MATH3961: Metric Spaces

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Abstract

Thank you for stopping by to read this. These are notes collated from lectures and tutorials as I took this course.

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1. Introduction

1.1 Introduction to spaces

1.1.1 Different types of spaces

The hierarchical relationships between spaces are as follows

1. Inner Product Space (Vector space with an inner product)
2. Normed Vector Space (Vector space with a norm)
3. Metric Space (Non-empty set with a metric)
4. Topological Space (Non-empty set with a topology)

1. A norm on a vector space is not induced by an inner product unless the *parallelogram law* is satisfied.
2. A metric on a set is not induced by a norm unless the metric is *translation invariant* and *positive homogenous.*
3. A topology on a set is not induced by a metric unless it is a *Hausdorff topology.*

Definition 1.1 *(Completeness). A metric space is called complete if every Cauchy sequence in the space is convergent.*

Lemma 1.2 *Every convergent sequence is a Cauchy sequence.*

Note that it doesn't make sense to define Cauchy sequences in topological spaces as the notion of distance is not defined.

Proposition 1.3 *Let X be a complete metric space. A sequence is convergent if and only if it is a Cauchy sequence.*

Remark 1.4 *Introducing completeness is useful because now convergence only depends on the terms of the sequence itself.*

So now when we introduce completeness, we get

1. Hilbert Space (Complete inner product space)
2. Banach Space (Complete normed vector space)
3. Complete Metric Space

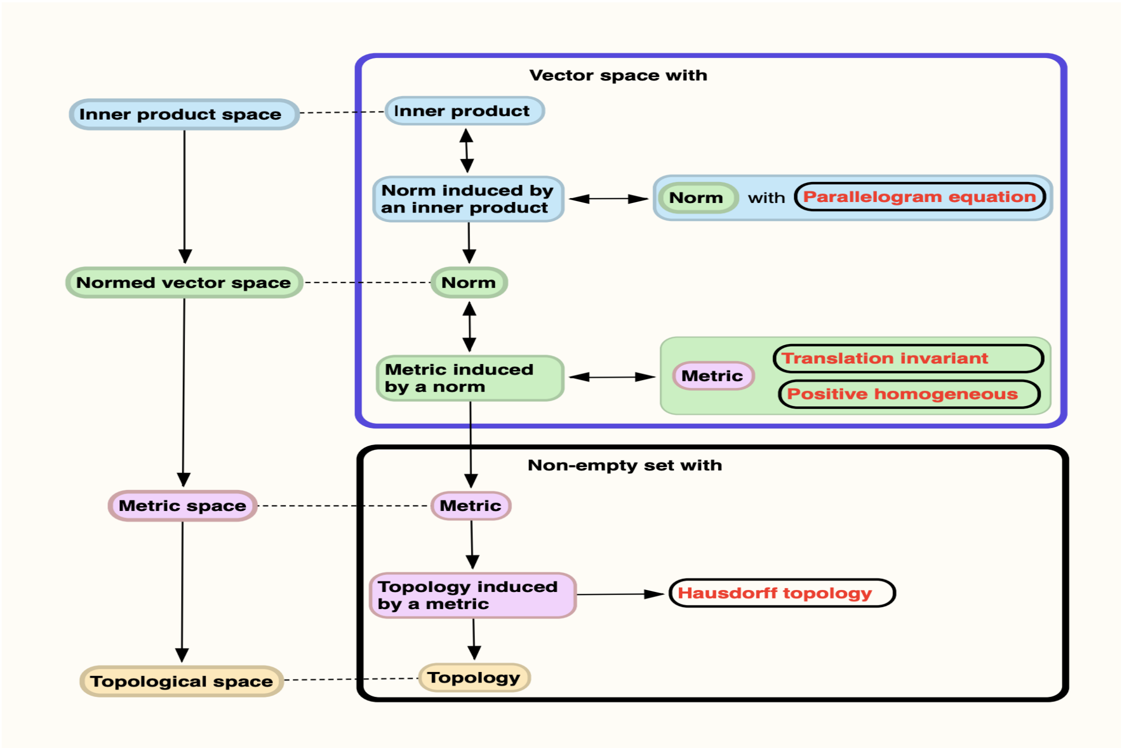

Figure 1.1: Relationships between spaces.

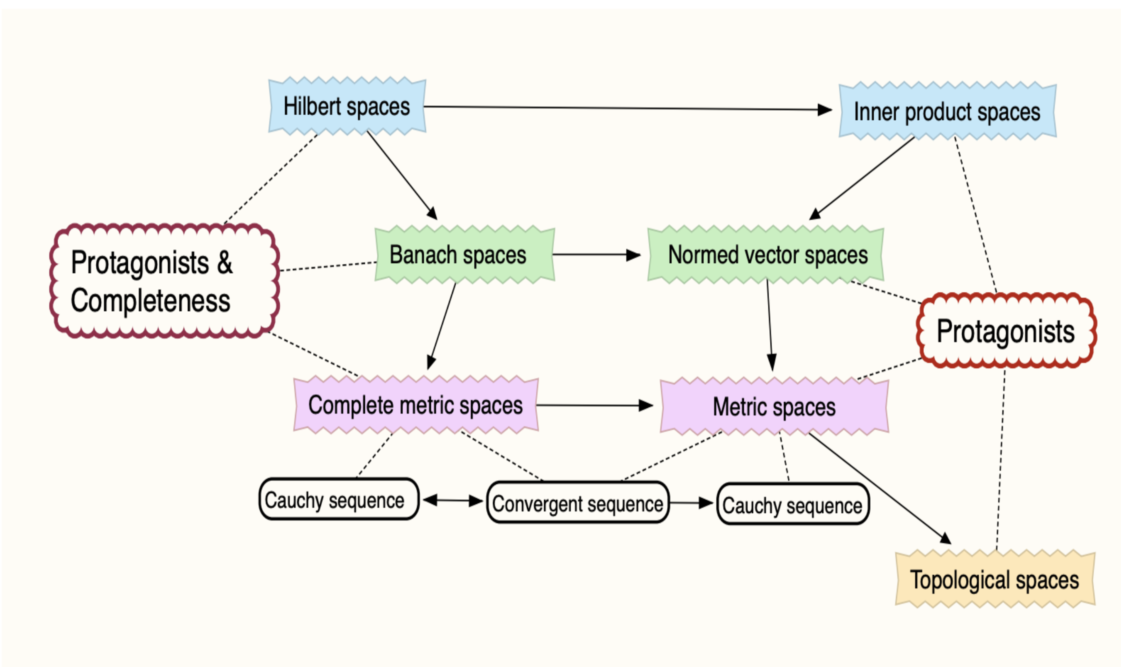

Figure 1.2: Relationships between complete spaces.

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2. Vector Spaces and Inner Product Spaces

2.1.2 Vector Spaces

Definition 2.5 *(Abelian Group). An abelian group is a set A together with an operation $\circ$ that combines any two elements $a, b \in A$ to form another element $a \circ b$. The set A and operation $\circ$ must satisfy*

1. *Closure;*
2. *Associativity;*
3. *Identity element;*
4. *Inverse element;*
5. *Commutativity.*

Definition 2.6 *(Vector Space). Let $(V, +)$ be an Abelian group, together with a scalar multiplication operation $\circ$, associating to each pair (α, v) in $\mathbb{K} \times V$ a vector αv in V. If the pair satisfies associativity, identity element and distributivity for all $\alpha, \beta \in \mathbb{K}$ and all vector $v, w \in V$, then V is called a vector space over $\mathbb{K}$.*

Definition 2.7 *We define $C([a, b]; \mathbb{K})$ to be the space of all $\mathbb{K}$-valued continuous functions on [a,b].*

Theorem 2.8 *(Extreme Value Theorem). Every function in $C([a, b])$ that is bounded, attains its maximum as well as its minimum.*

Definition 2.9 *(Bounded functions). We define $B([a, b]; \mathbb{K})$ as the space of bounded functions on [a,b].*

Definition 2.10 *(p-summable sequence)We denote ℓ_p the set of all p-summable sequences in $\mathbb{K}$. A sequence $\{x_j\}_{j\geq 1} \in \mathbb{K}$ belongs to ℓ_p if and only if*

$$\sum_{j=1}^{\infty} |x_j|^p < \infty.$$

Definition 2.11 *(Linear Transformation). A mapping $T : X \to Y$ between vector spaces X and Y (over the same field $\mathbb{K}$) is called linear if for all $\alpha_1, \alpha_2 \in \mathbb{K}$ and every $x_1, x_2 \in X$, we have*

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2).$$

When $Y = \mathbb{K}$, a linear transformation $T : X \to \mathbb{K}$ is called a ***linear functional*** *on X.*

2.2 Inner Product Spaces

2.2.1 Inner Product Spaces

The inner product on a vector space allows us to introduce the idea of the length and angle between vectors.

Definition 2.12 *(Law of cosines). If the sides of a triangle have lengths a, b, and c, denoting by ϕ the angle made by the sides of length a and b, we have*

$$c^2 = a^2 + b^2 - abcos\phi.$$

Definition 2.13 *(Triangle Inequality). For 2 vectors u and v, we have*

$$||u + v|| \leq ||u|| + ||v||.$$

Definition 2.14 *(Inner Product). Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be in $\mathbb{K}^n$. The inner product on a vector space is a mapping which associates to each ordered pair of vectors, a scalar. We define it as*

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j \bar{y}_j.$$

This mapping $\langle ., . \rangle$ satisfies

1. *Positive definiteness;*
2. *Conjugate symmetry;*
3. *Linearity in the first argument.*

Definition 2.15 *(Inner Product Space). Let V be a vector space and $\langle ., . \rangle$ be an inner product on V. Then $(V, \langle ., . \rangle)$ is known as an inner product space.*

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3. Normed Vector Spaces

3.3 Normed Vector Spaces

3.3.1 Normed Vector Spaces

A normed vector space is a vector space in which a length (norm) can be assigned to each vector.

Definition 3.16 *(Norm). Let* $||.||: V \to \mathbb{R}$*. If* $||.||$ *satisfies the 3 properties of*

1. *Positive definiteness;*
2. *Positive homogeneity;*
3. *Triangle inequality;*

then $||.||$ *is known as a norm.*

Definition 3.17 *(Normed Vector Space). A normed vector space is a pair of a vector space* V *and a norm* $||.||$.

Definition 3.18 *(Norm induced by an inner product). We define a norm induced by an inner product as*

$$||x||= \sqrt{\langle .,. \rangle}$$

for every $x \in V$.

Definition 3.19 *(Cauchy-Schwarz Inequality). For 2 vectors u and v, the norm induced by an inner product satisfies*

$$|\langle u, v\rangle| \leq ||u|| \circ ||v||.$$

Definition 3.20 *(Paralellogram Equation). For a triangle with sides u and v, the norm induced by an inner product satisfies*

$$||u+v||^2+||u-v||^2= 2(||u||^2+||v||^2).$$

Theorem 3.21 *Every inner product space with the induced norm is a normed vector space.*

Definition 3.22 *(Conjugate Exponent).*

Let $1 \leq p \leq \infty$*. We denote p' as the conjugate exponent of p, that is*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

For any $p \in (1, \infty)$*, we have that*

$$p' = \frac{p}{p-1}.$$

Lemma 3.23 *(Young's Inequality). For all $a, b \in [0, \infty)$, we have*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^{p'}.$$

Moreover, equality holds if and only if $a^p = b^{p'}$.

Lemma 3.24 *(Hölder's inequality).*

Let $n \geq 1$. If $x_j, y_j \geq 0$ for all $j = 1, ..., n$, then

$$\sum_{j=1}^{n} x_j y_j \leq (\sum_{j=1}^{n} x_j^p)(\sum_{j=1}^{n} y_j^{p'})$$

Lemma 3.25 *(Hölder's Inequality on C[a,b]).*

Let $a, b \in \mathbb{R}$ with $a < b$. If $f, g \in C[a, b]$, then we have

$$\int_a^b |f(x)g(x)|dx \leq (\int_a^b |f(x)|^p dx)(\int_a^b |g(x)|^{p'} dx).$$

Proposition 3.26 *(Minkowski's Inequalities). Assume that $p \in [1, \infty)$.*

1. *Let $n \geq 1$. If $x_j, y_j \in \mathbb{K}$ for all $j = 1, 2, ..., n$, then we have*

$$(\sum_{j=1}|x_j + y_j|^p)^{\frac{1}{p}} \leq (\sum_{j=1}|x_j|^p)^{\frac{1}{p}} + (\sum_{j=1}|y_j|^p)^{\frac{1}{p}}.$$

2. *If $x_j, y_j \in \mathbb{K}$ for all $j \geq 1$ such that $\sum_{j=1}^{\infty}|x_j|^p < \infty$ and $\sum_{j=1}^{\infty}|y_j|^p < \infty$, then we have*

$$(\sum_{j=1}|x_j + y_j|^p)^{\frac{1}{p}} \leq (\sum_{j=1}|x_j|^p)^{\frac{1}{p}} + (\sum_{j=1}|y_j|^p)^{\frac{1}{p}}.$$

3. *Let $a, b \in \mathbb{R}$ with $a < b$. Assume that $f, g \in C[a, b]$. Then $(f + g) \in C[a, b]$ and*

$$(\int_a^b |f(x) + g(x)|^p dx)^{\frac{1}{p}} \leq (\int_a^b |f(x)|^p dx)^{\frac{1}{p}} + (\int_a^b |g(x)|^p dx)^{\frac{1}{p}}.$$

Corollary 3.27 *Let $n \geq 1$ and $p \in (1, \infty)$. If $x_j, y_j \in \mathbb{K}$ for all $j = 1, ..., n$, then*

$$|\sum_{j=1}^{n} x_j y_j| \leq (\sum_{j=1}^{n}|x_j|^p)^{\frac{1}{p}} + (\sum_{j=1}^{n}|y_j|^{p*})^{\frac{1}{p*}}.$$

4.4 Metric Spaces

4.4.1 Metrics

Definition 4.28 *(Metric). A function d, assigning to every ordered pair of points x and y in a non-empty set X a real number d(x, y), is called a* ***metric*** *(or distance) on X if it satisfies*

1. $d(x, y) \geq 0$ *where d(x, y) = 0 if and only if x = y (Positive Definiteness) ;*
2. *d(x, y) = d(y, x) (Symmetry);*
3. $d(x, z) \leq d(x, y) + d(y, z)$ *(Triangle Inequality);*

for all points $x, y, z \in X$*.*

Definition 4.29 *(Metric Space). A metric space is a* ***non-empty*** *set with a metric.*

Remark 4.30 *Taking a non-empty subset Y of a metric space (X, d) with the original metric d will obtain you the* ***metric subspace*** *of the original metric space (X,* d_Y*).*

Example 4.31 *For all points* $x = (x_1, x_2, ..., x_n)$ *and* $y = (y_1, y_2, ..., y_n)$ *in* $\mathbb{K}^n$*, we define*

1. $d_1(x, y) = \sum_{j=1}^{n} (x_j - y_j)$ *(Taxicab Metric/Manhattan Metric);*
2. $d_p(x, y) = \sum_{j=1}^{n} (|x_j - y_j|^p)^{\frac{1}{p}}$ *(P-Metric);*
3. $d_\infty(x, y) = \max_{1 \leq j \leq n} |x_j - y_j|$ *(Supremum/Chebyshev Metric).*

Remark 4.32 *The power of the p-metric (1/p) is necessary for the p-metric to be a metric or else the triangle inequality can be violated.*

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5. Metrics and The Topology of a Metric Space

5.4.2 More Properties of Metrics

Unlike norms which requires a vector space, metrics only require a set.

Definition 5.33 *(Discrete metric). The discrete metric d_{dis} is defined by*

$$d_{dis}(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Lemma 5.34 *Any set admits a metric by defining the* ***discrete metric*** *on the set.*

Definition 5.35 *(Metric induced by a norm). The metric induced by a norm on a vector space is defined as*

$$d(x, y) = ||x - y||$$

Theorem 5.36 *Every normed vector space with its induced metric is a metric space.*

We look at ways to determine whether is a metric is induced by a norm. We look at the properties of vector addition and scalar multiplication in vector spaces.

Definition 5.37 *(Translation-invariance). A metric d is called* ***translation-invariant*** *if*

$$d(x, y) = d(x + z, y + z)$$

for every vectors x, y, and z.

Definition 5.38 *(Positive-homogeneity). We say a metric d on a vector space is* ***positive-homogeneous*** *if*

$$d(\alpha x, \alpha y) = |\alpha| d(x, y)$$

.

Theorem 5.39 *A metric on a non-trivial vector space is induced by a norm if and only if the metric is translation-invariant* ***and*** *positive-homogenous.*

Example 5.40 *The discrete metric is not induced by a norm as it does not satisfy positive homogeneity.*

Lemma 5.41 *Every translation-invariant and homogenous metric induces a norm by*

$$||x|| = d(x, 0).$$

5.5 Topology of a Metric Space

5.5.1 Definitions

Definition 5.42 *(Open Ball). Let x be a point in a metric space (X, d) and $\epsilon > 0$. Then the set*

$$B_d(x;\epsilon) = \{y \in X : d(x,y) < \epsilon\}$$

is known as the open ϵ-ball about x.

