Lecture Notes for

Algebra II

Linear Algebraic Groups

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I. Affine Varities

1. Affine Sets

Conventions 1.1. Throughout these notes k denotes an algebraically closed field.

1.2. In this Section we recall the notion of an affine set and some of the basic theorems about them. The missing proofs can, for example, be found in the [AlgI18b].

1.1. Definition

Definition 1.3. The *affine n-space* over k is $\mathbb{A}^n \coloneqq \mathbb{A}^n_k \coloneqq k^n$.

1.4. It follows from k being infinite that $f = g$ for all $f, g \in k[x_1, \dots, x_n]$ with $f(x) = g(x)$ for every $x \in \mathbb{A}^n$. We will therefore regard the polynomial ring $k[x_1, \dots, x_n]$ as the ring of polynomial functions on $\mathbb{A}^n$.

Definition 1.5. For every subset $S \subseteq k[x_1, \dots, x_n]$ the set

$$\mathrm{V}(S) \coloneqq \{x \in \mathbb{A}^n \mid f(x) = 0 \text{ for every } f \in S\}$$

is the *affine set* given by S.

Lemma 1.6.

1) It holds for every subset $S \subseteq k[x_1, \dots, x_n]$ that $\mathrm{V}(S) = \mathrm{V}((S))$.

2) It holds for all ideals $I_1 \trianglelefteq I_2 \trianglelefteq k[x_1, \dots, x_n]$ that $\mathrm{V}(I_1) \supseteq \mathrm{V}(I_2)$.

3) It holds for all ideals $I_1, I_2 \trianglelefteq k[x_1, \dots, x_n]$ that

$$\mathrm{V}(I_1) \cup \mathrm{V}(I_2) = \mathrm{V}(I_1 \cap I_2) = \mathrm{V}(I_1 \cdot I_2)\,.$$

4) It holds for every family $(I_\lambda)_{\lambda \in \Lambda}$ of ideals $I_\lambda \trianglelefteq k[x_1, \dots, x_n]$ that

$$\bigcap_{\lambda \in \Lambda} \mathrm{V}(I_\lambda) = \mathrm{V}\left(\bigcup_{\lambda \in \Lambda} I_\lambda\right) = \mathrm{V}\left(\sum_{\lambda \in \Lambda} I_\lambda\right)\,.$$

5) It holds that $\mathrm{V}(1) = \emptyset$.

6) It holds that $\mathrm{V}(0) = \mathbb{A}^n$. $\square$

Corollary 1.7. There exists a unique topology on $\mathbb{A}^n$ whose closed subsets are the subsets of the form $\mathrm{V}(S)$ for subsets $S \subseteq k[x_1, \dots, x_n]$. $\square$

Definition 1.8. The topology from Corollary 1.7 is the *Zariski topology* on $\mathbb{A}^n$. The induced subspace topology on a subset $X \subseteq \mathbb{A}^n$ is the *Zariski topology* on X.

Theorem 1.9 (Hilbert's basis theroem)**.** Every ideal $I \trianglelefteq k[x_1, \dots, x_n]$ is finitely generated. □

Corollary 1.10. It holds for every subset $S \subseteq k[x_1, \dots, x_n]$ that

$$\mathrm{V}(S) = \mathrm{V}(f_1, \dots, f_m) = \mathrm{V}(f_1) \cap \dots \cap \mathrm{V}(f_m).$$

for some finitely many $f_1, \dots, f_m \in S$. □

1.2. Coordinate Rings

Definition 1.11. For every subset $X \subseteq \mathbb{A}^n$ the set

$$\mathrm{I}(X) \coloneqq \{f \in k[x_1, \dots, x_n] \mid f(x) = 0 \text{ for every } x \in X\}$$

is the *vanishing ideal* of X.

Lemma 1.12.

1) For every subset $X \subseteq \mathbb{A}^n$ the vanishing ideal $\mathrm{I}(X)$ is an ideal in $k[x_1, \dots, x_n]$.
2) It holds for all subsets $X_1 \subseteq X_2 \subseteq \mathbb{A}^n$ that $\mathrm{I}(X_1) \trianglerighteq \mathrm{I}(X_2)$.
3) It holds for every family $(X_\lambda)_{\lambda \in \Lambda}$ of subsets $X_\lambda \subseteq \mathbb{A}^n$ that

$$\mathrm{I}\left(\bigcup_{\lambda \in \Lambda} X_\lambda\right) = \bigcap_{\lambda \in \Lambda} \mathrm{I}(X_\lambda).$$

4) It holds that $\mathrm{I}(\varnothing) = (1) = k[x_1, \dots, x_n]$.
5) It holds that $\mathrm{I}(\mathbb{A}^n) = 0$ (because k is infinite). □

Definition 1.13. The *coordinate ring* of an affine set $X \subseteq \mathbb{A}^n$ is the k-algebra

$$\mathrm{A}(X) \coloneqq k[x_1, \dots, x_n]/\mathrm{I}(X).$$

1.14. For an affine set $X \subseteq \mathbb{A}^n$ any two polyomial functions $f, g \in k[x_1, \dots, x_n]$ coincide on X in the sense that $f(x) = g(x)$ for all $x \in X$ if and only if $f - g \in \mathrm{I}(X)$, i.e. if and only if f and g are identified in $\mathrm{A}(X)$. We will therefore regard the coordinate ring $\mathrm{A}(X)$ as the ring of polynomial functions on X.

We can then define for every ideal $I \trianglelefteq \mathrm{A}(X)$ the *vanishing set*

$$\mathrm{V}(I) \coloneqq \mathrm{V}_X(I) \coloneqq \{x \in X \mid f(x) = 0 \text{ for every } f \in I\}$$

and can define for every subset $Y \subseteq X$ the *vanishing ideal*

$$\mathrm{I}(Y) \coloneqq \mathrm{I}_X(Y) \coloneqq \{f \in \mathrm{A}(X) \mid f(y) = 0 \text{ for every } y \in Y\}.$$

For $X = \mathbb{A}^n$ this agrees with the previous definitions of vanishing sets and vanishing ideals. The properties from Lemma 1.6 and Lemma 1.12 also hold for this more general definitions.

Lemma 1.15. The Zariski closed subsets of an affine set X are precisely the subsets of the form $\mathrm{V}(I)$ with $I \trianglelefteq \mathrm{A}(X)$.

Corollary 1.16. Let X be an affine set. For a subset $X' \subseteq X$ the following conditions are equivalent:

1) The set is Zariski dense in X.

2) It follows for every $f \in \mathrm{A}(X)$ from $f|_{X'} = 0$ that $f = 0$.

3) It follows for all $f, g \in \mathrm{A}(X)$ from $f|_{X'} = g|_{X'}$ that $f = g$.

Proof. It holds that

$$\begin{aligned} & X' \text{ is Zariski dense in } X \\ \iff & \text{if } C \subseteq X \text{ is a Zariski closed with } X' \subseteq C \text{ then } C = X \\ \iff & \text{if } f \in \mathrm{A}(X) \text{ with } X' \subseteq \mathrm{V}(f) \text{ then } \mathrm{V}(f) = X \\ \iff & \text{if } f \in \mathrm{A}(X) \text{ with } f|_{X'} = 0 \text{ then } f = 0 \\ \iff & \text{if } f, g \in \mathrm{A}(X) \text{ with } (f-g)|_{X'} = 0 \text{ then } f - g = 0 \\ \iff & \text{if } f, g \in \mathrm{A}(X) \text{ with } f|_{X'} = g|_{X'} \text{ then } f = g \end{aligned}$$

as claimed. □

Corollary 1.17. If X is an affine set then the sets

$$\mathrm{D}(f) \coloneqq \mathrm{D}_X(f) \coloneqq \{x \in X \mid f(x) \neq 0\}$$

with $f \in \mathrm{A}(X)$ form a basis for the Zariski topology of X.

Proof. The sets $\mathrm{D}(f)$ are Zariski open because $\mathrm{D}(f) = X \smallsetminus \mathrm{V}(f)$. If $U \subseteq X$ is any Zariski open subset then the complement $X \smallsetminus U$ is Zariski closed and thus of the form $X \smallsetminus U = \mathrm{V}(I)$ for some ideal $I \trianglelefteq \mathrm{A}(X)$. If $f_\lambda \in I$, $\lambda \in \Lambda$ is a generating set of I then it follows from

$$\mathrm{V}(I) = \mathrm{V}(f_\lambda \mid \lambda \in \Lambda) = \bigcap_{\lambda \in \Lambda} \mathrm{V}(f_\lambda)$$

that

$$U = X \smallsetminus \mathrm{V}(I) = X \smallsetminus \bigcap_{\lambda \in \Lambda} \mathrm{V}(f_\lambda) = \bigcup_{\lambda \in \Lambda} \big(X \smallsetminus \mathrm{V}(f_\lambda)\big) = \bigcup_{\lambda \in \Lambda} \mathrm{D}(f_\lambda)$$

as desired. □

Definition 1.18. For an affine set X the open subsets $\mathrm{D}(f) \subseteq X$ with $f \in \mathrm{A}(X)$ are the *standard open subsets* of X.

Notation 1.19. If more generally X' is any set, $f \colon X' \longrightarrow k$ is a function and $X \subseteq X'$ is a subset then we will use the notation

$$\mathrm{D}_X(f) \coloneqq \{x \in X \mid f(x) \neq 0\}.$$

Definition 1.20. Let R be a commutative ring.

1) The ring R is *reduced* if $0 \in R$ is the only nilpotent element of R.

2) The *radical* of an ideal $I \trianglelefteq R$ is

$$\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some } n \geq 0\}.$$

3) An ideal $I \trianglelefteq R$ is *radical* if $I = \sqrt{I}$.

Lemma 1.21. Let R be a commutative ring.

1) For every ideal $I \trianglelefteq R$ its radical $\sqrt{I}$ is again an ideal in R.

2) An ideal $I \trianglelefteq R$ is radical if and only if the quotient R/I is reduced. □

Lemma 1.22. The ideal $\mathrm{I}(X) \trianglelefteq k[x_1, \dots, x_n]$ is radical for every subset $X \subseteq \mathbb{A}^n$. □

Corollary 1.23. For every affine set X its coordinate ring $\mathrm{A}(X)$ is a finitely generated, commutative, reduced k–algebra. □

Theorem 1.24 (Hilbert's Nullstellensatz, version 1)**.** If X is an affine set then holds for every ideal $I \trianglelefteq \mathrm{A}(X)$ that $\mathrm{I}(\mathrm{V}(I)) = \sqrt{I}$. □

Theorem 1.25 (Hilbert's Nullstellensatz, version 2)**.** If X is an affine set and $I \triangleleft \mathrm{A}(X)$ is a proper ideal then its vanishing set $\mathrm{V}(I)$ is nonempty. □

1.3. Irreducibility

Definition 1.26. Let X be a topological space.

1) The space X is *reducible* if $X = C_1 \cup C_2$ for some proper closed subsets $C_1, C_2 \subsetneq X$.

2) The space X is *irreducible* if it is nonempty and not reducible.

Lemma 1.27. For a nonempty topological space X the following conditions are equivalent:

1) The space X is irreducible.

2) Every two nonempty open subsets of X intersect nontrivially.

3) Every nonemtpy open subset of X is dense. □

Lemma 1.28. Every irreducible space is connected. □

Example 1.29. The affine set $\mathrm{V}(xy) \subseteq \mathbb{A}^2$ is connected but reducible, which shows that the converse to Lemma 1.28 does not hold.

Proposition 1.30. For every topological space X there exist a unique collection $(C_\lambda)_{\lambda \in \Lambda}$ of closed irreducible subsets $C_\lambda \subseteq X$ with $X = \bigcup_{\lambda \in \Lambda} C_\lambda$ and $C_\lambda \not\subseteq C_\mu$ for $\lambda \neq \mu$. □

Definition 1.31. The sets C_λ, $\lambda \in \Lambda$ from Proposition 1.30 are the *irreducible components* of X.

Definition 1.32. A topological space X is *noetherian* if every descending sequence

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$$

of closed subsets $C_i \subseteq X$ stabilizes, or equivalently if every ascending sequence

$$U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$$

of open subsets $U_i \subseteq X$ stabilizes.

Lemma 1.33. Subspaces of noetherian topological spaces are again noetherian. ☐

Lemma 1.34. Any affine set is noetherian. ☐

Lemma 1.35. A noetherian topological space has only finitely many irreducible components. ☐

Corollary 1.36. An affine set has only finitely many irreducible components. ☐

Theorem 1.37 (Hilbert's Nullstellensatz, version 3)**.** For every affine set X, the maps $\mathrm{V}_X, \mathrm{I}_X$ restrict to the following bijections:

$$\begin{array}{ccc}
\left\{\begin{matrix}\text{affine algebraic}\\ \text{sets } Y \subseteq X\end{matrix}\right\} & \underset{\mathrm{V}_X}{\overset{\mathrm{I}_X}{\rightleftarrows}} & \left\{\begin{matrix}\text{radical ideals}\\ I \trianglelefteq \mathrm{A}(X)\end{matrix}\right\} \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\left\{\begin{matrix}\text{irreducible affine}\\ \text{algebraic set}\\ Y \subseteq X\end{matrix}\right\} & \underset{\mathrm{V}_X}{\overset{\mathrm{I}_X}{\rightleftarrows}} & \left\{\begin{matrix}\text{prime ideals}\\ \mathfrak{p} \trianglelefteq \mathrm{A}(X)\end{matrix}\right\} \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\{\text{points } p \in X\} & \underset{\mathrm{V}_X}{\overset{\mathrm{I}_X}{\rightleftarrows}} & \left\{\begin{matrix}\text{maximal ideals}\\ \mathfrak{m} \trianglelefteq \mathrm{A}(X)\end{matrix}\right\}
\end{array}$$

For every point $p = (p_1, \dots, p_n) \in X$ the corresponding maximal ideal $\mathfrak{m}_p \trianglelefteq \mathrm{A}(X)$ is given by $\mathfrak{m}_p = (\overline{x_1} - p_1, \dots, \overline{x_n} - p_n)$. ☐

Corollary 1.38. If X is an affine set then it holds for all $f, g \in \mathrm{A}(X)$ that $\mathrm{D}(f) \subseteq \mathrm{D}(g)$ if and only if $f \in \sqrt{(g)}$.

Proof. It holds that

$$\mathrm{D}(f) \subseteq \mathrm{D}(g) \iff \mathrm{V}(f) \supseteq \mathrm{V}(g) \iff \mathrm{I}(\mathrm{V}(f)) \subseteq \mathrm{I}(\mathrm{V}(g)) \iff \sqrt{(f)} \subseteq \sqrt{(g)} \iff f \in \sqrt{(g)},$$

as desired. ☐

1.4. Morphisms of Affine Sets

Definition 1.39. Let X, Y be affine sets with $Y \subseteq \mathbb{A}^m$.

1) A function $f\colon X \longrightarrow k = \mathbb{A}^1$ is *regular* if it is an element of $\mathrm{A}(X)$.

2) A map $f\colon X \longrightarrow \mathbb{A}^n$ is *regular* if it is regular in each coordinate.

3) A map $f\colon X \longrightarrow Y$ is *regular* if it is the restriction of a regular map $X \longrightarrow \mathbb{A}^m$.

A map $X \longrightarrow Y$ is a *morphism* of affine sets if it is regular. The set of morphisms $X \longrightarrow Y$ is denoted by $\mathrm{Mor}(X, Y)$.

Lemma 1.40. Let X, Y, Z be affine sets.

1) The identity map $\mathrm{id}_X\colon X \longrightarrow X$ is a morphism.

2) For every two morphisms $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ their composition $g \circ f\colon X \longrightarrow Z$ is again morphism. □

Lemma 1.41. Let X, Y be affine sets and let $f\colon X \longrightarrow Y$ be a morphism of affine sets.

1) It holds for every $\varphi \in \mathrm{A}(Y)$ that $f^{-1}(\mathrm{D}_Y(\varphi)) = \mathrm{D}_X(\varphi \circ f)$.

2) The map f is continuous with respect to the Zariski topologies on X, Y. □

Lemma 1.42. Let X, Y, Z be affine sets.

1) If $f\colon X \longrightarrow Y$ is a morphism of affine sets then the map

$$f^*\colon \mathrm{A}(Y) \longrightarrow \mathrm{A}(X), \quad \varphi \longmapsto \varphi \circ f$$

is a well-defined homomorphism of k-algebras.

2) It holds that $\mathrm{id}_X^* = \mathrm{id}_{\mathrm{A}(X)}$.

3) It holds for any two composable morphisms of affine sets $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$ that $(g \circ f)^* = f^* \circ g^*$. □

1.43 (Finite sets as affine sets)**.** If X is a finite affine set then every function $X \longrightarrow k$ is regular. It follows that for every affine set Y every map $X \longrightarrow Y$ is regular. If X' is another finite affine set with $|X| = |X'|$ then it follows that every bijection $X \longrightarrow X'$ is already an isomorphism of affine sets.

This shows that any two affine sets of the same cardinality are isomorphic as affine sets. This allows us to regard every finite set as an affine set by identifying it with an (up to isomorphicsm unique) affine set of suitable cardinality.

Lemma 1.44. Let X, Y be affine sets and let $f\colon X \longrightarrow Y$ be a morphism of affine sets with image $X' \coloneqq \mathrm{im}(f)$. The induced morphism f^* is injective if and only if the image X' is dense in Y.

Proof. It holds that

$$
\begin{aligned}
& f^* \text{ in injective} \\
\iff & \text{it follows for all } \varphi \in \mathrm{A}(G) \text{ from } f^*(\varphi) = 0 \text{ that } \varphi = 0 \\
\iff & \text{it follows for all } \varphi \in \mathrm{A}(G) \text{ from } \varphi \circ f = 0 \text{ that } \varphi = 0 \\
\iff & \text{it follows for all } \varphi \in \mathrm{A}(G) \text{ from } \varphi|_{X'} = 0 \text{ that } \varphi = 0 \\
\iff & X' \text{ is dense in } Y
\end{aligned}
$$

by Corollary 1.16. □

Proposition 1.45. For any two affine sets X, Y the map

$$
\mathrm{Mor}(X, Y) \longrightarrow \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(\mathrm{A}(Y), \mathrm{A}(X)), \quad f \longmapsto f^*
$$

is a well-defined bijection.

Proof. We prove the claim by constructing an inverse to $(-)^*$.

With $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ the coordinate rings $\mathrm{A}(X)$ and $\mathrm{A}(Y)$ are given by

$$
\mathrm{A}(X) = k[x_1, \dots, x_n]/\mathrm{I}(X) \quad \text{and} \quad \mathrm{A}(Y) = k[y_1, \dots, y_m]/\mathrm{I}(Y).
$$

For any homomorphism of k-algebras $F\colon \mathrm{A}(Y) \longrightarrow \mathrm{A}(X)$ we associate a morphism of affine sets $\tilde{F}^\circ : X \longrightarrow \mathbb{A}^m$ with coordinates $\tilde{F}^\circ = (\tilde{F}^\circ_1, \dots, \tilde{F}^\circ_m)$ given by

$$
\tilde{F}^\circ_j = F(\overline{y_j}) \in \mathrm{A}(X)
$$

for every $j = 1, \dots, m$.

The morphism $\tilde{F}^\circ : X \longrightarrow \mathbb{A}^m$ restrict to a morphism $F^\circ : X \longrightarrow Y$: The affine set Y is given by $Y = \mathrm{V}(\mathrm{I}(Y))$ so needs to be shown that $p(\tilde{F}^\circ(x)) = 0$ for all $p \in \mathrm{I}(Y)$, $x \in X$. For this we calculate

$$
\begin{aligned}
p(\tilde{F}^\circ(x)) &= p(\tilde{F}^\circ_1(x), \dots, \tilde{F}^\circ_m(x)) && (1.1) \\
&= p(F(\overline{y_1})(x), \dots, F(\overline{y_m})(x)) && (1.2) \\
&= p(F(\overline{y_1}), \dots, F(\overline{y_m}))(x) && (1.3) \\
&= F(p(\overline{y_1}, \dots, \overline{y_m}))(x) && (1.4) \\
&= F\left(\overline{p(y_1, \dots, y_m)}\right)(x) && (1.5) \\
&= F(\overline{p})(x) \\
&= F(0)(x) && (1.6) \\
&= 0.
\end{aligned}
$$

Equation (1.1) uses the definition of $\tilde{F}^\circ$, Equation (1.2) uses the definiton of the components $\tilde{F}^\circ_j$, Equation (1.3) uses that the k-algebra structure on $\mathrm{A}(X)$ is given pointwise, Equation (1.4) uses that F is a k-algebra homomorphism, Equation (1.5) uses that $\overline{(-)}$ is a k-algebra homomorphism, and Equation (1.6) uses that $p \in \mathrm{I}(Y)$.

The constructions $(-)^*$ and $(-)^\circ$ are mutually inverse: If $f\colon X \longrightarrow Y$ is a morphism of affine sets with coordinates $f = (f_1, \dots, f_m)$ then

$$(f^*)^\circ_j = (f^*)(\overline{y_j}) = \overline{y_j} \circ f = f_j$$

for every $j = 1, \dots, m$ and therefore $(f^*)^\circ = f$. If $F\colon \mathrm{A}(Y) \longrightarrow \mathrm{A}(X)$ is a homomorphism of k-algebras then

$$(F^\circ)^*(\overline{y_j}) = \overline{y_j} \circ F^\circ = F^\circ_j = F(\overline{y_j})$$

for every $j = 1, \dots, m$ and therefore $(F^\circ)^* = F$. □

Lemma 1.46. For every finitely generated, commutative, reduced k-algebra A there exists an affine set X with $A \cong \mathrm{A}(X)$ as k-algebras.

Proof. Let $a_1, \dots, a_n \in A$ be generating set of A as a k-algebra. Then there exists a unique homomorphisms of k-algebras $f\colon k[x_1, \dots, x_n] \longrightarrow A$ with $f(x_i) = a_i$ for every $i = 1, \dots, n$, and f is surjective. It follows that f induces an isomorphism of k-algebras $k[x_1, \dots, x_n]/I \longrightarrow A$ for $I \coloneqq \ker(f)$.

The ideal I is a radical ideal because the quotient $k[x_1, \dots, x_n]/I \cong A$ is reduced. It follows from the second version of Hilbert's Nullstellensatz that $\mathrm{I}(X) = I$ for the affine set $X \coloneqq \mathrm{V}(I)$. It follows that

$$A \cong k[x_1, \dots, x_n]/I = k[x_1, \dots, x_n]/\mathrm{I}(X) = \mathrm{A}(X)$$

as desired. □

Corollary 1.47. The coordinate ring $\mathrm{A}(-)$ gives rise to a contravariant equivalence

$$\begin{aligned} \{\text{affine sets}\} &\longrightarrow \left\{ \begin{matrix} \text{finitely generated,} \\ \text{commutative,} \\ \text{reduced } k\text{-algebras} \end{matrix} \right\}, \\ X &\longmapsto \mathrm{A}(X), \\ f &\longmapsto f^*. \end{aligned}$$

Proof. It follows from Corollary 1.23 and Lemma 1.42 that $\mathrm{A}(-)$ defines a functor as claimed. It follows from Proposition 1.45 that $\mathrm{A}(-)$ is fully faithful and it follows from Lemma 1.46 that $\mathrm{A}(-)$ is dense. □

Corollary 1.48. Let X, Y be affine sets

1) The affine sets X, Y are isomorphic if and only if their coordinate rings $\mathrm{A}(X)$, $\mathrm{A}(Y)$ are.

Let $f\colon X \longrightarrow Y$ be a morphism of affine sets with image $X' \coloneqq \mathrm{im}(f)$.

2) The morphism f is an isomorphism if and only if the induced algebra homomorphism f^* is an isomorphism.

3) The induced map f^* is surjective if and only if X' is closed in Y and f is an isomorphism onto X'.[1]

[1] The author linkes to think about f as a closed embedding, but is not sure if this is how algebraic geometers use this term.

Proof. Parts 1) and 2) follows from Corollary 1.47.

To show part 3) first suppose that X' is closed in $Y \subseteq \mathbb{A}^m$ and that f restricts to an isomorphism $X \longrightarrow X'$. To show that f^* surjective we may assume that $X = X' \subseteq Y$ and that f is the inclusion $f\colon X \hookrightarrow Y$. The induced algebra homomorphism $f\colon \mathrm{A}(Y) \longrightarrow \mathrm{A}(X)$ maps every $\varphi \in \mathrm{A}(Y)$ to $\varphi \circ f$, which is the restriction of φ to X. The homomorphism f^* is therefore surjective because every regular function on X is the restriction of a regular function on $\mathbb{A}^m$, and therefore also of a regular function on Y.

Suppose now that f^* is surjective. The ideal $I \coloneqq \ker(f^*)$ is radical because the quotient $\mathrm{A}(Y)/I \cong \mathrm{A}(X)$ is reduced. It follows from Hilbert's Nullstellensatz that $X'' \coloneqq \mathrm{V}(I)$ is an affine set $X'' \subseteq Y$ with $\mathrm{A}(X'') = \mathrm{A}(Y)/I \cong \mathrm{A}(X)$. We show that $X' = X''$ and that f restricts to an isomorphism $X \longrightarrow X'$:

That $0 = f^*(\varphi) = \varphi \circ f$ for every $\varphi \in I$ means that

$$X' = \mathrm{im}(f) \subseteq \mathrm{V}(I) = X''$$

It follows that f restricts to a morphism of affine varieties $\tilde{f}\colon X \longrightarrow X''$ which fits in the following commutative diagram:

$$\begin{tikzcd} & Y \\ X \arrow[ur, "f"] \arrow[r, "\tilde{f}"'] & X'' \arrow[u, hook] \end{tikzcd}$$

This induces on coordinate rings the following commutative diagram:

$$\begin{tikzcd} & \mathrm{A}(Y) \arrow[dl, "f^*"'] \arrow[d] \\ \mathrm{A}(X) & \mathrm{A}(X'') \arrow[l, "\tilde{f}^*"] \end{tikzcd}$$

The homomorphism $\mathrm{A}(Y) \longrightarrow \mathrm{A}(X'')$ is the restriction homomorphism $\varphi \longmapsto \varphi|_{X''}$, which is by the above observations surjective with kernel $\mathrm{I}(X'') = \mathrm{I}(\mathrm{V}(I)) = I$.

Both f^* and the restriction homomorphism $\mathrm{A}(Y) \longrightarrow \mathrm{A}(X'')$ are surjective algebra homomorphisms with the same kernel so it follows that there exists a unique algebra homomorphism $\mathrm{A}(X'') \longrightarrow \mathrm{A}(X)$ which makes the above diagram commute, and that it is an isomorphism. This shows that $\tilde{f}^*$ is an isomorphism. It follows from part 2) that $\tilde{f}$ is an isomorphism from X to X''. The inclusion

$$\mathrm{im}(f) = X' \subseteq X'' = \mathrm{im}(\tilde{f}) = \mathrm{im}(f)$$

is therefore already an equality $X' = X''$, and f restricts to the isomorphism $\tilde{f}$. □

1.5. Products of Affine Sets

Lemma 1.49. For any two affine sets $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ the set

$$X \times Y \subseteq \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$$

is again an affine set.

Proof. We may label the coordinates of $\mathbb{A}^n$ by $x_1, \dots, x_n$ while labeling the coordinates of $\mathbb{A}^m$ by $x_{n+1}, \dots, x_{n+m}$. The affine set X is then cut out by an ideal $I \trianglelefteq k[x_1, \dots, x_n]$ while Y is cut out by an ideal $J \trianglelefteq k[x_{n+1}, \dots, x_{n+m}]$. The set $X \times Y$ is then cut out by the generated ideal $(I, J) \trianglelefteq k[x_1, \dots, x_{n+m}]$. □

Definition 1.50. For any two affine sets $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ the affine set $X \times Y \subseteq \mathbb{A}^{n+m}$ is the *product* of X and Y.

Example 1.51. It holds that $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ as affine sets.

Warning 1.52. The Zariski topology on $X \times Y$ is finer than the product topology (i.e. it has more open sets) and in general strictly so.

To show that the Zariski topology on $X \times Y$ is finer than the product topology it sufficies to consider the case $X = \mathbb{A}^n$, $Y = \mathbb{A}^m$ because both the Zariski topology and product topology on $X \times Y$ are inherited from $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$. It further sufficies to show that the sets $U \times V$ for open subsets $U \subseteq \mathbb{A}^n$, $V \subseteq \mathbb{A}^m$ are open in the Zariski topology because these form a basis of the product topology. This holds because

$$\mathbb{A}^{n+m} \setminus (U \times V) = ((\mathbb{A}^n \setminus U) \times \mathbb{A}^m) \cup (\mathbb{A}^n \times (\mathbb{A}^m \setminus V))$$

is Zariski closed by Lemma 1.49.

