Felippa's Gauss Integration Rules for Tetrahedra

Tetrahedron Gauss Quadrature Module

(Felippa: AFEM, Chapter 17 Page 17-22)

Note: we return the weight divided by 6 here whereas Felippa defines $J = \frac{1}{6}\det(\boldsymbol{J})$ (see, e.g., Equation 17.31)

In[1]:=
```
TetrGaussRuleInfo[{rule_, numer_}, point_] :=
   Module[{jk6 = {{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}}, jk12 = {{1, 2}, {1, 3}, {1, 4},
       {2, 3}, {2, 4}, {3, 4}, {2, 1}, {3, 1}, {4, 1}, {3, 2}, {4, 2}, {4, 3}}, i = point, j, k, g1,
     g2, g3, g4, h1, w1, w2, w3, eps = 10.^(-16), info = {{Null, Null, Null, Null}, 0}},
    If[rule == 1, info = {{1/4, 1/4, 1/4, 1/4}, 1}];
    If[rule == 4, g1 = (5 - Sqrt[5])/20; h1 = (5 + 3*Sqrt[5])/20;
     info = {{g1, g1, g1, g1}, 1/4}; info[[1, i]] = h1];
    If[rule == 8, j = i - 4;
     g1 = (55 - 3*Sqrt[17] + Sqrt[1022 - 134*Sqrt[17]])/196;
     g2 = (55 - 3*Sqrt[17] - Sqrt[1022 - 134*Sqrt[17]])/196;
     w1 = 1/8 + Sqrt[(1 715 161 837 - 406 006 699*Sqrt[17])/23 101]/3120;
     w2 = 1/8 - Sqrt[(1 715 161 837 - 406 006 699*Sqrt[17])/23 101]/3120;
      If[j ≤ 0, info = {{g1, g1, g1, g1}, w1}; info[[1, i]] = 1 - 3*g1];
      If[j > 0, info = {{g2, g2, g2, g2}, w2}; info[[1, j]] = 1 - 3*g2]];
    If[rule == -8, j = i - 4;
     If[j ≤ 0, info = {{0, 0, 0, 0}, 1/40}; info[[1, i]] = 1];
     If[j > 0, info = {{1, 1, 1, 1}/3, 9/40}; info[[1, j]] = 0]];
    If[rule == 14, (*g1,g2+roots of P(g)=0,
     P=9+96*g-1712*g^2-30464*g^3-127232*g^4+86016*g^5+1060864*g^6*)
     g1 = 0.0927352503108912264023239137370306;
     g2 = 0.3108859192633006097973457337634578;
     g3 = 0.4544962958743503505081194737206605;
     If[!numer, {g1, g2, g3} = Rationalize[{g1, g2, g3}, eps]];
     w1 = (-1 + 6*g2*(2 + g2*(-7 + 8*g2)) + 14*g3 -
           60*g2*(3 + 4*g2*(-3 + 4*g2))*g3 + 4*(-7 + 30*g2*(3 + 4*g2*(-3 + 4*g2)))*g3^2)/
        (120*(g1 - g2)*(g2*(-3 + 8*g2) + 6*g3 + 8*g2*(-3 + 4*g2)*g3 - 4*
             (3 + 4*g2*(-3 + 4*g2))*g3^2 + 8*g1^2*(1 + 12*g2*(-1 + 2*g2) + 4*g3 - 8*g3^2) +
           g1*(-3 - 96*g2^2 + 24*g3*(-1 + 2*g3) + g2*(44 + 32*(1 - 2*g3)*g3))));
```

```
    w2 = (-1 - 20 * (1 + 12 * g1 * (2 * g1 - 1)) * w1 + 20 * g3 * (2 * g3 - 1) * (4 * w1 - 1)) /
       (20 * (1 + 12 * g2 * (2 * g2 - 1) + 4 * g3 - 8 * g3 ^ 2));
   If[i < 5, info = {{g1, g1, g1, g1}, w1}; info[[1, i]] = 1 - 3 * g1];
   If[i > 4 && i < 9, info = {{g2, g2, g2, g2}, w2}; info[[1, i - 4]] = 1 - 3 * g2];
   If[i > 8, info = {{g3, g3, g3, g3}, 1 / 6 - 2 * (w1 + w2) / 3};
    {j, k} = jk6[[i - 8]]; info[[1, j]] = info[[1, k]] = 1 / 2 - g3]];
  If[rule == -14, g1 = (243 - 51 * Sqrt[11] + 2 * Sqrt[16 486 - 9723 * Sqrt[11] / 2]) / 356;
   g2 = (243 - 51 * Sqrt[11] - 2 * Sqrt[16 486 - 9723 * Sqrt[11] / 2]) / 356;
   w1 = 31 / 280 + Sqrt[(13 686 301 - 3 809 646 * Sqrt[11]) / 5965] / 600;
   w2 = 31 / 280 - Sqrt[(13 686 301 - 3 809 646 * Sqrt[11]) / 5965] / 600;
   If[i < 5, info = {{g1, g1, g1, g1}, w1}; info[[1, i]] = 1 - 3 * g1];