Definition 5.43 *(Open Set). A subset of X is called open if it contains an open ball about each of its points.*

Definition 5.44 *(Closed Ball). Let x be a point in a metric space (X, d) and $\epsilon > 0$. Then the set*

$$\bar{B}_d(x;\epsilon) = \{y \in X : d(x,y) \leq \epsilon\}$$

is known as the closed ϵ-ball about x.

Example 5.45 *Let X be a set with the discrete metric d_{dis}. Then,*

$$B_{d_{dis}}(x;r) \begin{cases} \{x\} & r \leq 1 \\ X & r > 1 \end{cases}$$

Theorem 5.46 *Every open ball in a metric space is an open set.*

Definition 5.47 *(Closed Set). A set in a metric space (X, d) is called closed if its complement in X is open.*

Theorem 5.48 *Every closed ball is a closed set in X.*

Theorem 5.49 *For every metric space (X, d), each of the following is an open set.*

1. *The empty set;*
2. *The set X;*
3. ***Arbitrary** unions of open sets;*
4. ***Finite intersections** of open sets.*

Theorem 5.50 *(Metric Topology). The collection of all open sets in a metric space is called the topology induced by the metric or the **metric topology**.*

Remark 5.51 *Note that a metric induces the idea of open balls, which induces the idea of open sets, which then induces a topology.*

Theorem 5.52 *A set in a metric space is open if and only if it can be written as a union of open balls.*

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6. Topology of Metric Spaces and Topologically Equivalent Metrics

6.5.2 Topology of Metric Spaces

Definition 6.53 *(Topology). A topology on a set X is a collection $\mathcal{T}$ of subsets of X such that*

1. *Both the empy set and X are elements of $\mathcal{T}$;*
2. *Any union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$;*
3. *Any intersection of finitely many elements of $\mathcal{T}$ is an element of $\mathcal{T}$.*

The members of $\mathcal{T}$ are called open sets.

Definition 6.54 *(Topological Space). If $\mathcal{T}$ is a topology on X, then the pair $(X, \mathcal{T})$ is called a topological space.*

Example 6.55 *Example of arbitrary intersection of open sets is $(-1/n, 1)$ which gives us the closed set [0,1) as $n \to \infty$.*

Definition 6.56 *($\mathcal{T}$-open). Let $(X, \mathcal{T})$ be a topological space. Members of the topology $\mathcal{T}$ are called open relative to $\mathcal{T}$ or simply $\mathcal{T}$-open.*

Definition 6.57 *(Base of a topology). The base (or basis) $\boldsymbol{B}$ for a topological space X with topology τ is a collection of open sets in X such that every open set X can be written as a union of elements of $\boldsymbol{B}$.*

Example 6.58 *The family of all open balls in a metric space forms a base for the metric topology. Every open set can be expressed as the union of open balls.*

Theorem 6.59 *A set in a metric space is open if and only if it is a union of open balls.*

Definition 6.60 *(Open set in a topology). Let (X, τ) be a topological space. A set $A \subseteq X$ is open if for every point $a \in A$, there exists a neighbourhood U of x such that $x \in U \subseteq A$.*

6.5.3 Topologically Equivalent Metrics

Definition 6.61 *(Topological Equivalence). Two metrics on the same set are **topologically equivalent** if they induce the same topology. Equivalently, two metrics are topologically equivalent if the open sets in one metric topology coincides with the open sets in the other metric topology.*

Lemma 6.62 *(Nesting condition). Let there be two metrics on the same set X. Every open ball about any point in the set with respect to one metric contains an open ball about the same point in the other metric.*

Symbolically, let d and ρ be metrics on the same set X. The two metrics are topologically equivalent if and only if for every $x \in X$ and every $r > 0$, there exists r', r" > 0 such that

$$\begin{cases} B_\rho(x, r') \subseteq B_d(x, r) \\ B_d(x, r'') \subseteq B_\rho(x, r). \end{cases}$$

Lemma 6.63 *Two metrics on the same set are topologically equivalent if and only if the nesting condition holds.*

Example 6.64 *Every metric space is homeomorphic to a bounded metric space. That is, for a metric space (X, d), it is homeomorphic to the metric space (X, d') where $d(x, y)' = min1, d(x, y)$.*

Definition 6.65 *(Strongly equivalent). Two metrics d and ρ on the same set X are called **strongly equivalent** if there exists a $c > 0$ such that*

$$\frac{\rho(x, y)}{c} \leq d(x, y) \leq c\rho(x, y)$$

for all $x, y \in X$.

Lemma 6.66 *A sufficient condition for two metrics d and ρ on the same set X to be **topologically equivalent** if for all $x \in X$, there exists a c_x such that*

$$\frac{\rho(x, y)}{c_x} \leq d(x, y) \leq c_x\rho(x, y)$$

for all $x, y \in X$.

Remark 6.67 *Strong equivalence implies the above lemma which implies topological equivalence.*

Corollary 6.68 *All p-metrics on $\mathbb{K}^n$ are strongly equivalent, where $1 \leq p \leq \infty$.*

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7. Topological Spaces

7.5.4 Examples of topological spaces

Many topologies can be placed on the same set X.

Definition 7.69 *(Indiscrete Topology). Let (X, $\mathcal{T}$) be a topological space. The indiscrete topology is such that only the empty set and set X is $\mathcal{T}$-open.*

Definition 7.70 *(Discrete Topology). Let (X, $\mathcal{T}$) be a topological space. The discrete topology is such that every subset of X is $\mathcal{T}$-open. Equivalently, it is the power set $\mathcal{P}(X)$ of X.*

Definition 7.71 *(Cofinite Topology). Let (X, $\mathcal{T}$) be a topological space. The cofinite topology on X is the collection of all subsets of X with finite complements, together with the empty set.*

7.5.5 Topological Spaces

Definition 7.72 *(Isolated point of a subset). Let $(X, \mathcal{T})$ be a topological space. Let $H \subseteq X$. Then $x \in H$ is an isolated point of H if there exists an open set $\Omega \in \mathcal{T}$ such that*

$$\{x\} = H \cap \Omega.$$

In other words, there exists an open set of X that contains no other points of H other than x.

Definition 7.73 *(Isolated point of a space). Let $(X, \mathcal{T})$ be a topological space. A point $x \in X$ is an isolated point if there exists an open set $\Omega \in \mathcal{T}$ such that*

$$\Omega = \{x\}.$$

Remark 7.74 *A point x in a topological space is isolated if $\{x\}$ is open.*

Remark 7.75 *If $x \in X$ is an isolated point, then there exists a neighbourhood of x which does not contain any other point of X. In terms of metric spaces, there exists an open ball around the point containing only the point, hence the singleton set is open.*

Theorem 7.76 *A topological space is discrete if and only if each of its points is isolated.*

Proof:(Sketch). Every set $U \subseteq X$ is open as $U = \cup_{x \in U}\{x\}$ is the union of open sets which gives us an open set. ∎

Lemma 7.77 *Let (X, τ) be a discrete topological space. Then every open set is a clopen set.*

Proof: By definition, we have that $A \in \tau$ for every subset $A \subseteq X$. By definition of the topology, $A^c \in \tau$ as well. Hence, A is also closed. ■

Lemma 7.78 *Every finite metric space is a discrete space.*

Lemma 7.79 *The co-finite topology coincides with the discrete topology on a finite set.*

7.5.6 Metrization of topological spaces

Definition 7.80 *(Metrizable Space). A topological space is called metrizable if its topology can be induced by a metric.*

Remark 7.81 *A metrizable space is a topological space that is* ***homeomorphic*** *to a metric space.*

Definition 7.82 *(Neighbourhood of a point). Let (X, $\mathcal{T}$) be a topological space and x is a* ***point*** *in X. A neighbourhood of x is a subset $V \subset X$ that includes an* ***open set*** *U containing x,*

$$x \in U \subseteq V.$$

This is equivalent to $x \in X$ being in the interior of V.

Definition 7.83 *(Hausdorff Property). Let (X, $\mathcal{T}$) be a topological space. The topological space satisfies the Hausdorff property if for every $x, y \in X$ such that $x \neq y$, there exists a neighbourhood U of x and a neighbourhood V of y such that*

$$U \cap V = \emptyset.$$

Definition 7.84 *(Hausdorff Space/T_2 space). A topological space (X, $\mathcal{T}$) satisfying the Hausdorff property is known as a Hausdorff space.*

Theorem 7.85 *Every metric space is a Hausdorff space.*

Theorem 7.86 *(T_2 axiom). A necessary condition for a topology to be metrizable is for the topological space to be a Hausdorff space.*

Lemma 7.87 *Every singleton is a closed set in a Hausdorff space.*

Lemma 7.88 *Any set endowed with the indiscrete topology is not Hausdorff.*

Lemma 7.89 *An infinite topological space with the cofinite topology is not Hausdorff.*

7.5.7 Comparing Topologies

Definition 7.90 *(Comparable topologies). Let X be a set and τ_1 and τ_2 be two topologies defined on X. If either $\tau_1 \subseteq \tau_2$ or $\tau_1 \supseteq \tau_2$, then τ_1 and τ_2 are said to be* ***comparable****. If $\tau_1 \subseteq \tau_2$, then τ_2 is said to be the* ***finer/stronger topology*** *and τ_1 is said to be the* ***coarser/weaker topology****.*

Remark 7.91 *Two topologies on the same set need not be comparable.*

Remark 7.92 *In general, the union of two topologies is not necessarily a topology.*

Remark 7.93 *The indiscrete topology is the coarsest topology and the discrete topology is the finest topology. The intersection of topologies is the finest topology in all topologies.*

7.5.8 Relative Topology

Definition 7.94 *(Relatively Open). Let (X,d) be a metric space. Let $Y \subseteq X$ be a subset and we obtain the metric subspace (Y, d_Y). A subset G of Y is called relatively open in Y if G is open in (Y, d_Y).*

Remark 7.95 *A set that is open in the metric subspace (Y, d_Y) need not be open in the metric space (X, d).*

Definition 7.96 *(Relative Topology/Topological Subspace). Let (X, $\mathcal{T}$) be a topological space. Let $Y \subset X$ be a non-empty subset of X. Then the* ***relative topology*** *is defined as*

$$\mathcal{T}_Y = \{\Omega \cap Y : \Omega \in \mathcal{T}\}.$$

Lemma 7.97 *(Characterisation of open sets in relative topology). Let Y be a metric subspace of (X, d). A subset of $A \subseteq Y$ is relatively open in Y if and only if $A = Y \cap \Omega$ for an open set Ω in (X, d).*

Corollary 7.98 *Let Y be a metric subspace of (X, d). A subset of $A \subseteq Y$ is closed (in X) if there exists a closed set U in X such that $A = U \cap Y$.*

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8. Convergent Sequences and Equivalent Formulations of a Topology

8.5.9 Convergent Sequences

Definition 8.99 *(Sequence). A sequence $\{x_n\}_{n\geq 1}$ in a set X is a **function** from the set of positive integers to X.*

Definition 8.100 *(Lies eventually). A sequence converges to x if for every open set U such that $x \in U$, there exists $N_U \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N_u$. A sequence is said to lies eventually if all but **finitely** many terms belong to U.*

Definition 8.101 *(Convergence). A sequence $\{x_n\}_{n\geq 1}$ in a topological space X converges to a point x in X if the sequence lies eventually in every open set containing x, which is called the limit of $\{x_n\}$.*

Definition 8.102 *(Eventually constant). Let (X, d) be a metric space. A sequence is eventually constant if there exists a $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n = c$ for some $c \in X$.*

Example 8.103 *A sequence in X with the discrete topology converges to a point x in X if and only if the sequence is eventually constant.*

Example 8.104 *A sequence in the same set X with the indiscrete topology converges to every point in X. If X has more than one point, then the limits of the sequences are not unique.*

Remark 8.105 *Different topologies placed on a given set can cause the convergent sequences to differ and the limits may be different too as seen from the above examples.*

Proposition 8.106 *(Uniqueness of limits in Hausdorff spaces). Every convergent sequence in a Hausdorff space has a unique limit.*

Corollary 8.107 *Every convergent sequence in a metric space has an unique limit.*

Lemma 8.108 *Two metrics are topologically equivalent if and only if they have the same convergent sequences.*

Lemma 8.109 *A sequence $\{x_n\}_{n\geq 1}$ converges to x in a metric space (X, d) if and only if the sequence of distances $\{d(x_n, x)\}_{n\geq 1}$ converges to 0 as $n \to \infty$.*

8.5.10 Equivalent Formulations of a Topology

Definition 8.110 *(Closed Sets Axioms). A topological space is a set X together with a collection $\mathcal{A}$ of subsets of X, the members of which are called closed sets such that $\mathcal{A}$ contains $\emptyset$, X and is closed under both arbitrary intersections and finite unions.*

Definition 8.111 *(Closure). The closure $\bar{A}$ of a set A is the smallest closed set that contains A.*

Definition 8.112 *(Interior). The interior int(U) of a set U is the largest open set that is contained within U.*

Lemma 8.113 *A set is closed if and only if it is equal to its closure.*

Lemma 8.114 *A set is open if and only if it is equal to its interior.*

Definition 8.115 *(Interior, Closure and Boundary). Let U and A be subsets of a topological space X.*

1. *The interior of U, denoted by int(U), is the union of all open sets contained in U.*
2. *The closure of A, denoted by $\bar{A}$, is the intersection of all closed sets that contain A.*
3. *The boundary of A, denoted by ∂A, is the closure of A without the interior of A, that is $\partial A := \bar{A} - int(A)$.*

Definition 8.116 *(Kuratowski's Closure Axioms). A topological space is a pair consiting of a set X and a function (closure) from $\mathcal{P}(X)$ to itself satisfying for every subsets A and B of X with the following axioms:*

1. *The closure of the empty set is the empty set;*
2. *The set A is a subset of its closure;*
3. *The closure of the closure of A is the closure of A;*
4. *The closure of the union $A \cup B$ is the union of the closure of A and the closure of B.*

Definition 8.117 *(Neighbourhood of a point). A subset of a topological space is called a neighbourhood of a point x if it contains an open set containing x.*

Definition 8.118 *(Neighbourhood Axioms). A topological space consists of a set X together with a family $\mathcal{U} = \{\mathcal{U}_x\}_{x \in X}$ of sets $\mathcal{U}_x$ of subsets of X, called neighbourhoods of x such that*

1. *Every neighbourhood of x contains x and X is a neighbourhood of each of its points;*
2. *Every subset of X that contains a neighbourhood of x is itself a neighbourhood of x;*
3. *The intersection of any two neighbourhoods of x yields again a neighbourhood of x;*
4. *Within every neighbourhood of x lies a neighbourhood of x that is a neighbourhood of each of its points.*

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9. Closure of a set and Limit Points

9.5.11 Closure of a set and Limit Points

Proposition 9.119 *Let Y be a subspace of a topological space X. If $A \subset Y$, then the closure of A in Y is*

$$Y \cap \bar{A}.$$

Remark 9.120 *The closure of a set refers to the smallest closed set with respect to the whole space that covers the set. The above formulation refers to the smallest closed set in the space Y that covers our set of interest A. Note that the notation $\bar{A}$ refers to the closure with respect to the whole space X. We need to be careful to distinguish between closed sets in Y and closed sets in X.*

Definition 9.121 *(Intersection of sets). Two sets intersect if their intersection is not empty.*

Definition 9.122 *(Limit point/Accumulation points/Cluster point). Let $A \subset X$ in (X, τ). A point $x \in X$ is called a limit point of A if every neighbourhood of x intersects $A \backslash \{x\}$.*

Remark 9.123 *Every neighbourhood of x in A contains another point from A. Hence, we can approximate x with points in A. This generalises what a limit is.*

Remark 9.124 *Note that in a metric space (X, d), a point $x \in A \subseteq X$ is a limit point of A if for every ball $B_d(x, \epsilon)$, there exists an $a \in A$ such that $a \in B_d(x, \epsilon)$ for all $\epsilon > 0$.*

Definition 9.125 *(Derived set). The derived set of A, written as A', is the set of all limit points of A.*

Theorem 9.126 *(Characterization of closure). Let A be a subset of a topological space (X, τ).*

1. *A point $x \in X$ belongs to $\bar{A}$ if and only if every neighbourhood of x intersects A.*
2. *$\bar{A} = A \cup A'$.*

Corollary 9.127 *A set in a topological space is closed if and only if it contains all its limit points.*

Definition 9.128 *(Dense sets for topological spaces). A subset A of a topological space X is called **dense** if $\bar{A} = X$.*

Remark 9.129 *This means that every point in X is either in the set A or a limit point of A.*

Remark 9.130 *For a metric space (X, d), a set $A \subseteq X$ is dense in X if for every $x \in X$ and for all $\epsilon > 0$, there exist an element $a \in A$ such that $a \in B_d(x; \epsilon)$.*

Theorem 9.131 *Let (X, τ) be a topological space and let $A \subseteq X$. Then A is dense in X if and only if for every $U \in \tau \backslash \{\emptyset\}$, we have that $A \cap U \neq \emptyset$.*

Example 9.132 *(The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$). The set set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ is dense in the topological space $(\mathbb{R}, \tau)$ where τ is the topology of open intervals. Recall from Analysis that there is always a natural number in between two real numbers through the Archmidean property.*

Definition 9.133 *(Countable set). A set is countable if there exists an injective map from it to the set $\mathbb{N}$ of natural numbers.*

Definition 9.134 *(Separable). A topological space is called separable if it admits a countable dense subset.*

Lemma 9.135 *Every singleton in a Hausdorff space is a closed set.*

Definition 9.136 *(T_1-space). A topological space in which every singleton is a closed set is called a T_1-space. In other words, for every pair of points in X, there exists disjoint neighbourhoods of the points.*

Theorem 9.137 *(Characterisation of limit points in T_1-space). Let A be a subset of a T_1-space X. A point $x \in X$ is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A.*

Theorem 9.138 *For every subset A of a T_1-space, the derived set A' is closed.*

Definition 9.139 *(Local base). Let (X, τ) be a topological space and let $x \in X$. A local base of the element x is a collection of open neighbourhoods of x, $\mathcal{B}_x$, such that for every $U \in \tau$ where $x \in U$, there exists a $B \in \mathcal{B}_x$ such that $\mathcal{B}_x \in U$.*

In other words, a local base at a point x in a topological space is a collection $\mathcal{B}_x$ of open neighbourhoods of x such that every neighbourhood of x contains a member of $\mathcal{B}_x$.

