Homework 1

1. Let V be a finite dimensional vector space and $L(V)$ linear maps from V to itself. Then $L(V)$ is also a finite dimensional vector space and so is the product $L(V) \times L(V)$. Define a map
$$F: L(V) \times L(V) \to L(V)$$
by
$$F(A,B) = A \circ B.$$
Show that F is differentiable at all $(A,B) \in L(V) \times L(V)$ and calculate the derivative.

   One way to do this is to fix a basis and the write A and B in terms of this basis. Then the composition $A \circ B$ is just matrix multiplication and the derivative can be taken using the usual rules of calculus. **DO NOT DO THIS!!!** Understanding how to do with this without choosing a basis (i.e. fixing coordinates) is key to many of things we will do in this class.

2. For $A \in L(V)$, define A^k to be the k-fold composition $A \circ \cdots \circ A$ and define $P: L(V) \to L(V)$ by
$$P(A) = c_n A^n + \cdots + c_0 \cdot \mathrm{Id}.$$
Show that P is differentiable and that the derivative at A is given by
$$P_*(A) = nc_n A^{n-1} + (n-1)c_{n-1}A^{n-2} + \cdots + c_1 \cdot \mathrm{Id}.$$

3. Problem #7 from Chapter 1 in Lee.

4. Problem #9 from Chapter 1 in Lee.

5. Let V be an n-dimensional vector space and $G_k(V)$ the Grassmannian of k-dimensional subspaces of V. Given $X \in GL(V)$ we have a map
$$X_*: G_k(V) \to G_k(V)$$
given by
$$X_*(S) = X(S).$$
Show that X_* is a diffeomorphism.

   Choose an $P \in G_k(V)$ and define a map
$$\pi: GL(V) \to G_k(V)$$
by
$$\pi(X) = X(P).$$
Show that π is a submersion. (**Hint:** You should first prove this at the identity in $GL(V)$ and use this to prove the general case. To calculate the

map at the identity, first pick a $(n-k)$-dimensional subspace Q such that $P \cap Q = \{0\}$ and then, as in class, $L(P;Q)$ is a neighborhood of P in $G_k(V)$. If we choose a sufficiently small neighborhood U of the identity then $\pi(U)$ will lie in the $L(P;Q)$. We can then choose a basis $E_1, \ldots, E_n$ for V such that $E_1, \ldots, E_k$ spans P and $E_{k+1}, \ldots, E_n$ spans Q. This defines coordinates for both $GL(V)$ and $L(P;Q)$ and the map π has a very simple form in these coordinates. After first doing this calculation in coordinates, you may also try to show this without coordinates using the fact that inclusions of vector spaces and projections to subspaces are both smooth maps.)