To show that the Zariski toplogy on $X \times Y$ is in general strictly finer than the product topology we consider the case $X = Y = \mathbb{A}^1$. The diagonal

$$\Delta = \{(x, x) \mid x \in \mathbb{A}^1\} = V(x_1 - x_2) \subseteq \mathbb{A}^2$$

is then Zariski closed. But Δ cannot be closed in the product topology because $\mathbb{A}^1$ is not Hausdorff[2], as it is an infinite set endowed with the cofinite topology.

Proposition 1.53. Let X, X_1, X_2, Y_1, Y_2 be affine sets.

1) The projections $\pi_i \colon X_1 \times X_2 \longrightarrow X_i$ are morphisms of affine sets.

2) A map $f \colon X \longrightarrow Y_1 \times Y_2$ given by $f = (f_1, f_2)$ with $f_i \colon X \longrightarrow Y_i$ is a morphism of affine sets if and only if both f_1, f_2 are morphisms of affine sets.

This shows that the product of two affine sets is their categorical product in the category of affine sets.

3) If $f \colon X \longrightarrow X'$ and $g \colon Y \longrightarrow Y'$ are two morphisms of affine sets then the induced map $f \times g \colon X \times Y \longrightarrow X' \times Y'$ is again a morphism of affine sets. □

Proposition 1.54. For any two affine sets X, Y the map

$$\mathrm{A}(X) \otimes_k \mathrm{A}(Y) \longrightarrow \mathrm{A}(X \times Y), \quad f \otimes g \longmapsto \big[(x, y) \longmapsto f(x)g(y)\big]$$

is a well-defined natural isomorphism of k-algebras. □

[2]Here we use the well-known fact from point set topology that a toplogical space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is a closed subset of $X \times X$.

2. Quasi-Affine Sets

2.1. Definition

Definition 2.1. If X is an affine set and $X' \subseteq X$ is a Zariski open subset then X' is a *quasi-affine set*. The *Zariski topology* on X' is the subspace topology induced by the Zariski topology on X.

Remark 2.2. A subset $X \subseteq \mathbb{A}^n$ is a quasi-affine set if and only if it is of the form $X = C \cap U$ for some Zariski closed subset $C \subseteq \mathbb{A}^n$ and Zariski open subset $U \subseteq \mathbb{A}^n$.

Corollary 2.3. Let $X, X_1, \dots, X_n \subseteq \mathbb{A}^n$ be quasi-affine sets.

1) Every Zariski open subsets of X is again a quasi-affine set.

2) Every Zariski closed subsets of X is again a quasi-affine set.

3) The finite intersection $X_1 \cap \cdots \cap X_n$ is again a quasi-affine set.

Proof. Let $C, C_1, \dots, C_n \subseteq \mathbb{A}^n$ be Zariski closed and $U, U_1, \dots, U_n \subseteq \mathbb{A}^n$ Zariski open such that $X = C \cap U$ and $X_i = C_i \cap U_i$ for every i.

1) If $V \subseteq X$ is Zariski open then there exists a Zariski open subset $V' \subseteq \mathbb{A}^n$ with $V = V' \cap X$. It then follows that
$$V = V' \cap X = V' \cap C \cap U = C \cap (V' \cap U)$$
with $V' \cap U \subseteq \mathbb{A}^n$ being Zariski open.

2) This can be shown in the same way as above.

3) It follows that
$$X_1 \cap \cdots \cap X_n = (C_1 \cap \cdots \cap C_n) \cap (U_1 \cap \cdots \cap U_n)$$
with $C_1 \cap \cdots \cap C_n \subseteq \mathbb{A}^n$ being Zariski closed and $U_1 \cap \cdots \cap U_n \subseteq \mathbb{A}^n$ being Zariski open. □

Example 2.4.

1) Every affine set is a quasi-affine set.

2) If X is an affine set then $\mathrm{D}_X(f)$ is a quasi-affine set for every $f \in \mathrm{A}(X)$.

3) It follows from the previous example with $X = \mathbb{A}^{n^2}$ and $f = \det$ that $\mathrm{GL}_n(k) = \mathrm{D}(\det)$ is a quasi-affine set.

Lemma 2.5. A quasi-affine set has only finitely many irreducible components. □

2.2. Morphisms of Quasi-Affine Sets

Definition 2.6. Let X be a quasi-affine set and let $f\colon X \longrightarrow k$ be a function.

1) The function f is *regular at* $x \in X$ if f is a rational function in some neighbourhood of x, i.e. if there exist $U \subseteq X$ open and $g, h \in \mathrm{A}(X)$ such that $x \in U$, $h(y) \neq 0$ for every $y \in U$ and $f(y) = g(y)/h(y)$ for all $y \in U$.

2) The function f is *regular* if it is regular at every point $x \in X$. The set of all regular functions $X \longrightarrow k$ is denoted by $\mathcal{O}(X)$.

Notation 2.7. If X is a set, $U \subseteq X$ is a subset and $f, f', g, g' : X \longrightarrow k$ are functions with $g(x), g'(x) \neq 0$ for every $x \in U$ and $f(x)/g(x) = f'(x)/g'(x)$ for all $x \in U$ then we say that $f/g \equiv f'/g'$ on U.

Proposition 2.8. Let X be an affine set and let $f\colon X \longrightarrow k$ be a regular function in the sense of Definition 2.6. The map f is then already a polynomial, and thus regular in the sense of Definition 1.39. Thus both definitions agree for affine sets, and $\mathcal{O}(X) = \mathrm{A}(X)$.

Proof. There exists an open cover $(U_i)_{i\in I}$ of X such that f is given on each U_i by a rational function f_i/g_i for suitable $f_i, g_i \in \mathrm{A}(X)$. We may assume that each U_i is of the form $U_i = \mathrm{D}(h_i)$ for some $h_i \in \mathrm{A}(X)$ as these sets form a basis for the Zariski topology on X.

It follows for all $i, j \in I$ that $f_i/g_i \equiv f_j/g_j$ on $U_i \cap U_j$, and therefore that $f_i g_j \equiv f_j g_i$ on $U_i \cap U_j$. The set $U_i \cap U_j$ is given by

$$U_i \cap U_j = \mathrm{D}(h_i) \cap \mathrm{D}(h_j) = \mathrm{D}(h_i h_j)\,.$$

It follows that by multiplying the above equality with $h_i h_j$ we arrive at the equality

$$h_i h_j f_i g_j = h_i h_j f_j g_i\,, \tag{2.1}$$

which holds both on $U_i \cap U_j = \mathrm{D}(h_i h j)$ and on $X \setminus (U_i \cap U_j) = \mathrm{V}(h_i h_j)$, i.e. on the whole of X.

We may assume that $h_i = g_i$: It holds that $\mathrm{D}(h_i) \subseteq \mathrm{D}(g_i)$ because the rational function f_i/g_i is defined on $U_i = \mathrm{D}(h_i)$. It follows from Corollary 1.38 that $h_i \in \sqrt{(g_i)}$ and therefore that $h_i^n = a g_i$ for some $a \in \mathrm{A}(X)$, $n \geq 0$. It follows that $a(x) \neq 0$ forevery $x \in U_i = \mathrm{D}(h_i)$ and therefore

$$\frac{f_i}{g_i} \equiv \frac{a f_i}{a g_i} \equiv \frac{a f_i}{h_i^n}$$

on U_i. By replacing f_i with $a f_i$ and both g_i and h_i with h_i^n the claim follows. Note that Equation (2.1) can now be rewritten as

$$f_i g_j \cdot g_i g_j = f_j g_i \cdot g_i g_j\,. \tag{2.2}$$

So we may swap the indices in the term $f_i g_j$ if the factor $g_i g_j$ is present.

The denominators g_i, $i \in I$ have no common zeros because

$$\mathrm{V}(g_i \mid i \in I) = \bigcap_{i\in I} \mathrm{V}(g_i) = X \setminus \bigcup_{i \in I} \mathrm{D}(g_i) = X \setminus X = \emptyset\,,$$

The squares g_i^2, $i \in I$ do therefore also have no common zeroes (because g_i and g_i^2 have the same zeroes). It follows from Hilbert's Nullstellensatz that there exists a linear combination

$$1 = \sum_{i \in I} a_i g_i^2 \tag{2.3}$$

for suitable $a_i \in \mathrm{A}(X)$. By using Equation (2.2) and Equation (2.3) it follows that it holds on U_j that

$$f \equiv \frac{f_j}{g_j} \equiv \frac{f_j}{g_j} \sum_{i \in I} a_i g_i^2 \equiv \sum_{i \in I} \frac{a_i f_j g_i^2}{g_j} \equiv \sum_{i \in I} \frac{a_i f_j g_i^2 g_j}{g_j^2} \equiv \sum_{i \in I} \frac{a_i f_i g_j^2 g_i}{g_j^2} \equiv \sum_{i \in I} a_i f_i g_i \,.$$

This shows that $f \equiv \sum_{i \in I} a_i f_i g_i$ on U_j for every $j \in I$ and therefore $f = \sum_{i \in I} a_i f_i g_i$. □

Definition 2.9. Let X, Y be quasi-affine sets.

1) A map $f\colon X \longrightarrow \mathbb{A}^m$ is *regular* if it is regular in each coordinate.

2) A map $f\colon X \longrightarrow Y$ is *regular* if it is the restriction of a regular map $X \longrightarrow \mathbb{A}^m$.

A map $X \longrightarrow Y$ is a *morphism* of quasi-affine sets if it is regular.

Remark 2.10. Proposition 2.8 shows that Definition 2.9 agrees with Definition 1.39 for affine sets.

Lemma 2.11. Let $X \subseteq X' \subseteq \mathbb{A}^n$ be a quasi-affine set with X' an affine set.

1) The sets $\mathrm{D}_X(f)$ with $f \in \mathrm{A}(X')$ are a basis of the Zariski topology on X.

2) The sets $\mathrm{D}_X(f)$ with $f \in \mathscr{O}(X)$ are open in X, and also form a basis of the Zariski topology on X.

Proof.

1) This follows from the fact that the sets $\mathrm{D}_{X'}(f)$ with $f \in \mathrm{A}(X')$ are a basis for the Zariski topology on X' and that $\mathrm{D}_X(f) = \mathrm{D}_{X'}(f) \cap X$ for every $f \in \mathrm{A}(X)$.

2) It remains to show that every $f \in \mathscr{O}(X)$ is continuous. If f is a globally defined rational function given by $f = g/h$ for $f, g \in \mathbb{A}(X')$ then $\mathrm{D}_X(f) = \mathrm{D}_X(g)$ is open.

   If more generally f is regular then there exists an open cover $(U_i)_{i \in I}$ of X such that $f|_{U_i}$ is rational for every $i \in I$. It then follows by the above that $f|_{U_i}$ is continuous for every $i \in I$ (where we use that U_i is again quasi-affine and $f|_{U_i}$ is again regular) and therefore that f is continuous (because continuity is a local property). □

Corollary 2.12. Let X, Y be quasi-affine sets and let $f\colon X \longrightarrow Y$ be a morphism of quasi-affine sets.

1) It holds for every $\varphi \in \mathscr{O}(Y)$ that $f^{-1}(\mathrm{D}_Y(\varphi)) = \mathrm{D}_X(\varphi \circ f)$.

2) The map f is continuous with respect to the Zariski topologies on X, Y. □

Lemma 2.13. Let X, Y, Z be quasi-affine sets.

1) The identity map $\mathrm{id}_X : X \longrightarrow X$ is a morphism.

2) For every two morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ their composition $g \circ f : X \longrightarrow Z$ is again a morphism.

Proof.

2) We need to show that $g \circ f$ is regular at every point $x \in X$. It follows from the regularity of g that there exists an open neighbourhood $V \subseteq Y$ of $f(x)$ on which g is given by a rational function g_1/g_2. It follows from the continuity of f that there exists an open neighbourhood U of x with $f(U) \subseteq V$. By using the regularity of f and shrinkening V if necessary we may assume that f is given by a rational function f_1/f_2 on U. It then follows that
$$(g \circ f)(x) = \frac{g_1\left(\frac{f_1(x)}{f_2(x)}\right)}{g_2\left(\frac{f_1(x)}{f_2(x)}\right)}$$
for every $x \in U$, which shows that $g \circ f$ is given by a rational function on U. (Recall that the composition of rational functions is again rational.) □

Lemma 2.14. Let X, Y, Z be quasi-affine sets

1) If $f : X \longrightarrow Y$ is a morphism of quasi-affine sets then the map
$$f^* : \mathscr{O}(Y) \longrightarrow \mathscr{O}(X), \quad \varphi \longmapsto \varphi \circ f$$
is a homomorphism of k-algebras.

2) It holds that $\mathrm{id}_X^* = \mathrm{id}_{\mathscr{O}(X)}$.

3) If $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ are morphisms of quasi-affine sets then $(g \circ f)^* = f^* \circ g^*$. □

2.3. Products of Quasi-Affine Sets

Lemma 2.15. If $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are quasi-affine sets then the set $X \times Y \subseteq \mathbb{A}^{n+m}$ is again quasi-affine.

Proof. Let $X' \subseteq \mathbb{A}^n$ and $Y' \subseteq \mathbb{A}^m$ be affine sets such that $X \subseteq X'$ and $Y \subseteq Y'$ are open subsets. Then $X \times Y$ is an open subset of the affine set $X' \times Y' \subseteq \mathbb{A}^{n+m}$ because the Zariski topology on $X' \times Y'$ is finer than the product topology. □

Definition 2.16. For quasi-affine sets $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ their *product* is the quasi-affine set $X \times Y \subseteq \mathbb{A}^{n+m}$.

2.17. Note that for affine sets X, Y their product as affine sets is the same as their product of quasi-affine sets. We therefore do not need to specify which kind of product of we are talking about when dealing with affine sets.

Lemma 2.18. Let $X, X_1, X_2, X_2, Y_1, Y_2$ be quasi-affine sets.

1) The projections $\pi_i\colon X_1 \times X_2 \longrightarrow X_i$ are morphisms of quasi-affine sets.

2) A map $f\colon X \longrightarrow Y_1 \times Y_2$ given by $f = (f_1, f_2)$ with $f_i\colon X \longrightarrow Y_i$ is a morphism of quasi-affine sets if and only if both f_1, f_2 are morphisms of quasi-affine sets.

This shows that the product of two quasi-affine sets is their categorical product in the category of quasi-affine sets.

3) If $f\colon X \longrightarrow X'$ and $g\colon Y \longrightarrow Y'$ are to morphisms of quasi-affine sets then the induced map $f \times g\colon X \times Y \longrightarrow X' \times Y'$ is again a morphism of quasi-affine sets. □

2.4. Affine Varieties

Definition 2.19. An *affine variety* is a quasi-affine set which is isomorphic to an affine set (as a quasi-affine set). A *quasi-affine variety* is just a quasi-affine set. A *morphism* of (quasi-)affine varieties is just a morphism of quasi-affine sets.[3]

Notation 2.20. For an affine variety X we often write $\mathrm{A}(X) \coloneqq \mathcal{O}(X)$. Proposition 2.8 shows that this is well-defined when X is an affine set.

Remark 2.21. Many authors additionally require affine varieties to be irreducible. The author tries to avoid this restriction.

Example 2.22.

1) Every affine set is an affine variety.

2) Let X be an affine variety and let $f \in \mathrm{A}(X)$. Then the open subset $\mathrm{D}_X(f)$ is again an affine variety:

   We may assume that $X \subseteq \mathbb{A}^n$ is an affine set. For the set

$$Y = \left\{(x,t) \in \mathbb{A}^n \times \mathbb{A}^1 = \mathbb{A}^{n+1} \,\middle|\, x \in X, xt = 1\right\}$$

   the map

$$\varphi\colon \mathrm{D}_X(f) \longrightarrow Y, \quad x \longmapsto \left(x, \frac{1}{f(x)}\right)$$

   is a bijection. If X is cut out by some ideal $I \trianglelefteq k[x_1, \dots, x_n]$ then Y is cut out by the ideal I together with the polynomial $fx_{n+1} - 1$, which shows that Y is again an affine set. The map φ is rational in each coordinate and therefore a morphism. The inverse of φ is given by projection onto the first n-th coordinates, which is also a morphism. This shows that φ is an isomorphism, which shows that claim.

[3]We will later on extend this notions to projective and quasi-projective sets.

It follows in particular that the isomorphism φ of affine varieties induces an isomorphism of k-algebras $\varphi^*\colon\ \mathrm{A}(Y) \longrightarrow \mathrm{A}(\mathrm{D}_X(f))$. It follows that

$$\begin{aligned}\mathrm{A}(\mathrm{D}_X(f)) &\cong \mathrm{A}(Y) \\ &= k[x_1, \dots, x_n, x_{n+1}]/(I, f x_{n+1} - 1) \\ &\cong (k[x_1, \dots, x_n]/I)[x_{n+1}]/(f x_{n+1} - 1) \\ &= \mathrm{A}(X)[x_{n+1}]/(f x_{n+1} - 1) \\ &\cong \mathrm{A}(X)[f^{-1}]\end{aligned}$$

is the localization of $\mathrm{A}(X)$ at f. This shows that every regular function on $\mathrm{D}_X(f)$ is of the form g/f^n for some $g \in \mathrm{A}(X)$ and $n \geq 0$.

3) As an instance of the previous example we find that $\mathrm{GL}_n(k) = \mathrm{D}(\det) \subseteq \mathbb{A}^{n^2}$ is an affine variety with

$$\mathrm{A}(\mathrm{GL}_n(k)) = \mathrm{A}\left(\mathbb{A}^{n^2}\right)\left[\det^{-1}\right] = k\left[x_{11}, \dots, x_{nn}, \det^{-1}\right] .$$

As an example we have for $n = 2$ that

$$\mathrm{A}(\mathrm{GL}_2(k)) = k\left[a, b, c, d, \frac{1}{ad - bc}\right] .$$

Lemma 2.23. Let X, X_1, X_2 be affine varieties.

1) Every closed subset $C \subseteq X$ is again an affine variety.

2) If X_1 and X_2 are both affine

Lemma 2.24. Every Zariski closed subset of an affine variety is again an affine variety.

Proof. Let X be an affine variety and let $Y \subseteq X$ be Zariski closed. Then there exists an affine set $X' \subseteq \mathbb{A}^n$ and an isomorphism $\varphi\colon X \longrightarrow X'$. The set $Y' := \varphi(X')$ is Zariski closed in X' because φ is a homeomorphism, and it follows that $Y' \subseteq \mathbb{A}^n$ is an affine set. The isomorphism $\varphi\colon X \longrightarrow X'$ restricts to an isomorphism $Y \longrightarrow Y'$, showing that Y is an affine variety. □

Lemma 2.25. The coordinate ring $\mathrm{A}(-)$ gives rise to a contravariant equivalence

$$\begin{aligned}\{\text{affine varieties}\} &\longrightarrow \left\{\begin{matrix}\text{finitely generated,} \\ \text{commutative,} \\ \text{reduced } k\text{-algebras}\end{matrix}\right\} , \\ X &\longmapsto \mathrm{A}(X) , \\ f &\longmapsto f^* .\end{aligned}$$

Proof. This follows from Corollary 1.47. □

Lemma 2.26. For affine varieties X, Y the map

$$\mathrm{A}(X) \otimes_k \mathrm{A}(Y) \longrightarrow \mathrm{A}(X \times Y) , \quad f \otimes g \mapsto [(x, y) \mapsto f(x)g(y)]$$

is a well-defined natural isomorphism of k-algebras.

Proof. To see that the proposed map $\Phi = \Phi_{X,Y}$ is well-defined let $f \in \mathrm{A}(X)$ and $g \in \mathrm{A}(Y)$. We need to show that $\Phi(f \otimes g)$ is regular at every point $(x, y) \in X \times Y$. There exist a neighbourhood $U \subseteq X$ of x on which f is given by a rational function f_1/f_2, and similarly a neighbourhood $V \subseteq Y$ of y on which g is given by a rational function g_1/g_2. It follows that $U \times V$ is an open neighbourhood of (x, y) in $X \times Y$ because the Zariski topology on $X \times Y$ is finer than the product topology. In this neighbourhood the map $\Phi(f \otimes g)$ is given by the rational function $(f_1 g_1)/(f_2 g_2)$. This shows that $\Phi(f \otimes g)$ is regular at (x, y).

To show that naturality of Φ let X', Y' be another pair of affine varieties and consider a pair of morphisms $\varphi\colon X \longrightarrow X'$, $\psi\colon Y \longrightarrow Y'$. For the naturialty of Φ we need the diagram

$$\begin{tikzcd} \mathrm{A}(X) \otimes_k \mathrm{A}(Y) \arrow[r, "\Phi_{X,Y}"] & \mathrm{A}(X \times Y) \\ \mathrm{A}(X') \otimes_k \mathrm{A}(Y') \arrow[u, "\varphi^* \otimes \psi^*"] \arrow[r, "\Phi_{X',Y'}"] & \mathrm{A}(X' \times Y') \arrow[u, "(\varphi \times \psi)^*"'] \end{tikzcd} \tag{2.4}$$

to commutes. This holds because

$$\begin{aligned} \Phi_{X,Y}((\varphi^* \times \psi^*)(f \otimes g))(x, y) &= \Phi_{X,Y}(\varphi^*(f) \otimes \psi^*(g))(x, y) \\ &= \varphi^*(f)(x)\psi^*(g)(y) \\ &= (f \circ \varphi)(x)(g \circ \psi)(y) \\ &= f(\varphi(x))g(\psi(y)) \\ &= \Phi_{X',Y'}(f \otimes g)(\varphi(x), \psi(y)) \\ &= \Phi_{X',Y'}(f \otimes g)((\varphi \times \psi)(x, y)) \\ &= (\varphi \times \psi)^*(\Phi_{X',Y'}(f \otimes g))(x, y) \end{aligned}$$

for all $(x, y) \in X \times Y$, and therefore

$$\Phi_{X,Y}((\varphi^* \otimes \psi^*)(f \otimes g)) = (\varphi \times \psi)^*(\Phi_{X',Y'}(f \otimes g))$$

for every simple tensor $f \otimes g \in \mathrm{A}(X') \otimes \mathrm{A}(Y')$, and thus overall

$$\Phi_{X,Y} \circ (\varphi^* \otimes \psi^*) = (\varphi \times \psi)^* \circ \Phi_{X',Y'}\,.$$

To show that Φ is an isomorphism let $X' \subseteq \mathbb{A}^n$ and $Y' \subseteq \mathbb{A}^m$ be affine sets for which there exists isomorphisms $\varphi\colon X \longrightarrow X'$ and $\psi\colon Y \longrightarrow Y'$. Then in the resulting commutative diagram (2.4) the vertical arrows are both isomorphisms and the lower horizontial map $\Phi_{X',Y'}$ is an isomorphism by Proposition 1.54. It follows that the upper horizontal arrow, which can be expressed as

$$\Phi_{X,Y} = (\varphi \times \psi)^* \circ \Phi_{X',Y'} \circ (\varphi^* \otimes \psi^*)^{-1}\,, \tag{2.5}$$

is also an isomomorphism. □

Remark 2.27. That $\Phi_{X,Y}$ is well-defined for affine varieties X, Y also follows from Equation (2.5). The above argumentation actually shows that $\Phi_{X,Y}$ is well-defined whenever X, Y are quasi-affine varietes, and it is explained in [MO17] that $\Phi_{X,Y}$ is then again an isomorphism. (One uses that that the isomorphism holds for affine varieties, and then uses that quasi-affine varities are 'locally affine' and that regularity is a local condition.)

Remark 2.28. Just as we can generalize Corollary 1.47 to Lemma 2.25 and Proposition 1.54 to Lemma 2.26 we can generalize most of our previous results about affine sets from Section 1 to affine varieties, as these are just affine sets up to isomorphism. We will not do this explicitely for every such result but instead also refer to the result about affine sets even when we are using the generalized result about affine varieties.

2.29. The notion of quasi-affine sets and the above generalization of affine sets to affine varieties were originally not given in the lecture. Quasi-affine sets were only introduced much later on together with projective and quasi-projective sets. We have choosen to include these concept earlier on so that $\mathrm{GL}_n(k)$ becomes an affine variety. This places the upcoming discussion of affine algebraic groups on a more formally solid foundation.

The idea of using quasi-affine sets and the regular maps between them was inspired by [AlgI18a, 2.2]. The proof of Proposition 2.8 is taken from [Mil17, Lemma 3.10].

II. Affine Algebraic Groups

3. Definition and First Examples

Definition 3.1. An *affine algeberaic group* is an affine variety G together with a group structure such that both the multiplication map $G \times G \longrightarrow G$, $(g_1, g_2) \mapsto g_1 g_2$ and inversion map $G \longrightarrow G$, $g \mapsto g^{-1}$ are morphisms of affine varieties.

If G, H are affine algebraic groups then a map $f\colon G \longrightarrow H$ is a *homomorphism of affine algebraic groups* if it is both a morphism of affine varieties and a group homomorphism.

Example 3.2. The *additive groups* $\mathbb{G}_a := (\mathbb{A}^1, +)$ is an affine algebraic group since the addition

$$+\colon \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2 \longrightarrow \mathbb{A}^1, \quad (x, y) \mapsto x + y$$

and inversion

$$-\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^1, \quad x \mapsto -x$$

are regular.

Example 3.3. The *general linear group* $\mathrm{GL}_n(k)$ together with the usual matrix multiplication is an affine algebraic group. We have seen in Example 2.22 that $\mathrm{GL}_n(k)$ is an affine variety, the multiplication map $\mathrm{GL}_n(k) \times \mathrm{GL}_n(k) \longrightarrow \mathrm{GL}_n(k)$ is polynomial and therefore regular, and the inversion map $\mathrm{GL}_n(k) \longrightarrow \mathrm{GL}_n(k)$ is a rational function (because the entries of A^{-1} are rational functions in the entries of A by Cramer's rule) and therefore also regular.

It follows for $n = 1$ that the *multiplicative group* $\mathbb{G}_m := \mathrm{GL}_1(k) = k^\times$ is an affine algebraic group.

Lemma 3.4. Every Zariski closed subgroup of an affine algebraic group is again an affine algebraic group.

Proof. Let G be an affine algebraic group and let $H \leq G$ be a Zariski closed subgroup. It follows from Lemma 2.24 that H is again an affine variety, and the multiplication $H \times H \longrightarrow H$ and inversion $H \longrightarrow H$ are morphisms because they are restrictions of the multiplication $G \times G \longrightarrow G$ and inversion $G \longrightarrow G$. ☐

Corollary 3.5. Every Zariski closed subgroup of $\mathrm{GL}_n(k)$ is an affine algebraic group. ☐

Example 3.6. It follows from Corollary 3.5 that the following are affine algebraic groups:

1) The *special linear group* $\mathrm{SL}_n(k) = \{A \in \mathrm{GL}_n(k) \mid \det A = 1\}$.

2) The *orthogonal group* $\mathrm{O}_n(k) = \{A \in \mathrm{GL}_n(k) \mid A^T A = I\}$.

3) The *special orthogonal group* $\mathrm{SO}_n(k) = \mathrm{SL}_n(k) \cap \mathrm{O}_n(k)$.

4) The *symplectic group* $\mathrm{Sp}_{2n}(k) = \{A \in \mathrm{GL}_n(k) \mid A^T J A = J\}$ where $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$.

5) Every finite group, when regarded as an affine set and thus an affine variety, is an affine algebraic group. The multiplication map $G \times G \longrightarrow G$ and inversion map $G \longrightarrow G$ are morphisms of afffine varieties because both G and $G \times G$ are finite, and therefore all maps $G \times G \longrightarrow G$ and $G \longrightarrow G$ are morphisms.

6) The *group of diagonal matrices* $\mathrm{D}_n(k) = \left\{\left(\begin{smallmatrix} * & & \\ & \ddots & \\ & & * \end{smallmatrix}\right) \in \mathrm{GL}_n(k)\right\}$.

7) The *group of upper triangular matrices* $\mathrm{T}_n(k) = \left\{\left(\begin{smallmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{smallmatrix}\right) \in \mathrm{GL}_n(k)\right\}$.

8) The *group of unitriangular matrices* $\mathrm{U}_n(k) = \left\{\left(\begin{smallmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ & & 1 \end{smallmatrix}\right) \in \mathrm{GL}_n(k)\right\}$.

