VE281

Data Structures and Algorithms

Binary Search Trees and AVL Trees

Announcement

- We will give you a chance to pre-test your code.
  - It will be available to you this Saturday.
  - See the TA announcement later.

- Written Homework Three was posted on Sakai.
  - Due by 11:40 am on Nov. 13^{th}.

Review

- Binary Search Trees
  - search, insertion, removal
- Average Case Time Complexity
  - The average case time complexity for a **successful** search is $\Theta(\log n)$

Outline

- Average Case Time Complexity
- Rank Search
- Range Search
- AVL Trees

Average Case Time Complexity

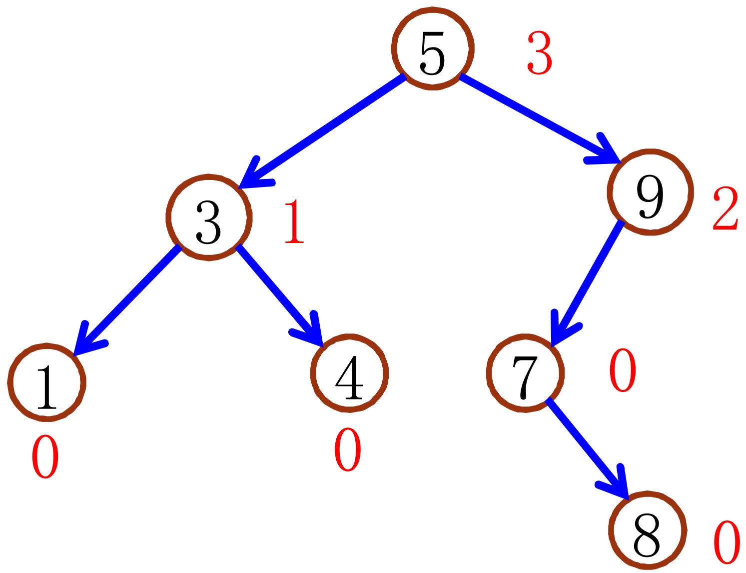
- Given n nodes, the average-case time complexity for an **unsuccessful search** is $O(\log n)$.
- Given n nodes, the average-case time complexities for search, insertion, and removal are all $O(\log n)$.
  - Insertion and removal include "search".

	Search	Insert/Remove
Linked List	$O(n)$	$O(n)$
Sorted Array	$O(\log n)$	$O(n)$
Hash Table (Separate Chaining)	$O(L)$	$O(L)$
BST	$O(\log n)$	$O(\log n)$

BST with **leftSize**

- Each node has an additional field **leftSize**, indicating the number of nodes in its left subtree.

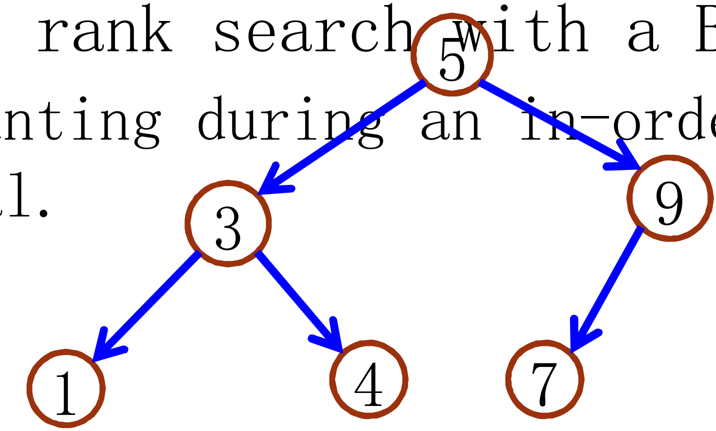
```
struct node {
   Item item;
   int leftSize;
   node *left;
   node *right;
};
```



Should change insertion and removal methods.

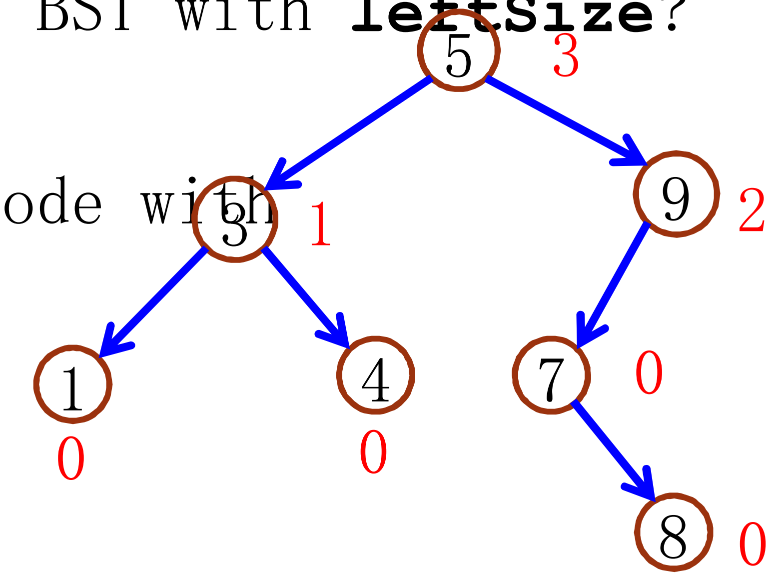
Rank Search

- **Rank**: the index of the key in the **ascending order**.
  - We assume that the smallest key has rank 0.
- **Rank search**: get the key with rank k (i.e., the k-th smallest key).
- Hash table does not support efficient rank search.
- How to do rank search with a BST?
  - Keep counting during an in-order depth-first traversal.



BST Rank Search

- Can we increase the efficiency of rank search with a BST with **`leftSize`**?
- What is the node with
  - rank = 3?
  - rank = 2?
  - rank = 5?



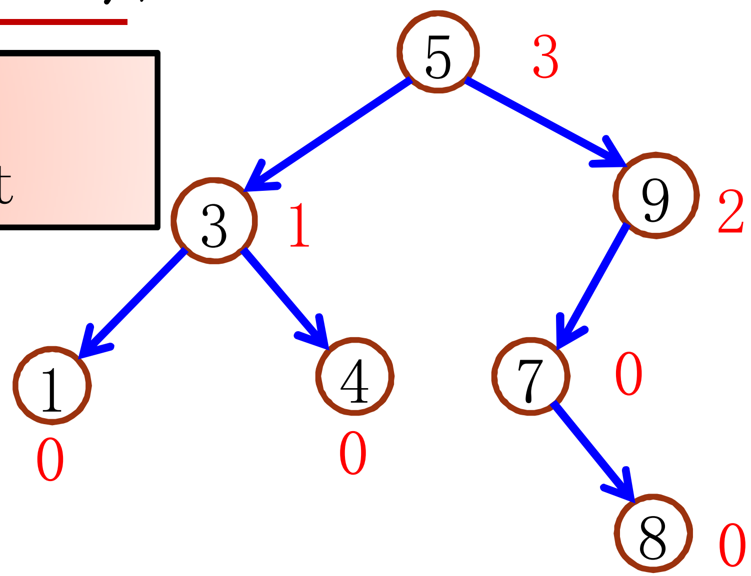
- Observation: **`x.leftSize`** = the rank of **`x`** in the tree rooted at **`x`**.
  - The rank of node 9 is 2 in the tree rooted at node 9.

BST Rank Search

```
node *rankSearch(node *root, int rank) {
  if(root == NULL) return NULL;
  if(rank == root->leftSize) return root;
  if(rank < root->leftSize)
    return rankSearch(root->left, rank);
  else
    return rankSearch(root->right,
      rank - 1 - root->leftSize);
}
```

The number of nodes including the current **root** and its subtree.

What will **rankSearch(root,5)** return?

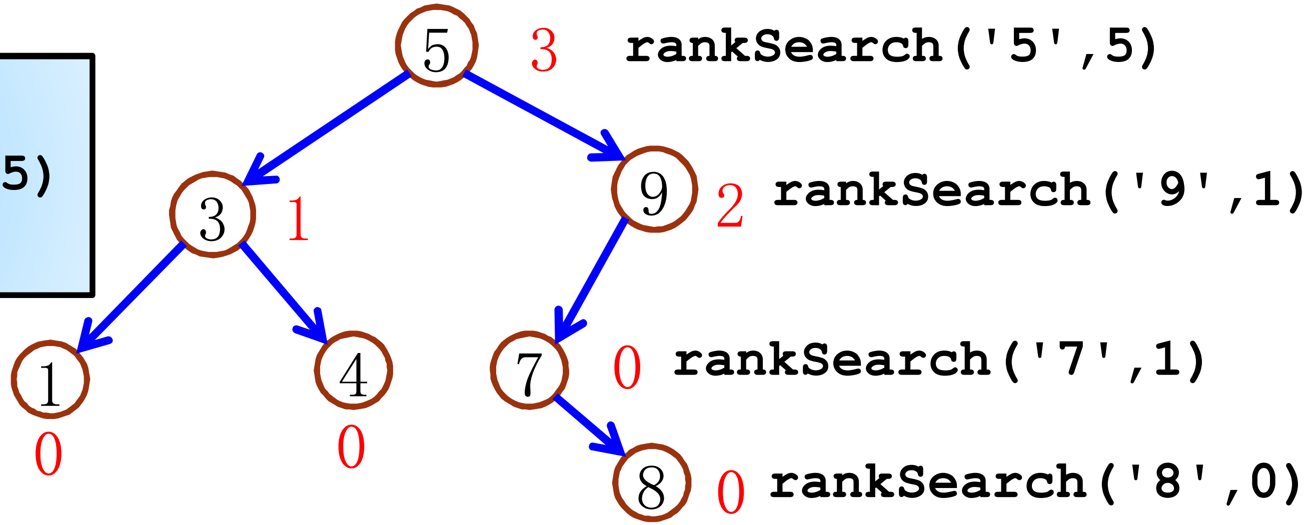


BST Rank Search

Example

```
node *rankSearch(node *root, int rank) {
  if(root == NULL) return NULL;
  if(rank == root->leftSize) return root;
  if(rank < root->leftSize)
    return rankSearch(root->left, rank);
  else
    return rankSearch(root->right,
      rank - 1 - root->leftSize);
}
```

What will **rankSearch(root,5)** return?