   If[i > 4 && i < 9, info = {{g2, g2, g2, g2}, w2}; info[[1, i - 4]] = 1 - 3 * g2];
   If[i > 8 && i < 15, info = {{0, 0, 0, 0}, 2 / 105};
    {j, k} = jk6[[i - 8]]; info[[1, j]] = info[[1, k]] = 1 / 2]];
  If[rule == 15, g1 = (7 - Sqrt[15]) / 34;
   g2 = 7 / 17 - g1;
   g3 = (10 - 2 * Sqrt[15]) / 40;
   w1 = (2665 + 14 * Sqrt[15]) / 37 800; w2 = (2665 - 14 * Sqrt[15]) / 37 800;
   If[i < 5, info = {{g1, g1, g1, g1}, w1}; info[[1, i]] = 1 - 3 * g1];
   If[i > 4 && i < 9, info = {{g2, g2, g2, g2}, w2}; info[[1, i - 4]] = 1 - 3 * g2];
   If[i > 8 && i < 15, info = {{g3, g3, g3, g3}, 10 / 189};
    {j, k} = jk6[[i - 8]]; info[[1, j]] = info[[1, k]] = 1 / 2 - g3];
   If[i == 15, info = {{1 / 4, 1 / 4, 1 / 4, 1 / 4}, 16 / 135}]];
 If[rule == -15, g1 = (13 - Sqrt[91]) / 52;
    If[i < 5, info = {{1, 1, 1, 1} / 3, 81 / 2240}; info[[1, i]] = 0];
    If[i > 4 && i < 9, info = {{1, 1, 1, 1} / 11, 161 051 / 2 304 960};
     info[[1, i - 4]] = 8 / 11];
    If[i > 8 && i < 15, info = {{g1, g1, g1, g1}, 338 / 5145};
     {j, k} = jk6[[i - 8]]; info[[1, j]] = info[[1, k]] = 1 / 2 - g1];
    If[i == 15, info = {{1 / 4, 1 / 4, 1 / 4, 1 / 4}, 6544 / 36 015}]];
 If[rule == 24, g1 = 0.21460287125915202928883921938684991;
    g2 = 0.040673958534611353115579448956410059;
    g3 = 0.32233789014227551034399447076249 2125;
    If[! numer, {g1, g2, g3} = Rationalize[{g1, g2, g3}, eps]];
    w1 = (85 + 2 * g2 * (-319 + 9 * Sqrt[5] + 624 * g2) - 638 * g3 - 24 * g2 * (-229 + 472 * g2) * g3 +
        96 * (13 + 118 * g2 * (-1 + 2 * g2)) * g3 ^ 2 + 9 * Sqrt[5] * (-1 + 2 * g3)) /
       (13 440 * (g1 - g2) * (g1 - g3) * (3 - 8 * g2 + 8 * g1 * (-1 + 2 * g2) - 8 * g3 + 16 * (g1 + g2) * g3));
    w2 = -(85 + 2 * g1 * (-319 + 9 * Sqrt[5] + 624 * g1) - 638 * g3 - 24 * g1 * (-229 + 472 * g1) * g3 +
        96 * (13 + 118 * g1 * (-1 + 2 * g1)) * g3 ^ 2 + 9 * Sqrt[5] * (-1 + 2 * g3)) /
       (13 440 * (g1 - g2) * (g2 - g3) * (3 - 8 * g2 + 8 * g1 * (-1 + 2 * g2) - 8 * g3 + 16 * (g1 + g2) * g3));
    w3 = (85 + 2 * g1 * (-319 + 9 * Sqrt[5] + 624 * g1) - 638 * g2 - 24 * g1 * (-229 + 472 * g1) * g2 +
        96 * (13 + 118 * g1 * (-1 + 2 * g1)) * g2 ^ 2 + 9 * Sqrt[5] * (-1 + 2 * g2)) /
       (13 440 * (g1 - g3) * (g2 - g3) * (3 - 8 * g2 + 8 * g1 * (-1 + 2 * g2) - 8 * g3 + 16 * (g1 + g2) * g3));
```

```
     g4 = (3 - Sqrt[5]) / 12; h4 = (5 + Sqrt[5]) / 12; p4 = (1 + Sqrt[5]) / 12;
     If[i < 5, info = {{g1, g1, g1, g1}, w1}; info[[1, i]] = 1 - 3 * g1];
     If[i > 4 && i < 9, info = {{g2, g2, g2, g2}, w2}; info[[1, i - 4]] = 1 - 3 * g2];
     If[i > 8 && i < 13, info = {{g3, g3, g3, g3}, w3};
       info[[1, i - 8]] = 1 - 3 * g3];
     If[i > 12, info = {{g4, g4, g4, g4}, 27 / 560};
       {j, k} = jk12[[i - 12]]; info[[1, j]] = h4; info[[1, k]] = p4]];
    info[[2]] = info[[2]] / 6;
    (* we include the division by 6 directly here *)
  If[numer, Return[N[info, 20]], Return[Simplify[info]]];];
```

