VE281

Data Structures and Algorithms

Priority Queues, Heaps, and Trie

Outline

- Priority Queue
- Min Heap and Its Operations
- Initializing a Min Heap
- Trie

Priority Queues

- Two kinds of priority queues:
  - Min priority queue.
  - Max priority queue.
- We will focus on **min priority queue**.
  - The max priority queue is similar.

Min Priority Queue

- Collection of items.
- Each item has a key (or "**priority**").
- Support the following operations:
  - **isEmpty**
  - **size**
  - **enqueue**: put an item into the priority queue
  - **dequeueMin**: remove element with **min** key.
  - **getMin**: get item with **min** key

Complexity Of Operations

- Priority queues are most commonly implemented using **Binary Heaps**.

- **`isEmpty`**, **`size`**, and **`getMin`** are $O(1)$ time complexity in the worst case.
- **`enqueue`** and **`dequeueMin`** are $O(\log n)$ time complexity in the worst case, where n is the size of the priority queue.

Application of Priority Queue: Sorting

- Sorting elements (in ascending order):
  1. **enqueue** elements to be sorted into a min priority queu
     Complexity: $O(n \log n)$
  2. Repeatedly ca xtract elements out of the queue.
     Complexity: $O(n \log n)$
- The resulting elements are sorted by their keys.
  $O(n \log n)$
- What is the time complexity?

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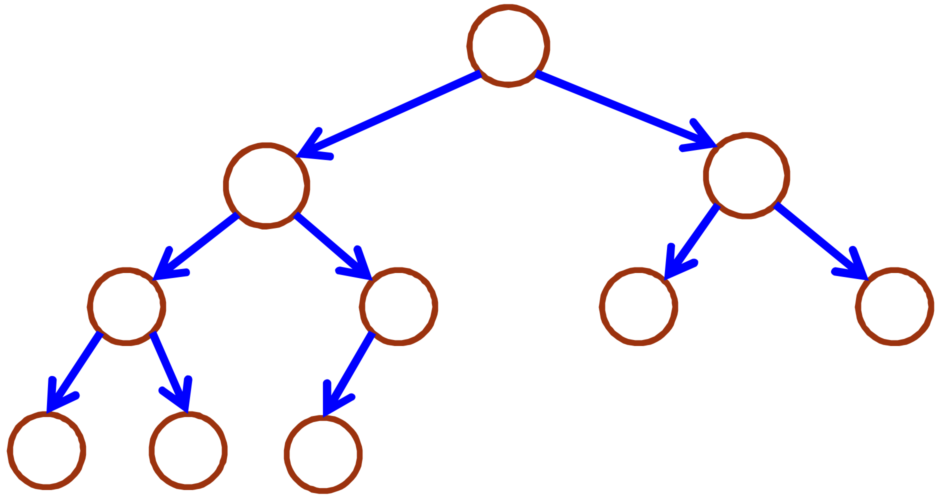
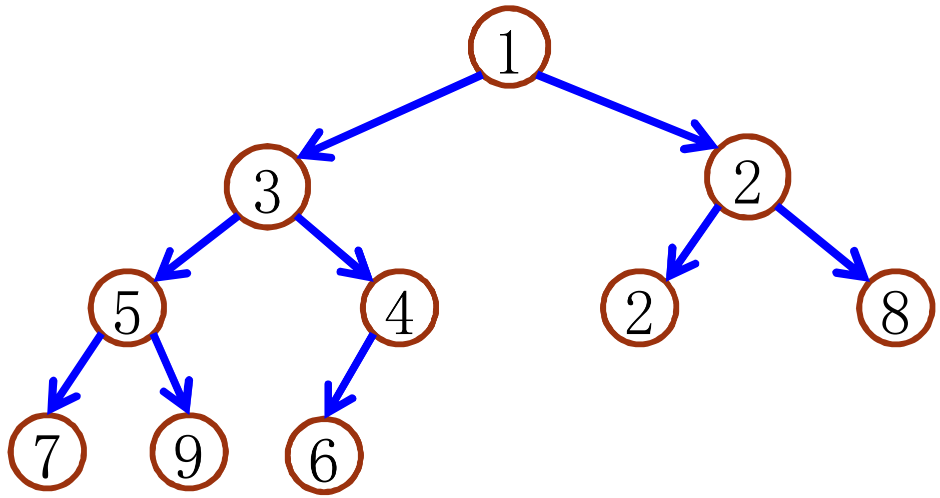
Binary Heap

- A binary heap is a complete binary tree.

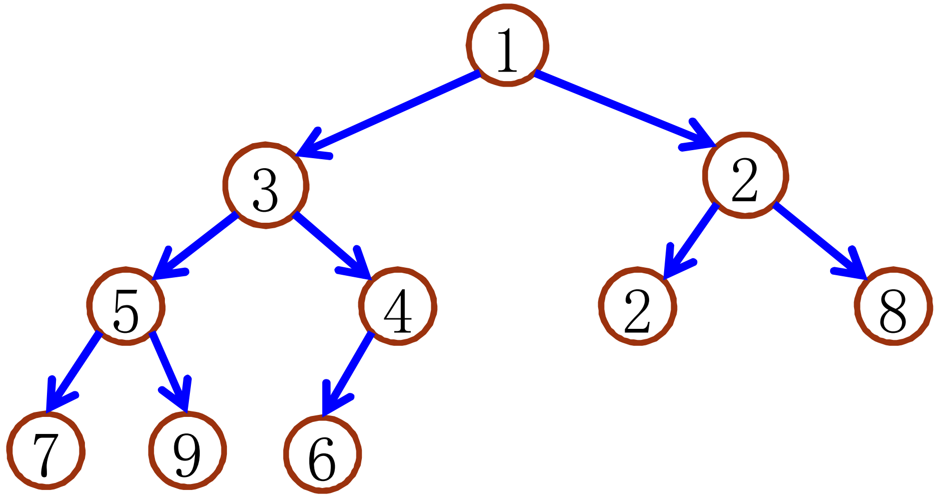


Min Heap

- A min heap is
  - a binary heap, and
  - a tree where for **any** node v, the key of v is smaller than or equal to ($\leq$) the keys of any **descendants** of v.
    - The key of the root of **any** subtree is always the smallest among all the keys in that subtree.
- Example