Remark 3.7. One may rephrase the definition of an affine algebraic groups as saying that G is an affine variety together with morphisms

$$m\colon G \times G \longrightarrow G \quad \text{and} \quad i\colon G \longrightarrow G$$

and an element $e \in G$ such that the following diagrams commute:

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

associativity

$$\begin{tikzcd} & G \times G \arrow[dd, "m"] & \\ G \times \{e\} \arrow[ur, hook] \arrow[dr, "\mathrm{pr}_1"'] & & \{e\} \times G \arrow[ul, hook'] \arrow[dl, "\mathrm{pr}_2"] \\ & G & \end{tikzcd}$$

neutral element

$$\begin{tikzcd} G \arrow[rr, "(i, \mathrm{id})"] \arrow[dd, "(\mathrm{id}, i)"'] \arrow[dr] & & G \times G \arrow[dd, "m"] \\ & \{e\} \arrow[dr] & \\ G \times G \arrow[rr, "m"'] & & G \end{tikzcd}$$

inverse

If more generally $\mathscr{C}$ is any category with finite products, including a terminal object $*$, then a *group object* in $\mathscr{C}$ is an object $G \in \mathscr{C}$ together with morphisms

$$m\colon G \times G \longrightarrow G, \qquad i\colon G \longrightarrow G, \qquad e\colon * \longrightarrow G$$

such that the following diagrams commute:

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

associativity

$$\begin{tikzcd} & G \times G \arrow[dd, "m"] & \\ G \times * \arrow[ur, "\mathrm{id} \times e"] \arrow[dr, "\mathrm{pr}_1"'] & & * \times G \arrow[ul, "e \times \mathrm{id}"'] \arrow[dl, "\mathrm{pr}_2"] \\ & G & \end{tikzcd}$$

neutral ‘element’

$$\begin{tikzcd} G \arrow[rr, "(i, \mathrm{id})"] \arrow[dd, "(\mathrm{id}, i)"'] \arrow[dr] & & G \times G \arrow[dd, "m"] \\ & * \arrow[dr, "e"] & \\ G \times G \arrow[rr, "m"'] & & G \end{tikzcd}$$

inverse

Example 3.8.

1) The group objects in the category of sets are just groups.

2) The group objects in the category of topological spaces are topological groups.

3) The group objects in the category of smooth real manifolds are real Lie groups.

4) The group objects in the category of affine varieties are affine algebraic groups.

Warning 3.9. An affine algebraic group G is in general not a topological group. While the multiplication $m\colon G \times G \longrightarrow G$ is a morphism of affine varieties and therefore continuous, it is so with respect to the Zariski topology on $G \times G$. For G to be a topological group we would need m to be continuous with respect to the product topololgy of $G \times G$, which is in general coarser than the Zariski topology (see Warning 1.52).

Definition 3.10. For an affine algebraic group G the connected component of the identity $1 \in G$ is denoted by G^0.

Proposition 3.11. Let G be a linear algebraic group.

1) The connected components of G coincide with its irreducible components.

2) The connected/irreducible component G^0 is a normal subgroup of G.

3) The connected/irreducible components of G are the cosets of G^0.

4) The group G^0 has finite index in G.

Proof.

1) It follows from Lemma 2.5 that G has only finitely many irreducible components $C_1, \dots, C_n$. It holds that $C_1 \nsubseteq C_2 \cup \dots \cup C_n$ because it would otherwise follows from the irreducibility of C_1 that $C_1 \subseteq C_i$ for some $i \geq 2$, which would contradict $C_1, \dots, C_n$ being the irreducible components of G. Let $g \in C_1$ with $x \notin C_2 \cup \dots \cup C_n$.

   The element x is contained in precisely one irreducible component. For every $g \in G$ the map
   $$G \longrightarrow G, \quad h \longmapsto ghx^{-1}$$
   is regular and therefore a homeomorphism, and maps x to g. It follows that every $g \in G$ is contained in precisely one irreducible component. The irreducible components of G are therefore disjoint.

   Each irreducible component C is closed. The complement of C is also closed because it is the union of all other irreducible components, and therefore a finite union of closed sets. This shows that each irreducible component C of G is actually clopen.

   It follows that each connected component of G is contained in an irreducible component. It also holds that every irreducible component is contained in a connected component because irreducible topological spaces are connected. It follows that the connected and irreducible components coincide.

2) It holds that $1 \in G^0$ by the definition of G^0. For every $g \in G^0$ the left multiplication
   $$\lambda_g\colon G \longrightarrow G, \quad h \longmapsto gh$$

is an isomorphism of varieties and thus a homeomorphism. It therefore maps the component G^0 of the identity 1 onto the component of $\lambda_g(1) = g$. It follows from g being contained in G^0 that this component is G^0 and therefore that $gG^0 = \lambda_g(G^0) = G^0$.

It follows similarly that $(G^0)^{-1}$ is the component of G which contains $1^{-1} = 1$ and therefore that $(G^0)^{-1} = G^0$.

Together this shows that G^0 is a subgroups of G. To show that G^0 is normal in G let $g \in G$. The conjugation map

$$c_g \colon G \longrightarrow G, \quad h \longmapsto ghg^{-1}$$

is regular and therefore a homeomorphism. It follows that $gG^0g^{-1} = c_g(G^0)$ is the component containing $c_g(1) = 1$ and therefore that $gG^0g^{-1} = G^0$.

3) We find in the above notation that the component of $g \in G$ is given by $\lambda_g(G^0) = gG^0$.

4) The index $[G : G^0]$ is the number of components of G. □

Corollary 3.12. An affine algebraic group is connected if and only if it is irreducible. □

3.13. If G is an affine algebraic group then the connected component G^0 is closed (as this holds for the connected components of any topological space) and therefore again an affine algebraic group. We thus get a short exact sequence of affine algebraic groups

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow G/G^0 \longrightarrow 1$$

with G^0 connected and G/G^0 finite.

4. Hopf Algebra Structure on the Coordinate Ring

4.1. If X is an affine variety then the coordinate ring $\mathrm{A}(X)$ is a k-algebra, with addition and multiplication coming from k. In this section we will see that if G is an affine algebraic group then the additional group structure of G gives the coordinate ring $\mathrm{A}(G)$ the additional structure of a Hopf algebra.

4.2. A k-algebra A may be defined as a k-vector space A together with two k-linear maps

$$m \colon A \otimes A \longrightarrow A \quad \text{and} \quad u \colon k \longrightarrow A$$

such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes m"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[dd, "m"] & \\
A \otimes k \arrow[ur, "\mathrm{id} \otimes u"] \arrow[dr, "\sim"'] & & k \otimes A \arrow[ul, "u \otimes \mathrm{id}"'] \arrow[dl, "\sim"] \\
 & A &
\end{tikzcd}$$

The commutativity of the first diagram encodes the associativity of the multiplication m and the commutativity of the second diagram encodes that the elemente $u(1)$ is the unit with respect to this multiplication.

By reversing the arrows in these diagrams we arrive at the notion of a coalgebra:

Definition 4.3. A (k-)coalgebra is a k-vector space C together with k-linear maps

$$\Delta\colon A \longrightarrow A \otimes A \quad \text{and} \quad \varepsilon\colon A \longrightarrow k$$

such that the diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathrm{id}"'] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, "\sim"'] \arrow[d, "\Delta"] \arrow[dr, "\sim"] & \\ C \otimes k & C \otimes C \arrow[l, "\mathrm{id}\otimes\varepsilon"] \arrow[r, "\varepsilon\otimes\mathrm{id}"'] & k \otimes C \end{tikzcd}$$

commute. The map Δ is the *comultiplication* of C and ε is its *counit*. The commutativity of the first diagram is the *coassociativity* of Δ and the commutativity of the second diagram states that ε is *counitial*.

Definition 4.4. A *k-bialgebra* is a k-algebra B together with k-linear maps $\Delta\colon B \longrightarrow B \otimes B$ and $\varepsilon\colon B \longrightarrow k$ such that (B, Δ, ε) is a k-coalgebra and Δ, ε are algebra homomorphisms.

Definition 4.5. A k-Hopf algebra is a k-bialgebra H together with a k-linear map $S\colon H \longrightarrow H$ which makes the diagram

$$\begin{tikzcd} & H \otimes H \arrow[r, "S\otimes\mathrm{id}"] & H \otimes H \arrow[dr, "m"] & \\ H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & k \arrow[r, "u"] & H \\ & H \otimes H \arrow[r, "\mathrm{id}\otimes S"] & H \otimes H \arrow[ur, "m"'] & \end{tikzcd}$$

commutes. The map S is the *antipode* of H.

Remark 4.6. If C is a k-coalgebra and A is a k-algebra then one can define on $\mathrm{Hom}_k(C, A)$ the structure of a k-algebra via the *convolution product* $*$ given by

$$f * g \coloneqq m_A \circ (f \otimes g) \circ \Delta_C$$

for all $f, g \in \mathrm{Hom}_k(C, A)$. The convolution unit of $\mathrm{Hom}_k(C, A)$ is given by $u_A \circ \varepsilon_C$. If B is a k-bilalgebra then it follows that $\mathrm{End}_k(B)$ is a k-algebra with respect to the convolution product, and an antipode for B is precisely a convolution inverse to the identity $\mathrm{id}_B \in \mathrm{End}_k(B)$. It follows in particular that for every k-bialgebra B there exists at most one possible antipode map which makes it into a Hopf algebra.

The antipode of a Hopf algebra is an antimorphism of both k-algebras and k-coalgebras.

4.7. Let G be an affine algebraic group with multiplication $m\colon G \times G \longrightarrow G$, inversion $G \longrightarrow G$ and neutral element $e \in G$; let $j\colon \{e\} \longrightarrow G$ be the inclusion. The induced algebra homomorphism $m^*\colon \mathrm{A}(G) \longrightarrow \mathrm{A}(G \times G)$ is given by

$$m^*(f)(x_1, x_2) = f(x_1 x_2) \tag{4.1}$$

for all $f \in \mathrm{A}(G)$, $x_1, x_2 \in G$, the induced homomorphism $i^* : \mathrm{A}(G) \longrightarrow \mathrm{A}(G)$ is given by

$$i^*(f)(x) = f(x^{-1})$$

for all $f \in \mathrm{A}(G)$, $x \in G$

We define $\Delta : \mathrm{A}(G) \longrightarrow \mathrm{A}(G) \otimes \mathrm{A}(G)$ to be the composition of the algebra homomorphism $m^* : \mathrm{A}(G) \longrightarrow \mathrm{A}(G \times G)$ with the natural isomorphism $\mathrm{A}(G \times G) \cong \mathrm{A}(G) \otimes \mathrm{A}(G)$ from Lemma 2.26. The homomorphism Δ is thus given by $\Delta(f) = \sum_{i=1}^{n} f_1 \otimes f_2$ such that

$$m^*(f)(x_1, x_2) = \sum_{i=1}^{n} f_1(x_1) f_2(x_2)$$

for all $x_1, x_2 \in G$. Equation (4.1) then becomes

$$f(x_1 x_2) = \sum_{i=1}^{n} f_1(x_1) \otimes f_2(x_2)$$

for all $f \in \mathrm{A}(G)$, $x_1, x_2 \in G$. We further define ε to be the composition of $j^* : \mathrm{A}(G) \longrightarrow \mathrm{A}(\{e\})$ with the (unique) isomorphism (of k-algebras) $\mathrm{A}(\{e\}) \cong k$. The homomorphism ε is given by

$$\varepsilon(f) = f(e)$$

for all $f \in \mathrm{A}(G)$. Lastly we define $S \coloneqq i^*$, which is given by

$$S(f)(x) = f(x^{-1})$$

for all $f \in \mathrm{A}(G)$, $x \in G$.

That m, i, e give a group structure on G can be encoded in the commutativity of the following diagrams:

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd} \qquad \begin{tikzcd} & G \times G \arrow[dd, "m"] & \\ G \times \{e\} \arrow[ur, "\mathrm{id} \times j"] \arrow[dr, "\mathrm{pr}_1"'] & & \{e\} \times G \arrow[ul, "j \times \mathrm{id}"'] \arrow[dl, "\mathrm{pr}_2"] \\ & G & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[rr, "(i,\mathrm{id})"] \arrow[dr] \arrow[dd, "(\mathrm{id},i)"'] & & G \times G \arrow[dd, "m"] \\ & \{e\} \arrow[dr, "j"] & \\ G \times G \arrow[rr, "m"'] & & G \end{tikzcd}$$

By applying the contravariant functor $\mathrm{A}(-)$ to these diagrams we get commutative diagrams involving $\mathrm{A}(G)$, Δ, ε and S, and which will show that these homomorphisms give $\mathrm{A}(G)$ the structure of a Hopf-algebra.

- By applying $\mathrm{A}(-)$ to the first diagram we get the following commutative diagram:

$$\begin{tikzcd} \mathrm{A}(G) \arrow[r, "m^*"] \arrow[d, "m^*"'] & \mathrm{A}(G \times G) \arrow[d, "(\mathrm{id} \times m)^*"] \\ \mathrm{A}(G \times G) \arrow[r, "(m \times \mathrm{id})^*"'] & \mathrm{A}(G \times G \times G) \end{tikzcd}$$

By using the natural (!) isomorphism $\mathrm{A}(G \times G) \cong \mathrm{A}(G) \otimes \mathrm{A}(G)$ this becomes the following commutative diagram:

$$\begin{tikzcd} \mathrm{A}(G) \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathrm{A}(G) \otimes \mathrm{A}(G) \arrow[d, "\mathrm{id} \otimes \Delta"] \\ \mathrm{A}(G) \otimes \mathrm{A}(G) \arrow[r, "\Delta \otimes \mathrm{id}"'] & \mathrm{A}(G) \otimes \mathrm{A}(G) \otimes \mathrm{A}(G) \end{tikzcd} \tag{4.2}$$

This diagram gives the coassociativity of Δ.

- By applying $\mathrm{A}(-)$ to the second diagram we get the following commutative diagram:

$$\begin{tikzcd} & \mathrm{A}(G) \arrow[dl, "\mathrm{pr}_1^*"'] \arrow[dd, "m^*"] \arrow[dr, "\mathrm{pr}_2^*"] & \\ \mathrm{A}(G \times \{e\}) & & \mathrm{A}(\{e\} \times G) \\ & \mathrm{A}(G \times G) \arrow[ul, "(\mathrm{id} \times i)^*"] \arrow[ur, "(i \times \mathrm{id})^*"'] & \end{tikzcd}$$

By applying the natural isomorphism $\mathrm{A}(G \times \{e\}) \cong \mathrm{A}(G) \otimes k$ the algebra homomorphisms pr_1^* becomes the natural isomorphism $\mathrm{A}(G) \cong \mathrm{A}(G) \otimes k$, and similarly for pr_2^*. We thus get the following commutative diagram:

$$\begin{tikzcd} & \mathrm{A}(G) \arrow[dl, "\sim"'] \arrow[dd, "\Delta"] \arrow[dr, "\sim"] & \\ \mathrm{A}(G) \otimes k & & k \otimes \mathrm{A}(G) \\ & \mathrm{A}(G) \otimes \mathrm{A}(G) \arrow[ul, "\mathrm{id} \otimes \varepsilon"] \arrow[ur, "\varepsilon \otimes \mathrm{id}"'] & \end{tikzcd}$$

This diagram gives that ε is a counit.

Together this shows that Δ, ε endow $\mathrm{A}(G)$ with the structure of a k-bilalgebra.

- Before we apply $\mathrm{A}(-)$ to the third diagram we rewrite this diagram as follows, where d denotes the diagonal map:

$$\begin{tikzcd} & G \times G \arrow[r, "i \times \mathrm{id}"] & G \times G \arrow[dr, "m"] & \\ G \arrow[ur, "d"] \arrow[rr] \arrow[dr, "d"'] & & \{e\} \arrow[r, "j"] & G \\ & G \times G \arrow[r, "\mathrm{id} \times S"'] & G \times G \arrow[ur, "m"'] & \end{tikzcd}$$

By applying the functor $\mathrm{A}(-)$ to this diagram we get the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{A}(G \times G) \arrow[r, "(i\times \mathrm{id})^*"] & \mathrm{A}(G \times G) \arrow[dr, "d^*"] & \\
\mathrm{A}(G) \arrow[ur, "m_*"] \arrow[r, "j^*"] \arrow[dr, "m_*"'] & \mathrm{A}(\{e\}) \arrow[rr] & & \mathrm{A}(G) \\
 & \mathrm{A}(G \times G) \arrow[r, "(\mathrm{id}\times i)^*"] & \mathrm{A}(G \times G) \arrow[ur, "d^*"'] &
\end{tikzcd}$$

(The unlabeled arrow is the unique homomorphism of k-algebras.) By applying the natural isomorphism $\mathrm{A}(G \times G) \cong \mathrm{A}(G) \otimes \mathrm{A}(G)$ and the isomorphism $\mathrm{A}(\{e\}) \cong k$ the induced morphism d^* becomes the multiplication $\mathrm{A}(G) \otimes \mathrm{A}(G) \longrightarrow \mathrm{A}(G)$, $f \otimes g \longmapsto fg$. We thus get the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{A}(G) \otimes \mathrm{A}(G) \arrow[r, "S\otimes \mathrm{id}"] & \mathrm{A}(G) \otimes \mathrm{A}(G) \arrow[dr, "\mathrm{mult}"] & \\
\mathrm{A}(G) \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & k \arrow[rr] & & \mathrm{A}(G) \\
 & \mathrm{A}(G) \otimes \mathrm{A}(G) \arrow[r, "\mathrm{id}\otimes S"] & \mathrm{A}(G) \otimes \mathrm{A}(G) \arrow[ur, "\mathrm{mult}"'] &
\end{tikzcd}$$

The commutativity of this diagram shows that S is an antipode for the bialgebra $(\mathrm{A}(G), \Delta, \varepsilon)$.

Altogether we have seen and shown that the groups structure on the affine variety G induces on its coordinate ring $\mathrm{A}(G)$ the structure of a Hopf algebra with

- the comultiplication Δ of $\mathrm{A}(G)$ being induced by the multiplication and G and given by $\Delta(f) = \sum_{i=1}^n f_1 \otimes f_2$ such that
$$f(x_1 x_2) = \sum_{i=1}^n f_1(x_1) f_2(x_2) \tag{4.3}$$
for all $f \in \mathrm{A}(G)$, $x_1, x_2 \in G$;
- the counit ε of $\mathrm{A}(G)$ being induced by by the neutral element of G and given by evaluation at e, i.e. by
$$\varepsilon(f) = f(e) \tag{4.4}$$
for all $f \in \mathrm{A}(G)$;
- and the antipode S of $\mathrm{A}(G)$ being induced by the inversion of G and given by
$$S(f)(x) = f(x^{-1}) \tag{4.5}$$
for all $f \in \mathrm{A}(G)$, $x \in G$.

Example 4.8. Let us consider the additive groups $\mathbb{G}_{\mathrm{a}} = (\mathbb{A}^1, +)$. The coordinate ring of $\mathbb{G}_{\mathrm{a}}$ is given by

$$\mathrm{A}(\mathbb{G}_{\mathrm{a}}) = \mathrm{A}(\mathbb{A}^1) = k[x] \,.$$

To determine the action of the comultiplication $\Delta \colon k[x] \longrightarrow k[x] \otimes k[x]$ on the algebra generator $x \in k[x]$ we note that

$$x(y_1 + y_2) = y_1 + y_2 = x(y_1) \cdot 1 + 1 \cdot x(y_2)$$

for all $y_1, y_2 \in G$. It follows from the explicit description of the comultiplication given in (4.3) that

$$\Delta(x) = x \otimes 1 + 1 \otimes x \,.$$

The counit $\varepsilon \colon k[x] \longrightarrow k$ is by (4.4) given by $\varepsilon(f) = f(0)$ for every $f \in k[x]$. It is on the algebra generator $x \in k[x]$ given by

$$\varepsilon(x) = x(0) = 0 \,.$$

The antipode $S \colon k[x] \longrightarrow k[x]$ is by (4.5) given by $S(f(x)) = f(-x)$ for all $f \in k[x]$. It is on the algebra generator $x \in k[x]$ given by

$$S(x) = -x \,.$$

Example 4.9. The coordinate ring of the general linear group $\mathrm{GL}_n(k)$ is given by

$$\mathrm{A}(\mathrm{GL}_n(k)) = k\left[x_{11}, \dots, x_{nn}, \det{}^{-1}\right]$$

where the element $\det \in k[x_{11}, \dots, x_{nn}]$ is given by

$$\det = \sum_{\sigma \in \mathrm{S}_n} \mathrm{sgn}(\sigma) x_{1\sigma(1)} \cdots x_{n\sigma(n)} \,.$$

To determine the action of the comultiplication Δ on the algebra generators x_{ij} we note that

$$x_{ij}(A_1 A_2) = \sum_{\ell=1}^{n} (A_1)_{i\ell} (A_2)_{\ell j} = \sum_{\ell=1}^{n} x_{i\ell}(A_1) x_{\ell j}(A_2)$$

for all $A_1, A_2 \in \mathrm{GL}_n(k)$. It follows from the explicit description of the comultiplication given in (4.3) that

$$\Delta(x_{ij}) = \sum_{\ell=1}^{n} x_{i\ell} x_{\ell j} \,.$$

To determine the action of Δ on the generator $\det^{-1}$ we note that

$$\det{}^{-1}(A_1 A_2) = \frac{1}{\det(A_1 A_2)} = \frac{1}{\det(A_1)\det(A_2)} = \frac{1}{\det(A_1)} \cdot \frac{1}{\det(A_2)} = \det{}^{-1}(A_1) \cdot \det{}^{-1}(A_2)$$

for all $A_1, A_2 \in \mathrm{GL}_n(k)$. It follows that

$$\Delta\left(\det{}^{-1}\right) = \det{}^{-1} \otimes \det{}^{-1} \,.$$

The action of the counit ε is by (4.4) given by $\varepsilon(f) = f(I)$ for all $f \in k[x_{11}, \dots, x_{nn}, \det^{-1}]$. It follows that the action of ε on the algebra generators x_{ij} is given by

$$\varepsilon(x_{ij}) = x_{ij}(I) = \delta_{ij}$$

and that the action of ε on the algebra generator $\det^{-1}$ is given by

$$\varepsilon\left(\det^{-1}\right) = \det^{-1}(I) = \frac{1}{\det(I)} = \frac{1}{1} = 1\,.$$

The action of the antipode S is by (4.5) given by $S(f)(A) = f(A^{-1})$ for all $f \in \mathrm{A}(\mathrm{GL}_n(k))$ and all $A \in \mathrm{GL}_n(k)$. It therefore follows from

$$\det^{-1}\left(A^{-1}\right) = \frac{1}{\det(A^{-1})} = \frac{1}{\det(A)^{-1}} = \det(A)$$

that $S(\det^{-1}) = \det$. The action of S on the algebra generators x_{ij} is messier to write down, as it boils down to explicitely expressing the coordinates of A^{-1} in terms of the coordinates of A via Cramer's rule. One finds that

$$\begin{aligned} S(x_{ij}) = (-1)^{ij} \det^{-1} \sum_{\sigma \in \mathrm{S}_{n-1}} \operatorname{sgn}(\sigma) \cdot \Bigg(& \sum_{\substack{p,q=1,\dots,i-1 \\ p<i,\sigma(q)<j}} x_{i,\sigma(j)} + \sum_{\substack{p,q=1,\dots,n-1 \\ p \geq i,\sigma(q)<j}} x_{i+1,\sigma(j)} \\ & + \sum_{\substack{p,q=1,\dots,n-1 \\ p<i,\sigma(q)\geq j}} x_{i,\sigma(j)+1} + \sum_{\substack{p,q=1,\dots,n-1 \\ p\geq i,\sigma(q)\geq j}} x_{i+1,\sigma(j)+1} \Bigg) \end{aligned}$$

Example 4.10. As a special case of the previous example we find that the coordinate ring of the multilplicative group $\mathbb{G}_{\mathrm{m}} = (k^\times, \cdot) = \mathrm{GL}_1(k)$ is given by

$$\mathrm{A}(\mathbb{G}_{\mathrm{m}}) = k[x, x^{-1}]$$

with the comultiplication Δ, counit ε and antipode S being given on the generators x, x^{-1} by

$$\Delta\left(x^{\pm 1}\right) = x^{\pm 1} \otimes x^{\pm 1}\,, \qquad \varepsilon\left(x^{\pm 1}\right) = 1\,, \qquad S\left(x^{\pm 1}\right) = x^{\mp 1}\,.$$

Remark 4.11. One can show that the contravariant equivalence of categories

$$\mathrm{A}(-) : \{\text{affine varieties}\} \longrightarrow \left\{\begin{matrix}\text{finitely generated,} \\ \text{commutative, reduced} \\ k\text{-algebras}\end{matrix}\right\}$$

from Lemma 2.25 induces a contravariant equivalence of categories

$$\mathrm{A}(-) : \{\text{affine algebraic groups}\} \longrightarrow \left\{\begin{matrix}\text{finitely generated,} \\ \text{commutative, reduced} \\ k\text{-Hopf algebras}\end{matrix}\right\}.$$

In this sense a group structure on an affine variety G (which is given by morphisms of affine varieties) is 'the same' as a Hopf algebra structure on its coordinate ring $\mathrm{A}(G)$.

5. Embedding Theorems

5.1. We have seen in Corollary 3.5 that every closed subgroup of $\mathrm{GL}_n(k)$ is an affine algebraic group. In this Section we show the converse: Every affine algebraic group is isomorphic to a closed subgroup of some $\mathrm{GL}_n(k)$.

5.2. We motivate the next few statements by first considering a finite group G. Then G acts on the group algebra $k[G]$ via left multplication. This makes $k[G]$ into a faithful representation of G, resulting in an injective group homorophism $G \hookrightarrow \mathrm{GL}(k[G])$. The image of this inclusion is a finite, and thus closed subgroups of $\mathrm{GL}_n(k)$ which is isomorphic to G.

For an affine algebraic group G modify this approach by replacing the group algebra $k[G]$ with the coordinate ring $\mathrm{A}(G)$: For every $g \in G$ the right multiplication

$$r_g \colon G \longrightarrow G, \quad h \longmapsto hg$$

is an isomorphism of affine varieties, which induces an isomorphism of k-algebras

$$\rho_g \colon \mathrm{A}(G) \longrightarrow \mathrm{A}(G), \quad f \longmapsto f \circ r_g = f((-)g).$$

It holds for all $g_1, g_2 \in G$ that

$$\rho_{g_1}\rho_{g_2} = r_{g_1}^* r_{g_2}^* = (r_{g_2} r_{g_1})^* = r_{g_1 g_2}^* = \rho_{g_1 g_2}$$

so $\rho \colon G \longrightarrow \mathrm{GL}(\mathrm{A}(G))$ is a group homomorphism.

Since the coordinate ring $\mathrm{A}(G)$ is in general infinite-dimensional (unless G is finite) we start by showing that $\mathrm{A}(G)$ contains a finite-dimensional G-invariant subspace.

We will use the following observation: Dually to how the right multiplication multiplication r_g can be expressed via the multiplication $m \colon G \times G \longrightarrow G$ we can express the action $\rho_g = r_g^*$ via m^* und thus via Δ. If $\Delta(f) = \sum_{i=1}^n h_i \otimes a_i$ with $f_i, a_i \in \mathrm{A}(G)$ then it follows from (4.3) that

$$\rho_g(f) = r_g^*(f) = f \circ r_g = f((-)g) = \sum_{i=1}^n h_i(-)a_i(g) = \sum_{i=1}^n a_i(g)h_i \tag{5.1}$$

is a linear combination of $h_1, \dots, h_n$ with coefficients $a_1(g), \dots, a_n(g)$.

Lemma 5.3. Let $V \subseteq \mathrm{A}(G)$ be a linear subspace.

1) The linear subspace V is G-invariant if and only if $\Delta(V) \subseteq V \otimes \mathrm{A}(G)$.

2) If V is finite-dimensional then there exists a finite-dimensional linear subspace $W \subseteq \mathrm{A}(G)$ with $V \subseteq W$.

Proof.

