 *(Conditionally Convergent). A series is conditionally convergent if $\sum_{k=0}^{\infty} a_k$ converges but $\sum_{k=0}^{\infty} ||a_k||$ diverges.*

We then show that rearranging the order of a conditionally convergent series changes the limit.

Theorem 14.130 *(Riemann Series Theorem). If an infinite series of real numbers is conditionally convergent, then its terms can be arranged in a permutation so that the new series converges to an arbitrary real number, or diverges.*

Example The alternating Harmoic series is an example of this

$$\sum_{i=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

as it converges to ln(2). However, let M be a real number, then there exists a permutation of the series such that the sum is M.

Definition 14.131 *(Convergent Series). A series in $\mathbb{K}^N$ is convergent if it converges to a vector in $\mathbb{K}^N$.*

Theorem 14.132 *A series that is absolutely convergent in $\mathbb{K}^N$ is a convergent series in $\mathbb{K}^N$ and we have the infinite triangle inequality*

$$||\sum_{k=0}^{\infty} a_k|| \leq \sum_{k=0}^{\infty} ||a_k||$$

Proof:(Sketch). By definition of absolute convergence, the sequence of norms converges. By the Cauchy Criterion, the partial sums $S_n = \sum_{k=0}^{\infty} ||a_k||$ converges, and $\{S_n\}$ is a Cauchy sequence. This means that for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ where

$$\sum_{k=n+1}^{m} ||a_k|| < \epsilon$$

for all $m > n > n_\epsilon$. Then look at $t_n = \sum_{k=0}^{m} a_k$ and use the triangle inequality to derive both conditions alongside taking limits of partial sums. ■

Criteria for absolute convergence/divergence for series in $\mathbb{K}^N$

We now look at a series of vectors now.

Theorem 14.133 *(The Ratio Test). Let $\sum_{k=0}^{\infty} a_k$ be a series in $\mathbb{K}^N$. We define*

$$R := \lim_{n\to\infty} sup \frac{||a_{n+1}||}{||a_n||};$$

$$r := \lim_{n\to\infty} inf \frac{||a_{n+1}||}{||a_n||}.$$

(i) If $R < 1$, then the series is absolutely convergent;

(ii) If $r > 1$, then the series diverges;

*(iii) If $R \geq 1$ **and** $r \leq 1$, then the test is inconclusive as the series either converges or diverges.*

Proof: (i) First assume $R < 1$ and $\beta \in (R, 1)$. Recall that by definition of limit superior of R, that is the same as

$$\lim_{n\to\infty} b_n = R$$

where $b_n = \sup_{k \geq n} \frac{||a_{k+1}||}{||a_k||}$. Additionally, we can always find such a β by the density of $\mathbb{Q}$ in $\mathbb{R}$. Hence, as $\lim_{n\to\infty} b_n = R < \beta$, there exists $n_0 \in \mathbb{N}$ such that

$$b_n \leq \beta \leftrightarrow sup_{k\geq n} \frac{||a_{k+1}||}{||a_k||} \leq \beta$$

for all $n \geq n_0$. Hence, for each level of $n \geq n_0$, we can multiply the denominator in the above inequality out so that we now have inequalities for each rank as

$$||a_{n_0+1}|| \leq \beta ||a_{n_0}||$$

$$||a_{n_0+2}|| \leq \beta ||a_{n_0+1}||$$

$$...$$

$$||a_n|| \leq \beta ||a_{n-1}||$$

We can cancel out terms to arrive at

$$||a_n|| \leq \beta^{n-n_0} ||a_{n_0}||$$

for all $n \geq n_0$. We can rewrite the right hand term as $\frac{||a_{n_0}||}{\beta^{n_0}} \beta^n$ where the coefficient is just a non-negative constant. As $0 < \beta < 1$, by the comparison test, we compare this to the geometric series $\sum_{n=0}^{\infty} \beta^n$ which converges and hence by the comparison test, this implies that $\sum_{n=0}^{\infty}||a_n||$ converges. This then implies that $\sum_{n=0}^{\infty} a_n$ converges.

(ii) We have that

$$r == \lim_{n\to\infty} inf_{k\geq n} \frac{||a_{k+1}||}{||a_k||} > 1$$

and we denote $d_n = inf_{k\geq n} \frac{||a_{k+1}||}{||a_k||}$. Then there exists $n_0 \in \mathbb{N}$ such that $d_n > 1$ for all $n \geq n_0$. We have that $||a_{n+1}||> ||a_n||$ for all $n \geq n_0$. Hence, the sequence of norms $\{a_n\}_{n\geq n_0}$ is strictly increasing. Hence, this sequence of norms cannot converge to 0 as $n \to \infty$ as it is a non-negative sequence that is strictly increasing. Then $a_n \not\to 0$ as $n \to \infty$ which implies that $\sum_{n=0}^{\infty} a_n$ diverges. ■

Theorem 14.134 *(The Root Test). Let $\sum_{k=0}^{\infty} a_k$ be a series in $\mathbb{K}^N$. We define*

$$\alpha := \lim_{n\to\infty} sup \sqrt[n]{||a_n||}$$

(i) If $\alpha < 1$, then the series is absolutely convergent;

(ii) If $\alpha > 1$, then the series diverges;

(iii) If $\alpha = 1$, then the test is inconclusive as the series either converges or diverges.

Proof: (i) We assume that $\alpha := \lim_{n\to\infty} sup \sqrt[n]{||a_n||} < 1$. We show that the series is absolutely convergent. We define the sequence of norms $B_n = \sup_{k\geq n} \sqrt[k]{||a_k||}$. Hence, $\alpha := \lim_{n\to\infty} B_n$. We then choose a $\beta \in (\alpha, 1)$. As $B_n \to \alpha$ as $n \to \infty$ and $\alpha < \beta < 1$, we have that there exists $n_0 \in \mathbb{N}$ such that

$$\sqrt[n]{||a_n||} \leq sup_{k\geq n} \sqrt[k]{||a_k||} = B_n \leq \beta$$

for all $n \geq n_0$. Hence, $||a_n||< \beta^n$ for all $n \geq n_0$. By comparison test with Geometric series where $\beta < 1$, we have that $\sum_{n=0}^{\infty}||a_n||$ converges, which implies that $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

(ii) Similar proof to ratio test. ■

Corollary 14.135 *A series is unconditionally convergent in $\mathbb{K}^N$ if it is an absolutely convergent series in $\mathbb{K}^N$.*

Remark 14.136 *Note that the converse does not remain true in general for* ***infinite*** *dimensional spaces.*

Corollary 14.137 *A series that is unconditionally convergent in $\mathbb{K}^N$ is a convergent series in $\mathbb{K}^N$.*

Here is a summary of all the tests.

The table below shows to wich classes of series $\sum_{k=0}^{\infty} a_k$ the tests are applicable in terms of properties of a_k and the partial sums (s_n).

✔ means that the test is applicable, ✗ means that the test is not in general applicable.

	boundedness of (s_n)	Comparison test	Limit comparison test	Cauchy condensation test	Integral test	Leibniz test	Ratio test	Root test	Cauchy criterion
$a_k \geq 0$	✔	✔	✔	✗	✗	✗	✔	✔	✔
$a_k \geq 0$ and a_k decreasing	✔	✔	✔	✔	✔	✗	✔	✔	✔
$a_k = (-1)^k \lvert a_k \rvert$, $\lvert a_k \rvert$ decreasing	✗	✗	✗	✗	✗	✔	✔	✔	✔
$a_k \in \mathbb{R}$ (arbitrary signs)	✗	✗	✗	✗	✗	✗	✔	✔	✔
$a_k \in \mathbb{R}^N$ or $\mathbb{C}^N$	✗	✗	✗	✗	✗	✗	✔	✔	✔
$a_k \in \mathbb{R}^N$ or $\mathbb{C}^N$, test on $\sum_{k=1}^{\infty} \lVert a_k \rVert$	✔	✔	✔	✔	✔	✗	✔	✔	✔

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15. Power Series and their Convergence Properties

15.3.6 Power Series and their Convergence Properties

Definition 15.138 *(Power Series). Let $\{a_k\}_{k\geq 0}$ be a sequence in $\mathbb{K}^N$ and $x \in K$. Then the series*

$$\sum_{k=0}^{\infty} a_k x^k$$

*is called a **power series** in $\mathbb{K}^N$. This is also known as the power series centered at 0. Furthermore, we can also look at the power series*

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

centered at $x_0 \in \mathbb{K}$.

Here, we have a scalar x, which we raise to the power and multiply it by sequence. Power series has really nice convergence properties.

A power series will converge for certain values of x. There is an interval $(-\rho, \rho)$, where if $x \in (-\rho, \rho)$, then the power series converges. ρ is known as the radius of convergence and if ρ is within this interval, then the power series always converges absolutely, which then implies it also converges and hence it is unconditionally Convergent.

Theorem 15.139 *(Cauchy-Hadamard Theorem). Every power series $\sum_{k=0}^{\infty} a_k x^k$ in $\mathbb{K}^N$ either converges for all $x \in \mathbb{C}$ or there exists a $\rho \in [0, \infty)$ such that the series*

1. *converges absolutely for all $x \in \mathbb{C}$ if $|x| < \rho$;*
2. *diverges for all $x \in \mathbb{C}$ if $|x| > \rho$;*
3. *converges absolutely for every $x \in \mathbb{C}$ (where we set $\rho := \infty$);*
4. *diverges for every $x \in \mathbb{C}/\{0\}$ (where we set $\rho = 0$).*

Additionally, for $|x| = \rho$, anything can happen and hence the behaviour is not defined and depends on the series.

Morever, we define

$$\rho = \frac{1}{\lim\limits_{n\to\infty} sup \sqrt[n]{||a_n||}}$$

where we set $1/\infty := 0$ and $1/0 := \infty$.

Definition 15.140 *(Radius of Convergence). The number $\rho \in [0, \infty)$ defined above is known as the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k x^k$. The radius of convergence can be derived in multiple ways.*

(Root Test).

$$\rho = \frac{1}{\lim_{n\to\infty} sup \sqrt[n]{||a_n||}}$$

(Ratio Test).

$$\rho = \lim_{n\to\infty} \frac{||a_n||}{||a_{n+1}||}$$

Remark 15.141 *Note that when using the root test, the n-th root must match the power of z. That is, if we had z^{2n}, then we need to use $\sqrt[2n]{||a_n||}$*

Remark 15.142 *We have that a power series converges uniformly and absolutely in any region that* ***lies entirely*** *inside its circle of convergence. So we can express convergence in 2 ways. We can that it converges uniformly on any ball $b(0, r)$ with $r < \rho$ OR it converges locally uniformly on $b(0, \rho)$.*

Remark 15.143 *Using ratio test for ρ, if it is 1 or ∞. However, for root test, if we get 1/0, we say it is ∞, so it converges everywhere and if we get $1/\infty = 0$, it diverges everywhere.*

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16. Double Series

16.3.7 Double Series

Definition 16.144 *(Double Series). A double series is defined to be*

$$\sum_{j,k=0}^{\infty} x_{jk}.$$

This can be thought of as an infinite 2 dimensional array.

There are many ways to add them up. This matters as the limit may depend on the order of summation.

1. Sum by rows
$$\sum_{j=0}^{\infty}(\sum_{k=0}^{\infty} x_{jk})$$
2. Sum by columns
$$\sum_{k=0}^{\infty}(\sum_{j=0}^{\infty} x_{jk})$$
3. Sum by a bjiection $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$
$$\sum_{i=0}^{\infty} x_{\sigma(i)}$$
.

Theorem 16.145 *(Absolute Convergence for Double Series). Suppose that $x_{jk} \in \mathbb{K}^N$ are such that*

$$M := \sup_{m,n \in \mathbb{N}} \sum_{j=0}^{m}(\sum_{k=0}^{n} ||x_{jk}||) < \infty.$$

Then the series of summing over rows/columns/bijection converges absolutely and

$$\sum_{j=0}^{\infty}(\sum_{k=0}^{\infty} x_{jk}) = \sum_{k=0}^{\infty}(\sum_{j=0}^{\infty} x_{jk}) = \sum_{i=0}^{\infty} x_{\sigma(i)}$$

for every bijection $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

Proof:(Sketch). To prove the absolute convergence of the three series, we have that $\sum_{j=0}^{m} ||x_{j,k}|| \leq M$ for every $m, k \in \mathbb{N}$. The series converges as the sequence of partial sums here is bounded and hence the series converges absolutely. We repeat this argument for other 2 series.

To prove equality of the limits, we can look at partial sums and use the definition of absolute convergence. We then take limits. ■

A series that is absolutely convergent is unconditionally convergent, that is, the terms can be rearranged without changing the limit. Recall that the converse does not hold since taking the norm of vectors can stop things from cancelling out with each other and hence we get a different value. We can apply our double series absolute convergence to our original series.