Outline

- Average Case Time Complexity
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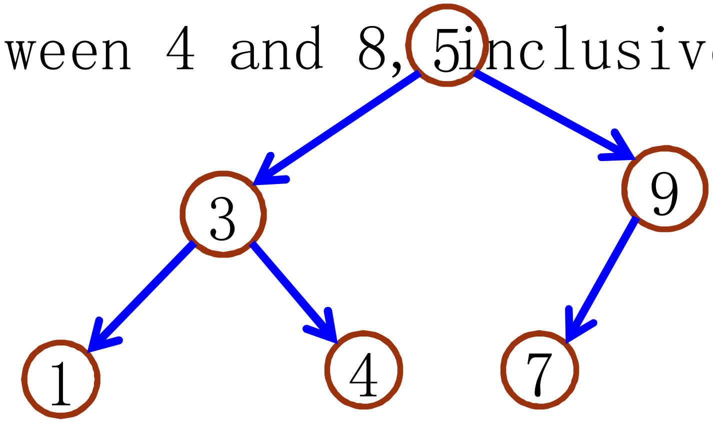
- Range Search
- AVL Trees

BST Range Search

- Instead of finding an exact match, find all items whose keys fall **between a range of values, inclusive**.
  - E.g., between 4 and 8, inclusive.



How could you implement range search?

- Example applications:
  - Buy ticket for travel between certain dates.
  - List the top-10 most popular songs.

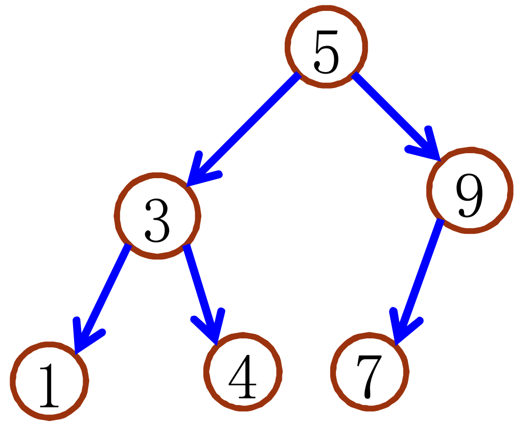
BST Range Search

Algorithm

1. If node is in search range add node to results.
2. Compute range of left subtree.
   - If search range covers all or part of left subtree, search left. (**recursive call**)
3. Compute range of right subtree.
   - If search range covers all or

```
void rangeSearch(node *root, Key searchRange[],
  Key subtreeRange[], List results)
```

   right. (**recursive call**)



4. Return results.

BST Range Search

Example

rangeSearch('5', [4,8], [-∞,+∞], results)

searchRange subtreeRange

Is 5 in [4,8]? results ← 5

Does [-∞,4] overlap [4,8]? Yes

Is 3 in [4,8]? No

Does [-∞,2] overlap [4,8]? No

Does [4,4] overlap [4,8]? Yes

Is 4 in [4,8]? results ← 4

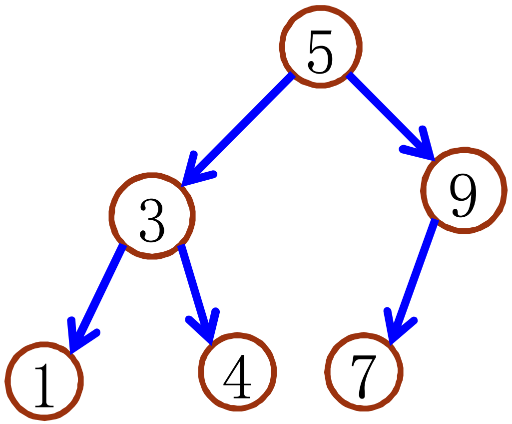
Does [6,+∞] overlap [4,8]? Yes

Is 9 in [4,8]? No

Does [6,8] overlap [4,8]? Yes

Is 7 in [4,8]? results ← 7

Does [10,+∞] overlap [4,8]? No



results:
5,4,7

BST Range Search

Supporting Functions

- If node is in the search range, add node to the **results** list.

- Compute subtree’ s range:
  - Replace upper bound of left subtree by node’ s value (If possible, node’ s value “minus one”).
  - Replace lower bound of right subtree by node’ s value (If possible, node’ s value “plus one”).

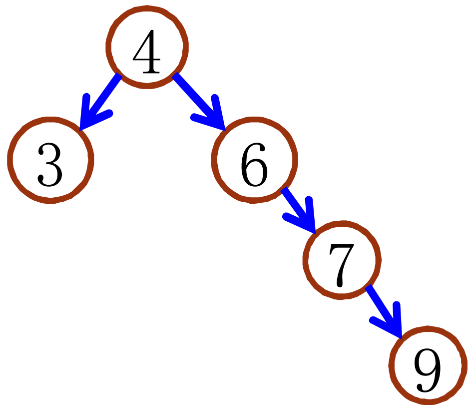
- If search range covers all or part of subtree, search subtree.

Outline

- Average Case Time Complexity
- Rank Search
- Range Search
- AVL Trees

Motivation

- Given n nodes, the **average case** time complexities for search, insertion, and removal on BST are all $O(\log n)$.
- However, the **worst case** time complexities are still $O(n)$.
  - The reason is that a tree could become "**unbalanced**" after a number of insertions and removals.



- We want to maintain the tree as a "**balanced**" tree.

Balanced Search Trees

- What are the requirements to call a tree a balanced tree?

- Would you require a tree to be perfect/complete to call it balanced?
  - No! They ar

Balanced Search Trees

- We need another definition of "balanced condition."
- We want the definition to satisfy the following two criteria:
  1. Height of a tree of n nodes = $O(\log n)$.
  2. Balance condition can be maintained efficiently: $O(1)$ time to rebalance a tree.
- Several balanced search trees, each with its own balance condition
  - AVL trees
  - 2-3 trees
  - red-black trees

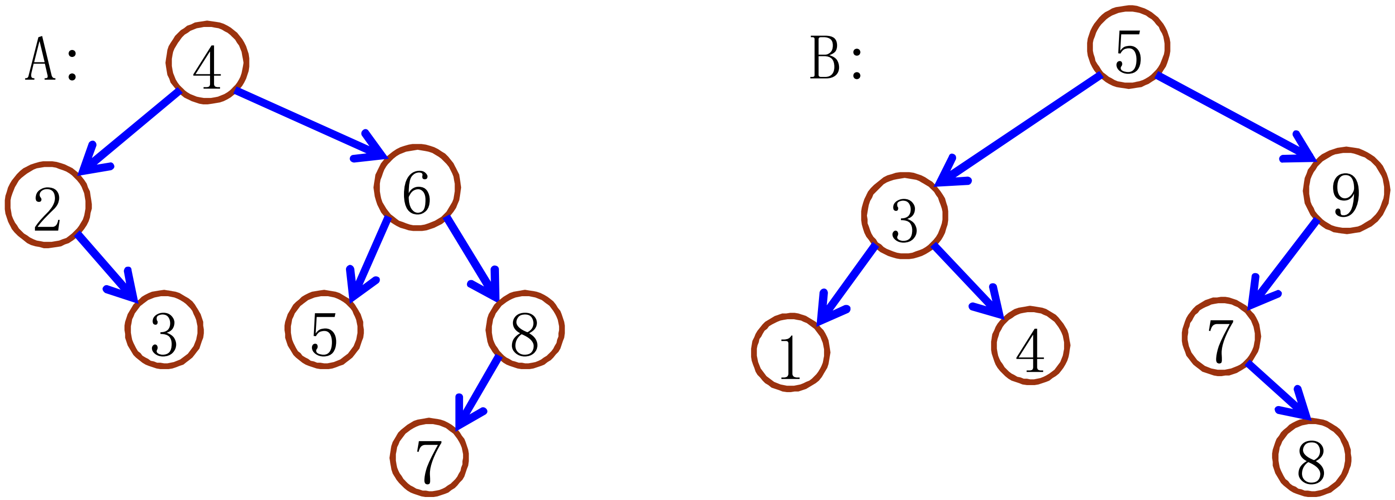
AVL Trees

- Adelson-Velsky and Landis trees
  - AVL tree is a binary search tree.

