Solve for global min var portfolio

Begin with portfolio covariance Σ and returns z. The global minimum variance portfolio is given by

$$\begin{aligned} &\min w'\Sigma w \\ s.t. 1'w =& 1 \end{aligned}$$

where the last term imposes a unique solution w/o loss of generality. Then:

$$\begin{aligned} 0 =& \Sigma w_g - \lambda 1 \\ w_g =& \lambda \Sigma^{-1} 1 \\ 1 =& \lambda 1' \Sigma^{-1} 1 \end{aligned}$$

Let $A = 1'\Sigma^{-1}1$. Then $w_g = \frac{\Sigma^{-1}1}{A}$

Proof that $w_p \Sigma w_g = \frac{1}{A}$

Pick any portfolio s.t. wlog $w_p'1 = 1$. Then

$$\begin{aligned} w_g =& \frac{\Sigma^{-1}1}{A} \\ w_p \Sigma w_g =& \frac{w_p' 1}{A} = \frac{1}{A} \end{aligned}$$

Delta Method: Derivation of Asymtotic Sample Covariance

- Note: Heavily adapted from (https://stats.stackexchange.com/questions/105337/asymptotic-distribution-of-sample-variance-of-non-normal-sample)

$$
\begin{aligned}
nS_{XY} =& \sum_I \left[\left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)\right] \\
=& \sum_I \left[\left(X_i - \mu_X - \left(\overline{X} - \mu_X\right)\right)\left(Y_i - \mu_Y - \left(\overline{Y} - \mu_Y\right)\right)\right] \\
=& \sum_I \left[\left(X_i - \mu_X\right)\left(Y_i - \mu_Y\right)\right] - \sum_I \left(Y_i - \mu_Y\right)\left(\overline{X} - \mu_X\right) \\
& - \sum_I \left(X_i - \mu_X\right)\left(\overline{Y} - \mu_Y\right) + n\left(\overline{X} - \mu_X\right)\left(\overline{Y} - \mu_Y\right) \\
=& \sum_I \left[\left(X_i - \mu_X\right)\left(Y_i - \mu_Y\right)\right] - n\left(\overline{Y} - \mu_Y\right)\left(\overline{X} - \mu_X\right) \\
\sqrt{n}\left(S_{XY} - \sigma_{XY}\right) =& \frac{\sqrt{n}}{n} \sum_I \left[\left(X_i - \mu_X\right)\left(Y_i - \mu_Y\right)\right] - \sqrt{n}\left(\overline{Y} - \mu_Y\right)\left(\overline{X} - \mu_X\right) - \sqrt{n}\sigma_{XY} \\
=& \sqrt{n} \sum_I \frac{1}{n} \left[\left(X_i - \mu_X\right)\left(Y_i - \mu_Y\right) - \sigma_{XY}\right] - \sqrt{n}\left(\overline{Y} - \mu_Y\right)\left(\overline{X} - \mu_X\right)
\end{aligned}
$$

- Next note that since by the CLT $\sqrt{n}\left(\overline{Y} - \mu_Y\right) \xrightarrow{d} N(\cdot)$ and by WLLN $\overline{X} - \mu_X \xrightarrow{p} 0$.
  - Thus by Slutsky's Theorem $\sqrt{n}\left(\overline{Y} - \mu_Y\right)\left(\overline{X} - \mu_X\right) \xrightarrow{p} 0$
- Hence

$$
\sqrt{n}\left(S_{XY} - \sigma_{XY}\right) \approx \sqrt{n} \sum_I \left[\frac{1}{n}\left(X_i - \mu_X\right)\left(Y_i - \mu_Y\right) - \sigma_{XY}\right]
$$

  - Since $E\left[\left(X_i - \mu_X\right)\left(Y_i - \mu_Y\right)\right] = \sigma_{XY}$ and

$$
V\left[\left(X_i - \mu_X\right)\left(Y_i - \mu_Y\right)\right] = E\left[\left(X_i - \mu_X\right)^2\left(Y_i - \mu_Y\right)^2\right] - \sigma_{XY}^2
$$

– Thus

$$\sqrt{n}\left(S_{XY}-\sigma_{XY}\right)\xrightarrow{d}N\left(0,\ \sigma_{XXYY}-\sigma_{XY}^2\right)$$

* Where σ_{XXYY} is defined as $E\left[(X_i-\mu_X)^2\,(Y_i-\mu_Y)^2\right]$. Use the plug-in estimators to get the estimated distribution.