Corollary 16.146 *(Absolutely Convergent Series are Unconditionally Convergent). Suppose that $\sum_{k=0}^{\infty} a_k$ is an absolutely convergent series in $\mathbb{K}^N$ and that $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection. Then $\sum_{k=0}^{\infty} a_{\sigma(k)}$ is absolutely convergent and*

$$\sum_{k=0}^{\infty} a_k = \sum_{j=0}^{\infty} a_{\sigma(j)}.$$

Any permutation of a absolutely convergent series still converges to the same limit.

We now look at another application by looking at the product of two convergent series a_j and b_k. We can have that

$$ab = \sum_{j=0}^{\infty} (\sum_{k=0}^{\infty} a_j b_k) = \sum_{k=0}^{\infty} (\sum_{j=0}^{\infty} a_j b_k).$$

Here, we have that the first row of our new array is $a_0 b_k$ for $k \in \mathbb{N}$. Additionally, we have the first column of our array is $a_j b_0$ for $j \in \mathbb{N}$. We can also construct products by multiplying via diagonals instead.

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17. Cauchy Products

17.3.8 Cauchy Products

Definition 17.147 *(Cauchy Products). For 2 series a_j and b_k, we can express the two series as*

$$c_n := \sum_{k=0}^{n} a_k b_{n-k}.$$

where c_n is known as the ***Cauchy product*** *of the series a and b. We collect the new series by its diagonal.*

Theorem 17.148 *(Convergence of Cauchy Products). Suppose that $a = \sum_{j=0}^{\infty}$ and $b = \sum_{k=0}^{\infty} b_k$ are absolutely convergent series in $\mathbb{K}$. Then their Cauchy product converges absolutely and*

$$(\sum_{j=0}^{\infty} a_j)(\sum_{k=0}^{\infty} b_k) = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k b_{n-k}).$$

Proof: The conditions of the double series theorem is satisfied and hence this is also satisfied. In particular,

$$\sum_{j=0}^{\infty}(\sum_{k=0}^{n}|a_j b_k|) = \sum_{j=0}^{m}(|a_j|)(\sum_{k=0}^{n}|b_k|) \leq \sum_{j=0}^{\infty}(|a_j|)(\sum_{k=0}^{\infty}|b_k|) := M < \infty.$$

■

Example 17.149 *Recall that for the exponential function, we have*

$$exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and this converges absolutely for all $x \in \mathbb{C}$. We compute the Cauchy product of exp(x) and exp(y).

We use the Binomial theorem after introduction $n!/n!$.

$$c_n = \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} x^k y^{n-k} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \frac{(x+y)^n}{n!}$$

for all $n \in \mathbb{N}$. Hence,

$$exp(x)exp(y) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = exp(x+y).$$

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18. Closed and Open Sets

18.3.9 Closed and Open Sets

We now look at Topology of sets. First, we describe functions using open and closed subsets of $\mathbb{K}^N$. We use these sets to also help us describe limits and continuity. To define open sets, we use **open balls**.

Definition 18.150 *(Open Ball). Let $x \in \mathbb{K}^N$ and pick a $r > 0$. We call*

$$B(x,r) := \{y \in \mathbb{K}^N : ||y - x|| < r\}$$

the open ball about x with radius r.

From this, we can define open sets.

Definition 18.151 *(Open Sets). A set $U \subseteq \mathbb{K}^N$ is called open if either $U = \emptyset$ or for every $x \in U$, there exists $r > 0$ such that*

$$B(x,r) \subseteq U.$$

In other words, for every element in the set, we can find a radius r in order to fit a ball around the element and still be contained within the set.

Definition 18.152 *(Closed Sets). A set is closed if its complement is open.*

Lemma 18.153 *An open ball B(x,r) is an open set.*

Proof: We show that for every $z \in B(x,r)$, there exists $\epsilon > 0$ such that $B(z,\epsilon) \subseteq B(x,r)$. Set distance ϵ to be the size of ball subtract the distance from x to z. First fix an element $z \in B(x,r)$. Then set $\epsilon := r - ||x - z||$. If $y \in B(z,\epsilon)$, then $||y - z|| < \epsilon$. Hence, we have that by triangle inequality

$$||y - x|| = ||y - z + z - x|| \leq ||y - z|| + ||z - x||$$

$$< \epsilon + ||z - x|| = r - ||x - z|| + ||z - x|| = r.$$

Therefore, $||y - x|| < r$, that is, $y \in B(x,r)$. Hence, $B(z,\epsilon) \subseteq B(x,r)$ as required which shows that $B(x,r)$ is open. ∎

We now derive properties of open sets in $\mathbb{K}^N$.

Proposition 18.154 *(Basic properties of open sets). Open sets have the following properties:*

(i) $\emptyset$ and $\mathbb{K}^N$ are open sets;

(ii) The union of an arbitrary collection of open sets is open;

*(iii) The intersection of **finitely** many open sets is open.*

Proof: (i) This is true by definition.

(ii) Choose an $x \in \bigcup_{i\in I} U_i$. This means that $x \in U_j$ for some $j \in I$. Then that means there exists $r > 0$ such that $B(x,r) \subset U_j$ which is a subset of $\bigcup_{i\in I} U_i$. Hence, this works for every choice x and therefore $\bigcup_{i\in I} U_i$ is open.

(iii) Let $x \in \bigcap_{i=1}^n U_i$. This means that $x \in U_i$ for every i=1,2,...,n. Since U_i is open, there exists $r_i > 0$ such that $B(x,r_i) \subseteq U_i$. We let

$$r := min\{r_1, ..., r_n\}.$$

Note that this is the reason we require a finite number of open sets as a list of infinite numbers is not guaranteed to have a minimum. Hence, then that means $B(x,r) \subset \bigcap_{i=1}^n U_i$ and as this works for every $x \in \bigcap_{i=1}^n U_i$, this means that $\bigcap_{i=1}^n U_i$ is open. ■

Remark 18.155 *An example of the intersection of infinitely many open sets not being open can be*

$$\bigcap_{i=0}^{\infty}(0, 1 + \frac{1}{i}) = (0, 1]$$

which is not open nor closed.

Proposition 18.156 *(Basic properties of closed sets). Closed sets have the following properties:*

(i) $\emptyset$ and $\mathbb{K}^N$ are closed sets;

(ii) The intersection of an arbitrary collection of closed sets is closed;

(iii) The union of an arbitrary collection of finitely many closed sets is closed.

Proof:(i) This holds from the definition of closed sets as the complement $\mathbb{K}^N$ and $\emptyset$ is open.

(ii) Let $(A_i)_{i\in I}$ be an arbitrary collection of closed subsets of $\mathbb{K}^N$. Let $U_i := A_i^c$ be open by the definition of closed sets for all $i \in I$. Hence, $\cup_{i\in I} U_i$ is open from earlier statement. Using De Morgan's Law

$$(\bigcap_{i\in I} A_i)^c = \bigcup_{i\in I} A_i^c = \bigcup_{i\in I} U_i$$

is open. Therefore, by definition of closed sets, $\cup_{i=1}^n A_i$ is closed.

(iii) Let A_i for i=1,...,n be closed subsets of $\mathbb{K}^N$. Then $U_i := A_i^c$ is open by definition of closed sets. Then, $\cap_{i=1}^n U_i$ is open. Be De Morgan's Law,

$$(\bigcup_{i\in I} A_i)^c = \bigcap_{i\in I} A_i^c = \bigcap_{i\in I} U_i$$

is open. Hence, $\bigcup_{i=1}^n A_i$ is closed by definition of a closed set. ■

Remark 18.157 *An example of the union of infinitely many closed sets not being closed can be*

$$\bigcup_{i=0}^{\infty}[\frac{1}{i}, 1] = (0, 1]$$

which is not closed nor open.

Note that the reason we need a finite union of closed sets, is because we look at the complement, which is the intersection of open sets, which requires to be finite in order for earlier argument to apply.

Note that sets can be both open and closed. Also, sets that are not closed does not mean they are open and vice versa. To check whether a set is open, make sure you can always find a r for each point in the set such that the ball around that set is contained within the set.

Proposition 18.158 *A set A that is closed and bounded from above(below) admits a maximum(minimum).*

Proof: Since the set A is bounded from above, by the least upper bound axiom, it has a supremum $\mathcal{M}$. Then, by definition of a supremum, for every $\epsilon > 0$, there exists $a \in A$ such that

$$\mathcal{M} - \epsilon < a < \mathcal{M}.$$

SO then, $B(\mathcal{M}, \epsilon) \cap A \neq \emptyset$ for all $\epsilon > 0$. So then $\mathcal{M}$ is in the closure of A. However, the closure of a closed set is the set itself. Hence, $\mathcal{M}$ is in A and is therefore the maximum. ■

Sometimes, it is more convenient to replace the base set $\mathbb{K}^N$ by a subset $D \subseteq \mathbb{K}^N$. This may be because we want the "universe" of points we want to analyse be from D rather than $\mathbb{K}^N$.

Definition 18.159 *(Relatively Open Sets). Let $D \subseteq \mathbb{K}^N$, which is our new domain. The set $U \subseteq D$ is called relatively open in D if $U \neq \emptyset$ or for every $x \in U$, there exists $r > 0$ such that $B(x, r) \cap D \subseteq U$.*

Remark 18.160 *Just for intution on why we need $B(x, r) \cap D$, is because for every point x, we look at all the points in $\mathbb{K}^{\mathbb{N}}$ that is within the ball of x. However, the point of relatively open sets is that D is now "our universe" rather than $\mathbb{K}^N$ and hence there isn't any purpose to looking at points outside of our universe, which is D. Hence, the reason we put a restriction at only looking at points in the ball that are also in our universe.*

Definition 18.161 *(Relatively Closed Sets). $A \subseteq D$ is called relatively closed in D if the complement of $A^c \cap D$ is relatively open in D.*

Here, we say U is open in D or A is closed in D. If $D = \mathbb{K}^N$, then open/closed is equivalent to relatively open/closed.

Proposition 18.162 *(Basic properties of relatively open sets). Let $D \subseteq \mathbb{K}^N$. Then,*

(i) $\emptyset$ and D are open in D;

(ii) Arbitrary unions of relatively open sets are open in D;

(iii) Finite intersection of relatively open sets are open in D.

These properties can be used to prove many properties of continuous functions. Furthermore, any collection of subsets of D with the above properties holding for D, is known as a **topology**.

Definition 18.163 *(Diameter of a set). Let $K_n \subset \mathbb{R}^N$ be a closed and non-empty set. The diameter of the set is*

$$diam(K_n) := \sup_{x,y \in K_n} ||x - y|| \to 0.$$

Proposition 18.164 *(Cantor's Intersection Theorem). Let K_n be a decreasing nested sequence of non-empty, closed and bounded sets of $\mathbb{R}$. The intersection is non-empty, that is*

$$\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset.$$

Proof:(Sketch). Show that $\{x_n\}$ is a Cauchy sequence. From that, show that the limit of the sequence is in every set K_n and hence it is in the intersection. ■

Remark 18.165 *Note that the intersection of open sets will be an empty set.*

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19. Closures and their sequential characeterisation

19.3.10 Closures

We first look at the standard definitions.

Definition 19.166 *(Interior Point). A point $x \in A$ is called an interior point of A if there exists $r > 0$ such that $B(x, r) \subseteq A$.*

Definition 19.167 *(Interior). The interior of the set A, is the set of all interior points of A.*

$$int(A) = \{x \in A : \text{ } x \text{ is an interior point of } A\}.$$

Definition 19.168 *(Closure). The closure of the set A is the set of all points that are "close" to A.*

$$\bar{A} = \{x \in \mathbb{K}^N : B(x, r) \cap A \neq \emptyset \textbf{ for all } \boldsymbol{r} > 0\}$$

Notice that the closure requires for all radius r.

Definition 19.169 *(Boundary). The boundary of the set A are all the points in the closure of A that are **not** in the interior of A.*

$$\partial A = \bar{A} - int(A).$$

However, these definitions can make building up other properties more cumbersome, hence we introduce more generalised definitions.

Definition 19.170 *(Interior of a set). Let $A \subseteq \mathbb{K}^N$. The interior of A, denoted by int(A), is the **largest open set** in $\mathbb{K}^N$ that is **included** in A. By definition, int(A) is open and $int(A) \subseteq A$. More precisely,*

$$int(A) = \bigcup \{U \subseteq \mathbb{K}^N : \text{ } U \text{ is open and } U \subseteq A\}$$

Hence, if $int(A) = A \leftrightarrow A$ is open.

We note that for something to be in the interior, we need to be able to find a radius such that the open ball around that point is also in the set A.

Definition 19.171 *(Closure of a set). The closure of a set A, denoted by $\bar{A}$ is defined as the **smallest closed set** in $\mathbb{K}^N$ that **contains** A. More precisely,*

$$\bar{A} = \bigcap \{F \subseteq \mathbb{K}^N : \text{ } F \text{ is closed and } A \subseteq F\}.$$

Hence, if $\bar{A} = A \leftrightarrow A$ is closed.

Proposition 19.172 *Let $A \subseteq \mathbb{K}^N$. Then the following assertions are true.*

(i) The set int(A) is open. Moreover, A is open if and only if int(A) = A.

(ii) The set $\bar{A}$ is closed. Moreover, A is closed if and only if $\bar{A} = A$.