- AVL trees' balance condition:
  - An empty tree is AVL balanced.
  - A non-empty binary tree is AVL balanced if
  1. Both its left and right subtrees are AVL balanced, and
  2. The height of left and right subtrees differ by at most 1.

AVL Trees

- Are the following trees AVL balanced?



Properties of AVL Trees

- The height h of an AVL balanced tree with n internal nodes satisfies

$$\log_2(n+1) - 1 \leq h \leq 1.44 \log_2(n+2)$$

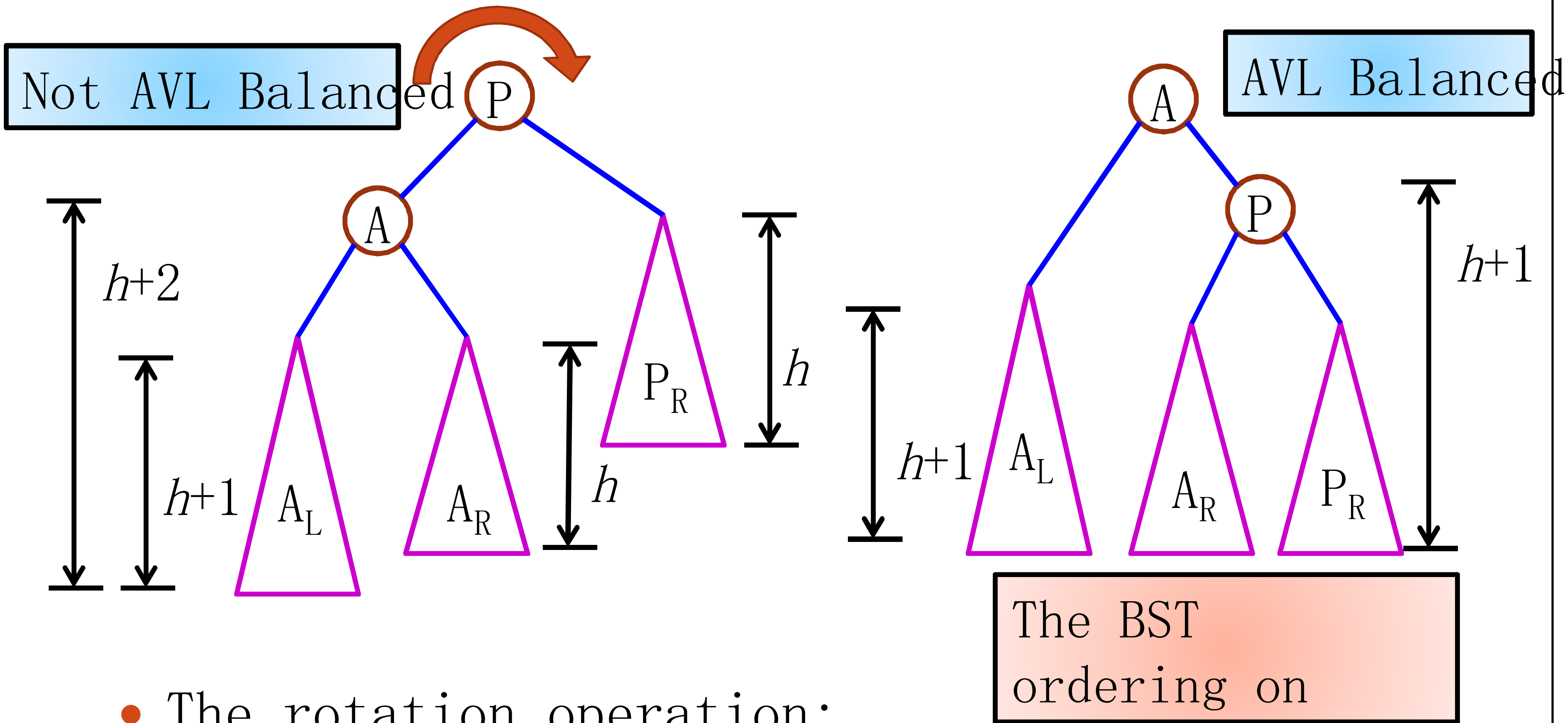
- AVL trees satisfies the general "balanced condition" 1:
  - The height of a tree of n nodes is $O(\log n)$.
  - Search is guaranteed to always be $O(\log n)$ time!
- We will also show that AVL trees satisfy the general "balance condition" 2:
  - Balance condition can be maintained efficiently.

AVL Trees Operations

- Search, insertion, and removal all work exactly the same as with BST.

- However, after each insertion or removal, we must check whether the tree is still AVL balanced.
  - If not, we need to “re-balance” the tree.

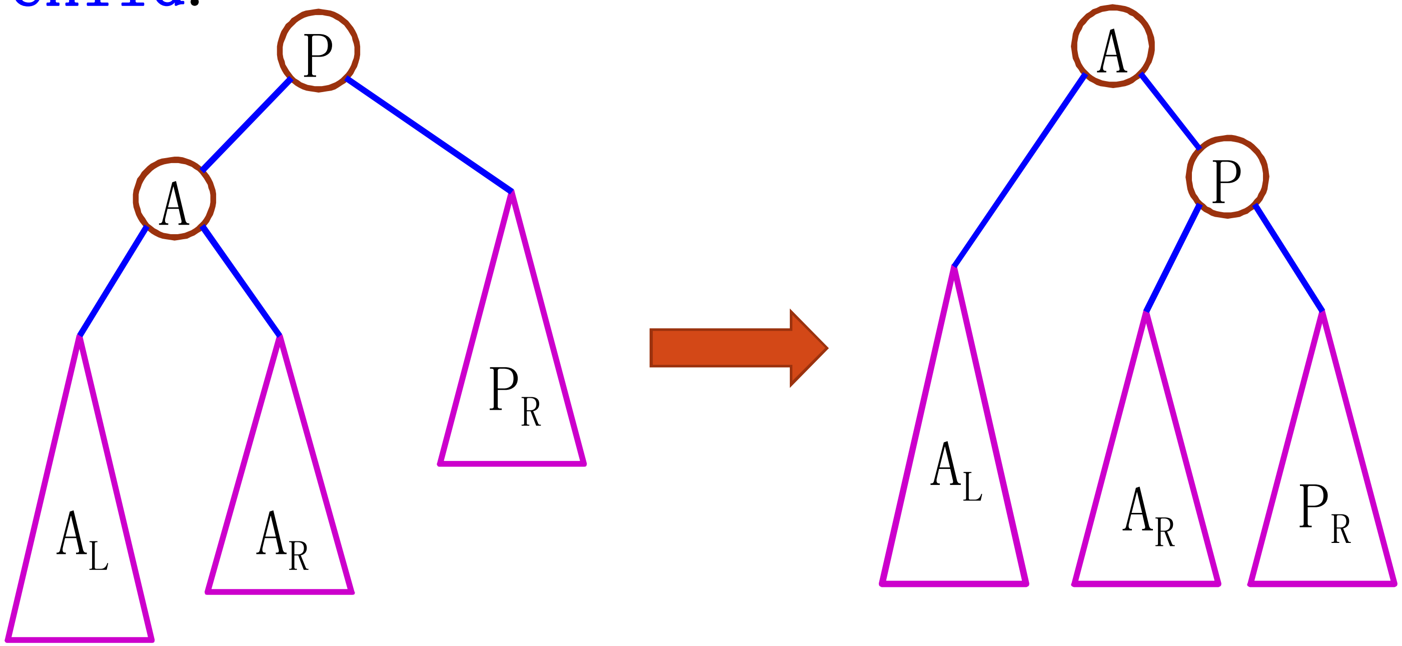
Re-Balance the Tree via Rotation



- The rotation operation:
  - Interchange the role of a parent and one of its children, while still preserving the BST ordering on the keys.