1) Let $(h_i \mid i \in I)$ be a basis of V and extend this to a basis $(h_j \mid j \in J)$ of $\mathrm{A}(G)$ with $I \subseteq J$. For $f \in \mathrm{A}(G)$ the element $\Delta(f)$ can be written as $\Delta(f) = \sum_{j \in J} h_j \otimes a_{f,j}$ for unique elements $a_{f,j} \in \mathrm{A}(g)$, and it follows from (5.1) that

$$\rho_g(f) = \sum_{j \in J} a_{f,j}(g) h_j.$$

It follows that

$$\begin{aligned}
&V \text{ is } G\text{-invariant} \\
\iff\ & \rho_g(f) \in V \text{ for all } g \in G,\ f \in V \\
\iff\ & a_{f,j}(g) = 0 \text{ for all } j \in J \setminus I,\ g \in G,\ f \in V \\
\iff\ & a_{f,j} = 0 \text{ for all } j \in J \setminus I,\ f \in V \\
\iff\ & \Delta(f) \in V \otimes \mathrm{A}(G) \text{ for all } f \in V \\
\iff\ & \Delta(V) \subseteq V \otimes \mathrm{A}(G)\,,
\end{aligned}$$

as desired.

2) We may assume that V is one-dimensional. Let $f \in V$ be nonzero. Then $\Delta(f) = \sum_{i=1}^{n} f_i \otimes a_i$ for some $f_i, a_i \in \mathrm{A}(G)$ and it follows from (5.1) that

$$\rho_g(f) = \sum_{i=1}^{n} a_i(g) f_i$$

for every $g \in G$. This shows for the finite-dimensional linear subspace $W' \subseteq \mathrm{A}(G)$ given by $W' \coloneqq \langle f_1, \dots, f_n \rangle_k$ that $\rho_g(f) \in W'$ for every $g \in G$. It follows for $W \coloneqq \langle \rho_g(f) \mid g \in G \rangle_k$, which is the G-invariant subspace generated by f, from $W \subseteq W'$ that W is also finite-dimensional. □

Theorem 5.4 (Embedding theorem)**.** Every affine algebraic group G is isomorphic to a closed subgroup of some $\mathrm{GL}_n(k)$.

Proof. The coordinate ring $\mathrm{A}(G)$ is finitely generated as a k-algebra. Let $h_1, \dots, h_r \in \mathrm{A}(G)$ be a set of k-algebra generators and set $V \coloneqq \langle h_1, \dots, h_r \rangle_k$. It follows from Lemma 5.3 that V is contained in a finite-dimensional G-invariant subspace $W \subseteq \mathrm{A}(G)$. We may assume that $h_1, \dots, h_r$ are linearly independent and extend this family to a basis $B = (h_1, \dots, h_n)$ of W. For every $g \in G$ let $R(g) \in \mathrm{GL}_n(k)$ be the matrix representation of ρ_g with respect to the basis B. In this way the representation $\rho \colon G \longrightarrow \mathrm{GL}(W)$ becomes a matrix representation (i.e. group homomorphism) $R \colon G \longrightarrow \mathrm{GL}_n(k)$.

It follows by Lemma 5.3 from the G-invariance of W that $\Delta(W) \subseteq W \otimes \mathrm{A}(G)$. We may therefore write for every $j = 1, \dots, n$ the element $\Delta(h_j)$ as

$$\Delta(h_j) = \sum_{i=1}^{n} h_i \otimes a_{ij}$$

for unique elements $a_{ij} \in \mathrm{A}(G)$. It follows from (5.1) that

$$\rho_g(h_j) = \sum_{i=1}^{n} a_{ij}(g) h_i$$

This show shows that the entries of R are given by the regular functions a_{ij}, which shows that the group homomorphism R is regular and thus a morphism of affine algebraic groups.

It remains to show that the image of R is closed in $\mathrm{GL}_n(k)$ and that R restricts to an isomorphism of affine varieties $G \longrightarrow \mathrm{im}(R)$. According to Corollary 1.48 it sufficies to show that the induced algebra homomorphism f^* is surjective. It follows from $f^*(x_{ij}) = a_{ij}$ that the coefficient functions a_{ij} are contained in the image of f^*. It follows with

$$h_j = h_j(1 \cdot (-)) = \sum_{i=1}^{n} h_i(1)a_{ij}(-) = \sum_{i=1}^{n} h_i a_{ij}$$

that $h_1, \dots, h_n$ are contained in the image of f^*. As the elements $h_1, \dots, h_r$ generate $\mathrm{A}(G)$ as a k-algebra it further follows that f^* is surjective. □

5.5. We have now seen that affine algebraic groups are (up to isomorphism) the same as closed subgroups of $\mathrm{GL}_n(k)$. Affine algebraic groups are therefore also know as *linear algebraic groups*. We will therefore use this term (instead of 'affine algebraic group') throughout the rest of this notes.

Remark 5.6. One may think about the embedding theorem as anagalogous to

- Cayley's theorem, which ensures that every finite group can be embedded into some S_n;
- Whitney's theorem, which ensures that every manifold M can be embedded into some $\mathbb{R}^n$;
- and Ado's theorem, which ensures that every finite-dimensional Lie algebra over a field k can be embedded into some $\mathfrak{gl}_n(k)$.

Corollary 5.7 (Sharpening of the embedding)**.** Let G be a linear algebraic group and let $H \leq G$ be a closed subgroup. Then G is isomorphic to a closed subgroup G' of some $\mathrm{GL}(W)$, for W a finite-dimensional-k-vector space, such that

$$H = \mathrm{Stab}(W_H) = \{g \in G \mid \rho_g(W_h) \subseteq W_H\}.$$

for some k-linear subspace $W_H \subseteq W$.

Proof. Let $I \coloneqq \mathrm{I}_G(H)$. Let $f_1, \dots, f_r$ be a linearly independent generating set for the ideal I. We may extend this to a linearly independent algebra generating set $f_1, \dots, f_s$ for $\mathrm{A}(G)$. By Lemma 5.3 there exists a finite-dimensional G-invariant subspace $W \subseteq \mathrm{A}(G)$ which contains $f_1, \dots, f_s$. We may extend $f_1, \dots, f_s$ to a basis $f_1, \dots, f_n$ of W. We set $W_H \coloneqq W \cap I$.

We find as in the proof of the embedding theorem that $\rho\colon G \longrightarrow \mathrm{GL}(W)$ restrict to an isomorphism $G \longrightarrow \mathrm{im}(\rho)$ with $\mathrm{im}(\rho)$ being closed in $\mathrm{GL}(W)$

It follows from $\mathrm{V}(I) = \mathrm{V}(\mathrm{I}(H)) = H$ (because H is closed) for every $x \in G$ that $x \in H$ if and only if $\varphi(x) = 0$ for every $\varphi \in I$. It follows for every $g \in G$ that

$$\begin{aligned}
& g \in H \\
\iff & gh \in H \text{ for every } h \in H \\
\iff & r_g(h) \in H \text{ for every } h \in H \\
\iff & \varphi(r_g(h)) = 0 \text{ for all } \varphi \in I, h \in H \\
\iff & \rho_g(\varphi)(h) = 0 \text{ for all } \varphi \in I, h \in H
\end{aligned}$$

$$\begin{aligned}
&\iff \rho_g(\varphi) \in I \text{ for every } \varphi \in I \\
&\iff \rho_g(I) \subseteq I \\
&\iff \rho_g(W_H) \subseteq W_H \qquad\qquad (5.2)\\
&\iff g \in \operatorname{Stab}(W_H).
\end{aligned}$$

Note for the equivalence (5.2) that if $\rho_g(I) \subseteq I$ then also

$$\rho_g(W_H) = \rho_g(W \cap I) \subseteq \rho_g(W) \cap \rho_g(I) \subseteq W \cap I = W_H$$

because W is G-invariant. If on the other hand $\rho_g(W_H) \subseteq W_H$ then the generators $f_1, \dots, f_r$ of the ideal I, which are contained in both W and I and therefore also in W_H, are mapped into $W_H \subseteq I$ by the algebra automorphism ρ_g. It then follows that the whole of I is again mapped into I by ρ_g. □

Recall 5.8. Let V, W be k–vector spaces and let $d \geq 0$.

1) If $e_1, \dots, e_n$ is a basis of V then the elements $e_{i_1} \wedge \dots \wedge e_{i_d}$ with $1 \leq i_1 < \dots < i_d \leq n$ form a basis for $\bigwedge^d V$. It follows in particular that $\dim \bigwedge^d V = \binom{n}{d}$ (where $\binom{n}{d} = 0$ for $d > n$).

2) It also follows that if $U \subseteq V$ an an n-dimensional linear subspace then $\bigwedge^d U$ can be regarded as a linear subspace of $\bigwedge^d V$ of dimension $\binom{n}{d}$ (with $\binom{n}{d} = 0$ for $d > n$). If $d \leq n$ then the map

$$\begin{aligned}
\{n\text{-dimensional subspaces of } V\} &\longrightarrow \left\{\binom{n}{d}\text{-dimensional subspaces of } \bigwedge^d V\right\}, \\
U &\longmapsto \bigwedge^d U
\end{aligned}$$

in injective.

To show this injectivity let U_1 and U_2 be two distinct d-dimesional linear subspaces of V. Let $e_1, \dots, e_r$ with $r < n$ be a basis of $U_1 \cap U_2$ and extend this to a basis $e_1, \dots, e_r, e'_{r+1}, \dots, e'_n$ of U_1 and also to a basis $e_1, \dots, e_r, e''_{r+1}, \dots, e''_n$ of U_2. Then $e_1, \dots, e_r, e'_{r+1}, \dots, e'_n, e''_{r+1}, \dots, e''_n$ is a basis for $U_1 + U_2$. The induced basis for $\bigwedge^d (U_1 + U_2)$ then contains the induced bases for $\bigwedge^d U_1$ and $\bigwedge^d U_2$ as two distinct subsets. Any two distinct subsets of a linearly independent family of vectors span distinct subspaces, so it follows that $\bigwedge^d U_1$ and $\bigwedge^d U_2$ are different.

3) Let $f\colon V \longrightarrow W$ be a k-linear map and let A be the matrix which represents f with respect to a basis $v_1, \dots, v_n$ of V and a basis $w_1, \dots, w_m$ of W. Then the coefficients of the induced linear map $\bigwedge^d f$ with respect to the induced bases of $\bigwedge^d V$ and $\bigwedge^d W$ are given by the $d \times d$ minors of the matrix A.

Let's be a bit more precise: It holds for all $1 \leq j_1 < \dots < j_d \leq n$ that

$$\left(\bigwedge^d f\right)\left(v_{j_1} \wedge \dots \wedge v_{j_d}\right)$$

$$
\begin{aligned}
&= f(v_{j_1}) \wedge \cdots \wedge f(v_{j_d}) \\
&= \left(\sum_{i=1}^{m} A_{ij_1} w_i \right) \wedge \cdots \wedge \left(\sum_{i=1}^{m} A_{ij_d} w_i \right) \\
&= \sum_{i_1,\dots,i_d=1}^{m} A_{i_1 j_1} \cdots A_{i_d j_d} w_{i_1} \wedge \cdots \wedge w_{i_d} \\
&= \sum_{\substack{i_1,\dots,i_d=1,\dots,m \\ \text{pairwise different}}} A_{i_1 j_1} \cdots A_{i_d j_d} w_{i_1} \wedge \cdots \wedge w_{i_d} \\
&= \sum_{1 \leq i_1 < \cdots < i_d \leq m} \sum_{\sigma \in \mathrm{S}_d} \operatorname{sgn}(\sigma) A_{i_{\sigma(1)} j_1} \cdots A_{i_{\sigma(d)} j_d} w_{i_1} \wedge \cdots \wedge w_{i_d} \,.
\end{aligned}
$$

This shows that for all $1 \leq i_1 < \cdots < i_d \leq m$ the coefficient of the basis element $w_{i_1} \wedge \cdots \wedge w_{i_d}$ in the image element $f(v_{j_1} \wedge \cdots \wedge v_{j_d})$ is given by

$$
\sum_{\sigma \in \mathrm{S}_d} \operatorname{sgn}(\sigma) A_{i_{\sigma(1)} j_1} \cdots A_{i_{\sigma(d)} j_d} \,,
$$

which is precisely the $d \times d$ minor of A which corresponds to the rows $i_1, \dots, i_d$ and the colums $j_1, \dots, j_d$.

It follows in particular that the group homomorphism

$$
\mathrm{GL}(V) \longrightarrow \mathrm{GL}\left(\bigwedge^d V \right), \quad f \longmapsto \bigwedge^d f
$$

is regular, and thus a homorphism of linear algebraic groups.

Lemma 5.9 (Chevalley)**.** Let G be a linear algebra group and let $H \leq G$ be a closed subgroup. Then there exists for some finite-dimenisonal k-vector space V a homomorphism of linear algebraic groups $G \longrightarrow \mathrm{GL}(V)$ such that $H = \mathrm{Stab}(L)$ for some one-dimensional linear subspace $L \subseteq V$.

Proof. By Corollary 5.7 we may assume that $G \subseteq \mathrm{GL}(W)$ is a closed subgroup for some finite-dimensional k-vector space W and that $H = \mathrm{Stab}(H)$ for some linear subspace $W_H \subseteq W$ of dimension $d \coloneqq \dim(W_H)$. We then consider $V \coloneqq \bigwedge^d W$ and the homomorphism of linear algebraic groups

$$
G \longrightarrow \mathrm{GL}(V), \quad g \longmapsto \bigwedge^d g \,.
$$

Then linear subspace $L \coloneqq \bigwedge^d W_H$ of V is one-dimensional and if $h \in H = \mathrm{Stab}(W_H)$ then $h \in \mathrm{Stab}(L)$. Suppose on the other hand that $g \in G$ with $g \in \mathrm{Stab}(L)$. Then W_H and $g(W_H)$ are two d-dimensional linear subspaces of W with

$$
\bigwedge^d W_H = L = gL = \left(\bigwedge^d g \right)(L) = \left(\bigwedge^d g \right) \left(\bigwedge^d W_H \right) = \bigwedge^d g(W_H)
$$

it follows that $W_H = gW_H$. This then shows that $g \in \mathrm{Stab}(W_H) = H$. Together this shows that $\mathrm{Stab}(V) = \mathrm{Stab}(W_H) = H$ as desired. □

Proposition 5.10. Let G be a linear algebraic group and let $H \trianglelefteq G$ be a closed normal subgroup. Then there exists for some finite-dimensional k-vector space V a homomorphism of affine algebraic groups $G \longrightarrow \mathrm{GL}(V)$ with kernel H.

Proof. By Chevalley's lemma we may start with a homomorphism $\varphi\colon G \longrightarrow \mathrm{GL}(W)$ such that $H = \mathrm{Stab}(L)$ for some one-dimensional linear subspace $L \subseteq W$. It follows that every nonzero vector $w \in L$ is a common eigenvector of all H, i.e. there exists for every $h \in H$ a scalar $\chi(h) \in k^\times$ with $hv = \chi(h)v$. (The map $\chi_w\colon G \longrightarrow k^\times$ is a group homomorphism but we won't need this here.)

Let $W' \subseteq W$ be the linear subspace generated by the common eigenvectors for H. If w' is a common eigenvectors for all $h \in H$ then it follows for every $g \in G$ that gw' is again a common eigenvector for all $h \in G$ since

$$hgw' = g\underbrace{g^{-1}hg}_{\in H}w' = g\chi_{w'}(g^{-1}hg)w' = \chi_{w'}(g^{-1}hg)gw' \,. \tag{5.3}$$

This shows that W' is G-invariant and that the set of common eigenvectors for H is generating set for W'. We may use the G-invariance of W' and that $L \subseteq W'$ to replace W by W'. It then follows from W being k-generated by common eigenvectors of H that there exists a decomposition

$$W = W_1 \oplus \cdots \oplus W_n$$

into common eigenspaces $W_1, \dots, W_n$ of the $h \in H$, i.e. every V_i is a k-linear subspace of W and every $h \in H$ acts on every V_i by some scalar $\chi_i(h) \in k^\times$. The map $\chi_i\colon H \longrightarrow k^\times$ is a group homomorphism and it follows from $k^\times$ being abelian that χ_i is constant on conjugacy classes. It therefore follows from the calculation done in (5.3) that

$$gV_i \subseteq V_i \tag{5.4}$$

for all i.

The action of G on W induces an action of G on $\mathrm{End}_k(W)$ by conjugation given by

$$g \cdot f = \varphi(g) \circ f \circ \varphi(g)^{-1}$$

for all $f \in \mathrm{End}_k(W)$. It follows from $\varphi\colon G \longrightarrow \mathrm{GL}(W)$ being regular that the corresponding group homomorphism $\psi\colon G \longrightarrow \mathrm{GL}(\mathrm{End}_k(W))$ is also regular.

We now consider the linear subspace $V \subseteq \mathrm{End}_k(W)$ given by

$$V = \{f \in \mathrm{End}_k(W) \mid f(V_i) \subseteq V_i \text{ for every } i\} = \prod_{i=1}^n \mathrm{End}_k(V_i) \,.$$

This linear subspace V of $\mathrm{End}_k(W)$ is G-invariant: If $f \in V$ and $g \in G$ then it follows from (5.4) that

$$(g \cdot f)(V_i) = gf(g^{-1}V_i) \subseteq gf(V_i) \subseteq gV_i = V_i \,.$$

The conjugation action of G on $\mathrm{End}_k(W)$ therefore restricts to an aciton of G on V, which corresponds to a regular group homomorphism $\rho\colon G \longrightarrow \mathrm{GL}(V)$.

We claim that $\ker(\rho) = H$. It holds that $H \subseteq \ker(\rho)$ because every $h \in H$ acts on every V_i by multiplication with some scalar The needed equality $\varphi(h) \circ f \circ \varphi(h)^{-1}$ for $h \in H$ and $f \in V$

therefore holds on every V_i, and thus on all of V. If on the other hand $g \in G$ with $g \in \ker(\rho)$ then it follows that $\varphi(g)$ must be in the center of $\prod_{i=1}^{n} \mathrm{End}_k(V_i)$, which is given by

$$\mathrm{Z}\left(\prod_{i=1}^{n} \mathrm{End}_k(V_i)\right) = \prod_{i=1}^{n} \mathrm{Z}(\mathrm{End}_k(V_i)) = \prod_{i=1}^{n} k\,.$$

This means that $\varphi(g)$ acts on every V_i by some scalar. This holds in particular for the one-dimensional subspace L, which is contained in some V_i (namely the one for $\chi_i = \chi_w$ with $w \in L$ nonzero). It then holds that $g \in \mathrm{Stab}(L) = H$. $\square$

III. Jordan–Chevalley Decomposition

6. Jordan–Chevalley for (Finite) Endomorphisms

6.1. Semisimple Endomorphisms

Definition 6.1. An endomorphism $g\colon V \longrightarrow V$ of a k-vector space V is *semisimple* if it is diagonalizable.[1]

Lemma 6.2. Let $g\colon V \longrightarrow V$ be an endomorphism of a k-vector space V.

1) If g is semisimple and $W \subseteq V$ is a g-invariant subspace then the restriction $g|_W$ is again semisimple.

2) If g is semisimple and $W \subseteq V$ is a g-invariant subspace then the induced endomorphism $V/W \longrightarrow V/W$ is again semisimple.

3) If $V = \bigcup_{i \in I} W_i$ is a cover by g-invariant subspaces $W_i \subseteq V$ such that the restriction $g|_{W_i}$ is semisimple for every $i \in I$ then g is semisimple. $\square$

Lemma 6.3. If $g_1, g_2\colon V \longrightarrow V$ are two semisimple endomorphisms of a k-vector space V which commute with each other then $g_1 + g_2$ and $g_1 \circ g_2$ are again semisimple.

Proof. This holds because g_1 and g_2 are simultaneously diagonalizable. $\square$

6.2. Additive Jordan–Chevalley Decomposition

Proposition 6.4 (Additive Jordan–Chevalley decomposition)**.** Let $g\colon V \longrightarrow V$ be an endomorphism of a finite-dimensional k-vector space V.

1) There exists unique endomorphisms $g_s, g_n\colon V \longrightarrow V$ with $g = g_s + g_n$ such that g_s is semisimple, g_n is nilpotent and g_s and g_n commute with each other.

2) There exist polynomials $P, Q \in k[t]$ without constant part such that $g_s = P(g)$ and $g_n = Q(g)$.

3) An endomorphism $h\colon V \longrightarrow V$ commutes with g if and only if it commutes with both g_s and g_n.

[1] One can more generally define an endomorphism $g\colon V \longrightarrow V$ to be semisimple if it makes V into a semisimple $k[x]$-module when x acts via g. If k is algebraically closed then this is equivalent to g being diagonalizable, so we will work with this definition.

Proof. Let $\chi_g \in k[t]$ be the characteristic polynomials of g and let

$$\chi_g(t) = (t - \lambda_1)^{n_1} \cdots (t - \lambda_r)^{n_r}$$

be the decomposition of g into linear factors with $\lambda_1, \dots, \lambda_r \in k$ being the pairwise different eigenvalues of g. For every $i = 1, \dots, r$ let $V_i \coloneqq \ker(g - \lambda_i \operatorname{id}_V)^{n_i}$ be the generalized eigenspace of g with respect to the eigenvalues λ_i. It then holds that

$$V = V_1 \oplus \cdots \oplus V_r . \tag{6.1}$$

The polynomials $(t - \lambda_1)^{n_1}, \dots, (t - \lambda_r)^{n_r}$ are pairwise coprime, so it follows from the chinese remainder theorem that there exist some polynomial $P \in k[t]$ with

$$P(t) \equiv \lambda_i \mod (t - \lambda_i)^{n_i} \tag{6.2}$$

for all $i = 1, \dots, r$. We may also assume that

$$P(t) \equiv 0 \mod t;$$

if 0 is an eigenvalue of g then this follows from (6.2) and otherwise we may add t to the list of coprime polynomials $(t - \lambda_1)^{n_1}, \dots, (t - \lambda_r)^{n_r}$. We thus have $P(0) = 0$.

We set $g_s \coloneqq P(g)$. It follows from (6.2) for every i that

$$P(t) = \lambda_i + P_i'(t)(t - \lambda_i)^{n_i}$$

for some $P' \in k[t]$, and therefore that

$$g_s = P(g) = \lambda_i \operatorname{id}_V + P_i'(g)(g - \lambda_i \operatorname{id}_V)^{n_i} .$$

It follows that g_s acts on V_i by multiplication with the scalar λ_i because V_i is annihilated by $(g - \lambda_i \operatorname{id}_V)^{n_i}$. This shows that (6.1) is the decomposition of V into eigenspaces of g_s and therefore that g_s is diagonalizable.

We also set $Q(t) \coloneqq t - P(t)$ and $g_n \coloneqq Q(g) = g - g_s$. It follows from $P(0) = 0$ that also $Q(0) = 0$. It holds that $g = g_s + g_n$ and the endomorphisms g_s and g_n commute because they are both polynomials in g. As g_s acts on V_i by multiplication with λ_i it follows that $g_n = g - g_s$ acts on V_i by $g - \lambda \operatorname{id}_V$, and thus nilpotent. It does so for every $i = 1, \dots, n$, which shows by (6.1) that g_n is nilpotent.

Altogether this shows part 2) and the existence for part 1). Part 3) follows from part 2).

To show the uniquenes for part 1) let $g_s', g_n' \colon V \longrightarrow V$ be another pair of endomorphisms satisfying $g = g_s' + g_n'$ with g_s' being semisimple, g_n' being nilpotend and g_s' and g_n' commuting. It then follows that g_s' and g_n' commute wtih $g = g_s' + g_n'$ and therefore also with g_s and g_n by part 3). It also follows from

$$g_s + g_n = g = g_s' + g_n'$$

that

$$g_s - g_s' = g_n' - g_n .$$

The left hand side of this equation is again semisimple as both g_s and g'_s are semisimple and commute and are therefore simultaneously diagonalizable. The right hand side of the equation is nilpotent as both g'_n and g_n and nilpotent and they commute. The only diagonalizable nilpotent endomorphism is the zero endomorphism, so it follows that

$$g_s - g'_s = g'_n - g_n = 0$$

and thus $g_s = g'_s$ and $g_n = g'_n$. □

Definition 6.5. Let $g\colon V \longrightarrow V$ be an endomorphism of a finite-dimensional k-vector space V. The unique decomposition $g = g_s + g_n$ from Proposition 6.4 is the *Jordan–Chevalley decomposition* of g. The summand g_s is the *semisimple part* of g and the summand g_n is the *nilpotent part* of g.

Lemma 6.6. Let $g\colon V \longrightarrow V$ be an endomorphism of a finite-dimensional k-vector space V and let $W \subseteq V$ be a g-invariant subspace.

1) The subspace W is also g_s-invariant and g_n-invariant.

2) It holds that $(g|_W)_s = g_s|_W$ and $(g|_W)_n = g_n|_W$.

3) It holds for the induced endomorphisms $\overline{g}, \overline{g_s}, \overline{g_n}\colon V/W \longrightarrow V/W$ that $\overline{g} = \overline{g_s} + \overline{g_n}$ is the Jordan–Chevalley decomposition of $\overline{g}$.

4) Let $g'\colon V' \longrightarrow V'$ be another endomorphism of a finite-dimensional k-vector space V and let $f\colon V \longrightarrow V'$ be a k-linear map with $f \circ g = g' \circ f$, i.e. such that the diagram

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ V' \arrow[r, "g'"] & V' \end{tikzcd} \tag{6.3}$$

commutes. It then holds that $f \circ g_s = g'_s \circ f$ and $f \circ g_n = g'_n \circ f$, so that the diagram (6.3) splits up in the following two commutative diagrams:

$$\begin{tikzcd} V \arrow[r, "g_s"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ V' \arrow[r, "g'_s"] & V' \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "g_n"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ V' \arrow[r, "g'_n"] & V' \end{tikzcd}$$

Proof.

1) The subspace W is g_s-invariant and g_n-invariant because g_s and g_n are polynomials in g.

2) It follows from $g = g_s + g_n$ that $g|_W = g_s|_W + g_n|_W$. The endomorphisms $g_s|_W$ and $g_n|_W$ commute with each other because g_s and g_n commute, the restriction $g_s|_W$ is again semisimple by Lemma 6.2 and the restriction $g_n|_W$ is again nilpotent.

3) It follows from $g = g_s + g_n$ that $\overline{g} = \overline{g_s} + \overline{g_n}$. The endomorphisms $\overline{g_s}$ and $\overline{g_n}$ commute with each other because g_s and g_n commute, the induced endomorphism $\overline{g_s}$ is again semisimple by Lemma 6.2 and $\overline{g_n}$ is again nilpotent.

4) It follows from part 2) that the claim holds when f in injective and it follows from part 3) that the claim holds when f is surjective. For $V'' \coloneqq \operatorname{im}(f)$ the k-linear map $f\colon V \longrightarrow V'$ factorizes as
$$V \xrightarrow{i} V'' \xrightarrow{p} V'$$
with i injective and p surjective. Together with the restriction $g'' : V'' \longrightarrow V''$ of g' to V' this results in the following commutative diagram:
$$\begin{tikzcd}
V \arrow[r, "g"] \arrow[d, "i"'] \arrow[dd, bend right, "f"'] & V \arrow[d, "i"] \arrow[dd, bend left, "f"] \\
V'' \arrow[r, "g''"] \arrow[d, "p"'] & V'' \arrow[d, "p"] \\
V' \arrow[r, "g'"] & V'
\end{tikzcd}$$
Since i is injective and p is surjective this gives the following two commutative diagrams:
$$\begin{tikzcd}
V \arrow[r, "g_s"] \arrow[d, "i"'] \arrow[dd, bend right, "f"'] & V \arrow[d, "i"] \arrow[dd, bend left, "f"] \\
V'' \arrow[r, "g''_s"] \arrow[d, "p"'] & V'' \arrow[d, "p"] \\
V' \arrow[r, "g'_s"] & V'
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "g_u"] \arrow[d, "i"'] \arrow[dd, bend right, "f"'] & V \arrow[d, "i"] \arrow[dd, bend left, "f"] \\
V'' \arrow[r, "g''_u"] \arrow[d, "p"'] & V'' \arrow[d, "p"] \\
V' \arrow[r, "g'_u"] & V'
\end{tikzcd}$$
The commutativity of the outer diagrams shows the claim. □

6.3. Multiplicative Jordan–Chevalley Decomposition

Definition 6.7. An endomorphism $g : V \longrightarrow V$ of a k-vector space V is *unipotent* if $g - \mathrm{id}_V$ is nilpotent.