– Moreover, the sample covariance matrix is distributed as

$$\sqrt{n}\left(\boldsymbol{S}_{XY}-\Sigma_{XY}\right)\xrightarrow{d}N\left(\mathbf{0},\ \begin{bmatrix}\mu_{4X}-\sigma_X^4 & \sigma_{XXYY}-\sigma_{XY}^2\\ \sigma_{XXYY}-\sigma_{XY}^2 & \mu_{4Y}-\sigma_Y^4\end{bmatrix}\right)$$

- Finally, it will be helpful to have an alternative definition of sample covariance which can be formed via convex partitions of the data.

  – Suppose X and Y are vectors of data and Z is their sum. Then $\sigma_Z^2=\sigma_X^2+\sigma_Y^2+2\sigma_{XY}$, so we should be able to estimate σ_{XY} as

$$\begin{aligned}2nS_{XY}=&\sum_I\left[\left(Z_i-\overline{Z}\right)^2\right]-\sum_I\left[\left(Y_i-\overline{Y}\right)^2\right]-\sum_I\left[\left(X_i-\overline{X}\right)^2\right]\\ &\sum_I\left[\left(\left(X_i-\overline{X}\right)+\left(Y_i-\overline{Y}\right)\right)^2\right]-\sum_I\left[\left(Y_i-\overline{Y}\right)^2\right]-\sum_I\left[\left(X_i-\overline{X}\right)^2\right]\\ =&2\sum_I\left[\left(X_i-\overline{X}\right)\left(Y_i-\overline{Y}\right)\right]\ \checkmark\end{aligned}$$

  – Thus

$$\begin{aligned}\sqrt{n}\left(\frac{1}{2}\left(S_Z^2-S_X^2-S_Y^2\right)-\sigma_{XY}\right)\xrightarrow{d}&N\left(0,\ \sigma_{XXYY}-\sigma_{XY}^2\right)\\ \equiv&N\left(0,\ \mu_{4Z}-\mu_{4X}-\mu_{4Y}-\left(\sigma_Z^4-\sigma_X^4-\sigma_Y^4\right)\right)\end{aligned}$$

MCMC

Overview

- Use a Bayesian MCMC approach, with Gibbs sampling
  - This approach relies heavily on the central limit theorem and other asymptotics
  - Suppose we pick a test portfolio P of mx1 weights w_P from which to test our candidate weights for the minimum variance portfolio w_G
    * Define S_G as the sample variance of R_G, the returns of all assets weighted by w_G
    * Define S_{GP} as the sanoke covariance of the minimum variance portfolio and the test portfolio. For shorthand, designate $S \equiv \{S_G, S_{GP}\}$
      · Note given w_G, the test portfolio weights w_P, and the data D, S is fully specified.
      · Since w_P, D, w_G only enter the model via S, conditioning on S is equivelent to conditioning on w_P, w_G, and D
    * Define $\zeta_G^2 \equiv \frac{\mu_{4G}-\sigma_G^4}{n}$ and $\zeta_P^2 \equiv \frac{\mu_{4P}-\sigma_P^4}{n}$ (both unobserved). For shorthand, designate $\zeta^2 \equiv \{\zeta_G^2, \zeta_P^2\}$
    * We must estimate w_G, σ_G^2 and ζ, implying m+1 parameters.
  - Unfortunately, directly evaluating the weights leads to intractable posteriors. This leads to the following general "almost MCMC" algorithm:
    1. Draw a random test portfolio P with overall returns R_P.
    2. Draw from $p(\sigma_G^2|\cdot)$, $p(\zeta_G|\cdot)$, $p(\zeta_P|\cdot)$, $p(S_G|\cdot)$, $p(S_P|\cdot)$ that is, draw from the parameter posteriors for these parameters given all other parameters. Note this fully specifies a new vector of weights for w_G, as shown in the following steps.