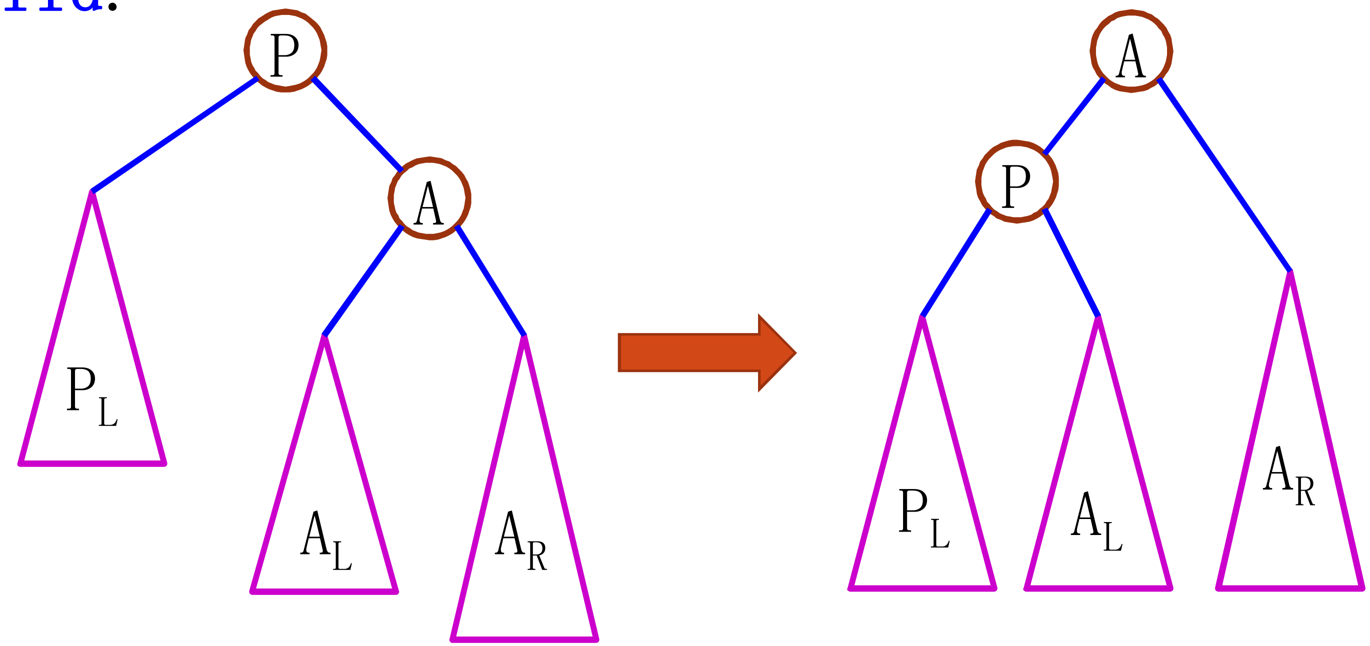
Right Rotation

1. The right link of the left child becomes the left link of the parent.
2. Parent becomes right child of the old left child.



Left Rotation

- The left link of the **right child** becomes the right link of the **parent**.
- **Parent** becomes left child of the **old right child**.

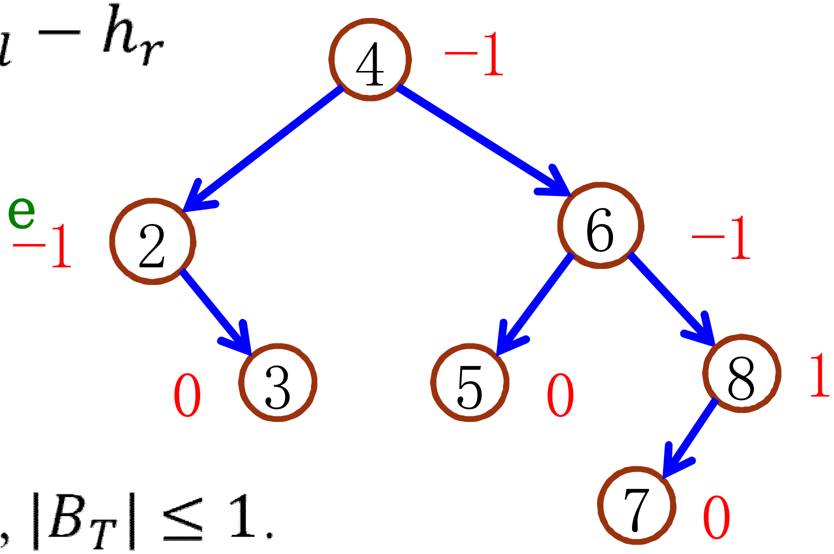


Balance Factor

- Let T_l and T_r be the left and right subtrees of a tree rooted at node T.
- Let h_l be the height of T_l and h_r the height of T_r.
- Define the **balance factor** (B_T) of node T as

$$B_T = h_l - h_r$$

Balance Factor Example



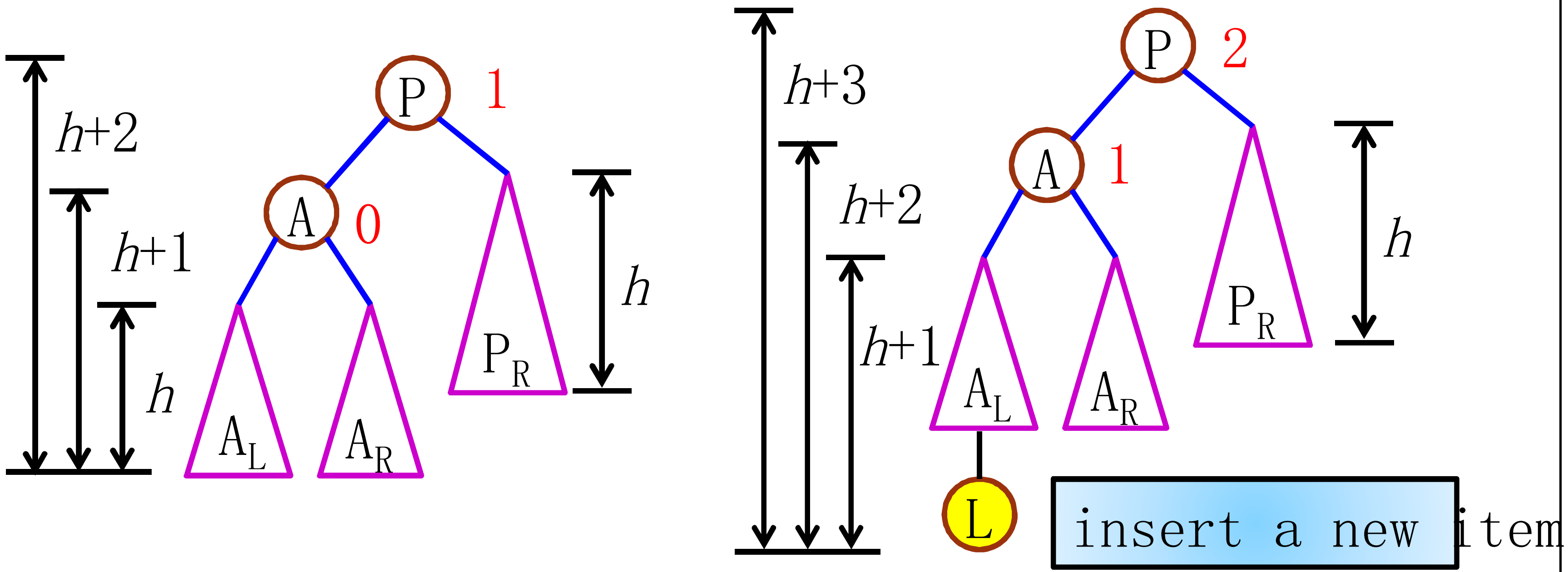
- AVL tree's balance condition:
  - For **every node** T in the tree, $|B_T| \leq 1$.

Insertion

- Inserting an item in a tree affects potentially the heights of all of the nodes along the **access path**, i.e., the path from the root to that leaf.
- When an item is inserted in a tree, the height of any node on the access path may increase by one.
- To ensure the resulting tree is still AVL balanced, the heights of all the nodes along the access path must be **recomputed** and the AVL balance condition must be **checked**.
  - Sometimes, increasing the height by one does not violate the AVL balance condition.
  - In other cases, the AVL balance condition is violated

Breaking AVL Balance Condition

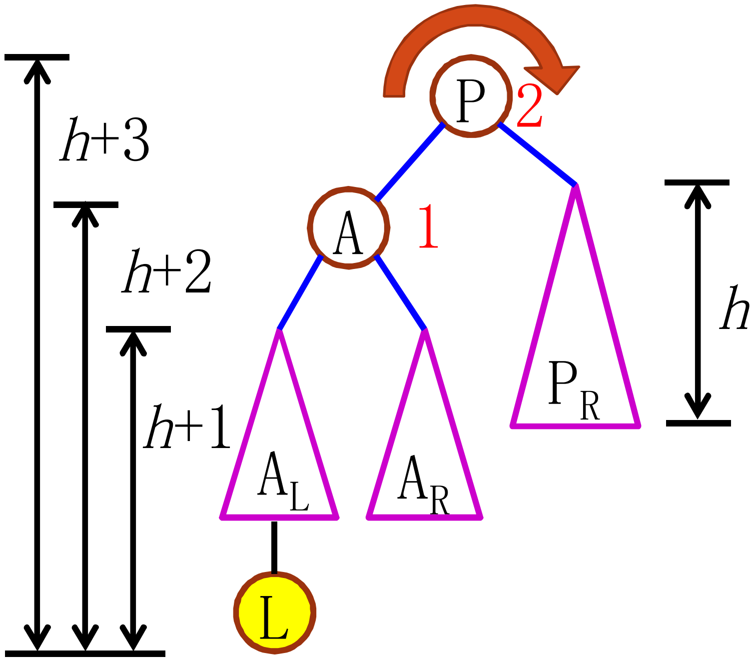
Left-Left Insertion



Left-left insertion: the first two edges in the insertion path from node P both go to the left.