Remark 6.8. If more generally $\lambda \in k$ then an endomorphism $g : V \longrightarrow V$ is *λ-potent* if $g - \lambda \, \mathrm{id}_V$ is nilpotent. Then nilpotent is equivalent to 0-potent and unipotent is equivalent to 1-potent. Note that if V is finite-dimensional then g is nilpotent if and only if 0 is the only eigenvalue of g (because k is algebraically closed) and thus g is λ-potent if and only if λ is the only eigenvalue of g.

Lemma 6.9. Let V be a k-vector space and let $g, h : V \longrightarrow V$ be endomorphisms such that g is invertible. Then g commutes with h if and only if g^{-1} commutes with h.

Proof. The equality $gh = hg$ can be transformed into the equivalent equation $hg^{-1} = g^{-1}h$ by multiplying it from both sides with g^{-1}. □

Lemma 6.10. Let V be a k-vector space, let $u\colon V \longrightarrow V$ be invertible and let $n\colon V \longrightarrow V$ be nilpotent such that u and n commute. Then $u - n$ is again invertible.

Proof. It holds for $u = \mathrm{id}_V$ that $\mathrm{id}_V - n$ in invertible with

$$(\mathrm{id}_V - n)^{-1} = \sum_{i=0}^{\infty} n^i .$$

It follows in the general case that u^{-1} and n also commute, from which it then follows that $u^{-1}n$ is again nilpotent and therefore that

$$u - n = u(\mathrm{id}_V - u^{-1}n)$$

is again invertible. ☐

Corollary 6.11. Unipotent endomorphisms are invertible. ☐

Corollary 6.12. Let $g\colon V \longrightarrow V$ be an endomorphism of a finite-dimensional k-vector space V with Jordan–Chevalley decomposition $g = g_s + g_n$. Then g is invertible if and only if g_s is invertible.

Proof. This follows from Lemma 6.10 because $g_s = g - g_n$ and $g = g_s - (-g_n)$ and $\pm g_n$ commutes with both g and g_s. ☐

Lemma 6.13. Let $g \in \mathrm{GL}(V)$ for a finite-dimensional k-vector space V. Then g^{-1} is a polynomial in g, i.e. there exists some $P \in k[t]$ with $g^{-1} = P(g)$.

Proof. It holds for the characteristic polynomial $\chi_g(t) = \sum_{i=1}^{n} a_i t^i$ that $\chi_g(g) = 0$ by the Cayley–Hamilton theorem. It also holds that $a_0 = \pm\det(g) \neq 0$. It follows that

$$g^{-1} = -\frac{1}{a_0}\sum_{i=1}^{n} a_i g^{i-1} = P(g)$$

for the polynomial $P(t) \coloneqq -\frac{1}{a_0}\sum_{i=0}^{n-1} a_{i+1} t^i$. ☐

Proposition 6.14 (Multiplicative Jordan–Chevalley decomposition)**.** Let $g \in \mathrm{GL}(V)$ where V is a finite-dimensional k-vector space.

1) There exist unique endomorphisms $g_s, g_u\colon V \longrightarrow V$ with $g = g_s g_u$ such that the factor g_s is semisimple, the factor g_u is unipotent and the factors g_s and g_u commute with each other.

2) The factors g_s and g_u are again invertible.

3) The factor g_s is the semisimple part of g, and the factor g_u is given by $g_u = \mathrm{id}_V + g_s^{-1} g_n$.

4) There exist polynomials $P, Q \in k[t]$ with $g_s = P(g)$ and $g_u = Q(g)$.

5) An element $h \in \mathrm{GL}(V)$ commutes with g if and only if it commutes with both g_s and g_u.

Proof. It follows for the additive Jordan–Chevalley decomposition $g = g_s + g_n$ from Corollary 6.12 that g_s is invertible. We may therefore write

$$g = g_s + g_n = g_s ae(\mathrm{id}_V + g_s^{-1} g_n) = g_s g_u$$

for $g_u \coloneqq \mathrm{id}_V + g_s^{-1} g_n$. The endomorphism g_u is unipotent because $g_s^{-1} g_n$ is again nilpotent since g_s^{-1} and g_n again commute, and the elements g_s and g_u commute because

$$g_s g_u = g_s + g_n = (\mathrm{id}_V + g_n g_s^{-1}) g_s = (\mathrm{id}_V + g_s^{-1} g_n) g_s = g_u g_s \,.$$

This shows the claimed existence for part 1), and also shows part 2) and part 3).

To show the uniqueness for part 1) let $g = g'_s g'_u$ be another decomposition with g'_s semisimple, g'_u unipotent and g'_s and g'_u commuting. We may write

$$g = g'_s g'_u = g'_s + g'_s(g'_u - \mathrm{id})$$

with $g'_n \coloneqq g'_s(g'_u - \mathrm{id})$ being nilpotent (because $g'_u - \mathrm{id}_V$ is nilpotent and commutes with g') and commuting with g'_s. It then follows from the uniqueness of the additive Jordan–Chevalley decomposition that $g'_s = g_s$ and that $g'_n = g_n$, and therefore also that $g'_u = g_u$.

Both g_s and g_n are polynomials in g. It then follows that g_s^{-1} is a polynomial in g because it is a polynomial in g_s, and it further follows that g_u is a polynomial in g. This shows part 4)

Part 5) follows from part 4). □

Definition 6.15. Let V be a finite-dimensional k-vector space and let $g \in \mathrm{GL}(V)$. The decomposition $g = g_s g_u$ from Proposition 6.14 is the *multiplicative Jordan–Chevalley decomposition* of g. The factor g_u is the *unipotent part* of g.

Lemma 6.16. Let V be a finite-dimensional k-vector space and let $W \subseteq V$ be a g-invariant subspace for $g \in \mathrm{GL}(V)$.

1) The subspace W is also g_s-invariant and g_u-invariant.

2) The restriction $g|_W$ is again invertible and it holds that $(g|_W)_s = g_s|_W$ and $(g|_W)_u = g_u|_W$.

3) It holds for the induced endomorphisms $\overline{g}, \overline{g_s}, \overline{g_u} \colon V \longrightarrow V$ that $\overline{g}, \overline{g_s}, \overline{g_u} \in \mathrm{GL}(V/W)$ and that the Jordan–Chevalley decomposition decompositon of $\overline{g}$ is given by $\overline{g} = \overline{g_s} \circ \overline{g_u}$.

4) Let $g' \in \mathrm{GL}(V')$ for another finite-dimensional k-vector space V' and let $f \colon V \longrightarrow V'$ be a k-linear map with $f \circ g = g' \circ f$, i.e. such that the diagram

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ V' \arrow[r, "g'"] & V' \end{tikzcd} \tag{6.4}$$

commutes. Then it holds that $f \circ g_s = g_s \circ f'$ and $f \circ g_u = g'_u \circ f$, so that the diagram (6.4) factorizes as follows:

$$\begin{tikzcd}
V \arrow[r, "g_u"] \arrow[d, "f"'] \arrow[rr, bend left, "g"] & V \arrow[r, "g_s"] \arrow[d, "f"] & V \arrow[d, "f"] \\
V' \arrow[r, "g'_u"] \arrow[rr, bend right, "g"'] & V' \arrow[r, "g'_s"] & V'
\end{tikzcd}$$

Proof.

1) That W is also g_s-invariant and g_n-invariant holds because g_s and g_u are polynomials in g.

2) The restriction $g|_W$ is again injective and thus an isomorphism by the finite-dimensionality of W. It follows from $g = g_s g_u$ that $g|_W = g_s|_W g_u|_W$, and the restrictions $g_s|_W$ and $g_u|_W$ commute because g_s and g_u commute. The restriction $g|_s$ is again semisimple by Lemma 6.2 and the restriction g_u is again unipotent. The claimed equalities now follow from the uniqueness of the multiplicative Jordan–Chevalley decomposition.

3) The induced endomorphisms $\overline{g}, \overline{g_s}, \overline{g_u} : V/W \longrightarrow V/W$ are again surjective and therefore isomorphisms by the finite-dimensionality of V/W. It follows from $g = g_s g_u$ that $\overline{g} = \overline{g_s} \circ \overline{g_u}$, the endomorphisms $\overline{g_s}$ and $\overline{g_u}$ commute with each other because g_s and g_u commute, the endomorphism $\overline{g_s}$ is again semisimple by Lemma 6.2 and the endomorphism $\overline{g_u}$ is again unipotent.

4) The claim holds when f is injective by part 2) and when f is surjective by part 3). We can therefore proceed as in the proof of Lemma 6.6. □

7. Jordan–Chevalley for Locally Finite Endomorphisms

7.1. Locally Finite Endomorphisms

Lemma 7.1. For an endomorphism $g : V \longrightarrow V$ of a k-vector space V the following conditions are equivalent:

1) Every $v \in V$ is contained in some finite-dimensional g-invariant subspace.

2) V is the union of some of its finite-dimensional g-invariant subspaces.

3) V is the sum of some of its finite-dimensional g-invariant subspaces.

4) V is the union of all of its finite-dimensional g-invariant subspaces.

5) V is the sum of all of its finite-dimensional g-invariant subspaces.

Proof.

1) $\iff$ 2): This is just a reformulation.

2) $\iff$ 4), and 3) $\iff$ 5): These is clear.

5) $\iff$ 4) The set S of all finite-dimensional g-invariant subspaces of V is directed with respect to the inclusion $\subseteq$, meaning that for any two $W_1, W_2 \in S$ there exists some $W \in S$ with $W_1, W_2 \subseteq W$ (for example $W = W_1 + W_2$). It follows that $\sum_{W \in S} W = \bigcup_{W \in S} W$. □

Definition 7.2. An endomorphism $g \colon V \longrightarrow V$ of a k-vector space V which satisfies the equivalent conditions from Lemma 7.1 is *locally finite*.

Example 7.3. Semisimple endomorphisms are locally finite because for every semisimple endomorphism $g \colon V \longrightarrow V$ the vector space V is a sum one-dimensional g-invariant subspaces (namely the ones spanned by eigenvectors).

7.2. Locally Nilpotent and Locally Unipotent Endomorphisms

Definition 7.4. An endomorphism $g \colon V \longrightarrow V$ of a k-vector space V is *locally nilpotent* (resp. *locally unipotent*) if it acts nilpotent (resp. unipotent) on every $v \in V$, i.e. if there exists for every $v \in V$ some power n such that $g^n(v) = 0$ (resp. $(g - \mathrm{id}_V)^n(v) = 0$).

Remark 7.5. An endomorphism $g \colon V \longrightarrow V$ of a k-vector space V is locally unipotent if and only if $g - \mathrm{id}_V$ is locally nilpotent.

Lemma 7.6. For an endomorphism $g \colon V \longrightarrow V$ of a k-vector space V the following conditions are equivalent:

1) The endomorphism g is locally nilpotent (resp. locally unipotent).
2) The endomorphism g is locally finite and the restriction $g|_W$ is nilpotent (resp. unipotent) for every finite-dimensional g-invariant subspace $W \subseteq V$.
3) There exists a covering $V = \bigcup_{i \in I} W_i$ by finite-dimensional g-invariant subspaces $W_i \subseteq V$ such that the restriction $g|_{W_i}$ is nilpotent (resp. unipotent) for every $i \in I$.

Proof. We first consider the (locally) nilpotent case:

1) $\implies$ 2) Suppose that g is locally nilpotent. Then for every $v \in V$ there exists some $n \geq 1$ with $g^n(v) = 0$ and it follows that $\langle v, g(v), \dots, g^{n-1}(v) \rangle_k$ is a finite-dimensional g-invariant subspace of V which contains v. This shows that g is locally finite.

If $W \subseteq V$ is finite-dimensional g-invariant subspace with finite generating set $w_1, \dots, w_r \in W$ then there exist $n_1, \dots, n_r \geq 1$ with $g^{n_i}(w_i) = 0$ for all i. It follows for $n \coloneqq \max(n_1, \dots, n_r)$ that $(g|_W)^n = 0$, which shows that $g|_W$ is nilpotent.

2) $\implies$ 3) One can choose the covering by all finite-dimensional g-invariant subspaces.

3) $\implies$ 1) There exists for every $v \in V$ some $i \in I$ with $v \in W_i$, and it follows from $g|_{W_i}$ being nilpotent that g acts nilpotent on v.

The (locally) unipotent case follows from the (locally) unipotent one by using that g is locally unipotent if and only if $g - \mathrm{id}_V$ is locally nilpotent, and that g and $g - \mathrm{id}_V$ have the same invariant subspaces. $\square$

Corollary 7.7. Let $g \colon V \longrightarrow V$ is a locally nilpotent (resp. locally unipotent) endomorphism of a finite-dimensional k-vector space V then g is nilpotent (resp. unipotent). $\square$

Lemma 7.8. Let $g \colon V \longrightarrow V$ be an endomorphism of a k-vector space V and let $W \subseteq V$ be a g-invariant subspace. If g is locally finite, locally nilpotent or locally unipotent then the restriction $g|_W$ has the same property.

Proof. If g is locally finite then there exists for every $w \in W$ a finite-dimensional g-invariant subspace $W' \subseteq V$ with $w \in W'$. Then $W \cap W'$ is a finite-dimensional $(g|_W)$-invariant subspace of W which contains w. f If g is locally nilpotent (resp. locally unipotent) then g acts nilpotent (resp. unipotent) on every $v \in V$ and therefore also on every $v \in W$. $\square$

Lemma 7.9. Let $g_1, g_2 \colon V \longrightarrow V$ be two endomorphisms of a k-vector space V which commute with each other.

1) If both g_1 and g_2 are locally nilpotent then the sum $g_1 + g_2$ is again locally nilpotent.

2) If g_1 or g_2 is locally nilpotent then the composition $g_1 \circ g_2$ is again locally nilpotent.

3) If both g_1 and g_2 are locally unipotent then the composition $g_1 \circ g_2$ is again locally unipotent.

Proof.

1) There exist for every $v \in V$ powers n and m with $g_1^n(v) = 0$ and $g_2^m(v) = 0$. It follows that
$$(g_1 + g_2)^{n+m}(v) = \sum_{\ell=0}^{n+m} \left(g_1^\ell \circ g_2^{n+m-\ell}\right)(v) = \sum_{\ell=0}^{n} g_1^\ell\big(\underbrace{g_2^{n+m-\ell}(v)}_{=0}\big) + \sum_{\ell=n+1}^{n+m} g_2^{n+m-\ell}\big(\underbrace{g_1^\ell(v)}_{=0}\big) = 0$$

2) We may assume that g_2 is locally nilpotent. For every $v \in V$ there then exists some power n with $g_2^n(v) = 0$, for which it then follows that
$$(g_1 \circ g_2)^n(v) = g_1^n\big(\underbrace{g_2^n(v)}_{=0}\big) = 0\,.$$

3) It holds that
$$g_1 \circ g_2 - \mathrm{id}_V = (g_1 - \mathrm{id}_V) \circ g_2 + (g_2 - \mathrm{id}_V)$$
The terms $g_1 - \mathrm{id}_V$ and $g_2 - \mathrm{id}_V$ are locally nilpotent, and it follows from part 2) that the composition $(g_1 - \mathrm{id}_V) \circ g_2$ is again locally nilpotent. It further follows from part 1) that $(g_1 - \mathrm{id}_V) \circ g_2 + (g_2 - \mathrm{id}_V)$ is locally nilpotent. This shows that $g_1 \circ g_2 - \mathrm{id}_V$ is locally nilpotent, and therefore that $g_1 \circ g_2$ is locally unipotent. $\square$

7.3. Additive Jordan–Chevalley Decomposition

Lemma 7.10. The only endomorphism $g \colon V \longrightarrow V$ which is both semisimple and locally nilpotent is the zero endomorphism.

Proof. It follows from g being locally nilpotent that 0 is the only (possible) eigenvalue of g. □

Proposition 7.11 (Additive Jordan–Chevalley decomposition)**.** Let $g \colon V \longrightarrow V$ be a locally finite endomorphism of a k-vector space V.

1) There exists unique endomorphisms $g_s, g_n \colon V \longrightarrow V$ with $g = g_s + g_n$ such that g_s is semisimple, g_n is locally nilpotent and g_s and g_n commute with each other.

2) An endomorphism $h \colon V \longrightarrow V$ commutes with g if and only if it already commutes with both g_s and g_n.

3) If $V = \bigcup_{i \in I} W_i$ is any covering by finite-dimensional g-invariant subspaces then the endomorphisms g_s and g_n are uniquely determined by the property that every W_i is also g_s-invariant and g_n-invariant and that $g|_{W_i} = g_s|_{W_i} + g_n|_{W_i}$ is the Jordan–Chevalley decomposition of $g|_{W_i}$ for every $i \in I$.

Proof. It holds for the set

$$\mathscr{F} \coloneqq \{W \subseteq V \mid W \text{ is a finite-dimensional } g\text{-invariant subspace of } V\}$$

that $V = \bigcup_{W \in \mathscr{F}} W$ because g is locally finite. For every $W \in \mathscr{F}$ let $g|_W = g_{s,W} + g_{n,W}$ be the Jordan–Chevalley decomposition of $g|_W$. It holds for all $W_1, W_2 \in \mathscr{F}$ that

$$g_{s,W_1}\big|_{W_2} = g_{s,W_1 \cap W_2} = g_{s,W_2}\big|_{W_1}$$

by Lemma 6.6. It follows that there exists a unique map $g_s \colon V \longrightarrow V$ with $g_s|_W = g_{s,W}$ for every $W \in \mathscr{F}$. This map is k-linear since there exists for all $v_1, v_2 \in V$ some $W \in \mathscr{F}$ with $v_1, v_2 \in W$, with the restriction $g_s|_W = g_{s,W}$ being k-linear. It follows from Lemma 6.2 that the endomorphism g_s is semisimple

It follows similarly that there exists a unique map $g_n \colon V \longrightarrow V$ with $g_n|_W = g_{n,W}$ for all $W \in \mathscr{F}$, which is then k-linear and by Lemma 7.6 locally nilpotent. It holds that $g = g_s + g_n$ and the endomorphisms g_s and g_n commute with each other, because this holds on every subspace $W \in \mathscr{F}$ and V is covered by them. This shows the existence for part 1).

To show part 2) suppose that h commutes with g. We show that the claimed equalities hold at every point $v \in V$. There exists a finite-dimensional g-invariant subspace $W \subseteq V$ with $v \in W$ because g is locally finite, and the subspace $h(W)$ is again g-invariant because g and h commmute. The subspaces W, $h(W)$ together with the actions of g and h on them fit into the following commutative diagram:

$$\begin{tikzcd}
W \arrow[r, "g|_W"] \arrow[d, "h|_W"'] & W \arrow[d, "h|_W"] \\
h(W) \arrow[r, "g|_{h(W)}"] & h(W)
\end{tikzcd}$$

It follows from Lemma 6.6 that the restriction $h|_W$ satisfies the equations

$$h\big|_W \circ \left(g\big|_W\right)_s = \left(g\big|_{h(W)}\right)_s \circ h|_W \quad \text{and} \quad h\big|_W \circ \left(g\big|_W\right)_n = \left(g\big|_{h(W)}\right)_n \circ h|_W .$$

By evaluating these equations at $v \in W$ and using the above construction of g_s and g_n it follows that

$$h(g_s(v)) = g_s(h(v)) \quad \text{and} \quad h(g_n(v)) = g_n(h(v)) ,$$

as desired.

To show the uniqueness for part 1) let $g = g'_s + g'_n$ be another decomposition with g'_s semisimple and g'_n locally nilpotent such that g'_s and g'_n commute. It then follows from part 3) that g'_s and g'_n commute with g_s and g_n because they commute with g. It then follows from

$$g_s + g_n = g = g'_s + g'_n$$

that

$$g_s - g'_s = g'_n - g_n .$$

The endomorphism $g_s - g'_s$ is again semisimple by Lemma 6.3 and the endomorphism $g'_n - g_n$ is again locally nilpotent by Lemma 7.9. It follows from Lemma 7.10 that

$$g_s - g'_n = g'_n - g_n = 0$$

and therefore that $g_s = g'_s$ and $g_n = g'_n$.

To show part 3) let $V = \bigcup_{i \in I} W_i$ be a cover of V by finite-dimensional g-invariant subspaces. Then for every $i \in I$ the restriction $g_s|_{W_i} = g_{s,W_i}$ is semisimple and the restriction $g_n|_{W_i} = g_{n,W_i}$ is nilpotent. It follows for every $i \in I$ from $g = g_s + g_n$ that $g|_{W_i} = g_s|_{W_i} + g_n|_{W_i}$, and it follows that $g_s|_{W_i}$ and $g_n|_{W_i}$ commute because g_s and g_n commute. Together this shows that $g|_{W_i} = g_s|_{W_i} + g_n|_{W_i}$ is the Jordan–Chevalley decomposition of $g|_{W_i}$ for every $i \in I$. The maps g_s and g_n are uniquely determined by this because their actions on the W_i are uniquely determined and these subspaces cover V. □

Definition 7.12. If $g\colon V \longrightarrow V$ is a locally finite endomorphism of a k-vector space V then the decomposition $g = g_s + g_n$ from Proposition 7.11 is the (*additive*) *Jordan–Chevalley decomposition* of g. The summand g_s is the *semisimple part* of g and the summand g_n is the *nilpotent part* of g.

Lemma 7.13. Let $g\colon V \longrightarrow V$ be a locally finite endomorphism of a k-vector space V and let $W \subseteq V$ be a g-invariant subspace.

1) The subspace W is also g_s-invariant and g_n-invariant
2) The Jordan–Chevalley decomposition of the restriction $g|_W$ is given by $g|_W = g_s|_W + g_n|_W$.
3) The Jordan–Chevalley decomposition of the induced endomorphism $\overline{g}\colon V/W \longrightarrow V/W$ is given by $\overline{g} = \overline{g_s} + \overline{g_n}$.

4) Let $g' : V' \longrightarrow V'$ be an endomorphism of another k-vector space V' and let $f\colon V \longrightarrow V'$ be a k-linear map with $f \circ g = g' \circ f$, i.e. such that the diagram

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ V' \arrow[r, "g'"] & V' \end{tikzcd}$$

commutes. Then $f \circ g_s = g_s' \circ f$ and $f \circ g_n = g_n' \circ f$, i.e. the following diagrams commute:

$$\begin{tikzcd} V \arrow[r, "g_s"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ V' \arrow[r, "g_s'"] & V' \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "g_n"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ V' \arrow[r, "g_n'"] & V' \end{tikzcd}$$

Proof.

1) Let $V = \bigcup_{i\in I} V_i$ be a cover by finite-dimensional g-invariant subspaces, and for every $i \in I$ let $W_i \coloneqq V \cap W_i$. Then $W = \bigcup_{i\in I} W_i$ is a cover of W by finite-dimensional g-invariant subspaces because

$$W = W \cap V = W \cap \bigcup_{i\in I} V_i = \bigcup_{i\in I} (W \cap V_i) = \bigcup_{i\in I} W_i \,.$$

It then holds for every $i \in I$ that $g|_{W_i} = g_s|_{W_i} + g_n|_{W_i}$ is the Jordan–Chevalley decomposition of $g|_{W_i}$. It follows in particular that every W_i is both g_s-invariant and g_n-invariant and therefore that $W = \bigcup_{i\in I} W_i$ is both g_s-invariant and g_n-invariant.

2) By continuing the above line of thought we find that $(g_s|_W)|_{W_i} = (g|_{W_i})_s$ for every $i \in I$. It follows from Proposition 7.11 that $g_s|_W = (g|_W)_s$ and similarly for the nilpotent part.

3) It follows from $g = g_s + g_n$ that $\overline{g} = \overline{g_s} + \overline{g_n}$, the restrictions $\overline{g_s}$ and $\overline{g_n}$ commute because g_s and g_n commute, the restriction $\overline{g_s}$ is again semisimple by Lemma 6.2 and the restriction $\overline{g_n}$ is again locally nilpotent.

4) If f is injective then the claim holds by part 2) and if f is surjective then it holds by part 3). The claim follows for general f as in the proof of Lemma 6.6. □

Lemma 7.14. If $g_1, g_2 : V \longrightarrow V$ are two locally finite endomorphisms of a k-vector space V which commute with each other then

$$(g_1 + g_2)_s = (g_1)_s + (g_2)_s \quad \text{and} \quad (g_1 + g_2)_n = (g_1)_n + (g_2)_n \,.$$

Proof. It holds that

$$g_1 + g_2 = (g_1)_s + (g_1)_n + (g_2)_s + (g_2)_n = \big((g_1)_s + (g_2)_s\big) + \big((g_1)_n + (g_2)_n\big) \,.$$

The summands $(g_1)_s + (g_2)_s$ and $(g_1)_n + (g_2)_n$ commute with each other because g_1 and g_2 commute, the summand $(g_1)_s + (g_2)_s$ is semisimple by Lemma 6.3 and the summand $(g_1)_n + (g_2)_n$ is locally nilpotent by Lemma 7.9. □

7.4. Multiplicative Jordan–Chevalley Decomposition

Lemma 7.15. Let g be an endomorphism of a k-vector space V.

1) If g is invertible then for every finite-dimensional g-invariant subspace $W \subseteq V$ the restriction $g|_W$ is again invertible. It then holds that $(g|_W)^{-1} = g^{-1}|_W$.

2) If $V = \bigcup_{i \in I} W_i$ is a cover by g-invariant subspaces then g is invertible if and only if the restriction $g|_{W_i}$ is invertible for every $i \in I$. It then holds that $(g|_{W_i})^{-1} = g^{-1}|_{W_i}$ for every $i \in I$.

Proof.

1) If g is an invertible then the restriction $g|_W$ is injective and thus an isomorphism by the finite-dimensionality of W. The equality $(g|_W)^{-1} = g^{-1}|_W$ holds because both $(g|_W)^{-1}$ and g^{-1} act the same on W.

2) If g is invertible then every restriction $g|_{W_i}$ is invertible by part 1). It holds on the other hand that
$$\ker(g) = \bigcup_{i \in I} \ker\left(g\big|_{W_i}\right) \quad \text{and} \quad \operatorname{im}(g) = \bigcup_{i \in I} \operatorname{im}\left(g\big|_{W_i}\right),$$
from which it follows that g is injective if and only if every restriction $g|_{W_i}$ is injective, and that g is surjective if every restriction $g|_{W_i}$ is surjective. □

Corollary 7.16. Let $g\colon V \longrightarrow V$ be endomorphism of a k-vector space V.

1) If g is locally unipotent then g is invertible.

2) If g is locally finite then g is invertible if and only if its semisimple part g_s is invertible.

Proof. Let $V = \bigcup_{i \in I} W_i$ be a cover of V by finite-dimensional g-invariant subspaces.

1) The restriction $g|_{W_i}$ is unipotent and thus invertible for every $i \in I$, so the invertibility of g follows from Lemma 7.15.

2) Let $g = g_s + g_n$ be the Jordan–Chevalley decomposition of g. Then the Jordan–Chevalley decomposition decompositon of $g|_{W_i}$ is for every $i \in I$ given by $g|_{W_i} = g_s|_{W_i} + g_n|_{W_i}$. It follows from Lemma 7.15 and Corollary 6.12 that
$$\begin{aligned} & g \text{ is invertible} \\ \iff & g|_{W_i} \text{ is invertible for every } i \in I \\ \iff & g_s|_{W_i} \text{ is invertible for every } i \in I \\ \iff & g_s \text{ is invertible,} \end{aligned}$$
as claimed. □

Proposition 7.17 (Multiplicative Jordan–Chevalley decomposition)**.** Let V be a k-vector space and let $g \in \mathrm{GL}(V)$ be locally finite.

1) There exist unique endomorphisms $g_s, g_u \colon V \longrightarrow V$ with $g = g_s g_u$ such that the factor g_s is sempisimple, the factor g_u is locally unipotent and g_s and g_u commute with each other.

2) The factors g_s and g_u are invertible.

3) The factor g_s is the semisimple part of g, and the factor g_u is given by $g_u = \mathrm{id}_V + g_s^{-1} g_n$.

4) An endomorphism $h \colon V \longrightarrow V$ commutes with g if and only if it already commutes with both g_s and g_u.

5) If $V = \bigcup_{i \in I} W_i$ is any covering by finite-dimensional g-invariant subspaces then the factors g_s and g_u are uniquely determined by the property that every W_i is also g_s-invariant and g_u-invariant and that $g|_{W_i} = g_s|_{W_i} g_u|_{W_i}$ is the multiplicative Jordan–Chevalley decomposition of $g|_{W_i}$ for every $i \in I$.

Proof. Part 1), part 2) and part 3) can be shown in same way as in the finite-dimensional case by using that g_s in invertible. Part 4) follows from part 3).