Results

In[2]:=
```
ToRSTW[info_] := Module[
     {transformMat, sumAndRst, zetas, sum, r, s, t, w},
     {zetas, w} = info;
     transformMat = {
       {1, 1, 1, 1},
       {0, 1, 0, 0},
       {0, 0, 1, 0},
       {0, 0, 0, 1}
      };
     sumAndRst = transformMat.zetas;
     {sum, r, s, t} = sumAndRst;
     Return[{r, s, t, w}];];
```

Rule 1

In[3]:=
```
ToRSTW[TetrGaussRuleInfo[{1, False}, 1]]
```

Out[3]= $\left\{ \dfrac{1}{4}, \dfrac{1}{4}, \dfrac{1}{4}, \dfrac{1}{6} \right\}$

Rule 4

In[4]:=
```
Table[ToRSTW[TetrGaussRuleInfo[{4, True}, i]], {i, 4}]
```

Out[4]= {{0.13819660112501051518, 0.13819660112501051518,
     0.13819660112501051518, 0.041666666666666666667}, {0.58541019662496845446,
     0.13819660112501051518, 0.13819660112501051518, 0.041666666666666666667},
    {0.13819660112501051518, 0.58541019662496845446, 0.13819660112501051518,
     0.041666666666666666667}, {0.13819660112501051518,
     0.13819660112501051518, 0.58541019662496845446, 0.041666666666666666667}}