3. Now we partition the portfolio into three components. Assign each index $i \in 1:m$ randomly to one of sets $G1$, $G2$, or $G3$. Then define the following mx1 vectors:

$$\Omega_{G1} \equiv \omega_{G1} \left\{ \iota \left(i \in G1 \right) \right\}_{i \in 1:m}$$
$$\Omega_{G2} \equiv \omega_{G2} \left\{ \iota \left(i \in G2 \right) \right\}_{i \in 1:m}$$
$$\Omega_{G3} \equiv \omega_{G3} \left\{ \iota \left(i \in G3 \right) \right\}_{i \in 1:m}$$

where $\iota$is an indicator function, and ω_{G1}, ω_{G2}, ω_{G3} are scalars. That is, each vector contains a constant value for all assigned indices and zero for all other indices.

4. Define $w'_G \equiv \left\{ \left(\Omega_{iG1} + \Omega_{iG2} + \Omega_{iG3} \right) w_{iG} \right\}_{i \in 1:m}$. Then solve for $\omega \equiv \left\{ \omega_{G1}, \omega_{G2}, \omega_{G3} \right\}$. Note that these parameters are fully specified by the following three conditions:
   (a) The sample variance of the new vector of weights is S_G. That is, $V\left(R'_G\right) = S_G$
   (b) The sample covariance of the new vector of weights with the test portfolio is S_{GP}, or $cov\left(R'_G, R_P\right) = S_{GP}$.
   (c) The weights of the new portfolio add to 1. This can be expressed as $\left(\Omega_{G1} + \Omega_{G2} + \Omega_{G3}\right) \cdot w_G = 1$.
5. Repeat steps 1-5 many many times.

Posteriors

- The likelihood is proportional to

$$p\left(S|w_G, \zeta^2, \sigma_G^2, w_P\right) \propto \left(\frac{1}{\zeta_G^2}\right)^{\frac{1}{2}} \exp\left[-\frac{\left(S_G - \sigma_G^2\right)^2}{2\zeta_G^2}\right] \left(\frac{1}{\zeta_P^2}\right)^{\frac{1}{2}} \exp\left[-\frac{\left(S_{GP} - \sigma_G^2\right)^2}{2\zeta_P^2}\right]$$

- Consider the following priors (IG is the inverse gamma distribution):

$$\sigma_G^2 \sim N\left(\theta_G,\, \delta_G^2\right)$$

$$\zeta_G^2 \sim IG\left(\alpha_G,\, \beta_G\right)$$

$$\zeta_P^2 \sim IG\left(\alpha_P,\, \beta_P\right)$$

$$S_G \sim N\left(\theta_{SG},\, \delta_{SG}^2\right)$$

$$S_{GP} \sim N\left(\theta_{SGP},\, \delta_{SGP}^2\right)$$

- Now derive the posteriors:
  - Start with σ_G^2
    - * Use the property that the convolution of normals is a normal $N(a, b)$ where a is the precision weighted average of the source means and b is the inverse sum of the source precisions.

$$\begin{aligned}
p\left(\sigma_G^2 | S,\, \zeta^2\right) \propto & p\left(S|\zeta^2,\, \sigma_G^2\right) p\left(\sigma_G^2;\, N\left(\theta_G,\, \delta_G^2\right)\right) \\
& \propto \left(\frac{1}{\zeta_G^2}\right)^{\frac{1}{2}} \exp\left[-\frac{\left(S_G-\sigma_G^2\right)^2}{2\zeta_G^2}\right] \times \left(\left(\frac{1}{\zeta_P^2}\right)^{\frac{1}{2}} \exp\left[-\frac{\left(S_{GP}-\sigma_G^2\right)^2}{2\zeta_P^2}\right]\right) \times \left(\frac{1}{\delta_G^2}\right)^{\frac{1}{2}} \text{ex} \\
& \propto p\left(\sigma_G^2,\, N\left(\left[\frac{S_G}{\zeta_G^2}+\frac{S_{GP}}{\zeta_P^2}+\frac{\theta_G}{\delta_G^2}\right]\zeta_G^{2*},\, \zeta_G^{2*}\right)\right) \\
s.t. & \\
\zeta_G^{2*} = & \left[\frac{1}{\zeta_G^2}+\frac{1}{\zeta_P^2}+\frac{1}{\delta_G^2}\right]^{-1}
\end{aligned}$$