To show part 5) let $V = \bigcup_{i \in I} W_i$ be a cover by finite-dimensional g-invariant subspaces. Then it holds for every $i \in I$ that the additive Jordan–Chevalley decomposition of $g|_{W_i}$ is given by $g|_{W_i} = g_s|_{W_i} + g_n|_{W_i}$. It follows that the unipotent part of $g|_{W_i}$ is given by

$$\left(g\big|_{W_i}\right)_u = \mathrm{id}_{W_i} + \left(g\big|_{W_i}\right)_s^{-1} \left(g\big|_{W_i}\right)_n = \left(\mathrm{id}_V + g^{-1} g_n\right)\big|_{W_i} = g_u\big|_{W_i}$$

as desired. That g is uniquely determined by its restrictions $g|_{W_i}$ holds because the W_i cover V. □

Definition 7.18. If V is a k-vector space and $g \in \mathrm{GL}(V)$ is locally finite then the decomposition $g = g_s g_u$ from Proposition 7.17 is the (*multiplicative*) *Jordan–Chevalley decomposition decompositon* of g. The factor g_u is the *unipotent part* of g.

Lemma 7.19. Let V be a k-vector space and let $g \in \mathrm{GL}(V)$ be locally finite. Let $W \subseteq V$ be a g-invariant subspace.

1) The subspace W is also g_s-invariant and g_u-invariant.

2) The restriction $g|_W$ is again invertible and it holds that $(g|_W)_s = g_s|_W$ and $(g|_W)_u = g_u|_W$.

3) It holds for the induced endomorphisms $\overline{g}, \overline{g_s}, \overline{g_u} \colon V \longrightarrow V$ that $\overline{g}$, $\overline{g_s}$, and $\overline{g_u}$ are invertible and locally finite, and the Jordan–Chevalley decomposition decompositon of $\overline{g}$ is given by $\overline{g} = \overline{g_s} \circ \overline{g_u}$.

4) Let V' be another k-vector space, let $g' \in \mathrm{GL}(V')$ be locally finite and let $f \colon V \longrightarrow V'$ be a k-linear map with $f \circ g = g' \circ f$, i.e. such that the diagram

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ V' \arrow[r, "g'"] & V' \end{tikzcd} \tag{7.1}$$

commutes. Then it holds that $f \circ g_s = g_s \circ f'$ and $f \circ g_u = g'_u \circ f$, so that the diagram (7.1) factorizes as follows:

$$\begin{tikzcd}
V \arrow[r, "g_u"] \arrow[d, "f"'] \arrow[rr, bend left, "g"] & V \arrow[r, "g_s"] \arrow[d, "f"] & V \arrow[d, "f"] \\
V' \arrow[r, "g'_u"] \arrow[rr, bend right, "g"'] & V' \arrow[r, "g'_s"] & V'
\end{tikzcd}$$

Proof.

2) The restriction $g|_W$ is again locally finite and so there exists a cover $W = \bigcup_{i \in I} W_i$ by finite-dimensional g-invariant subspaces $W_i \subseteq V$. The restriction $g|_{W_i}$ is invertible with $(g|_{W_i})^{-1} = g^{-1}|_{W_i}$ for every $i \in I$, which shows that very W_i is (g^{-1})-invariant. It follows that W is (g^{-1})-invariant, and therefore that $g|_W$ is invertible with $(g|_W)^{-1} = g^{-1}|_W$.
   If $g = g_s + g_n$ is the additive Jordan–Chevalley decomposition of g then it follows from Lemma 7.13 that the subspace W is invariant under both g_s and g_n and that the additive Jordan–Chevalley decomposition of the restriction $g|_W$ is given by $g|_W = g_s|_W + g_n|_W$. It follows from the invertibility of $g|_W$ that $(g|_W)_s = g_s|_W$ is invertible, and it holds that $(g_s|_W)^{-1} = g_s^{-1}|_W$. It follows that

$$\left(g\middle|_W\right)_u = \mathrm{id}_W + \left(g_s\middle|_W\right)^{-1}\left(g_n\middle|_W\right) = \left(\mathrm{id}_V + g_s^{-1} g_n\right)\Big|_W = g_u\big|_W \,.$$

1) This follows from the above.

3) It follows from $g = g_s g_u$ that $\overline{g} = \overline{g_s}\,\overline{g_n}$, the induced endomorphisms $\overline{g_s}$ and $\overline{g_n}$ commute because g_s and g_n commute, the endomorphism $\overline{g_s}$ is again semisimple by Lemma 6.2 and the endomorphism $\overline{g_u}$ is again locally unipotent.

4) If f is injective then the claim follows from part 2) and if f is surjective then the claim follows from part 3). The claim follows for general f as in the proof of Lemma 6.6. ☐

Lemma 7.20. Let V be a k-vector space and let $g_1, g_2 \in \mathrm{GL}(V)$ be locally finite and commuting with each other. Then

$$(g_1 g_2)_s = (g_1)_s (g_2)_s \quad \text{and} \quad (g_1 g_2)_u = (g_1)_u (g_2)_u \,.$$

Proof. The factors $(g_1)_s$, $(g_1)_u$, $(g_2)_s$ and $(g_2)_u$ commute with each other because g_1 and g_2 commute. It therefore holds that

$$g_1 g_2 = (g_1)_s (g_1)_u (g_2)_s (g_2)_u = \big((g_1)_s (g_2)_s\big)\big((g_1)_u (g_2)_u\big) \,.$$

The factors $(g_1)_s(g_2)_s$ and $(g_1)_u(g_2)_u$ commute with each other, the factor $(g_1)_s(g_2)_s$ is again semisimple by Lemma 6.3 and the factor $(g_1)_u(g_2)_u$ is again locally unipotent by Lemma 7.9. ☐

8. Jordan–Chevalley for Linear Algebraic Groups

Lemma 8.1. Let $g : V \longrightarrow V$ and $h : W \longrightarrow W$ be endomorphism of k-vector spaces V and W.

1) If both g and h are semisimple then the endomorphism $g \otimes h$ is again semisimple.

2) If g or h is locally nilpotent then $g \otimes h$ is again locally nilpotent.

3) If g and h are locally unipotent then $g \otimes h$ is again locally unipotent.

Proof.

1) If V and W have eigenspace decomposition $V = \bigoplus V_\lambda(g)$ and $W = \bigoplus_\mu W_\mu(h)$ then

$$V \otimes W = \left(\bigoplus_\lambda V_\lambda(g) \right) \otimes \left(\bigoplus_\mu W_\mu(h) \right) = \bigoplus_\kappa \underbrace{\left(\bigoplus_{\lambda+\mu=\kappa} V_\lambda(g) \otimes W_\mu(h) \right)}_{=(V\otimes W)_\kappa(g\otimes h)}$$

is the decomposition of $V \otimes W$ into eigenspaces of $g \otimes h$.

2) We consider only the case that g is locally nilpotent. For every simple tensor $v \otimes w \in V \otimes W$ there then exists some power n for which $f^n(v) = 0$, and for which it then follows that

$$(g \otimes h)^n(v \otimes h) = g^n(v) \otimes h^n(v) = 0\,.$$

An arbitrary element $x \in V \otimes W$ can be written as a sum of simple tensors $x = \sum_{i=1}^r v_i \otimes w_i$. There then exists some power n with $g^n(v_i) = 0$ for all $i = 1, \dots, r$, and it follows from the above that $(g \otimes h)^n(x) = 0$.

3) It holds that

$$\begin{aligned} &g \otimes h - \mathrm{id}_{V\otimes W} \\ &= g \otimes h - \mathrm{id}_V \otimes \mathrm{id}_W \\ &= g \otimes h - \mathrm{id}_V \otimes h + \mathrm{id}_V \otimes h - \mathrm{id}_V \otimes \mathrm{id}_W \\ &= (g - \mathrm{id}_V) \otimes h + \mathrm{id}_V \otimes (h - \mathrm{id}_W)\,. \end{aligned}$$

The endomorphism $g - \mathrm{id}_V$ and $h - \mathrm{id}_W$ are locally nilpotent, and it follows from the above that the summand $(g - \mathrm{id}_V) \otimes h$ is again locally nilpotent. The summands $(g - \mathrm{id}_V) \otimes h$ and $\mathrm{id}_V \otimes (h - \mathrm{id}_W)$ commute factorwise and therefore commute with each other. It follows from Lemma 7.9 that $g \otimes h - \mathrm{id}_{V\otimes W}$ is locally nilpotent. □

Lemma 8.2. Let V, W be k-vector spaces and let $g \in \mathrm{GL}(V)$, $h \in \mathrm{GL}(W)$ be locally finite. Then $g \otimes h \in \mathrm{GL}(V \otimes W)$ is locally finite with

$$(g \otimes h)_s = g_s \otimes h_s \quad \text{and} \quad (g \otimes h)_u = g_u \otimes h_u\,.$$

Proof. It holds that
$$g \otimes h = (g_s g_u) \otimes (h_s h_u) = (g_s \otimes h_s)(g_u \otimes h_u)$$
and the factors $g_s \otimes h_s$ and $g_u \otimes h_u$ commute with each other because they do so factorwise. It follows from Lemma 8.1 that the factor $g_s \otimes h_s$ is again semisimple and that the factor $g_u \otimes h_s$ is again locally unipotent. □

Lemma 8.3. Let G be a group and let $\varphi\colon G \longrightarrow G$ be a homomorphism of left G-sets, i.e. it holds for all $g, x \in G$ that $\varphi(gx) = g\varphi(x)$. Then there exists a unique element $h \in G$ such that φ is given by right multiplication with h, and h is given by $h = \varphi(1)$.

Proof. The uniqueness of h follows from $\varphi(1) = 1 \cdot h = h$ and the existence follows from
$$\varphi(g) = \varphi(g \cdot 1) = g \cdot \varphi(1) = g \cdot h\,,$$
as claimed. □

Theorem 8.4. Let G be a linear algebraic group.

1) There exists for every $g \in G$ unique elements $g_s, g_u \in G$ with $(\rho_g)_s = \rho_{g_s}$ and $(\rho_g)_u = \rho_{g_u}$.
2) It holds that $g = g_s g_u$ and the elements g_s and g_u commute.

Proof.

1) Let $\mu\colon \mathrm{A}(G) \otimes \mathrm{A}(G) \longrightarrow \mathrm{A}(G)$ be the multiplication map. The map ρ_g is an algebra automorphism so that
$$\mu \circ (\rho_g \otimes \rho_g) = \rho_g \circ \mu\,.$$
It follows from Lemma 7.19 that also
$$\mu \circ (\rho_g \otimes \rho_g)_s = (\rho_g)_s \circ \mu$$
and by Lemma 8.1 therefore that
$$\mu \circ \big((\rho_g)_s \otimes (\rho_g)_s\big) = (\rho_g)_s \circ \mu\,.$$
This shows that $(\rho_g)_s$ is again an algebra automorphism.[2] It follows that there exists a unique homomorphism of affine varieties $\varphi\colon G \longrightarrow G$ with $(\rho_g)_s = \varphi^*$.

The map φ is already a homomorphism of left G-sets: For every $h \in G$ let $l_h\colon G \longrightarrow G$ denote the left multiplication with h. Then l_h commutes with r_g for every $h \in G$, from which it follows that $\lambda_h = l_h^*$ commutes with $\rho_g = r_g^*$ for every $h \in G$. It then follows that λ_h also commutes with the semisimple part $(\rho_g)_s$ for every $h \in G$, from which it then follows that φ commutes with l_h for every $h \in G$.

It follows from Lemma 8.3 that there exists a unique element $g_s \in G$ with $\varphi = r_{g_s}$, which is equivalent to $(\rho_g)_s = \varphi^* = r_{g_s}^* = \rho_{g_s}$.

It follows in the same way that there exists a unique element $g_u \in G$ with $(\rho_g)_u = \rho_{g_u}$.

[2] That $(\rho_g)_s$ preserves the unit 1 follows from $(\rho_g)_s$ being additive, multiplicative and bijective, and thus an isomorphism of not-necessarily-commutative rings.

2) It holds that

$$\rho_{g_s g_u} = r^*_{g_s g_u} = (r_{g_u} r_{g_s})^* = r^*_{g_s} r^*_{g_u} = \rho_{g_s} \rho_{g_u} = (\rho_g)_s (\rho_g)_u = \rho_g ,$$

therefore $r_{g_s g_u} = r_g$ and thus $g = g_s g_u$. It also follows in the same way that $g = g_u g_s$.

□

Definition 8.5. Let G be a linear algebraic group and let $g \in G$ The decomposition $g = g_s g_u$ from Theorem 8.4 is the *Jordan–Chevalley decomposition* of g. The factor g_s is the *semisimple part* of g and the factor g_u is the *unipotent part* of g.

Proposition 8.6. For $g \in \mathrm{GL}(V)$ the 'abstract' Jordan–Chevalley decomposition of Definition 8.5 coincides with the 'concrete' one from Definition 6.15.

Proof. Let $G \coloneqq \mathrm{GL}(V)$.

We start by embedding V into $\mathrm{A}(G)$. Let $f \in V^*$ be nonzero and consider the k-linear map

$$\varphi \colon V \longrightarrow \mathrm{A}(G) , \quad v \longmapsto [g \longmapsto f(gv)] .$$

This map is injective: There exists some $v' \in V$ with $f(v') \neq 0$. Because $G = \mathrm{GL}(V)$ acts transitively on $V \setminus \{0\}$ it follows that there exists for every $v \in V$ with $v \neq 0$ some $g \in \mathrm{GL}(V)$ with $gv = v'$; it then holds that

$$\varphi(v)(g) = f(gv) = f(v') \neq 0$$

and therefore that $\varphi(v) \neq 0$.

The map φ is also G-equivariant because it holds for all $g, h \in G$, and $v \in V$ that

$$\varphi(gv)(h) = f(hgv) = \varphi(v)(hg) = \rho_g(\varphi(v))(h)$$

and therefore that $\varphi(gv) = \rho_g(\varphi(v))$. Together this shows that φ is an embedding of representations of G. Let $W \coloneqq \varphi(V)$.

Let $g = g_s g_u$ be the 'abstract' Jordan–Chevalley decomposition of $g \in G$. Then $\rho_g = \rho_{g_s} \rho_{g_u}$ is the Jordan–Chevalley decomposition of ρ_g. It follows that $\rho_g|_W = (\rho_{g_s}|_W)(\rho_{g_u}|_W)$ is the Jordan–Chevalley decomposition of $\rho_g|_W$, so that g_s acts semisimple on W and g_u acts unipotent on W. It follows from φ being an isomorphism $V \longrightarrow W$ of representations of G that g_s also acts semisimple on V and that g_u acts unipotent on V. But this means that g_s is semisimple as an endomorphism $V \longrightarrow V$ and that g_u is unipotent as an endomorphism $V \longrightarrow V$. The decomposition $g = g_s g_u$ is therefore also the 'concrete' Jordan–Chevalley decomposition of g. □

Lemma 8.7 (Functoriality of the Jordan–Chevalley decomposition)**.** Let G and H be linear algebraic groups and let $f \colon G \longrightarrow H$ be a homomorphism of linear algebraic groups. Then

$$f(g)_s = f(g_s) \quad \text{and} \quad f(g)_u = f(g_u)$$

for every $g \in G$.

Proof. For every $g \in G$ the diagram

$$\begin{array}{ccc} H & \xrightarrow{r_{f(g)}} & H \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ G & \xrightarrow{r_g} & G \end{array}$$

commutes, which by dualizing results in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{A}(H) & \xleftarrow{\rho_{f(g)}} & \mathrm{A}(H) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{A}(G) & \xleftarrow{\rho_g} & \mathrm{A}(G) \end{array}$$

It follows from Lemma 7.19 that the diagram

$$\begin{array}{ccc} \mathrm{A}(H) & \xleftarrow{(\rho_{f(g)})_s} & \mathrm{A}(H) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{A}(G) & \xleftarrow{(\rho_g)_s} & \mathrm{A}(G) \end{array}$$

also commutes, which is by construction of the semisimple parts $f(g)_s$ and g_s the same as the following diagram:

$$\begin{array}{ccc} \mathrm{A}(H) & \xleftarrow{\rho_{f(g)_s}} & \mathrm{A}(H) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{A}(G) & \xleftarrow{\rho_{g_s}} & \mathrm{A}(G) \end{array} \tag{8.1}$$

This is the dual of the following diagram:

$$\begin{array}{ccc} H & \xrightarrow{r_{f(g)_s}} & H \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f} \\ G & \xrightarrow{r_{g_s}} & G \end{array} \tag{8.2}$$

It follows from the commutativity of the diagram (8.1) that the diagram (8.2) commutes (because the functor $\mathrm{A}(-)$ is faithful). By evaluting the identity $f \circ r_{g_s} = r_{f(g)_s} \circ f$ at the identity $1 \in G$ it follows that

$$f(g_s) = f(1 \cdot g_s) = (f \circ r_{g_s})(1) = (r_{f(g)_s} \circ f)(1) = f(1) \cdot f(g)_s = 1 \cdot f(g)_s = f(g)_s .$$

The equality $f(g_u) = f(g)_u$ can be shown in the same way. □

Corollary 8.8. Let V be a finite-dimensional k-vector space.

1) If $G\colon G \hookrightarrow \mathrm{GL}(V)$ is an embedding of linear algebraic groups then the Jordan–Chevalley decomposition in G coincides with the 'concrete' Jordan–Chevalley decomposition (from Definition 6.15) in $\mathrm{GL}(V)$.

2) If $G \subseteq \mathrm{GL}(V)$ is a closed subgroup then the Jordan–Chevalley decomposition in G coincides with the 'concrete' Jordan–Chevalley decomposition in $\mathrm{GL}(V)$.

Lemma 8.9. Let G be a linear algebraic group and let $g_1, g_2 \in G$ be two group elements which commute with each other. Then

$$(g_1 g_2)_s = (g_1)_s (g_2)_s \quad \text{and} \quad (g_1 g_2)_u = (g_1)_u (g_2)_u \,.$$

Proof. We may assume that G is a closed subgroup $\mathrm{GL}(V)$ for some finite-dimensional k-vector space V. The claim then follows from Lemma 7.20 thanks to Corollary 8.8. ☐

Definition 8.10. For a linear algebraic group G let

$$G_s \coloneqq \{g_s \mid g \in G\} \quad \text{and} \quad G_u \coloneqq \{g_u \mid g \in G\} \,.$$

Lemma 8.11. For every linear algebraic group G the set G_u is closed in G.

Proof. We may assume that G is a closed subgroup of some $\mathrm{GL}_n(k)$. It then holds that

$$G_u = \{g \in G \mid (g - I)^n = 0\} \,,$$

which shows that G_u is cut out by a polynomial condition. ☐

IV. Triangularization Results

9. Some Notions From Group Theory

9.1. We review some notions from group theory.

Definition 9.2. The *center* of a group G is

$$\mathrm{Z}(G) = \{g \in G \mid gh = hg \text{ for every } h \in G\}.$$

Lemma 9.3. For every group G its center $\mathrm{Z}(G)$ is a normal subgroup. □

Example 9.4. It holds that $\mathrm{Z}(\mathrm{GL}_n(k)) = k^\times \cdot I \cong k^\times = \mathbb{G}_{\mathrm{m}}$.

Definition 9.5. Let G be a group.

1) The *commutator* of two elements $g, h \in G$ is given by

$$[g, h] \coloneqq ghg^{-1}h^{-1}.$$

2) For any two subgroups $H, K \subseteq G$ their *commutator* $[H, K]$ of G is given by

$$[H, K] \coloneqq \langle [h, k] \mid h \in H, k \in K \rangle.$$

3) The *commutator subgroup* or *derived* (*sub*)*group* is given by

$$\mathrm{D}(G) \coloneqq [G, G].$$

9.6. Both the center $\mathrm{Z}(G)$ and the commutator subgroup $\mathrm{D}(G)$ measure how non-commutative the group G is: It holds that

$$\mathrm{Z}(G) = G \iff G \text{ is abelian} \iff \mathrm{D}(G) = 1.$$

Lemma 9.7. If $H, K \leq G$ are subgroups of a group G then $[H, K]$ is again a subgroup of G. If both H and K are normal in G then $[H, K]$ is again normal in G.

Proof. It holds for all $h \in H$, $k \in K$ and every $g \in G$ that

$$g[h, k]g^{-1} = [ghg^{-1}, gkg^{-1}] \in [H, K]$$

and therefore $g[H, K]g^{-1} \subseteq [H, K]$ because conjugation by g is a group homomorphisms. □

Corollary 9.8. For every group G its commutator subgroup $\mathrm{D}(G)$ is a normal. □

Lemma 9.9. Let G be a group.

1) For a normal subgroup $N \trianglelefteq G$ the quotient G/N is abelian if and only if N contains the commutator subgroup $\mathrm{D}(G)$.

2) Every subgroup H of G which contains the commutator subgroup $\mathrm{D}(G)$ is normal.

Proof.

1) For $g \in G$ we denote the corresponding residue class by $\overline{g} \in G/N$. It then holds that

$$
\begin{aligned}
& G/N \text{ is abelian} \\
\iff & \text{every two elements } h_1, h_2 \in G/N \text{ commute} \\
\iff & [\overline{g_1}, \overline{g_2}] = 1 \text{ for all } \overline{g_1}, \overline{g_2} \in G/N \\
\iff & \overline{[g_1, g_2]} = 1 \text{ for all } g_1, g_2 \in G \\
\iff & [g_1, g_2] \in N \text{ for all } g_1, g_2 \in G \\
\iff & \mathrm{D}(G) \subseteq N.
\end{aligned}
$$

2) The subgroup $H/\mathrm{D}(G)$ of $G/\mathrm{D}(G)$ is normal because $G/\mathrm{D}(G)$ is abelian. This is equivalent to H being normal in G. □

Definition 9.10. The *derived series* of a group G is inductively defined by $\mathrm{D}^0(G) \coloneqq G$ and

$$\mathrm{D}^{n+1}(G) \coloneqq \mathrm{D}\left(\mathrm{D}^n(G)\right) = [\mathrm{D}^n(G), \mathrm{D}^n(G)]$$

for all $n \geq 0$. The group G is *solvable* if $\mathrm{D}^n(G) = 1$ for n sufficiently large.

Lemma 9.11. A group G is solvable if and only if there exists a decreasing chain of subgroups

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = 1$$

such that every G_{i+1} is normal in G_i with abelian quotient G_i/G_{i+1}.

Proof. If G is solvable then one can choose $G_i = \mathrm{D}^i(G)$ for every i.

If such a descreasing chain exists then it follows from G_i/G_{i+1} being abelian that G_{i+1} contains $\mathrm{D}(G_i)$. It then follows inductively that G_i contains $\mathrm{D}^i(G)$ for every i, and therefore that $\mathrm{D}^n(G) = 1$. □

Example 9.12.

1) The alternating group A_n is not solvable for $n \geq 5$.

2) The group of upper triangular matrices $\mathrm{T}_n(k)$ is solvable.

Lemma 9.13. Let G be a group.

1) If G is solvable then every subgroup H of G is again solvable.

2) If N is a normal subgroup of G then G is solvable if and only if both G and G/N are solvable.

3) If H and K are normal solvable subgroups of G then HK is again solvable.

Proof. 1) It holds inductively that $\mathrm{D}^n(H) \leq \mathrm{D}(G)$ for all n. It therefore follows for sufficiently large n from $\mathrm{D}^n(G) = 1$ that also $\mathrm{D}^n(H) = 1$.

2) Suppose that G is solvable. It has already been shown that N is solvable. If $\pi\colon G \longrightarrow G/N$ denotes the canonical projection then it holds that $\mathrm{D}^n(G/N) = \mathrm{D}^n(\pi(G)) = \pi(\mathrm{D}^n(G))$ for all n. It therefore follows for n sufficiently large from $\mathrm{D}^n(G) = 1$ that also $\mathrm{D}^n(G/N) = 1$.

   Suppose on the other hand that both N and G/N are solvable. It follows for n sufficiently large that
$$\pi(\mathrm{D}^n(G)) = \mathrm{D}^n(\pi(G)) = \mathrm{D}^n(G/N) = 1\,,$$
   and it then holds that $\mathrm{D}^n(G) \subseteq N$. It follows from N being solvable that $\mathrm{D}^n(G)$ is solvable, and therefore also that G is solvable.

3) We consider the short exact sequence
$$1 \longrightarrow K \longrightarrow HK \longrightarrow HK/K \longrightarrow 1\,.$$
   The group K is solvable by assumption, and the quotient $HK/K \cong H/(H \cap K)$ is solvable because H is solvable. It follows that HK is solvable. □

Definition 9.14. The *central series* of a group G is inductively defined by $\mathrm{C}^0(G) \coloneqq G$ and $\mathrm{C}^{n+1}(G) \coloneqq [G, \mathrm{C}^i(G)]$ for all $i \geq 0$. The group G is *nilpotent* if $\mathrm{C}^n(G) = 1$ for n sufficiently large.

Lemma 9.15. If G is a group then it holds that $\mathrm{D}^n(G) \leq \mathrm{C}^n(G)$ for all n. □

Corollary 9.16. Every nilpotent group is solvable. □

Example 9.17.

1) Every abelian group is nilpotent.
2) The group of unitriangular matrices $\mathrm{U}_n(k)$ is nilpotent.
3) The group $\mathrm{T}_n(k)$ of upper triangular $(n \times n)$-matrices is not nilpotent (even though it is solvable).

Lemma 9.18. Let G be a group.

1) If G is nilpotent then every subgroup $H \leq G$ is again nilpotent.
2) If G is nilpotent and N is a normal subgroup of G then G/N is again nilpotent.
3) The group G is nilpotent if and only if both $\mathrm{Z}(G)$ and $G/\mathrm{Z}(G)$ are nilpotent.
4) If G is nilpotent and nontrivial then the center $\mathrm{Z}(G)$ is again nontrivial.

Proof.

1) It holds inductively that $\mathrm{C}^n(H) \leq \mathrm{C}^n(G)$ for every n. It therefore follows for n sufficiently large from $\mathrm{C}^n(G) = 1$ that also $\mathrm{C}^n(H) = 1$.

2) If $\pi\colon G \longrightarrow G/N$ denotes the canonical projection then it holds inductively that

$$\pi(\mathrm{C}^n(G)) = \mathrm{C}^n(G/N)$$

for all n. It therefore follows for n sufficiently large from $\mathrm{C}^n(G) = 1$ that also $\mathrm{C}^n(G/N) = 1$.

3) It follows from $G/\mathrm{Z}(G)$ being nilpotent that $\mathrm{C}^n(G) \leq \ker(\pi) = \mathrm{Z}(G)$ for n sufficiently large. It then follows that

$$\mathrm{C}^{n+1}(G) = [G, \mathrm{C}^n(G)] \leq [G, \mathrm{Z}(G)] = 1\,.$$

4) Let n be the power for which $\mathrm{C}^n(G) \neq 1$ but $\mathrm{C}^{n+1}(G) = 1$. It follows from

$$1 = \mathrm{C}^{n+1}(G) = [G, \mathrm{C}^n(G)]$$

that $\mathrm{C}^n(G) \leq \mathrm{Z}(G)$, and from $\mathrm{C}^n(G) \neq 1$ therefore $\mathrm{Z}(G) \neq 1$. □

10. Unipotent Groups

Definition 10.1. A linear algebraic group G is *unipotent* if $G = G_u$.

Warning 10.2. There also exists the notion of a *semisimple* algebraic group G, which does *not* mean that $G = G_s$.

Recall 10.3. Let V be a k-vector space. For every k-linear subspace $W \subseteq V$ its *annihilator* or *orthogonal complement* is given by

$$W^\perp = \{f \in V^* \mid f|_W = 0\} = \{f \in V^* \mid f(w) = 0 \text{ for every } w \in W\}\,,$$

and for every k-linear subspace $U \subseteq V^*$ let

$$U^\perp = \{v \in V \mid f(v) = 0 \text{ for every } f \in U\} = \bigcap_{f \in U} \ker(f)\,.$$

If V is finite-dimensional then these constructions result in mutually inverse, order-reversing bijections

$$\{k\text{-linear suspaces } W \subseteq V\} \underset{(-)^\perp}{\overset{(-)^\perp}{\rightleftarrows}} \{k\text{-linear suspaces } U \subseteq V^*\}$$

It follows in particular for every k-linear subspace $U \subseteq V^*$ that $U = V^*$ if and only if $U^\perp = 0$.

Lemma 10.4 (Burnside)**.** Let V be a finite-dimensional k-vector space and let $A \subseteq \mathrm{End}_k(A)$ be a subalgebra such that the only A-invariant subspaces of V are 0 and V. Then $A = \mathrm{End}_k(V)$.