Definition 9.140 *(First-countable). A topological space is called* ***first-countable*** *if at each of its points, there exists a countable local base.*

We look at a few basic theorems relating local bases and basis of the topology.

Theorem 9.141 *Let (X, τ) be a topological space and let $\mathcal{B}$ be a basis of τ. Then for each $x \in X$, we have that $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$ is a local basis of x.*

Theorem 9.142 *Let (X, τ) be a topological space. Let $\{\mathcal{B}_x\}_{x \in X}$ be a collection of local bases for each $x \in X$. Then $\mathcal{B} = \cup_{x \in X} \mathcal{B}_x$ is a basis for τ.*

Theorem 9.143 *Let A be a subset of a* ***first-countable space*** *X.*

1. *A point $x \in X$ is a limit point of A if and only if x is the limit of a sequence of points in $A \setminus \{x\}$.*
2. *A point $x \in X$ belongs to the closure of A if and only if it is the limit of a sequence in A.*

Definition 9.144 *(Sequentially open sets/Sequentially closed sets). Let X be a topological space. A subset $U \subset X$ is called sequentially open if for every sequence in X that converges to a point in U, lies eventually in U. We say that a subset $F \subset X$ is sequentially closed if whenever a sequence in F converges in X, then its limit belongs to F.*

Theorem 9.145 *In any topological space, every open set is sequentially open and every closed set is sequentially closed.*

Theorem 9.146 *In a* ***first-countable space****, a set is open if and only if it is sequentially open and, similarly, a set is closed if and only if it is sequentially closed.*

Remark 9.147 *In a topological space, an open set is a sequentially open set. In a Hausdorff space, a set is open if and only if it is sequentially open.*

Remark 9.148 *In an arbitrary topological space, a sequentially closed set* ***does not*** *imply a closed set. However, it does hold in a metric space as it is a first countable space.*

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10. Cauchy Sequence and Completeness of Metric Spaces

10.5.12 Cauchy Sequence and Completeness of Metric Spaces

Definition 10.149 *(Cauchy Sequence). A sequence $\{x_n\}_{n\geq 1}$ in a metric space (X, d) is called a Cauchy sequence if for every $\epsilon > 0$, there exists $n_\epsilon \geq 1$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq n_\epsilon$.*

Definition 10.150 *(Diameter of a set). Given a metric space (X, d) and a non-empty subset A of X, the diameter of A, denoted by diam A, is defined by*

$$diam A \coloneqq \sup\{d(x, y) : x, y \in A\}.$$

Definition 10.151 *(Bounded set). We say that $A \subset X$ is bounded if diam $A < \infty$.*

Lemma 10.152 *Let (X, d) be a metric space and $A \subseteq X$. The set A is bounded if and only if for every $x \in A$, there exists $r > 0$ so that $A \subseteq B_d(x, r)$.*

Lemma 10.153 *Let (X, d) be a metric space and $A \subseteq X$. Every finite set A is bounded.*

Lemma 10.154 *Every Cauchy sequence in (X, d) is a bounded set.*

Lemma 10.155 *Every metric space is a Hausdorff space so that each convergent sequence in a metric space has a unique limit.*

Definition 10.156 *(Complete Metric Space). A metric space is called complete if every of its Cauchy sequence converges to a point in the space.*

Lemma 10.157 *Let (X, d) be a metric space.*

1. *Any convergent sequence is a Cauchy sequence.*
2. *If a Cauchy sequence contains a convergent subsequence, then the whole sequence converges to the same limit.*

Corollary 10.158 *A metric space is complete if and only if every Cauchy sequence has a convergent subsequence.*

Remark 10.159 *Not every Cauchy sequence is convergent. A Cauchy sequence is convergent if it contains a convergent subsequence.*

Proposition 10.160 *Every Cauchy sequence in discrete space is eventually constant.*

Proof:(Sketch). Every Cauchy sequence in discrete space is eventually constant. Every eventually constant sequence is convergent. ■

10.5.13 Examples of Complete Metric Spaces

We recall from Analysis the completeness of $\mathbb{K}^N$.

Theorem 10.161 *(Bolzano-Weierstrass theorem). Let $\{x_n\}_{n\geq 1}$ be a bounded sequence. Then, the sequence has a convergent subsequence.*

Theorem 10.162 *(Completeness of $\mathbb{K}^N$). Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence in $\mathbb{K}^N$. Then,*

1. *Every Cauchy sequence is bounded in $\mathbb{K}^N$;*
2. *Every Cauchy sequence contains a convergent subsequence by the Bolzano-Weierstrass theorem.*
3. *Every Cauchy sequence that contains a convergent subsequence converges to the limit of its convergent subsequence.*

Hence, every Cauchy sequence in $\mathbb{K}^N$ converges in $\mathbb{K}^N$.

Theorem 10.163 *(Completeness of $(C([a,b],\mathbb{K}),d_\infty)$). Let $C[a,b]$ be the space of all $\mathbb{K}$-valued continuous functions on [a,b]. Let $d_\infty(f,g)=\sup_{x\in[a,b]}|f(x)-g(x)|$ be the supremum norm.*
$(C([a,b],\mathbb{K}),d_\infty)$ is a Banach space.

Theorem 10.164 *(Completeness of $(B([a,b],\mathbb{K}),d_\infty)$). Let $B[a,b]$ be the space of all $\mathbb{K}$-valued **bounded** functions on [a,b]. Let $d_\infty(f,g)=\sup_{x\in[a,b]}|f(x)-g(x)|$ be the supremum norm.*
Then $(B([a,b],\mathbb{K}),d_\infty)$ is a Banach space.

Theorem 10.165 *(Completeness of ℓ_p). Let $\ell_p=\{x=\{x_j\}_{j\geq 1}:\sum_{j=1}^{\infty}|x_j|^p<\infty\}$ for $1\leq p<\infty$. Define $d(x,y)=\left(\sum_{j=1}^{\infty}|x_j-y_j|^p\right)^{1/p}$.*
Then (ℓ_p,d) is a Banach space.

Theorem 10.166 *(Completeness of ℓ_∞). Let $\ell_\infty=\{x=\{x_j\}_{j\geq 1}:\sup_{i\in\mathbb{N}}|x_i|<\infty\}$. Define $d_\infty(x,y)=\left(\sup_{i\in\mathbb{N}}|x_i-y_i|\right)$.*
Then (ℓ_∞,d_∞) is a Banach space.

Theorem 10.167 *Let (X,d) be a complete metric space. A subspace of a complete metric space is complete if and only if it is closed.*

Remark 10.168 *If we have a space which we know is a subspace of a complete metric space, to show that it is complete, it suffices to show that the subspace is closed.*

Theorem 10.169 *(Completeness of c). Let c be the space of all convergent sequences in $\mathbb{K}$. Let $d_\infty(x,y)=\sup_{i\in\mathbb{N}}|x_i-y_i|$.*
Then (c,d_∞) is a Banach space.

Theorem 10.170 *(Completeness of c_0). Let c_0 be the space of all sequences in $\mathbb{K}$ that converges to 0. Let $d_\infty(x,y)=\sup_{i\in\mathbb{N}}|x_i-y_i|$.*
Then (c_0,d_∞) is a Banach space.

10.5.14 Examples of Incomplete Metric Spaces

Theorem 10.171 *Let* $C[0,1]$ *be the space of continuous functions on [0,1]. We define the p-norm as* $\int_0^1 |f(x) - g(x)dx$*. The space C[0,1] with the p-norm is* ***incomplete****.*

Theorem 10.172 *Let* c_{00} *be the space of all sequences in* $\mathbb{K}$ *with at most finitely many non-zero terms. Then* $c_0 0$ *is an incomplete subspace of the Banach space* ℓ_∞ *as it is* ***not*** *a closed subspace.*

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11. Cantor's Intersection Theorem

11.5.15 Cantor's Intersection Theorem

Remark 11.173 *Completeness is not topologically invariant. That is, if a metric space is complete and its topology induced is topologically equivalent to another metric space's topology, that does not mean that the other metric space is complete.*

Lemma 11.174 *We have the descending order of hierarchy for properties of a sequence.*

1. *Convergent sequence;*
2. *Cauchy sequence;*
3. *Bounded sequence.*

Definition 11.175 *(Nested sequence). Let (X, d) be a metric space. Then a sequence of $\{X_n\}$ of sets in X is said to be a nested sequence if $X_n \supseteq X_{n+1}$ for all n.*

The following is a characterization of a complete metric space.

Theorem 11.176 *(Cantor's Intersection Theorem). A metric space (X, d) is complete if and only if whenever a sequence $\{F_n\}_{n\geq 1}$ of non-empty subsets of X satisfies*

1. *F_n is a closed set for all $n \geq 1$;*
2. *$F_1 \supseteq F_2 \supseteq ... \supseteq F_n \supseteq F_{n+1}$ for all $n \geq 1$;*
3. *diam $(F_n) \to 0$ as $n \to \infty$,*

then $\bigcap_{n=1}^{\infty} F_n = \{x\}$ is a ***single point*** *where x is a unique point common to all F_n.*

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12. Baire Category Theorem

12.5.16 Baire Category Theorem

Definition 12.177 *Let (X, d) be a metric space. We say that $A \subseteq X$ is dense in X if $\bar{A} = X$.*

Lemma 12.178 *Let (X, τ) be a topological space. Then let $\{D_1, ..., D_m\}$ be a sequence of open **dense** sets in X. Then the **finite** intersection*

$$\bigcap_{i=1}^{m} D_i$$

*is also an **open** and dense set.*

Remark 12.179 *In **general**, a countable intersection of open dense sets in a topological space is **not** dense.*

Definition 12.180 *(Baire Space). A topological space is called a Baire space if every countable intersection of open dense sets in X is also dense in X.*

We look at the sufficient conditions for a topological space to be a Baire space.

Theorem 12.181 *(Baire Category Theorem). Every complete metric space is also a Baire space.*

Proposition 12.182 *Every open subspace of a Baire space is also a Baire space.*

Definition 12.183 *(Nowhere Dense). Let (X, d) be a metric space. A subset Y of X is called nowhere dense if $\bar{Y}$ has no interior points, that is, int($\bar{Y}$) $= \emptyset$. Alternatively, for every $U \in \tau$, where τ is the topology induced by the metric d, there exists a $A \subseteq U$ such that $\bar{A} \cap Y = \emptyset$.*

Lemma 12.184 *The subset Y of X is nowhere dense if and only if $X \setminus \bar{Y}$ is a dense open subset of X.*

Remark 12.185 *A nowhere dense set can be thought of a dense that is so small that it has no open ball inside of it.*

Theorem 12.186 *Let $A \subseteq X$ be nowhere dense. Then for $B \subseteq A$, we have that B is nowhere dense.*

Theorem 12.187 *Let (X, d) be a complete metric space. Let $\{E_n\}_{n\geq 1}$ be a sequence of nowhere dense subsets of X. Then $\bigcup_{n=1}^{\infty} E_n$ is nowhere dense in X.*

Remark 12.188 *If we have a collection of sets that contain no open ball inside of them, then the union will not contain an open ball.*

Corollary 12.189 *The set $\mathbb{R}$ is uncountable.*

Definition 12.190 *(First Category/Meagre). Let (X, d) be a metric space. A subset $Y \subseteq X$ is called of the first category if Y can be expressed as a countable union of nowhere dense sets. Otherwise, it is called of the second category.*

Theorem 12.191 *Let (X, τ) be a topological space. Let $\{A_i\}_{i \in I}$ be a collection of sets of the first category. Then $\bigcup_{i \in I} A_n$ is of the first category.*

Theorem 12.192 *(Baire Category theorem). A complete metric space is of the second category.*

Remark 12.193 *Let X be a complete metric space and we express it as the union of closed sets i.e. $X = \cup_{i=1}^{\infty} A_i$. By the Baire category theorem, at least one of these sets must contain an open ball.*

Theorem 12.194 *A subspace of a complete metric space is complete if and only if the subset is a closed set.*

Remark 12.195 *Baire category theorem shows that every complete metric space is a Baire space. However, a Baire space need not be a metric space nor complete.*

Proposition 12.196 *Any open subspace of a Baire space is a Baire space.*

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13. Continuous Functions

13.6 Continuous Functions

13.6.1 Continuous Functions

Definition 13.197 *(Global continuous function). Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is called continuous if whenever V is open in Y, we have that $f^{-1}[V]$ is open in X.*

Proposition 13.198 *A function $f : X \to Y$ is continuous if and only if for every closed set $B \subseteq Y$, $f^{-1}[B]$ is closed in X.*

We look at different formulations of continuity.

Proposition 13.199 *Define the function between topological spaces $f : X \to Y$. The following are equivalent.*

1. *f is continuous.*
2. *$f(\bar{A}) \subseteq \overline{f(A)}$ for every $A \subseteq X$.*
3. *$\overline{f^{-1}[B]} \subseteq f^{-1}[\bar{B}]$ for every $B \subseteq Y$.*
4. *$f^{-1}[int(B)] \subseteq int(f^{-1}[B])$.*

Definition 13.200 *(Pointwise continuous). Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is continuous at a point $x \in X$ if for all open sets V in Y such that $f(x) \in V$, there exists an open set U in X such that $x \in U$ and $f(U) \subseteq V$. In otherwords, the preimage of any neighbourhood of f(x) is a neighbourhood of x.*

Theorem 13.201 *Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if and only if f is continuous at every point of X.*

Theorem 13.202 *Let X, Y, Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then $g \circ f : X \to Z$ is also continuous.*

Lemma 13.203 *Let the function f map between the topological spaces $f : X \to Y$.*

1. *Every function f is continuous if X is the discrete topology;*
2. *Every function f is continuous if Y is the indiscrete topology.*

13.6.2 Homeomorphisms

Definition 13.204 *(Homeomorphism). Let X and Y be topological spaces. A function $f : X \to Y$ is called a homeomorphism if*

1. *f is bijective;*
2. *f is continuous;*
3. *f^{-1} is continuous.*

Remark 13.205 *$f : X \to Y$ is also a homeomorphism if f is bijective and U is open in X if and only if $f(U)$ is open in Y.*

Lemma 13.206 *A bijective continuous map is a homeomorphism if and only if it an open map if and only if it is a closed map.*

Definition 13.207 *(Homeomorphic spaces). We say that X and Y are homeomorphic (topologically equivalent) if there exists a homeomorphism $f : X \to Y$.*

Homeomorphic spaces are topologically equivalent.

Example 13.208 *The space $[0, r)$ and $[0, \infty)$ is homeomorphic by the homeomorphism $f(t) = \frac{x}{1+t}$ for $t \in [0, r)$.*

Lemma 13.209 *The following are homeomorphisms:*

1. *The identity map $\phi : X \to X$;*
2. *The inverse of a homeomorphism is also a homeomorphism;*
3. *The composition of homeomorphism is also a homeomorphism.*

Theorem 13.210 *Let (X, τ) be a topological space and (Y, d) be a metric space. A function $f : X \to Y$ is continuous at $x \in X$ if and only if for all $\epsilon > 0$, there exists a neighbourhood U of x such that $d(f(x), f(y)) < \epsilon$ for all $y \in U$.*

13.6.3 Topological Properties

Definition 13.211 *(Topological property). A property of a topological space is called topological if it is preserved under homeomorphisms.*

Proposition 13.212 *The following are topological properties.*

1. *Discrete.*
2. *Separable.*

3. *Metrizable.*
4. *Finite/Countable/Uncountable.*
5. *Hausdorff Property.*

Proposition 13.213 *Let $f : X \to Y$ be a homeomorphism.*

1. $f(\bar{A}) = \overline{f(A)}$.
2. *f(int(A)) = int(f(A)).*
3. $f(\partial A) = \partial f(\partial A)$.
4. *If $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y.*

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14. Finite product of metric spaces

14.6.4 Initial Topology

Often, we start with a topology and ask if this function continuous. Now we are interested in the reverse. Starting with a function, what type of topology will make the function continuous.

Definition 14.214 *(Initial Topology). Let X and Y_i be topological spaces where $i \in I$. The initial topology induced by $\{f_i\}_{i\in I}$ on X is the* ***coarsest*** *topology on X that makes each map $f_i : X \to Y_i$ continuous.*

Starting off with a topology on each space Y_i, we require that the preimage of every open set on every topological space Y_i is open in X. Hence, we construct a topology on X that satisfies this. We define a set to be open in X by selecting the preimage of every open set on every topological space. This gives us the coarsest topology and hence the inital topology.