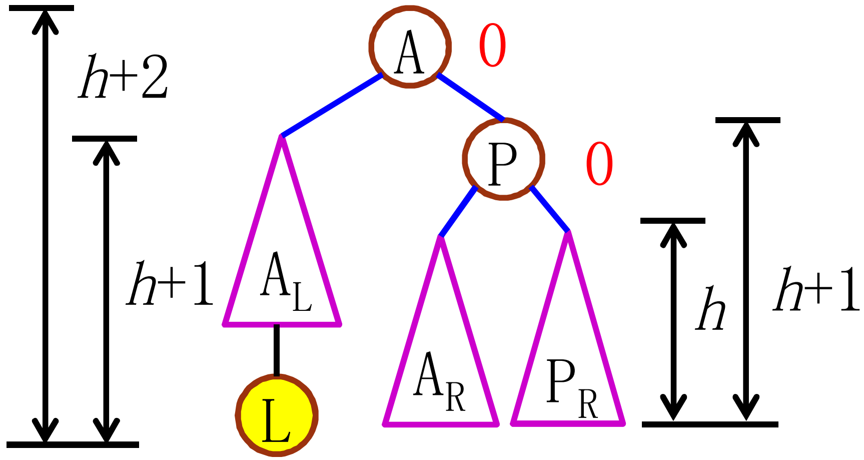
Restoring AVL Balance Condition

Left-Left Insertion



How to restore AVL balance?

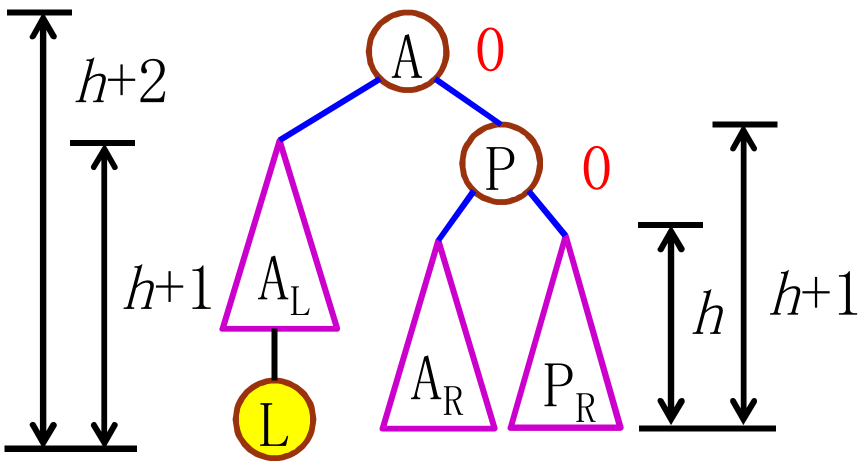
Do a right rotation at node P.



The rotation is also called **left-left (LL) rotation**.

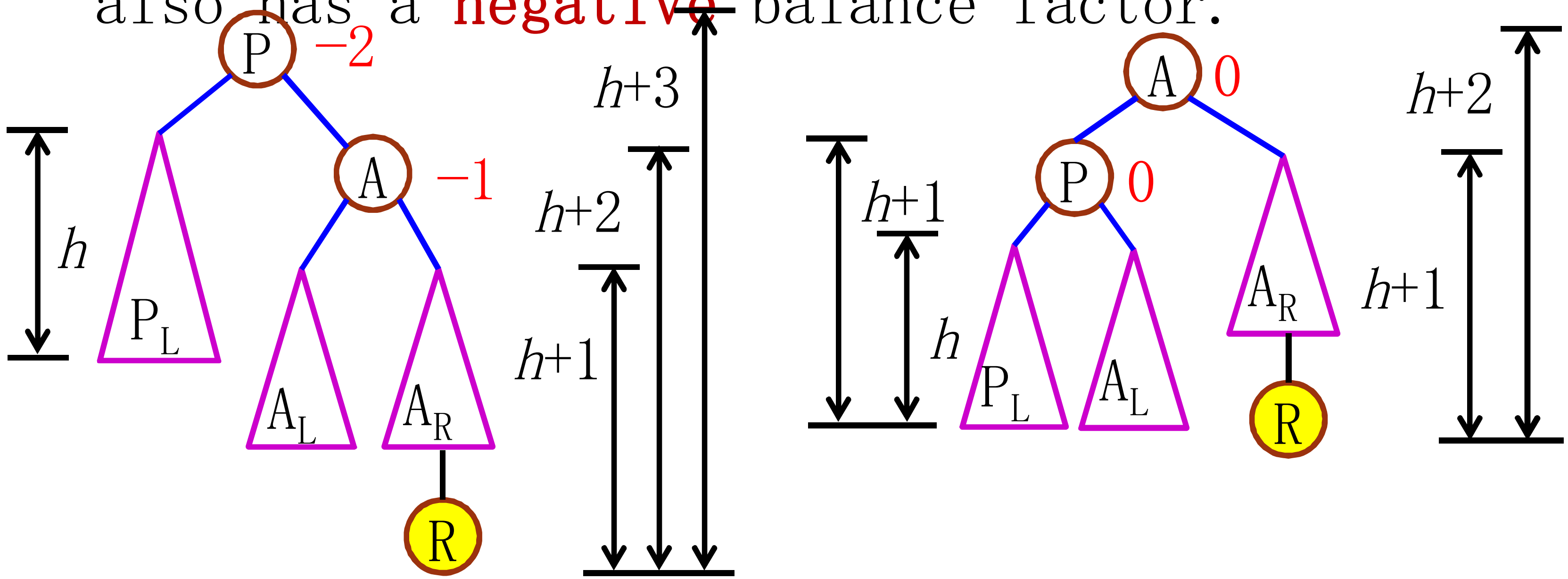
Properties of Left-Left Rotation

- The ordering property of BST is kept.
- Both nodes A and P have balance factor of 0.
- The height of the tree **after the rotation** is the same as the height of the tree before insertion.



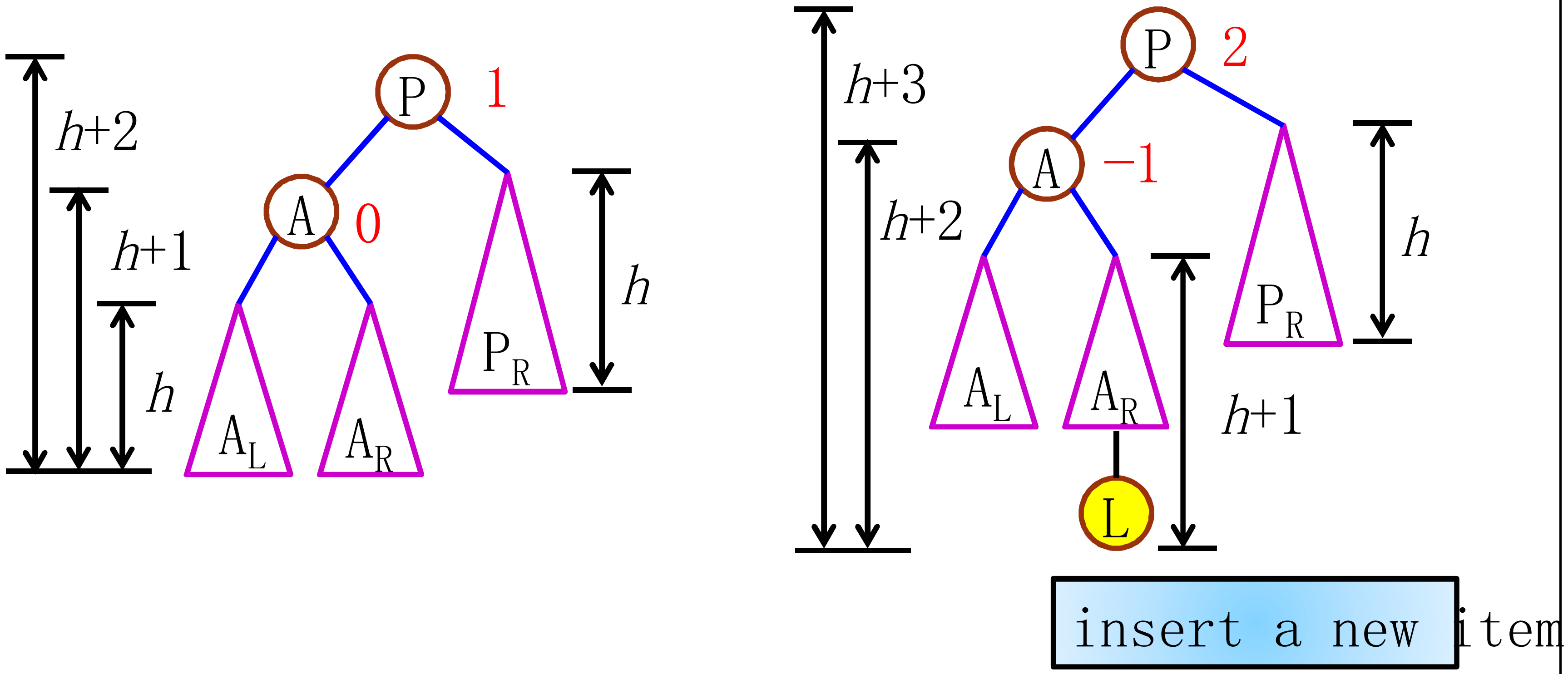
Right-Right (RR) Rotation

- Symmetric to left-left rotation.
- An RR rotation is called for when the node becomes unbalanced with a **negative** balance factor and the right subtree of the node also has a **negative** balance factor.