– Now for ζ_G^2

$$
\begin{aligned}
p\left(\zeta_G^2|S,\, \zeta_P^2\, \sigma_G^2\right) \propto & p\left(S|\, \zeta^2,\, \sigma_G^2\right) p\left(\zeta_G^2;\; IG\left(\alpha_G,\, \beta_G\right)\right) \\
& \propto \left(\frac{1}{\zeta_G^2}\right)^{\frac{1}{2}} \exp\left[-\frac{\left(S_G-\sigma_G^2\right)^2}{2\zeta_G^2}\right] \times \left(\left(\frac{1}{\zeta^2}\right)^{\frac{1}{2}} \exp\left[-\frac{\left(S_{GP}-\sigma_G^2\right)^2}{2\zeta_P^2}\right]\right) \times p\left(\zeta_G,\, IG\left(\alpha\right.\right. \\
& \propto \left(\frac{1}{\zeta_G^2}\right)^{\frac{-1}{2}} \exp\left[-\frac{\left(S_G-\sigma_G^2\right)^2}{2\zeta_G^2}\right] \times \left(\left(\zeta_G^2\right)^{-\alpha_G-1} \exp\left(-\frac{\beta_G}{\zeta_G^2}\right)\right) \\
& \propto \left(\frac{1}{\zeta_G^2}\right)^{\alpha_G+\frac{1}{2}} \exp\left(-\frac{\beta}{\zeta_G^2}-\frac{\left(S_G-\sigma_G^2\right)^2}{2\zeta_G^2}\right) \\
& \propto p\left(\zeta_G,\, IG\left(\alpha_G+\frac{1}{2},\, \beta_G+\frac{\left(S_G-\sigma_G^2\right)^2}{2}\right)\right)
\end{aligned}
$$

– Apply the same logic to ζ_P^2

$$
\begin{aligned}
p\left(\zeta_P^2|S,\, \sigma_G^2\right) \propto & p\left(S|\zeta^2,\, \sigma_G^2\right) p\left(\zeta_P^2;\, IG\left(\alpha_P,\, \beta_P\right)\right) \\
& \propto \left(\frac{1}{\zeta_G^2}\right)^{\frac{-1}{2}} \exp\left[-\frac{\left(S_G-\sigma_G^2\right)^2}{2\zeta_G^2}\right] \times \left(\left(\frac{1}{\zeta_P^2}\right)^{\frac{-1}{2}} \exp\left[-\frac{\left(S_{GP}-\sigma_G^2\right)^2}{2\zeta_P^2}\right]\right) \times p\left(\zeta_P;\, IG\left(\alpha\right.\right. \\
& \propto \left(\frac{1}{\zeta_P^2}\right)^{\frac{-1}{2}} \exp\left[-\frac{\left(S_{GP_j}-\sigma_G^2\right)^2}{2\zeta_P^2}\right] \times \left(\left(\zeta_P^2\right)^{-\alpha_P-1} \exp\left(-\frac{\beta_P}{\zeta_P^2}\right)\right) \\
& \propto \left(\frac{1}{\zeta_P^2}\right)^{\alpha_P+\frac{1}{2}} \exp\left(-\frac{\beta_P}{\zeta_P^2}-\frac{\left(S_{GP}-\sigma_G^2\right)^2}{2\zeta_P^2}\right) \\
& \propto \left(\zeta_P^2;\, IG\left(\alpha_P+\frac{1}{2},\, \beta_P+\frac{\left(S_{GP}-\sigma_G^2\right)^2}{2}\right)\right)
\end{aligned}
$$