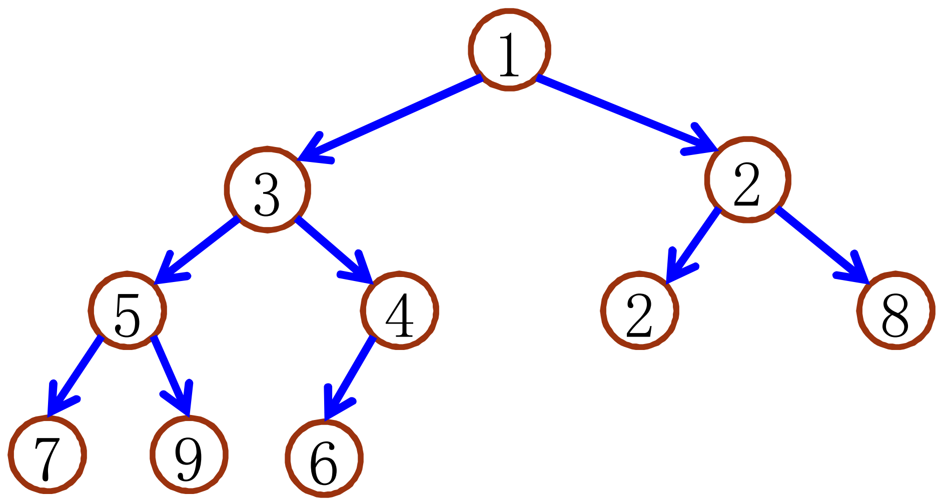
Min Heap



- However, the keys of nodes across subtrees have no required relationship.

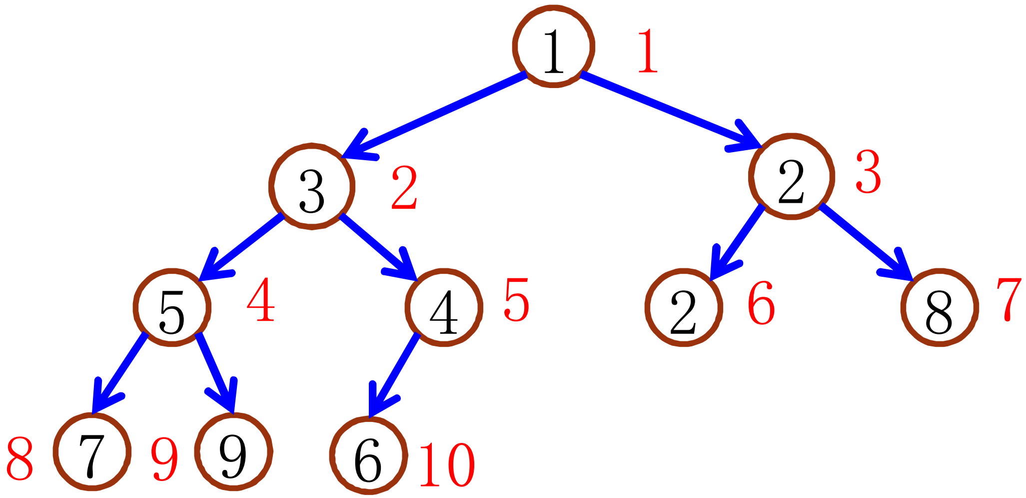
Heap Height

- Assume the heap has n nodes, the height of the heap is
$$\lceil \log_2(n+1) \rceil - 1$$



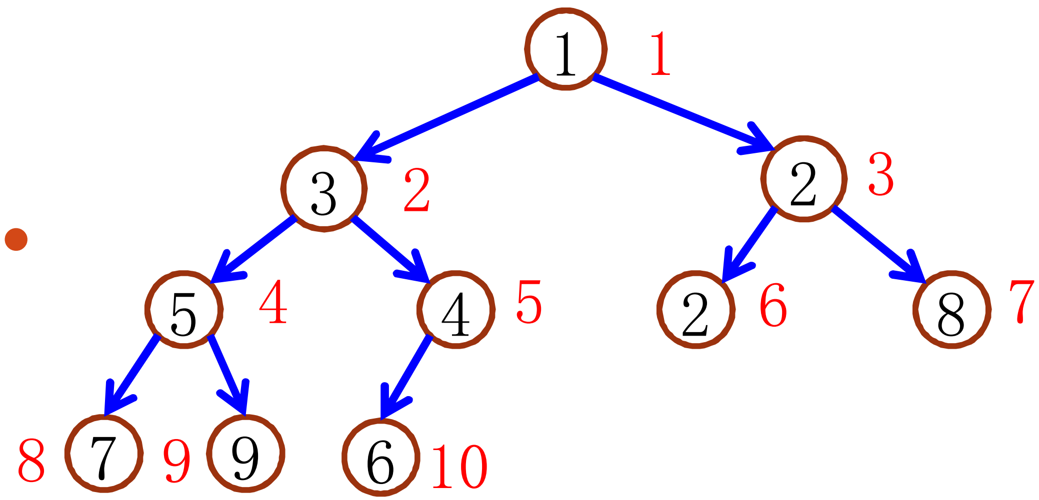
Binary Heap Implementation as an Array

- Store the elements in an array in the order produced by a level-order traversal.
- The first element is stored at index 1.



