CSCI 663 Homework 12

Chuong Ngo

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1 Overview

A feedback vertex set of an undirected or directed graph is a subset of vertices that intersects with the vertex set of each cycle in the graph. In other words, it is a set of vertices whose removal leaves a graph without cycles because the set contains at least one vertex of any cycle in the graph. The associated decision problem for this problem is:

INSTANCE: A graph $G = (V, E)$ and a positive integer K
QUESTION: Is there a subset $X \subseteq V$ with $|X| \leq k$ such that G with the vertices from X deleted is cycle-free?

Variants of this problem includes the weighted feedback vertex set (WFVS) and the unweighted feedback vertex set (UFVS). The WFVS is defined as finding a minimum feedback vertex set of a given weighted graph (G, w). When $w = 1$, a special case of WFVS appears known as UFVS. Bar-Yehuda et. al introduces an approximation algorithms that has a linear complexity.[1]

2 Unweighted Feedback Vertex Set

For this section G is a unweighted graph and the $2-3-subgraph$ is a subgraph H of G such that the degree in H of every vertex in $A(G)$ is either 2 or 3. The degree of a vertex is the number of edges incident to the vertex. The algorithm presented by the authors for this case is as follows:

if G is a forest, then

$F \leftarrow \emptyset$

else begin:

Using Depth First Search, find a maximal 2-3-subgraph H of G;

Using Breadth First Search, find the set X of critical linkpoints in H;

Let Y be the set of allowed branchpoints in H;

Find a set W that covers all branchpoint-free cycles of H which are not covered by X; $F \leftarrow X \cup Y \cup W$;

end.

[1]Citation: Bar-Yehuda, R., Geiger, D., Naor, J., & Roth, R. M. (1998). Approximation algorithms for the feedback vertex set problem with applications to constraint satisfaction and Bayesian inference. SIAM journal on computing, 27(4), 942-959.

In other words, let H' be a graph obtained from H by removing the set X along its incident edges. Let H_b be a subgraph of H' induced by the allowed linkpoints and blackout vertices of H'. A *linkpoint* is a 2-degree vertex and a *blackout vertex* is a prescribed vertex that is not allowed to participate in any feed vertex set. For every isolated cycle in H_b, an allowed linkpoint from the cycle to W is arbitrarily chosen. Next, each maximal chain of allowed linkpoints in H_b are replaced by an edge, creating the graph H_b^*. Unit costs are then assigned to all edges corresponding to a chain of allowed linkpoints, and a zero cost to all other edges. A minimum-cost spanning forest, T of H_b^* is then calculated. This algorithm has a linear time complexity of $O(4-(2/n))$. To prove that the algorithm outputs a feedback vertex set, two lemmas can be used.

Lemma 1 Let H be a maximal 2-3-subgraph of a valid graph G and let Γ be a simple cycle in G. Then, one of the following holds:

1. Γ is a witness cycle of some critical linkpoint of H
2. Γ passes through some allowed branchpoint of H
3. Γ is a cycle in H that consists only of blackout vertices or allowed linkpoints of H.

Assume that none of the above points in the lemma hold and that implies in particular that Γ cannot be entirely contained in H. This yields two cases:

1. Γ does not intersect with H
2. Γ intersects with H only in blackout vertices and allowed linkpoints of H.

For the first case, Γ and H can be joined to obtain a 2-3 graph H^* of G that contains H as a proper subgraph and this contradicts the maximality of H. For the second case, Γ intersects with H only in blackout vertices, Γ and H can be joined, contradicting the maximality of H. If Γ intersects with H in some allowed linkpoints of H and since Γ is assumed to not be contained in H, two allowed linkpoints v_1 and v_2 in $V(\Gamma)\cap V(H)$ that are connected by path P along Γ such that $V(P)\cap V(H)\cap A(G)=v_1,v_2$. P is also not entirely contained in H. Joining P and H results in 2-3-subgraph of G and contains H as a proper subgraph, contradicting the maximality of H.

Lemma 2 Let H be a maximal 2-3 subgraph of G and let Γ_1 and Γ_2 be witness cycles in G of two distinct critical linkpoints in H. That implies Γ_1 and Γ_2 are independent cycles like $V(\Gamma_1)\cap V(T_2)\subseteq B(G)$.

To prove lemma 2, assume v_1 and v_2 be critical linkpoints associated with Γ_1 and Γ_2, respectively. Also assume that $V(\Gamma_1)\cap V(\Gamma_2)$ does not contain an allowed vertex $u\in A(G)$. There is a path P in G that runs along parts of cycles Γ_1 and Γ_2, starting from v_1, passing through u, and ending at v_2 and implies $V(P)\cap V(H)\cap A(G)=\{v_1,v_2\}$ since Γ_1 and Γ_2 are witness cycles. v_1 and v_2 are both distinct critical linkpoints and vertex u cannot possibly coincide with either of them, implying P is not entirely contained in H. Joining both P and H results in a 2-3 subgraph of G. Thus, a contradiction.