Remark 14.215 *Note that as a last resort, letting X be a discrete space will ensure that every $\{f_i\}_{i\in I}$ is continuous.*

14.6.5 Finite Product of topological spaces

The product topology, is a special case of the initial topology with respect to the family of projection maps.

Definition 14.216 *(Product Topology/Tychonoff topology). Let $\{X_1, ..., X_n\}$ be finite collection of topology sets with the resulting topological product being the set $\prod_{i=1}^n X_i$. The product topology is then defined to be*

$$\mathcal{B} = \{\prod_{i=1}^{n} U_i : U_i \text{ is open in } X_i : \text{ for } i \in \{1, ..., n\}\}.$$

Theorem 14.217 *The topological products $X \times Y$ and $Y \times X$ are homeomorphic to each other.*

Theorem 14.218 *Let $\{X_1, ..., X_n\}$ be finite collection of topology sets with the resulting topological product being the set $\prod_{i=1}^n X_i$. Define p_j for j=1,...,n to be the projection maps such that $p_j : \prod_{i=1}^n X_i \to X_j$. Then*

1. *Each p_j is surjective, open, and continuous.*
2. *The initial topology on $\prod_{i=1}^n X_i$ is the product topology.*

Theorem 14.219 *Let each X_i for $i \in \{1, 2, ..., n\}$ hold property $\mathcal{P}$. Then property $\mathcal{P}$ also holds for $\prod_{i=1}^n X_i$. Property $\mathcal{P}$ include*

1. *First countable*
2. *Second countable*

3. *Separable*
4. *Hausdorff*
5. *Metrizable.*

Theorem 14.220 *Let $A_i \subseteq X_i$ for $i \in \{1, 2, ..., n\}$ be dense subsets. Then $\prod_{i=1}^{n} A_i$ is dense in $\prod_{i=1}^{n} X_i$.*

14.6.6 Finite product of metric spaces

Definition 14.221 *(Finite product of metric spaces). Let $(X_1, d_1), ..., (X_n, d_n)$ be metric spaces. Let the product set be $X = X_1 \times X_2 \times ... \times X_n$ which consists of all n-tuples $(x_1, ..., x_n)$ where $x_k \in X_k$ for every k = 1,...,n. Define d to be the* ***product metric*** *on this space. Then (X, d) is known as the finite product of metric spaces.*

Definition 14.222 *(Properties of product metric). The two properties we would want the metric d on the product set X to have are listed below:*

1. *A sequence $\{x^{(j)}\}_{j\geq 1} = (x_1^{(j)}, ..., x_n^{(j)})_{j\geq 1}$ converges to $x = (x_1, ..., x_n)$ in (X,d) if and only if for each k=1,...,n, the sequence of component entries $\{x_k^{(j)}\}_{j\geq 1}$ converges to x_k in (X_k, d_k).*
2. *$d_k(x_k, y_k) \leq d(x, y)$ for all $x, y \in X$ for all $k = 1, ..., n$.*

Remark 14.223 *Property (2) neither implies nor is implied by property (1). However, if property (2) holds, then whenever a sequence $\{x^{(j)}\}_{j\geq 1}$ converges to $x = (x_1, ..., x_n)$ in (X,d), then each of the component sequences $\{x_k^j\}_{j\geq 1}$ converges to x_k in (X_k, d_k) for each k=1,...,n.*

Theorem 14.224 *Let $(X_1, d_1), ..., (X_n, d_n)$ be complete metric spaces. Let d be a product metric on $X = X_1 \times ... \times X_n$ that satisfies property (1) and property (2). Then (X,d) is complete.*

Theorem 14.225 *Suppose that d is a metric on $X = X_1 \times ... \times X_n$ which satisfies property (1). Define*

$$\mathcal{B} = \{\Omega \subseteq X : \text{ there exists an open set } U_k \in (X_k, d_k) \text{ for } 1 \leq k \leq n \text{ such that } \Omega = U_1 \times ... \times U_n\}.$$

Then the open sets in (X,d) are the unions of sets in $\mathcal{B}$.

For any topological space Y, we can construct a continuous function from Y to the finite product of metric spaces.

Theorem 14.226 *Let (X_k, d_k) be metric spaces for k=1,...,n. Let $X = X_1 \times ... \times X_n$ be endowed with a metric d which satisfies condition (1). Define the projection $\pi_k : X \to X_k$ by $\pi_k(x) = x_k$ where $x = (x_1, ..., x_n) \in X$ for each k=1,...,n.*

1. *Each $\pi_k : X \to X_k$ is continuous for all k=1,...,n.*
2. *Let Y be any topological space. Then a function $f : Y \to X$ is continuous if and only if $\pi_k \circ f : Y \to X_k$ is continuous for all k=1,...,n.*

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15. Sequential continuity

15.6.7 Sequential continuity

15.6.8 Sequential Continuity

Recall that any open set is sequentially open but **not the converse for arbitrary topological spaces**. However, in a first countable space, it holds both ways.

Definition 15.227 *(Sequentially continuous function). A function $f : X \to Y$ between topological spaces X and Y is called sequentially continuous if whenever $\{x_n\}$ converges to x in X, we have $\{f(x_n)\}$ converges to f(x) in Y.*

Sequential continuity is a weaker form of continuity.

Theorem 15.228 *(Relation between continuity and sequential continuity). Every continuous map between topological spaces is sequentially continuous. Conversely, every sequentially continuous map from a **first-countable** space to a topological space is continuous.*

Remark 15.229 *If we have a first-countable space, to show that the preimage of an open set is open, it suffices to show that the preimage is sequentially open, that is, every sequence that converges to a point in the set lies eventually in the set.*

It is much easier to show sequentially continuity of a function which we should utilise if a space is first-countable or a metric space.

Proposition 15.230 *(Continuity of the norm). Every norm on a vector space is continuous.*

Proposition 15.231 *(Continuity of the inner product). Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Then the inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$ is continuous with respect to the induced norm.*

Theorem 15.232 *Let f and g be continuous real-valued functions on a topological space X. The following real-valued functions are all continuous on X:*

1. $f + g$
2. $max\{f, g\}$
3. $min\{f, g\}$
4. $|f|$
5. fg
6. f/g

Recall for topological equivalence, two metrics are topologically equivalent if the topology they induce are the same or equivalently, a sequence converges to an element of the set in one metric if and only if the sequence converges to teh same limit in the other metric. A set with a metric is homeomorphic to itself with a topologically equivalent metric.

Theorem 15.233 *Let d and p be two metrics on the same set X. The following are equivalent:*

1. *The metrics d and p are topologically equivalent;*
2. *We have $x_n \to x$ in (X,d) if and only if $x_n \to x$ in (X,p);*
3. *The identity id: $(X, d) \to (X, p)$ is a homeomorphism.*

With this theorem, we can check whether are 2 metrics on the same set topologically equivalent.

Example 15.234 *We can show that the supremum metric on C[0,1] is not topologically equivalent to any p-metric for $1 \leq p < \infty$ by constructing sequences of functions and showing that they converge to different limits depending on which metric we use.*

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16. Uniform Continuity

16.7 Uniform Continuity

16.7.1 Uniform Continuity

Definition 16.235 *(Continuity). Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. The function f is continuous if and only if for every $x \in X$, and every $\epsilon > 0$, there exists $\delta = \delta(x, \epsilon) > 0$ such that*

$$d_Y(f(x), f(y)) < \epsilon$$

for all $y \in X$ such that $d_X(x, y) < \delta$.

We can strengthen this definition.

Definition 16.236 *(Uniform continuity). Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called uniformly continuous if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that*

$$d_Y(f(x), f(y)) < \epsilon$$

for all $x, y \in X$ with $d_X(x, y) < \delta$.

Remark 16.237 *Uniformly continuous functions only apply do not apply for functions on topological spaces.*

Lemma 16.238 *A uniformly continuous function is continuous.*

Example 16.239 *The norm on any vector space V is uniformly continuous with respect to the induced metric.*

Theorem 16.240 *Let (X, d_x) and (Y, d_y) be metric spaces. Let $f : X \to Y$ be a uniformly continuous function. Let $\{x_n\}$ be a Cauchy sequence in X. Then, the sequence $\{f(x_n)\}$ is a Cauchy sequence in Y.*

Remark 16.241 *Uniformly continuous functions preserve Cauchy sequences. Continuous/homeomorphism do not preserve this as Cauchy sequences are not a topological property.*

Example 16.242 *It is possible to have a continuous function that preserve Cauchy sequences that is not uniformly continuous. An example would be $f(x) = x^2$.*

Lemma 16.243 *Every Cauchy sequence is bounded.*

Proposition 16.244 *Let (X, d_x) and (Y, d_y) be metric spaces. We have that $f : X \to Y$ a uniformly continuous function if and only if whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $d_x(x_n, y_n) \to 0$, then $d_y(f(x_n), f(y_n)) \to 0$ as $n \to \infty$.*

Remark 16.245 *We can use the contrapositive of this statement to show a function is NOT uniformly continuous.*

Theorem 16.246 *(Heine-Cantor Theorem). Any continuous function from a compact interval $[a, b] \subset \mathbb{R}$ to a metric space is uniformly continuous.*

Theorem 16.247 *Every uniformly continuous function f from a dense subset D of a metric space X into a complete metric space Y admits a unique continuous extension F from X to Y (that is, F restricted to D coincides with f). Moreover, F is uniformly continuous on X.*

Definition 16.248 *(Isometry). Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called an isometry if*

1. *f is bijective;*
2. *(distance preserving) $d_X(x, y) = d_Y(f(x), f(y))$ for all $x, y \in X$.*

Remark 16.249 *Any distance preserving function is injective. Hence an isometry strengthens this by ensuring it is also surjective.*

Definition 16.250 *(Isometric metric spaces). We say that X and Y are isometric metric spaces if there exists an isometry $f : X \to Y$.*

Lemma 16.251 *Let $X \sim Y$ denote that (X, d_X) and (Y, d_Y) are isometric metric spaces. We have that $\sim$ is an equivalence relation on the class of metric spaces.*

Two metric spaces are considered identical if there exists an isometry between them. The reason for this is that properties such as completeness is not preserved by homeomorphisms as they are not topological properties.

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17. Completion of metric space

17.7.2 Completion of metric space

We can take **any** metric space and construct a "complete" version of it.

Definition 17.252 *A metric space (X, d_X) admits a completion if there exists a complete metric space $(\tilde{X}, \tilde{d})$ such that X is isometric to a dense subspace of $(\tilde{X}, \tilde{d})$.*

Theorem 17.253 *A metric space (X, d) admits a completion if there exists a complete metric space $(\tilde{X}, \tilde{d})$ such that X is isometric to a dense subspace of (Y, d_Y). That is, there exists an isometry $\eta : X \to \tilde{X}$ such that $\overline{\eta(X)} = \tilde{X}$.*

We have that $\tilde{X}$ is bigger than X. Furthermore, we want to ensure that $\tilde{X}$ is the **smallest** complete metric space that has an isometry from X to it.

Theorem 17.254 *(Existence and uniqueness of completion for a metric space). Any metric space (X,d) admits a completion. A completion of (X, d) is a complete metric space $(\tilde{X}, \tilde{d})$ with an isometry $\eta : (X, d) \to (\tilde{X}, \tilde{d})$ satisfying the property that for any other complete metric space (Z, ρ), with another isometry $\phi : (X, d) \to (Z, \rho)$, there exists an unique isometry $h : (\tilde{X}, \tilde{d}) \to (Z, \rho)$ such that $h = \phi \circ \eta^{-1}$.*

Remark 17.255 *The unique isometry h is what helps to establish that $(\tilde{X}, \tilde{d})$ is the smallest complete metric space with an isometry with (X,d).*

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18. Completion of metric space continue

18.7.3 Completion of metric space continued

Corollary 18.256 *Every distance-preserving function f from dense subset $D \subseteq X$ to a complete metric space Y admits a unique continuous extension $F : X \to Y$ which is also distance-preserving. Moreover, if X is a complete metric space, then F is an isometry between X and the closure $f(D)$ in Y.*

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19. Base of a topology

19.8 Base of a topology

19.8.1 Base of a topology

Working with bases gives us more flexibility in defining topologies. We don't have to work with all open sets, we can just work with a smaller collection of open sets such that we can recover back any open sets by taking unions. Furthermore, we can also look at second countable properties.

Definition 19.257 *(Base of a topology). A collection $\mathcal{B}$ of open sets in a topological space X is a base for the topology of X if every open set of X is a union of sets in $\mathcal{B}$.*

Example 19.258 *(Usual topology of $\mathbb{R}$). The collection $\mathcal{B}$ of all open intervals (a, b) is a good candidate for the base of the set $\mathbb{R}$.*

Example 19.259 *(Base for discrete topology). The base of singleton sets in the discrete topology is the smallest base in the topology as every subset of X can be expressed as the union of singleton sets.*

Remark 19.260 *Every topology admits a base since the topology itself is always a base for the topology.*

Remark 19.261 *We need not include the empty set in a base for the topology as the empty set is the empty union of open sets.*

Remark 19.262 *We can reformulate the definition of a continuous function now. To check if a function is continuous, it is sufficient to check that for a function $f : X \to Y$, $f^{-1}(U)$ is an open subset of X for every $u \in \mathcal{B}$ where $\mathcal{B}$ is the base of Y. We can simply express every open set in Y as the union of the base.*

Definition 19.263 *(Local base). Let (X, τ) be a topological space and let $x \in X$. A local base of the element x is a collection of neighbourhoods B_x such that for all $U \in \tau$ such that $x \in U$, there exists a $B \in B_x$ such that $x \in B \subseteq U$.*

Not every collection of open sets form a base for a topology. If τ is a topology on X, we can verify whether or not $\mathcal{B}$ is a base for the topology τ by the following theorem.

Theorem 19.264 *(Characterisation of the base of a topology). Let (X, τ) be a topological space. A collection $\mathcal{B}$ of open sets in a topological space X is a base* ***for its topology*** *if and only if $\mathcal{B}$ is a local base for every $x \in X$.*

However, what if we don't know what τ is? The following theorem checks whether does the collection of subsets $\mathcal{B}$ form the base for **some** topology on X.

Theorem 19.265 *(Generating topologies from a collection of subsets of a set). A collection $\mathcal{B}$ of* ***subsets of a set*** *X forms the base* ***for a*** *topology on X if and only if the following two conditions are satisfied:*

1. *Every point in X belongs to at least one member of the collection $\mathcal{B}$.*
2. *If x belongs to the intersection $U \cap V$ of two members U and V of the collection $\mathcal{B}$, then x belongs to a member W of $\mathcal{B}$ that is contained in $U \cap V$.*

   If the collection $\mathcal{B}$ satisfies the two conditions above, the generated topology is:

$$\tau_{\mathcal{B}} = \{ \bigcup_{\Omega \in \mathcal{B}^*} \Omega : \mathcal{B}^* \subseteq \mathcal{B} \}.$$

Remark 19.266 *The first condition is for arbitrary unions of open sets to be in the topology whilst the second condition is for finite intersection of open sets to be in the topology. This is so that for every intersection, we can express the set of points in the intersection as the union of base elements from the the 1st condition and hence the union of these elements will be closed in the topology generated by $\mathcal{B}$.*

Remark 19.267 *The first theorem refers to the conditions for a collection of open sets in a topology to be the base for the topology. The second theorem refers to how to take an arbitrary collection of* ***subsets*** *to form a topology and for that arbitrary collection to then be the base for that topology.*

Example 19.268 *(The lower limit topology of $\mathbb{R}$). The collection of all intervals on the real line $[a, b)$ for [a,b) is a base for a topology on $\mathbb{R}$. The topology on $\mathbb{R}$ generated by this base is called the* ***Sorgenfrey line****. The set $\mathbb{R}$ with this topology is denoted by $\mathbb{R}_\ell$.*

Example 19.269 *(The K-topology of $\mathbb{R}$). Let K denote the set of all numbers 1/n, where n is a positive integer. The collection of all intervals*

Definition 19.270 *(Subbase). A collection of subsets of a set X whose union equals X is called a subbase for a topology on X. In other words, a collection $S \subseteq \tau$ is called a Subbase for τ if the finite interesection of elements from S forms a basis of τ i.e. $\mathcal{B}_S = \{U_1 \cap ... \cap U_n : U_i \in S \quad \text{for all } i \geq 1\}$.*

Theorem 19.271 *Given a subbase for a topology on a set, the collection of all unions of finite intersection of elements in the subbase forms a topology on the same set.*

Remark 19.272 *This says that every open set in the topology can be expressed as arbitrary unions of finite intersections of elements in the subbase. So, given a collection of topologies on the same set, we can collect all the subsets of all the topologies to arrive at a subbase. From that, we can take the collection of finite intersections to get a a base and hence a topology. This topology is the smallest topology that contains all the family of topologies on a set.*

Remark 19.273 *This collection of elements of the subbase forms a base due to it satisfying the two conditions mentioned earlier for a collection of subsets to be the base for a topology.*

Theorem 19.274 *(Crtierion for Coarser topology). Let (X, τ_1) and (X, τ_2) be topological spaces. Then τ_1 is coarser than τ_2 ($\tau_1 \subseteq \tau_2$) if and only if for each $x \in X$ and every $U_x \in \tau_1$ such that $x \in U_x$, there exists a $V_U \in \tau_2$ such that $x \in V_U \subseteq U_x$.*