–	1	3	2	5	4	2	8	7	9	6
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

Index Relation

- 

Index relation allow to move up and down heap easily.

- A node at index i ($i \neq 1$) has its parent at index $\lfloor i/2 \rfloor$.
- Assume the number of nodes is n. A node at index i ($2i \leq n$) has its left child at $2i$.
  - If $2i > n$, it has no left child.
- A node at index i ($2i + 1 \leq n$) has its right child at $2i + 1$.
  - If $2i + 1 > n$, it has no right child.

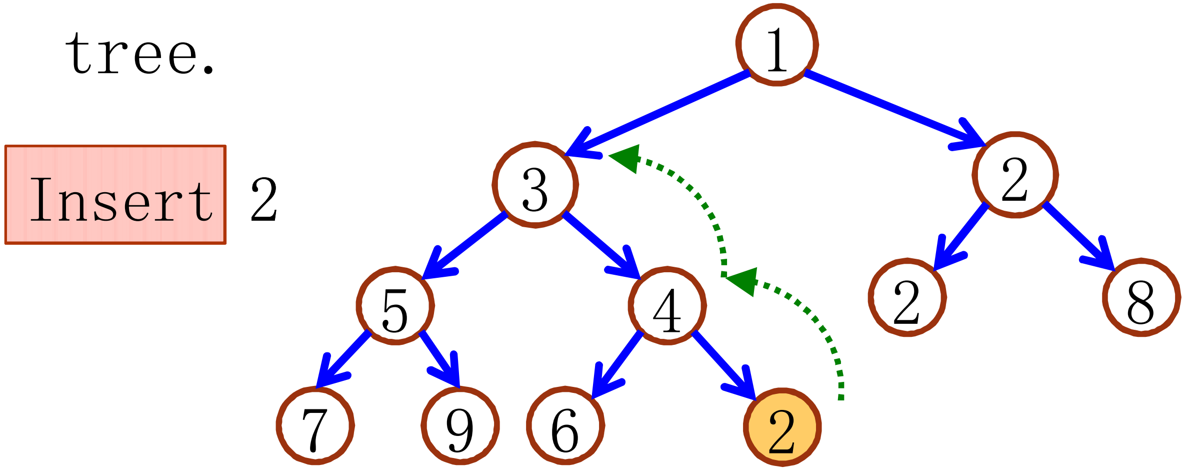
Binary Heap Implementation

- We also have a **size** variable to keep the number of nodes in the heap.

- Operations
  - **isEmpty: return size==0;**
  - **size: return size;**
  - **getMin: return heap[1];**

enqueue

- Insert **newItem** as the rightmost leaf of the tree.