Claim 19.173 *The complement of the closure of A is the interior of the complement of A.*

$$(\bar{A})^c = Int(A^c).$$

Proof: We have that

$$\begin{aligned}
\bar{A} &= \cap\{F \subseteq \mathbb{K}^N : \text{F is closed, } A \subseteq F\} \\
(\bar{A})^c &= \cup\{F^c \subseteq \mathbb{K}^N : F^c \text{ is closed, } F^c \subseteq A^c\} \\
&= \cup\{U \subseteq \mathbb{K}^N : \text{U is open, } U \subseteq A^c\} \\
&= int(A^c).
\end{aligned}$$

where we have that $U = F^c$ and the last equality arising by definition. ∎

19.3.11 Sequential characeterisation of closure

We now make a connection between the closure of a set A and sequences in A. We want to look at the fact that all accumulation points of sequences in A lie in the closure $\bar{A}$. Furthermore, we can also show that every point in the closure $\bar{A}$ is the limit of a sequence in A. Hence, we can actually characterise the closure $\bar{A}$ as the set of points which are accessible by a sequence of points in A.

Proposition 19.174 *(Sequential Characterisation of closure). Let $A \subseteq \mathbb{K}^N$. Then we have*

(i) If $\{x_n\}$ is a sequence in A, which converges to $x \in \mathbb{K}^N$, then $x \in \bar{A}$;

(ii) If $x \in \bar{A}$, then there exists a sequence $\{x_n\}$ in A converging to x.

Proof: (i) First recall that the closure is $\bar{A} = \{x \in \mathbb{K}^N : \forall\, r > 0, B(x,r) \cap A \neq \emptyset\}$. As $x_n \to x$ as $n \to \infty$, combined with the fact that $x_n \in A$ for all $n \geq 0$, we have that for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $||x_n - x||< \epsilon$ for all $n \geq n_\epsilon$. We can re-express this as $x_n \in B(x,\epsilon)$ for all $n \geq n_\epsilon$. We let $\epsilon = r$ and hence we have that $x_n \in B(x,r) \cap A$ for all $n \geq n_\epsilon$. Therefore, we get the claim that $x \in \bar{A}$ as $B(x,\epsilon) \cap A \neq \emptyset$.

(ii) If $x \in \bar{A}$, then by definition of $\bar{A}$, the sets $B(x,r) \cap A \neq \emptyset$ for all $r > 0$. In particular, for every $n \in \mathbb{N}$, we can choose $x_n \in B(x,1/n) \cap A \neq \emptyset$. Hence, there exists $x_n \in A$ and $||x_n - x||\to 0$, that is, $x_n \to x$. Hence, we have found the sequence needed.

∎

We can now use this proposition on closed sets.

Proposition 19.175 *(Sequential characterisation of closed sets). For a set $A \subset \mathbb{K}^N$, then we have that A is closed if and only if every convergent sequence in A has its limit in $A \subseteq \mathbb{K}^N$.*

Proof: We use that the closure of a closed set is itself.

$\rightarrow$

If A is closed, then $A = \bar{A}$. Then, let $\{x_n\}$ be a sequence in A which converges to x. From the sequential characterisation of closures, the limit x is in $\bar{A}$. However, as the closure of A is itself, then that means that $x \in A$.

$\leftarrow$

First, we note that $A \subseteq \bar{A}$ by definition of $\bar{A}$. Then, we assume that every sequence in A has its limit in A. We need to show that $\bar{A} \subseteq A$ to show $A = \bar{A}$ and hence A is closed. We pick an arbitrary $x \in \bar{A}$ and from the sequential characteristic of closures, then there exists a sequence in A that converges to x. However, we already assumed that the limit $x \in A$. Therefore, $\bar{A} \subseteq A$, which then shows that $\bar{A} = A$ and can only be true if A is closed. $\blacksquare$

This characterisation makes sense, as we know that the limit point of a sequence is in the closure of the set and since a closed set's closure is itself, then the limit point must also be in the closed set.

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20. Limits and Continuity of Functions

20.4 Limits and Continuity

20.4.1 Limits for Functions

Definition 20.176 *(Limits for functions). Let $f : D \to \mathbb{K}^N$ be a function with domain $D \subseteq \mathbb{K}^d$ and $x_0 \in \bar{D}$ Let $b \in \mathbb{K}^N$.*

(i) We say that f(x) converges to b as x approaches x_0 if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$||f(x) - b|| < \epsilon$$

for all $x \in D/\{x_0\}$ with $0 < ||x - x_0|| < \delta(\epsilon)$. We write

$$f(x) \to b \text{ as } x \to x_0 \quad \text{or} \quad b = \lim_{x \to x_0} f(x).$$

(ii) Suppose that $D \subseteq \mathbb{R}$ is not bounded from above. Then $b = \lim_{x \to \infty} f(x)$ if for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$||f(x) - b|| < \epsilon$$

for all $x \in D$ with $x > \delta(\epsilon)$.

Here, note that we have a sequence in the domain approaching x_0 with is in the closure $\bar{D}$, not necessarily in D. Additionally, (ii) looks at finding the rank δ for which $f(x)$ is within an ϵ neighbourhood of b.

We can formulate the definition of a limit in many ways.

Proposition 20.177 *The following assertions are equivalent.*

(i) $\lim_{x \to x_0} f(x) = b$;

(ii) For every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $f(x) \in B(b, \epsilon)$ for all $x \in B(x_0, \delta(\epsilon)) \cap D/\{x_0\}$ with $x \neq x_0$;

(iii) $\lim_{n \to \infty} f(x_n) = b$ ***for every sequence*** *$\{x_n\}$ in $D/\{x_0\}$ with $x_n \to x_0$ and $x_n \neq x_0$ for all $n \in \mathbb{N}$.*

Proof: (i) $\to$ (ii) by the definition of the limit and the open ball.

(i) $\to$ (iii) Assume that $\lim_{x \to x_0} f(x) = b$. Then, by definition, we have that for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$||f(x) - b|| < \epsilon$$

for all $x \in D/\{x_0\}$ with $||x - x_0|| < \delta$. Let $\{x_n\}$ be a sequence in $D/\{x_0\}$ with $\lim_{n \to \infty} x_n = x_0$. Fix $\epsilon > 0$ to be arbitrary. Then there exists $n_\delta \geq n$ such tht

$$||x_n - x_0|| < \delta$$

for all $n \geq n_\delta$. Hence, we have that $||f(x_n) - b|| < \epsilon$ for all $n \geq n_\delta$. This gives us that $\lim_{n \to \infty} f(x_n) = b$.

(iii) $\rightarrow$ (i). (Contradiction). Assume that $\lim_{x \to x_0} f(x) \neq b$. Then, there exists $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists $x_\delta \in D/\{x_0\}$ with $||x_\delta - x_0||< \delta$ such that $||f(x_\delta - b)||> \epsilon_0$. Let $\{\delta_n\}$ be a sequence of positive numbers converging to 0. (For example, $\delta_n = \frac{1}{n}$). Then, there exists $x_n \in D/\{x_0\}$ with $0 \leq ||x_n - x_0||< \delta_n = \frac{1}{n}$ and $||f(x_n) - b|| \geq \epsilon_0$. By the squeeze law, $x_n \to x_0$ as $n \to \infty$. We get a contradiction as $\lim_{n \to \infty} f(x_n) = b$, but we assumed that the limit was not b. ■

Remark 20.178 *For (ii), we require x to be in the domain D and hence why we take the intersection with D. We are not interested in points outside of the domain, similar to the example with the relatively open sets.*

The equivalence of (i) and (iii) allows us to apply results on limits of sequences to limits of functions.

20.4.2 Continuity of Functions

Definition 20.179 *(Continuity). The function $f : D \to \mathbb{K}^N$ is said to be continuous at $x_0 \in D$ if $\lim_{x \to x_0} f(x) = f(x_0)$. Moreover, f is said to be continuous if it is continuous at every $x \in D$. We define the set*

$$C(D, \mathbb{K}^N) := \{f : D \to \mathbb{K}^N : f \text{ is continuous}\}$$

and call it the space of continuous functions from D into $\mathbb{K}^N$.

Note that we require $x \in D$, not just $x \in \bar{D}$ in order for $f(x_0)$ to be defined and make sense. However, if $\lim_{x \to x_0} f(x)$ exists for a $x_0 \notin D$ but in $x \in \bar{D}$, we can extend the domain to be $D \cup \{x_0\}$ and hence the new function is continuous after we define $f(x_0) := b$.

Proposition 20.180 *(Characterisation of continuity of functions). If for all U open in $\mathbb{K}^N$, then $f^{-1}(U) = \{x \in D : f(x) \in U\}$ is relatively open in D, then f is continuous.*

If for all A closed in $\mathbb{K}^N$, then $f^{-1}(A)$ is relatively closed in D.

The set $C(D, \mathbb{K}^N)$ is a vector space over $\mathbb{K}$ if we define addition and mutliplication by scalars **pointwise**. By limit laws, f+g and αf are continuous if f and g are continuous with $\alpha \in \mathbb{K}$. Note that this vector is only finite dimensional.

20.4.3 Properties of Continuous Functions

Here, we will prove some properties of continuous functions on closed and bounded sets of $\mathbb{K}^N$. We will need to use the Bolzano-Weierstrass Theorem that asserts that every bounded sequence in $\mathbb{K}^N$ has a convergent subsequence.

Definition 20.181 *(Bounded set). A set $A \subseteq \mathbb{K}^N$ is called bounded if there exists $R \geq 0$ such that $||x|| \leq R$ for all $x \in A$.*

Geometrically, you can think of it as the set A being able to be fit into a large enough ball. We can use the following result that helps us to determine whether a set is bounded based on its sequences.

Proposition 20.182 *The set $A \subseteq \mathbb{K}^N$ is bounded if and only if every sequence in A has a convergent subsequence.*

Proof:

$\rightarrow$

Suppose that A is bounded and has a sequence (x_n) in A. This sequence will also be bounded. By the Bolzano-Weierstrass theorem, this bounded sequence will have a convergent subsequence.

$\leftarrow$(Contrapositive).

Suppose that A is not bounded. Then, for every $n \in \mathbb{N}$, there exists $x_n \in A$ such that $||x_n|| \geq n$. Hence, we have found a sequence (x_n) in A where every subsequence is unbounded and therefore divergent. Hence it has no divergent subsequence. By the contrapositive, this means that a sequence with a convergent subsequence implies that the set is bounded. ■

We can further characterise **compact** sets, which in this case are **closed and bounded** sets.

Definition 20.183 *(Sequentially Compact Sets). A set A, where every sequence in A has a convergent subsequence with limit in A.*

Corollary 20.184 *For a set $A \subseteq \mathbb{K}^N$, we have that a set A is closed and bounded if and only if A is sequentially compact.*

Continuous functions carries compact sets to compact sets.

Theorem 20.185 *Let $f \in C(D, \mathbb{K}^N)$ with $D \subseteq \mathbb{K}^d$. If $A \subseteq D$ is closed and bounded, then its image*

$$f(A) = \{y \in \mathbb{K}^N : \textit{there exists } x \in A \ \textit{ s.t. } f(x) = y\}$$

is also closed and bounded.

Proof: To prove that $f(A)$ is a compact subset of $\mathbb{K}^N$, it is enough to show that any sequence $\{y_n\}$ in f(A) contains a convergent subsequence with the limit in f(A).

Let A be closed and bounded. Let $\{y_n\}$ be a sequence in f(A). Then for every $n \in \mathbb{N}$, there exists $x_n \in A$ with $f(x_n) = y_n$. Since A is bounded, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with $x_{n_k} \rightarrow x$ as $k \rightarrow \infty$. Since A is closed, $x \in A$.

By continuity of f,

$$y_{n_k} = f(x_{n_k}) \rightarrow f(x) \in f(A) \quad \textit{as } k \rightarrow \infty.$$

Hence, $\{y_n\}$ has a convergent subsequence with limit in f(A). Hence, f(A) is sequentially compact. This shows that f(A) is closed and bounded. ■

Using the characterisation of continuity involving closed sets, we can prove the following fact on inverse functions.

Proposition 20.186 *(Continuity of inverse functions). Let $D \subseteq \mathbb{K}^d$ be compact (closed and bounded) and $f \in C(D, \mathbb{K}^N)$ injective. Then the inverse function $f^{-1} : f(D) \rightarrow \mathbb{K}^d$ is continuous.*

Proof: Suppose $A \subseteq \mathbb{K}^d$ is closed. Then we know that the image of $(f^{-1})^{-1}[A] = f(A \cap D)$ is closed and bounded since $A \cap D$ is closed and bounded. Hence, the pre-image of every closed subset $A \subseteq \mathbb{K}^d$ under f^{-1} is closed. Hence, the inverse is continuous. ■

We also derive another property of functions on closed and bounded sets.