Breaking AVL Balance Condition

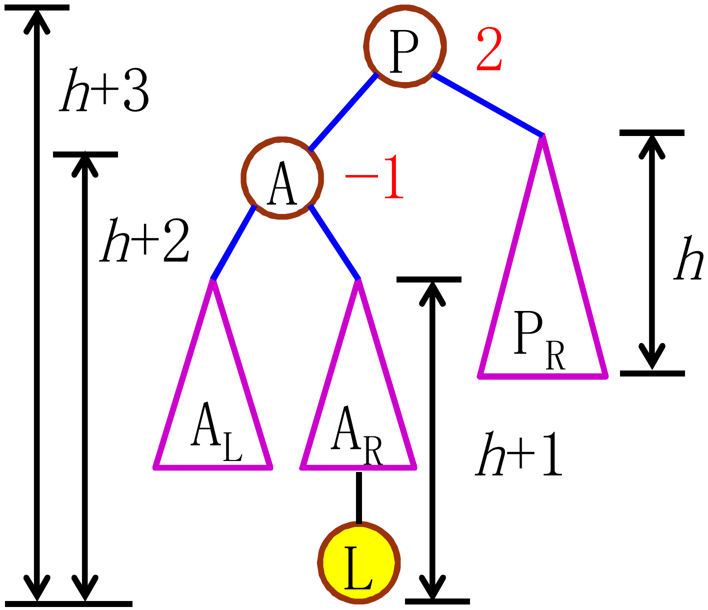
Left-Right Insertion



Left-Right insertion: the first edge in the insertion path goes to the left and the second edge goes to the right.

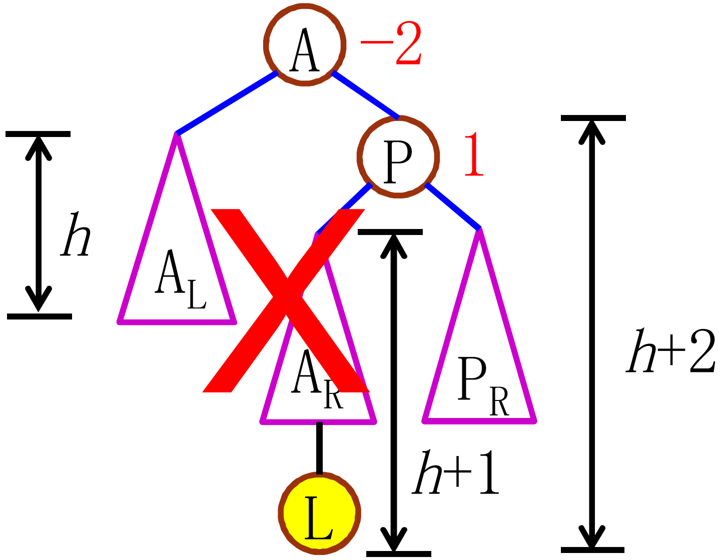
Restoring AVL Balance Condition

Left-Right Insertion

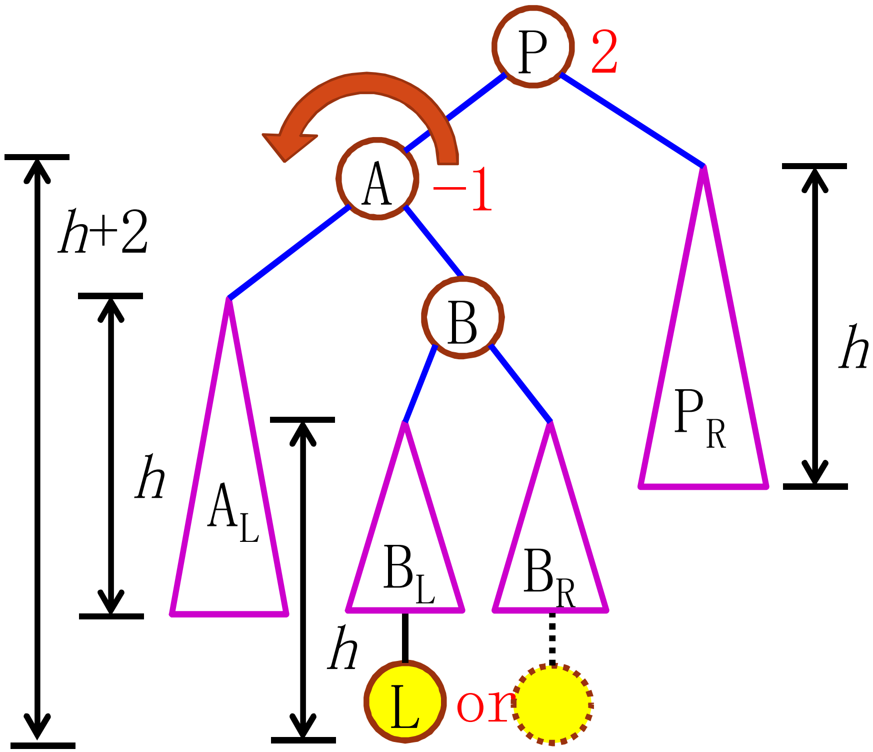


How to restore AVL balance?

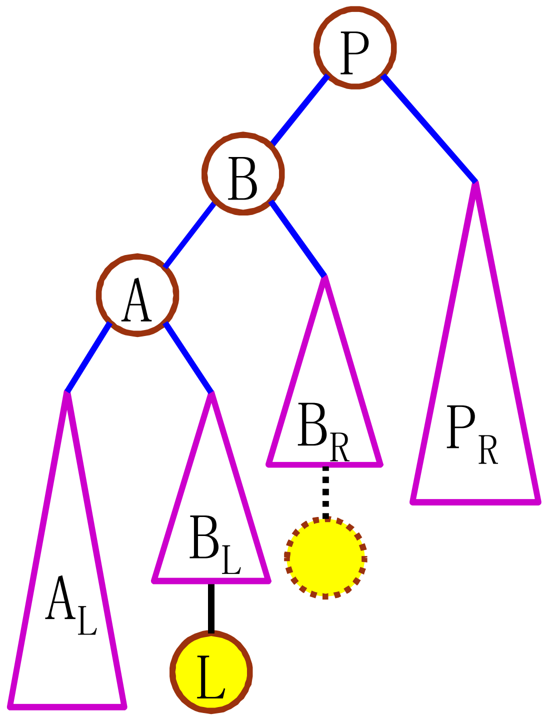
A right rotation at node P does not work!