Remark 19.275 *A finer/stronger topology is one where the open neighbourhoods are* ***smaller*** *than the open neighbourhoods of the coarser/weaker topology.*

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20. Second countable spaces are separable and Lindelöf

20.8.2 Second-countable spaces, separability, and Lindelöf

Definition 20.276 *(First-countable). A topological space is called first countable if for every $x \in X$, there exists a* ***countable*** *local base for x.*

Example 20.277 *The topological space $(\mathbb{R}, \tau_{\mathbb{R}})$ is first-countable if for each $x \in \mathbb{R}$, we let $\beta_x = \{(x - \frac{1}{n}, x + \frac{1}{n}) : n \in \mathbb{N}\}$.*

Theorem 20.278 *All metric spaces are first countable topological spaces.*

Proof: Construct the countable base of open balls

$$\mathcal{B} = \{B_{1/n}(x) : n \in \mathcal{N}_{>0}\}.$$

Each $B \in \mathcal{B}$ is an open neighbourhood of x. For a open neighbourhood ball of size ϵ, use the Archmidean principle that for every real number ϵ, there exists a natural number n such that $n > \epsilon$ to find a ball 1/n that is smaller than ϵ. ∎

Proposition 20.279 *Every topological space with a finite set is first-countable.*

Definition 20.280 *(Second-countable). A topological space is called second-countable if it admits a countable base of open sets.*

Theorem 20.281 *Every second-countable topological space is a first-countable topological space.*

Definition 20.282 *(Hereditary property). If P is a property of a topological space X and every subspace also have property P, then we say that property P is hereditary. Recall that for a topological space (X, τ), the topological subspace is defined to be (A, τ_A) where*

1. *$A \subseteq X$;*
2. *$\tau_A = \{\Omega \cap A : \Omega \in \tau\}$.*

Lemma 20.283 *Every subspace of a second countable topological space is second countable.*

Definition 20.284 *(Open cover). An open cover of the topological space X is a family of open sets of X whose union is X.*

Definition 20.285 *(Subcover). A subcover of an open cover $\mathcal{C}$ is a subset of $\mathcal{C}$ whose union is X.*

Definition 20.286 *(Lindelöf space). A topological space X is called Lindelöf if every open cover of X admits a* ***countable*** *subcover.*

Definition 20.287 *(Hereditarily Lindelöf). A Lindelöf space is called hereditarily Lindelöf if each of its subspaces is Lindelöf.*

Lemma 20.288 *Every **closed** subspace of a Lindelöf space X is Lindelöf.*

Example 20.289 *(The set of natural numbers with the discrete topology is Lindelöf). For any open cover of $\mathbb{N}$, we can choose a countable collection of sets from the open cover that covers $\mathbb{N}$.*

Theorem 20.290 *(Urysohn's metrization theorem). Every second countable regular topological space is metrizable.*

Theorem 20.291 *For topological spaces, the following assertions hold.*

1. *Every second-countable space is separable.*
2. *(Lindelöf Lemma). Every second-countable space is (hereditarily) Lindelöf.*

Proof:(Sketch).

1) Construct a dense set from the countable base $\mathcal{B} = \{B_1, ..., B_n, ...\}$ by

$$B' = \{x_n : x_n \text{ is ANY element in } B_n : \quad \text{for i=1, 2, ...}\}.$$

Then for any $U \in \tau \backslash \{\emptyset\}$, we have that

$$U \cap B' \neq \emptyset$$

as U can be expressed as union of subcollection of $\mathcal{B}$ and B' contains an element from **every** base element.

2) Construct a collection of collections where for each open set in the cover, we gather all the base elements contained in that open set. We then choose a base element from each cover and this new collection is a countable subcover. Furthermore, every subspace of a secound countable space is second countable and hence Lindelöf as well. ■

Remark 20.292 *If we can find a Lindelöf space that is not separable for a topological space, then we know that the space is not metrizable.*

The lower limit topology on $\mathbb{R}$ provides an example of a Lindelöf and separable topological space NOT being second countable.

Example 20.293 *(The lower limit topology on $\mathbb{R}$ is Lindelöf, separable, and only first countable). Let $\mathcal{B}$ be a basis for $\mathbb{R}_\ell$. Then for each $x \in \mathbb{R}_\ell$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x$ where $B_x \subset [x, x+1)$. If $x \neq y$, then $B_x \neq B_y$. The mapping $x \to B_x$ of $\mathcal{R}_\ell$ is one to one and hence $|\mathcal{B}| = |\mathbb{R}_\ell|$ which means that the base is uncountable. Hence, the lower limit topology is NOT second countable.*

Lemma 20.294 *Every open subspace of a separable space is separable.*

Lemma 20.295 *Continuous images of separable spaces are separable.*

Lemma 20.296 *Every closed subspace of a Lindelöf space is Lindelöf.*

Lemma 20.297 *Continuous images of Lindelöf spaces are Lindelöf.*

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21. Separable Metric spaces

21.8.3 Second countable, separable, and Lindelöf metric spaces.

Theorem 21.298 *For a **metrizable** space, we have that*

$$\text{Separable} \quad \leftrightarrow \quad \text{Second-Countable} \quad \leftrightarrow \quad \text{Lindelöf.}$$

Example 21.299 *The space ℓ_p for $1 \leq p < \infty$ is a second-countable, separable, and Lindelöf space.*

Example 21.300 *The metric space ℓ_∞ is an example of a metric space that is **not** separable and hence doesn't satisfy any of the properties above.*

Lemma 21.301 *Every separable metric space is second-countable.*

Proof: (Sketch). Let $S \subseteq X$ be a countable dense subset. The countable base for X will be

$$\mathcal{B} = \{B_d(x; 1/n) : x \in S, n \geq 1\}.$$

■

Lemma 21.302 *Every Lindelöf metric space is second-countable.*

Proof: For every $n \geq 1$, we define the open cover as

$$\mathcal{A}_n = \{B_d(x; 1/n) : x \in X\}.$$

The countable subcover $\mathcal{B}_n$ will be

$$\mathcal{B}_n = \{B_d(x_k^{(n)}; 1/n) : x_k^{(n)} \in X, k \geq 1)\}.$$

Then, the countable collection $\{\mathcal{B}_n\}_{n \geq 1}$ is a base of open sets for X. ■

Definition 21.303 *(Polish space). A metric space (X, d) is called a Polish space if it is separable and there exists a metric ρ equivalent to d such that (X, ρ) is complete.*

21.8.4 Examples of separable spaces

Theorem 21.304 *(ℓ^p-space for $p < \infty$ is separable). Let $\ell^p = \{\{x_1, x_2, x_3, ...\} : x_i \in \mathbb{K}, \sum_{i=1}^{\infty} |x_i|^p < \infty\}$. Let the metric be $d_p(x, y) = (\sum_{i=1}^{\infty} |x_i - y_i|^p)^{\frac{1}{p}}$. The metric space (ℓ^p, d) is separable.*

Theorem 21.305 *The space (ℓ^∞, d) is not separable.*

21.8.5 Totally Bounded Metric Spaces

Definition 21.306 *(Bounded space). A metric space is called bounded if there exists a point $x \in X$ and $r > 0$ such that $X \subseteq B_d(x; r)$.*

Definition 21.307 *(Totally bounded space). A metric space X is called totally bounded if for every $\epsilon > 0$, we can cover X by **finitely** many open balls with centers in X and radius ϵ.*

Remark 21.308 *Note that topological spaces cannot be totally bounded as we require a metric to construct open balls.*

Lemma 21.309 *The property of a metric space to be totally bounded is hereditary.*

Lemma 21.310 *Every totally bounded metric space is separable and, hence, Lindelöf.*

Proof:(Sketch). Chose the separable dense subset to be the center points of the balls that cover the space the radius $\epsilon = 1/n$. ■

Lemma 21.311 *Every set that is totally bounded is bounded but not the converse.*

Lemma 21.312 *In $\mathbb{R}^n$, a set is totally bounded if and only if it is bounded.*

Theorem 21.313 *(Characterization of totally bounded spaces). A metric space is totally bounded if and only if each of its sequences contains a Cauchy subsequence.*

Proof:(Sketch). $\rightarrow$
Assume that every sequence contains a Cauchy subsequence. Since there exists an $\epsilon > 0$ such that we cannot cover X by finitely many open balls, we can construct a sequence inductively to be the point that is not yet covered by the previous open balls. Any 2 points x_m, x_n in this sequence will have that $d(x_n, x_m) > \epsilon$. Hence, this contradicts the assumption that every sequence contains a Cauchy subsequence. ■

Definition 21.314 *(Sequentially compact space). A topological space X is called sequentially compact if every sequence in X contains a convergent subsequence.*

Lemma 21.315 *If a Cauchy sequence has a convergent subsequence, then the Cauchy sequence converges to the limit of the subsequence.*

Proof:(Sketch). We use the Cauchy property and the fact that the subsequence converges to x. We then use the triangle inequality to show that the Cauchy sequence therefore converges to the limit point x. ■

Using the previous lemma, for a metric space that is sequentially compact, every Cauchy sequence will have a convergent subsequence, and hence, the Cauchy sequence converges to the limit of the subsequence. Recall that a complete metric space is one where every Cauchy sequence converges.

Theorem 21.316 *(Characterization of totally bounded and complete metric spaces). A metric space is totally bounded and complete if and only if it is sequentially compact.*

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22. Compactness

22.9 Compactness

22.9.1 Compactness

Definition 22.317 *(Compact Topological Space). A topological space (X, τ) is called compact if every open cover of X admits a finite subcover, that is, for every family $\{U_j\}_{j\in J}$ of open subsets of X with $\bigcup_{j\in J} U_j = X$, there exist finitely many U_j's whose union is X.*

Definition 22.318 *(Compact Subset). A subset S of a topological space X is called compact if S is compact in the relative topology inherited from X. This holds if and only if for every family $\{U_j\}_{j\in J}$ of open subsets of X that covers S ($S \subseteq \cup_{j\in J} U_j$), there exists a finite subfamily $\{U_{j_k}\}_{1\le k\le n}$ that covers S ($S \subseteq \cup_{k=1}^{n} U_{j_k}$),*

Theorem 22.319 *Let X be any topological space. If $A \subseteq X$ is a finite set, then A is compact in X.*

Corollary 22.320 *If X is a finite topological space, then every subset $A \subseteq X$ is compact in X.*

Theorem 22.321 *Let X and Y be topological spaces, $A \subseteq X$, and $f : A \to Y$ be a continuous map. If A is compact in X, then $f(A)$ is compact in Y.*

Lemma 22.322 *A discrete topological space is compact if and only if it is finite.*

Lemma 22.323 *Every nonempty set with the cofinite topology is compact.*

Lemma 22.324 *Any finite union of compact subsets of a topological space is compact.*

Lemma 22.325 *Every intersection of compact subsets of a Hausdorff space is compact.*

Definition 22.326 *(Finite intersection property). A family of nonempty sets is said to have the finite intersection property if every finite subfamily has nonempty intersection.*

Proposition 22.327 *(Characterisation of compactness). A topological space is compact if and only if every family of nonempty closed subsets with the finite intersection property has nonempty intersection.*

Theorem 22.328 *A closed subspace of a compact topological space is compact.*

Theorem 22.329 *A metric space is compact if and only if it is sequentially compact.*

Theorem 22.330 *(Characterization of compact metric spaces). Let X be a metric space. Then the following assertions are equivalent:*

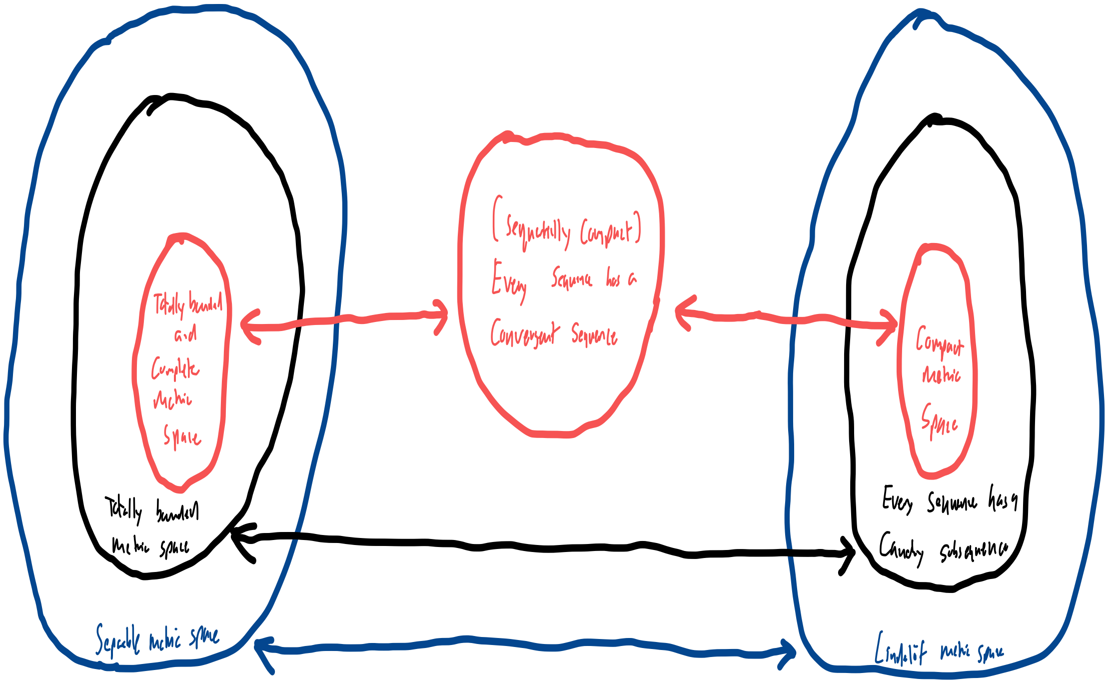

Figure 22.3: Relationships for compact metric spaces.

1. *X is compact;*
2. *X is sequentially compact;*
3. *X is totally bounded **and** complete.*

Remark 22.331 *For non-metrizable topological space, compactness neither implies nor is implied by sequential compactness.*

Theorem 22.332 *(Heine-Borel Theorem). A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Remark 22.333 *In general for metric spaces, a closed and bounded set does not imply a compact set.*

Remark 22.334 *In topological spaces, a compact set does not imply a closed set.*

Proposition 22.335 *Every continuous function from a compact metric space to a metric space is uniformly continuous.*

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23. Compactness in Topological Spaces

23.9.2 Compactness in Topological Spaces

Lemma 23.336 *Every compact metric space is totally bounded and complete.*

Theorem 23.337 *Let X be a compact topological space and let $A \subseteq X$. If A is closed in X, then A is compact in X.*

Remark 23.338 *The intuition that we need $A \subseteq X$ to be closed is that A^c will be open and hence for an open covering of A denoted by $\{U_i\}_{i\in I}$, we have that $\{U_i\}_{i\in I} \cup A^c$ is an open cover of X. Since X is compact, we can find a finite subcover for X and as a result, a finite subcover for A too.*

Lemma 23.339 *Let S be a compact subset of a Hausdorff topological space X. Then, for every $x \in X \setminus S$, there exists disjoint open neighbourhoods U of x and V of S.*

Theorem 23.340 *Let X be a Hausdorff topological space and let $A \subseteq X$. If A is compact in X, then A is closed in X.*

Remark 23.341 *The intuition is that since X is Hausdorff, we can construct a collection of neighbourhoods to cover points in A and another collection to cover points not in A that do not intersect. Then since A is compact, we can find a finite subcover of A and we can find a finite set for the points not in A. The intersection of these sets not covering A will be open and hence the complement of A is open, meaning that A is closed.*

Theorem 23.342 *(Characterisation of compact topological spaces). Let (X, τ) be a topological space. The space X is compact if and only if for a family of closed sets of X $\{F_j\}_{j\in J}$, we have that*

$$\bigcap_{j\in J} F_j \neq \emptyset.$$

Definition 23.343 *(Regular Space). A topological space is called regular if every closed subset $A \subset X$ and a point $x \notin A$ admits disjoint open neighbourhoods.*

Definition 23.344 *(Normal Space). A topological space is called normal if every disjoint nonempty closed subsets of X admits disjoint open neighbourhoods.*

Theorem 23.345 *Every compact Hausdorff space is normal.*

Remark 23.346 *Compact subsets of a non-Hausdorff topological space need not be closed.*

Proposition 23.347 *Any finite union of compact subsets of a topological space is compact.*

Proof: Use induction. For n=2, let $U_1, U_2 \subseteq X$ be compact subsets. Let C be an open cover of $U_1 \cup U_2$. Then C is an open cover of U_1 and U_2. Since they are compact, there exists a countable subcovers $U_1 \subseteq C_1$ and $U_2 \subseteq C_2$. We have that $U_1 \cup U_2 \subseteq C_1 \cup C_2$ is a countable subcover. ■

Proposition 23.348 *Every intersection of compact subsets of a Hausdorff space is compact.*

Proof: Use induction. For n=2, let A and B be compact subsets of the Hausdorff space X. Then A and B must be closed sets. Their intersection is closed. $A \cap B$ is a closed subset of the compact sets A and B, hence it is compact as closed subsets of a compact set is compact. ■