- Again for S_G:

$$
\begin{aligned}
p\left(S_G|\zeta^2,\, \sigma_G^2,\, S_P\right) \propto & p\left(D|S,\, \zeta^2,\, \sigma_G^2\right) p\left(S_G;\, N\left(\theta_{SG},\, \delta_{SG}^2\right)\right) \\
& \propto \left(\frac{1}{\zeta_G^2}\right)^{\frac{1}{2}} \exp\left[-\frac{\left(S_G - \sigma_G^2\right)^2}{2\zeta_G^2}\right] \times \left(\left(\frac{1}{\zeta_P^2}\right)^{\frac{1}{2}} \exp\left[-\frac{\left(S_{GP} - \sigma_G^2\right)^2}{2\zeta_P^2}\right]\right) \times \left(\frac{1}{\delta_{SG}^2}\right)^{\frac{1}{2}} \\
& \propto p\left(\sigma_G^2,\, N\left(\left[\frac{\sigma_G^2}{\zeta_G^2} + \frac{\theta_G}{\delta_G^2}\right]\zeta_{SG}^{2*},\, \zeta_{SG}^{2*}\right)\right)
\end{aligned}
$$

$$
s.t.
$$

$$
\zeta_{SG}^{2*} = \left[\frac{\sigma_G^2}{\zeta_G^2} + \frac{\theta_{SG}}{\delta_{SG}^2}\right]^{-1}
$$

- Again for S_P:

$$
\begin{aligned}
p\left(S_P|\zeta^2,\, \sigma_G^2,\, S_G\right) \propto & p\left(D|S,\, \zeta^2,\, \sigma_G^2\right) p\left(S_P;\, N\left(\theta_{SGP},\, \delta_{SGP}^2\right)\right) \\
& \propto \left(\frac{1}{\zeta_G^2}\right)^{\frac{1}{2}} \exp\left[-\frac{\left(S_G - \sigma_G^2\right)^2}{2\zeta_G^2}\right] \times \left(\left(\frac{1}{\zeta_P^2}\right)^{\frac{1}{2}} \exp\left[-\frac{\left(S_{GP} - \sigma_G^2\right)^2}{2\zeta_P^2}\right]\right) \times \left(\frac{1}{\delta_{SGP}^2}\right) \\
& \propto p\left(\sigma_G^2,\, N\left(\left[\frac{\sigma_G^2}{\zeta_P^2} + \frac{\theta_{SGP}}{\delta_{SGP}^2}\right]\zeta_{SGP}^{2*},\, \zeta_{SGP}^{2*}\right)\right)
\end{aligned}
$$

$$
s.t.
$$

$$
\zeta_{SGP}^{2*} = \left[\frac{\sigma_G^2}{\zeta_P^2} + \frac{\theta_{SGP}}{\delta_{SGP}^2}\right]^{-1}
$$

Mapping draws to weights

- Solving the system for the three weight
  - Define the following. The key here is each of these quantities is known given our

previous guess of w_G.

$$w_s = \left\{ \sum_{i \in 1:m} w_{iG} \iota \left(i \in Gk \right) \right\}_{k \in 1:3} \quad (3x1)$$

$$\Psi_G = \begin{bmatrix} S_{G1} & S_{G12} & S_{G13} \\ S_{G12} & S_{G2} & S_{G23} \\ S_{G13} & S_{G23} & S_{G3} \end{bmatrix} \quad (3x3)$$

$$\Psi_{PG} = \begin{bmatrix} S_{PG1} \\ S_{PG2} \\ S_{PG3} \end{bmatrix} \quad (3x1)$$

* Here, S_{G1} is the sample variance of the G1 partion, S_{G13} is the sample covariance between the G1 and G3 portfolios, while $S_{PG1}$is the covariance between portfolio P and the G1 portfolio. w_s is the sum of the weights of the G1, G2, and G3 portfolios (3x1 vector)

- Then we solve:

$$\begin{aligned} \omega' \Psi_G \omega &= S_G \\ \omega' \Psi_{PG} &= S_{GP} \\ \omega' w_s &= 1 \end{aligned}$$

- (See the mathematica file Algebra for some solution strategies)

Important References

- Wikipedia
  - Gamma distribution
  - Inverse gamma distribution

  - Wishart Distribution
  - Inverse Wishart Distribution
  - Estimation of Covariance Matrices

- Other web sites