Rule 8

In[5]:= `Table[ToRSTW[TetrGaussRuleInfo[{8, True}, i]], {i, 8}]`

Out[5]= {{0.32805469671142664734, 0.32805469671142664734,
    0.32805469671142664734, 0.023087994418643690387}, {0.015835909865720057993,
    0.32805469671142664734, 0.32805469671142664734, 0.023087994418643690387},
   {0.32805469671142664734, 0.015835909865720057993, 0.32805469671142664734,
    0.023087994418643690387}, {0.32805469671142664734, 0.32805469671142664734,
    0.015835909865720057993, 0.023087994418643690387},
   {0.10695227393293068277, 0.10695227393293068277, 0.10695227393293068277,
    0.018578672248022976279}, {0.67914317820120795168,
    0.10695227393293068277, 0.10695227393293068277, 0.018578672248022976279},
   {0.10695227393293068277, 0.67914317820120795168, 0.10695227393293068277,
    0.018578672248022976279}, {0.10695227393293068277,
    0.10695227393293068277, 0.67914317820120795168, 0.018578672248022976279}}

Rule 14

In[6]:= `Table[ToRSTW[TetrGaussRuleInfo[{14, True}, i]], {i, 14}]`

Out[6]= {{0.092735250310891226402, 0.092735250310891226402,
    0.092735250310891226402, 0.012248840519393658257}, {0.72179424906732632079,
    0.092735250310891226402, 0.092735250310891226402, 0.012248840519393658257},
   {0.092735250310891226402, 0.72179424906732632079, 0.092735250310891226402,
    0.012248840519393658257}, {0.092735250310891226402,
    0.092735250310891226402, 0.72179424906732632079, 0.012248840519393658257},
   {0.31088591926330060980, 0.31088591926330060980, 0.31088591926330060980,
    0.018781320953002641800}, {0.067342242210098170608,
    0.31088591926330060980, 0.31088591926330060980, 0.018781320953002641800},
   {0.31088591926330060980, 0.067342242210098170608, 0.31088591926330060980,
    0.018781320953002641800}, {0.31088591926330060980, 0.31088591926330060980,
    0.067342242210098170608, 0.018781320953002641800},
   {0.045503704125649649492, 0.45449629587435035051, 0.45449629587435035051,
    0.0070910034628469110730}, {0.45449629587435035051,
    0.045503704125649649492, 0.45449629587435035051, 0.0070910034628469110730},
   {0.45449629587435035051, 0.45449629587435035051, 0.045503704125649649492,
    0.0070910034628469110730}, {0.045503704125649649492,
    0.045503704125649649492, 0.45449629587435035051, 0.0070910034628469110730},
   {0.045503704125649649492, 0.45449629587435035051, 0.045503704125649649492,
    0.0070910034628469110730}, {0.45449629587435035051,
    0.045503704125649649492, 0.045503704125649649492, 0.0070910034628469110730}}

Rule 15

In[7]:= **Table[ToRSTW[TetrGaussRuleInfo[{15, True}, i]], {i, 15}]**

Out[7]= {{0.091971078052723032789, 0.091971078052723032789,
  0.091971078052723032789, 0.011989513963169770002}, {0.72408676584183090163,
  0.091971078052723032789, 0.091971078052723032789, 0.011989513963169770002},
 {0.091971078052723032789, 0.72408676584183090163, 0.091971078052723032789,
  0.011989513963169770002}, {0.091971078052723032789,
  0.091971078052723032789, 0.72408676584183090163, 0.011989513963169770002},
 {0.31979362782962990839, 0.31979362782962990839, 0.31979362782962990839,
  0.011511367871045397547}, {0.040619116511110274837,
  0.31979362782962990839, 0.31979362782962990839, 0.011511367871045397547},
 {0.31979362782962990839, 0.040619116511110274837, 0.31979362782962990839,
  0.011511367871045397547}, {0.31979362782962990839, 0.31979362782962990839,
  0.040619116511110274837, 0.011511367871045397547},
 {0.44364916731037084426, 0.056350832689629155741, 0.056350832689629155741,
  0.0088183421516754850088}, {0.056350832689629155741,
  0.44364916731037084426, 0.056350832689629155741, 0.0088183421516754850088},
 {0.056350832689629155741, 0.056350832689629155741, 0.44364916731037084426,
  0.0088183421516754850088}, {0.44364916731037084426, 0.44364916731037084426,
  0.056350832689629155741, 0.0088183421516754850088},
 {0.44364916731037084426, 0.056350832689629155741, 0.44364916731037084426,
  0.0088183421516754850088}, {0.056350832689629155741, 0.44364916731037084426,
  0.44364916731037084426, 0.0088183421516754850088}, {0.25000000000000000000,
  0.25000000000000000000, 0.25000000000000000000, 0.019753086419753086420}}