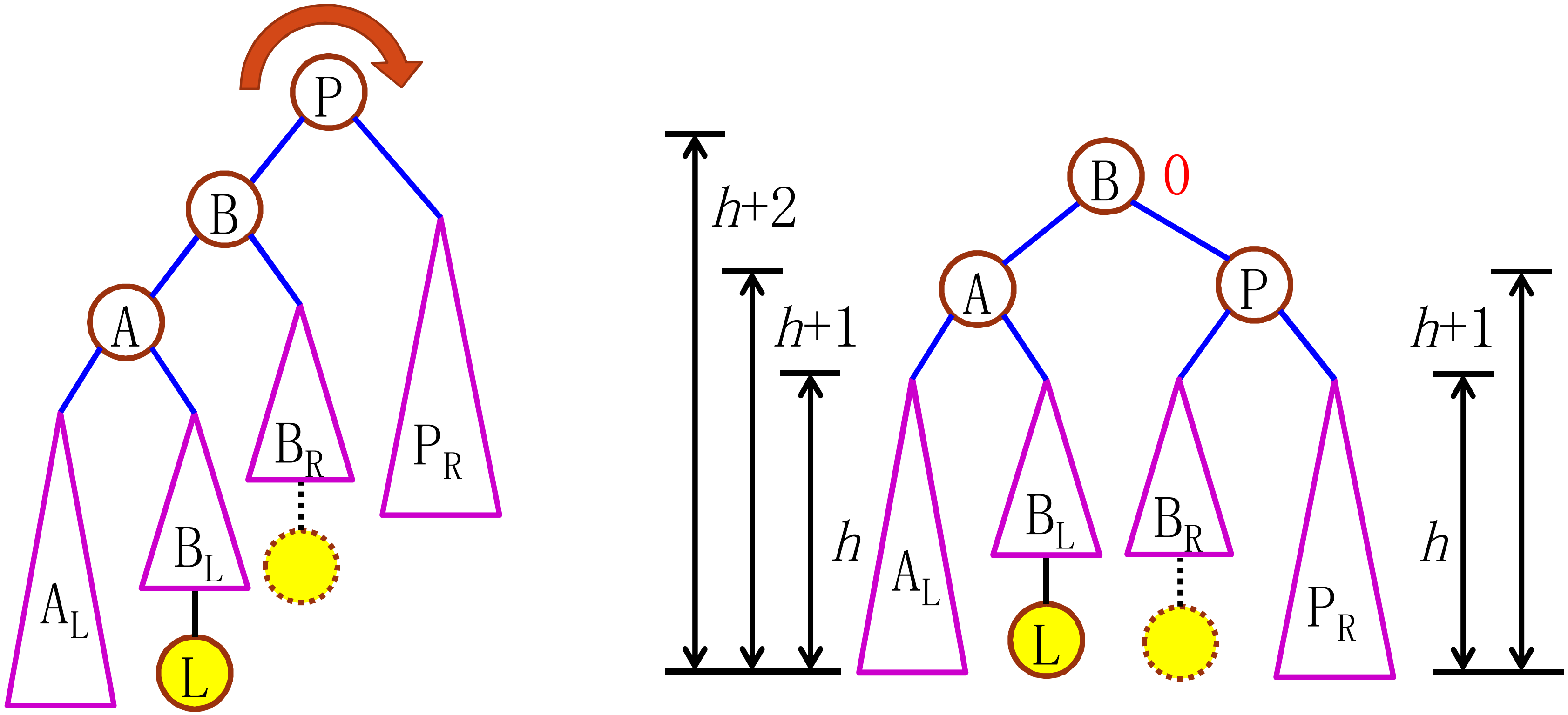
Left-Right (LR) Rotation



A **double rotation** to re-bal
Do a **left** rotation on node
then a **right** rotation on no
(next slide).

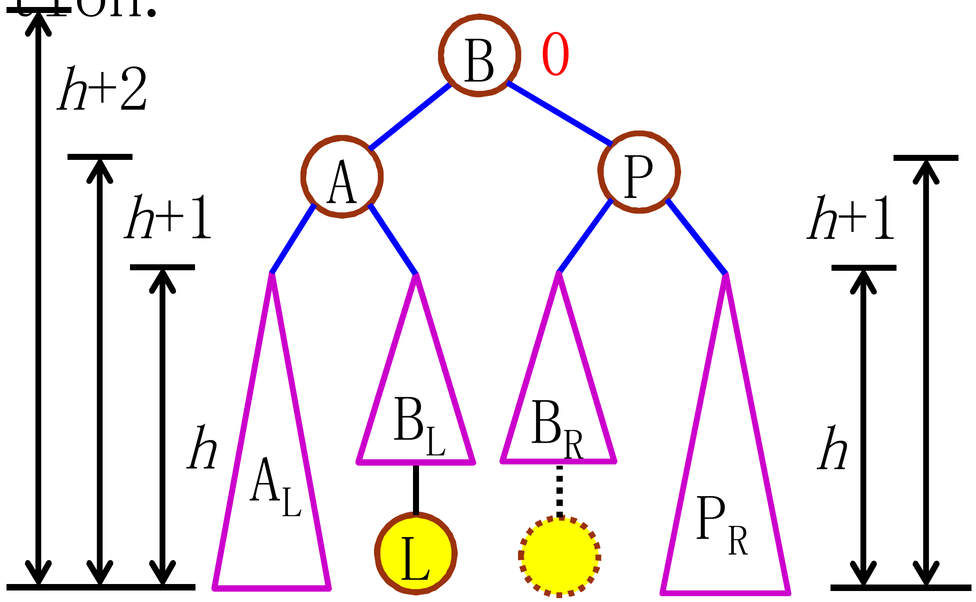


Left-Right (LR) Rotation



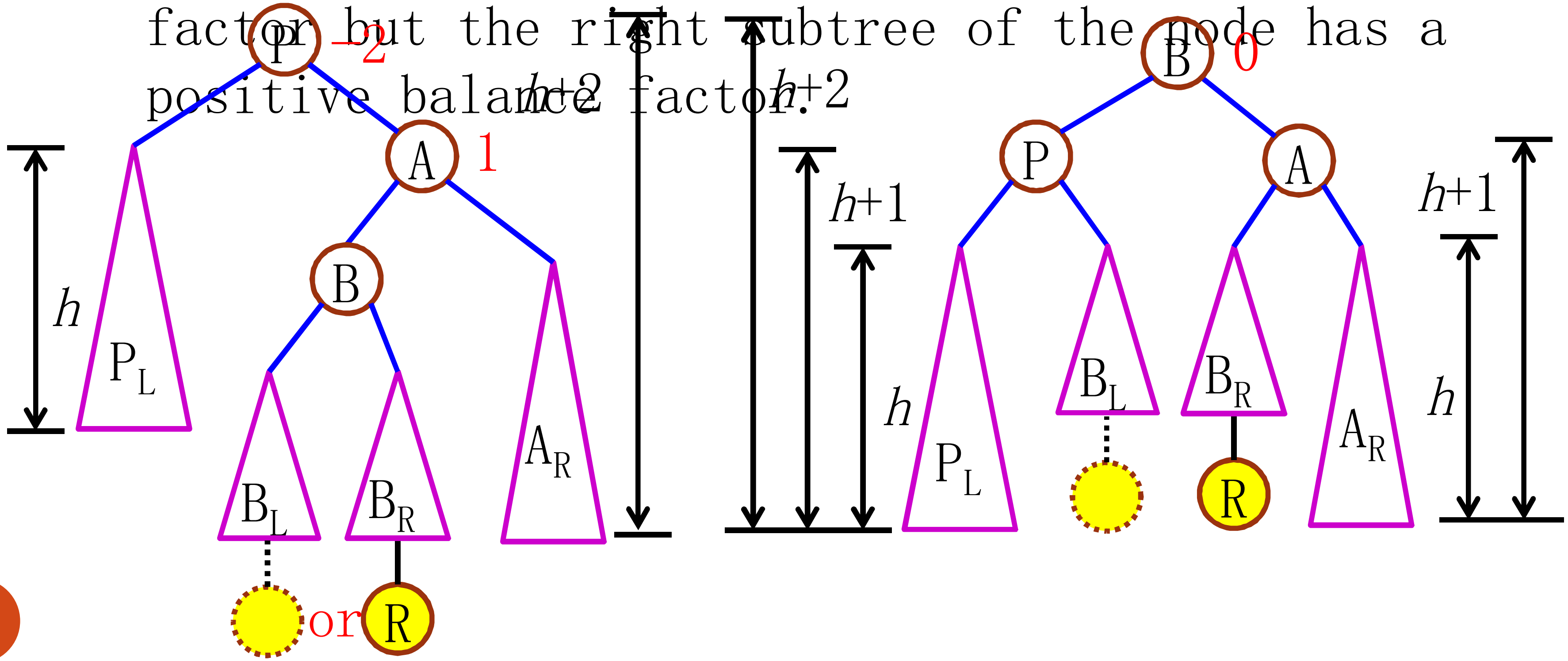
Properties of Left-Right Rotation

- The ordering property of BST is kept.
- Node B has a balance factor of 0.
- The height of the tree **after the rotation** is the same as the height of the tree before insertion.



Right-Left (RL) Rotation

- Symmetric to left-right rotation; also a double rotation.
- An RL rotation is called for when the node becomes unbalanced with a negative balance factor but the right subtree of the node has a positive balance factor.



Rotation Summary

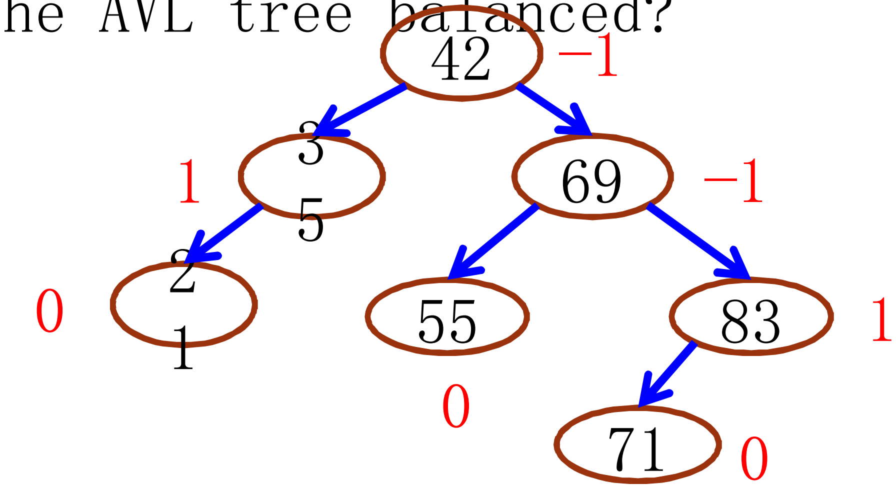
- When an AVL tree becomes unbalanced, there are four cases to consider depending on the **direction** of the first two edges on the insertion path from the **unbalanced node**:
  - Left-left — LL Rotation
  - Right-right — RR Rotation
  - Left-right — LR Rotation
  - Right-left — RL Rotation