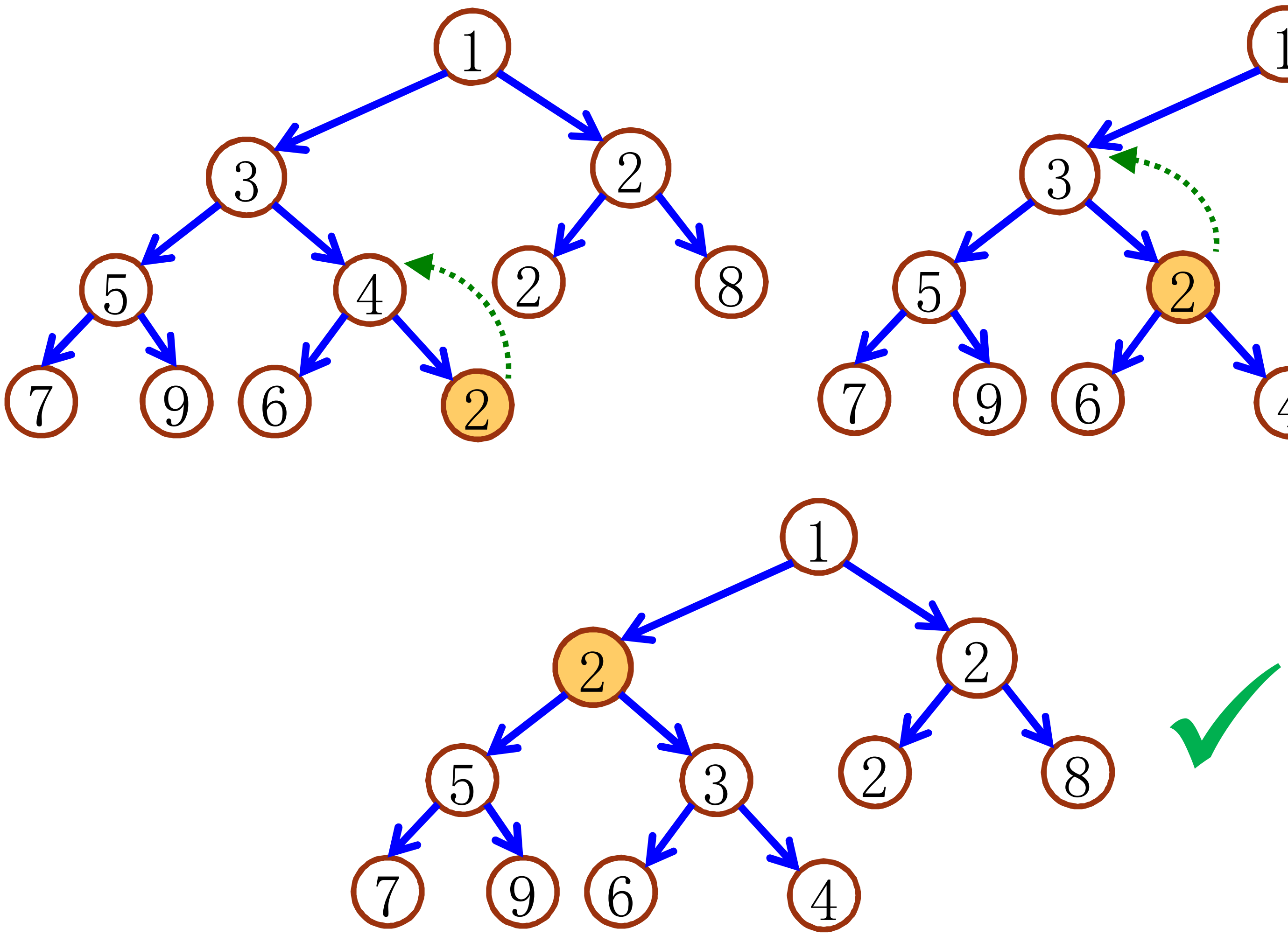


`heap[++size] = newItem;`

- The tree may no longer be a heap at this point!
- Percolate up **newItem** to an appropriate spot in the heap to restore the heap property.

Percolate Up

Illustration



Percolate Up

```
void minHeap::percolateUp(int id) {
  while(id > 1 && heap[id/2] > heap[id]) {
    swap(heap[id], heap[id/2]);
    id = id/2;
  }
}
```

- Pass index (**id**) of array element that needs to be percolated up.
- Swap the given node with its parent and move up to parent until:
  - we reach the root at position 1, or
  - the parent has a smaller (or equal) key.

enqueue

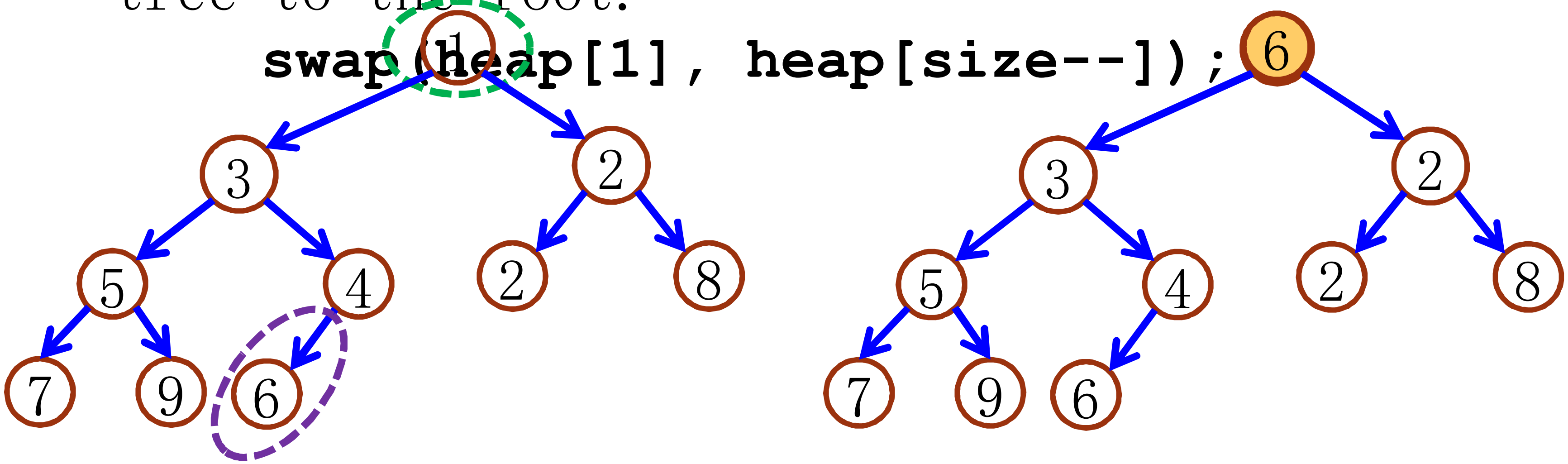
```
void minHeap::enqueue(Item newItem) {
  heap[++size] = newItem;
  percolateUp(size);
}
```

- What is the time complexity?
  - $O(\log n)$

dequeueMin

- The min item is at the root. Save that item to be returned.
- Move the item in the rightmost leaf of the tree to the root.