Rule 24

In[8]:= **Table[ToRSTW[TetrGaussRuleInfo[{24, True}, i]], {i, 24}]**

Out[8]= {{0.21460287125915202929, 0.21460287125915202929,
    0.21460287125915202929, 0.0066537917096945820166}, {0.35619138622254391213,
    0.21460287125915202929, 0.21460287125915202929, 0.0066537917096945820166},
  {0.21460287125915202929, 0.35619138622254391213, 0.21460287125915202929,
    0.0066537917096945820166}, {0.21460287125915202929,
    0.21460287125915202929, 0.35619138622254391213, 0.0066537917096945820166},
  {0.040673958534611353116, 0.040673958534611353116, 0.040673958534611353116,
    0.0016795351758867738247}, {0.87797812439616594065, 0.040673958534611353116,
    0.040673958534611353116, 0.0016795351758867738247},
  {0.040673958534611353116, 0.87797812439616594065, 0.040673958534611353116,
    0.0016795351758867738247}, {0.040673958534611353116,
    0.040673958534611353116, 0.87797812439616594065, 0.0016795351758867738247},
  {0.32233789014227551034, 0.32233789014227551034, 0.32233789014227551034,
    0.0092261969239424536825}, {0.032986329573173468968,
    0.32233789014227551034, 0.32233789014227551034, 0.0092261969239424536825},
  {0.32233789014227551034, 0.032986329573173468968, 0.32233789014227551034,
    0.0092261969239424536825}, {0.32233789014227551034, 0.32233789014227551034,
    0.032986329573173468968, 0.0092261969239424536825},
  {0.26967233145831580803, 0.063661001875017525299, 0.063661001875017525299,
    0.0080357142857142857143}, {0.063661001875017525299,
    0.26967233145831580803, 0.063661001875017525299, 0.0080357142857142857143},
  {0.063661001875017525299, 0.063661001875017525299, 0.26967233145831580803,
    0.0080357142857142857143}, {0.60300566479164914137, 0.26967233145831580803,
    0.063661001875017525299, 0.0080357142857142857143},
  {0.60300566479164914137, 0.063661001875017525299, 0.26967233145831580803,
    0.0080357142857142857143}, {0.063661001875017525299,
    0.60300566479164914137, 0.26967233145831580803, 0.0080357142857142857143},
  {0.60300566479164914137, 0.063661001875017525299, 0.063661001875017525299,
    0.0080357142857142857143}, {0.063661001875017525299,
    0.60300566479164914137, 0.063661001875017525299, 0.0080357142857142857143},
  {0.063661001875017525299, 0.063661001875017525299, 0.60300566479164914137,
    0.0080357142857142857143}, {0.26967233145831580803, 0.60300566479164914137,
    0.063661001875017525299, 0.0080357142857142857143},
  {0.26967233145831580803, 0.063661001875017525299, 0.60300566479164914137,
    0.0080357142857142857143}, {0.063661001875017525299,
    0.26967233145831580803, 0.60300566479164914137, 0.0080357142857142857143}}