Proposition 23.349 *A discrete topological space is compact if and only if it is finite.*

Proposition 23.350 *Any non-empty subset with the cofinite topology is compact.*

Theorem 23.351 *(Extreme value theorem). Let $f : X \to \mathbb{R}$ be a continuous function where X is a compact topological space. Then f is bounded and, moreover, f attains its maximum value and minimum value.*

Proof:X is compact so $f(X) \subseteq \mathbb{R}$ is compact. Recall that a set in $\mathbb{R}^N$ is compact if and only if it is closed and bounded. Hence f is closed and bounded in $\mathbb{R}$. Hence f is bounded.
Now recall 2 things

1. Every bounded set admits a supremum/infimum.
2. Every closed set contains its supremum/infimum.

Hence f(X) admits a supremum and infimum in $\mathbb{R}$ contained in itself. Hence f attains its maximum and minimum value. ■

Theorem 23.352 *Let $f : X \to Y$ where X is a compact metric space. Then any continuous function f is uniformly continuous.*

Definition 23.353 *(T_3-Space). A topological space is a T_3 space if it is a T_1 space and regular.*

Definition 23.354 *(T_4-Space). A topological space is a T_4 space if it is a T_1 space and normal.*

Lemma 23.355 *Every T_4 space is a T_3 space.*

23.9.3 Separation Axioms

23.9.4 Characterisation of regular/normal spaces

Proposition 23.356 *(Characterisation of a normal space). A topological space X is normal if and only if each open neighbourhood of any nonempty closed subset A of X contains the closure of an open neighbourhood of A.*

Proposition 23.357 *(Characterisation of a regular space). A topological space X is regular if and only if each open neighbourhood of any $x \in X$ contains the closure of an open neighbourhood of x.*

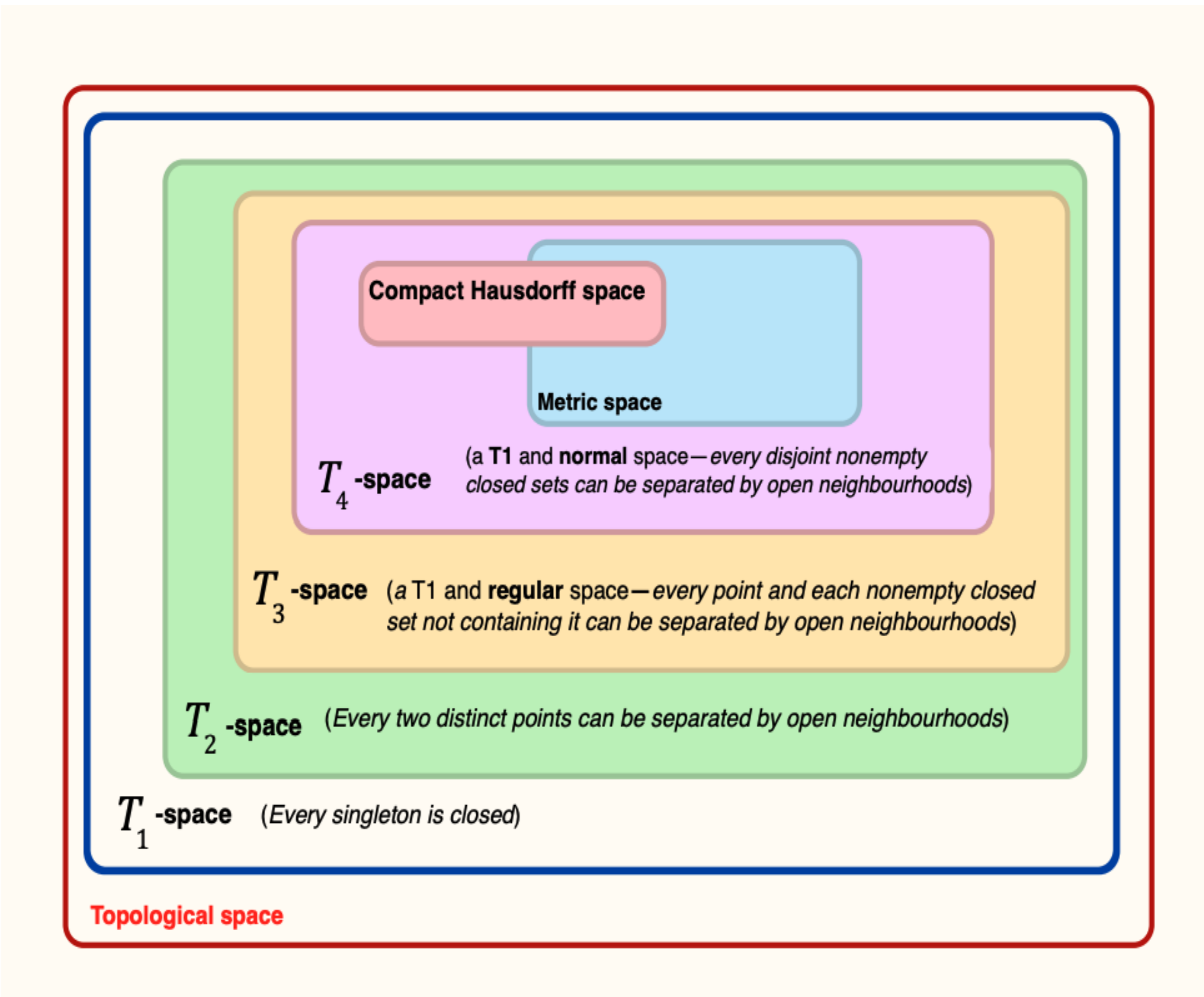

Figure 23.4: Separation Axioms.

23.9.5 Important examples of T_4 spaces

Theorem 23.358 *Every metric space is a T_4 space.*

Proposition 23.359 *Let X be a Hausdorff space and let S be any compact subset of X.*

1. *For every $x \in X \setminus S$, there exist disjoint open neighbourhoods U of x and V of S.*
2. *The compact subset S is closed.*

Remark 23.360 *Compact subsets of a non-Hausdorff space need not be closed.*

Theorem 23.361 *Every compact Hausdorff space is a T_4 space.*

Corollary 23.362 *A compact space is a T_2-space if and only if it is a T_3-space if and only if it is a T_4 space.*

23.9.6 Baire Category Theorem Revisited

Theorem 23.363 *Every compact Hausdorff space is a Baire space.*

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24. Urysohn's Lemma

24.9.7 Urysohn's Lemma

Urysohn's lemma is useful for us to establish whether is a topological space is normal.

Theorem 24.364 *(Urysohn's Lemma). Let X be a topological space. X is a normal space if and only if for every non-empty closed set A and B that are disjoint, there exists a continuous function $f : X \to [0, 1]$ such that*

$$\begin{cases} f = 0 & \textit{on } A \\ f = 1 & \textit{on } B \end{cases}$$

Lemma 24.365 *A topological space is normal if and only if any two disjoint closed subsets can be separated by a continuous function.*

Remark 24.366 *Separated here refers to the fact that the closure of two sets are disjoint.*

Uryson's lemma is useful for formulating conditions for a topological space to be metrizable. Furthermore, Urysohn's lemma is useful as all metric spaces and all compact Hausdorff spaces are normal.

Remark 24.367 *We can extend from having an interval of [0, 1] to an interval of [a, b] where $a < b$. We have that $\tilde{f} : X \to [a, b]$ such that $\tilde{f} = a$ on A and $\tilde{f} = b$ on B.*

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25. Tietze-Urysohn Extension Theorem

25.9.8 Revision from analysis on sequences of functions

We look at sequences of functions $f_n : D \to \mathbb{K}^N$ with domain $D \subseteq \mathbb{K}^d$ and $n \in \mathbb{N}$. If we fix $x \in D$, we then have that $f_n(x)$ is a sequence in $\mathbb{K}^N$.

Definition 25.368 *(Pointwise Convergence). We say that the sequence of functions f_n converges pointwise to f on D if for all $x \in D$, the sequence $\{f_n(x)\}$ converges to f(x) as $n \to \infty$. That is, for all $x \in D$ and for all $\epsilon > 0$, there exists $n_{\epsilon,x} \geq 1$ such that*

$$||f_n(x) - f(x)||< \epsilon$$

for all $n \geq n_{\epsilon,x}$.

In other words, we have that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for all $x \in D$. We write $f_n \to f$ pointwise.

Definition 25.369 *(Uniform Convergence). We say $f_n \to f$ uniformly on D if for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that*

$$||f_n(x) - f(x)||< \epsilon$$

for all $n > n_\epsilon$ and all $x \in D$. We say that $f_n(x) \to f(x)$ uniformly with respect to $x \in D$.

Definition 25.370 *(Supremum Norm). Let $f : D \to \mathbb{K}^N$ be a function. We define its* ***supremum norm*** *by*

$$||f||_{\infty,D}= \sup_{x\in D}||f(x)||.$$

Lemma 25.371 *The supremum norm of a function is finite if and only if f is a bounded function.*

Proposition 25.372 *(Characterisation of uniform convergence). Let $f_n : D \to \mathbb{K}^N$ be functions. Then $f_n \to f$ uniformly on D if and only if $||f_n - f||_{\infty,D}\to 0$ as $n \to \infty$.*

Definition 25.373 *(Absolute Convergence for series of functions). Let $g_k : D \to \mathbb{K}^N$ be a* ***sequence*** *of functions, where $D \subseteq \mathbb{K}^d$. The series $\sum_{k=0}^{\infty} g_k$ is called absolutely convergent on D if for every $x \in D$, the series $\sum_{k=0}^{\infty} g_k(x)$ converges absolutely. In other words,*

$$\sum_{k=0}^{\infty}||g_k(x)||_{\infty,D}$$

converges in $\mathbb{R}$ for all $x \in D$.

Remark 25.374 *So what this means is that we fix $x \in D$ and then we get a series of vectors by evaluating each function at x. Recall that to check for convergence of a series of vectors, you look at the series of norms and see does that converge in $\mathbb{R}$. Then see does the series of norms converge in $\mathbb{R}$ for every $x \in D$. If it does, then it is absolutely convergent.*

Definition 25.375 *(Uniform Convergence for series of functions). Let $g_k : D \to \mathbb{K}^N$ be a* ***sequence*** *of functions, where $D \subseteq \mathbb{K}^d$. If the sequence of $\{f_n\}$ of partial sums converges uniformly on D, where $f_n(x) = \sum_{k=0}^{n} g_k(x)$ for all $x \in D$, the series $\sum_{k=0}^{\infty} g_k$ converges uniformly.*

Remark 25.376 *A series converges if the series of partial sums converges. However, we want uniform convergence, so we need to have that the series of partial sums to converge uniformly on D. So the sequence of partial sums is a sequence of functions $\{f_n\}$. Recall that the uniform convergence of a sequence of functions is when the supremum norm $||f_n - f||_{\infty,D} \to 0$ as $n \to \infty$.*

We can now introduce a criterion to check for uniform convergence of a series.

Theorem 25.377 *(Weierstrass M-Test). Let $g_n : D \to \mathbb{K}^N$ be a sequence of functions. If*

$$\sum_{k=0}^{\infty} ||g_k||_{\infty,D}$$

converges, then the original series

$$\sum_{k=0}^{\infty} g_k$$

converges absolutely and therefore uniformly on D.

Remark 25.378 *Here, we look at the series of supremum norms of each function g_k, which is finding the largest value of $g_k(x)$ for all $x \in D$. We get a series of non-negative numbers when we take the supremum norms and if this series of non-negative numbers converges (where we can use many of the tests for non-negative series), then the series converges absolutely and uniformly on the domain D.*

25.9.9 Tietze-Urysohn Extension Theorem

The Tietze-Urysohn extension theorem is useful to help us in determining whether is a space a normal space.

Theorem 25.379 *(Tietze-Urysohn Extension Theorem). Let X be a normal topological space and let Y be a closed subspace of X. Let $f : Y \to \mathbb{R}$ be a bounded and continuous function on Y. Then there exists a function $h : X \to \mathbb{R}$ that is bounded and continuous on X such that $h|_Y = f$.*

Proof:(Sketch). We denote $f : Y \to \mathbb{R}$ to be a bounded and continuous function on Y where $C_0 := ||f||_{\infty,Y} = \sup_{y \in Y} |f(y)|$. We will construct a bounded and continuous function $h : X \to \mathbb{R}$ such that $h|Y = f$.
To construct h, we define

$$h = \sum_{n=0}^{\infty} g_n$$

where $g_n : X \to \mathbb{R}$ are bounded and continuous functions. We then construct the sequence

$$h_n = \sum_{k=0}^{\infty} g_k$$

for all $n \in \mathbb{N}$. We require g_n to satisfy 2 properties

1. $||g_n||_{\infty,X} \leq C_0 \frac{2^n}{3^n+1}$ for all $n \geq 0$,
2. $||f - \sum_{k=0}^{\infty} g_k||_{\infty,Y} \leq C_0 \frac{2^{n+1}}{3^{n+1}}$ for all $n \geq 0$.

Property 1 ensures that our function h is the sum of a uniformly convergent sequence $\{h_n\}_{n\geq 1}$ and hence h will be continuous by invoking the Weierstrass M-test.
Property 2 ensures the uniform limit of h coincides with f so that h=f on Y by letting $n \to \infty$.
We construct the sequence $\{g_n\}_{n\geq 1}$ by constructing the closed sets where $f(y) \leq |\frac{C_0}{3}$ which will be closed in Y and also be closed in X due to Y being closed. We can then apply Urysohn's lemma to construct g_0. We repeat the step inductively to construct our sequence $\{g_n\}_{n\geq 1}$. ■

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26. Connected Spaces

26.10 Connected Spaces

26.10.1 Connected Spaces

Definition 26.380 *(Separation). A separation of a topological space X is a pair U, V of disjoint non-empty open subsets of X such that*

1. $U \cup V = X$
2. $U \cap V = \emptyset$
3. $U \neq \emptyset$ *and* $V \neq \emptyset$
4. $U, V \in \tau$

Definition 26.381 *(Disconnected). Let X be a topological space. We say that X is separated/disconnected if it can be broken up into 2 disjoint non-empty open subsets of X.*

Definition 26.382 *(Connected). A topological space X is connected if there is no separation of X.*

Theorem 26.383 *(Clopen set criterion for Connectdness). X is connected if and only if the only clopen subsets of X is the empty set and X.*

Remark 26.384 *To show a topological space is not connected, find a clopen set that is not $\emptyset$ or X.*

Definition 26.385 *(Connected subspace). Let X be a topological space. A subset $E \subseteq X$ is called a connected subspace if E is connected in the relative topology τ_E; i.e. there is no separation with open sets from τ_E*

Remark 26.386 *Connectdness is a topological property which is preserved by homeomorphisms. This is because it is formulated entirely by open space.*

Remark 26.387 *To show that 2 spaces are **not** homeomorphic, we need to show that one space is not connected/compact.*

Theorem 26.388 *Let X and Y be topological spaces. Let f be a continuous function. If X is connected, then $f(X)$ is a connected subset of Y.*

Remark 26.389 *This relaxes the condition for f to be a homeomorphism as f does not need to be a bijection.*

Corollary 26.390 *If X and Y are topoloigcal spaces and $f : X \to Y$ is a homeomorphism between X and Y, then if X is connected, then Y is connected.*

Theorem 26.391 *Let X be a topological space and let $A \subseteq X$. If the subspace A is connected, then the closure $\overline{A}$ is also connected.*

Definition 26.392 *(Totally disconnected). A topological space in which singletons are the only connected subsets is called totally disconnected.*

Remark 26.393 *A totally disconnected space does not imply a discrete space.*

Theorem 26.394 *Let X and Y be topological spaces. If for every $x \in X$, we have that $X \setminus \{x\}$ is connected with regards to the subspace topology, and if there exists a $y \in Y$ such that $Y \setminus \{y\}$ is disconnected with respect to the subspace topology, then X is not homeomorphic to Y.*

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27. Applications of Connected Spaces

27.10.2 Applications of Connected Spaces

Theorem 27.395 *(Intermediate Value Theorem). Let $f : X \to \mathbb{R}$ be a continuous function where X is a connected topological space. If $a, b \in X$, then for all M between $f(a)$ and $f(b)$, there exists a $c \in X$ such that $f(c) = M$.*

Theorem 27.396 *All intervals of $\mathbb{R}$ with the usual topology on $\mathbb{R}$ are connected and these are the only connected subsets of $\mathbb{R}$.*

The arbitrary union of connected topological spaces need not be connected. However, if we have a point common to all the connected spaces, then the union is indeed connected. We introduce a sufficient condition for the union of connected spaces to be connected.

Theorem 27.397 *(Common point criterion for connected unions). Let X be a topological space and let $\{A_i\}_{i\in I}$ be an arbitrary collection of connected topological spaces. If $\bigcap_{i\in I} A_i \neq \emptyset$ then $\bigcup_{i\in I} A_i$ is connected.*

We can develop a similar result to the theorem above.

Theorem 27.398 *Let $\{E_j\}_{j\in J}$ be a family of connected subsets of a topological space X such that $E_j \cap E_k \neq \emptyset$ for all $j, k \in J$. Then*

$$\bigcup_{j\in J} E_j$$

is also connected.