Theorem 20.187 *(Extreme Value Theorem). Any real-value function that is continuous on a compact (closed and bounded) subset of* $\mathbb{K}^d$ *attains a maximum and minimum. In other words, if* $f \in C(D, \mathbb{R})$*, then there exists a,b* $\in D$ *such that*

$$m := f(a) \leq f(x) \leq f(b) := M$$

for all $x \in D$*. Here,* m *is the* ***minimum*** *and* M *is the* ***maximum****. So the domain D is closed and bounded.*

Proof: Since D is a compact subset of $\mathbb{K}^d$ and $f : D \to \mathbb{R}$ is continuous, we know that f(D) is also compact from the previous theorem. Hence, f(D) is a closed and bounded subset of $\mathbb{R}$. By the completeness of $\mathbb{R}$, we infer that a sup f(D) and inf f(D) exists in $\mathbb{R}$ (as we have a bounded set). f(D) is closed so that f(D) contains all its accumulation points. As the infimum and supremum is also an accumulation point, this implies that

$$\sup f(D) \in f(D) \quad \textit{and} \quad \inf f(D) \in f(D).$$

As the supremum and infimum is within the set, this means that we have a **maximum** and **minimum** for f(D). ■

Continuous functions can be characterised by open or closed sets.

Proposition 20.188 *Let* $D \subseteq \mathbb{R}^d$ *and* $f : D \to \mathbb{R}^N$ *be a function. Given the set* $U \subseteq \mathbb{R}^N$*, we define the* ***inverse image*** *or the* ***pre-image*** *to be the set*

$$f^{-1}[U] := \{x \in D : f(x) \in U\}.$$

Furthermore, the following statements are equivalent.

(i) $f : D \to \mathbb{R}^N$ *is continuous;*

(ii) $f^{-1}[U]$ *is open in D for every open set* $U \subseteq \mathbb{R}^N$*;*

(iii) $f^{-1}[A]$ *is closed in D for every closed set* $A \subseteq \mathbb{R}^N$*.*

Remark 20.189 *Note that the image of an open or closed set does not need to be open or closed respectively!*

Remark 20.190 *If the set A is compact and f is continuous, then f(A) is compact. Resultantly, a continuous function on a compact set into* $\mathbb{R}$ *has a maximum and a minimum.*

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21. Uniform Continuity and Convergence

21.4.4 Uniform Continuity

Another consequence of compactness of sets is the idea of uniform continuity. We can now look at stronger definitions of continuity. Usually for continuity, δ depends on ϵ and x_0. However, if the domain is compact, δ is now independent of the point $x_0 \in D$.

Definition 21.191 *(Uniform Continuous). Let $D \subseteq \mathbb{K}^d$ and $f : D \to \mathbb{K}^N$. We say that f is uniformly continuous on D if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that*

$$||f(x) - f(y)|| < \epsilon$$

for all x,y $\in D$ with $||x - y|| < \delta$.

Remark 21.192 *The difference between this and continuous function is that for a continuous function we had that for a function $f : D \to \mathbb{K}^N$, then it is continuous on D if it is continuous at each point in D, that is, for all $x \in D$ and for all $\epsilon > 0$, there exists $\delta = \delta(x, \epsilon) > 0$ such that*

$$||f(x) - f(y)|| < \epsilon$$

for all x,y $\in D$ with $||x - y|| < \delta$.

In the case of a continuous function, the δ depends on both x and ϵ, whilst for uniform continuous functions, it only depends on ϵ.

Therefore, we have the statement that

A uniform continuous function implies a continuous function.

Here, δ must be the same for all x for uniform continuity. The converse does not hold.

Theorem 21.193 *Any continuous function from a compact subset of $\mathbb{K}^d$ to $\mathbb{K}^N$ is uniformly continuous.*

Proof: Let $f : D \to \mathbb{K}^N$ be continuous and $D \subseteq \mathbb{K}^d$ is a compact subset. Assume by contradiction, that f is not uniformly continuous. That is, there exists an $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists $x_\delta, y_\delta \in D$ with

$$||x_\delta - y_\delta|| < \delta$$

for which

$$||f(x_\delta) - f(y_\delta)|| \geq \epsilon_0.$$

Let $\{\delta_n\}_{n\geq 1}$ be a sequence of positive numbers converging to 0. (For example, $\delta_n = \frac{1}{n}$ for $n \geq 1$). Then, there exists $\{x_n\}$ and $\{y_n\}$ in D such that

$$||x_n - y_n|| < \frac{1}{n} \quad \forall n \geq 1$$

$$||f(x_n) - f(y_n)|| \geq \epsilon_0 \quad \forall n \geq 1.$$

Since D is compact, we have that D is sequentially compact. Then, there exists a subsequence $\{x_{n_k}\}$ converging to x in D. Using that

$$0 < ||x_{n_k} - y_{n_k}|| < \frac{1}{n_k}$$

for all $k \geq 1$ and $\lim_{k\to\infty} x_{n_k} = x$. We conclude that

$$\lim_{k\to\infty} y_{n_k} = x.$$

We arrive at a contradiction of

$$0 \leq ||y_{n_k} - x|| \leq ||y_{n_k} - x_{n_k}|| + ||x_{n_k} - x||$$

where the last 2 terms goes to 0 as $k \to \infty$.

Recall that

$$||f(x_n) - f(y_n)|| \geq \epsilon_0$$

for all $n \geq 1$. Tus, taking n_k instead of n, we get that

$$||f(x_{n_k} - f(y_{n_k}))|| \geq \epsilon_0$$

for all $k \geq 1$.

Take $k \to \infty$ in the above inequality to obtain that

$$||f(x) - f(x)|| \geq \epsilon_0.$$

We use the continuity of f, where by

$$f(x_{n_k}) \to f(x) \quad k \to \infty;$$
$$f(y_{n_k}) \to f(x) \quad k \to \infty.$$

This contradicts the claim that $||f(x) - f(x)|| \geq \epsilon_0$. Hence, the claim is proven. ■

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22. Pointwise and Uniform Convergence and the Supremum Norm

22.5 Uniform Convergence

22.5.1 Pointwise and Uniform Convergence

We look at sequences of functions $f_n : D \to \mathbb{K}^N$ with domain $D \subseteq \mathbb{K}^d$ and $n \in \mathbb{N}$. If we fix $x \in D$, we then have that $f_n(x)$ is a sequence in $\mathbb{K}^N$.

Definition 22.194 *(Pointwise Convergence). We say that the sequence of functions f_n converges pointwise to f on D if for all $x \in D$, the sequence $\{f_n(x)\}$ converges to f(x) as $n \to \infty$. That is, for all $x \in D$ and for all $\epsilon > 0$, there exists $n_{\epsilon,x} \geq 1$ such that*

$$||f_n(x) - f(x)|| < \epsilon$$

for all $n \geq n_{\epsilon,x}$.

In other words, we have that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for all $x \in D$. We write $f_n \to f$ pointwise.

Definition 22.195 *(Uniform Convergence). We say $f_n \to f$ uniformly on D if for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that*

$$||f_n(x) - f(x)|| < \epsilon$$

for all $n > n_\epsilon$ and all $x \in D$. We say that $f_n(x) \to f(x)$ uniformly with respect to $x \in D$.

Something important to note is that

Uniform convergence implies pointwise convergence.

The differences between pointwise and uniform convergence, is that for uniform convergence, we note that for **every** $\epsilon > 0$, the **same** n_0 (rank) can be chosen for all $x \in D$, so the rank n_ϵ only depends on ϵ. In other words, $n_{\epsilon,x}$ for pointwise convergence depends on ϵ and x in the domain.

If $\{f_n\}$ were to converge uniformly on D, then its uniform limit would coincide with the pointwise limit on D. One thing we can do is first to find a pointwise limit, and then see if we can find a way to make the rank independent so that we get the uniform convergence limit. If $\{f_n\}$ does not converge pointwise to f on D, then $\{f_n\}$ does not converge uniformly to f on D (this is the contrapositive of the previous sentence).

So why are interested in this? Suppose we had $f_n \in [a, b]$ and $f_n \to f$ pointwise on [a,b]. Then if we had

$$\lim_{n\to\infty} \int_a^b f_n(x)dx$$

We can't just pass the limit inside the integral in general if we only have pointwise convergence, we require uniform convergence!

Practical Applications When studying uniform convergence, there are 2 steps.

1. Find what is the pointwise limit.

2. See if you can "upgrade" it to a uniform limit, where the rank now does not depend on x.

We can do this because the uniform limit is the same as the pointwise limit. Furthermore, make sure you separate out special cases where the limit differs. In (2), to show that $\{f_n\}$ converges uniformly to f (pointwise limit) on some domain D, we need to look for a sequence of positive numbers $\{g_n\}_{n\geq 1}$ such that it bounds it from above

$$||f_n(x) - f(x)|| \leq g_n$$

for all $n \geq 1$ and $x \in D$. If we can find such a sequence, then we can express $||f_n(x) - f(x)||$ as bounded by g_n and therefore show it converges uniformly, where the rank does not depend on x anymore, just ϵ.

Remark 22.196 *The word uniform/uniformly can only be used when we discuss sequences of functions. For a sequences of functions, once you fix x, you get a sequences of vectors in $\mathbb{K}^N$. However, if you change x, you get a different sequence of vectors. Hence, uniformly helps here as the property we are interested in holds uniformly if it holds for all values of x in the domain D.*

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23. Supremum Norm and Uniform Cauchy Sequences

23.5.2 Supremum Norm

We now want to be able to introduce the idea of a norm so that we can bring together concepts on sequences to concepts on uniform convergence.

Definition 23.197 *(Supremum Norm). Let $f : D \to \mathbb{K}^N$ be a function. We define its* ***supremum norm*** *by*

$$||f||_{\infty,D} = \sup_{x \in D} ||f(x)||.$$

Remark 23.198 *We have that $||f||_{\infty} < \infty$ if and only if f is a bounded function. The reason this is is that we plug a vector x into the function, take the norm of it, and do this for all $x \in D$. From this, we get a list of non-negative numbers (as the norm takes a vector and gives a non-negative number), and now we take the supremum of this list. If the function is bounded, then that means when we take the list of non-negative numbers, a supremum must exist by completeness of $\mathbb{R}$.*

Remark 23.199 *If $f : D \to \mathbb{K}^N$ is continuous and the domain D is a compact subset of $\mathbb{K}^d$, then we know that f is bounded, that is $||f||_{\infty,D} < \infty$. This holds from previous theorems we've shown as continuous functions on compact domains preserves compactness.*

We now show how the supremum norm has many similar properties to the Euclidean norm.

Proposition 23.200 *(Basic properties of $||.||_{\infty,D}$). Let $f, g : D \to \mathbb{K}^N$ be functions. Then we have that*

(i) $||f||_{\infty,D} \geq 0$ with equality if and only if f(x) = 0 for all $x \in D$;

(ii) $||\alpha f||_{\infty,D} = |\alpha| ||f||_{\infty,D}$ for all $\alpha \in \mathbb{K}$;

(iii) $||f + g||_{\infty,D} \leq ||f||_{\infty} + ||g||_{\infty,D}$ (Triangle property).

Proof: (i) This holds as we have a list of non-negative numbers to take the supremum over. The supremum can only be 0 if every element of the list is the number 0. Hence, f(x) = 0 for all $x \in D$.

(ii) This is immediate from norm in $\mathbb{K}^N$ to pull α out.

(iii) $||f(x)+g(x)|| \leq ||f(x)|| + ||g(x)||$ for all $x \in D$ by triangle inequality. $||f(x)|| + ||g(x)|| \leq ||f||_{\infty,D} + ||g||_{\infty,D}$. By definition, this is an upper bound so we have that

$$||f + g||_{\infty,D} \leq ||f||_{\infty,D} + ||g||_{\infty,D}$$

■

Corollary 23.201 *(Reversed Triangle Inequality). If f and g are bounded functions, then*

$$|||f||_{\infty,D} - ||g||_{\infty,D}| \leq ||f - g||_{\infty,D}.$$

We can now express uniform convergence in terms of the supremum norm.

Proposition 23.202 *Let $f_n : D \to \mathbb{K}^N$ be functions. Then*

$$f_n \to f \text{ uniformly on } D \text{ if and only if } ||f_n - f||_{\infty,D} \to 0 \text{ as } n \to \infty.$$

23.5.3 Uniform Cauchy Sequences

We now want to extend the ideas of Cauchy sequences to functions.

Definition 23.203 *(Uniform Cauchy Sequences). Let $f_n : D \to \mathbb{K}^N$ be a sequence of functions with $D \subseteq \mathbb{K}^d$. We say that $\{f_n\}$ is uniformly Cauchy on D if for all $\epsilon > 0$, there exists $n_\epsilon \geq 1$ such that $||f_m - f_n||_{\infty,D} < \epsilon$ for all $m > n \geq n_\epsilon$.*

Remark 23.204 *This means that $\{f_n(x)\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{K}^N$ and that the rank n_ϵ in the definition of Cauchy sequences does not depend on x in D.*

Furthermore, note that we omit the argument x when taking the supremum norm of $||f_m - f_n||_{\infty,D}$ as by definition, the supremum norm evaluates over all $x \in D$.