Proof. We first show that A contains an endomorphism of rank 1 and then show that A contains every endomorphism of rank 1. It then follows that $A = \mathrm{End}_k(V)$ as every endomorphism of V is a linear combination of rank 1 endomorphisms.

To show that A contains an endomorphism of rank 1 let $\alpha \in A$ be a nonzero endomorphism of minimal rank $r > 1$. We show that already $r = 1$.

Suppose that $r \geq 2$. Then there exist vectors $v_1, v_2 \in V$ such that the images $\alpha(v_1)$ and $\alpha(v_2)$ are linearly independent. There exists some $\beta \in A$ with $\beta\alpha(v_1) = v_2$ for which it then follows that $\alpha\beta\alpha(v_1) = \alpha(v_2)$ and $\alpha(v_1)$ are linearly independent. This shows that α and $\alpha\beta\alpha$ are linearly independent. It therefore holds that

$$\alpha\beta\alpha - \lambda\alpha \neq 0$$

for every $\lambda \in k$. We show that this linear combination has a strictly smaller rank than α for some suitable λ, which then contradicts the choice of α.

It holds that $\mathrm{im}(\alpha\beta\alpha - \lambda\alpha) \subseteq \mathrm{im}(\alpha)$ and therefore $\mathrm{rank}(\alpha\beta\alpha - \lambda\alpha) \leq \mathrm{rank}(\alpha) = r$ for every $\lambda \in k$. To find a suitable λ we note that $U \coloneqq \mathrm{im}(\alpha)$ is $\alpha\beta$-invariant because

$$\alpha\beta(U) = \alpha\beta\alpha(V) = \alpha(\beta\alpha(V)) \subseteq \alpha(V) = U.$$

It follows from U being nonzero that $\alpha\beta$ has an eigenvector $u \in U$ with eigenvalue $\lambda \in k$. The eigenvector u is of the form $u = \alpha(v)$ for some $v \in V$ with $v \notin \ker(\alpha)$, for which it then follows that

$$v \in \ker(\alpha\beta\alpha - \lambda\alpha).$$

The existence of v shows that $\ker(\alpha)$ is a proper subspace of $\ker(\alpha\beta\alpha - \lambda\alpha)$, which shows that the inequality $\mathrm{rank}(\alpha\beta\alpha - \lambda\alpha) \leq \mathrm{rank}(\alpha)$ is strict.

We have shown α has rank 1, and therefore that A contains an endomorphism of rank 1.

We now show that A contains every endomorphism of rank 1. Let $\varphi_0 \in A$ be of rank 1 and let $\varphi \in \mathrm{End}_k(V)$ be any other endomorphism of rank 1. We show that there exists $\alpha, \beta \in A$ with $\varphi = \beta\varphi_0\alpha$; we will (roughly speaking) use α to adjusted the kernel of φ_0 to the one of φ, and then use β to adjust the image of φ_0 to the one of φ.

It follows from φ_0 and φ having rank 1 that there exist nonzero functionals $f_0, f \in V^*$ and nonzero vectors $v_0, v \in V$ with $\varphi_0 = f_0(-)v_0$ and $\varphi = f(-)v$.

We start by showing that there exists some $\alpha \in A$ with $f = f_0\alpha$. We do so by showing that $fA = V^*$, where $fA = \{f\alpha \mid \alpha \in A\}$. For this we show that $(fA)^\perp = 0$. This holds because

$$(fA)^\perp = \{v \in V \mid f(\alpha(v)) = 0 \text{ for every } \alpha \in A\} = \bigcap_{\alpha \in A} \ker(f\alpha)$$

is an A-invariant subspace of V, which is contained in $\ker(f_0)$ and is thus a proper A-invariant subspace of V. That $(fA)^\perp = 0$ thus follows from 0 and V being the only A-invariant subspaces of V.

It follows from $f = f_0\alpha$ that

$$\varphi_0\alpha = (f_0\alpha)(-)v_0 = f(-)v_0.$$

It holds that $Av_0 = V$ because Av_0 is a nonzero A-invariant subspace of V, so there exists some $\beta \in A$ with $\beta v_0 = v$. It then follows that

$$\beta\varphi_0\alpha = \beta(f(-)v_0) = f(-)\beta(v_0) = f(-)v = \varphi,$$

which shows that φ is contained in A. □

Theorem 10.5 (Kolchin)**.** Let V be a finite-dimensional k-vector space of positive dimension and let $G \leq \mathrm{GL}(V)$ be a subgroup which consists of unipotent elements. Then there exists a fixed vector for G, i.e. there exists some $v \in V$ with $gv = v$ for every $g \in G$.

Proof. We show the theorem by induction overy $n \coloneqq \dim(V)$. For $n = 1$ every nonzero vector $v \in V$ does the job.

Suppose now that $n \geq 2$ and that the claim holds for all strictly smaller dimensions. If V has a proper nonzero G-invariant subspace W then it follows from the induction hypothesis that there exists a nonzero fixed vector $w \in W$ for G because the restriction $g|_W$ is for every $g \in G$ again unipotent.

In the following we therefore only consider the case that no such subspace W exists. It then follows from Lemma 10.4 that G generates $\mathrm{End}_k(V)$ as a k-algebra. The group G is closed under products and contains the identity id_V, so the subalgebra generated by G coincides with the k-linear subspace of $\mathrm{End}_k(V)$ spanned by G. It therefore follows that $\mathrm{End}_k(V)$ is spanned by G as a k-vector space.

It holds for every $g \in G$ that $\mathrm{tr}(g) = n$ because 1 is the only eigenvalue of g. It follows for all $g, g' \in G$ that

$$\mathrm{tr}((g - \mathrm{id}_V)g') = \mathrm{tr}(gg' - g') = \mathrm{tr}(gg') - \mathrm{tr}(g') = n - n = 0\,,$$

and therefore that

$$\mathrm{tr}((g - \mathrm{id}_V)f) = 0$$

for all $f \in \mathrm{End}_k(V)$ because $\mathrm{End}_k(V)$ is spanned by G as a k-vector space. It follows that $g - \mathrm{id}_V = 0$ because the trace form in nongenerate.[1] This shows that $g = \mathrm{id}_V$ for every $g \in G$ and thus $G = 1$. But this contradicts V having no nonzero proper G-invariant subspace. □

Definition 10.6. Let V be a k-vector space.

1) A *flag* of V is a strictly increasing chain of k-linear subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V.$$

2) A flag $(V_i)_{i=0}^n$ of V is *complete* if it has no proper refinement, i.e. if $\dim V_i = i$ for every i.

3) If a group G acts on V then a flag $(V_i)_{i=0}^n$ is *G-invariant* if every V_i is G-invariant.

Lemma 10.7. Let V be a finite-dimensional k-vector space and let $G \subseteq \mathrm{End}_k(A)$ be a set of endomorphisms. Then the following two conditions are equivalent:

1) There exists a basis of V with respect to which every $g \in G$ is represented by an upper triangular matrix

2) There exists a G-invariant complete flag of V.

[1] This means that for every $h \in \mathrm{End}_k(V)$ with $h \neq 0$ there exists some $h' \in \mathrm{End}_k(V)$ with $\mathrm{tr}(hh') \neq 0$.

Proof. Suppose there exists a G-invariant complete flag $(V_i)_{i=0}^n$ of V. Then let $B = (b_1, \dots, b_n)$ be a basis of V for which $(b_1, \dots, b_i)$ is a basis of V_i for every i. Then with respect to the basis B every group element $g \in G$ is represented by an upper triangular matrix. Since 1 is the only eigenvalue of g it follows that this upper triangular matrix is already unitriangular.

Suppose on the other hand that there exists a basis $B = (b_1, \dots, b_n)$ of V with respect to which G is represented by unitriangular matrices. Then $V_i \coloneqq \langle b_1, \dots, b_i \rangle$ is a G-invariant subspace of V for every i, which result in a G-invariant complete flag $(V_i)_{i=0}^n$ of V. □

Corollary 10.8 (Kolchin)**.** Let V be a finite-dimensional k-vector space and let $G \leq \mathrm{GL}(V)$ be a subgroup which consists of unipotent elements. Then there exists a basis of V with respect to which G is given by unitriangular matrices.

Proof. We show the claim by induction over $n \coloneqq \dim(V)$. If $n = 1$ then $G = 1$ and the claim holds.

Let $n \geq 2$. There exists by Kolchin's theorem a nonzero common fixed vector $v \in V$ for G. Every $g \in G$ then induces an endomorphism $\overline{g} \colon V/\langle v \rangle \longrightarrow V/\langle v \rangle$ which is again unipotent. It follows from the induction hypothesis that there exists $b_2, \dots, b_n \in V$ such that $\overline{b_1}, \dots, \overline{b_n}$ is a basis of $V/\langle v \rangle$ with respect to which $\overline{g}$ is for every $g \in G$ given by an unitriangular matrix $A(g)$.

It follows with $b_1 \coloneqq v$ that $b_1, \dots, b_n$ is a basis for V with respect to which every $g \in G$ is given by a matrix of the form

$$\begin{pmatrix} 1 & * \\ 0 & A(g) \end{pmatrix},$$

which is then again unitriangular. □

Corollary 10.9. A subgroup $G \subseteq \mathrm{GL}_n(k)$ consists of unipotent matrices if and only if there exists some $\alpha \in \mathrm{GL}_n(k)$ such that $\alpha G \alpha^{-1} \leq \mathrm{U}_n(k)$. □

Corollary 10.10.

1) If V is a finite-dimensional k-vector space then every subgroup $G \subseteq \mathrm{GL}(V)$ whose elements are unipotent is nilpotent.

2) Every unipotent linear algebraic group is nilpotent.

Proof.

1) It follows from Corollary 10.8 that G is for $n \coloneqq \dim(V)$ isomorphic to a subgroup of $\mathrm{U}_n(k)$. It therefore follows from $\mathrm{U}_n(k)$ being nilpotent that G is also nilpotent.

2) We may assume that G is a (closed) subgroup of $\mathrm{GL}(V)$ for a finite-dimensional k-vector space V. The group G then consists of unipotent endomorphisms and the claim follows from part 1). □

11. Commutative Linear Algebraic Groups

11.1. Some Linear Algebra

Lemma 11.1. Let $g, h : V \longrightarrow V$ be two endomorphisms of a k-vector space V which commute with each other. Then the eigenspaces of g are h-invariant.

Proof. If $v \in V$ is an eigenvector of g for the eigenvalue $\lambda \in k$ then it follows that

$$g(h(v)) = h(g(v)) = h(\lambda v) = \lambda h(v) .$$

This shows that $h(v)$ is again contained in the λ-eigenspace of g. □

Definition 11.2. Let $(g_i)_{i \in I}$ be a family of endomorphisms $g_i : V \longrightarrow V$ of a k-vector space V. A vector $v \in V$ is a *common eigenvector* of the family $(g_i)_{i \in I}$ if it is an eigenvector of every g_i, i.e. if the vector v is nonzero and there exists for every $i \in I$ some scalar $\lambda_i \in k$ with $g_i(v) = \lambda_i v$. The *common eigenspace* of the family of endomorphisms $(g_i)_{i \in I}$ with respect to a family of scalars $(\lambda_i)_{i \in I}$ is given by

$$V\big((g_i)_{i \in I}, (\lambda_i)_{i \in I}\big) \coloneqq V_{(\lambda_i)_{i \in I}}((g_i)_{i \in i}) = \bigcap_{i \in I} V(g_i, \lambda_i) ,$$

where $V(g_i, \lambda_i)$ denotes the eigenspace of g_i with respect to the scalar λ_i.

Lemma 11.3. Let $(g_i)_{i \in I}$ be a family of endomorphisms $g_i : V \longrightarrow V$ of a k-vector space V. Then the sum $\sum_{(\lambda_i)_{i \in I}} V((g_i)_{i \in I}, (\lambda_i)_{i \in I})$ is direct.

Corollary 11.4 (Characterization of simultaneous diagonalization)**.** For a family $(g_i)_{i \in I}$ of endomorphisms $g_i : V \longrightarrow V$ of a k-vector space V, the following conditions are equivalent:

1) It holds that $V = \bigoplus_{(\lambda_i)_{i \in I}} V((g_i)_{i \in i}, (\lambda_i)_{i \in I})$.

2) It holds that $V = \sum_{(\lambda_i)_{i \in I}} V((g_i)_{i \in i}, (\lambda_i)_{i \in I})$.

3) There exists a k-basis of V consisting of common eigenvector of the endomorphisms $(g_i)_{i \in I}$.

4) The k-vector space V is generated by common eigenvector of the endomorphisms $(g_i)_{i \in I}$.

If V is finite-dimensional then the following condition is also equivalent to the above ones:

5) There exists a basis of V with respect to which every g_i is given by a diagonal matrix. □

Corollary 11.5. If V is a finite-dimensional nonzero k-vector space and $\mathscr{G} \subseteq \mathrm{End}_k(V)$ is a set of pairwise commuting endomorphisms then there exists a common eigenvector vor $\mathscr{G}$, i.e. a nonzero vector $v \in V$ such that v is an eigenvector for every $g \in \mathscr{G}$.

Proof. We may assume that $\mathscr{G}$ is finite: If v is a common eigenvector for $g_1, \dots, g_n \in \mathscr{G}$ then v is a common eigenvector for all $g \in \langle g_1, \dots, g_n \rangle_k$. We may choose $g_1, \dots, g_n \in \mathscr{G}$ with $\langle g_1, \dots, g_n \rangle_k = \langle \mathscr{G} \rangle_k$ because $\mathrm{End}_k(V)$ is finite-dimensional, for which it then follows that v is a common eigenvector for all $g \in \mathscr{G}$.

We now show the claim for $\mathscr{G} = \{g_1, \dots, g_n\}$ by induction over n. It holds for $n = 1$ (because k is algebraically closed).

For $n \geq 2$ let λ be an eigenvalue for g_n (which exists because because k is algebraically closed). The eigenspace $V_\lambda(g_n)$ is then a subspace of V which is invariant under $g_1, \dots, g_{n-1}$ and it follows from the induction hypothesis that there exists a common eigenvector $v \in V_\lambda(g_n)$ for $g_1, \dots, g_{n-1}$. This is then also an eigenvector for g_n. □

Corollary 11.6. Let V be a finite-dimensional k-vector space and let $\mathscr{G} \subseteq \operatorname{End}_k(V)$ be a set of pairwise commuting endomorphisms. Then there exists a basis V which respect to which $\mathscr{G}$ is given by upper triangular matrices.

Proof. We show the claim by induction over $n \coloneqq \dim(V)$. It holds for $n = 0$ so let $n \geq 1$.

It follows from Corollary 11.5 that there exists a common eigenvector $v \in V$ for $\mathscr{G}$. Every $g \in \mathscr{G}$ induces an endomorphism $\overline{g} \colon V/\langle v \rangle \longrightarrow V/\langle v \rangle$, and these endomorphism commute pairwise with eath other. It follows from the induction hypothesis that there exists $b_2, \dots, b_n \in V$ such that the residue classes $\overline{b_2}, \dots, \overline{b_n}$ form a basis of $V/\langle v \rangle$ with respect to which $\overline{g}$ is represented by a upper triangular matrix $A(g)$ for every $g \in \mathscr{G}$.

It follows with $b_1 \coloneqq v$ that $b_1, \dots, b_n$ is a basis of V with respect to which every $g \in G$ is given by a matrix of the form

$$\begin{pmatrix} \lambda(g) & * \\ 0 & A(g) \end{pmatrix}$$

where $\lambda(g)$ is the eigenvalue of v with respect to g. □

Lemma 11.7. Let V be a k-vector space.

1) If $g_1, \dots, g_n \colon V \longrightarrow V$ are diagonalizable endomorphisms which pairwise commute with each other then they are simultaneously diagonalizable.

2) Let V be finite-dimensional and let $\mathscr{G} \subseteq \operatorname{End}_k(V)$ be a set of commuting diagonalizable endomorphisms. Then $\mathscr{G}$ is simultaneously diagonalizable, i.e. there exists a basis of V with respect to which $\mathscr{G}$ is represented by diagonal matrices.

Proof.

1) We show the claim by induction over n. It holds for $n = 0$ and $n = 1$.

   Let $n \geq 2$. It follows from the induction hypothesis that $g_1, \dots, g_{n-1}$ are simultaneously diagonalizable, so we may write

   $$V = \bigoplus_{\lambda_1, \dots, \lambda_{n-1}} \left(V_{\lambda_1}(g_1) \cap \dots \cap V_{\lambda_{n-1}}(g_{n-1}) \right) .$$

   The summand

   $$V_{\lambda_1}(g_1) \cap \dots \cap V_{\lambda_{n-1}}(g_{n-1}) \eqqcolon V(\lambda_1, \dots, \lambda_{n-1})$$

   is the simultaneous eigenspace of $g_1, \dots, g_{n-1}$ with respect to the eigenvalues $\lambda_1, \dots, \lambda_{n-1}$. The eigenspaces $V_{\lambda_i}(g_i)$ are g_n-invariant because g_n commutes with $g_1, \dots, g_{n-1}$. The common eigenspace $V(\lambda_1, \dots, \lambda_{n-1})$ is therefore also g_n-invariant because it is the intersection of g_n-invariant subspaces. The restriction $g_n|_{V(\lambda_1,\dots,\lambda_{n-1})}$ is again diagonalizable and so there exists a decomposition

$$V(\lambda_1, \dots, \lambda_{n-1}) = \bigoplus_{\lambda_n} V(\lambda_1, \dots, \lambda_{n-1})_{\lambda_n}(g_n)$$

into g_n-eigenspaces. This then results overall in a decomposition of common eigenspaces

$$V = \bigoplus_{\lambda_1,\dots,\lambda_{n-1}} \bigoplus_{\lambda_n} V(\lambda_1, \dots, \lambda_{n-1})_{\lambda_n}(g_n) = \bigoplus_{\lambda_1,\dots,\lambda_n} V_{\lambda_1,\dots,\lambda_n}(g_1, \dots, g_n)\,.$$

This shows that $g_1, \dots, g_n$ are again simultaneously diagonalizable.

2) The subspace $\langle \mathscr{G} \rangle_k \subseteq \mathrm{End}_k(V)$ is finite because $\mathrm{End}_k(V)$ is finite. It follows that there exist $g_1, \dots, g_n \in \mathscr{G}$ with $\langle \mathscr{G} \rangle_k = \langle g_1, \dots, g_n \rangle_k$. The endomorphisms $g_1, \dots, g_n$ are simultaneously diagonalizable by part 1) and so there exists a basis B of V with respect to which every g_i is represented by a diagonal matrix. It then follows that every endomorphism $g \in \langle \mathscr{G} \rangle_k$ is given by a diagonal matrix with respect to B, and this holds in particular every endomorphism $g \in \mathscr{G}$. □

11.2. Decomposition of Commutative Linear Algebraic Groups

Corollary 11.8. If G is a commutative linear algebraic group then there exists for suitable n a closed embedding $G \hookrightarrow \mathrm{T}_n(k)$ of linear algebraic groups for which G_s is given by diagonal matrices, and G_s is therefore given by '$G \cap \mathrm{D}_n(k)$'.

Proof. We may assume that G is a subgroup of some $\mathrm{GL}(V)$ for a finite-dimensional k-vector space V.

The set G_s is simultaneously diagonalizable by Lemma 11.7 so there exists a decomposition $V = \bigoplus_i V_i$ into common eigenspaces for G_s. Every common eigenspace V_i is G-invariant because it is of the form $V_i = \bigcap_{g \in G_s} V_{\lambda_{i,g}}(g)$ for scalars $\lambda_{i,g} \in k$, with each eigenspace $V_{\lambda_{i,g}}(g)$ being G-invariant since G is commutative.

It follows from Corollary 11.6 that for every V_i there exists a basis B_i with respect to which every restriction $g|_{V_i}$ is given by an upper triangular matrix. For the semisimple elements $g \in G_s$ the restrictions $g|_{V_i}$ are by construction given by scalar matrices with respect to B_i (and more generally with respect to any basis of V_i). By combining the bases B_i we arrive at a basis B for V with respect to which every $g \in G$ is given by an upper triangular matrix and every $g \in G_s$ is already given by a diagonmal matrix.

That G_s is given by '$G \cap \mathrm{D}_n(k)$' follows from every element of G_s being given by a diagonal matrix, and every element of '$G \cap \mathrm{D}_n(k)$' being semisimple. □

Theorem 11.9. Let G be a commutative linear algebraic group. Then G_s and G_u are closed subgroups of G and the map

$$G_s \times G_u \longrightarrow G\,, \quad (g_s, g_u) \longmapsto g_s g_u$$

is an isomorphism of linear algebraic groups.

Proof. By Corollary 11.8 we may assume that G is a closed subgroup of $\mathrm{T}_n(k)$ for some n such that G_s is given by $G_s = G \cap \mathrm{D}_n(k)$.

This description of G_s shows that G_s is closed in G, and G_u is closed in G by Lemma 8.11.

The maps $(-)_s, (-)_u \colon G \longrightarrow G$ are group homomorphisms by Lemma 8.9 because G is commutative. It follows that G_s and G_u are subgroups of G and that the map

$$\psi \colon G \longrightarrow G_s \times G_u, \quad g \mapsto (g_s, g_u)$$

is a group homomorphism. That the given map $\varphi \colon G_s \times G_u \longrightarrow G$ is a group homomorphism follows from G being commutative.

It holds that $\varphi\psi = \mathrm{id}_G$ because $g = g_s g_u$ for every $g \in g$, and it holds that $\psi\varphi = \mathrm{id}_{G_s \times G_u}$ because for every $g \in G$ the decomposition $g = g_s g_u$ is the Jordan–Chevalley decomposition of g because g_s and g_u commute with each other by the commutativity of G.

The map φ is a morphism of affine varieties because it is given by matrix multiplication, and the map ψ is a morphism of affine varieties because both g_s and g_u are polynomials in g by Proposition 6.14. ☐

Corollary 11.10. If a commutative linear algebraic group G is connected then the subgroups G_s and G_u are again connected.

Proof. Both G_s and G_u are images of G under the continuous maps $(-)_s$ and $(-)_u$. ☐

12. Diagonalizable Linear Algebraic Groups

Definition 12.1. A linear algebraic group G is *diagonalizable* if it is isomorphic to a closed subgroup of some $\mathrm{D}_n(k) \cong \mathbb{G}_\mathrm{m}^n$. It is a torus if $G \cong \mathrm{D}_n(k) \cong \mathbb{G}_\mathrm{m}^n$ for some n.

Lemma 12.2. For a linear algebraic group G the following conditions are equivalent:

1) G is a diagonalizable.

2) G is commutative with $G = G_s$.

Proof.

1) $\Longleftrightarrow$ 2): The group $\mathrm{D}_n(k)$ is commutative and every element of $\mathrm{D}_n(k)$ is semisimple, so the same follows for every of its closed subgroups.

2) $\Longleftrightarrow$ 1): This follows from Corollary 11.8. ☐

Definition 12.3. A linear algebraic group G is a *torus* if it is isomorphic to $\mathrm{D}_n(k) \cong \mathbb{G}_\mathrm{m}^n$ for some n.

12.4. We will show that tori and diagonalizable linear algebraic groups can be characterized via their *character group*.

Definition 12.5. A *character* of a group G (over k) is a group homomorphism $\chi \colon G \longrightarrow k^\times$.

Lemma 12.6 (Dedekind–Artin)**.** For any group G the set of characters $G \longrightarrow k$ is linearly independent in $\mathrm{Maps}(G, k)$.

Proof. We show that pairwise different characters $\chi_1, \dots, \chi_n \colon G \longrightarrow k^\times$ are linearly independent by induction over n. The linear independence holds for $n = 0$ and it holds for $n = 1$ because every character is nonzero (as it maps $1 \in G$ onto $1 \in k^\times$).

Let $n \geq 2$ and let $\lambda_1\chi_1 + \cdots + \lambda_n\chi_n = 0$ be a linear combination. It holds for all $g, h \in G$ that

$$0 = \lambda_1\chi_1(gh) + \cdots + \lambda_n\chi_n(gh) = \lambda_1\chi_1(g)\chi_1(h) + \cdots + \lambda_n\chi_n(g)\chi_n(h)$$

and therefore

$$0 = \lambda_1\chi_1(g)\chi_1 + \cdots + \lambda_n\chi_n(g)\chi_n\,. \tag{12.1}$$

It also holds that

$$0 = \chi_n(g) \cdot (\lambda_1\chi_1 + \cdots + \lambda_n\chi_n) = \lambda_1\chi_n(g)\chi_1 + \cdots + \lambda_n\chi_n(g)\chi_n\,. \tag{12.2}$$

By subtracting the relation (12.2) from the relation (12.1) it follows that

$$0 = \lambda_1\,(\chi_1(g) - \chi_n(g))\,\chi_1 + \cdots + \lambda_{n-1}(\chi_{n-1}(g) - \chi_n(g))\chi_{n-1}\,.$$

It follows from the induction hypothesis that for every $i = 1, \dots, n-1$, $\lambda_i = 0$ or $\chi_i(g) = \chi_n(g)$. There exists for every $i = 1, \dots, n$ some $g \in G$ with $\chi_i(g) \neq \chi_n(g)$ because χ_i and χ_n are distinct. It follows in combination that $\lambda_1 = \cdots = \lambda_{n-1} = 0$. It then also follows that $\lambda_n = 0$ because $\lambda_n\chi_n = 0$. □

Definition 12.7. A *rational character* of a linear algebraic group G is a morphism of linear algebraic groups $\chi\colon G \longrightarrow \mathbb{G}_{\mathrm{m}}$.

12.8. If G is a group and $f_1, f_2 \colon G \longrightarrow \mathbb{G}_{\mathrm{m}}$ are two group homomorphisms then their pointwise product

$$f_1f_2 \colon G \longrightarrow \mathbb{G}_{\mathrm{m}}\,, \quad g \longmapsto f_1(g)f_2(g)$$

is again a group homomorphism because $\mathbb{G}_{\mathrm{m}}$ is abelian. It also holds for every group homomorphism $f\colon G \longrightarrow \mathbb{G}_{\mathrm{m}}$ that the map

$$f^{-1} \colon G \longrightarrow \mathbb{G}_{\mathrm{m}}\,, \quad g \longmapsto f(g)^{-1}$$

is again a group homomorphism. It follows that the set of group homomorphisms $G \longrightarrow \mathbb{G}_{\mathrm{m}}$ forms a group with respect to pointwise multiplication. The neutral element is given by the trivial group homomorphism and the inverse of a group homomorphism $f\colon G \longrightarrow \mathbb{G}_{\mathrm{m}}$ is given by f^{-1} as above.

If G is a linear algebraic group and $\chi_1, \chi_2 \colon G \longrightarrow \mathbb{G}_{\mathrm{m}}$ are rational characters then the group homomorphism $\chi_1\chi_2 \colon G \longrightarrow \mathbb{G}_{\mathrm{m}}$ is again a rational character because it is given by the composition

$$\chi_1\chi_2 \colon G \xrightarrow{\Delta} G \times G \xrightarrow{\chi_1 \times \chi_2} \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \xrightarrow{\text{mult}} \mathbb{G}_{\mathrm{m}}\,.$$

It similarly follow that for every rational character $\chi\colon G \longrightarrow \mathbb{G}_{\mathrm{m}}$ its inverse $\chi^{-1} \colon g \longrightarrow \mathbb{G}_{\mathrm{m}}$ is again a rational character becaus it is given by the composition

$$\chi^{-1} \colon G \xrightarrow{\chi} \mathbb{G}_{\mathrm{m}} \xrightarrow{(-)^{-1}} \mathbb{G}_{\mathrm{m}}\,.$$

It also holds that the trivial group homomorphism $G \longrightarrow \mathbb{G}_{\mathrm{m}}$ is a rational character. It follows that the set of rational characters $G \longrightarrow \mathbb{G}_{\mathrm{m}}$ forms a group with respect to pointwise multiplication.

Definition 12.9. The *character group* of a linear algebraic group G is the group of rational characters $G \longrightarrow \mathbb{G}_{\mathrm{m}}$ together with pointwise multiplication. It is denoted by $\mathrm{X}^*(G)$.

Remark 12.10. Other common notations for the character group are $\widehat{G}$ and $\chi(G)$.