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28. Path Connectdness

28.10.3 Path Connectdness

Definition 28.399 *(Connected Component). Let X be a topological space and pick a point $x \in X$. The connected component of $x \in X$, denoted by $C(x)$, is the union of all connected subsets of X that contains x.*

Remark 28.400 *We have that*

1. *$C(x)$ is connected.*
2. *$C(x)$ is the largest connected subset of X that contains x.*

Theorem 28.401 *Any 2 connected components of a topological space X either coincides or are disjoint.*

Theorem 28.402 *The connected component of X forms a partition of X into maximal connected subsets, that is,*

$$X = \bigcup_{x \in X} C(x)$$

where for any $x, y \in X$, we have either that

1. *$C(x) = C(y)$ or*
2. *$C(x) \cap C(y) = \emptyset$.*

Definition 28.403 *(Totally disconnected). A topological space is totally disconnected if the singletons are the only connected subsets.*

Lemma 28.404 *Every discrete/countable metric space is totally disconnected.*

Lemma 28.405 *If E is a connected subset of a topological space, then $\bar{E}$ is also connected.*

Theorem 28.406 *Every connected component of a topological space is closed.*

Definition 28.407 *(Path). Let $x_0, x_1 \in A$. A path in X from x_0 to x_1 is a continuous function γ such that*

$$\gamma : [0, 1] \to X$$

such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Definition 28.408 *(Path-Connected). The space X is called path-connected if for every $x_0, x_1 \in X$, there exists a path from x_0 to x_1.*

Lemma 28.409 *The relation "there is a path from x to y" is an equivalence relation which satisfies the properties of*

1. *Reflexive,*
2. *Symmetric,*
3. *Transitive.*

Definition 28.410 *(Path Components). The equivalent classes corresponding to the above relation are called the path components of X.*

Definition 28.411 *(Path-Connected Space). A topological space is called path-connected if there is one path component, that is, any 2 points of X can be joined by a path in X.*

Remark 28.412 *Path components of X are the maximal path-connected subsets of the space X.*

Remark 28.413 *The path components of a space need not be open. Path components need not coincide with the connected components.*

Lemma 28.414 *A fixed path component always lies within a necessarily unique connected component.*

Theorem 28.415 *A path connected space X is connected.*

Lemma 28.416 *An open ball in $\mathbb{R}^d$ for $d \geq 2$ is path connected and therefore connected.*

Remark 28.417 *A connected space does NOT imply a path-connected space.*

Lemma 28.418 *Only 1 connected component contains the path component A.*

Corollary 28.419 *Let X be a topological space. Then each connected component of X is an union of path components of X.*

Remark 28.420 *Path components of a topological space need not coincide with the connected components if X is locally path-connected.*

Definition 28.421 *(Cut point). A point p of a topological space is called a cut point if $X - \{p\}$ is not connected.*

Remark 28.422 *To see whether 2 spaces are homeomorphic, you can count to see if there number of non-cut points are the same in both spaces. Homeomorphism preserves the connected structure of the space and hence the number of cut points.*

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29. Contraction Mapping Theorem

29.11 Contraction Mapping Theorem

29.11.1 Contraction Mapping Theorem

The purpose of the contraction mapping theorem is to determine what are the sufficient conditions for the existence of fixed points for mappings.

Definition 29.423 *(Fixed point). Let (X, d) be a metric space and $\phi : X \to X$ be a function. Then $x \in X$ is a fixed point of ϕ if $\phi(x) = x$.*

Definition 29.424 *(Contraction). A map $\phi : X \to X$ is called a contraction if there exists $c \in (0, 1)$ such that*

$$d(\phi(x), \phi(y)) \leq cd(x, y)$$

for all $x, y \in X$.

Lemma 29.425 *If $\phi : X \to X$ is a contraction, then there is at most one fixed point for ϕ.*

Theorem 29.426 *(Contraction Mapping Theorem). Every contraction ϕ on a complete metric space (X, d) admits a unique fixed point x_*. Moreover, this fixed point can be constructed as follows, pick an arbitrary $x \in X$ and define $x_n = \phi(x_{n-1})$. Then, $\lim_{n\to\infty} x_n = x_*$.*

Proof:(Sketch). We only need to show existence of a fixed point as from the previous lemma, we are guaranteed the uniqueness of a fixed point.
1) Construct a sequence $x_{n+1} = \phi^n(x_0)$ and show that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence. By the completion of X, $\{x_n\}_{n\geq 1}$ converges to a point x_* in X.

2) Show that x_* is the fixed point by using triangle inequality, definition of contraction, and taking limits. ■

29.11.2 Applications of the Contraction Mapping Theorem

Definition 29.427 *(Lipschitz continuous). We assume that F is Lipschitz continuous in the first variable, there exists a constant $L > 0$ such that*

$$|F(x, t) - F(y, t)| \leq L|x - y|$$

for $x, y \in \overline{B}(\psi, r)$ and every $t \in [0, T]$.

Theorem 29.428 *(Cauchy-Picard Theorem). Let U be an open subset of $\mathbb{R}^n$. Fix $\psi \in U$. For $T > 0$, let $F : U \times [0, T] \to \mathbb{R}^n$ be a continuous functions. Choose $r > 0$ small enough such that the closed ball $\overline{B}(\psi, r) \subseteq U \subseteq \mathbb{R}^n$. By continuity of F and compactness of $\overline{B}(\psi, r) \times [0, T]$, we have*

$$M := \sup\{|F(x, t)| : (x, t) \in \overline{B}(\psi; r) \times [0, T]\} < \infty.$$

We assume that F is Lipschitz continuous.
Suppose the above condition hold. Then, there exists $\beta \in [0, T]$ such that the following initial value problem has a unique solution u on $[0, \beta]$:

$\frac{du}{dt} = F(u(t), t)$ *for* $0 \leq t \leq \beta$,
$u(0) = \psi$.

Furthermore, β depends only on r, M and the Lipschitz constant L.

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30. Normed Vector Space

30.12 Hilbert Spaces

30.12.1 Normed Vector Space

Proposition 30.429 *(Continuity of the norm). Let* $(E, ||.||)$ *be a normed vector space. Then the norm* $||.||$: $E \to \mathbb{R}$ *is a continuous function.*

Proof: Since E is a normed vector space, then it is also a metric space. Recall that to show a function is continuous, we can show it is sequentially continuous for a metric space. ■

Proposition 30.430 *(Continuity of inner product). Let* E *be an inner product space. Then the inner product* $\langle .,. \rangle : E \times E \to \mathbb{K}$ *is continuous with respect to the induced norm.*

Proposition 30.431 *(Paralellogram identity). Let* $(E, \langle .,. \rangle)$ *be an inner product space and let* $||.||_\infty$ *be the induced norm. Then*

$$||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2)$$

for all $u, v \in E$.

Lemma 30.432 *Let* $||.||$ *be a norm on a vector space* E *satisfying the parallelogram identity. Then there exists an inner product on* E *inducing the norm* $||.||$ *where* $\langle .,. \rangle = \sqrt{||.||}$.

Theorem 30.433 *(Cauchy-Schwarz Inequality). Let* E *be an inner product space with an inner product* $\langle .,. \rangle)$. *Then*

$$|\langle u, v \rangle| \leq ||u||.||v||$$

for all $u, v \in E$ *with equality if and only if* u *and* v *are linearly dependent.*

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31. Projections

31.12.2 Projections

Definition 31.434 *(Distance of a set). Let $S \subset X$ where X is a* ***normed space****. For every $x \in X$, we define the distance of the point x to the set S itself as*

$$d(x, S) = \inf_{s \in S} ||x - s||.$$

Lemma 31.435 *Let $S \subset X$ where X is a* ***normed space****. Then for every $x \in X$, $d(x, S) = 0$ if and only if $x \in \overline{S}$.*

Definition 31.436 *(Projection). Let E be a normed vector space and M be a* ***non-empty closed*** *subset of E. We define the set of projections for a point $x \in E$ onto M by*

$$P_M(x) = \{m \in M : ||x - m|| = dist(x, M) = inf\{||x - y|| : y \in M\}\}.$$

Remark 31.437 *If the non-empty closed subset $M \subset E$ is not a convex subset of the plane, then $P_M(x)$ may contain more than one point.*

Definition 31.438 *(Proximal). The set M is called proximal if the set $P_M(x)$ is non-empty for each $x \in X$.*

Definition 31.439 *(Chebychev set). A subset S of X is called a Chebychev set if for every $x \in X$, the set $P_M(x)$ contains a unique point.*

Lemma 31.440 *Let X admit a Chebychev subset S, then we can define a function $\phi : X \to S$ from $x \to P_S(x)$.*

Proposition 31.441 *Every Chebychev set in a normed space is proximal.*

Proposition 31.442 *A necessary condition for a subset S to be proximal is for S to be closed.*

Proposition 31.443 *Every nonempty closed subset of a finite-dimensional normed vector space is proximal.*

Remark 31.444 *Being closed is also a sufficient condition for a set to be proximal in* ***finite-dimensional*** *normed spaces. This does not hold for an infinite-dimensional normed space.*

Proposition 31.445 *Let X be a normed space. Every compact subset of X is proximal but not necessarily Chebychev.*

Proposition 31.446 *Every nonempty closed subset $M \subset \mathbb{R}^n$ with the Euclidean norm $||.||$ is proximal.*

Definition 31.447 *(Convex set). A set M is convex if for all $\lambda \in [0,1]$, we have that*

$$\lambda x + (1-\lambda)y \in M$$

for all $x, y \in M$.

Theorem 31.448 *(Existence and uniqueness of projection onto a convex set). Let H be a finite-dimensional Hilbert space and $M \subset H$. Then, M is a Chebychev set if and only if M is closed and convex. That is, for all $x \in X, P_M(x)$ contains a unique point.*

Theorem 31.449 *(Characterisation of projection). Let H be a Hilbert space and M be a nonempty, closed, and convex subset of H. Then for a point $m_x \in M$, the following assertions are equivalent:*

1. *A point $m_x \in M$ coincides with the projection $P_M(x)$ of x onto M i.e. $m_x = P_M(x)$;*
2. *$Re\langle x - m_x, m - m_x\rangle \leq 0$ for all $m \in M$.*

Remark 31.450 *By convexity of M, we expect the angle between x and a projection point m_x compared to all other points to be larger or equal to $\pi/2$.*

Definition 31.451 *(Vector subspace). Let V be the vector space over the field $\mathbb{K}$ and let W be a subset of V. Then W is a vector subspace if for all $x, y \in W$ and for all $\alpha, \beta \in \mathbb{K}$, we have that $\alpha x + \beta y \in W$.*

Proposition 31.452 *For a* ***normed vector space****, we have that*

Normed Vector Space $\supset$ Proximal Set $\supset$ Chebychev Set $\supset$ Nonempty, closed, and convex subset.

Proposition 31.453 *For an* ***inner product space****, we have that*

Inner Product Space $\supset$ Closed Vector Space $\supset$ Proximal Subspace $\leftrightarrow$ Chebychev Subspace $\supset$ Complete Subspace

Proposition 31.454 *For an* ***Hilbert space****, we have that*

Hilbert Space $\supset$ Proximal Subspace $\leftrightarrow$ Chebychev Subspace $\leftrightarrow$ Complete Subspace $\leftrightarrow$ Closed Vector Subspace

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32. Orthogonal Complements

32.12.3 Orthogonal Complements

Let M be a nonempty subset of an inner product space $(X, \langle .,. \rangle)$. Recall that the definition of a projection is $P_M(x) = \{m \in M : ||x - m|| = dist(x, M) = inf\{||x - y||: y \in M\}\}$.

Definition 32.455 *(Orthogonal complement). The orthogonal complement of M in X, denoted by $M^\perp$, is defined by*

$$M^\perp := \{x \in X : \langle x, m \rangle = 0 \text{ for all } m \in M\}.$$

We want to look at properties of orthogonal complements to help us characterise projections in Hilbert spaces.

Lemma 32.456 *The orthogonal complement $M^\perp$ of M in X is a closed vector subspace of X.*

Lemma 32.457 *Let $M^\perp$ be the orthogonal complement of M in X. We have that*

$$M^\perp = \overline{M}^\perp = (spanM)^\perp = (span\overline{M})^\perp = \overline{spanM}^\perp.$$

Lemma 32.458 *Every vector **subspace** M of a Hilbert space is convex.*

Lemma 32.459 *Let H be a Hilbert space and let $M \subset H$. For any $x \in H$, we can express as the sum of an element in its projection and orthogonal complement*

$$x = P_M(x) + (x - P_M(x))$$

where $P_M(x) \in M$ and $(x - P_M(x)) \in M^\perp$.

Theorem 32.460 *Let M be a closed subspace of a finite dimensional Hilbert space H. Then M is a Chebychev subspace. Moreover, for every $x \in H$, we have $m_x \in M$ coincides with $P_M(x)$ if and only if $x - m_x \in M^\perp$.*

Theorem 32.461 *(Orthogonal complements). Let M be a closed vector subspace of the Hilbert space H. The orthogonal projector $P_M : H \to M$ satisfies the properties*

1. *$H = M \bigoplus M^\perp$ and $M^{\perp\perp} = M$.*
2. *$P_M : H \to M$ is a bounded linear operator with $||P_M|| \leq 1$. Moreoever, if M is at least one-dimensional, then $||P_M|| = 1$.*
3. *$P_M(M^\perp) = \{0\}$.*

Remark 32.462 *We explain the implications of this theorem.*

1. *We can decompose the Hilbert space H into a direct sum of a closed subspace M and its orthogonal subspace $M^{\perp}$. Resultantly, $M \cap M^{\perp} = \{0\}$. Furthermore, any element $x \in H$ can be expressed as the sum of an element in M and an element in $M^{\perp}$. In particular, $x = P_M(x) + (x - P_M(x))$ where $P_M(x) \in M$ is the projection and $(x - P_M(x)) \in M^{\perp}$. Finally, we generally have the case that $M \subset M^{\perp\perp}$ but for closed sets in Hilbert spaces, we have equality.*

2. *The projection operator P_M has a norm of 1 for any non-trivial closed subspaces M. For $||P_M|| \leq 1 \leftrightarrow ||P_M x|| \leq ||x||$ for all $x \in H$, so the norm of the projection is less than the norm of the vector.*

3. *The projection of any element in $M^{\perp}$ onto M is the zero vector.*

Lemma 32.463 *Let $S \subset H$ where H is a Hilbert space and S is a closed subspace. Then, $S^{\perp}$ is a complete subspace.*

We now change the focus from Hilbert spaces to inner product spaces, whereby a closed vector subspace **does not** imply a Chebychev subspace.

Theorem 32.464 *(Pre-projection theorem). Every proximal vector subspace M of an inner product space X is a Chebychev subspace. Moreover, for every $x \in X$, we have $m_x \in M$ coincides with $P_M(x)$ if and only if $x - m_x \in M^{\perp}$. Furthermore, the orthogonal projector $P_M : X \to M$ satisfies the properties of:*

1. $X = M \bigoplus M^{\perp}$ *and* $M^{\perp\perp} = M$.

2. $P_M : X \to M$ *is a bounded linear operator with* $||P_M|| \leq 1$. *Moreoever, if M is at least one-dimensional, then* $||P_M|| = 1$.

3. $P_M(M^{\perp}) = \{0\}$.

Theorem 32.465 *Let X be an inner product space over $\mathbb{K}$. Let M be a vector subspace of X. Then M is a closed and convex subset of X. Furthermore, every proximal subspace coinces with a Chebychev subspace.*

Proposition 32.466 *(Characterisation for a dense subset in a Hilbert space). A vector subspace M of a Hilbert space H is dense in H if and only if $M^{\perp} = \{0\}$. In particular, $H = \overline{M} \bigoplus M^{\perp}$.*

Corollary 32.467 *Suppose that M is a non-empty subset of the Hilbert space H. Then*

$$M^{\perp\perp} := (M^{\perp})^{\perp} = \overline{Span(M)}.$$

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33. Hilbert Spaces

33.12.4 Hilbert Spaces

Lemma 33.468 *Any finite-dimensional normed space is complete and separable.*

Proposition 33.469 *In a finite-dimensional normed space E, any two norms are equivalent, that is, for every pair of norms $||.||_1$ and $||.||_2$ on E, there exist a positive constant C such that*

$$C||x||_2 \leq ||x||_1 \leq C||x||_2$$

for all $x \in E$.

Corollary 33.470 *(Generalisation of Heine Borel Theorem). In a finite-dimensional normed space, a set is compact if and only if it is closed and bounded.*

Definition 33.471 *(Isometric isomorphism). A linear transformation T from a normed space $(X, ||.||_X)$ to a normed space $(Y, ||.||_Y)$ is called an isometric isomorphism if T is bijective and $||T(x)||_Y = ||x||_X$ for all $x \in X$.*

Theorem 33.472 *Every finite dimensional inner product spaces is a separable Hilbert space.*

Proof: Every finite dimensional inner product space has an isometric isomorphism to $\mathbb{K}^N$ with the usual dot product, which itself is a complete inner product space. ■

There are 3 types of Hilbert spaces.

1. Finite dimensional Hilbert spaces.
2. Infinite dimensional separable Hilbert spaces.
3. Infinite dimensional non-separable Hilbert spaces.