Theorem 23.205 *Let $f_n : D \to \mathbb{K}^N$ be a sequence of functions where $D \subseteq \mathbb{K}^d$. Then we have that*

$$\{f_n\} \text{ converges uniformly on } D \leftrightarrow \{f_n\} \text{ is uniformly Cauchy on } D.$$

Theorem 23.206 *Let $D \subseteq \mathbb{K}^d$. Suppose that $f_n : D \to \mathbb{K}^N$ is continuous on D for all $n \geq 1$.*

If f_n converges uniformly to f on D, then $f : D \to \mathbb{K}^N$ must be continuous on D.

In other words, for a sequence of continuous functions that converges uniformly, its uniform limit must also be continuous.

We have a very useful and practical corollary.

Corollary 23.207 *Assume that $f_n \in \mathcal{C}(D, \mathbb{K}^N)$ where $D \subseteq \mathbb{K}^d$ and $f_n \to f$ pointwise on D as $n \to \infty$. If f is **not continuous** on D, then f_n **cannot converge** uniformly to f on D.*

Remark 23.208 *So if the pointwise limit f is not continuous, then we can't have that f_n converges to f **uniformly**. This result does not actually say anything if the pointwise limit f is continuous on D. So, if each f_n is continuous and it converges pointwise to a continuous function f, the corollary doesn't tell us anything. We can only conclude whether does it not converge uniformly.*

Remark 23.209 *It may be the case that $f_n \in \mathcal{C}(D, \mathbb{K}^N)$, where $f_n \to f$ pointwise on D and $f \in \mathcal{C}(D, \mathbb{K}^N)$ but then f_n **does not** converge uniformly on D.*

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24. Absolute and Uniform Convergence for series of functions

24.5.4 Absolute and Uniform Convergence for series of functions

Definition 24.210 *(Absolute Convergence for series of functions). Let $g_k : D \to \mathbb{K}^N$ be a* ***sequence*** *of functions, where $D \subseteq \mathbb{K}^d$. The series $\sum_{k=0}^{\infty} g_k$ is called absolutely convergent on D if for every $x \in D$, the series $\sum_{k=0}^{\infty} g_k(x)$ converges absolutely. In other words,*

$$\sum_{k=0}^{\infty} ||g_k(x)||_{\infty,D}$$

converges in $\mathbb{R}$ for all $x \in D$.

Remark 24.211 *So what this means is that we fix $x \in D$ and then we get a series of vectors by evaluating each function at x. Recall that to check for convergence of a series of vectors, you look at the series of norms and see does that converge in $\mathbb{R}$. Then see does the series of norms converge in $\mathbb{R}$ for every $x \in D$. If it does, then it is absolutely convergent.*

Definition 24.212 *(Uniform Convergence for series of functions). Let $g_k : D \to \mathbb{K}^N$ be a* ***sequence*** *of functions, where $D \subseteq \mathbb{K}^d$. If the sequence of $\{f_n\}$ of partial sums converges uniformly on D, where $f_n(x) = \sum_{k=0}^{n} g_k(x)$ for all $x \in D$, the series $\sum_{k=0}^{\infty} g_k$ converges uniformly.*

Remark 24.213 *A series converges if the series of partial sums converges. However, we want uniform convergence, so we need to have that the series of partial sums to converge uniformly on D. So the sequence of partial sums is a sequence of functions $\{f_n\}$. Recall that the uniform convergence of a sequence of functions is when the supremum norm $||f_n - f||_{\infty,D} \to 0$ as $n \to \infty$.*

We can now introduce a criterion to check for uniform convergence of a series.

Theorem 24.214 *(Weierstrass M-Test). Let $g_n : D \to \mathbb{K}^N$ be a sequence of functions. If*

$$\sum_{k=0}^{\infty} ||g_k||_{\infty,D}$$

converges, then the original series

$$\sum_{k=0}^{\infty} g_k$$

converges absolutely and therefore uniformly on D.

Remark 24.215 *Here, we look at the series of supremum norms of each function g_k, which is finding the largest value of $g_k(x)$ for all $x \in D$. We get a series of non-negative numbers when we take the supremum norms and if this series of non-negative numbers converges (where we can use many of the tests for non-negative series), then the series converges absolutely and uniformly on the domain D.*

Remark 24.216 *When you have a power series, you can check whether does it converge uniformly within the radius of convergence. First, you need to find the radius of convergence using the Cauchy-Hadamard theorem. You know that the power series converges absolutely, and therefore also unconditionally within this radius of convergence. Now we want to check does it converge uniformly within this radius of convergence. We can use the Weierstrass M-test to check is this the case by seeing does the series of the norm for values of* $z < |\rho|$ *converge in* $\mathbb{R}$.

Remark 24.217 *So if* $\sum_{k=0}^{\infty} ||g||_{\infty,D}$ *converges, then* $\sum_{k=0}^{\infty} g_k(x)$ *converges uniformly on D. In practice,* $\sum_{k=0}^{\infty} g_k(x)$ *converges uniformly on* D *if we can find* $a_k \geq 0$ *independent of* $x \in D$ *such that* $||g_k(x)|| \leq a_k$ *for all* $x \in D$ *and* $\sum_{k=0}^{\infty} a_k$ *converges.*

Remark 24.218 *Note that this condition is* ***sufficient*** *but not necessary for the uniform convergence of the series of functions* $\sum_{k=0}^{\infty} g_k$. *So the series of supremum norm diverges but the series of functions may converge. Hence, if this doesn't hold, then we would look at the sequence of partial sums and see if that converges uniformly by looking at* $||f_n - f||_{\infty,D} \to 0$ *as* $n \to \infty$.

Corollary 24.219 *(Continuity of power series). Suppose that* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *is a power series in* $\mathbb{K}^N$ *with radius of convergence* $\rho > 0$. *Then it converges absolutely on* $B(0,\rho)$ *and it converges uniformly on* $B(0,r)$ *for every* $0 < r < \rho$. *Moreover,* $f : B(0,\rho) \to \mathbb{K}^N$ *is continuous.*

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25. Differentiation and Integration

25.5.5 Differentiation and Integration

We now review derivatives except we also allow for differentiation of complex numbers. First recall that a function f is continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$.

Definition 25.220 *(Differentiation). Let $D \subseteq \mathbb{K}$, be an open set and $f : D \to \mathbb{K}^N$. We say that*

1. *f is differentiable at $x_0 \in D$ if there exists $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$;*
2. *f is differentiable on D if f is differentiable at each point in D;*
3. *f is continuously differentiable on D if f is differentiable on D and its derivative $f' : D \to \mathbb{K}^N$ is continuous on D.*

Remark 25.221 *The reason we require D to be an open set is that when we look at x, we know for sure that there exists a $h > 0$ such that $x + h$ in the domain D.*

We use the notation of $C^1(D, \mathbb{K}^N) = \{f : D \to \mathbb{K}^N, \text{ f is continuously differentiable}\}$, where D is the domain and $\mathbb{K}^N$ is the codomain. Additionally, we let $C^0(D, \mathbb{K}^N) := C(D, \mathbb{K}^N)$, so it is not continuously differentiable, it is just continuous.

Remark 25.222 *$C^1(D, \mathbb{K}^N)$ is a vector subspace of $C(D, \mathbb{K}^N)$. (f+g)' = f' + g' and $(\alpha f)' = \alpha f'$.*

Proposition 25.223 *For a function $f : D \to \mathbb{K}^N$, where D is open in $\mathbb{K}^N$, that is differentiable at $x_0 \in D$ then f is continuous at x_0.*

Proof: So if $f'(x_0)$ exists, then we have that

$$||f(x) - f(x_0)|| = |x - x_0| \left|\left|\frac{f(x) - f(x_0)}{x - x_0}\right|\right| \to 0 \circ ||f'(x_0)|| = 0$$

as $x \to x_0$. ∎

Now, if f is differentiable at $x_0 \in D$, we let

$$\phi(x) := \frac{f(x) - f(x_0)}{x - x_0}$$

for all $x \neq x_0$ and we also have that $\phi(x_0) = f'(x_0)$. Then $\phi : D \to \mathbb{K}^N$ and by definition of $f'(x_0)$ being differentiable, then ϕ is continuous at x_0.

Furthermore, by how we construct ϕ, we can rearrange the terms to get

$$f(x) = f(x_0) + \phi(x)(x - x_0)$$

for all $x \in D$. Hence, if there exists a ϕ such that the above statement holds, then ϕ is continuous at x_0 and as $\phi(x_0) = f'(x_0)$, this means that f is differentiable at x_0. This leads to Carathéodory's characterisation of the derivative. We only need to find a function ϕ which is continuous at x_0 and hence it shows that the function f is differentiable at x_0.

Proposition 25.224 *(Catathéodory Criteria for differentiability). Let $f : D \to \mathbb{K}^N$ be a function with $D \subseteq \mathbb{K}$ is open and $x_0 \in D$. Then we have*

(i) The function f is differentiable at $x_0 \in D$ if and only if

(ii) There exists a function $\phi : D \to \mathbb{K}^N$ that is continuous at $x_0 \in D$ such that $f(x) = f(x_0) + \phi(x)(x - x_0)$ for all $x \in D$.

Moreover, if (ii) holds, then $f'(x_0) = \phi(x_0)$.

Hence, we can now show some rules on differentiation.

Proposition 25.225 *(Product and Quotient Rule). Suppose that $f : D \to \mathbb{K}$ and $g : D \to \mathbb{K}^N$ are functions which are differentiable at x_0. Then we have that*

(i) (Product Rule). $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$;

(ii) (Quotient Rule). $(\frac{1}{f})'(x_0) = -\frac{f'(x_0)}{(f(x_0))^2}$ if $f(x_0) \neq 0$.

Proposition 25.226 *(Chain Rule). Suppose that $U, V \subseteq \mathbb{K}$ are open sets, that $f : V \to \mathbb{K}^N$ and $g : U \to V$. Further, suppose that g is differentiable at $x_0 \in U$ and that f is differentiable at $g(x_0) \in V$. Then $f \circ g$ is differentiable at x_0 and*

$$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0).$$

Now assume that $D \subseteq K$ is open and that $f : D \to \mathbb{K}$ is injective. Then it has an inverse function $f^{-1} : f(D) \to \mathbb{K}$. By the chain rule, we compute the derivative of inverse function by differentiating the identity $f(f^{-1}(y)) = y$.

Proposition 25.227 *(Differentiability of inverse function). Suppose that $D \subseteq \mathbb{K}$ is open and that $f : D \to \mathbb{K}$ is injective and that f is differentiable at $f^{-1}(y_0)$. If $f'(f^{-1}(y_0)) \neq 0$, then f^{-1} is differentiable at $y_0 \in f(D)$ and the derivative is given by the quotient rule.*

To integrate a complex function, if $f(t) = u(t) + iv(t)$ with $u, v \in C([a, b], \mathbb{R})$, then we integrate the real and imaginary parts separately

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt.$$

For a vector valued function $f = (f_1, f_2, ..., f_N) \in C([a, b], \mathbb{K}^N)$, we define its integral as the integral of each component in the vector valued function.

The Fundamental Theorem of calculus provides a link between integration and differentiation.

Theorem 25.228 *(Fundamental Theorem of Calculus). The following assertions hold:*

(i) If $f \in C^1([a,b], \mathbb{K}^N)$*, then*

$$f(b) - f(a) = \int_a^b f'(t)dt;$$

(ii) If $f \in c([a,b], \mathbb{K}^N)$*, then for every* $c \in [a,b]$*,*

$$\frac{d}{dt}\int_c^t f(s)ds = f(t)$$

for all $t \in [a,b]$*.*

Here, the domain is the compact interval [a,b]. In the case of vector valued functions, we use the mean value theorem.

Theorem 25.229 *(Mean Value Theorem). Suppose that* $D \subseteq \mathbb{K}$ *is open and* $f \in C^1(D, \mathbb{K}^N)$*. If* $a, b \in D$ *are such that the line segment joining a and b lies within D (i.e.* $(1-t)a + b \in D$ *for all* $t \in [0,1]$*), then we have*

$$f(b) - f(a) = (b-a)\int_0^1 f'((1-t)a + tb)dt$$

We now integrate over the line segment connecting a and b which we require to be entirely in D. Hence, we get something similar to a triangle inequality.

Lemma 25.230 *If* $f \in C([a,b], \mathbb{K}^N)$*, then*

$$||\int_a^b f(t)dt|| \leq \int_a^b ||f(t)||dt.$$

Theorem 25.231 *Let* $f_n \in C([a,b], \mathbb{K}^N)$ *for every* $n \geq 1$ *and assume that* $f_n \to f$ ***uniformly*** *on [a,b]. Then* $f \in C([a,b], \mathbb{K}^N)$ *and*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx.$$

Remark 25.232 *This result is not true if we replace the uniform convergence of* $f_n \to f$ *by pointwise convergence.*

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26. Integral with Parameters

26.5.6 Integral with Parameters

We now look at parameter integrals.