12.11. The character group $\mathrm{X}^*(-)$ defines a contravariant functor

$$\begin{aligned} \{\text{linear algebraic groups}\} &\longrightarrow \{\text{abelian groups}\}\,, \\ G &\longmapsto \mathrm{X}^*(G)\,, \\ [f\colon\, G \longrightarrow H] &\longmapsto \left[f^*:\; \mathrm{X}^*(H) \longrightarrow \mathrm{X}^*(G), \chi \mapsto \chi \circ f\right]. \end{aligned}$$

Lemma 12.12. For every linear algebraic group G its character group $\mathrm{X}^*(G)$ is a subgroup of the unit group $\mathrm{A}(G)^\times$. □

Lemma 12.13. Let G and H be linear algebraic groups. Then $\mathrm{X}^*(G \times H) \cong \mathrm{X}^*(G) \times \mathrm{X}^*(H)$.

Proof. The canonical projections $p:\; G \times H \longrightarrow G$ and $q:\; G \times H \longrightarrow H$ are homomorphisms of linear algebraic groups and therefore induce group homomorphisms $p^*:\; \mathrm{X}^*(G) \longrightarrow \mathrm{X}^*(G\times H)$ and $q^*:\; \mathrm{X}^*(H) \longrightarrow \mathrm{X}^*(G \times H)$. These two group homomorphisms can be combined into a single group homomorphism

$$\varphi:\; \mathrm{X}^*(G) \times \mathrm{X}^*(H) \longrightarrow \mathrm{X}^*(G \times H) \quad (\chi_1, \chi_2) \mapsto p^*(\chi_1)q^*(\chi_2)$$

because the group $\mathrm{X}^*(G\times H)$ is abelian. The rational character $\varphi(\chi_1, \chi_2)$ is on elements of $G \times H$ given by

$$\varphi(\chi_1,\chi_2)(g,h) = p^*(\chi_1)(g,h)q^*(\chi_2)(g,h) = \chi_1(p(g,h))\chi_2(q(g,h)) = \chi_1(g)\chi_2(h)\,.$$

To construct the inveres ψ of φ we consider the 'inclusions' $i:\; G \longrightarrow G\times H$ and $j:\; H \longrightarrow G\times H$ which are homomorphisms of linear algebraic groups. The induced group homomorphisms $i^*:\; \mathrm{X}^*(G \times H) \longrightarrow \mathrm{X}^*(G)$ and $j^*:\; \mathrm{X}^*(G \times H) \longrightarrow \mathrm{X}^*(H)$ result in a single group homomorphism

$$\psi \coloneqq (i^*, j^*):\; \mathrm{X}^*(G \times H) \longrightarrow \mathrm{X}^*(G) \times \mathrm{X}^*(H)\,, \quad \chi \mapsto (i^*(\chi), j^*(\chi)) = (\chi \circ i, \chi \circ j)\,.$$

The two group homomorphisms φ and ψ are mutually inverse to each other: It holds for every pair (χ_1, χ_2) of rational characters $\chi_1 \in \mathrm{X}^*(G)$, $\chi_2 \in \mathrm{X}^*(H)$ that

$$\begin{aligned} &\psi(\varphi(\chi_1,\chi_2)) \\ &= \psi(p^*(\chi_1)q^*(\chi_2)) \\ &= (i^*(p^*(\chi_1)q^*(\chi_2)), j^*(p^*(\chi_1)q^*(\chi_2))) \\ &= (i^*(p^*(\chi_1))i^*(q^*(\chi_2)), j^*(p^*(\chi_1))j^*(q^*(\chi_2))) \\ &= ((pi)^*(\chi_1)(qi)^(\chi_2), (pj)^*(\chi_1)(qj)^*(\chi_2)) \\ &= (\chi_1, \chi_2) \end{aligned} \tag{12.3}$$

where we use for the step (12.3) that $pi = \mathrm{id}_G$ and $qj = \mathrm{id}_H$ while the compositions pj and qi are the trivale homomorphisms. This shows that $\varphi\psi = \mathrm{id}$. It also holds for every rational character $\chi \in \mathrm{X}^*(G \times H)$ and all group elements $(g,h) \in G \times H$ that

$$\begin{aligned}
&\varphi(\psi(\chi))(g,h) \\
&= \varphi(\chi \circ i, \chi \circ j)(g,h) \\
&= (\chi \circ i)(g)(\chi \circ j)(h) \\
&= \chi((g,1))\chi((1,h)) \\
&= \chi((g,1)(1,h)) \\
&= \chi(g,h)\,,
\end{aligned}$$

which shows that $\varphi\psi = \mathrm{id}$. □

Lemma 12.14. If G is a linear algebraic group and $\mathrm{char}(k) = p > 0$ then the character group $\mathrm{X}^*(G)$ has only trivial p-torsion.

Proof. If $\mathrm{char}(k) = p > 0$ then it holds for every $x \in \mathbb{G}_\mathrm{m}$ that

$$x^p - 1 = (x-1)^p$$

and it therefore therefore holds that $x^p = 1$ if and only if $x = 1$. This shows that the multiplicative group $\mathbb{G}_\mathrm{m}$ has only trivial p-torsion. It follows that for every linear algebraic group G its character group $\mathrm{X}^*(G)$ has only trivial p-torsion. □

12.15. We will now show that tori and diagonalizable linear algebraic groups can be characterized via they character groups. We start by showing that the character group of the torus $\mathrm{D}_n(k)$ is the free abelian group $\mathbb{Z}^n$.

Lemma 12.16. Let R be an integral domain. Then the unit group of the ring of Laurant polynomials $R[t,t^{-1}]$ is given by $R[t,t^{-1}]^\times = \{at^n \mid a \in R^\times, n \in \mathbb{Z}\}$.

Proof. For every nonzero Laurant polynomial $p = \sum_{i\in\mathbb{Z}} p_i t^i \in k[t,t^{-1}]$ its *lower degree* is given by

$$\deg^-(p) \coloneqq \min\{i \in \mathbb{Z} \mid p_i \neq 0\}$$

and its *upper degree* is given by

$$\deg^+(p) \coloneqq \max\{i \in \mathbb{Z} \mid p_i \neq 0\}\,.$$

It holds that $\deg^-(p) \leq \deg^+(p)$ with equality if and only if p is of the form $p = at^n$ for some nonzero $a \in R$ and some $n \in \mathbb{Z}$.

It follows from R being an integral domain that both $\deg_-$ and $\deg_+$ are additive in the sense that

$$\deg_\pm(p_1p_2) = \deg_\pm(p_1) + \deg_\pm(p_2)$$

for any two nonzero Laurant polynomials $p_1, p_2 \in R[t,t^{-1}]$.

If $p \in R[t,t^{-1}]$ is a unit with inverse $q \in R[t,t^{-1}]$ then it follows that

$$\deg^+(p) + \deg^+(q) = \deg^+(pq) = \deg^+(1) = 0$$

and therefore that $\deg^+(p) = -\deg^+(q)$. It follows similarly that $\deg^-(p) = -\deg^-(q)$. This shows together with $\deg^-(q) \leq \deg^+(q)$ that

$$\deg^+(p) = -\deg^+(q) \leq -\deg^-(q) = \deg^-(p)\,,$$

which together with $\deg^-(p) \leq \deg^+(p)$ shows that $\deg^-(p) = \deg^+(p)$. The Lauraunt polynomial p is therefore of the form $p = at^n$ for some $a \in R$, $n \in \mathbb{Z}$. By switching the roles of p and q it also follows that q is of the form $q = bt^m$ for some $b \in R$, $m \in \mathbb{Z}$.

It follows from p and q being inverse to each other that $m = -n$ and that $a, b \in R^\times$ with $b = a^{-1}$. □

Lemma 12.17. It holds for the group of diagonal matrices $\mathrm{D}_n(k)$ that

$$\mathrm{A}(\mathrm{D}_n(k)) = k[T_1, T_1^{-1}, \dots, T_n, T_n^{-1}]$$

where

$$T_i\left(\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}\right) = d_i\,,$$

for every i, and it holds that

$$\mathrm{A}\big(\mathrm{D}_n(k)\big)^\times = \bigcup_{a \in \mathbb{Z}^n} k^\times T_1^{a_1} \cdots T_n^{a_n} \cong k^\times \times \mathbb{Z}^n\,,$$

and

$$\mathrm{X}^*\big(\mathrm{D}_n(k)\big) = \bigcup_{a \in \mathbb{Z}^n} T_1^{a_1} \cdots T_n^{a_n} \cong \mathbb{Z}^n\,.$$

It holds in particular that $\mathrm{X}^*(\mathrm{D}_n(k))$ is a k-basis of $\mathrm{A}(\mathrm{D}_n(k))$.

Proof. It holds that

$$\begin{aligned} & \mathrm{A}(\mathrm{D}_n(k)) \\ = {}& \mathrm{A}(\mathbb{G}_\mathrm{m} \times \cdots \times \mathbb{G}_\mathrm{m}) \\ \cong {}& \mathrm{A}(\mathbb{G}_\mathrm{m}) \otimes \cdots \otimes \mathrm{A}(\mathbb{G}_\mathrm{m}) \\ = {}& k\big[T_1, T_1^{-1}\big] \otimes \cdots \otimes k\big[T_n, T_n^{-1}\big] \\ \cong {}& k\big[T_1, T_1^{-1}, \dots, T_n T_n^{-1}\big] \end{aligned}$$

as claimed. This equality $\mathrm{A}(\mathrm{D}_n(k))^\times = \bigcup_{a \in \mathbb{Z}^n} k^\times T_1^{a_1} \cdots T_n^{a_n}$ follows from Lemma 12.16 by induction over n on

$$k\big[T_1, T_1^{-1}, T_2, T_2^{-1}, \dots, T_n, T_n^{-1}\big] = k\big[T_1, T_1^{-1}\big]\big[T_2, T_2^{-1}\big] \cdots \big[T_n, T_n^{-1}\big]\,.$$

The inclusion $\bigcup_{a \in \mathbb{Z}^n} T_1^{a_1} \cdots T_n^{a_n} \subseteq \mathrm{X}^*(\mathrm{D}_n(k))$ holds because every $T_i \colon \mathrm{D}_n(k) \longrightarrow \mathbb{G}_\mathrm{m}$ is a homomorphism of linear algebraic groups (namely the projection of $\mathrm{D}_n(k) = \mathbb{G}_\mathrm{m}^n$ onto the i-th factor). To convince ourselves of the inclusion $\mathrm{X}^*(\mathrm{D}_n(k)) \subseteq \bigcup_{a \in \mathbb{Z}^n} T_1^{a_1} \cdots T_n^{a_n}$ we note that every $\chi \in \mathrm{X}^*(\mathrm{D}_n(k))$ is a unit in $\mathrm{A}(\mathrm{D}_n(k))^\times$ and thus of the form $\chi = \lambda T_1^{a_1} \cdots T_n^{a_m}$ for some scalar $\lambda \in k^\times$ and some multiindex $a \in \mathbb{Z}^n$. It holds that

$$1 = \chi(1) = \lambda$$

and therefore that $\chi = T_1^{a_1} \cdots T_n^{a_n}$. □

Theorem 12.18 (Structure of diagonalizable groups)**.** For a linear algebraic group G the following conditions are equivalent:

1) The group G is diagonalizable.

2) The character group $\mathrm{X}^*(G)$ is finitely generated and a k-basis of $\mathrm{A}(G)$.

3) It holds that $G \cong \mu_{d_1} \times \cdots \times \mu_{d_n} \times \mathrm{D}_r(k)$ where $\mu_d \coloneqq \{g \in \mathbb{G}_\mathrm{m} \mid g^d = 1\}$ is the group of the d-th roots of unity.

Proof. The proof proceeds in six steps.

Step 1 Let G be a diagonalizable linear algebraic group and let $i \colon G \hookrightarrow \mathrm{D}_n(k)$ be a closed embedding for suitable n. The induced algebra homomorphism $i^* \colon \mathrm{A}(\mathrm{D}_n(k)) \longrightarrow \mathrm{A}(G)$ is then a surjection (for example by Corollary 1.48) and we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{A}(\mathrm{D}_n(k)) \arrow[r, "i^*"] & \mathrm{A}(G) \\ \mathrm{X}^*(\mathrm{D}_n(k)) \arrow[u, hook] \arrow[r, "i^*"] & \mathrm{X}^*(G) \arrow[u, hook] \end{tikzcd}$$

It follows from Lemma 12.17 that $\mathrm{X}^*(\mathrm{D}_n(k))$ is a k-basis of $\mathrm{A}(\mathrm{D}_n(k))$, from which it then follows with the surjectivity of $i^* \colon \mathrm{A}(\mathrm{D}_n(k)) \longrightarrow \mathrm{A}(G)$ and the commutativity of the diagram that $\mathrm{A}(G)$ is generated by $\mathrm{X}^*(G)$ as a k-vector space. It follows that the set $\mathrm{X}^*(G)$ is a k-basis for $\mathrm{A}(G)$ because it is linearly independent by the Dedekind–Artin lemma.

This shows the implication 1) $\implies$ 2).

Step 2 Note that in the above situation the image $i^*(\mathrm{X}^*(\mathrm{D}_n(k)))$ is is a k-generating set of $\mathrm{A}(G)$ (because $\mathrm{X}^*(\mathrm{D}_n(k))$ is a k-basis for $\mathrm{A}(\mathrm{D}_n(k))$ and the algebra homomorphism $i^* \colon \mathrm{A}(\mathrm{D}_n(k)) \longrightarrow \mathrm{A}(G)$ is surjective). This image is contained in the k-basis $\mathrm{X}^*(G)$ of $\mathrm{A}(G)$, from which it follows that $i^*(\mathrm{X}^*(\mathrm{D}_n(k))) = \mathrm{X}^*(G)$, i.e. that i^* is surjective. This shows that every character of G is the restriction of a character of $\mathrm{D}_n(k)$.

Step 3 Let G be any linear algebraic groups and H is a diagonalizable linear algebraic group then the map

$$\varepsilon \coloneqq (-)^* \colon \operatorname{Hom}_{\text{lin. alg. groups}}(G, H) \longrightarrow \operatorname{Hom}_{\mathbf{Ab}}(\mathrm{X}^*(H), \mathrm{X}^*(G))$$

is bijective.

To see that the map ε is injective we note that we have for every homomorphism of linear algebraic groups $f \colon G \longrightarrow H$ the following commutative diagram:

$$\begin{tikzcd} \mathrm{A}(H) \arrow[r, "f^*"] & \mathrm{A}(G) \\ \mathrm{X}^*(H) \arrow[u, hook] \arrow[r, "f^*"] & \mathrm{X}^*(G) \arrow[u, hook] \end{tikzcd}$$

The homomorphism f is uniquely determined by the induced algebra homomorphism $f^* : \mathrm{A}(H) \longrightarrow \mathrm{A}(G)$ (because $\mathrm{A}(-)$ is faithful), which in turn is uniquely determined by its action on the k-basis $\mathrm{X}^*(H)$ of $\mathrm{A}(H)$. This action is given by th induced group homomorphism $\varepsilon(f) = f^* : \mathrm{X}^*(H) \longrightarrow \mathrm{X}^*(G)$, which shows that ε is injective.

To show that ε is surjective let $f\colon \mathrm{X}^*(H) \longrightarrow \mathrm{X}^*(G)$ be a group homomorphism. It follows from $\mathrm{X}^*(H)$ being a basis of $\mathrm{A}(H)$ that χ extends uniquely to a k-linear map $F\colon \mathrm{A}(H) \longrightarrow \mathrm{A}(G)$. The map F is multiplicative on the basis $\mathrm{X}^*(H)$ of $\mathrm{A}(H)$ and satisfies the equality $F(1) = \chi(1) = 1$; this shows that F is an algebra homomorphism. It follows that there exists a unique morphism of affine varieties $h : G \longrightarrow H$ with $F = f^*$. This morphim h is a group homomorphism: It holds that

$$\begin{aligned} & h(g_1g_2) = h(g_1)h(g_2) \text{ for all } g_1, g_2 \in G \\ \iff\ & \varphi(h(g_1g_2)) = \varphi(h(g_1)h(g_2)) \text{ for all } g_1, g_2 \in G,\ \varphi \in \mathrm{A}(H) \\ \iff\ & \chi'(h(g_1g_2)) = \chi'(h(g_1)h(g_2)) \text{ for all } g_1, g_2 \in G,\ \chi' \in \mathrm{X}^*(H)\,, \end{aligned} \tag{12.4}$$

where the equivalence (12.4) holds because $\mathrm{X}^*(H)$ is a k-basis for $\mathrm{A}(H)$. The needed equality $\chi'(f(g_1g_2)) = \chi'(f(g_1)f(g_2))$ holds true because

$$\begin{aligned} \chi'(h(g_1g_2)) &= h^*(\chi')(g_1g_2) \\ &= F(\chi')(g_1g_2) \\ &= f(\chi')(g_1g_2) \\ &= f(\chi')(g_1)f(\chi')(g_2) \qquad (12.5) \\ &= F(\chi')(g_1)F(\chi')(g_2) \\ &= h^*(\chi')(g_1)h^*(\chi')(g_2) \\ &= \chi'(h(g_1))\chi'(h(g_2)) \\ &= \chi'(h(g_1)h(g_2))\,, \end{aligned}$$

where we use for the equality (12.5) that $f(\chi') \in \mathrm{X}^*(G)$ is again a character, and thus a group homomorphism. With this we have shown that f is a group homomorphism, and therefore already a homomorphism of linear algebraic groups. It follows from the commutativity of the diagram

$$\begin{tikzcd} \mathrm{A}(H) \arrow[r, "F=h^*"] & \mathrm{A}(G) \\ \mathrm{X}^*(H) \arrow[u, hook] \arrow[r, "f"] & \mathrm{X}^*(G) \arrow[u, hook] \end{tikzcd}$$

that $f = h^* = \varepsilon(h)$, which shows that ε is surjective.

Step 4 Suppose that G is a linear algebraic group for which its character group $\mathrm{X}^*(G)$ is finitely generated, and a basis of $\mathrm{A}(G)$. It follows from $\mathrm{X}^*(G)$ being finitely generated that there exists for some n a surjective grouphomorphism $\mathbb{Z}^n \longrightarrow \mathrm{X}^*(G)$ and thus a surjective grouphomomorphism $h : \mathrm{X}^*(\mathrm{D}_n(k)) \longrightarrow \mathrm{X}^*(G)$. It then follows from

Step 3 that there exists a homomorphism of linear algebraic groups $f\colon G \longrightarrow \mathrm{D}_n(k)$ with $h = f^*$.

The homomorphism f is a closed embedding: We consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{A}(\mathrm{D}_n(k)) \arrow[r, "f^*"] & \mathrm{A}(G) \\
\mathrm{X}^*(\mathrm{D}_n(k)) \arrow[u, hook] \arrow[r, "h=f^*"] & \mathrm{X}^*(G) \arrow[u, hook]
\end{tikzcd}$$

The lower horizontral arrow $f^*\colon \mathrm{X}^*(\mathrm{D}_n(k)) \longrightarrow \mathrm{X}^*(G)$ is surjective and $\mathrm{X}^*(G)$ is a k-basis of $\mathrm{A}(G)$, so it follows that the image of the composition $\mathrm{X}^*(\mathrm{D}_n(k)) \longrightarrow \mathrm{A}(G)$ is a k-generating set of $\mathrm{A}(G)$. It follows that the induction algebra homomorphism $f^*\colon \mathrm{A}(\mathrm{D}_n(k)) \longrightarrow \mathrm{A}(G)$ is surjective. This shows that f is closed embedding by Corollary 1.48.

This shows the implication 2) $\implies$ 1).

Step 5 To show the implication 1) $\implies$ 3) let G be a diagonalizable linear algebraic group, which we may assume to be a closed subgroup of some $\mathrm{D}_n(k) =: T$.

We have seen in Step 2 that every character of G is the restriction of a character of T, i.e. that the group homomorphism $i^*\colon \mathrm{X}^*(T) \longrightarrow \mathrm{X}^*(G)$ induced by the inclusion $i\colon G \hookrightarrow T$ is surjective. Let $\Gamma := \ker(i^*) \leq \mathrm{X}^*(T)$ be its kernel. It then holds that $G = \bigcap_{\chi\in\Gamma} \ker(\chi)$:

The inclusion $G \subseteq \bigcap_{\chi\in\Gamma} \ker(\chi)$ holds by definition of Γ. To convince ourselves of the other inclusion we note that $G' := \bigcap_{\chi\in\Gamma} \ker(\chi)$ is a closed subgroup of T for which a character $\chi \in \mathrm{X}^*(T)$ is trivial on G (i.e. contained in Γ) if and only if it is trivial on G'. It follows that both G and G' have the same characters, i.e. that the inclusion $j\colon G \hookrightarrow G'$ induces an isomorphism of group $j^*\colon \mathrm{X}^*(G') \longrightarrow \mathrm{X}^*(G)$, since the characters on G and G' are the restrictions of the characters of T by Step 2. This shows that the induced algebra homorphism $j\colon \mathrm{A}(G') \longrightarrow \mathrm{A}(G)$ maps the k-basis $\mathrm{X}^*(G')$ of $\mathrm{A}(G')$ bijectively onto the k-basis $\mathrm{X}^*(G)$ of $\mathrm{A}(G)$, and is therefore an algebra isomorphism. It follows that j is an isomorphism and therefore that $G = G'$, as claimed.

The group

$$\mathrm{X}^*(T) = \left\{T_1^{a_1} \cdots T_n^{a_n} \mid a \in \mathbb{Z}^n\right\} \cong \mathbb{Z}^n$$

is free abelian of finite rank so we may apply the theory of elementary divisors. With find that there exists a $\mathbb{Z}$-basis $\tilde{T}_1, \dots, \tilde{T}_n$ of $\mathrm{X}^*(T)$ with respect to which the subgroup Γ of $\mathrm{X}^*(T)$ is generated by $\tilde{T}_1^{d_1}, \dots, \tilde{T}_n^{d_n}$ for suitable $d_1, \dots, d_n \in \mathbb{Z}$.

Let $f\colon \mathrm{X}^*(T) \longrightarrow \mathrm{X}^*(T)$ be the unique group automorphism which maps T_i to $\tilde{T}_i$ for every i. It follows from Step 3 that there exists a (unique) homomorphism of linear algebraic groups $h\colon T \longrightarrow T$ with $h^* = f$. The homomorphism h is again an automorphism: The inverse $f^{-1}\colon \mathrm{X}^*(T) \longrightarrow \mathrm{X}^*(T)$ is again a group homomorphism, and thus there exists a (unique) homomorphism of linear algebraic groups $h'\colon T \longrightarrow T$

with $(h')^* = f^{-1}$. It follows from

$$(hh')^* = (h')^* h^* = f^{-1} f = \mathrm{id}_{\mathrm{X}^*(T)} = \mathrm{id}_T^*$$

that $hh' = \mathrm{id}_T$. We find similarly that $h'h = \mathrm{id}_T$.

The image $G' \coloneqq h(G)$ is again a closed subgroup of T, and the linear algebraic groups G and G' are isomorphic via (the restriction of) h. Let Γ' be the kernel of the restriction homomorphism $\mathrm{X}^*(T) \longrightarrow \mathrm{X}^*(G')$; it holds that

$$G' = \bigcap_{\chi \in \Gamma'} \ker(\chi).$$

It holds for every $\chi \in \mathrm{X}^*(G)$ that

$$\begin{aligned} \chi \in \Gamma' &\iff \chi|_{G'} = 1 \\ &\iff \chi|_{h(G)} = 1 \\ &\iff (\chi \circ h)|_G = 1 \\ &\iff \chi \circ h \in \Gamma \\ &\iff h^*(\chi) \in \Gamma \\ &\iff \chi \in (h^*)^{-1}(\Gamma), \end{aligned}$$

which shows that

$$\Gamma' = (h^*)^{-1}(\Gamma) = f^{-1}(\Gamma).$$

It holds that $f^{-1}(\tilde{T}_i) = T_i$ for every i, and the group Γ is generated by $\tilde{T}_1^{d_1}, \dots, \tilde{T}_n^{d_n}$. It follows that the group Γ' is generated by $T_1^{d_1}, \dots, T_n^{d_n}$.

Hence we may replace G by G', and Γ by Γ', and assume that

$$\Gamma = \left\langle T_1^{d_1}, \dots, T_n^{d_n} \right\rangle = \left\{ T_1^{a_1} \cdots T_n^{a_n} \,\middle|\, a \in d_1\mathbb{Z} \times \cdots \times d_n\mathbb{Z} \right\}.$$

It then follows that

$$\begin{aligned} G = \bigcap_{\chi \in \Gamma} \ker(\Gamma) &= \ker\left(T_1^{d_1}\right) \cap \cdots \cap \ker\left(T_n^{d_n}\right) \\ &= (\mu_{d_1} \times \mathbb{G}_{\mathrm{m}} \times \cdots \times \mathbb{G}_{\mathrm{m}}) \cap \cdots \cap (\mathbb{G}_{\mathrm{m}} \times \cdots \times \mathbb{G}_{\mathrm{m}} \times \mu_{d_n}) \\ &= \mu_{d_1} \times \cdots \times \mu_{d_n} \end{aligned}$$

with $\mu_d = \mathbb{G}_{\mathrm{m}}$ for $d = 0$.

This shows the implication 1) $\implies$ 3).

Step 6 To show the implication 3) $\implies$ 1) we may assume that $g = \mu_{d_1} \times \cdots \times \mu_{d_n} \times \mathrm{D}_r(k)$. It then follows from μ_{d_i} being a closed subgroup of $\mathbb{G}_{\mathrm{m}}$ for every i that G is a closed subgroup of

$$\underbrace{\mathbb{G}_{\mathrm{m}} \times \cdots \times \mathbb{G}_{\mathrm{m}}}_{n} \times \mathrm{D}_r(k) = \mathrm{D}_{n+r}(k).$$

This shows that G is diagonalizable. □

Corollary 12.19. The character group $\mathrm{X}^*(-)$ defines a contravariant equivalence of categories

$$\begin{Bmatrix} \text{diagonalizable} \\ \text{linear algebraic groups} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{finitely generated abelian groups} \\ \text{(with trivial } p\text{-torsion if } \operatorname{char}(k) = p > 0) \end{Bmatrix},$$
$$G \longmapsto \mathrm{X}^*(G),$$
$$f \longmapsto f^*,$$

which restrict to a contravariant equivalence

$$\{\text{tori}\} \longrightarrow \{\text{free abelian groups of finite rank}\}.$$

Proof. We denote the first of the proposed equivalences by F. That F is well-defined follows from Theorem 12.18 and Lemma 12.14. We have seen in Step 3 of the above proof that F is fully faithful.

To see that F is essentially surjective let A be a finitely generated abelian group, with trivial p-torsion if $\operatorname{char}(k) = p > 0$. It follows from the classification of finitely generated abelian groups that

$$A \cong \mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_n \oplus \mathbb{Z}^r$$

for suitable integers $d_1, \dots, d_n \geq 2$ and some $r \geq 0$; if $\operatorname{char}(k) = p > 0$ then it follows from A having trivial p-torsion that none of the integers $d_1, \dots, d_n$ is a multiple of p. The linear algebraic group

$$G \coloneqq \mu_{d_1} \times \cdots \times \mu_{d_n} \times \mathrm{D}_r(k)$$

is a closed subgroup of $\mathrm{D}_{n+r}(k)$ and it follows from Lemma 12.13 that

$$\mathrm{X}^*(G) \cong \mathrm{X}^*(\mu_{d_1}) \times \cdots \times \mathrm{X}^*(\mu_{d_n}) \times \mathrm{X}^*(\mathrm{D}_r(k)).$$

The factor $\mathrm{X}^*(\mathrm{D}_r(k))$ is given by $\mathbb{Z}^r$. It follows from p not dividing the integer d_i that $\mu_{d_i} \cong \mathbb{Z}/d_i$: The group μ_{d_i} is cyclic because it is a finite subgroup of the multiplicative group $k^\times$, and it has order d_i because the polynomial $x^{d_i} - 1 \in k[x]$ has no multiple roots since it is separable. It follows that

$$\mathrm{X}^*(\mu_{d_i}) = \mathrm{X}^*(\mathbb{Z}/d_i) \cong \mathbb{Z}/d_i$$

because every group homomorphism $\mathbb{Z}/d_i \longrightarrow \mathbb{G}_\mathrm{m}$ is already a homomorphism of linear algebraic groups. It follows altogether that

$$\mathrm{X}^*(G) \cong \mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_n \oplus \mathbb{Z}^r \cong A,$$

which shows that F is essentially surjective.

It follows from Lemma 12.17 that F restrict so the second equivalence of categories. □

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