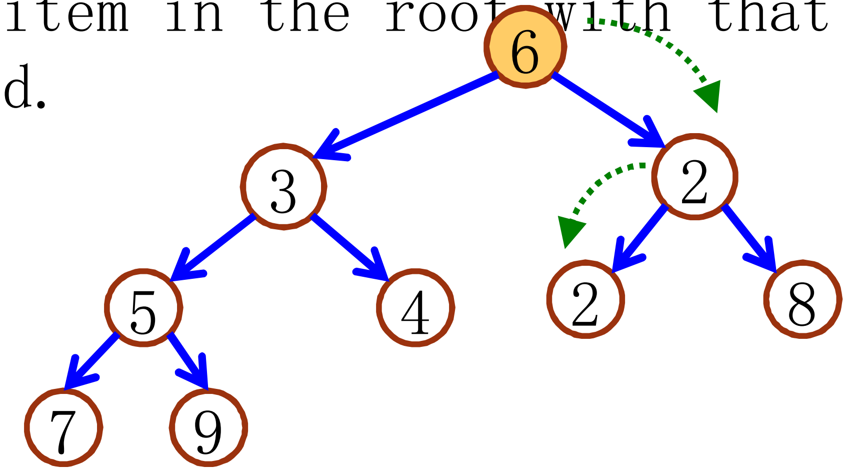
```
swap(heap[1], heap[size--]);
```



- The tree may no longer be a heap at this point!

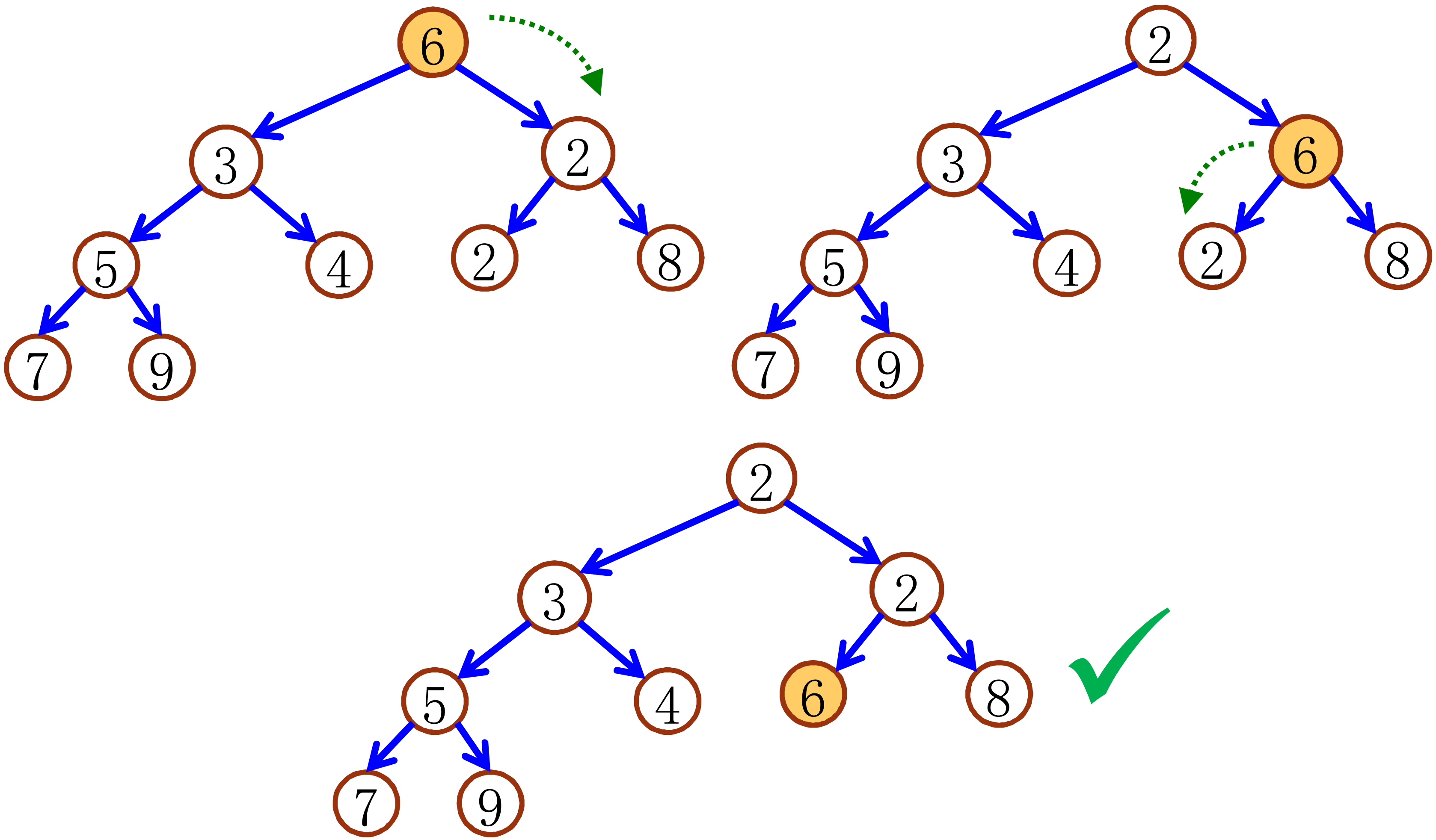
dequeueMin

- Percolate down the recently moved item at the root to its proper place to restore heap property.
  - For each subtree, if the root has a larger search key than either of its children, swap the item in the root with that of the smaller child.



Percolate Down

Illustration



Percolate Down

```
void minHeap::percolateDown(int id) {
  for(j = 2*id; j <= size; j = 2*j) {
    if(j < size && heap[j] > heap[j+1]) j++;
    if(heap[id] <= heap[j]) break;
    swap(heap[id], heap[j]);
    id = j;
  }
}
```

find the smaller c

- Pass index of array element that needs to be percolated down.
- Swap the key in the given node with the smallest key among the node’s children, moving down to that child, until:
  - we reach a leaf node, or
  - both children have larger (or equal) key

dequeueMin

```
Item minHeap::dequeueMin() {
  swap(heap[1], heap[size--]);
  percolateDown(1);
  return heap[size+1];
}
```

- What is the time complexity?
  - $O(\log n)$

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Initializing a Min Heap

- How do we initialize a min heap from a set of items?