Example 33.473 *The prototype for the separable infinite dimensional Hilbert space is the Hilbert space ℓ_2 with the inner product $\langle x, y\rangle_{\ell_2} = \sum_{j=1}^{\infty} x_j \bar{y}_j$.*

Proposition 33.474 *Every finite-dimensional Hilbert space is separable.*

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34. Orthonormal Sequences

34.12.5 Orthonormal Sequences

Definition 34.475 *(Orthogonal Vectors). Two vectors u, v of a Hilbert space H are orthogonal if $\langle u, v\rangle = 0$.*

Theorem 34.476 *(Pythagoras' Theorem). Let $k \geq 2$. If $u_1, ..., u_k$ are mutually orthogonal vectors in an inner product space X, then we have*

$$||\sum_{j=1}^{k} u_j||^2 = \sum_{j=1}^{k} ||u_j||^2.$$

Definition 34.477 *(Orthonormal Sequence). A sequence $\{e_j\}_{j\geq 1}$ in a Hilbert space H is called orthonormal if*

$$\begin{cases} \langle e_j, e_k\rangle = 0 & \text{if } j \neq k \\ \langle e_j, e_j\rangle = ||e_j||^2 = 1 & \text{for all } j \geq 1. \end{cases}$$

Definition 34.478 *(Span). Assume that $\{e_j\}_{j\geq 1}$ is an orthonormal sequence in X. For every $N \geq 1$, let E_N denote the vector space spanned by $e_1, ..., e_N$, which we denote by*

$$E_N = span\{e_1, ..., e_N\}.$$

The finite-dimensional vector space E_N with the induced inner product from X is a Hilbet space. Furthermore, we have that

$$span\{e_j\}_{j\geq 1} = \bigcup_{N=1}^{\infty} E_N.$$

Lemma 34.479 *Every complete vector subspace of an inner product space is a Chebychev subspace. Hence, E_N is a Chebychev subspace and any projection onto E_N will be a unique projection.*

We first look at the representation of a vector in E_N. We then look at the representation of a vector in X that is projected onto E_N.

Proposition 34.480 *Let $\{e_j\}_{j\geq 1}$ be an orthonormal sequence in an inner product space $(X, \langle ., .\rangle)$.*

1. *If $v \in E_N = span\{e_1, ..., e_N\}$, then $v = \sum_{j=1}^{N} \langle v, e_j\rangle e_j$ and $||v||^2 = \sum_{j=1}^{N} |\langle v, e_j\rangle|^2$.*
2. *For every $N \geq 1$, the **unique** projection of $u \in X$ onto E_N is $U_N = \sum_{j=1}^{N} \langle u, e_j\rangle e_j$ and, moreover,*

$$||u||^2 = ||u - U_N||^2 + ||U_N||^2.$$

Remark 34.481 *The first result in the above proposition states that the scalar coefficient λ_j of the linear combination of the orthonormal sequence for a given vector v is the inner product of v and corresponding orthonormal vector e_j. The second result looks at the unique projection of u onto X and we can decompose u by an element in E_N and its orthogonal complement $E_N^{\perp}$ and then apply Pythagoras theorem.*

Lemma 34.482 *A non-negative series is convergent if and only if the sequence of partial sums is bounded from above.*

Remark 34.483 *The sequence of non-negative partial sums is monotonically increasing and hence we only need to show it is bounded in order to show convergence.*

Proposition 34.484 *(Bessel's Inequality). If $\{e_j\}_{j\geq 1}$ is an orthonormal sequence in an inner product space $(X, \langle ., . \rangle)$, then*

$$\sum_{j=1}^{\infty} |\langle u, e_j \rangle|^2 \leq ||u||^2$$

for all $u \in X$ where it is a bounded and convergent series.

Remark 34.485 *Bessel's inequality is a statement of an element u in an inner product space X with respect to an orthonormal sequence.*

We now seek to turn Bessel's inequality to an equality by imposing more assumptions. We will now impose Hilbert spaces rather than inner product spaces.

We want to imitate the inner product structure in ℓ_2 for a Hilbert space.

Proposition 34.486 *Let $\{e_j\}_{j\geq 1}$ be an orthonormal sequence in a Hilbert space $(H, \langle ., . \rangle)$. If $a = \{a_j\}_{j\geq 1}$ and $b = \{b_j\}_{j\geq 1}$ belong to ℓ_2, then $u_a = \sum_{j=1}^{\infty} a_j e_j$ and $v_b = \sum_{j=1}^{\infty} b_j e_j$ are convergent series* ***in H*** *and, moreover,*

$$\langle u_a, v_b \rangle_H = \langle a, b \rangle_{\ell_2} = \sum_{j=1}^{\infty} a_j \bar{b}_j.$$

Proof:(Sketch). We construct the sequence of partial sums and show that it is a Cauchy sequence and by the completeness of H, then it converges where $U_N = \sum_{j=1}^{N} a_j e_j$ for all $N \geq 1$. ■

We want to ask the question when does the Fourier Bessel's series $\sum_{j=1}^{\infty} \langle u, e_j \rangle e_j$ converge? This occurs when we upgrade the space from an inner product space to a Hilbert space.

Corollary 34.487 *Let H be a Hilbert space and let $\{e_j\}_{j\geq 1}$ be an orthonormal sequence in H. Then, for every $u \in H$, the series $\sum_{j=1}^{\infty} \langle u, e_j \rangle e_j$ converges to U in H and $||U|| \leq ||u||$.*

We now want to ask the question that for a $u \in H$ where H is a Hilbert space, we know from above that the Fourier Bessel series converges to an $U \in H$ which we don't know anything about, when does $U = u$? In order to answer this, we require that the orthonormal sequence $\{e_j\}_{j\geq 1}$ be **maximal**.

Definition 34.488 *(Maximal/Complete). An orthonormal sequence $\{e_j\}_{j\geq 1}$ in an inner product space X is said to be maximal or complete if whenever u in X satisfies $u \perp e_j$ for each $j \geq 1$, then $u = 0$.*

Theorem 34.489 *Let $\{e_j\}_{j\geq 1}$ be an orthonormal sequence in a Hilbert space H. The following statements are equivalent.*

1. *The vector space span $\{e_j\}_{j\geq 1}$ of finite linear combinations of e_j's is dense in H.*

2. *The orthonormal sequence* $\{e_j\}_{j\geq 1}$ *is maximal.*
3. *For every* $u \in H$*, the Fourier-Bessel series* $\sum_{j=1}^{\infty}\langle u, e_j\rangle e_j$ ***converges to u in H***.
4. *(Parseval's identity). We have* $||u||^2 = \sum_{j=1}^{\infty}|\langle u, e_j\rangle|^2$ *for every* $u \in H$.

Remark 34.490 *From point 3, we finally managed to get our Fourier-Bessel series to converge to the vector* $u \in X$*!*

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35. Orthonormal Basis

35.12.6 Orthonormal Basis

We have seen that maximal orthonormal sequences are extremely useful to characterise the convergence of the Fourier Bessel series. We now try find the conditions such that we can guarantee the existence of such sequences.

Theorem 35.491 *Every separable inner product space contains a maximal orthonormal sequence.*

Remark 35.492 *We use the Gram-Schmidt procedure on the countable dense subset of the inner product space.*

Lemma 35.493 *An orthonormal sequence is maximal if and only if it is a countable orthonormal basis.*

Remark 35.494 *The term sequence imposes a condition that the cardinality of the set is countable.*

Definition 35.495 *(Orthogonal/orthonormal). Let H be a Hilbert space. A nonempty subset S of H is called*

1. ***Orthogonal set** if every distinct vectors x and y in S are orthogonal;*
2. ***Orthonormal set** if it is an orthogonal set and $\langle x, x\rangle = ||x||^2 = 1$ for every $x \in S$.*

Definition 35.496 *(Complete orthonormal basis). A complete orthonormal basis of H is an orthonormal set S such that the span S is dense in H.*

Proposition 35.497 *An orthonormal set $S \subset H$ is a complete orthonormal basis if and only if for every $u \in H$ that satisfies $u \perp S$, we have that $u = 0$.*

Corollary 35.498 *Every separable Hilbert space admits a **countable** orthonormal basis.*

Theorem 35.499 *Let H be a Hilbert space. The following assertions are equivalent:*

1. *The Hilbert space H is separable.*
2. *There exists a countable orthonormal basis for H.*
3. *Every orthonormal basis of H is countable.*

Remark 35.500 *To show that a Hilbert space is not separable, it suffices to find a orthonormal basis that is not countable.*

Theorem 35.501 *Every separable infinite-dimensional Hilbert space H can be identified with ℓ_2 via an isometric isomorphism preserving the inner products, meaning that there exists a bijective linear map $T : H \to \ell_2$ such that $\langle Tu, Tv\rangle_{\ell_2} = \langle u, v\rangle$ for every $u, v \in H$.*

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36. Linear Operators

36.13 Banach Spaces

36.13.1 Linear Operators

Definition 36.502 *(Banach space). A normed vector space which is complete with respect to the metric induced by the norm on the space.*

Definition 36.503 *(Linear Functional). Let X be a vector space. A linear functional T from X to $\mathbb{K}$ is a function $T : X \to \mathbb{K}$ such that*

1. $T(x_1, x_2) = T(x_1) + T(x_2)$ *for* $x_1, x_2 \in X$.
2. $T(\alpha x) = \alpha T(x)$ *for all* $x \in X$ *and* $\alpha \in \mathbb{K}$.

Definition 36.504 *(Linear Operator). Let X and Y be vector spaces over the **same field**. A linear operator T from X to Y is a function $T : X \to Y$ such that*

1. $T(x_1, x_2) = T(x_1) + T(x_2)$ *for* $x_1, x_2 \in X$.
2. $T(\alpha x) = \alpha T(x)$ *for all* $x \in X$.

Definition 36.505 *(Set of linear operators). We denote the **set of all linear operators** from X to Y by $\mathcal{L}(X, Y)$.*

Definition 36.506 *(Bounded Linear Operator). Let T be a linear operator from the normed space $(X, ||.||_X)$ to a normed space $(Y, ||.||_Y)$. T is said to be **bounded** if there exists an $M > 0$ such that*

$$||T(x)||_Y \leq M||x||_X$$

for every $x \in X$. Equivalently, T is bounded if there exists $M > 0$ such that

$$||T(x)||_Y \leq M$$

for all $x \in X$ with $||x|| \leq 1$.

Lemma 36.507 *A bounded linear operator maps bounded sets in X to bounded sets in Y.*

Note that a bounded linear operator is not the same as bounded functions.

Proposition 36.508 *Let $(X, ||.||_X)$ and $(Y, ||.||_Y)$ be normed vector spaces and let T be a linear operator from X to Y. Then the following statements are equivalent:*

1. *T is a bounded linear opeartor.*
2. *T is continuous on X.*
3. *T is continuous at $0 \in X$.*

Definition 36.509 *(Set of bounded linear operators). Let $B(X,Y)$ denote the space of bounded and linear operators from X to Y.*

Remark 36.510 *The space $B(X,Y)$ is a vector space that is closed under pointwise addition and scalar multiplication.*

Theorem 36.511 *Let $B(X,Y)$ be the space of bounded and linear operators from X to Y. Then, $B(X,Y)$ is a normed vector space with the norm $||T||= ||T||_{B(X,Y)}= \sup_{x\in X;||x||\leq 1}||T(x)||_Y$. We call $||T||_{B(X,Y)}||$ the operator norm of T.*

Remark 36.512 *Notice that we now have 3 norms involved in the construction of B(X,Y). The norm on X, Y, and the norm on the linear opeator itself. Hence, the norm of a linear operator itself is the smallest value M such that $||T(x)||_Y\leq M$ for all $x \in X$ where $||x||\leq 1$. Such a supremum exists due to the linear operator T being bounded.*

Lemma 36.513 *For every $T \in B(X,Y)$, we have that*

$$||T(x)||_Y\leq ||T||||x||_X$$

for all $x \in X$.

Theorem 36.514 *If X is a normed vector space and Y is a Banach space, then the space $B(X,Y)$ of bounded linear operators from X to Y is a Banach space where*

$$||T||_{B(X,Y)}= \sup_{x\in X;||x||\leq 1}||T(x)||_Y.$$

Remark 36.515 *We already knew that B(X,Y) is a normed vector space. By imposing the restriction that Y is now a Banach space, the completeness property allows us to upgrade B(X,Y) to be a Banach space.*

Definition 36.516 *(Group). A group is a set G, with a binary operation . such that the pair (G,.) satisfies the group axioms of:*

1. *Closure.*
2. *Associativity.*
3. *Identity element exists.*
4. *Inverse element exists.*

Definition 36.517 *(Homomorphism). Let $(G,.)$ and $(H,*)$ be two groups with different group operations. A homomorphism is a function $\phi : G \to H$ such that*

$$\phi(g.h) = \phi(g) * \phi(h)$$

for all $g \in G$ and $h \in H$.

Definition 36.518 *(Isomorphism). An isomorphism is a homomorphism that is bijective. Two groups are said to be isomorphic if there exists an isomorphism between them.*

Definition 36.519 *(Topological isomorphism). Let* $(X.||.||_X)$ *and* $(Y,||.||_Y)$ *be normed linear spaces and let* T *be a linear operator from* X *to* Y*.* T *is a* ***topological isomorphism*** *if* T *is an isomorphism and* T *is continuous with a continuous inverse.*

If the normed linear spaces X and Y are topologically isomorphic, then they are identical for topological purposes (open sets, convergence, continuity etc).

Definition 36.520 *(Isometric isomorphism). Let* $(X.||.||_X)$ *and* $(Y,||.||_Y)$ *be normed linear spaces and let* T *be a linear operator from* X *to* Y*.* T *is a* ***topological isomorphism*** *if* T *is an isomorphism and* T *is an isometry, i.e.* $||Tx||_Y = ||x||_X$ *for all* $x \in X$*.*

If the normed linear spaces X and Y are isometrically isomorphic, then they are identical for metric purposes (distances, balls etc).

Theorem 36.521 *Let* X *be a normed vector space and* Y *be a Banach space. Let* $\{T_n\}_{n\geq 1}$ *be a sequence in B(X,Y) that is bounded in B(X,Y)*

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36. Principle of Uniform Boundedness

36.13.2 Principle of Uniform Boundedness

Theorem 36.522 *(Principle of uniform boundedness). Let X and Y be Banach spaces. Let $\{T_j\}_{j\in J}$ be a family of bounded linear operators from X to Y, that is, $T_j \in B(X,Y)$ for each $j \in J$. Assume that $\sup_{j\in J}||T_j(x)||< \infty$ for every $x \in X$. Then we have that*

$$\sup_{j\in J}||T_j||_{B(X,Y)}< \infty.$$

Proof:(Sketch). As X is a complete metric space, we can express it as a countable union of closed sets. From the Baire Category theorem, that every complete metric space **cannot** be written as a countable union of nowhere dense sets, then there exists a closed set E_n such that $int(\overline{E}_n) = int(E_n) \neq \emptyset$. We are then able to analyse an open ball in this set and pick a point in the ball that is not the center of the ball. We put a bound on the point by the definition of the set we constructed which the balls sits in. We then use the assumption of pointwise boundedness in addition to the reverse triangle inequality and linearity of the linear operator to derive a constant that bounds the family of linear operators for each point in the open unit ball. ■

The principle of uniform boundedness states that if we have a family of pointwise bounded linear operators, i.e. if we fix a point $x \in X$, we can bound the family of linear operators, we can in fact extend this to uniform boundedness. Pointwise boundedness can be expressed as for every $x \in X$, we have that $\sup_{j\in J}||T_j(x)||_Y< \infty$ or equivalently there exists $C_x > 0$ such that

$$||T(x)||_Y< C_x$$

for all $j \in J$.
When we refer to uniform boundedness, we refer to the fact that $\sup_{j\in J}||T_j||_{B(X,Y)}< \infty$ which is equivalent to there exists $M > 0$ such that

$$||T_j||_{B(X,Y)}\leq M$$

for all $j \in J$, which is equivalent to

$$\sup_{x\in X;||x||\leq 1}||T_j(x)||_Y.$$

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37. Compactness of sets in separable Hilbert spaces

37.13.3 Compactness of sets in separable Hilbert spaces

We know that a closed and bounded set is not compact for infinite dimensional metric spaces. An example would be the closed unit ball in a Banach space is not compact. We investigate the characterisation for compact sets in infinite dimensional spaces.

Lemma 37.523 *(Equi-smallness). Let H be a Hilbert space. If $\{e_k\}_{k\geq 1}$ is an orthonormal sequence in H and $\{u_n\}_{n\geq 1}$ converges to U in H, then for every $\epsilon > 0$, there exists $N \geq 1$ such that*

$$\sum_{k\geq N} |\langle u_n, e_n\rangle|^2 < \epsilon^2$$

for every $n \geq 1$.

We can now characterise compact sets in infinite dimensional Hilbert spaces. In addition to the set being closed and bounded, we require that the set satisfies equi-smallness property. Recall that any separable Hilbert space admits a countable orthonormal basis.

Theorem 37.524 *Let H be a separable infinite-dimensional Hilbert space. A set K in H is compact if and only if K is closed, bounded, and for any orthonormal basis $\{e_k\}_{k\geq 1}$ of H and every $\epsilon > 0$, there exists $N \geq 1$ such that*

$$\sum_{k>N} |\langle u, e_k\rangle|^2 < \epsilon^2$$

for all $u \in K$.