Definition 26.233 *(Parameter Integral). Let $D \subseteq \mathbb{K}$ be open and $f \in C([a,b], X, D, \mathbb{K}^N)$. We set*

$$g(x) := \int_a^b f(t,x)dt$$

for all $x \in D$. g is called a ***parameter integral*** *as x is a parameter with respect to integration.*

Proposition 26.234 *(Parameter Integral Theorem). Let $g : D \to \mathbb{R}^N$ be defined as above. Then the following assertions are true.*

(i) $g \in C(D, \mathbb{K}^N)$;

(ii) *If*

$$\frac{\partial f}{\partial x} \in C([a,b]\ X\ D, \mathbb{K}^N),$$

then $g \in C^1(D, \mathbb{K}^N)$ and

$$g'(x) = \int_a^b \frac{\partial}{\partial x} f(t,x)dt$$

for all $x \in D$.

26.5.7 Uniform Convergence, Integration, and Differentiation

We now consider integrals of a sequence of functions and see when can we interchange limit and integral. In particular, we require uniform convergence in order to do so.

Theorem 26.235 *Let $f_n \in C([a,b], \mathbb{K}^N)$, where $D \subseteq \mathbb{K}^d$ and assume that $f_n \to f$ uniformly on [a,b]. Then $f \in C([a,b], \mathbb{K}^N)$ and we can interchange the limit and integral*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx.$$

Uniform convergence preserves continuity in the limit and allows us to interchange integration and limit. However, other properties of f_n such as differentiability does not necessarily hold in the uniform limit function f.

However, if we asumme that derivatives converge to a function g locally uniformly on D and f_n converges pointwise to f on D, then the limit function is differentiable. First, we require **locally uniform convergence**.

Definition 26.236 *(Locally Uniform Convergence). We say that $f_n \to f$ locally uniformly on D if for all $x \in D$, there exists $r > 0$ such that $f_n \to f$ uniformly on $B(x, r) \cap D$.*

Lemma 26.237 *If $f_n \to f$ locally uniformly on D and $f_n \in C(D, \mathbb{K}^N)$ for all $n \in \mathbb{N}$, then $f \in C(D, \mathbb{K}^N)$. Hence, local uniform convergence preserves continuity.*

Theorem 26.238 *Let $D \subseteq \mathbb{K}$ be open and $f_n \in C^1(D, \mathbb{K}^N)$ for all $n \in \mathbb{N}$. If $f_n \to f$ pointwise on D* ***and*** *$f_n' \to g$ locally uniformly, then $f \in C^1(D, \mathbb{K}^N)$ and $f' = g$.*

Here, for a sequence of differentiable functions, we preserve differentiability under the pointwise limit.

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27. Introduction to Analytic Functions

27.6 Analytic Functions

27.6.1 Review of Taylor series

First, we review some stuff from calculus. A Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point.

Definition 27.239 *(Taylor Series). The Taylor series of a real/complex-valued function f(x) that is infinitely differentiable (smooth) at a real or complex number a is the power series*

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$$

which can be written compactly as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Remark 27.240 *Recall that when a = 0, the series is called a* ***Maclaurin series****.*

So what are Taylor series? First, we pick a point x_0 on a function. Then, the first derivative will get us a subtangent to that point, a second derivative will give us a parabola and so on. However, if we differentiate the function at the point x_0 infinitely many times and sum up the terms, this will approximate the actual function really well! Note that we already know what $f(x_0)$ but it is differentiating around this point is what helps us approximate the function. The $\frac{f^{(n)}(x_0)}{n!}$ term can be expressed as a sequence/coefficient a_n. Now, if we pick a different point to construct our Taylor series, such as x_1, we will get the same function approximation as the Taylor series around x_0 BUT the coefficients of our series b_k will be different to a_k. Hence, expanding around different points will give us different coefficients.

27.6.2 Analytic Functions

A function that can be represented by a power series is known as an **analytic function**. An analytic function is a function that is locally given by a convergent power series. A function is analytic if and only if its Taylor series about x_0 converges to the function in some neighborhood for every x_0 in its domain. This is a wide class of functions which includes exponential, logarithms, hyperbolic functions, polynomials, and more.

Definition 27.241 *(Analytic Function). Let $D \subseteq \mathbb{K}$ be open and $f : D \to \mathbb{K}^N$. We say that f is analytic on D if for every x_0, there exist $r > 0$ and a sequence $a_k \in \mathbb{K}^N$ such that*

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

for all $x \in B(x_0, r)$. If $D \subseteq \mathbb{R}$, we also say that f is ***real analytic****. Here, the coefficients $a_0, a_1, ...$ are real numbers and the series is convergent to $f(x)$ for x in a neighborhood of x_0.*

Remark 27.242 *Any power series is analytic on the open disk of convergence.*

Remark 27.243 *Any analytic function has derivatives of all orders and the power series in the right hand side coincides with the Taylor series of f about $x = x_0$.*

Intuitively, an analytic function is a function which can be represented by their Taylor series about every point in the domain.

We now look at more properties of power series, which are a form of analytic functions.

Theorem 27.244 *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series in $\mathbb{K}^N$ (i.e. $a_k \in \mathbb{K}^N, z \in \mathbb{K}$) with the radius of convergence $\rho > 0$. Define*

$$g(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} = \sum_{j=k-1}^{\infty} (j+1) a_{j+1} z^j$$

and

$$F(z) = \sum_{k=0}^{\infty} a_k \frac{z^{k+1}}{k+1}.$$

g is the derivative of f(g) whilst F(z) is the integral of f(z). The power series g and f have the same radius of convergence ρ as the power series f. Moreover, F and f are differentiable with $F' = f$ and $f' = g$ on $B(0, \rho)$.

Proof:(Sketch). Use the limit superior test. ■

Remark 27.245 *Here, in the radius of convergence, we can differentiate and integrate the series term by term. Furthermore, it is in fact infinitely differentiable.*

We can write an expression of what the power series looks like after differentiating it k times.

Lemma 27.246 *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series in $\mathbb{K}^N$ with radius of convergence $\rho > 0$. Then for all $z \in B(0, \rho)$, f has derivatives of all orders and*

$$f^{(k)}(z) = k! \sum_{n=k}^{\infty} \binom{n}{k} a_n z^{n-k}.$$

Moreover,

$$a_k = \frac{f^{(k)}(0)}{k!}$$

for all $k \in \mathbb{N}$.

Theorem 27.247 *(Uniquness Theorem for Power Series). Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be power series in $\mathbb{K}^N$, which converges for $|z| < r$ for some $r > 0$. Suppose that there exists $\{z_n\}_n$ in $\mathbb{K}/\{0\}$ such that $z_n \to 0$ as $n \to \infty$ and $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$. Then*

$$f(z) = g(z)$$

for all $z \in B(0, r)$.

Proof:(Sketch). Define a new power series h = f -g. We then need to show that h = 0 on $B(0, r)$ or the coefficients $c_k = 0$ for all $n \in \mathbb{N}$. We do this by induction. ■

Theorem 27.248 *Let* $g(z) = \sum_{k=0}^{\infty} a_k z^k$ *be a power series in* $\mathbb{K}^N$ *with radius of convergence* $\rho > 0$. *If* $z_0 \in B(0, \rho)$ *is arbitrary, then*

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} f^k(z_0)(z - z_0)^k$$

for all $z \in B(z_0, \rho - |z_0|)$.

Remark 27.249 *Some interesting things to note.*

1. *The above theorem gives the representation of* f *by the Taylor series of* f *about* z_0 *in the largest disk about* z_0 *that fits into* $B(0, \rho)$.
2. *It is possible that the power series* $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(z_0)(z-z_0)^k$ *converges in a larger disk centered at* z_0, *so* $B(z_0, \rho - |z_0|)$. *However, in the formula, we take* $z \in B(z_0, \rho - |z_0|)$ *since the power series* $\sum_{k=0}^{\infty} a_k z^k$ *representing* $f(z)$ *diverges for* $|z| > \rho$.
3. *This theorem shows that any power series is analytic on the open disk of convergence.*
4. *Any analytic function has derivatives of all orders and the power series of* $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ *coincides with the Taylor series of* f *about* $x = x_0$.

27.6.3 Intuition of analytic functions

We now look at the intution behind analytic functions.

So suppose we have a function f and it is smooth (infinitely differentiable and therefore continuous too). Then the power series of f at around 0 is

$$f = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

IF the above equation holds if there exists a $\rho > 0$ where for $|x| < \rho$, the function at 0 is equal to its Taylor series. We don't know whether does the series converge to f when x is **near** but not equal to 0. So we don't know if the function is actually represented by the Taylor series, but if it is, then it is analytic. Taking the infinite sum of derivatives makes sense since f is smooth!

We can now generalise it, so that a function has a power series representation at any point. This means that if we pick a point in the domain D, if we construct the Taylor series representation around that point, this approximate the function itself! Note that this is saying that the function itself is approximated by the Taylor series, not the function at the point a.

So a function is real analytic at a if there is some $\rho > 0$ such that

$$f = \sum_{k=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

when $|x - a| < \rho$. Hence, when we are near a (not exactly at a), we can represent the function f as a Taylor series expanded around the point a.

So in terms of descending hierarchy of generality, we have

1. All functions;
2. Continuous functions;
3. Differentiable functions;
4. Smooth functions;
5. Analytic functions.

So analytic functions are smooth functions but also if we write the Taylor series of the function around a point, then the Taylor series converges to the function itself **near** the point a but not exactly at a.

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28. Further properties of Analytic Function

28.6.4 Further properties of Analytic Function

Definition 28.250 *(Path). A path in a set D is a continuous function $\gamma : [a, b] \to D$. We write that $\gamma \in C([a, b], D)$.*

Definition 28.251 *(Path Connected set). An open set $D \subseteq \mathbb{K}$ is called **path connected** if for every pair $x, y \in D$, there exists a path $\gamma \in C([0, 1], D)$ with $\gamma(0) = x$ and $\gamma(1) = y$.*

Theorem 28.252 *(Uniqueness theorem for analytic functions). Let $D \subseteq \mathbb{K}$ be connected and open. Assume that $f, g : D \to \mathbb{K}^N$ are analytic functions on D. Moreover, suppose that $\{x_n\}$ is a sequence in D with $x_n \to x_0$, with $x_0 \in D$ and $x_n \neq x_0$ for all $n \in \mathbb{N}$. If $x_n \to x_0$ and $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$, then $f(x) = g(x)$ for all $x \in D$.*

Proof: (Sketch). There are 2 steps. First, construct a new power series h = f-g and show that this is 0 for x_0 in the ball $B(x_0, r)$.

Then, use the **continuation argument** to help prove that $h(y_0) = 0$ for an arbitrary $y_0 \in D$. ■

Proposition 28.253 *(Analytic Continuation). Suppose that we do not know the form of analytic function f(z), but we know its form inside a circle of convergence C_1. We can represent f(z) as a Taylor series inside this circle. Then, we can choose a point b inside C_1 and again construct a new series centered at b and finding the radius of convergence, we can get a new C_2, which may extend outside of C_1. We have more information regarding the function f(z) and we can continually repeat this so that f(z) can be extended analytically beyond C_1.*

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29. Logarithm

29.6.5 Logarithm

We are now interested in extending the real line analytic function logartihm onto the complex plane through its power series representation. From the uniqueness theorem of analytic functions, such an extension is unique.

Proposition 29.254 *(Real logarithm). The function log:* $(0, \infty) \to \mathbb{R}$ *is analytic. Moreover,*

(i) $log(1) = 0$ *and* $log(e) = 1$*;*

(ii) $log(xy) = logx + logy$ *for all* $x, y > 0$*;*

(iii) $\frac{d}{dx} log(x) = \frac{1}{x}$ *for all* $x > 0$*;*

(iv) *log:* $(0, \infty) \to \mathbb{R}$ *is strictly increasing and bijective;*

(v) The Taylor series expansion of log about $x_0 > 0$ *is given by*

$$logx = logx_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{kx_0^k}(x - x_0)^k$$

if $|x - x_0| < x_0$.

In order to construct a power series representing log(x), we want to find an F(z) such that $F'(z) = f(z)$ where $f(z) = 1/z$, in order to match the property of the logarithm. So first, find the power series representation of f(z) and inside the radius of convergence, we can integral it term by term to get F(z). To do so, we look at the ball centered around z_0 of radius $|z_0|$ for the point z. Add and subtract z_0 to the denomiator of 1/z and then rewrite it as a geometric series. The integrate it term by term to get

$$F(z) = C + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(z - z_0)^k}{(k+1)z_0^k}$$

for some $C \in \mathbb{C}$. For $F(x_0)$, we need $F(x_0) = logx_0$, so then if $z = x$ and $z_0 = x_0$, the summation term cancels out leaving only C, hence why we require it to be $logx_0$. Hence, we arrive at

$$logx = logx_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{kx_0^k}(x - x_0)^k$$

for $|x - x_0| < x_0$.

We now seek to extend the logarithm into the complex plane.

First, we extend it onto the right half of the complex plane, rather than just being resricted on the positive real line. We can define Log z to be

$$Logx = logx_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{kx_0^k}(z - x_0)^k$$

for all $z \in B(x_0, x_0)$, where we now look at complex number which has a real part greater than 0. Before, for log x, that was for the Taylor series around the interval but now for **L**og x, we are looking at the **ball** $B(x_0, x_0)$ for $\{z \in \mathbb{C} : Re\ z > 0\}$.