- Simple solution: insert each entry one by one.
  - The worst case time complexity for inserting the k-th item is $O(\log k)$, so creating a heap in this way is $O(n \log n)$.

- Instead, we can do better by putting the entries into a **complete** binary tree and running **percolate down** intelligently.

Initializing a Min Heap

- Put all the items into a complete binary tree.
  - Implemented using an array.

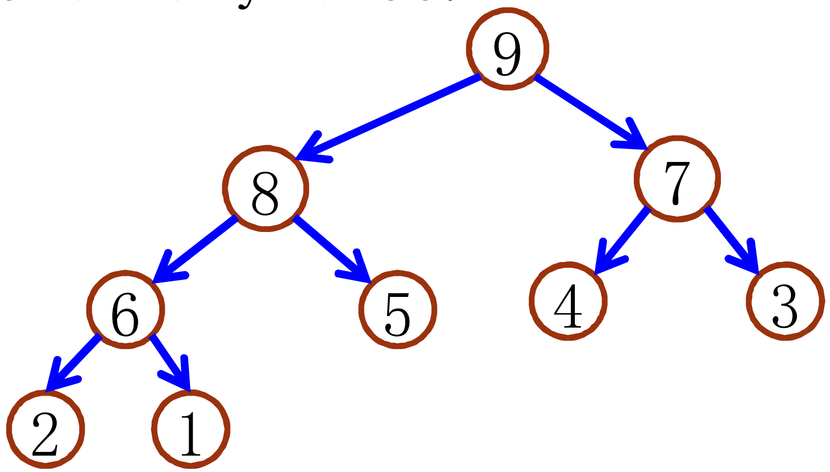
- Starting at the rightmost array position that has a child, percolate down all nodes in reverse level-order.
  - The rightmost array position that has a child is **`size/2`**.
  - For **`i = size/2`** down to **`1`**

```
percolateDown(i);
```

Initializing a Min Heap

Illustration

- Input items: 9, 8, 7, 6, 5, 4, 3, 2, 1
- First step: put all the items into a complete binary tree.

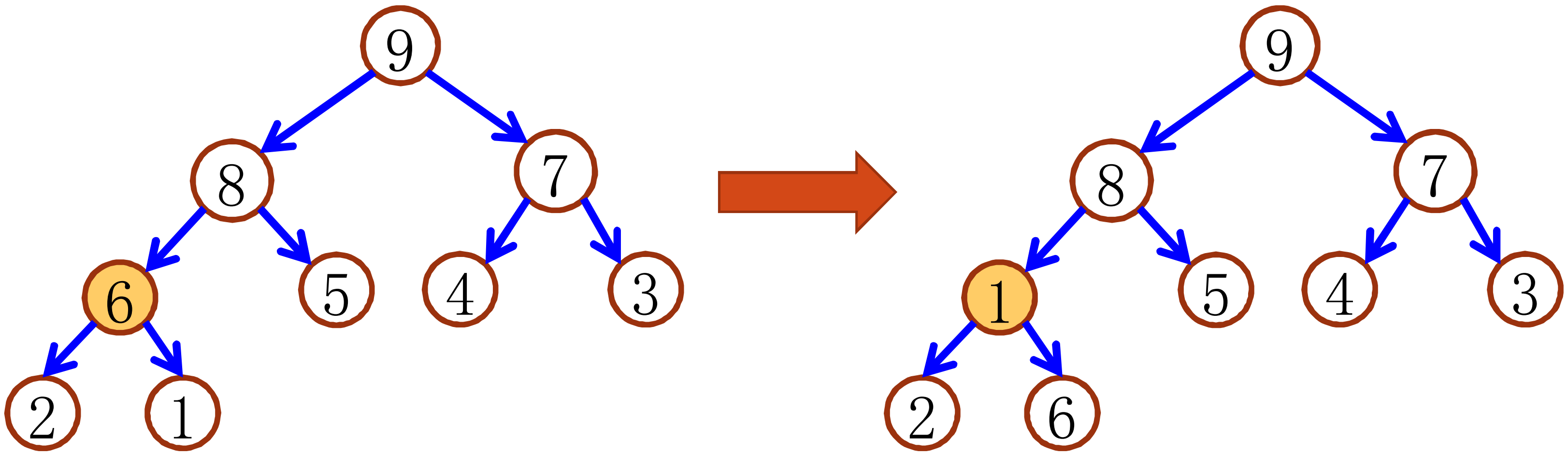


Initializing a Min Heap

Illustration

- Starting at the rightmost array position **that has a child**, percolate down all nodes in **reverse** level-order.