LL Rotation, RR Rotation: single rotation

LR Rotation, RL Rotation: double rotation

Exercises

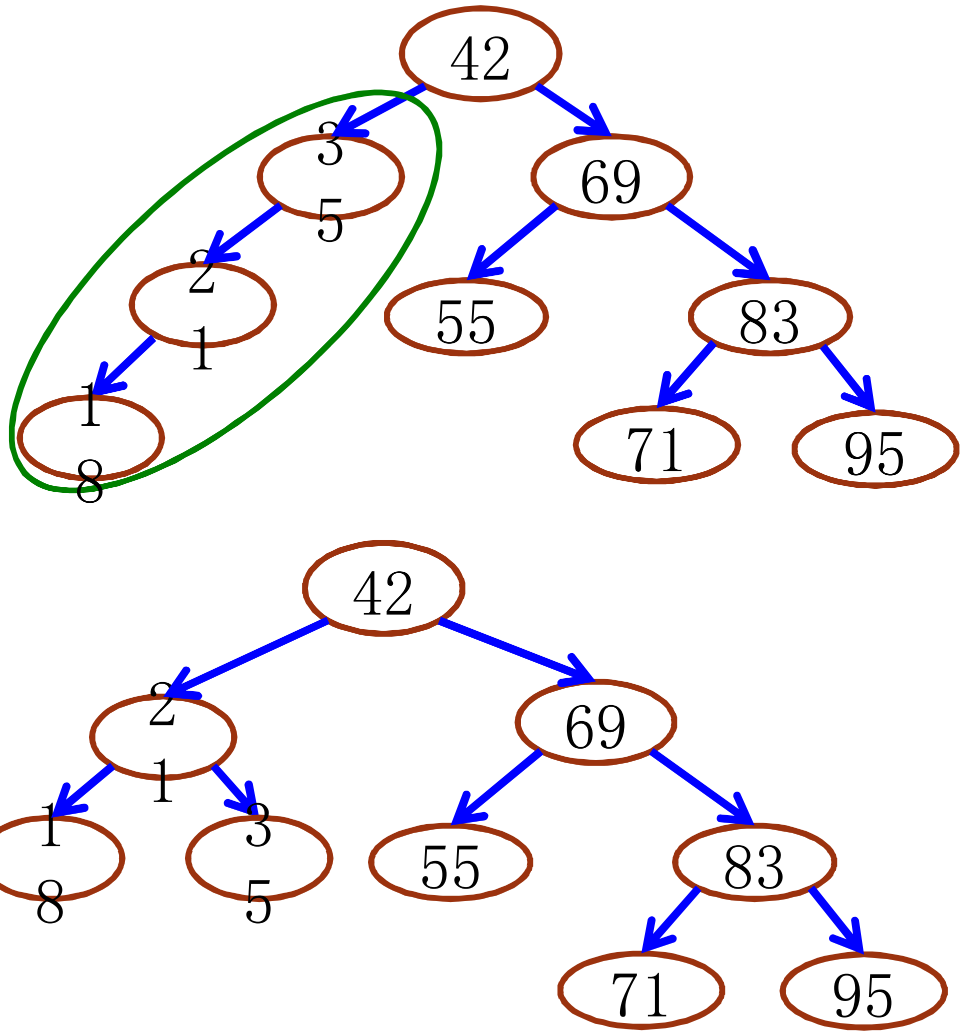
- Insert into an empty AVL tree: 42, 35, 69, 21, 55, 83, 71.
  - Compute the balance factors.
  - Is the AVL tree balanced?



- Insert 95, 18, 75?

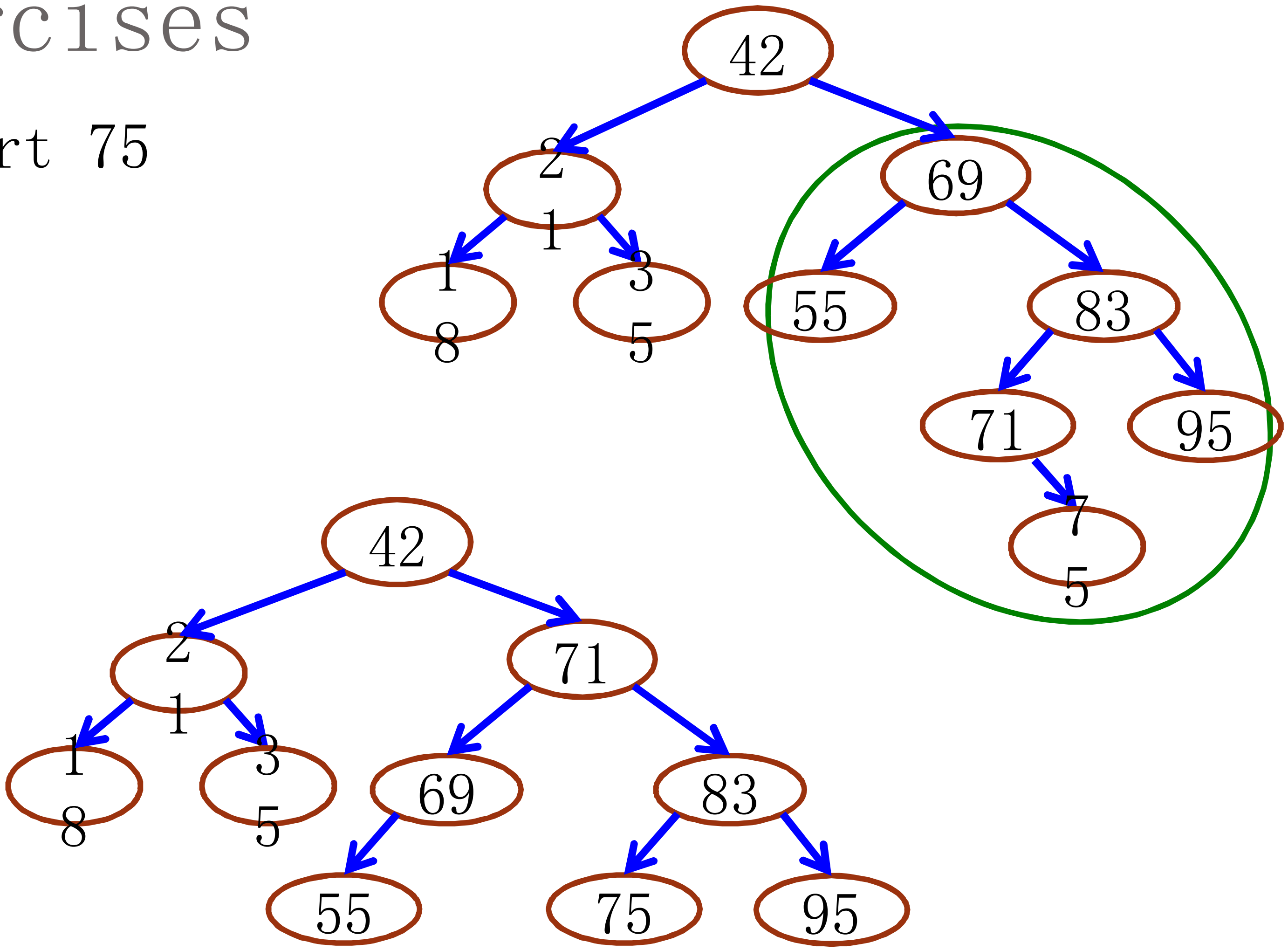
Exercises

- Insert 95, 18



Exercises

- Insert 75



The Number of Rotations Required

- When an AVL tree becomes unbalanced after an insertion, exactly one single or double rotation is required to balance the tree.
  - Before the insertion, the tree is balanced.
  - Only nodes on the access path of the insertion can be unbalanced. All other nodes are balanced.
  - We rotate at the first unbalanced node from the leaf.
  - By the properties of rotation, the height of the node after rotation is the same as that before insertion.
  - All ancestors of that node on the access path should now be balanced.

AVL Trees

Supporting Data Members and Functions

```
struct node {
  Item item;
  int height;
  node *left;
  node *right;
};
```

```
int Height(node *n) {
  if(!n) return -1;
  return n->height;
}

void AdjustHeight(node *n) {
  if(!n) return;
  n->height = max( Height(n->left),
    Height(n->right) ) + 1;
}

int BalFactor(node *n) {
  if(!n) return 0;
  return (Height(n->left) -
    Height(n->right));
}
```

AVL Trees

Supporting Functions

```
void LLRotation(node *&n);
void RRRotation(node *&n);
void LRRotation(node *&n);
void RLRotation(node *&n);


void Balance(node *&n) {
  if(BalFactor(n) > 1) {
    if(BalFactor(n->left) > 0) LLRotation(n);
    else LRRotation(n);
  }
  else if(BalFactor(n) < -1) {
    if(BalFactor(n->right) < 0) RRRotation(n);
    else RLRotation(n);
  }
}
```

AVL Trees

Changes to Insertion

```
void insert(node *&root, Item item)
{
  if(root == NULL) {
    root = new node(item);
    return;
  }
  if(item.key < root->item.key)
    insert(root->left, item);
  else if(item.key > root->item.key)
    insert(root->right, item);

  AdjustHeight(root);
  Balance(root);
}
```

Removal

- First remove node as with BST

- Then update the balance factors of those ancestors in the access path and rebalance as needed.