Now, if we take 2 different points to center the ball, x_0 and x_1, do their corresponding functions defined by their respective Taylor series coincide over the intersection of their balls? Yes! This is due to the uniqueness theorem and hence their Taylor series conincide. Therefore, we can define Log on the right half plane as

$$\bigcup_{x_0>0} B(x_0, x_0) = \{z \in \mathbb{C} : Re\ z > 0\}.$$

Note that any balls around points $z \in \mathbb{C}$ where $Re\ z > 0$, will not touch any points in the negative real line as the balls will have to touch the origin as the ball radius is $|z|$ which is just the length from the origin.

Finally, we can extend the extension to now looking at balls around points in the complex plane, although we need to be careful about the negative real line. This is because if you take a ball from the bottom left quadrant of the complex plane and a ball from the top left quadrant of the complex plane, they do not match up on the negative real line. This is because the top left quadrant's sequence's limit is π whilst it is $-\pi$ for the bottom left quadrant. Hence, we omit the negative real line. We have just constructed the complex logarithm.

Theorem 29.255 *(Complex Logarithm), There exists a unique function Log* $: \mathbb{C}/\{0\} \to \mathbb{C}$ *with the following properties*

1. $Log 1 = 0$;
2. $\frac{d}{dz}(Log z) = \frac{1}{z}$ *for all* $z \in \mathbb{C} - (-\infty, 0]$;
3. $exp(Log\ z) = z$ *for all* $z \in \mathbb{C} - \{0\}$.

Moreover, Log $: \mathbb{C} - (-\infty, 0] \to \mathbb{C}$ *is analytic.*

Definition 29.256 *(Principal Logarithm). The principal Logarithm Log* $: \mathbb{C}/\{0\} \to \mathbb{C}$ *is defined by*

$$LogZ := log|z| + i\ Arg\ z.$$

Definition 29.257 *(Principal Argument). This is the angle ψ of a complex number with respect to the real axis. This is called the principal argument of z and is denoted by Arg z.*

Definition 29.258 *(Principal Value). There is an unique number $t \in (-\pi, \pi]$ such that $z = |z|e^{it}$. This leads to only 1 unique values for principal arguments, rather than having multiple solutions as multiples of 2π.*

Remark 29.259 *The function Arg z:* $\mathbb{C}/\{0\} \to (-\pi, \pi]$ *is not continuous on* $(-\infty, 0]$.

Definition 29.260 *(Principal Powers). For $w \in \mathbb{C}$ and $z \in \mathbb{C} - \{0\}$, we define*

$$z^w = exp(wLogz)$$

and call it a principal power.

Definition 29.261 *(Principal Square root). A principal square root is a special case of principal power*

$$\sqrt{z} := exp(1/2\ \ log\ z).$$

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30. Complex Analysis: Introduction and Path Integrals

30.7 Complex Analysis

30.7.1 Introduction

First we describe why bother working with $\mathbb{C}$. Every $z \in \mathbb{C}$ can be represented as a $x \in \mathbb{R}^2$. Hence, all functions $\phi : \mathbb{C} \to \mathbb{C}$ has an isomorphism to all $\psi : \mathbb{R}^2 \to \mathbb{R}^2$. Hence, they are topoloigcally homeomorphic. We say that $\mathbb{R}^2$ is a real vector space whilst $\mathbb{C}$ is a **field algebra**.

The difference with working with $\mathbb{C}$ rather than $\mathbb{R}^2$ is that we can multiply complex numbers together, which we could not do in $\mathbb{R}^2$.

30.7.2 Complex Numbers Operation

Suppose $z = x + iy$ where $z \in \mathbb{C}$.

Definition 30.262 *(Modululs). The modulus is the length of the complex number.*

$$Mod\ z \coloneqq \sqrt{x^2 + y^2}.$$

Definition 30.263 *(Argument). The argument is the angle of the complex number with the positive real line.*

$$Arg\ z \coloneqq tan^{-1}(\frac{y}{x})$$

Definition 30.264 *(Complex-valued Function). We can write a complex valued as the sum of its real and imaginary parts.*

$$f(z) = u(x, y) + v(x, y)i.$$

30.7.3 Complex Functions Differentiation

Proposition 30.265 *(Complex Differentiable). When we differentiate a complex valued function, let $h \in \mathbb{C}$. Then, if the limit exists*

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

we say f is complex differentiable at z. Here, $f(z)$ must be differentiable in all directions and the derivative to coincide for every direction.

Complex differentiability implies it is differentiable when viewed as a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. Note that this does not hold vice versa!

Definition 30.266 *(Holomorphic Functions). A complex valued function is holomorphic on a region $\tilde{R}$ in the complex plane if at every point of its domain, it is complex differentiable at that point.*

Proposition 30.267 *A function is holomorphic if and only if it is analytic.*

Remark 30.268 *So if a function has a first derivative, it has derivatives of all orders. Then, for any point in the domain, we can expand it as a Taylor series.*

30.7.4 Path Integrals

Definition 30.269 *(Common Assumptions). We let $D \subseteq \mathbb{C}$ be open and $f \in C(D, \mathbb{C}^N)$.*

Remark 30.270 *The reason that we want open sets is that we generally look at neighborhoods around points. Hence, we require open sets.*

Definition 30.271 *(Complex Integral). When we are integrating complex functions, there are multiple paths to integrate over. We have $f(z) = u + vi$ and the path is parameterised by C where $x = x(t)$ and $y = y(t)$ for $t \in [a, b]$. Hence, we have*

$$\int_C f(z)dz = \int_C (u + vi)(dx + idy).$$

Definition 30.272 *(Primitive). We say that f admits a primitive on D, where $D \subseteq \mathbb{C}$ is open, if there exists $F : D \to \mathbb{C}^N$ a differentiable function on D such that $F' = f$ on D.*

Definition 30.273 *(Path Integrals). Let $f \in C(D, \mathbb{C}^N)$ where $D \subseteq \mathbb{C}$ is open. We have 2 cases depending on γ.*

(1) If $\gamma \in C^1([a, b], D)$ is a C^1-path in D, then

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

(2) If γ is a piecewise C^1-path in D, then using the partition in the definition of the piecewise C^1-path,

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(\gamma(t))\gamma'(t)dt.$$

Lemma 30.274 *We have that*

$$|\int_a^b w(t)dt| \leq \int_a^b |w(t)|dt.$$

Proposition 30.275 *(ML Inequality). Let $\gamma \in C([a, b], D)$ be a piecewise C^1-path in D, where $D \subseteq \mathbb{C}$ is open and $f \in C(D, \mathbb{C}^N)$. Then*

$$||\int_\gamma f(z)dz|| \leq ML$$

where $M = max_{t \in [a,b]} ||f(\gamma(t))||$ and $L = \int_a^b |\gamma'(t)|dt$.

Proposition 30.276 *(Path Independent Integrals). Let $f \in C(D, \mathbb{C}^N)$ where $D \subseteq \mathbb{C}$ is open.* ***Assume*** *that f admits a primitive F on D. Then for every piecewise C^1-path γ in D, we have*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

Corollary 30.277 *(Closed Path Integrals). If $f \in C(D, \mathbb{C}^N)$ has a primitive on the open set $D \subseteq \mathbb{C}$ and γ has a piecewise C^1-path in D and γ is a* ***closed path****, then*

$$\int_\gamma f(z)dz = 0.$$

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31. Path Dependence and Cauchy Riemann Equations

31.7.5 Path Dependence

Proposition 31.278 *Let γ be a closed C^1-path in $\mathbb{C}$ and $z_0 \in \mathbb{C}$. Then*

1. *$\int_\gamma (z - z_0)^n dz = 0$ for all $n \in \mathbb{Z} - \{-1\}$ provided that γ does not pass through z_0 when $n < 0$.*
2. *Assume that $n = -1$. Let γ be the positively oriented circle centered at 0 with radius $r > 0$. Then*

$$\int_\gamma \frac{dz}{z - z_0} = \begin{cases} 0 \;\; if \; |z_0| > \; r \\ 2\pi i \;\; if \; |z_0| < \; r \end{cases}$$

31.7.6 Cauchy Riemann Equations

Definition 31.279 *(Cauchy Riemann Equations). The Cauchy Riemann equations are defined to be system of equations of the partial derivatives of a complex function $f(z) = u(x, y) + v(x, y)i$*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Remark 31.280 *These are* ***necessary*** *conditions for a function to be holomorphic.*

We can extend this to make them necessary and sufficient conditions.

Proposition 31.281 *A function f is holomorphic on a region R if*

1. *Partial derivatives of u(x,y) and v(x,y) exist;*
2. *The partial derivatives are continuous;*
3. *The partial derivatives satisfies the Cauchy Riemann equations.*

Remark 31.282 *The continuity of partial derivatives can be relaxed to the function being real differentiable when viewed as $f : \mathbb{R}^2 \to \mathbb{R}^2$. In other words, the real partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}^2$ exists.*

We can then extend these equations.

Definition 31.283 *(Laplace's Equation). From the Cauchy-Riemann equations, we have the Laplace's equation being*

$$\nabla^2 u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$\nabla^2 v = \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Proposition 31.284 *A function that is holomorphic $f(z) = u(x,y)+v(x,y)i$ has u and v satisfying Laplace's equation.*

A holomorphic function is not only smooth, it is analytic (has a power series expansion). So the mixed partials are equal.

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32. Star Shaped Domains

32.7.7 Star Shaped Domains

We now move onto building the foundations in order to set the conditions for finding the primitive of a function.

Definition 32.285 *(Convex Set). A set $D \subseteq \mathbb{C}$ is called convex if*

$$(1-t)z_1 + tz_2 \in D$$

for all $z_1, z_2 \in D$ and for all $t \in [0, 1]$.

Definition 32.286 *(Star-shaped domain).*

1. *A set $D \subseteq \mathbb{C}$ is called* ***star-shaped*** *w.r.t $z_0 \in D$ for all $z \in D$, the line segment joining z_0 and z lies entirely in D, that is,*
$$(1-t)z_0 + tz$$
for all $t \in [0, 1]$ and for all $z \in D$.

2. *If D is star-shaped w.r.t each point $z_0 \in D$, then we say that D is* ***convex****.*

Remark 32.287 *Note that for (1), it says to pick a set and a point in that set and see if its star-shaped with respect to that one point. For (2), it says to pick a set and that it is star-shaped for any point in that set.*

Theorem 32.288 *(Existence of a primitive). Let $D \subseteq \mathbb{C}$ be open and star-shaped w.r.t $z_0 \in D$. If $f \in C(D, \mathbb{C}^N) \cap C^1(D - \{z_0\}, \mathbb{C}^N)$, then f admits a primitive on the set D of the form*

$$F(z) = (z - z_0) \int_0^1 f((1-t)z_0 + tz)dt$$

for all $z \in D$.

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33. Cauchy-Goursat Theorem and Formula

33.7.8 Cauchy Integral Theorem and Formula

Theorem 33.289 *(Cauchy-Goursat Theorem). Let $D \subseteq \mathbb{C}$ be open and star-shaped w.r.t $z_0 \in D$. If $f \in C(D, \mathbb{C}^N) \cap C^1(D - \{z_0\}, \mathbb{C}^N)$, then for every **closed piecewise** C^1-path γ in D, we have*

$$\int_\gamma f(z)dz = 0.$$

Theorem 33.290 *(Cauchy Integral Formula). Let $D \subseteq \mathbb{C}$ be open and $f \in C^1(D, \mathbb{C}^N)$. Let $z_0 \in D$ be arbitrary and $r > 0$ such that $\overline{B(z_0, r)} \subseteq D$. Let $\gamma(t)$ denote the boundary of $\overline{B(z_0, r)}$, that is, $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then we have that*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw$$

for all $z \in B(z_0, r)$.

Proof:(Sketch). First, construct a circle around the point of interest and break up the main circle. Take integrals over each of this path and since this is closed, the sum of all the integral should be 0. ■

Proposition 33.291 *Let $z_0 \in \mathbb{C}$ and $r > 0$. Denote*

$$S := \{z \in \mathbb{C} : |z - z_0| = r\}.$$

Here, S denotes every point on the boundary. Assume that $f \in C(S, \mathbb{C}^N)$. Then we have that

$$\int_S \frac{f(w)}{w - z} dw = \begin{cases} \sum_{k=0}^{\infty} a_k (z - z_0)^k & if\ |z - z_0| < r \\ \\ -\sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k} & if\ |z - z_0| > r \end{cases}$$

where each coefficient is given by the contour integral $a_k = \int_S \frac{f(w)}{(w - z_0)^{k+1}} dw$ for all $k \in \mathbb{Z}$.

*The first series converges **absolutely and uniformly** on **closed** subsets of $B(z_0, r)$, while the second series converges **absolutely and uniformly** on closed **and** bounded subsets of $\{z \in \mathbb{C} : |z - z_0| > r\}$.*

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34. Analyticity of Differentiable Functions

34.7.9 Analyticity of Differentiable Functions

We've arrived at one of the most important aspects of complex analysis. The idea that all C^1 functions in $\mathbb{C}$ are analytic.