Node at index 9/2 = 4

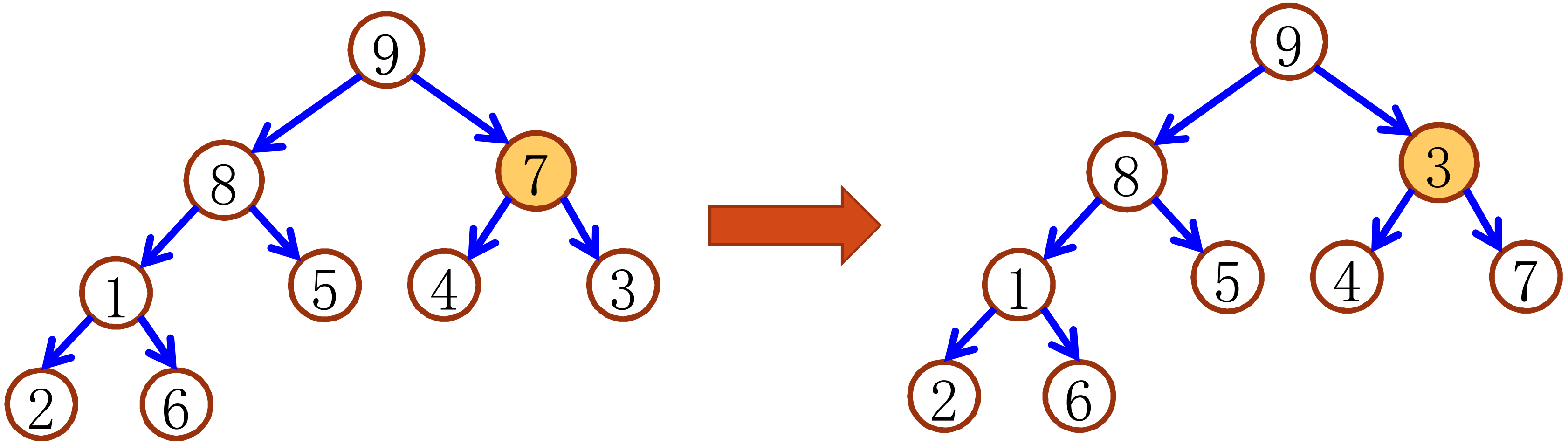


Move to next lower array pos

Initializing a Min Heap

Illustration

Node at index 3

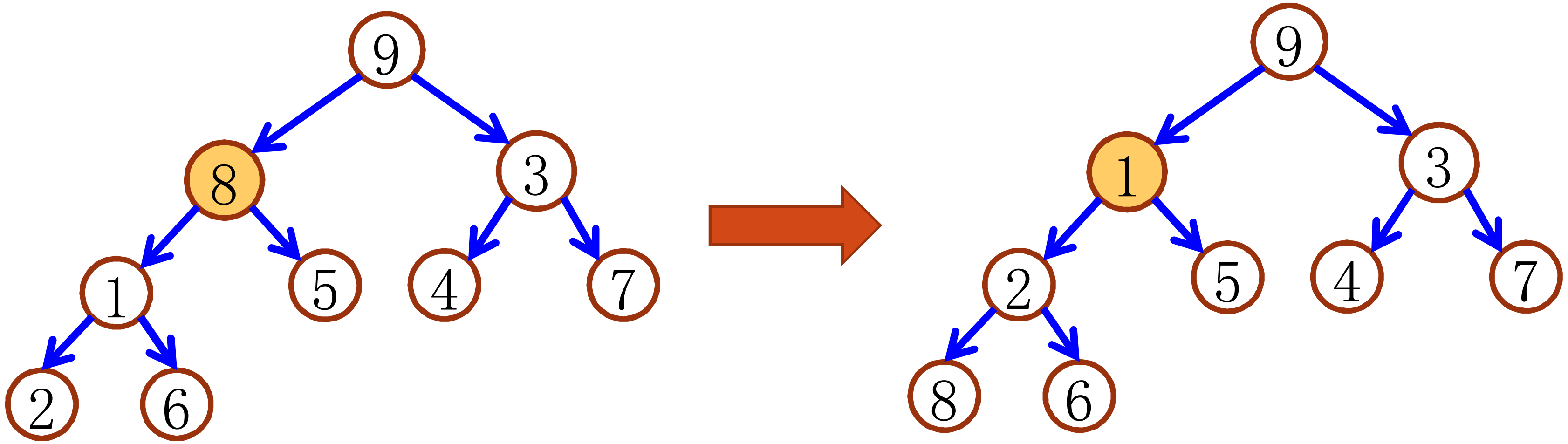


Move to next lower array pos

Initializing a Min Heap

Illustration

Node at index 2

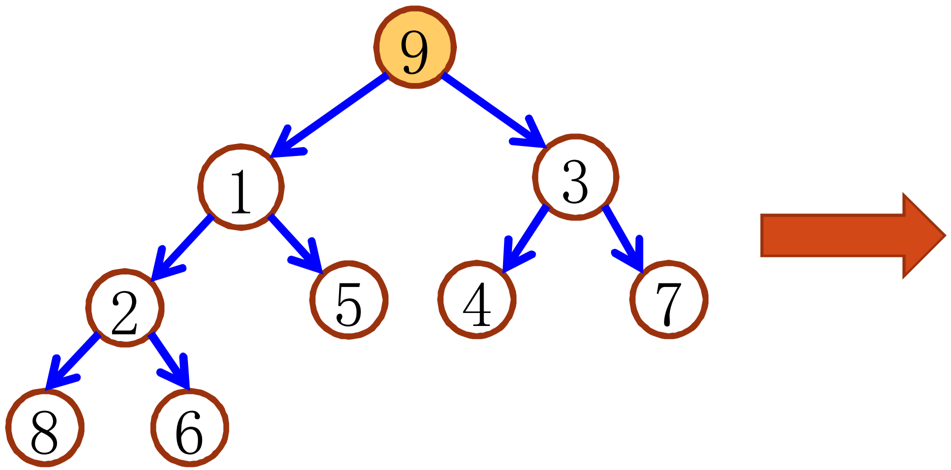


Move to next lower array pos

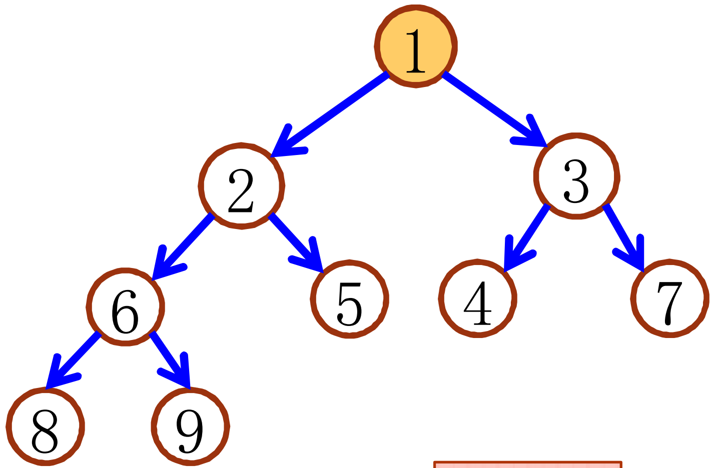
Initializing a Min Heap

Illustration

Node at index 1

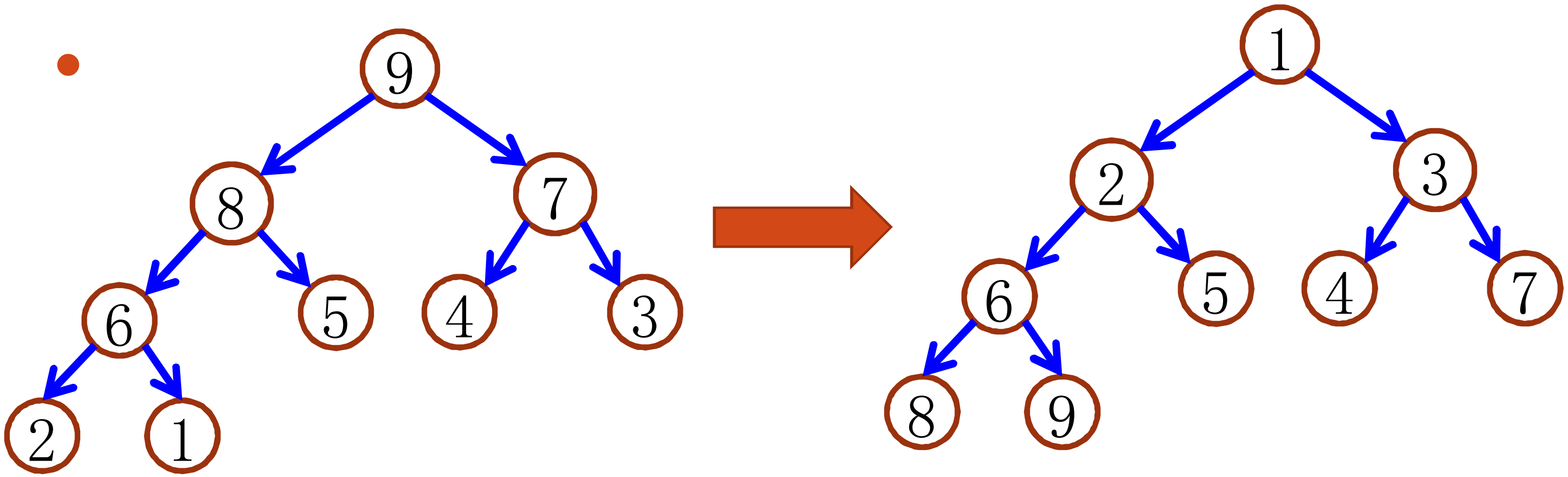


Exercise: What’ s the re



Done!

Time Complexity Analysis

- 

- Suppose the height of the heap is h.
- Number of nodes at level k $(0 \leq k \leq h)$ is $\leq 2^k$.
- The worst case time complexity of percolating down a node at level k is $O(h-k)$.

Time Complexity Analysis

- $$T(h) \le \sum_{k=0}^{h-1} 2^k O(h-k) = O\left(\sum_{k=0}^{h-1} 2^k (h-k)\right)$$
- What is $S(h) = \sum_{k=0}^{h-1} 2^k (h-k)$?

$$S(h) = 2^0 h + 2^1(h-1) + 2^2(h-2) + \cdots + 2^{h-1}\cdot 1$$

$$2S(h) = 2^1 h + 2^2(h-1) + \cdots + 2^{h-1}\cdot 2 + 2^h \cdot 1$$

$$2S(h) = \qquad 2^1 h \qquad + 2^2(h-1) + \cdots + 2^{h-1}\cdot 2 + 2^h \cdot 1$$

$$S(h) = 2^0 h + 2^1(h-1) + 2^2(h-2) + \cdots + 2^{h-1}\cdot 1$$

$$S(h) = 2S(h) - S(h) = 2^1 + 2^2 + \cdots + 2^h - h = 2^{h+1} - 2 - h$$

Time Complexity Analysis

- $$T(h) \leq O(2^{h+1} - 2 - h)$$
- For a complete binary tree, we have
$$h = \Theta(\log n)$$

where n is the number of nodes.

- Therefore, the algorithm for initializing a min heap with n nodes has worst case time complexity $T(n) = O(n)$.

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Trie

- The word “trie” comes from retrieval.
  - To distinguish with “tree”, it is pronounced as “try”.
- A trie is a tree that uses parts of the key, as opposed to the whole key, to perform search.
- A trie stores data records only in **leaf** nodes. Internal nodes serve as placeholders to direct the search process.