Theorem 34.292 *(Analyticity of C^1-functions on $\mathbb{C}$). Let $D \subseteq \mathbb{C}$ and $f \in C^1(D, \mathbb{C}^N)$. Then f is analytic on D. If $z_0 \in D$ is arbitrary and ρ is the radius of convergence of the Taylor series of f about $z = z_0$. Then*

$$sup\{r > 0 : \overline{B(z_0, r)} \subseteq D\} \leq \rho.$$

In particular, the radius of any closure of balls in the domain is less than the radius of convergence.

Corollary 34.293 *Let $D \subseteq \mathbb{C}$ be open.*

(i) If $f \in C(D, \mathbb{C}^N)$ admits a primitive F on D, then f is analytic on D;

(ii) If $z_0 \in D$ and $f \in C(D, \mathbb{C}^N) \cap C^1(D/\{z_0\}, \mathbb{C}^N)$ then f is analytic.

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35. Integrals on an annulus and Laurent Series

35.7.10 Integrals on an annulus and Laurent Series

Definition 35.294 *(Open annulus centered at z_0). Fix $z_0 \in \mathbb{C}$ and $0 \leq R_1 < R_2 \leq \infty$. We define the open annulus centered at z_0 to be the set*

$$A_{R_1,R_2}(z_0) = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}.$$

Remark 35.295 *We then define γ_r to be the positively oriented (anti-clockwise) circle centered at z_0 with radius r where $r \in (R_1, R_2)$.*

Lemma 35.296 *Suppose that g is analytic on A_{R_1,R_2} with values in $\mathbb{C}^N$. For $r \in (R_1, R_2)$, we define γ_r as the positively oriented circle centered at z_0 with radius r. Then,*

$$h(r) = \int_{\gamma_r} g(z)dz$$

is constant for all $r \in (R_1, R_2)$.

Theorem 35.297 *(Cauchy integral formula on an annulus). Let f be analytic on $A_{R_1,R_2}(z_0)$ with $0 \leq R_1 < R_2 \leq \infty$ with values in $\mathbb{C}^N$. Let $R_1 < r_1 < r_2 < R_2$ and let C_1, C_2 denote the positively extended circles centered at z_0 with radius r_1 and r_2 respectively. Then, for all $z \in A_{r_1,r_2}(z_0)$, we have*

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}dw.$$

Remark 35.298 *Intuitively what we have done is constructed 2 circles and took the difference of them.*

Theorem 35.299 *(Laurent Theorem). Let $D \subseteq \mathbb{C}$ be open and $f : D \to \mathbb{C}^N$ be analytic. Let $z_0 \in \mathbb{C}$ be such that $A_{R_1,R_2}(z_0) \subseteq D$ with $0 \leq R_1 < R_2 \leq \infty$. Then there exists $\tilde{a}_k \in \mathbb{C}^N$ (where $k \in \mathbb{Z}$) such that*

$$f(z) = \sum_{k=1}^{\infty} \tilde{a}_{-k}(z - z_0)^{-k} + \sum_{k=0}^{\infty} \tilde{a}_k(z - z_0)^k$$

for all $z \in A_{R_1,R_2}(z_0)$. The series converges absolutely and uniformly on closed and bounded subsets of $A_{R_1,R_2}(z_0)$. This annulus can be taken as the largest that fits within D.

Proof: We use Cauchy's theorem to help us get the coefficients. ■

Remark 35.300 *The Laurent series is just a generalisation of the Taylor series. The first part is known as the singular part whilst the second part is known as the regular part. The significance of this is that we can now center Laurent series on points of singularities.*

To get the coefficients, we require path integrals.

$$\tilde{a}_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n-1}}dz.$$

$$\tilde{a}_{-k} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}dz.$$

Remark 35.301 *To find Laurent series, we can use partial fraction decomposition or just simply recognising and doing algebra.*

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36. Singularity and poles

36.7.11 Singularity and poles

Definition 36.302 *(Isolated Singularity). Let $f : D \to \mathbb{C}^N$ be analytic on an open set $D \subseteq \mathbb{C}$. We say $z_0 \in \partial D$ (boundary) is an **isolated singularity** of f if there exists $r > 0$ such that $B(z_0, r)/\{z_0\} \subseteq D$.*

Definition 36.303 *(Nonisolated Singularity). A nonisolated singularity z_0 is when we take a ball around z_0 and there are other points z^* in the ball that are also a singularity.*

There are 3 types of isolated singularities, which we order in terms of increasing severity.

1. Removable singularity;
2. Pole of order m;
3. Essential singularity.

When figuring out singularities, we write out the Laurent series expansion and analyse the equation

$$f(z) = \sum_{k=1}^{\infty} \tilde{a}_{-k}(z - z_0)^{-k} + \sum_{k=0}^{\infty} \tilde{a}_k(z - z_0)^k$$

There are 3 cases to deal with.

1) There is no singular part, that is, $\tilde{a}_{-k} = 0$ for all $k \geq 1$. Hence, we are left with

$$f(z) = \sum_{k=0}^{\infty} \tilde{a}_k(z - z_0)^k$$

for all $z \in B(z_0, r)/\{z_0\}$. Then we have that f is analytic on $B(z_0, r)$ and $D \cup \{z_0\}$. In this case, z_0 is known as a **removable singularity**.

2) There exists $m \geq 1$ such that $\tilde{a}_{-m} \neq 0$ and $\tilde{a}_{-k} = 0$ for all $k > m$. We call z_0 a **pole of order m**. When m = 1, we say that z_0 is a **simple pole**.

3) There exists an infinite number of $\tilde{a}_{-k} = 0$ with $k \geq 1$. Then $z = z_0$ is called an **essential singularity**.

Theorem 36.304 *Let $f : D \to \mathbb{C}^N$ be analytic and z_0 an isolated singularity of f.*

(i) If z_0 is a removable singularity of f, then $\lim_{z \to z_0} f(z)$ exists and f has an analytic extension to $D \cup \{z_0\}$;

(ii) If f has a pole of order m at z_0, then there exists an analytic function $g : D \cup \{z_0\} \to \mathbb{C}^N$ with $g(z_0) \neq 0$ such that

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

for all $z \in D$.

36.7.12 Practical Applications

36.7.12.1 Removable Singularity

To check if the point z_0 is a removable singularity for the function f, we look at does

$$\lim_{z \to z_0} f(z) = f(z_0).$$

In other words, if taking the limit gives us something defined, then z_0 is a removable singularity. Furthermore, this means that the Laurent expansion about z_0 is just the Taylor expansion.

36.7.12.2 Poles

To check if z_0 is a pole of order m, we look at

$$\lim_{z \to z_0} (z - z_0)^m f(z) \neq 0.$$

If this is the case, then it **is** a pole of order m. We just need the limit to be nonzero.

36.7.12.3 Essential Singularity

An essential singularity arises if the test for poles and removable singularity fails.

MATH2923: Advance Analysis

37. Residues and Residue Theorem

37.7.13 Residues and Residue Theorem

All the singularities are isolated singularities in the next section.

Proposition 37.305 *Suppose that $D \subseteq \mathbb{C}$ is open and z_0 is an isolated singularity of the analytic function $f : D \to \mathbb{C}^N$. Let $r > 0$ be such that $B(z_0, r)/\{z_0\} \subseteq D$. We set $\gamma(t) := z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then*

$$\int_\gamma f(z)dz = 2\pi i\tilde{a}_{-1} = 2\pi i Res[f, z_0]$$

where $\tilde{a}_{-1}$ is the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion of f about z_0 in $B(z_0, r)/\{z_0\}$.

Remark 37.306 *So if we had a single point of singularity, the integral just requires the residue of the Laurent expansion about the singular point.*

Definition 37.307 *(Residue). The coefficient $\tilde{a}_{-1}$ of $(z - z_0)^{-1}$ in the Laurent expansion of f about z_0 in $0 < |z - z_0| < r$ (where z_0 is an isolated singularity for the analytic function $f : D \to \mathbb{C}^N$ such that $B(z_0, r)/\{z_0\} \subseteq D$) is called the **residue** of f at z_0. We denote this as*

$$Res[f, z_0] = \tilde{a}_{-1}.$$

Remark 37.308 *Just look at the coefficient of $\frac{1}{z-z_0}$ about the point of singularity z_0.*

Corollary 37.309 *(Computing Residues). We look at the different cases of how to compute the residue depending on the singularity.*

1. *(Removable Singularity) If z_0 is a removable singularity for f, then*

$$Res[f, z_0] = 0$$

.

2. *(Simple Pole) If z_0 is a simple pole (pole of order one) for f, then*

$$Res[f, z_0] = \lim_{z \to z_0} (z - z_0)f(z) = \tilde{a}_{-1} \neq 0$$

.

3. *(Pole of order m) If z_0 is a pole of order m ($m \geq 1$) for f, then*

$$Res[f, z_0] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz} [(z - z_0)^m f(z)]$$

.

4. *(Essential singularity) If z_0 is an essential singularity (so all singular terms in the Laurent series is non-negative), then we need to compute the Laurent expansion about the point of singularity and then look at the first coefficient.*

We arrive at one of the most useful theorems in the course.

Theorem 37.310 *(Residue Theorem). Suppose that $D \subset \mathbb{C}$ is open and $f : D \to \mathbb{C}^N$ is analytic on D. Let γ be positively oriented closed piecewise C^1-path in D such that f is analytic in the region enclosed by γ except a finite number of isolated singularities. Then, we have that*

$$\int_\gamma f(z)dz = 2\pi i \sum_{j=1}^{n} Res[f, z_j].$$

Proof:(Sketch). So to take the path integral, we just need to look at the residues of the singularities. Intuitively, the path integral of the analytic part will be zero whilst the path integrals over the points of singularities, we can write as Laurent series. Then, all the terms except the Residue of the singular part of the Laurent series will be integrated out, leaving us only the residues. So we end up applying Cauchy's integral formula combined with the residue for each of the singularity. Hence, we arrive at the equation. ■

37.7.14 Finding Laurent series

If a function is analytic on some domain, then that function is going to equal to its Laurent series on that domain.

Effectively, singularities are points where the function is not analytic. A nonisolated singularity (which we don't study in this course) is when taking the ball around the point of singularity, we have other points of singularity. However, isolated singularities are when we take balls around the point of singularity, we do not have any singularities, so the function is analytic.

First, we need to determine whether do we need a Laurent series or a Taylor series. If the point we are expanding about is singular, then we use a Laurent series. If not, we have 3 more separate cases. First, if the region is an annulus, then we use Laurent series. If the region is inside a circle, we use a Taylor series. Finally, if the region is outside the circle, we use a Laurent series.

Example 1 So for example, if we are given

$$f(z) = \frac{1}{(z-1)(z-2)}$$

and point of expansion about z=1. Here, the point of expansion is an isolated singularity, hence we need a Laurent series.

Example 2 Let $f(z) = \frac{1}{z-1}$ about the point z=0. This is not a point of singularity, so we need to consider it more. First, draw a circle centered at 0 passing through (1,0) as that is a point of singularity. Inside the circle, the point of expansion is NOT a singularity, hence we just need a Taylor series. Outside the circle, we will need a Laurent series.

Here, we have some known geometric series to keep in mind.

$$\frac{1}{1-z} = z^0 + z^1 + ... = \sum_{n=0}^{\infty} (z)^n$$

for $|z| < 1$.

$$\frac{1}{1-\frac{1}{z}} = \frac{1^0}{z^0} + \frac{1^1}{z^1} + \frac{1^2}{z^2} ... = \sum_{n=0}^{\infty} (\frac{1}{z})^n$$

for $|z| > 1$.

Furthermore, if we had

$$\frac{1}{1-z/c}$$

then this converges for $|z| < c$. Additionally, if $c = 2i$ for example, then it converges for $|z| < 2$.

Another trick to note is

$$\frac{1}{1-z} = -\frac{1}{z-1}.$$

Also,

$$\frac{1}{1-z} = \frac{1}{z} \frac{1}{\frac{1}{z} - 1} = -\frac{1}{z} \frac{1}{1 - \frac{1}{z}}.$$

37.7.15 Application of Residues to real integrals

We can now evaluate tough real integrals in the form of the residue theorem.

We can use the residue theorem to compute integrals of the form

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx$$

where $q(x) \neq 0$ for all $x \in \mathbb{R}$. Furthermore, we require that $Deg(q) \geq 2 + Deg(p)$.

Proposition 37.311 *Let p and q be polynomials with $q(x) \neq 0$ for all $x \in \mathbb{R}$ and $deg(q) \geq deg(p) + 2$. If $z_1, ..., z_n$ are zeroes of q with positive imaginary parts, then*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx = 2\pi i \sum_{k=1}^{n} Res[p/q, z_k]$$

.

Remark 37.312 *To actually apply this, first find all the points of singularity. Then, find the residues of the Laurent series expanded at these points. Then it is trivial to compute the integral. If we had no points of singularity, then the integral just evaluates to 0.*